Explorations of Quantum Decoherence Phenomena
by
Grum Teklemariam
Submitted to the Department of Physics
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in Physics
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
May 2002
© Massachusetts Institute of Technology 2002. All rights reserved.

Author

Department of Physics
May 20, 2002

Certified by

David G. Cory
Professor
Thesis Supervisor

Certified by

Edward H. Farhi
Professor
Thesis Supervisor

Read by

Jeffrey Goldstone
Professor
Thesis Reader

Accepted by

Thomas J. Greytak
Associate Department Head for Education
Explorations of Quantum Decoherence Phenomena

by

Grum Teklemariam

Submitted to the Department of Physics
on May 20, 2002, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy in Physics

Abstract

This thesis describes the experimental exploration of quantum decoherence using
discrete and continuous-time decoherence maps.

The experimental methodology uses liquid-state nuclear magnetic resonance spec-
troscopy techniques. Initially, a brief discussion of coherent control methods is given.
Then, a detailed discussion of the decoherent control methods is presented. These
methods describe how strong measurements can be emulated in an ensemble system
by using pulsed magnetic field gradients, and how NMR decoupling techniques can
be used to implement partial trace operations.

Next, using quantum erasers we explore the stability of three-particle systems
under different entangling interactions. With a two-spin system we illustrate the
essential features of quantum erasers. The extension to three-spins allows us to use
the pair of orthogonal decoherent operations used in quantum erasers to probe the
two classes of entanglement in three-particle systems: the GHZ state and the W state.

Finally, we develop a decoherence model of a decohering two-level system coupled
to an environment with a few degrees of freedom. The couplings are of the $\sigma_z\sigma_z$
type and only induce coherence damping. By introducing a stochastic evolution on
the environment, the resulting randomization of the environment phases causes loss
of information over the environment degrees of freedom and decoheres the system.
Control parameters in the stochastic driving of the environment were used to vary
the rates of decoherence on the system, thereby allowing the establishment of a scaling
law that related control parameters to decay rates.

Thesis Supervisor: David G. Cory
Title: Professor

Thesis Supervisor: Edward H. Farhi
Title: Professor
Acknowledgements

The experience of a PhD program represents a significant portion of ones life. I came to graduate school after being away for 6 years and a failed business that took a big toll on me. The initial years were very tough going. The friendships I made in those days were extremely important. All the members of NCBPS were very crucial to my success the first two years. I am particularly indebted to Kimani Stancil, Manyalibo Matthews, Tehani Finch, Sandra Brown, Marta Dark, Jermaine Massey, and Joshua Ehrlich. During those years Peggy Berkowitz, in the Physics Education Office, was extremely helpful with administrative manners and I deeply appreciate her support throughout the years.

Having passed all my qualifiers I am very grateful to David Cory who invited me to join his group. The field of quantum information processing has been a fun and exciting field to be in. I also thank David for his patience when the initial going was very slow for me. David’s creative and fresh ideas have kept the group very busy. I think he is a great asset to the field of QIP. His mastery of NMR is unparalleled and I have learnt very much from him. I look forward to a long friendship and collaboration.

Once in the group I have to thank Shyamal Somaroo for being a gifted physicist and for the infinite patience he showed in teaching me the basics of geometric algebra and quantum computing. We became friends soon after and I have no doubt our friendship will last our lifetimes. I also want to thank Yehuda Sharf for teaching me experimental techniques in the early goings.

Moving along, the most fruitful collaboration I enjoyed is with Evan Fortunato. Along with working with me on almost all my projects we have become very close friends. I hope this friendship also lasts our lifetime and many more fruitful endeavors will result. Next in the list of people to thank is Marco Pravia. Marco and Evan were instrumental in implementing the GHZ eraser. I think the three of us will never forget the SQUINT experience. I want to thank them both for the excellent job that they did on this project and also for not giving up despite the extremely grueling hours it required to complete it in time to present at SQUINT.
Of my other lab mates I want to thank Nicolas Boulant, Greg Boutis, Joseph Emerson, Sekhar Ramanathan and the rest of Cory group.

Other folks that have influenced my research are Tim Havel, who initially suggested I work on the quantum eraser, Lorenza Viola and Juan-Pablo Paz who deserve my special thanks.

I also want to thank the many friends and mentors who were very supportive throughout the years. In particular, Jay Kirsch, Ellory Schempp, the late Richard Rzezdian, Jianyu Lian and John Zhang. I also want to thank Makeda (Lily) Wubneh for coming into life at just the right time and making these last months a lot of fun.

Lastly, I want to thank my family to whom I dedicate this thesis.
Contents

1 Introduction .................................................. 13

2 NMR Control Methods .......................................... 15
   2.1 Metrics of Control ........................................... 16
   2.2 Coherent Control Methods ................................. 18
   2.3 Decoherent Control Methods ............................... 20

3 Quantum Erasers and Probing Entanglement Complexity ...... 25
   3.1 The Quantum Eraser ......................................... 26
   3.2 Some Properties of Tripartite Entanglement .............. 28
   3.3 Three-particle Quantum Erasers ........................... 32
      3.3.1 The GHZ State .......................................... 32
      3.3.2 The W State ............................................ 33
   3.4 Experimental Demonstrations ............................... 34
      3.4.1 The Two-spin Quantum Eraser ......................... 36
      3.4.2 Three-particle Quantum Erasers ....................... 38
   3.5 Conclusions .................................................. 46

4 A “New” Decoherence Model .................................. 50
   4.1 Introduction ................................................ 51
   4.2 Quantum Decoherence–A Simple Case Study ............. 52
   4.3 The Model System .......................................... 57
      4.3.1 Results ............................................... 62
4.3.2 Simulations ........................................ 63
4.4 The Implementation on a Physical System ............ 68
4.5 Discussion ............................................ 72
List of Figures

3-1  Proposed QIP application for the quantum eraser. Two qubits, A and B, which are part of a quantum communication network are entangled with arbitrary complex coefficients $a$ and $b$. This shared information can be transferred to either qubit by local operations if both qubits are accessible. If the qubits interact weakly or they are locally inaccessible, the information can still be transferred to either qubit by using the quantum eraser scheme. If it is required that the data be transferred to qubit B, this can be accomplished by a $\sigma_z^A$ measurement on qubit A projecting qubit B into the state $a|0\rangle_B \pm b|1\rangle_B$, where the sign depends on the measurement outcome of qubit A, $|\pm\rangle_A$ respectively. Furthermore, the state of qubit A subsequent to the measurement can be treated as an 'error syndrome' to correct the relative sign of qubit A by communicating this information via a classical channel. Once the two qubits get entangled the network is symmetric between them and the data can be equivalently be transferred to qubit A by switching the operations between the two qubits. ........................................ 29

3-2  The alanine molecule. The encircled labels on the $^{13}$Cs index the spins used in the experiment. ......................................................... 35
3-3 Logic network for the two-spin quantum eraser. Initially, a pseudo-pure state on spins 1 and 2 is created, $\rho_{PP} = |00\rangle\langle 00| \equiv E_1^4 E_2^3$. A $\frac{\pi}{2}$ y-pulse is then applied to spin 2, followed by a controlled-not (c-NOT) gate to create the Bell state (see text). Conditionality on the second spin being in the $|1\rangle$ state is represented in the network by a filled circle on its time line. Finally, one of the two complementary measurements, $\sigma_z^1$ or $\sigma_z^1$, are applied to spin 1. State tomography was performed to fully reconstruct the density matrices at the positions indicated.

3-4 Experimental density matrices reconstructed by tomography (in normalized units). The rows are enumerated in the standard computational basis, where for example 00 represents the state label $|00\rangle$. Although not shown, the columns are similarly labeled with the leftmost end representing $|00\rangle$ and the rightmost end representing $|11\rangle$. $\rho_{PP}$ is the three-spin pseudo-pure ground state, and $\rho_{BELL}$ is the pseudo-pure Bell state. The last two plots are $\rho_z$, which is $\rho_{BELL}$ after decohering spin 1 about the z-axis, and $\rho_x$, which is after decohering it about the x-axis. An amount of identity, chosen to optimize the input projection, was added to all experimentally measured density matrices.

3-5 Logic network for the disentanglement eraser. Initially, a pseudo-pure state on spins 1, 2 and 3 is created, $\rho_{ini} = |000\rangle\langle 000| \equiv E_1^4 E_2^3 E_3^2$. A $\frac{\pi}{2}$ y-pulse is then applied to spin 2, followed by two controlled-not (c-NOT) gates to create the GHZ state (see text). Conditionality on the second spin being in the $|1\rangle$ state is represented in the network by a filled circle on its time line. Finally, the two complementary measurements, $\sigma_z^1$ and $\sigma_z^1$, are applied to spin 1. State tomography was performed to fully reconstruct the density matrices at the positions indicated.
Experimental density matrices reconstructed by tomography (in normalized units). The rows and columns are enumerated in the standard computational basis as described in Fig. 4. $\rho_{ini}$ is the three-spin pseudo-pure ground state, and $\rho_{GHZ}$ is the pseudo-pure GHZ state. The last two plots are $\rho_z$, which is $\rho_{GHZ}$ after decohering spin 1 about the $z$-axis, and $\rho_x$, which is after decohering it about the $x$-axis. (Note: $\rho_{GHZ}$, $\rho_z$ and $\rho_x$ have been magnified by a factor of two for clarity). An amount of identity, chosen to optimize the input projection, was added to all experimentally measured density matrices.

The reconstructed conditional Bell and traced density matrices (in normalized units). The diagonal components have been shifted so unit trace is preserved. The top plot is the conditional Bell state (conditioned on the state of the ancilla qubit), while the bottom plot is the traced density matrix after applying the decoupling pulses to the ancilla qubit during the observation period.

Logic network for the creation of $\bar{W}$ state followed by complementary measurements. Starting with $\rho_{ini}$ the sequence of transformations shown creates $\rho_{\bar{W}}$. Subsequently, the set of complementary measurements described in the text are applied to spin 1 in the state $\rho_{\bar{W}}$.

Theoretical density matrices in a) and experimental density matrices in b) reconstructed by tomography (in normalized units). The rows and columns are enumerated in the standard computational basis as described in Fig. 4. In b) $\rho_{ini}$ is the three-spin pseudo-pure ground state, and $\rho_{\bar{W}}$ is the pseudo-pure $\bar{W}$ state. The last two plots are $\rho_z$, which is $\rho_{\bar{W}}$ after decohering spin 1 about the $z$-axis, and $\rho_x$, which is after decohering it about the $x$-axis. An amount of identity, chosen to optimize the input projection, was added to all experimentally measured density matrices. The plots in a) are the corresponding theoretical density matrices for comparison purposes.
4-1 A cartoon of the system, local quantum environment and the non-local classical environment. Note that the non-local environment only interacts with the local environment and not the system. 64

4-2 Some example simulated decays obtained on Matlab. The plots are averages over 50 realizations of the quantity \( \{ \rho_S^i \}_{01} = \langle 0 | T_{r_{LL}L} [\mathcal{U}_j \rho_{SLL}^{j-1} \mathcal{U}_j^{-1}] | 1 \rangle \) from Eq. 4.64. The kick rates on each plot are labeled and we note that the higher the kick frequency the faster the system decays. 66

4-3 The dependence of the decay constants on kick rate. This linear response can be simply modeled as discussed in the text. In particular, consult the discussions accompanying Section 4.3.1. 67

4-4 The decoupling limit. Beyond a kick frequency of about 900 kicks/ms the kicking is ineffective at inducing decoherence. After about 5000 kicks/ms the decays are no longer exponential. 69

4-5 The propyne molecule. The encircled labels on the \(^{13}\)Cs and the right-most hydrogen index the spins used in the experiment. The methyl group consists of the three hydrogens and an unlabeled carbon. In the experiments the field of the spectrometer was \( \sim 9.2 \) T and the hydrogen resonances were \( \sim 400 \) MHz while the carbon resonances were around \( \sim 100 \) MHz. The chemical shift difference between the two labeled carbons is 1.6 kHz. Using the indexing scheme in the figure the \( J \)-coupling constants are as follows: \( J_{12} = 246.5 \) Hz, \( J_{23} = 173.8 \) Hz and \( J_{13} = 51.8 \) Hz. The longitudinal relaxation times are \( T_1^1 = 8.7s \), \( T_1^2 = 23s \) and \( T_1^3 = 43s \), while the transverse relaxation times are \( T_2^1 = 1.1s \), \( T_2^2 = 1.9s \) and \( T_2^3 = 1.7s \). 70

4-6 Example decays from the experiment. The hydrogen signal was directly detected and the magnitude and real part of the complex signal is plotted. The fluctuations at the tail end of the higher kick rates are due to low statistics. This was confirmed with simulations at much higher averages. 73
4-7  The dependence of the experimental decay constants on kick rate. The error bars are small and are on the order of the size of the data points. Compared to the slope in the simulations of Fig. 4-3 the experimental one appears to decay faster. This disparity is due to a difference between the experiment and simulations. In the simulations, the pulse on time is infinitely narrow whereas in the experiments it is finite.  

4-8  A simulation to check revivals at longer times and higher averages. The times go out to 500ms and the averages are taken for 200 realizations. Note that the revivals that seem prevalent in Figs. 4-2 and 4-6 are diminished.
List of Tables

3.1 This table shows the transformations used to obtain the three-spin pseudo-pure state from the thermal equilibrium state, using geometric algebra notation [14, 32]. .................................................. 36

4.1 This table lists the parameters for the model Hamiltonian of Eqs. 4.57-4.61. ................................................................. 65
Chapter 1

Introduction

The field of quantum information processing (QIP) [1, 2, 3] has provided a new paradigm for investigating fundamental topics such as complexity theory, entanglement and decoherence. This new paradigm also holds the promise of computationally efficient devices such as a quantum computer that can execute algorithms exponentially fast compared to their classical analogs. One of the major obstacles to building a quantum computer is the effect of decoherence. Understanding decoherence and protecting the quantum computer against noise has become of paramount importance in QIP.

The process of quantum decoherence has two distinct steps. Initially, the system of interest gets entangled to a quantum apparatus. In the second stage of this process, the interference (or coherence) term of this joint state of the system and apparatus decays to zero with time. The state of the system is now in a classical, statistical mixture of possible outcomes. In this thesis we investigate these aspects of the decoherence program through the study of discrete and continuous-time decoherence maps. An example of a system that undergoes a discrete decoherent evolution is one where a system S, prepared in a superposition state, gets entangled to a measuring apparatus M, by a CNOT [4] interaction. Subsequently, the system density matrix can be recovered by a partial trace over the measuring apparatus M, resulting in a mixed state for the system S. Thus, the system will have undergone non-unitary dynamics.
With this example in mind, we experimentally investigate the following questions in this thesis:

1) How does the system S behave under measurements and partial traces of the apparatus M?

2) Under different entangling interactions, how much of its quantum nature does the system S retain following measurements, or partial traces of the apparatus M?

3) How do the coherences of the system S decay with time under a continuous-time decoherence map?

Our focus throughout this thesis was the practical considerations of physically realizable decoherence maps to understand more about the stability of quantum information in realistic settings. In chapter 2, we present the experimental background for these investigations. In chapter 3, we describe the quantum eraser effect and use it to probe entanglement complexity in three-spin systems which have undergone a discrete decoherent evolution. Finally, in chapter 4, our focus shifts from discrete to continuous-time maps. Here, we propose a "new" decoherence model that is practical and physically realizable in order to study how systems, in a continuous manner, loose their quantum nature due to interactions with their environment.
Chapter 2

NMR Control Methods

The recent development of precise coherent [7] and decoherent [8] control methods makes liquid-state NMR a particularly good test bed for exploring quantum effects. Liquid-state NMR implementations of QIP typically rely on chemically distinct spin-1/2 nuclei for qubits [3]. In a liquid NMR sample, the correlations between the spin states of nuclei in different molecules is averaged to zero by molecular motion, and hence we focus on the intramolecular correlations. For sensitivity purposes an ensemble of $\sim 10^{18}$ molecules is typically used. Therefore, a liquid-state NMR QIP has $\sim 10^{18}$ quantum processors, and a density matrix is needed to describe the statistics of measurements on this system.

When the spins are in thermal equilibrium their density matrix is given by the Boltzmann operator. In the high temperature approximation this is given by

$$\rho_{eq} = \frac{e^{-\beta \mathcal{H}}}{Z} \approx \frac{1}{2N} I + \frac{1}{2N} \beta \sum_{i=1}^{3} \gamma_i \sigma_i^z,$$  \hspace{1cm} (2.1)

where $N$ is the number of spins, $\beta \gamma_i \approx 10^{-5}$ at room temperatures [9], $\gamma_i$ is the gyromagnetic ratio of each nucleus and $\sigma_z$ is one of the Pauli matrices $\{I, \sigma_x, \sigma_y, \sigma_z\}$, $(I$ is the identity matrix). The density matrices of these systems is highly mixed, but

---

1This chapter is drawn from sections of [5, 6, 7]. The coherent control methods were primarily developed by E. M. Fortunato and M. A. Pravia.
can be transformed into a pseudo-pure state [10] having the form

$$\rho = \frac{(1-\epsilon)}{2N} I + \epsilon |\psi\rangle\langle\psi|,$$  \hspace{1cm} (2.2)

where $\epsilon = \frac{\beta N}{2N}$ for a homonuclear system. Under unitary transformations $\rho' = U\rho U^{-1}$, the identity part transforms trivially while the underlying state vector $|\psi\rangle$ transforms one-sidedly to $U|\psi\rangle$, exactly as it would for a true pure state.

In this chapter, we first establish a metric for quantifying how well a unitary transformation generates a desired end state. Next, we briefly discuss the coherent control techniques we used. Finally, we present the decoherent control techniques that were required for the implementation of the various quantum erasers in chapter 3.

### 2.1 Metrics of Control

In the standard model of quantum computing, an algorithm can be expressed as a series of unitary operations that maps a set of input states to a particular set of output states. A metric should describe the quality of a general transformation, including the possibility of non-unitary evolution. For an input state, $\rho_{in}$, the ideal transformation maps the system to a theoretical output state, $\rho_{th}$, i.e.,

$$\rho_{in} \xrightarrow{U_{th}} \rho_{th}. \hspace{1cm} (2.3)$$

On the other hand, an experimentally implemented control sequence will produce a different output state, $\rho_{exp}$, i.e.,

$$\rho_{in} \xrightarrow{U_{exp}} \rho_{exp}. \hspace{1cm} (2.4)$$

Noting that $\rho$ is Hermitian, the projection between these two states, defined as

$$P(\rho_{th}, \rho_{exp}) = \frac{\text{trace}(\rho_{th} \rho_{exp})}{\sqrt{\text{trace}(\rho_{th}^2) \text{trace}(\rho_{exp}^2)}}, \hspace{1cm} (2.5)$$

16
quantifies how similar in 'direction' the two states are. This metric is analogous to the dot product between two vectors, varying from $-1$ for anti-parallel states to 1 for identical states. A value of zero indicates orthogonal density matrices. In order to account for non-unitary evolution, a second term multiplies the projection yielding the attenuated correlation, namely,

$$C(\rho_{th}, \rho_{exp}) = P(\rho_{th}, \rho_{exp}) \sqrt{\frac{\text{trace}(\rho_{exp}^2)}{\text{trace}(\rho_{th}^2)}}$$

$$= \frac{\text{trace}(\rho_{th} \rho_{exp})}{\sqrt{\text{trace}(\rho_{th}^2)\text{trace}(\rho_{th}^2)}}. \quad (2.6)$$

Furthermore, since

$$\rho_{th} = U_{th}\rho_{in}U_{th}^{-1}, \quad (2.7)$$

then

$$\rho_{in} = U_{th}^{-1}\rho_{th}U_{th}. \quad (2.8)$$

Substituting this form for $\rho_{in}$ in Eq. 2.6 yields

$$C(\rho_{th}, \rho_{exp}) = \frac{\text{trace}(\rho_{th} \rho_{exp})}{\sqrt{\text{trace}(\rho_{th}^2)\text{trace}(\rho_{th}^2)}}, \quad (2.9)$$

upon recognizing that

$$\text{tr}(\rho_{in}^2) = \text{tr}(U_{th}^{-1}\rho_{th}U_{th}\rho_{th}U_{th}) = \text{tr}(U_{th}^{-1}\rho_{th}^2U_{th}) = \text{tr}(U_{th}\rho_{th}^{-1}\rho_{th}^2) = \text{tr}(\rho_{th}^2). \quad (2.10)$$

A cautionary note: $\rho_{in}$, the initial state, is the experimentally prepared pseudo-pure ground state which serves as the input state to both the theoretical and experimental transformations. Finally, we define the fidelity, $F$, of a transformation

$$F = \overline{C}(\rho_{th}, \rho_{out}), \quad (2.11)$$

where $\overline{C}$ represents the average attenuated correlation over an orthonormal set of input density operators (i.e., $\text{Trace}[\rho_j \rho_k] = \delta_{jk}$) that span the Hilbert space. (Although
any orthonormal set can be used, a good choice is a set corresponding to the basis spanning the system Hamiltonian). It should be noted that $F$ is maximized (with a value of one) when the implemented and ideal transformations are the same, and is insensitive to differences in the global phase between the ideal and implemented transformation.

### 2.2 Coherent Control Methods

In NMR, spins in a large static magnetic field (in our case, 7 T) are controlled via external RF pulses. The internal spin Hamiltonian is composed of both Zeeman interactions with the applied field modified by electron screening (chemical shift) and scalar couplings with other spins. Together these provide the QIP requirements of addressability and conditional logic respectively. In terms of spin operators, the internal Hamiltonian is

$$H_{\text{int}} = \sum_{k=1}^{n} \nu_k \sigma_z^k + \sum_{j>k}^{n} \sum_{k=1}^{n} J_{kj} \sigma_z^k \cdot \sigma_j^j,$$  \hspace{1cm} (2.12)

where $\nu_k$ represent the chemical shifts of the spins, $J_{kj}$ the coupling constant between spins $k$ and $j$, and $n$ is the number of spins. The external Hamiltonian describing the coupling between the spins and an oscillating RF field generated by a single transmitter is

$$H_{\text{ext}}(\omega_{RF}, \phi, \omega, t) = \sum_{k=1}^{n} e^{-i(\omega_{RF} t + \phi)\sigma_z^k} (-\omega \sigma_z^k) e^{i(\omega_{RF} t + \phi)\sigma_z^k},$$  \hspace{1cm} (2.13)

where $\omega_{RF}$ is the transmitter’s angular frequency, $\phi$ the initial phase, and $\omega$ the amplitude [11]. Of course, additional species can be added by including appropriate terms in $H_{\text{int}}$ and an additional $H_{\text{ext}}$ for each additional RF field.

Using this knowledge of the internal Hamiltonian and the form of the external Hamiltonian, the parameter values that generate the desired gate must be determined. Here, a quality factor $Q = 1 - \sqrt{F}$ is minimized by searching through the mathematical parameter space using the Nelder-Mead Simplex algorithm [12]. While
this function has many local minima, the Simplex algorithm often succeeds in finding satisfactory solutions. Our goal is to show that sufficient, implementable control sequences can be found. Finding the optimal solution is much more challenging and based on our system and control parameter values, is not expected to improve pulse performance significantly. We have parameterized the control sequence as a cascade of RF pulses with fixed power, transmitter frequency, initial phase, and pulse duration ($\tau$). As will be seen, this is a particularly convenient and completely general parameterization, but we make no claims that it is the only, nor necessarily the best choice. If the RF power is constant over the duration of a pulse, i.e., the pulse’s amplitude is square, the total Hamiltonian $H_{tot} = H_{int} + H_{ext}$ can be made time independent by transforming into the frame that rotates at the frequency of the transmitter. This allows the Liouville-von Neumann equation of motion to be solved by a single diagonalization. Initially, the starting density matrix is the same in both frames ($\hat{\rho}(0) = \rho(0)$), so that at the end of the pulse, the density matrix in the new frame is given by

$$\hat{\rho}(\tau) = e^{-iH_{eff}\tau}\rho(0)e^{iH_{eff}\tau},$$

(2.14)

where $H_{eff}$ is the effective Hamiltonian in the new frame [13]. Transforming this density matrix back to the original rotating frame gives

$$\rho = U_z(\tau)^{-1}e^{-iH_{eff}\tau}\rho(0)e^{iH_{eff}\tau}U_z(\tau),$$

(2.15)

where

$$U_z(\tau) = \left(e^{i\omega RF \sum_{k=1}^n \sigma^k_\tau}\right).$$

(2.16)

Therefore, in the original rotating frame, the transformation is given by

$$U_{period}(\tau) = U_z^{-1}(\tau)e^{-iH_{eff}\tau}.$$

(2.17)

Because the evolution under the whole sequence is given in the original rotating frame, no additional resources are required to concatenate pulses, nor is any mathematical correction required at the end of an experiment.
Cascading these periods yields the net transformation

\[ U_{net} = \prod_{m=1}^{N} U_z^{-1}(\tau_m)e^{-iH_{eff}^m(\omega^m,\omega^m_R,\phi^m)\tau_m} \]

(2.18)

where the index \( m \) refers to the \( m^{th} \) period, i.e., to the \( m^{th} \) square pulse, with a corresponding set of 4 parameters. In other words, \( N \) constant amplitude pulse periods, each with a different transmitter frequency and initial phase, are applied in series. Clearly, a single period is not sufficient to generate an arbitrary transformation; therefore the number of periods is increased until a suitable net transformation is found. Using desktop computing resources, this yields convergence times for three- and four-spin systems that are typically seconds to minutes.

In addition to the desired propagator, \( U_{ideal} \), an initial set of starting parameters for the pulse shape is required. While this initial guess must be reasonable (i.e., in the vicinity of the solution), many different starting points typically converge to equally deep minima. We have observed that the number of acceptable solutions for this parameterization is very large, allowing experimental implementation issues to be considered. For example, experimental limitations do not allow arbitrarily high powers or frequencies to be implemented. To keep the algorithm from returning infeasible solutions, a penalty function that increases as the parameter value moves towards infeasible solutions is added to the quality factor. Penalty functions are also used to guide the algorithm towards more favorable pulse solutions. In our case, penalties are placed on high powers, large frequencies, and negative- or long-time periods.

### 2.3 Decoherent Control Methods

In NMR, the signal detected is a spatial ensemble average given by the trace of the product of the transverse magnetic field components with the density matrix:

\[ <M_x(t) + iM_y(t)> = \gamma Tr\{ (\sigma_x + i\sigma_y) \rho(t) \}, \]

(2.19)
where $\rho(t) = \mathcal{U}_{int}(t)\mathcal{U}_{int}^{-1}$, $\mathcal{U}_{int} = e^{-i\mathcal{H}_{int}t}$, $\mathcal{H}_{int}$ is the internal Hamiltonian of the spins, and $\rho(t_0)$ is the state of the spin system right before detection starts. Alternatively, a partial trace operation can be implemented by NMR decoupling techniques. These techniques are normally used to simplify spectra of multiple spins, by removing the splittings caused by the decoupled spin. The $z$ component of the spin to be decoupled is rapidly modulated with respect to the timescale of the inverse coupling constant. This ensures that the decoupled spin has no effect upon the measurements and so implements a partial trace operation.

The effect of decoupling on the density matrix can be seen by first writing the density matrix of the spin system in terms of the subspaces of spin $i$, or the single element operators of spin $i$

$$
\rho = \begin{bmatrix}
|0\rangle_i\langle 0| \otimes \rho_{1\ldots i\ldots N}^{11} & |0\rangle_i\langle 1| \otimes \rho_{1\ldots i\ldots N}^{12} \\
|1\rangle_i\langle 0| \otimes \rho_{1\ldots i\ldots N}^{21} & |1\rangle_i\langle 1| \otimes \rho_{1\ldots i\ldots N}^{22}
\end{bmatrix},
$$

(2.20)

where $\hat{i}$ indicates the absence of spin $i$. The subspaces of spin $i$ can also be represented by the transition operators which are written here in terms of the geometric algebra (GA) notation [14]:

$$
|0\rangle_i\langle 0| = E^i_+ = \frac{1}{2}(I + \sigma^i_z),
$$

(2.21)

$$
|0\rangle_i\langle 1| = E^i_+\sigma^i_z = \frac{1}{2}(I + \sigma^i_z)\sigma^i_z,
$$

(2.22)

$$
|1\rangle_i\langle 0| = \sigma^i_z E^i_+ = \sigma^i_z\frac{1}{2}(I + \sigma^i_z),
$$

(2.23)

$$
|1\rangle_i\langle 1| = E^i_- = \frac{1}{2}(I - \sigma^i_z).
$$

(2.24)

Decoupling renders $\sigma^i_z$ and $\sigma^i_z$ unobservable thus reducing $E^i_\pm$ to the identity term $I$.

The density matrix then becomes

$$
\rho_r = \rho^{11}_{1\ldots \hat{i}\ldots N} + \rho^{22}_{1\ldots \hat{i}\ldots N},
$$

(2.25)
and the effective Hilbert space is reduced to that of the undecoupled spins. This may be compared with the mathematical definition of the partial trace over qubit $i$:

$$Tr_i\{\rho\} = \sum_k i\langle k|\rho|k\rangle_i = \rho^0_{i,1} + \rho^0_{i,2} + \rho^1_{1,i} + \rho^1_{2,i} + \rho^2_{1,i} + \rho^2_{2,i} + \cdots + \rho^N_{1,i} + \rho^N_{2,i}.$$  (2.26)

The implementation by NMR techniques proceeds as follows. Consider $N$ spins whose interaction Hamiltonian, in the weak coupling limit, is [9]

$$\mathcal{H}_{\text{int}} = \sum_{k=1}^{N} \nu_k \sigma_z^k + \sum_{j>k}^{N} \sum_{k=1}^{N} J_{kj} \sigma_z^k \sigma_z^j,$$  (2.27)

where $\nu_k$ is a chemical shift constant while $J_{kj}$ is a coupling constant. This system will exhibit $N2^{N-1}$ spectral lines (corresponding to transitions between energy levels). To remove the splittings, decoupling techniques are used to selectively "turn off" the interaction between the spins. This decoupling Hamiltonian [15] takes the form

$$\mathcal{H}_{\text{dec}} \propto \nu_{\text{RF}} \sigma_z^i,$$  (2.28)

where $\nu_{\text{RF}} >> J_{ij}$ for all $j$. $\mathcal{H}_{\text{dec}}$ is applied continuously during the signal observation period. Consequently, spin $i$ will nutate about the x-axis, and all terms of the density matrix involving spin $i$, such as $\sigma_z^i$, $\sigma_z^i \sigma_z^i$, $\sigma_+^i \sigma_-^i \cdots \sigma_+^i \cdots \sigma_-^i$, etc., will be rendered unobservable. More generally, the internal Hamiltonian is transformed to

$$\tilde{\mathcal{H}}_{\text{int}} = U_{\text{dec}} \mathcal{H}_{\text{int}} U_{\text{dec}}^\dagger = \sum_{k=1}^{N} \nu_k \sigma_z^k + \sum_{j>k}^{N} \sum_{k=1}^{N} J_{kj} \sigma_z^k \sigma_z^j.$$  (2.29)

Therefore, measurements of the spin system during this period yields no information about spin $i$, implementing the partial trace over the decoupled spin.

Another experimental means of implementing non-unitary operations is the use of linear magnetic field gradients. By taking advantage of the spatial extent of an
NMR sample, a magnetic field gradient, \( \nabla = \partial B_z / \partial z \), applied parallel to the static field direction, causes the spins' Larmor precession frequency to vary linearly with their spatial coordinates. This results in evolution under the Hamiltonian

\[
H_{\text{grad}} = \gamma \nabla z \frac{1}{2} \sum_j \sigma_j^z,
\]

(2.30)

according to the Liouville von Neumann equation

\[
i \frac{\partial \rho}{\partial t} = [\rho, \mathcal{H}].
\]

(2.31)

Subsequently, each coherence \( \rho_{k\ell} \) \((k \neq \ell)\) is multiplied by a spatially dependent phase \( \exp(-i \gamma m_{k\ell} \nabla z t / 2) \), where \( m_{k\ell} \) is the coherence order \([8, 9, 16]\) (i.e. the difference in the \( z \)-component of the angular momentum between the \( |k\rangle \) and \( |\ell\rangle \) states in units of \( h \)). After such a gradient pulse the density matrix, averaged over the sample volume, results in an incoherent sum, and satisfies \( \rho_{k\ell} = 0 \) for all \( k \neq \ell \) except for the zero quantum coherences \( m_{k\ell} = 0 \). This spatially dependent phase is reversible provided the spins have not changed location. If diffusion is allowed to take place, this incoherent evolution is rendered decoherent because the motion of the spins destroys the correlation between the phase and location.

In this study, linear magnetic field gradients were used to dephase the coherences of \( \rho \), to wipe out magnetization during the experiments, and as also happens in strong measurements of \( \sigma_z \) \([17]\). This dephasing operation was made spin specific by applying a series of \( \pi \) pulses to the other spins, interleaved by gradient pulses of the same amplitude and duration, to “refocus” all the other coherences and at the same time also refocus the evolution under the internal Hamiltonian. Such refocusing schemes are well known in NMR \([9]\) and apply here by recognizing that evolution under \( H_{\text{grad}} \) is similar to evolution under a chemical shift interaction term. In particular, a refocusing scheme that scales polynomially, in the number of time periods and \( \pi \) pulses as the number of spins grows, has been proposed in \([18]\).

To “measure” along any axis \((j)\), the spins frame is rotated such that \( j \) is now
along $z$. Then, the spins are subjected to a selective evolution under $\mathcal{H}_{\text{grad}}$. Finally, the frame is rotated back by the inverse of the rotation that took the spins from $j$ to $z$. 
Chapter 3

Quantum Erasers and Probing Entanglement Complexity

In this chapter we use the quantum eraser effect to study examples of discrete decoherence maps. Initially, we describe the quantum eraser effect, and then we use it to test the stability of three-particle entangled states. By stability we mean the degree to which one subsystem retains its quantum nature under decoherent operations on another.

The quantum eraser is one of the most intriguing effects in quantum mechanics. It was initially devised by Scully et. al. [19] in analyzing the quantum version of Young’s double slit experiment. The quantum eraser describes the loss or gain of interference or, more generally quantum information, in a subensemble, based on the measurement outcomes of two complementary observables. Thus, this is a way of testing how measurements on one system affects the other when they are mutually entangled. More recently, quantum erasers have undergone a revival because they serve to provide various measures of entanglement.

Quantum entanglement plays a critical role in decoherence. In many QIP applications, entanglement is central to the transfer and sharing of information. While two-particle entanglement is well classified, for three or more particles the situation is

\footnote{This chapter is drawn from sections of [5, 6] with contributions by E. M. Fortunato and M. A. Pravia.}
more complex. For example, while there are two classes of entanglement for three particles [20], there are nine types of entanglement for four particles [21]. Consequently, there has been a significant effort directed at quantifying these different types of entanglement and many metrics have been proposed [22, 23, 24, 25, 26]. In particular, a classification strategy, based on local operations assisted by classical communication (LOCC) [20], has been established.

In this chapter we study the extension of the quantum eraser to three qubits, thus enabling quantum information stored in a pair of qubits mutually entangled with a third to be manipulated. We use complementary measurements (i.e. a pair of orthogonal measurements) as a probe of entanglement classes in the case of three qubits. By this method we contrast the relative stability of the GHZ state with the W state, which differs from the GHZ state in the entanglement destroyed upon tracing over any single qubit.

3.1 The Quantum Eraser

Two-particle quantum erasers have been extensively discussed in the literature [19, 27, 28, 29]. The traditional quantum eraser effect can be demonstrated by considering a spin-1/2 particle placed in a magnetic field pointing along the z-axis. If we call this particle spin A, then the states corresponding to the two possible orientations can be designated as $|0\rangle_A$ and $|1\rangle_A$, which represent spin-up and spin-down, respectively. When a superposition state of $|0\rangle_A$ and $|1\rangle_A$ is formed the relative phase between the two eigenstates has observable consequences. For example, the states

$$|\pm\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle_A \pm |1\rangle_A)$$

form a superposition of $|0\rangle_A$ and $|1\rangle_A$ where the relative phase distinguishes between $|+\rangle_A$ and $|-\rangle_A$, which are the states of spin A pointing in the positive and negative x-axis, respectively. By contrast, a mixed state of spin A, for example the state of spin A described by the density matrix $\rho_A = \frac{1}{2}I^A$, has no observable relative phase.
information. In fact, the density matrix $\rho_A$ is what would result if spin A was prepared in the states $|+\rangle_A$ or $|−\rangle_A$ followed by a measurement along the z-axis.

This measurement process can be emulated by considering a second spin B that gets entangled to spin A in the state $|+\rangle_A$. If spin B is initialized in the state $|0\rangle_B$, then an entangling interaction can be effected by a controlled-NOT (c-NOT) gate given as:

$$C_A\text{NOT}_B = |0\rangle_A\langle 0| \otimes I^B + |1\rangle_A\langle 1| \otimes \sigma_x^B.$$  

This operation is an example of a discrete map because it has no time dependence. The result of this gate on the two spinors is

$$|\Psi_{AB}\rangle = C_A\text{NOT}_B |+\rangle_A|0\rangle_B = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B).$$  

From this combined state of the two spins the state of spin A is obtained by tracing over spin B,

$$\rho_A = Tr_B |\Psi_{AB}\rangle\langle \Psi_{AB}| = \frac{1}{2} I^A,$$

which yields the reduced density matrix of spin A. Therefore, entangling spin B with spin A destroys the superposition, or coherence, of spin A. The loss of coherence in spin A is a direct consequence of spin B getting entangled to spin A, because this means that it is possible to determine the state of spin A by suitable measurements on spin B. Even if we don’t measure spin B to find out the state of spin A, the mere fact that this is possible in principle decoheres spin A. Another way to view this situation is that spin B has gained ”which-way” information about the state of spin A irrespective of whether we choose to find out this information or not.

The quantum eraser prescribes a procedure by which this which-way information stored in spin B can be “erased”. It proceeds as follows. Starting with $|\Psi_{AB}\rangle$ we express the spin states of spin B in the x-basis:

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}|0\rangle_A \frac{1}{\sqrt{2}}(|+\rangle_B + |−\rangle_B) + |1\rangle_A \frac{1}{\sqrt{2}}(|+\rangle_B − |−\rangle_B)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)\frac{1}{\sqrt{2}}|+\rangle_B + \frac{1}{\sqrt{2}}(|0\rangle_A − |1\rangle_A)\frac{1}{\sqrt{2}}|−\rangle_B.$$  

27
It is now clear that $\sigma_z^B$ measurements on spin B will restore the coherence of spin A as either $|+\rangle_A$ or $|-\rangle_A$, depending on the measurement outcomes of spin B. Thus, entangling spins A and B creates a "measurement-like" scenario in which the coherence of spin A is lost. But the coherence of spin A can be recovered by measuring spin B along a complementary axis, whereby the information stored in spin B is erased. Consequently, the maximally mixed state of spin A, $\rho_A = \frac{1}{2}I^A$, has been purified by this procedure!

When the quantum eraser was initially devised it was used to demonstrate the principle of complementarity without invoking the underlying Heisenberg's uncertainty relations. In the example just discussed it is the non-commutativity of the operators $\sigma_z^B$ and $\sigma_z^B$ that was demonstrated. Spin A played the role of a probe spin in which the effects of a $\sigma_z^B$, or $\sigma_z^B$ measurement on spin B resulted in completely different outcomes for spin A. In this regard, it is also very interesting to note that once the joint state $|\Psi_{AB}\rangle$ was formed the state of spin A was being manipulated without ever interacting with it.

The quantum eraser scheme may find applications in communications networks that require the transfer of information amongst qubits that interact weakly, or where one qubit maybe locally inaccessible. Fig. 3-1 shows a network for two qubits that may be part of some communications network. Because the network is symmetric for the two qubits the information can equally be transferred to qubit A by performing an orthogonal measurement on qubit B. Furthermore, similar to quantum teleportation and error correction schemes, the measurement outcomes on the decohered qubit can be treated as an error syndrome, and this information can be used to recover the relative phase between the states of the data qubit. This application clearly illustrates that the "quantum eraser scheme" is an example of LOCC.

3.2 Some Properties of Tripartite Entanglement

Entanglement, which has been related to the non-local properties of a quantum state, is a characteristic feature of quantum systems. Presently, QIP research is revealing
Figure 3-1: Proposed QIP application for the quantum eraser. Two qubits, A and B, which are part of a quantum communication network are entangled with arbitrary complex coefficients $a$ and $b$. This shared information can be transferred to either qubit by local operations if both qubits are accessible. If the qubits interact weakly or they are locally inaccessible, the information can still be transferred to either qubit by using the quantum eraser scheme. If it is required that the data be transferred to qubit B, this can be accomplished by a $\sigma_z^A$ measurement on qubit A projecting qubit B into the state $a|0\rangle_B \pm b|1\rangle_B$, where the sign depends on the measurement outcome of qubit A, $|\pm\rangle_A$ respectively. Furthermore, the state of qubit A subsequent to the measurement can be treated as an 'error syndrome' to correct the relative sign of qubit A by communicating this information via a classical channel. Once the two qubits get entangled the network is symmetric between them and the data can be equivalently be transferred to qubit A by switching the operations between the two qubits.
that multipartite states can be entangled in different ways. For instance, there are two classes of entanglement for three qubit systems [20]: one represented by the GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),$$

(3.1)

and another by the W state

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$$

(3.2)

These two classes are inequivalent because states from one cannot be obtained from states of the other by single particle \(i.e.,\) local operations. By contrast, there are four mutually orthogonal entangled states for two particles and anyone of them can be obtained from the other by single particle operations. Consider the Bell state

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

We can obtain the other entangled states as follows:

$$|\psi_-\rangle = \sigma_z^1|\psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$

(3.3)

$$|\phi_+\rangle = \sigma_z^2|\psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

(3.4)

$$|\phi_-\rangle = \sigma_z^1\sigma_z^2|\psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

(3.5)

Another key difference between the GHZ and W states is that a single particle trace of the GHZ state results in a maximally mixed state compared with a non-maximally mixed result for the W state which still retains some entanglement.

Many different metrics have been proposed to quantify the "amount" of entanglement each state contains; each metric emphasizing a specific feature or property of multiparticle entanglement. Amongst the many proposed multiparticle entanglement metrics such as the concurrence [22], the \(n\)-tangle [23], the Schmidt rank [24], the negativity [25], etc., the one proposed by Meyer et. al. in [26] is most relevant to the present work because we are interested in the entanglement properties of a subsystem following local decoherent operations on a portion of a composite system. For a given state \(\psi\), the metric of Meyer et. al. is an average over each qubit subsequent to "tracing" over each binary state and comparing how orthogonal the remaining states
are. It is defined as

$$Q(\psi) = \frac{4}{n} \sum_{j=1}^{n} D(t_j(0)\psi, t_j(1)\psi),$$  \hspace{1cm} (3.6)$$

where $n$ is the number of qubits and the operation $t$ defines the map

$$t_j(b)|b_1...b_n\rangle = \delta_{bb_j}|b_1...\hat{b}_j...b_n\rangle$$  \hspace{1cm} (3.7)$$

for $b \in \{0, 1\}$ and the caret denotes absence. $D(u, v)$ is the norm squared of the wedge product between the states $u$ and $v$ ($D(u, v) = \sum_{x<y} |u_xv_y - u_yv_x|^2$), which measures how orthogonal they are. For example, $Q(|W\rangle) = 8/9$ and $Q(|GHZ\rangle) = 1$ (for reference, $Q(|000\rangle) = 0$), which shows that the GHZ state is the maximally entangled three-qubit state.

Complementary measurements also reveal differences between the GHZ and W states. While a measurement of one qubit of the GHZ state along the $z$-axis results in two separable subensembles, an orthogonal measurement (for instance, along the $x$ axis) results in maximally entangled subensembles. Alternatively, the same set of measurements applied to the W state does not result in such a stark difference in the states of the unmeasured qubits implying that the W state is more stable against particle losses. This fact is important for QIP applications which rely on multiparty communication schemes that use entangled qubits as resources. If anyone of the parties decides not to participate the communications link between the other parties can continue if they are using a W type state as a resource, whereas if they are using a GHZ type state then the link is severed. There are implications for decoherence models as well. A GHZ type entanglement is maximally effective at decohering subsystems, whereas a W type state is not as effective. Thus, a maximally entangled state is least robust against particle losses, but maximally effective in decohering a subsystem. This will be apparent in the next section.
3.3 Three-particle Quantum Erasers

While two qubits are the minimum required for quantum erasers, three qubits provide the interesting possibility of manipulating entanglement among two of the qubits. Garisto and Hardy [30] were the first to extend the quantum eraser to three-spins. They dubbed the GHZ version the "quantum disentanglement eraser".

3.3.1 The GHZ State

In the disentanglement eraser the components of a GHZ state are regarded as the components of a two-qubit Bell state labeled by the state of a third ancilla qubit A, as follows:

$$|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0_A|00\rangle + |1_A|11\rangle).$$

(3.8)

A projective measurement of the ancilla in this basis yields a pair of separable states $|00\rangle$ and $|11\rangle$ labeled by the ancilla qubit, realizing the ensemble

$$E_+^A|00\rangle\langle00| + E_-^A|11\rangle\langle11|,$$

(3.9)

where $E_{\pm} = \frac{1}{2}(I \pm \sigma_z)$. Alternatively, if the ancilla is measured along the x-axis a pair of complementary Bell states, $|\phi_{\pm}\rangle = 1/\sqrt{2}(|00\rangle \pm |11\rangle)$, labeled by the ancilla is obtained. To see this we rewrite the GHZ state in a basis with the ancilla qubit quantized along the x-axis

$$|\psi_{GHZ}\rangle = 1/\sqrt{2}[1/\sqrt{2}(|+A\rangle + |–A\rangle)|00\rangle$$

$$+ 1/\sqrt{2}(|+A\rangle – |–A\rangle)|11\rangle]$$

$$\equiv 1/\sqrt{2}(|+A\rangle|\phi_+\rangle + |–A\rangle|\phi_–\rangle).$$

(3.10)

A measurement of the ancilla along the x-axis, followed by a rotation of the ancilla to the z-axis thus results in the ensemble

$$E_+^A|\phi_+\rangle\langle\phi_+| + E_-^A|\phi_–\rangle\langle\phi_–|,$$

(3.11)
which is a mixture of complementary Bell states each labeled by the state of the ancilla. Thus, measurements of the ancilla qubit along the x-axis realizes a pair of subensembles of entangled states.

3.3.2 The W State

Like the GHZ state of the disentanglement eraser, the components of a W state can be regarded as the Bell states \(|\psi\pm\rangle = 1/\sqrt{2}(|01\rangle \pm |10\rangle)\) together with the classical state \(|00\rangle\), labeled by the state of a third ancilla qubit (given the subscript 'A'), as follows:

\[
|W\rangle = \frac{1}{\sqrt{3}}(|0_A\rangle(|01\rangle + |10\rangle) + |1_A\rangle|00\rangle). \tag{3.12}
\]

A measurement of the ancilla qubit along the z-axis yields the ensemble

\[
E^A_+|\tilde{\psi}_+\rangle\langle \tilde{\psi}_+| + E^A_-|00\rangle\langle 00|, \tag{3.13}
\]

where \(|\tilde{\psi}_+\rangle = \sqrt{2}|\psi_+\rangle\). Rewriting the W state with the ancilla qubit represented along the x-axis yields

\[
|W\rangle = \frac{1}{\sqrt{3}}[1/\sqrt{2}(|+A\rangle + |-A\rangle)|\tilde{\psi}_+\rangle \\
+1/\sqrt{2}(|+A\rangle - |-A\rangle)|00\rangle] \\
\equiv 1/\sqrt{2}(|+A\rangle(|\tilde{\psi}_+\rangle + |00\rangle) \\
+|-A\rangle(|\tilde{\psi}_+\rangle - |00\rangle)). \tag{3.14}
\]

Therefore, a measurement of the ancilla qubit along the x-axis then followed by a rotation of the ancilla back to the z-axis results in the ensemble

\[
E^A_+|\tilde{\psi}_+\rangle\langle \tilde{\psi}_+| + |\tilde{\psi}_+\rangle\langle \tilde{\psi}_+| + |00\rangle\langle 00| + |00\rangle\langle 00|) \\
+ E^A_-|\tilde{\psi}_+\rangle\langle \tilde{\psi}_+| - |\tilde{\psi}_+\rangle\langle \tilde{\psi}_+| + |00\rangle\langle 00|), \tag{3.15}
\]

33
which is a mixture of a Bell state plus a classical state and products of these states. Thus, the W state still retains entanglement even under a z-axis measurement of the ancilla qubit.

### 3.4 Experimental Demonstrations

The quantum erasers in this presentation were demonstrated on ensembles of molecules, each containing two spin-$-\frac{1}{2}$ nuclei for the two-particle quantum eraser effect and three spin-$-\frac{1}{2}$ nuclei for all the three-particle quantum eraser effects. All experiments were performed using standard Bruker spectrometers. The two-spin experiments were performed on the two-spin heteronuclear spin system $^{13}$C-labeled chloroform ($^{13}$CHCl$_3$) diluted with deuterated acetone. The two spins used were the $^{13}$C and H nuclei and they exhibited a weakly coupled spectrum corresponding to the internal Hamiltonian

\[
\mathcal{H}_{\text{int}} = \pi [\nu_C \sigma_z^C + \nu_H \sigma_z^H + \frac{J}{2} \sigma_z^C \sigma_z^H],
\]  

(3.16)

where the $\nu$'s are Larmor frequencies ($\nu_H/\nu_C \approx 4$) and $J$ the spin-spin coupling constant in Hertz. Due to the huge difference in the chemical shifts between $^{13}$C and H we use two channels thereby placing each nuclei in a corotating frame with no net zeeman evolution. Thus, $\nu_H$ and $\nu_C$ were set to zero in the respective frames of each nuclei, and $J = 214$ Hz.

The three spin experiments used the three spin 1/2 carbons in a $^{13}$C-labeled sample of alanine in deuterated water (Fig. 3-2). With decoupling of the protons [9], this spin system exhibits a weakly coupled spectrum corresponding to the Hamiltonian

\[
\mathcal{H}_{\text{int}} = \pi [\nu_1 \sigma_z^1 + \nu_2 \sigma_z^2 + \nu_3 \sigma_z^3 + \frac{1}{2} (J_{12} \sigma_z^1 \sigma_z^2 + J_{23} \sigma_z^2 \sigma_z^3 + J_{13} \sigma_z^1 \sigma_z^3)],
\]  

(3.17)

where the $\nu$'s are Larmor frequencies and the $J$'s the spin-spin coupling constants in Hertz. The frequency shifts of the carbon resonances with respect to the second are 9456.5 Hz (31.5216 ppm) for the first one and -2594.3 Hz (8.6476 ppm) for the third,
Figure 3-2: The alanine molecule. The encircled labels on the $^{13}\text{C}$s index the spins used in the experiment.

while the coupling constants are $J_{12} = 53.7$, $J_{23} = 34.6$ and $J_{13} = -1.4$ Hz. The $T_1$ relaxation times for the three spins are 21, 2.5 and 1.6 s, while the $T_2$ times are 550, 420 and 800 ms, respectively.

In the two spin case the equilibrium density matrix is given in the high temperature approximation by

$$\rho_{eq} \approx I + \beta(\sigma^C_z + \frac{\gamma^H}{\gamma^C}\sigma^H_z)$$

(3.18)

where the ratio of the gyromagnetic ratio is $\gamma^H / \gamma^C \approx 4$. By the procedure described in [31] $\rho_{eq}$ is transformed to

$$\rho = I + \beta(1 + \frac{\gamma^H}{\gamma^C})(\sigma^C_z + \sigma^H_z)$$

(3.19)

which has a balanced spin population compared to the equilibrium state. Because $\rho$ has a different eigenvalue structure relative to $\rho_{eq}$ a non-unitary operation was required, i.e. a gradient pulse, as part of the sequence of transformations. To transform $\rho$ to the desired pseudo-pure state, we use the two-spin pseudo-pure state preparation sequence described in [32], which yields

$$\rho_{PP} = \frac{1}{4}(I + \sigma^C_z + \sigma^H_z + \sigma^C_z \sigma^H_z)$$

(3.20)

ignoring non essential constants. We also designate $\dot{\rho}_{ini} = \rho_{PP} - \frac{1}{4}I$ as the traceless part of the initial state used in the experiment.

For the three spin case, the pseudo-pure ground state was prepared from the
Table 3.1: This table shows the transformations used to obtain the three-spin pseudopure state from the thermal equilibrium state, using geometric algebra notation [14, 32].

Thermal equilibrium state by the procedure summarized in Table 3.1, which uses magnetic field gradients (denoted by $\nabla$) to dephase off-diagonal elements of the density matrix along the way [10] just as in the two spin case. Letting $\rho_{eq} = \sigma_x^2 + \sigma_z^2 + \sigma_z^3$ be the traceless part of the equilibrium density matrix (which has a balanced spin population to start with and with all physical constants set to unity), the first two transformations in the table yield the state $(\sqrt{3}/\sqrt{2})\sigma_z^2 + (\sigma_x^1 + \sigma_z^3)E_+^2$. The third transformation in Table 3.1 swaps spins 1 and 2 yielding $(\sqrt{3}/\sqrt{2})\sigma_x^1 + (\sigma_z^2 + \sigma_z^3)E_+^1$. Spins 2 and 3 may then be transformed into the state $(\sqrt{3}/\sqrt{2})(\sigma_z^2 + \sigma_z^3 + \sigma_z^3\sigma_z^3)$ by the two-spin pseudo-pure state preparation procedure used in eq. 34, yielding the three-spin pseudo-pure ground state

$$\hat{\rho}_{ini} = \frac{\sqrt{3}}{\sqrt{2}} \left(E_+^1 E_+^2 E_+^3 - \frac{1}{8} I \right) \equiv \left(\langle 000|000\rangle - \frac{1}{8} I \right).$$  \hspace*{1cm} (3.21)

3.4.1 The Two-spin Quantum Eraser

The two-spin quantum eraser was implemented according to the logic network shown in Fig. 3. Starting with the two-spin initial state $\rho_{ini} \sim \pi/2$ y-rotation $R_y^0(\pi/2) \equiv \exp(-i\sigma_y^2\pi/4)$, transforms spin 2 into the superposition state $|+\rangle_2$. A c-NOT ([4]) gate, $N_1^{\dagger 2} \equiv e^{i(\sigma_y^2 E_+^2 \pi/2)}$, then entangles the two spins, where spin 2 is the control for the pair of spins, creating a pseudo-pure Bell state. The corresponding set of
transformations on the state vector $|00\rangle$ is:

$$
|00\rangle \xrightarrow{R_z^{\pi/2}} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle
\xrightarrow{N_{1,2}} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \equiv |\phi_{BELL}\rangle
$$

(3.22)

Implementations of these operations in NMR by RF (radio-frequency) pulse sequences may be found in [14, 32, 10]. The resulting traceless pseudo-pure Bell state can be written in GA notation as [14]

$$
\hat{\rho}_{BELL} = \left[ \frac{1}{2} (1 + \sigma_x^1 \sigma_x^2) (E_+^1 E_+^2 + E_-^1 E_-^2) - \frac{1}{4} I \right],
$$

(3.23)

where $\frac{1}{2} (1 + \sigma_x^1 \sigma_x^2) (E_+^1 E_+^2 + E_-^1 E_-^2) = |\phi_{BELL}\rangle \langle \phi_{BELL}|$. Note that switching the spin labels in $\hat{\rho}_{BELL}$ does not change the state which reflects the symmetry of the state $|\phi_{BELL}\rangle$ under particle exchange.

The coherences of $\hat{\rho}_{BELL}$ can be selectively dephased as described above with the pulse sequence

$$
P_z^1 = [\nabla]_z - [\pi]_z^2 - [\nabla]_z - [\pi]_z^2
$$

(3.24)

To dephase spin 1 in the same way as would a strong measurement of $\sigma_z^1$, we apply the pulse sequence

$$
P_z^1 = [\pi/2]_{-y} - P_z^1.
$$

(3.25)

Spin 1 is left along $z$ for subsequent tomography.

These states were confirmed by full state tomography [33]. Because only the single quantum ($m_{kl} = 1$) coherences give rise to observable (dipolar) magnetization, it is necessary to collect spectra not only following the dephasing operation, but also following additional $\pi/2$ pulses selective for single spins, to rotate the $m_{kl} = 0$ and $m_{kl} > 1$ coherences, as well as the populations (diagonal elements), into observable single quantum coherences.

The overall precision of quantum information transmission was quantified by the attenuated correlation, initially defined in [5, 7], and also see Sect. 2.1. It takes into
account not only systematic errors, but also the net loss of magnetization due to random errors. It is given by

\[
c(\hat{\rho}^{\text{exp}}) = \frac{Tr(\hat{\rho}^{\text{the}}\hat{\rho}^{\text{exp}})}{Tr(\hat{\rho}^{\text{the}})}. \tag{3.26}
\]

Here, \(\hat{\rho}^{\text{the}}\) is the measured initial ground state \(\hat{\rho}^{\text{exp}}_{\text{init}}\), transformed on a computer by the same sequence of unitary and non-unitary (measurement) operations to which it was subjected on the spectrometer to get \(\hat{\rho}^{\text{exp}}\).

The results of \(P^1_z\) and \(P^2_z\) applied to \(\hat{\rho}_{\text{BELL}}\) are

\[
\hat{\rho}_{\text{BELL}} \xrightarrow{P^1_z} \frac{1}{2}(E^1_+ E^2_+ + E^1_- E^2_- - \frac{1}{2} I), \tag{3.27}
\]

\[
\hat{\rho}_{\text{BELL}} \xrightarrow{P^2_z} \frac{1}{2}[(1 + \sigma^2_x) E^1_+ + (1 - \sigma^2_x) E^1_- - \frac{1}{2} I], \tag{3.28}
\]

where \(\frac{1}{2}(1 \pm \sigma_x) = |\pm\rangle \langle \pm|\). Tomography was performed at the points of the procedure indicated in Fig. 3-3; the real parts of these four density matrices are shown in Fig. 3-4 (the imaginary parts were essentially zero).

The values of the correlation for each of the four tomographic readouts were \(c(\hat{\rho}^{\text{exp}}_{\text{PP}}) = 1\) (by definition), \(c(\hat{\rho}^{\text{exp}}_{\text{BELL}}) = 0.98\), \(c(\hat{\rho}^{\text{exp}}_{\sigma_1}) = 0.97\) and \(c(\hat{\rho}^{\text{exp}}_{\sigma_2}) = 0.98\). The increases in \(c\) are not unexpected, since the additional \(\pi\) and gradient pulses needed to mimic measurements on \(\hat{\rho}_{\text{BELL}}\) are easily implemented with high precision, and the tomographic errors are estimated at \(\pm 1\%\). Thus, clearly the \(\sigma^1_x\) measurement on \(\hat{\rho}_{\text{BELL}}\) restores the coherence of spin 2 in the two subensembles distinguished by the state of spin 1.

### 3.4.2 Three-particle Quantum Erasers

#### The GHZ State

The logic network shown in Fig. 3-5 transforms the initial pseudo-pure state into the pseudo-pure GHZ state, and then decoheres the ancilla as indicated. The GHZ state is
Figure 3-3: Logic network for the two-spin quantum eraser. Initially, a pseudo-pure state on spins 1 and 2 is created, \( \rho_{PP} = |00\rangle \langle 00| = E^1_+ E^1_+ \). A \( \frac{\pi}{2} \) y-pulse is then applied to spin 2, followed by a controlled-not (c-NOT) gate to create the Bell state (see text). Conditionality on the second spin being in the \( |1\rangle \) state is represented in the network by a filled circle on its timeline. Finally, one of the two complementary measurements, \( \sigma^1_x \) or \( \sigma^1_z \), are applied to spin 1. State tomography was performed to fully reconstruct the density matrices at the positions indicated.

obtained by rotating spin 2 (since \( J_{13} \ll J_{12}, J_{23} \)) to the \( x \) axis in the rotating frame with a \( \pi/2 \) y-rotation \( R_y^2(\pi/2) \equiv \exp(-i\sigma^2_y \pi/4) \), and then using it as the control for a pair of c-NOT gates to the other two spins. This pair of c-NOT's was implemented by the propagator \( N^{13|2} \equiv e^{i(\sigma^1_x - \sigma^2_y) E^1_+ E^2_+ \pi/2} \). The overall sequence of transformations on the corresponding state vector is thus:

\[
|000\rangle \xrightarrow{\quad R_y^2(\pi/2) \quad} |0\rangle(|0\rangle + |1\rangle)|0\rangle/\sqrt{2} \\
\xrightarrow{N^{3|2}} |0\rangle(|00\rangle + |11\rangle)/\sqrt{2} \\
\xrightarrow{N^{1|2}} (|000\rangle + |111\rangle)/\sqrt{2} \equiv |\psi_{GHZ}\rangle
\]  

(3.29)

The resulting pseudo-pure GHZ state is written in GA notation as [14]

\[
\hat{\rho}_{GHZ} = \frac{\sqrt{3}}{4\sqrt{2}} \frac{1}{2} E^1_+ (E^1_+ E^3_+ + E^1_+ E^2_+ E^3_+) \\
+ \frac{1}{2} \sigma^1_x E^1_+ (\sigma^2_x E^2_+ + \sigma^3_x E^3_+) \\
+ \frac{1}{2} \sigma^1_x E^1_+ (\sigma^3_x E^2_+ - \sigma^1_x E^3_+) - \frac{1}{8} I.
\]  

(3.30)
Figure 3-4: Experimental density matrices reconstructed by tomography (in normalized units). The rows are enumerated in the standard computational basis, where for example 00 represents the state label |00⟩. Although not shown, the columns are similarly labeled with the leftmost end representing |00⟩ and the rightmost end representing |11⟩. \( \rho_{PP} \) is the three-spin pseudo-pure ground state, and \( \rho_{BELL} \) is the pseudo-pure Bell state. The last two plots are \( \rho_z \), which is \( \rho_{BELL} \) after decohering spin 1 about the z-axis, and \( \rho_x \), which is after decohering it about the x-axis. An amount of identity, chosen to optimize the input projection, was added to all experimentally measured density matrices.
Figure 3-5: Logic network for the disentanglement eraser. Initially, a pseudo-pure state on spins 1, 2 and 3 is created, $\rho_{\text{ini}} = |000\rangle\langle 000| \equiv E_+^1 E_+^2 E_+^3$. A $\frac{\pi}{2} y$-pulse is then applied to spin 2, followed by two controlled-not (c-NOT) gates to create the GHZ state (see text). Conditionality on the second spin being in the $|1\rangle$ state is represented in the network by a filled circle on its time line. Finally, the two complementary measurements, $\sigma_z^1$ and $\sigma_z^1$, are applied to spin 1. State tomography was performed to fully reconstruct the density matrices at the positions indicated.

The notation expresses $\hat{\rho}_{\text{GHZ}}$ in terms of the state of spin 1, and also shows the symmetry of the state under particle exchange. This state has previously been studied by NMR in [34].

The coherences of $\hat{\rho}_{\text{GHZ}}$ can be dephased as described above with the following sequence of RF and gradient pulses

$$P_z^1 = [\nabla]_z - [\pi]_z^2 - [\nabla]_z - [\pi]_z^{2,3} - [\nabla]_z - [\pi]_{z-x} - [\nabla]_z - [\pi]_{z-x}^{2,3}$$

(3.31)

The following pulse sequence dephases the ancilla in the same way as would a strong measurement of $\sigma_z^1$

$$P_z^1 = [\pi/2]_{-y} - P_z^1$$

(3.32)

For subsequent tomography the ancilla is left along $z$. 

41
The results of $P_x^1$ and $P_y^1$ applied to $\hat{\rho}_{GHZ}$ are

$$\hat{\rho}_{GHZ} \overset{P_x^1}{\rightarrow} \frac{\sqrt{3}}{4\sqrt{2}} \frac{1}{2} E_+^1 (E_+^2 E_+^3) + \frac{1}{2} E_-^1 (E_-^2 E_-^3) - \frac{1}{8} I]$$

\hspace{1cm} (3.33)

$$\hat{\rho}_{GHZ} \overset{P_y^1}{\rightarrow} \frac{\sqrt{3}}{4\sqrt{2}} \frac{1}{2} E_+^1 (1 + \sigma_z^2 \sigma_z^3) (E_+^2 E_+^3 + E_-^2 E_-^3) + \frac{1}{2} E_-^1 (1 - \sigma_z^2 \sigma_z^3) (E_+^2 E_+^3 + E_-^2 E_-^3) - \frac{1}{8} I]$$

\hspace{1cm} (3.34)

Thus, noting that the states $|\phi\pm\rangle\langle\phi\pm|$ have the GA representation $\frac{1}{2} (1 + \sigma_z^2 \sigma_z^3) (E_+^2 E_+^3 + E_-^2 E_-^3)$, the $P_x^1$ measurement realizes the ensemble in Eq. 3.33 while the $P_y^1$ measurement realizes the ensemble in Eq. 3.34. Tomography was performed at the points of the procedure indicated in Fig. 3-5; the real parts of these four density matrices are shown in Fig. 3-6 (the imaginary parts were again essentially zero).

The values of the attenuated correlation for each of the four tomographic readouts were $c(\hat{\rho}_{ini}^{exp}) = 1$ (by definition), $c(\hat{\rho}_{GHZ}^{exp}) = 0.88$, $c(\hat{\rho}_z^{exp}) = 0.92$ and $c(\hat{\rho}_x^{exp}) = 0.93$. Tomography on the intermediate state $|0\rangle(|00\rangle + |11\rangle)/\sqrt{2}$ yields an attenuated correlation of 0.93, showing that spins 2 and 3 were entangled before the GHZ state was created. The tomographic errors are estimated at $\pm 3\%$. As Fig. 3-6 shows, there is no entanglement present in $\rho_z$ whereas in $\rho_x$ the two-spin entanglement is recovered in the two subensembles of the ancilla spin following the $\sigma_z^1$ measurement.

Notice that if the result of the X-measurement were lost (i.e., the measured particle is traced over), the resulting reduced density matrix is maximally mixed. The state of the ancilla spin distinguishes these subensembles through its coupling to the other two spins. Thus, we performed an experiment where the ancilla spin is directly decoupled by the NMR decoupling techniques discussed in section 2.3. The result is shown in Fig. 3-7. This emphasizes the need of the measurement result to distinguish the two sub-ensembles and retain quantum information.
Figure 3-6: Experimental density matrices reconstructed by tomography (in normalized units). The rows and columns are enumerated in the standard computational basis as described in Fig. 4. $\rho_{\text{ini}}$ is the three-spin pseudo-pure ground state, and $\rho_{\text{GHZ}}$ is the pseudo-pure GHZ state. The last two plots are $\rho_z$, which is $\rho_{\text{GHZ}}$ after decohering spin 1 about the $z$-axis, and $\rho_x$, which is after decohering it about the $x$-axis. (Note: $\rho_{\text{GHZ}}$, $\rho_z$ and $\rho_x$ have been magnified by a factor of two for clarity). An amount of identity, chosen to optimize the input projection, was added to all experimentally measured density matrices.
Figure 3-7: The reconstructed conditional Bell and traced density matrices (in normalized units). The diagonal components have been shifted so unit trace is preserved. The top plot is the conditional Bell state (conditioned on the state of the ancilla qubit), while the bottom plot is the traced density matrix after applying the decoupling pulses to the ancilla qubit during the observation period.

The W State

The W state preparation pulse sequence was based on a similar sequence used in a concurrently running experiment by our group [35]. This sequence created the NOT of the W state, \(|\tilde{W}\rangle\), which is equivalent to the W state if logical 1 is the ground state:

\[
|\tilde{W}\rangle \equiv \sigma_z^1 \sigma_z^2 \sigma_z^3 |W\rangle = \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle). \tag{3.35}
\]

The logic network in Fig. 3-8 describes these experiments. The starting state is a three-spin pseudo-pure state, \(\tilde{\rho}_{init}\), and the set of gates shown creates the pseudo-pure \(\tilde{W}\) state:

\[
\tilde{\rho}_{W} = \frac{1}{3} \left\{ \frac{1}{8} I + \frac{1}{2} \sigma^1_z \sigma^2_z \sigma^3_z - E^1_+ E^2_+ E^3_+ \\
+ \frac{1}{2} \left[ E^1_- (\sigma^2_z \sigma^3_z + \sigma^2_y \sigma^3_y) + E^2_- (\sigma^1_z \sigma^3_z + \sigma^1_y \sigma^3_y) \right. \\
\left. + E^3_- (\sigma^1_z \sigma^2_z + \sigma^1_y \sigma^2_y) \right] \right\}. \tag{3.36}
\]
Figure 3-8: Logic network for the creation of $\tilde{W}$ state followed by complementary measurements. Starting with $\rho_{\text{ini}}$, the sequence of transformations shown creates $\rho_{\tilde{W}}$. Subsequently, the set of complementary measurements described in the text are applied to spin 1 in the state $\rho_{\tilde{W}}$. 

\[
U_1 = \frac{-1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \\
U_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \\
U_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
\]
While this expression highlights the symmetry of the state under particle exchange, the results are better explained by casting $\hat{\rho}_W$ in terms of the states of spin 1 as follows:

$$
\hat{\rho}_W = \frac{\sqrt{3}}{4\sqrt{2}} \left\{ \frac{1}{3} E_+^1 (E_-^2 E_3^3) \\
+ E_-^1 (1 + \sigma_z^2 \sigma_z^3)(E_+^2 E_3^3 + E_3^3 E_+^3) \\
+ \sigma_x^1 E_+^1 (\sigma_z^2 E_-^3 E_3^1 + E_3^2 \sigma_z^3 E_+^3) \\
+ \sigma_x^1 E_-^1 (\sigma_z^2 E_3^3 E_-^1 + E_3^2 \sigma_z^3 E_+^3) \right\} - \frac{1}{8} I. 
$$

(3.37)

Note that the $|\tilde{\psi}_+\rangle$ state Eq. 3.13 has the GA representation $(1 + \sigma_z^2 \sigma_z^3)(E_+^2 E_3^3 + E_3^2 E_+^3)$, which here is multiplying the $E_1^1$ term. Subsequently, the same set of measurements from the disentanglement eraser $P_1^1$ and $P_2^1$, Eqs. 3.31 and 3.32 respectively, are applied to $\hat{\rho}_W$. Appendix 1 shows the GA representation of these states. While the $P_1^1$ measurement realizes the ensemble in Eq. 3.14, the $P_2^1$ measurement realizes the ensemble in Eq. 3.15. Tomographic readouts of these states, performed at the points indicated in Fig. 3-8, yield the reconstructed density matrices shown in Fig. 3-9. Once again only the real parts are shown because the imaginary parts were effectively zero.

The values of the attenuated correlation for each of the four tomographic readouts were $c(\hat{\rho}_{int}^{exp}) = 1$ (by definition), $c(\hat{\rho}_W^{exp}) = 0.77$, $c(\hat{\rho}_1^{exp}) = 0.80$ and $c(\hat{\rho}_2^{exp}) = 0.73$. As Fig. 9 shows both $\hat{\rho}_{2}^{exp}$ and $\hat{\rho}_{2}^{exp}$ retain the entangled state $(1 + \sigma_z^2 \sigma_z^3)(E_+^2 E_3^3 + E_3^2 E_+^3)$ in a subsenseeme; this is in contrast to the results for the complementary measurements of the GHZ state.

### 3.5 Conclusions

In conclusion, we have used both a two-spin and a three-spin liquid-state NMR quantum information processor to obtain a precise implementation of the dynamics, both coherent and decoherent, underlying the quantum eraser scheme as applied to a Bell and GHZ state, and the $\tilde{W}$ state followed by complementary measurements. We have
Figure 3-9: Theoretical density matrices in a) and experimental density matrices in b) reconstructed by tomography (in normalized units). The rows and columns are enumerated in the standard computational basis as described in Fig. 4. In b) $\rho_{\text{ini}}$ is the three-spin pseudo-pure ground state, and $\rho_{W}$ is the pseudo-pure $W$ state. The last two plots are $\rho_{z}$, which is $\rho_{W}$ after decohering spin 1 about the $z$-axis, and $\rho_{x}$, which is after decohering it about the $x$-axis. An amount of identity, chosen to optimize the input projection, was added to all experimentally measured density matrices. The plots in a) are the corresponding theoretical density matrices for comparison purposes.
found that the experimental results confirm the theoretically predicted conditional expectation values. Our study of the two-spin quantum eraser suggests that it may be used to transfer information shared amongst qubits based only on local decoherent operations assisted by classical communication. The three-spin results confirm the inequivalency of the GHZ and \( \tilde{W} \) state under the action of complementary local decoherent operations. A \( \sigma_z \) measurement of the ancilla spin yields separable states in the subensembles of the GHZ state while the same measurement in the \( \tilde{W} \) state yields an entangled state in each subensemble. By contrast, a \( \sigma_x \) measurement of the ancilla spin in the GHZ state results in a maximally entangled state in both subensembles while the same measurements in the \( \tilde{W} \) state result in mixtures of entangled and separable states in both subensembles. Thus, the \( W \) state is more robust against particle losses while the GHZ state is most effective at decohering a subsystem.

A key feature of these experiments is the implementation of non-unitary control methods. Using gradient techniques we were able to judiciously and selectively render phase information macroscopically inaccessible in a way that precisely mimics the decoherence attendant on strong measurements. During this dephasing operation all interactions among the spins were refocused, and that only the macroscopically accessible information contained in the ancilla spin due to its earlier interactions with the other spins was changed.

With NMR decoupling methods we were able to perform a partial trace operation. By decoupling the ancilla spin, after the X-measurement on the GHZ state the reduced density matrix yields a maximally mixed state. This further emphasizes the need for the ancilla spin to distinguish the subensembles after the measurement. Unlike previous eraser implementations, it was not necessary to explicitly read out this information in each member of the ensemble in order to see the conditional coherence, because this was done for us by the coupling of the ancilla to the other spins while the spectra were being measured.
Appendix 1

The measurements $P_z^{\pm}$ and $P_z^{\mp}$, Eqs. 3.31 and 3.32 respectively, applied to $\hat{\rho}_W$ in GA notation:

$$\hat{\rho}_W \xrightarrow{P_z^{\pm}} \frac{\sqrt{3}}{4\sqrt{2}} \left\{ \frac{1}{3} E_+^1 (E_-^2 E_+^3) + E_-^1 (1 + \sigma_z^2 \sigma_\pm^3) (E_+^2 E_+^3 + E_-^2 E_+^3) \right\} - \frac{1}{8} I \}, \quad (A1)$$

$$\hat{\rho}_W \xrightarrow{P_z^{\pm}} \frac{\sqrt{3}}{4\sqrt{2}} \left\{ \frac{1}{3} \left( \frac{1}{2} E_+^1 [E_-^2 E_+^3 + E_-^2 \sigma_z^3 E_+^3] + (1 + \sigma_z^2 \sigma_\pm^3) (E_+^2 E_+^3 + E_-^2 E_+^3) \right) + \frac{1}{2} E_+^1 [E_-^2 E_+^3 + (1 + \sigma_z^2 \sigma_\pm^3) (E_+^2 E_+^3 + E_-^2 E_+^3)] + \sigma_z^2 E_+^2 E_-^3 - \sigma_z^2 E_-^2 E_+^3 \right\} - \frac{1}{8} I \}. \quad (A2)$$

49
Chapter 4

A “New” Decoherence Model

In this chapter, we develop a model for investigating continuous-time decoherence maps. Current theoretical models usually require infinite environments with non-vanishing couplings. (For example, see [37]). While these models have pedagogical value they are unrealistic. Motivated by practical considerations, we were led to construct a physically realizable and implementable model in a realistic setting. Therefore, we developed a decoherence model of a decohering two-level system coupled to an environment with a few degrees of freedom. Here, the couplings are of the $\sigma_z\sigma_z$ type and only induce coherence damping. Although we consider coherence damping, the model can be extended to other coupling Hamiltonians. By introducing a stochastic evolution on the environment, the resulting randomization of the environment phases causes loss of information over the environment degrees of freedom and decoheres the system. The model we developed is exactly solvable when the environment consists of one qubit. From this solution we can establish some limiting behaviors. In the regime of small stochastic phase kicks the system coherence undergoes an exponential decay with a decay rate that goes as the kick rate. A comparison of our model to established theoretical models suggests that the strength of our methodology lies in the ability to explore the boundary between a fully quantum and a fully classical environment.

This chapter is drawn from sections of [36] with significant contributions by E. M. Fortunato, C. Lopez, J. P. Paz and J. Emerson.
4.1 Introduction

As early as the 1930s von Neumann [38] recognized that quantum correlations are the principal mechanism by which decoherence results. He considered a system that was to be measured by first becoming correlated to a "quantum" apparatus through a unitary, information conserving quantum evolution. To complete the measurement a mechanism was needed by which this pure correlated state decayed into an appropriate mixture. In recent decades, Zurek [37] and others have extended von Neumann's work by employing an open-systems approach to an environment that is decohering a system coupled to it. Using these methods they have shown the emergence of superselection rules [39] and a transition from the quantum to the classical regime. In particular, Zurek has studied a two-level system coupled to N two-level systems through $\sigma_x\sigma_x$ type interactions and shown that the correlations which arise between system and environment lead to the damping of the system coherence, or off-diagonal elements of the density matrix.

Interest in this and other decoherence models (for example a two-level system coupled to a boson bath [40]), has grown over the last few years due to the development of QIP. A major challenge in QIP is the preservation of quantum coherence in the face of constant perturbations by an environment. While one could try to isolate the QIP device, this would make controlling the system difficult. Therefore, other strategies like QEC [41] and NSs [42, 43, 44] have been developed. The aim of this work is to develop methods to emulate decoherence in a physical setting, such as a QIP device, so that the nature and underlying physics of decoherence can be better understood and aid in the development of strategies to control it.

While Zurek's model captures many of the features of decoherence, its assumptions that an arbitrarily large number of environmental degrees of freedom will couple (with non-vanshing strength) to the system is unrealistic. We propose a model with a finite (and small) quantum environment that is stochastically modulated by classical fields to produce decoherence of the system. There are several advantages to this approach. First, this method allows the practical implementation of a physically
realizable model in a realistic setting such as a QIP device. The model is flexible in the sense that different size environments and coupling mechanisms can be considered. In particular, non-unital processes can be model by this method. (For example, a Heisenberg coupling can be modeled by this method). To our knowledge this is the first time classical fields have been used to mimic the actions of a larger quantum environment.

4.2 Quantum Decoherence—A Simple Case Study

In this section we would like to review the basic elements of quantum decoherence by presenting a case study of a concrete open-system model. The example was originally inspired by Zurek [37] and is simple enough to be solved analytically. Yet, inspite of its simplicity the model captures many of the elements of decoherence theories and allows considerations of irreversibility, loss of coherence and in particular, the role played by the “size” of the environment.

Consider $N$ two-level systems and focus on one system as the subsystem of interest. This subsystem interacts with the rest of the system through a bilinear interaction. The overall dynamics can then be described by the following Hamiltonian:

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_S + \mathcal{H}_E + \mathcal{H}_{SE},$$

(4.1)

where

$$\mathcal{H}_S = \nu_1 \sigma_x^1,$$

(4.2)

$$\mathcal{H}_E = \sum_{k=2}^{N} \nu_k \sigma_x^k,$$

(4.3)

$$\mathcal{H}_{SE} = \sum_{k=2}^{N} J_{1k} \sigma_x^1 \sigma_x^k.$$  

(4.4)

The terms in Eqs. 4.2 ($\mathcal{H}_S$) and 4.3 ($\mathcal{H}_E$), describe the free evolution of the system and environment. The terms in Eq. 4.4, ($\mathcal{H}_{SE}$) describe the bilinear interaction between the system and environment which are energy conserving and only cause phase
damping. Note that all three Hamiltonians mutually commute and share a common eigenbasis defined by the eigenstates of $\sigma_z$: $\{|e\alpha\rangle, e = 0, 1\}$ where $\alpha = 1, \ldots, N$. The prescription of the open-systems approach is to evolve the combined system and environment, represented by the density matrix $\rho_{\text{tot}}(t)$, and then recover the system density matrix from a partial trace over the environment degrees of freedom:

$$\rho_S(t) = \text{Tr}_E\{\rho_{\text{tot}}(t)\} = \begin{pmatrix} \rho_{00}(t) & \rho_{01}(t) \\ \rho_{10}(t) & \rho_{11}(t) \end{pmatrix}. \quad (4.5)$$

In Eq. 4.5 $\rho_{00}(t)$ and $\rho_{11}(t)$ represent the system population terms while $\rho_{01}(t) = \rho_{10}^*(t)$ represents the system coherence term. From the determinant of $\rho_S(t) \geq 0$ we have

$$0 \leq |\rho_{01}(t)| \leq \frac{1}{2}. \quad (4.6)$$

If the coherence terms vanish, the pure state is turned into a mixture in the computational basis ($\sigma_z$-basis), i.e. a preferred basis has been selected out by einselection. In fact, since $[\sigma_z, H_{\text{tot}}] = 0$, the interaction with the environment has two memory states $|0\rangle_S, |1\rangle_S$ as eigenstates. This is a purely phase damping mechanism and there is no energy exchange between system and environment. The populations remain unchanged throughout the system evolution.

In the interaction frame of $(H_S + H_E)$, the dynamics of the combined system are described by

$$\ddot{\rho}_{\text{tot}}(t) = e^{i(H_S + H_E)t} \rho_{\text{tot}}(t) e^{-i(H_S + H_E)t} \quad (4.7)$$

and the effective Hamiltonian in this frame becomes

$$\dot{H}_{\text{eff}} = H_{SE}. \quad (4.8)$$

The overall unitary propagator is given by

$$U_{SE}(t) = e^{-iH_{SE}t} = e^{-i \sum_{k=2}^{N} J_{ik} \sigma_i^+ \sigma_i^{+t}}. \quad (4.9)$$
Because the terms in $\mathcal{H}_{SE}$ commute for different values of $k$, $\mathcal{U}_{SE}$ can be written as

$$\mathcal{U}_{SE}(t) = \prod_{k=2}^{N} e^{-iJ_{kk}\sigma_{z}^{k}\sigma_{z}^{k}t}.$$  \hspace{1cm} (4.10)

At $t = 0$ the system is assumed to be in a pure state given by

$$|\psi(0)\rangle_{S} = a|0\rangle_{1} + b|1\rangle_{1}.$$ \hspace{1cm} (4.11)

A simple yet quite illustrative initial state of the environment is one where the environment states are each in a pure state:

$$|\psi(0)\rangle_{E} = \prod_{k=2}^{N} (\alpha_{k}|0\rangle_{k} + \beta_{k}|1\rangle_{k}).$$ \hspace{1cm} (4.12)

Thus, the initial state of the combined system is

$$|\Phi(0)\rangle_{SE} = |\psi(0)\rangle_{S} \otimes |\psi(0)\rangle_{E}$$

$$= (a|0\rangle_{1} + b|1\rangle_{1}) \prod_{k=2}^{N} (\alpha_{k}|0\rangle_{k} + \beta_{k}|1\rangle_{k}),$$ \hspace{1cm} (4.13)

and the evolution is given by

$$|\Phi(t)\rangle_{SE} = \mathcal{U}_{SE}[a|0\rangle_{1} \prod_{k=2}^{N} (\alpha_{k}|0\rangle_{k} + \beta_{k}|1\rangle_{k})$$

$$+ b|1\rangle_{1} \prod_{k=2}^{N} (\alpha_{k}|0\rangle_{k} + \beta_{k}|1\rangle_{k})]$$ \hspace{1cm} (4.14)

$$= a|0\rangle_{1} \prod_{k=2}^{N} e^{-iJ_{kk}\sigma_{z}^{k}\sigma_{z}^{k}t} |\phi\rangle_{k}$$

$$+ b|1\rangle_{1} \prod_{k=2}^{N} e^{iJ_{kk}\sigma_{z}^{k}\sigma_{z}^{k}t} |\phi\rangle_{k},$$ \hspace{1cm} (4.15)

where $|\phi\rangle_{k} = \alpha_{k}|0\rangle_{k} + \beta_{k}|1\rangle_{k}$. The effect of $\mathcal{U}_{SE}$ on the system states, $(|0\rangle_{1}, |1\rangle_{1})$, is to entangle the system with the environment. In the language of QIP the action of $\mathcal{U}_{SE}$ generates a conditional phase gate between the system and environment, conditioned
on the system states. The final state of the evolution is

$$\left| \tilde{\Phi}(t) \right\rangle_{SE} = a |0\rangle_1 \prod_{k=2}^{N} [\alpha_k e^{-iJ_{1k}t} |0\rangle_k + \beta_k e^{iJ_{1k}t} |1\rangle_k] $$

$$+ b |1\rangle_1 \prod_{k=2}^{N} [\alpha_k e^{iJ_{1k}t} |0\rangle_k + \beta_k e^{-iJ_{1k}t} |1\rangle_k]$$

(4.16)

and reflects the fact that the system and environment states are not factorizable. Finally, we are interested in calculating

$$\tilde{\rho}_{01}(t) = 1 |0\rangle_{SE} \left\langle \tilde{\Phi}(t)_{SE} |\tilde{\Phi}(t)_{SE} \right\rangle |1\rangle_1,$$

(4.17)

and directly obtain

$$\tilde{\rho}_{01}(t) = ab^* z(t)$$

(4.18)

where

$$z(t) = \prod_{k=2}^{N} \left[ |\alpha_k|^2 e^{-2iJ_{1k}t} + |\beta_k|^2 e^{2iJ_{1k}t}. \right]$$

(4.19)

Recall that $a$ and $b$ are the coefficients of the pure system.

The time-dependence of $z(t)$ contains the crucial information for understanding the behavior of the system coherence. In particular it is the magnitude of $z(t)$ that determines the damping of the phase information originally contained in $\tilde{\rho}_{01}(0)$. The evolution mapping $\rho_S(0) \to \rho_S(t)$ cannot be described by unitary dynamics if $|z(t)| \to 0$ and the one-time state change appears irreversible. However, $|z(t)|$ is quasi-periodic and one can define a recurrence time, $\tau_E$, such that

$$1 - |z(\tau_E)|^2 < \epsilon,$$

(4.20)

for every $\epsilon \ll 1$. The existence of such recurrences proves that in a finite environment, (when the couplings are time-independent and there are no couplings between the environment degrees of freedom), the information residing in the initial system coherence always comes back. For the suppression of coherence to become more and more effective we look at the size of the fluctuations of $z(t)$ around its time averaged
mean value \((< z(t) > = 0)\):

\[
(< |z(t)|^2 >)^{1/2} = \left( \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt' |z(t')|^2 \right)^{1/2} \approx \frac{1}{2^{N-1}} \prod_{k=2}^{N} [1 + (|\alpha_k|^2 - |\beta_k|^2)^2].
\] (4.21)

Thus, the fluctuations vary as \(\frac{1}{\sqrt{\text{dim} \mathcal{H}_d}}\), the size (or dimension) of the environment and this quantity determines how effective the decoherence process is whereby true irreversibility begins to emerge. In the continuum limit, that is \(N \rightarrow \infty\), \(z(t)\) is no longer quasi-periodic and \(\tau_E \rightarrow \infty\). Phase information is considered unrecoverably lost, displaced from the degrees of freedom of the system to the infinitely many degrees of freedom of the environment and the emergence of genuine irreversibility.

The example we just considered only involves a single system and doesn't require ensemble averages. Also, we see that the system coherence terms vanish in a natural way without having to introduce any additional procedures other than tracing away the environment degrees of freedom. A third crucial point is that the eigenstates that the system tends to were unambiguously selected out by the interaction Hamiltonian. Finally, we note that if the system-environment interaction was somehow reversed then the system coherence could be restored. To truly decoher a system (rather than dephase it) there needs to be another mechanism that destroys the information stored in the environment and makes it truly unrecoverable. Despite this point the example is an excellent pedagogical basis for understanding the program of decoherence.

To summarize, the steps in the quantum decoherence program within the setting of an open-systems approach are (1) the system of interest evolves through an entangling interaction with the environment, (2) the time evolution of the system alone is determined from a trace over the environment degrees of freedom after the combined system and environment evolve under the interaction Hamiltonian, (3) to extract the coherence term in the system density matrix which carries the phase information about the system, (4) the eigenstates that the system collapses to are determined as those states that commute with the interaction Hamiltonian and (5) to identify what determines the effectiveness of decoherence which in the example we considered is the dimension of the environment Hilbert space. Since the resulting correlations
between system and environment are from information preserving evolutions, there needs to be a mechanism by which the information transferred to the environment is unrecoverably lost. One way to achieve this decoherence is to spread the system phase information into the environment and then randomize the environment degrees of freedom. This approach is the essence of our proposal.

4.3 The Model System

The aim of this work is to experimentally study quantum decoherence in a physical setting. In particular, we want to simulate quantum decoherence on an NMR QIP which here consists of only three qubits. For practical reasons, we can't have arbitrarily many qubits so we have developed a method which is physically realizable while capturing the essential features of quantum decoherence. Our proposal calls for one qubit as the system and the other qubits as the quantum environment, with system and environment interacting through a bilinear coupling Hamiltonian. As discussed in the previous section, the environment size plays a critical role on the damping characteristics of the system. Thus, during the combined evolution of the system and environment we redress the environment states by applying a random phase kick to them through a stochastic Hamiltonian. This has the effect of scrambling the system phase information, stored in the environment, during the coupling interaction. As we shall show below, this scheme is effective at simulating quantum decoherence.

The model system is shown in Fig. 4-1 and is exactly solvable for a one-qubit environment. If we label the two qubits 1 and 2 their interaction Hamiltonian is given by

\[ H_{12} = \Omega \sigma_z^1 \sigma_z^2. \]  \hspace{1cm} (4.22)

Consider the evolution of these two qubits during a time \( T \). In the absence of any other interaction, the evolution operator is

\[ U_{12}(T) = \exp(-i\Omega T \sigma_z^1 \sigma_z^2). \]  \hspace{1cm} (4.23)
We will consider the evolution of this system subject to \( n \) kicks at times \( t_j = jT/n \) \((j = 1, \ldots, n)\). Every kick corresponds to applying a magnetic field that implements a rotation of the second qubit around the \( X \)-axis with an angle \( \epsilon_j \) chosen randomly in the interval \((-\alpha, +\alpha)\). The evolution operator for the \( j \)-th kick will be denoted as

\[
K_j = \exp(-i\epsilon_j \sigma_x^2). \tag{4.24}
\]

The evolution operator for the two qubit system, after \( n \) kicks, is:

\[
U_n = K_nU_{12}(\frac{T}{n})K_{n-1}U_{12}(\frac{T}{n}) \ldots K_1U_{12}(\frac{T}{n}). \tag{4.25}
\]

Obviously, the operator \( U_n \) depends on the values of the random variables \( \epsilon_j \) \((j = 1, \ldots, n)\).

Our goal is to obtain a closed expression for the reduced density matrix of the first qubit that characterizes an ensemble of realizations of the random variables \( \epsilon_j \). This density matrix is computed as:

\[
\bar{\rho}_1 = \int_{-\alpha}^{\alpha} \frac{d\epsilon_1}{2\alpha} \cdots \int_{-\alpha}^{\alpha} \frac{d\epsilon_n}{2\alpha} Tr_2(U_n \rho_{12}(0) U_n^\dagger). \tag{4.26}
\]

We will consider a factorizable initial state for the two qubits (this is not essential):

\[
\rho_{12}(0) = \rho_1(0) \otimes \rho_2(0). \tag{4.27}
\]

The initial density matrix of the first system can always be written in the basis of eigenstates of \( \sigma_x^1 \) as

\[
\rho_1(0) = \sum_{j,k=0,1} \rho_{jk}(0) |j\rangle \langle k|. \tag{4.28}
\]

Inserting (4.27) and (4.28) in (4.26) we obtain a simple expression if we note that the states \( |j\rangle \) are eigenstates of the operator \( U_n \). In fact, these states are eigenstates of the interaction Hamiltonian (and the operator implementing the kick does not change
them since it only acts on the second system). Therefore, we can write

$$U_n | j \rangle = U_n^{(j)}$$

(4.29)

where the operator $U_n^{(j)}$ acts on the second qubit and is defined as

$$U_n^{(j)} = K_n U_j \left( \frac{T}{n} \right) K_{n-1} U_j \left( \frac{T}{n} \right) \ldots K_1 U_j \left( \frac{T}{n} \right).$$

(4.30)

Here, we denoted $U_j(t) = \exp(-i\Omega t(-1)^j \sigma_z^j)$. Using this we can immediately obtain the following exact formula for the final density matrix of the first qubit

$$\tilde{\rho}_1 = \sum_{j,k=0,1} \rho_{jk}(0) f_{jk}(n, T) \ | j \rangle \langle k |.$$  

(4.31)

The function $f_{jk}(n, T)$ carries all the information about the effect of the second qubit (the environment) on the first one (the system) and is given by

$$f_{jk}(n, T) = \int_{-\alpha}^{\alpha} \frac{dk_1}{2\alpha} \ldots \int_{-\alpha}^{\alpha} \frac{dk_n}{2\alpha} \text{Tr}_2(U_n^{(j)} \rho_2(0) U_n^{(k)})).$$

(4.32)

It is clear from (4.32) that for $j = k$ the final trace over the second system is equal to one and therefore we always have $f_{jj} = 1$. Thus, this decoherence model only affects off diagonal terms in the $\sigma_z$-basis, which is also an obvious consequence of the fact that $\sigma_z$-eigenbasis is the pointer basis in this case.

The remaining task is to evaluate the function $f_{01}(n, T)$ since on general grounds one can show that $f_{jk}(n, T) = f_{kj}(n, T)^*$. To evaluate $f_{jk}(n, T)$ using (4.32) it is convenient to notice that, due to the decomposition of $U_n^{(j)}$ given in (4.25), we can think of it as being obtained by the successive application of a superoperator on the initial density matrix $\rho_2(0)$. Thus, we can write:

$$f_{01}(n, T) = \text{Tr}_2(O^n(\rho_2(0)))$$

(4.33)
where the superoperator $O$ is defined as

$$
O(\rho) = \int_{-\alpha}^{\alpha} \frac{de}{2\alpha} KU_0(T/n)\rho U_1(T/n)^\dagger K^\dagger.
$$

(4.34)

$$
= \int_{-\alpha}^{\alpha} \frac{de}{2\alpha} e^{-ie_\alpha} e^{-i\Omega T \sigma_z/n} \rho e^{-i\Omega T \sigma_z/n} e^{ie_\alpha},
$$

(4.35)

which, after integrating over the random variable turns out to be

$$
O(\rho) = c \left( e^{-i\Omega T \sigma_z/n} \rho e^{-i\Omega T \sigma_z/n} \right) + d \left( \sigma_z e^{-i\Omega T \sigma_z/n} \rho e^{-i\Omega T \sigma_z/n} \sigma_z \right),
$$

(4.36)

(4.37)

where $c - d = \sin(2\alpha)/2\alpha$ and $c + d = 1$. It is worth stressing that this superoperator is not trace preserving (hermitian, etc) but it is a useful tool for our analysis. It is easy to show that $O(\sigma_z) = \sigma_z$ and $O(\sigma_y) = (c - d)\sigma_y$. Thus, both $\sigma_x$ and $\sigma_y$ are eigenvectors of the superoperator $O$ (respectively with eigenvalue 1 and $c - d$). We will later need to find the other two eigenvectors which are linear combinations of the identity ($I$) and $\sigma_z$. After doing that we would be almost done. In fact, as we see in (4.32), to obtain the function $f_{01}$ we must compute a final trace over the second spin. So, it is clear that those terms in $\rho_2(0)$ that are proportional to $\sigma_z$ and $\sigma_y$ do not contribute to the final result (since they are traceless). Thus, to compute $f_{01}(n, T)$ we just have to apply the superoperator $O^n$ to the part of the initial state with components along the identity and $\sigma_z$. Writing the initial density matrix of the second qubit as $\rho_2(0) = (I + p_x \sigma_x + p_y \sigma_y + p_z \sigma_z)/2$ we obtain

$$
f_{01}(n, T) = \frac{1}{2} \text{Tr}(O^n(I)) + \frac{1}{2} p_z \text{Tr}(O^n(\sigma_z)).
$$

(4.38)

To evaluate the above two terms we first note that the action of $O$ both on the identity and $\sigma_z$ is

$$
O(I) = \cos(2\Omega T/n)I - i(c - d) \sin(2\Omega T/n)\sigma_z
$$

(4.39)

$$
O(\sigma_z) = -i \sin(2\Omega T/n)I + (c - d) \cos(2\Omega T/n)\sigma_z.
$$

(4.40)
The eigenvectors $v_1$ and $v_2$ and eigenvalues $\lambda_1$ and $\lambda_2$ of the superoperator $O$ satisfy the conditions $O(v_i) = \lambda_i v_i$. We find an exact solution in terms of $I$ and $\sigma_z$:

\begin{align}
  v_1 & = a_{11} I + a_{12} \sigma_z \\
  v_2 & = a_{21} I + a_{22} \sigma_z,
\end{align}

and conversely

\begin{align}
  I & = b_{11} v_1 + b_{12} v_2 \\
  \sigma_z & = b_{21} v_1 + b_{22} v_2.
\end{align}

Using this, we can write the final formula for $f_{01}(n,T)$ as

\[f_{01}(n,T) = \lambda_1^n a_{11}(b_{11} + p_2 b_{21}) + \lambda_2^n a_{21}(b_{12} + p_2 b_{22}).\]

From this we can read the final result. The only remaining task is to write the explicit form of these coefficients. Denoting

\[\gamma = (c - d) = \frac{\sin(2\alpha)}{2\alpha},\]

we can write the eigenvalues $\lambda_i$ as

\[\lambda_2 = \frac{1}{2}(1 + \gamma) \cos(2\Omega T/n) \pm \sqrt{\frac{(1 + \gamma)^2}{4} \cos^2(2\Omega T/n) - \gamma}.\]

It turns out that it is possible to rewrite the function $f_{01}$ as

\[f_{01}(n,T) = \frac{\cos(2\Omega T/n)(\lambda_1^n - \lambda_2^n) + \lambda_1 \lambda_2^n - \lambda_2 \lambda_1^n}{(\lambda_1 - \lambda_2)} - ip_2 \sin(2\Omega T/n) \frac{\lambda_1^n - \lambda_2^n}{(\lambda_1 - \lambda_2)}.\]

Notice that this formula is an explicit expression (obtained with no approximations) valid for all values of $n$ and $\gamma$. Below, we will analyze its predictions for some simple
4.3.1 Results

Let us consider some simple cases. First we will discuss the limit $\gamma = 1$ that corresponds to unitary evolution (no kicks) where we will recover simple expressions. Then we will consider the case $\gamma = 0$ that corresponds to averaging over angles between 0 and $2\pi$. Finally, we will analyze the case where $\gamma$ is close to one (averaging over small angles, which is the condition met in our simulations and experiments).

$\gamma = 1$, Unitary evolution

This is the simplest case. Here, we can show that the superoperator $O$ is such that $O(\sigma) = \sigma \exp(-i2\Omega T \sigma_z/n)$ for every operator $\sigma$ given as a linear combination of the identity and $\sigma_z$. Using this, it is easy to show that the function $f_{01}$ is

$$f_{01}(n, T) = \cos(2\Omega T) - i \, p_z \sin(2\Omega T)$$

(4.50)

This simple expression has a clear physical interpretation, the function becomes independent of $n$ (as it should since there are no kicks in this limit). Remember that $p_z$ is the initial polarization of the second qubit; therefore the above formula tells us simply that the first spin rotates in a way that depends on the polarization of the second one.

$\gamma = 0$, Complete randomization

Here we assume that the kick angles $\epsilon_j$ vary over the entire interval between 0 and $2\pi$. In this case the above formulae simplify substantially and we obtain

$$f_{01}(n, T) = \cos^n(2\Omega T/n) - i \, p_z \sin(2\Omega T/n) \cos^{n-1}(2\Omega T/n)$$

(4.51)
Here, in the large $n$ limit we clearly see a Zeno-like effect, (for an operator not a state), that can be obtained from the equation by noticing that

$$\cos^n(2\Omega T/n) \approx \exp(-2\Omega^2 T^2/n)$$  \hspace{1cm} (4.52)

$$\gamma \approx 1 - O(\alpha^2), \textbf{Average over small angles}$$

Here we consider the case where we average over small angles (the regime we consider in the simulations and experiments, $\alpha = \pi/20$). In such a case we have

$$\gamma \approx 1 - 2\alpha^2/3.$$  \hspace{1cm} (4.53)

In this case it is simple to expand both $\lambda_i$ in powers of $\alpha^2$ and obtain a very simple expression (which is valid also for small $n$). It is:

$$f_{01}(n,T) = (1 - \alpha^2/3)^n(1 + \alpha^2/3)$$  \hspace{1cm} (4.54)

$$\times (\cos(2\Omega T) - i p_z \sin(2\Omega T)) + O(\alpha^2).$$  \hspace{1cm} (4.55)

Thus, this shows that that the decay induced by the kicks is indeed exponential and that the rate is linear in $n$. The analysis of the exact formula shows that in this case for large $n$ a Zeno type effect is expected (as before).

### 4.3.2 Simulations

In this section we simulate a two-qubit environment. Consider the following Hamiltonian:

$$\mathcal{H}_{tot} = \mathcal{H}_S + \mathcal{H}_L + \mathcal{H}_{SL} + \mathcal{H}_{LL_2}.$$  \hspace{1cm} (4.56)

The system and local environments are described by the self Hamiltonians $\mathcal{H}_S$ and $\mathcal{H}_L$, while $\mathcal{H}_{SL}$ is the system-local environment interaction and $\mathcal{H}_{LL_2}$ is the local coupling between environment spins. In anticipation of the physical system we use,
Figure 4-1: A cartoon of the system, local quantum environment and the non-local classical environment. Note that the non-local environment only interacts with the local environment and not the system.

The terms in $\mathcal{H}_{\text{tot}}$ take the following form:

\begin{align*}
\mathcal{H}_S &= \nu_S \sigma_z^S, \\
\mathcal{H}_L &= \sum_{i=1}^{2} \nu_{Li} \sigma_z^{Li}, \\
\mathcal{H}_{SL} &= \sum_{i=1}^{2} J_{SLi} \sigma_z^S \sigma_z^{Li}, \\
\mathcal{H}_{L1L2} &= J_{L1L2} \sum_{i=x,y,z} \sigma_i^{L1} \sigma_i^{L2}, \\
\mathcal{H}_{\text{kick}} &= \sum_{i=1}^{2} \theta_i \sigma_y^{Li}.
\end{align*}

The $\mathcal{H}_{\text{kick}}$ term is the kick Hamiltonian. The angle $\theta$ is random and expresses the stochastic nature of the kicks to the local environment by the non-local environment. The steps in the simulation are as follows:

- Pick an initial state for the system and environment, $|\psi_0\rangle$.
- Let the system and environment evolve under $\mathcal{H}_{SL}$ for a time $\Delta$. 

64
\[ \nu_S = 0 \]
\[ \nu_{L1} = 800 \text{Hz} \]
\[ \nu_{L2} = -800 \text{Hz} \]
\[ J_{SL1} = 250 \text{Hz} \]
\[ J_{SL2} = 50 \text{Hz} \]
\[ J_{L1L2} = 174 \text{Hz} \]
\[ \theta = \{-\frac{\pi}{20}, +\frac{\pi}{20}\} \text{ (randomly chosen)} \]

Table 4.1: This table lists the parameters for the model Hamiltonian of Eqs. 4.57-4.61.

- Apply a kick to the local environment, through \( H_{\text{kick}} \), to redress those states.
- Repeat the previous two steps for a specified time that constitutes the measurement period.

The unitary map at every step takes on the form

\[ U_j = \prod_{l=1}^{k} e^{-iH_{\text{kick}}^l} e^{-iH_{\text{int}} \Delta / k}, \quad (4.62) \]

where \( H_{\text{kick}}^l = \sum_{i=1}^{2} \theta_i \sigma_y^i \) and \( \theta_i \) is drawn randomly, \( H_{\text{int}} = H_S + H_L + H_{SL} + H_{L1L2} \), \( \Delta \) is a cycle time and \( k \) defines the number of kicks per cycle time. At the end of some evolution time, \( T = N \Delta \), the combined system is in the state

\[ |\psi_T\rangle = \prod_{j=1}^{N} U_j |\psi_0\rangle. \quad (4.63) \]

For a given \( |\psi_0\rangle \) repeating the evolution given by Eq. 4.63 would result in different \( |\psi_T\rangle \)s. Although the system is decohered it is in a different state every time. To get the mean evolution we average over different realizations, or times \( T \).

Keeping in mind the physical system we used, the parameters in Eqs. 4.57-4.61 take on the values shown in Table 4.1. Initializing the state of the system and environment in the state \( \sigma_x^S E_1^{L1} E_1^{L2} \) we simulated the evolution of the system on Matlab. With a sample/cycle time of 1 ms we ran 10 different kick rates that ranged from 3 kicks/ms to 30 kicks/ms in steps of 3. The kicks were interleaved amongst delays during which time the system and environment evolved under the internal Hamiltonian.
Figure 4-2: Some example simulated decays obtained on Matlab. The plots are averages over 50 realizations of the quantity $\{\rho^j_{S}\}_{01} = \langle 0| Tr_{L1L2} (U_j^1 \rho^{-1}_{S L1L2} U_j^{-1}) |1\rangle_S$ from Eq. 4.64. The kick rates on each plot are labeled and we note that the higher the kick frequency the faster the system decays.
Figure 4-3: The dependence of the decay constants on kick rate. This linear response can be simply modeled as discussed in the text. In particular, consult the discussions accompanying Section 4.3.1.
The entire sequence of evolution consisted of a series of delays and pulses. A delay was given by the (sample time)/(kicks/sample) and the kicks were sampled from a uniform distribution of angles that ranged between $-\frac{\pi}{20}$ to $\frac{\pi}{20}$. The series was run for 150 sample points. For a given realization, the off-diagonal element of the system, at each sample point $j$, was recovered from the following procedure:

$$\{\rho^j_S\}_{01} = \langle 0 | T r_{L1L2} \left[ U_j \rho^{-1}_{SLL2} U^{-1}_j \right] | 1 \rangle_S \quad (4.64)$$

Then, $\{\rho^j_S\}_{01}$ was averaged over the realizations. We averaged over 50 realizations and obtain the plots shown in Fig. 4-2. The decays were fit to an exponential and the decay constants are plotted in Fig. 4-3.

The linear dependence on kick frequency implies the higher the kicks the more effective the induced decoherence. This dependence between kick frequency and decoherence rate may be used to control the effective environment size. Fig. 4-4 shows a plot of the decoupling limit and at about 900 kicks/ms the decays no longer continue growing. The trend reverses and the system starts to become decoupled from the environment as predicted earlier in Section 4.3.1. This is a well known phenomena in NMR [15] as well. If the rms angle by which the system gets kicked approaches a good fraction of a $\pi$ pulse, over a time frame that is comparable to the cycle time of the system-environment interaction, then the onset of decoupling is initiated. (See section 2.3).

### 4.4 The Implementation on a Physical System

In this section, we experimentally confirm our model. We chose Propyne (see Fig. 4-5 for internal Hamiltonian parameters) as the physical system. This molecule has many desirable features. The chemically distinct spin-1/2 nuclei, the one hydrogen which can be regarded as the system, and the two labeled carbons as the environment, have a large resonance frequency offset. This means hydrogen and carbon can be addressed, and detected separately, which as we shall see below has huge benefits.
Figure 4-4: The decoupling limit. Beyond a kick frequency of about 900 kicks/ms, the kicking is ineffective at inducing decoherence. After about 5000 kicks/ms the decays are no longer exponential.
Figure 4-5: The propyne molecule. The encircled labels on the $^{13}\text{C}$s and the rightmost hydrogen index the spins used in the experiment. The methyl group consists of the three hydrogens and an unlabeled carbon. In the experiments the field of the spectrometer was $\sim 9.2$ T and the hydrogen resonances were $\sim 400$ MHz while the carbon resonances were around $\sim 100$ MHz. The chemical shift difference between the two labeled carbons is 1.6 kHz. Using the indexing scheme in the figure the $J$-coupling constants are as follows: $J_{12} = 246.5$ Hz, $J_{23} = 173.8$ Hz and $J_{13} = 51.8$ Hz. The longitudinal relaxation times are $T_1 = 8.7$s, $T_2 = 23$s and $T_1 = 43$s, while the transverse relaxation times are $T_2 = 1.1$s, $T_2 = 1.9$s and $T_3 = 1.7$s.

Strong couplings present amongst these nuclei implies the interactions take place over short times, and the long relaxation times allow one to observe the hydrogen signals over a relatively long time span without significant natural decay. The simulations were carried out on ensembles of Propyne molecules using a standard Bruker Avance spectrometer. Neglecting the methyl group (because it couples in very weakly), the internal Hamiltonian for Propyne (to a good approximation) is given by

$$
\mathcal{H}_{\text{int}} = \pi [\nu_1 \sigma_x^1 + \nu_2 \sigma_x^2 + \nu_3 \sigma_x^3 + \frac{1}{2} (J_{12} \sigma_z^1 \sigma_z^2 + J_{23} \sigma_z^2 \sigma_z^3 + J_{13} \sigma_z^1 \sigma_z^3)],
$$

(4.65)

where the $\nu$'s are Larmor frequencies and the $J$'s the spin-spin coupling constants in Hertz. The various values are given in Fig. 4-5. In Eq. 4.65, the non-secular terms coupling the carbon nuclei are not included. There was evidence of there effect in the carbon spectra but they played a minor role in the experiments. Spin 1 is designated as the hydrogen nucleus (the system) and the other two are the carbon nuclei (the quantum environment).

A convenient choice for the initial state of system and environment is one where
hydrogen is in a superposition state and both carbons are in an eigenstate. By placing
the methyl hydrogens in an eigenstate as well, they can be eliminated from playing a
role in the hydrogen spin dynamics. This was accomplished by using a highly selective
rf pulse that irradiated a spectral line corresponding to the state

\[ \sigma_z^HE_+^{C1}E_+^{C2}E_+^M, \]  

(4.66)

where \( E_+ = \frac{1}{2}(I + \sigma_z) \), \( H \) represents hydrogen, \( C1 \) carbon 1, \( C2 \) carbon 2 and \( M \) the
methyl hydrogens. For this implementation we used a 5.5s EBURP1 [45, 9] pulse with
an rf power to generate as much signal as possible. The spectral resolution of this pulse
was .5 Hz and its design is such that it only generates a uniform excitation profile
in the specified bandwidth. Ultimately, only \( \sim 1/10 \)th of the maximum intensity
was excited. Nonetheless, this yielded sufficient signal-to-noise ratio to carry out the
experiments.

After the initial state preparation, the carbons were pulsed through the decou-
pling channel. Because the detection occurs on separate channels, just observing the
hydrogen signal directly yielded \( < \sigma_x^H(t) + i\sigma_y^H(t) > \), and was equivalent to tracing
away the carbons. A very homogeneous magnetic field combined with the long \( T_2s \)
of Propyne yielded a linewidth of \( \sim .4 \) Hz for the hydrogen spectra. Consequently,
the hydrogen signals decayed very slowly and we were able to pick a 150 ms portion
of the absolute magnitude that remained flat within a percent.

The carbon pulses, applied through the decoupling programs, were a series of
delays interleaved with pulses. During the delays the spins evolved under the internal
Hamiltonian generating the required interactions. The pulse flip angles were randomly
sampled from a uniform distribution that ranged between \( -\pi/20 \) to \( \pi/20 \) about the
\( y \)-axis. A cyctime of 1 ms was defined within which the kick frequency ranged
from 3 kicks/cycle to 30 kicks/cycle in steps of 3 for a total of 10 different kick
frequencies. The range of the kick frequency was limited by the shortest pulse the
spectrometer was capable of generating. In the present case that value was 100 ns.
The time allotted for a sequence of one delay period followed by a pulse was given by
the cycletime/(no. of kicks/cycle). Within this sequence the delay time is given by the total sequence time minus the pulse on time. The maximum pulse on time was 10 μs which corresponded to the maximum flip angle of π/20. The power setting for this flip angle was 2500Hz. (Compare this to the chemical shifts of the carbons which were each at 800Hz). For a given kick frequency, the length of the series of successive sequences of delay plus pulse, generated as described above, fit the total acquisition time of 150 ms.

The experiments were run for 10 different kick frequencies with an ensemble average over 50 realizations. A waiting time of 300s was used between successive realizations. Fig. 4-6 shows the result of the experiments. The absolute magnitude of these plots was fitted to an exponential. The χ² per degree of freedom for the 10 fits ranged from 1.1 to 8.5. The χ²’s fit the average decays well but don’t account for the details in the fine structure evident from the oscillations of the magnitudes. As the kick frequency increases the data demonstrates that the system is decohered faster. A plot of the decay constants as a function of kick frequency, Fig. 4-7, shows this trend clearly.

4.5 Discussion

The method we developed can be used to simulate a quantum environment with a few degrees of freedom. To mimick the actions of a larger environment we apply phase kicks to the quantum environment. The frequency with which we kick determines the decay rate and maybe related to an effective size of the quantum environment.

Another advantage of the model is that it has effectively infinite recurrence times. There is no means by which the system phase information can be restored to its original value once the phase kicking has commenced (assuming no record of the phase kicks is kept). The information residing in the quantum environment has been irrecoverably lost. The experiment results seem to exhibit revivals in the higher kick rates of Fig. 4-7. But this is due to low statistics and we have checked this by simulating out to longer times and higher statistics. See Fig. 4-8.
Figure 4-6: Example decays from the experiment. The hydrogen signal was directly detected and the magnitude and real part of the complex signal is plotted. The fluctuations at the tail end of the higher kick rates are due to low statistics. This was confirmed with simulations at much higher averages.
Figure 4-7: The dependence of the experimental decay constants on kick rate. The error bars are small and are on the order of the size of the data points. Compared to the slope in the simulations of Fig. 4-3 the experimental one appears to decay faster. This disparity is due to a difference between the experiment and simulations. In the simulations, the pulse on time is infinitely narrow whereas in the experiments it is finite.
Figure 4-8: A simulation to check revivals at longer times and higher averages. The times go out to 500ms and the averages are taken for 200 realizations. Note that the revivals that seem prevalent in Figs. 4-2 and 4-6 are diminished.
In conclusion, we have developed a model that is practical for simulating quantum decoherence effects on a physical system such as a QIP device. By varying the phase kicking rate in the stochastic Hamiltonian we can control the system decoherence rate. Furthermore, the method we developed can be used to model non-unital processes. In this presentation we have shown the effectiveness of the methodology in a two-spin environment. Using both Matlab simulations and an implementation on an NMR QIP device we have verified the results. Lastly, the analytical model we developed maybe used to show the connections between environment “size”, and both decay rates and recurrence times.
Bibliography


[11] Actually, $\omega$ equals a spin's nutation rate caused by an RF field. Because this parameter is experimentally controlled by attenuating the RF power, it is commonly referred to as the pulse power.


