D-branes, Gauge Theory and String Field Theory

by

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Abstract

In this thesis, we present several works done in last three years. They include three directions in the string theory. In the first direction, we use the brane setup to find mirror pairs of $SO(n)$ and $Sp(k)$ gauge groups for $N = 4$ three-dimensional gauge field theories. To reach this result, we analyze carefully the s-configuration and predict a nontrivial string dynamics, i.e., the splitting of branes on the orientifold planes. In the second direction, we develop the "inverse algorithm" and use it to get nontrivial world volume theories of D-branes probing more exotic singularities. In this process, we find the "toric duality" which relates different phases of D-brane probe theories. We realize later that the toric duality is an example of the more powerful Seiberg-duality so these different phases are related by the Seiberg duality. In the third direction, by using numerical calculation we get a strong evidence to support the second conjecture of Sen's three conjectures. We show that if the identity field is BRST exact state around the tachyon vacuum, the open string spectrum will decouple from the physics and leave only the closed string spectrum.

Thesis Supervisor: Amihay Hanany
Title: Professor of Physics

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Chapter 1

The introduction

String theory[1, 2, 3, 4], originally emerged as a description of strong interaction, has turned out to be the most promising candidate for unified theory. There are five consistent supersymmetric string theories: Type IIA, Type IIB, Type I $SO(32)$, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$. All of them contain the graviton naturally in their perturbative spectrum. They are also consistent theories of quantum gravity, at least in perturbation theory. Furthermore, all of them live on ten dimensional space-times so that these theories are big enough to include the known Standard model of lower energy description in four dimensional space-times. The way that string theory naturally unifies the Standard model and gravity is a very encouraging sign that we are in the right track.

Around 1995, two important works expanded our view about the relationship between the string theory and the unified theory. The first one is the emergence of M-theory[5, 6]. By the study of duality, especially the S-duality which relates the physics in strong interaction regions to the physics in weak interaction regions, we realized that all five string theories plus the eleven-dimensional supergravity theory are just a part of a more fundamental theory in eleven-dimensional space-times which is named as M-theory. In another word, all five string theories and the eleven-dimensional supergravity theory are just effective theories at some particular regions of the moduli space of M-theory. We can intuitively show this idea by a famous figure 1-1.

The second important work around 1995 was the understanding of D-branes. D-branes have been studied since 1989 in [7, 8] as a string propagating with Dirichilet boundary condition. However, only at 1995, Polchinski [9] realized that the D-brane is a BPS particle which carries Ramond-Ramond charge and breaks half of supersymmetries. As a Ramond-Ramond source required by string dualities, D-branes are intrinsic to Type II theories and can be used as a probe to investigate the string theory.

D-branes as a probe in string theory have a lot of merits. First they are heavier than strings so they could probe the distance smaller than the one seen by strings. Second, D-branes allow open strings ending on them. It has profound consequences. The massless spectrum of ended open strings provides a supersymmetric gauge field theory living on the world volume of D-branes. The study of relationships between the string theory and the lower energy world volume field theory of D-branes is the central task in recent few years.

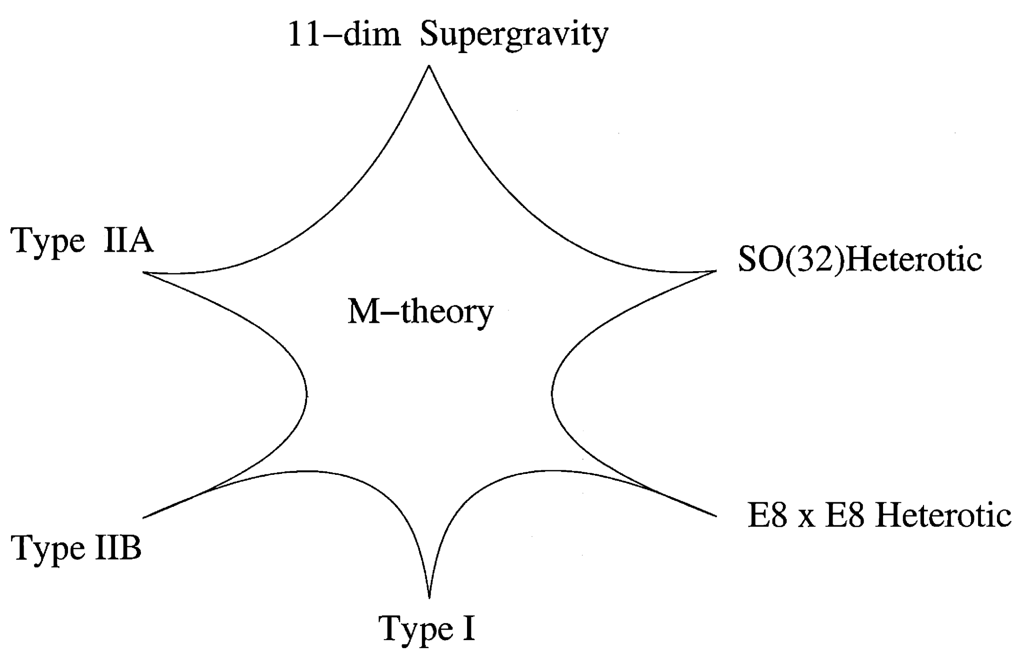


Figure 1-1: All string theories and the eleven-dimensional supergravity theory are limits of one theory, M-theory.

There are several works, which were along above main line and related to our studies, needed to be mentioned. The first one is the brane setup, i.e., the system of different kinds of branes(NS-branes, D-branes and orientifold planes). We will talk about it in more details in next chapter. The brane setup was first constructed in [10]. In that paper, Hanany and Witten used the brane system of NS-branes, D5-branes and D3-branes to exploit the three-dimensional supersymmetric gauge theory. Using the brane setup, it is easier to visualize the moduli space of gauge theories, the phase transition from the electric gauge theory to magnetic gauge theory as well as to derive the mirror symmetry of $N = 4$ three-dimensional gauge theories. More important, by consistent arguments from the field theory, they predicted the nontrivial s-configuration (i.e., there can not be more than one D3-brane between one NS-brane and one D5-brane) which was demonstrated later from various points of view. Brane setups serve as an dictionary between the string theory and the gauge field theory. Results in one part can be applied to another part. This two way communications have taught us a lot in recent years.

The second related work is the famous AdS/CFT corresponding. From the matching of low energy absorption cross-sections of various particles calculated by the string theory and the effective world-volume field theory of corresponding branes [11, 12], people wondered already that there may be a close relationship between field theories and string theories. However, it was Maldacena who put this conjecture into a clear form in [13]. In this paper, Maldacena suggested that " *Type IIB string theory on* $(AdS_5 \times S^5)_N$ *plus some appropriate boundary conditions (and possible also some boundary degrees of freedom) is dual to* $N = 4, d = 3 + 1$ $U(N)$ *super-Yang-Mills*". The basic idea behind this conjecture is that we have two different, but equivalent ways to describe a stack of Dp-branes. In one side, we have classical solutions in Type II closed string theory which are in terms of curved metric and other background fields including the Ramond-Ramond p-form potential. In another side, we have low energy effective $U(N)$ supersymmetric gauge theories living on the world volume of Dp-branes. Since these two different points of view describe same thing, they must equivalent to each other. However, these two pictures of D-branes are perturbatively valid for different regions in the moduli space. Perturbative field theory is valid when we keep $g_s N$ small while the low-energy gravitational description is perturbatively valid when the radius of curvature is much larger than the string scale, which is $g_s N$ very large. So AdS/CFT corresponding is a kind of strong-weak dualities.

The third related work is the study of geometries probed by D-branes [14]. The basic idea is that, from the D-brane perspective, the spacetime is a derived concept which emerges from the nontrivial moduli spaces of D-brane world volume gauge theories. In general, the analytic solutions for moduli spaces are difficult to get. However, in some particular cases we can have a powerful tool, i.e., toric geometry, to solve our problems. The authors of [14] explicitly demonstrated these ideas by world volume theories of D-branes probing orbifold singularities. They further found that corresponding linear sigma models are not generic in the sense that they probe only a part of the moduli space of the linear sigma model and see only geometric phases while project out non-geometric phases. A large part of our works will follow this line. We will show how to use the toric geometry to find corresponding gauge theories for more generic singularities which we do not know how to handle before. In this process we found the toric duality, which states that two different world volume theories are toric dual to each other if they have same toric variety as the moduli space. Later we realized that toric duality is nothing more, but the celebrating Seiberg-duality in toric cases. These studies expanded our understanding of the connection between geometry and physics.

The last related work is the study of string field theories, especially the Witten's Cubic String Field theory. String field theories have been studies about fifteen years already [15, 16, 17, 18]. However, recent interests on string field theories were intrigued by Sen's three conjectures [19, 20]. These three conjectures are:

- (i) The difference in energy between the perturbative and the tachyon vacuum exactly cancels the tension of the corresponding D-brane system;
- (ii) After the tachyon condenses, all open string degrees of freedom disappear, leaving us with the closed string vacuum; and
- (iii) Non-trivial tachyon field configurations correspond to lower-dimensional

D-branes.

Since the tachyon condensation is an off-shell process, we can not use the perturbative string theory to make the calculation. String field theory as an off-shell description provides us the tool to investigate above conjectures. In recent three years we have achieved a huge progress, where the first and third conjectures have been well understood. Now most pressing problems remained are to find the analytic solution for tachyon condensation and to understand the emergence of the closed string[120, 21].

The remainder part of this thesis is organized as follows. In the second chapter, we prepare some backgrounds, which are used extensively in later chapters. Starting from chapter three, we present works we have done in last a few years. In chapter three, the study of mirror symmetry of three dimensional $SO(n)$ and $Sp(k)$ gauge theories is showed. In chapter four, we develop the "inverse algorithm" and use it to get world volume theories of some toric singularities. In chapter five, we study the ambiguity in the inverse algorithm and present different phases for a given toric singularity. In chapter six, we give our understanding about the ambiguity found in chapter five and demonstrate that this ambiguity is a result of the well known Seiberg duality. In chapter seven, we move the study to the string field theory and show how to use the identity field to prove (numerically) the Sen's second conjecture. Finally we give a list of works done in these years.

Chapter 2

Some backgrounds

In this section, we will review some backgrounds which are needed for our works. These materials can be divided into three parts. In the first part we will introduce the brane setup and show how to use them to derive mirror pairs in three-dimensional $N = 4$ supersymmetric field theories. The second part is devoted to the "forward algorithm" presented in [14]. To understand the algorithm, we also introduce a little bit of the toric geometry. The third part is a brief introduction of string field theories, especially the bosonic string field theory established by Witten [15]. This will establish the frame which we will use to do some calculations.

2.1 The brane setup

Brane setup was first introduced in [10] (For a review, see reference [22]). Since then, the brane setup has been applied to discuss field theories in various dimensions. For example, in [10] the brane setup was used to study the relationship of certain moduli spaces of magnetic monopoles and Coulomb branches of certain three-dimensional supersymmetric theories. It was also used to derive the mirror symmetry of $N = 4$ three-dimensional supersymmetric theories in [10, 23], mirror symmetry of $N = 2$ three-dimensional supersymmetric theories in [24].

For the four-dimensional supersymmetric field theory, brane system has also a lot of applications. Witten used the $NS - D4$ brane system to discuss Seiberg-Witten curves of $N = 2$ theories by lifting it into the M-theory in [24]. This result was immediately generalized to the $NS - D4 - O4$ brane system in [26]. Elitzur et al [27, 28] used the $NS - NS' - D4$ system to study the Seiberg-duality of $N = 1$ $SU(N)$ gauge theories and by adding orientifold planes, Evans et al [29]managed to study the Seiberg-duality of $N = 1$ $SO(N)$ or $Sp(N)$ gauge theories. We can also discuss finite chiral theories by brane box models [30, 31]. There are many works done in this direction and we can not review them completely here.

For applications of brane setups to other dimensions we will just mention references here. The study of six-dimensional conformal theories can be found in [33, 32, 34]. The five dimensional gauge field theories and corresponding (p, q)-webs were investigated in [35, 36]. The two-dimensional field theory was discussed in [37].

From above brief survey, it is obviously that we can not discuss the brane setup and its applications thorough. In following we will introduce only the basic idea of brane setups.

2.1.1 Branes in string theory

In general we will have massless form-fields in the perturbative spectrum of string theories. Just as the 1-form field (gauge field A_μ) in QED coupling to electrons and monopoles, these form fields can couple to electron-like objects and monopole-like objects. These objects (BPS states) will have extended directions in general and are called "branes". We listed them in following table 2.1.1, where we have assumed the hodge dual relationship. For example, Neveu-Schwarz 6-form is hodge dual to Neveu-Schwarz 2-form so there is only one independent freedom. Also the D8-brane and D9-brane are little special [9] since there are no propagating states corresponding to 9-form and ten-form. This subtle point has been explained in detail in [9]. In loosing sense, the D9-brane is whole space-time and corresponds to Neveu-Schwarz boundary condition for strings while the D8-brane is the source of cosmological constants in Type IIB supergravity. In addition to above branes we have also the orientifold-planes which are fixed planes under a Z_2 action. This Z_2 symmetry reverses the spacetime coordinates as well as the orientation of strings, i.e.,

$$x^I(z,\bar{z}) \to -x^I(\bar{z},z); \quad I = p+1, ..., 9 \tag{2.1}$$

For every type of branes, we have corresponding orientifold planes, for example $D4 \leftrightarrow O4$, $M2 \leftrightarrow OM2$. Among them, Op planes are better understood while OM-planes and ON-planes still need more works.

Table 2.1: The summary of branes in string theories.

brane	Theory	form
F1(string)	IIA,IIB	Neveu-Schwarz 2-form
NS-brane	IIA,IIB	Neveu-Schwarz 6-form
D(-1)-brane	IIB	Ramond-Ramond 0-form
D0-brane	IIA	Ramond-Ramond 1-form
D1-brane	IIB	Ramond-Ramond 2-form
D2-brane	IIA	Ramond-Ramond 3-form
D3-brane	IIB	Ramond-Ramond 4-form (self dual)
D4-brane	IIA	Ramond-Ramond 5-form
D5-brane	IIB	Ramond-Ramond 6-form
D6-brane	IIA	Ramond-Ramond 7-form
D7-brane	IIB	Ramond-Ramond 8-form
D8-brane	IIA	Ramond-Ramond 9-form
D9-brane	IIB	Ramond-Ramond ten-form
M2-brane	M-theory	3-form
M5-brane	M-theory	6-form

The tension of a Dp-brane is

$$T_p = \frac{1}{g_s l_s^{p+1}} \tag{2.2}$$

where g_s is string coupling constants and l_s is the string length. Notice that in (2.2) we have chosen the normalization so that the tension of the fundamental string (F1) is $T_{F1} = \frac{1}{l_s^2}$. In this convention, the Dp-brane tension is equal to its RR-charge. The

RR-charge of the corresponding orientifold $Op^{\pm}$-planes are

$$Q_{Op^{\pm}} = \pm 2 \cdot 2^{p-5} Q_{Dp} \tag{2.3}$$

As we will discuss later, $Op^{\pm}$-planes exist on any dimension p. However, for $p \leq 6$ there are other types of orientifold-planes, for example $\widetilde{Op}^{\pm}$. The tension of NS-brane is

$$T_{NS} = \frac{1}{g_s^2 l_s^6} \tag{2.4}$$

Comparing it with the tension of the D5-brane, we see that

$$\frac{T_{D5}}{T_{NS}} = g_s \tag{2.5}$$

From it we conclude that in small coupling region, NS-branes are much heavier than D5-branes.

Type IIA string theory has $(1,1)$ spacetime supersymmetry, where the $(1,1)$ means one left and one right chiral supercharges in ten-dimensional space-times (see (2.6)). The spacetime supersymmetric charges generated by left and right moving Q_L, Q_R have opposite chirality:

$$\Gamma^0\Gamma^1...\Gamma^9 Q_L = +Q_L, \quad \Gamma^0\Gamma^1...\Gamma^0 Q_R = -Q_R. \tag{2.6}$$

Type IIB string theory has $(2,0)$ (two left chiral supercharges, see (2.7)) spacetime supersymmetry and chirality of charges are

$$\Gamma^0\Gamma^1...\Gamma^9 Q_L = +Q_L, \quad \Gamma^0\Gamma^1...\Gamma^0 Q_R = +Q_R. \tag{2.7}$$

We will focus on Type II theories since the Type I can be considered as the Type IIB string theory with orientifold planes and is a special case. Heterotic strings do not have D-branes, but do have NS-branes similar to these in Type II theories. D-branes are BPS saturated states and preserve half supersymmetries. More precisely, a Dp-brane with the world-volume extended along $X^{12..p}$ will preserve the supercharge of the form $\epsilon_L Q_L + \epsilon_R Q_R$ where

$$\epsilon_L = \Gamma^0\Gamma^1...\Gamma^p\epsilon_R \tag{2.8}$$

An anti Dp-brane will carry opposite charges and preserve another half supersymmetries. An NS five brane stretched in X^{12345} will preserve the supercharge of the form $\epsilon_L Q_L + \epsilon_R Q_R$ where for the Type IIA fivebrane

$$\epsilon_L = \Gamma^0\Gamma^1...\Gamma^5\epsilon_L, \qquad \epsilon_R = \Gamma^0\Gamma^1...\Gamma^5\epsilon_R, \tag{2.9}$$

while for the Type IIB fivebrane

$$\epsilon_L = \Gamma^0\Gamma^1...\Gamma^5\epsilon_L, \qquad \epsilon_R = -\Gamma^0\Gamma^1...\Gamma^5\epsilon_R, \tag{2.10}$$

Notice the projection (2.9) and (2.10) tell us that in non-chiral Type IIA theory the world volume theory in fivebrane will be chiral $(2,0)$ SUSY in six dimensions while in chiral Type IIB theory the world volume theory in fivebrane will be non-chiral $(1,1)$ SUSY in six dimensions.

The low energy worldvolume theory on an *infinite* Dp-brane is a $p+1$ dimensional field theory invariant under preserved 16 supercharges. The massless spectrum can be read out from the open string ending on this brane. They are a $p+1$ dimensional $U(1)$ gauge field A_μ, $9-p$ scalars X^I which parameterize fluctuations of the Dp-brane in transverse directions and fermions required by SUSY. The bosonic part of low energy worldvolume theory is

$$S = \frac{1}{g_{YM}^2}\int d^{p+1}x(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{l_s^4}\partial_\mu X^I\partial^\mu X_I) \tag{2.11}$$

with the $U(1)$ gauge coupling on the brane given by

$$g_{YM}^2 = g_s l_s^{p-3} \tag{2.12}$$

At high energy, we need to include massive modes of the open string as well as the interaction with the closed string in the $9+1$ dimensional bulk of spacetime. So to study the supersymmetric Yang-Mills theory living on the brane we must decouple gauge theory from the gravity and massive string modes. This requires the limit $l_s \to 0$ with fixed g_{YM}. Using (2.12), it means that (2.11) describes the UV behavior for $p \leq 3$ and IR behavior for $p \geq 4$.

There is a $U(1)$ gauge theory living on one Dp-brane. To get non-Abelian theory, we can put on a stack of nearby parallel Dp-branes and Chan-Paton indices to fields. In this case, the action becomes

$$S = \frac{1}{g_{YM}^2} \int d^{p+1}x Tr(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{l_s^4}\mathcal{D}_\mu X^I \mathcal{D}^\mu X_I) + V(X^I) \tag{2.13}$$

where X^I are adjoint fields of corresponding gauge group. One interesting thing is that the potential for X^I is

$$V \sim \frac{1}{l_s^8 g_{YM}^2} \sum_{I,J} Tr[X^I, X^J]^2 \tag{2.14}$$

which indicates that the moduli space of Higgs branch is parametered by eigenvalues of $\vec{X}$

$$\vec{x}_i = \langle \vec{X}_{ii} \rangle, \qquad i = 1, ..., N \tag{2.15}$$

In the brane setup, the $i-th$ vector $\vec{x}_i$ will geometrically denote the position of $i-th$ brane in transverse directions. The vacuum expectation values of (2.15) will generally break the $U(N)$ gauge symmetry to $U(1)^N$ and off-diagonal gauge fields will get masses

$$m = \frac{1}{l_s^2} |\vec{x}_i - \vec{x}_j| \tag{2.16}$$

Geometrically above masses will be the length of the fundamental string stretched between $i-th$ brane and $j-th$ brane in the unit of string length.

After discussing the low energy world volume theory of D-branes, we turn to the world volume theory of NS fivebranes. For a single Type IIA NS fivebrane the massless field is the tensor multiplet of six dimensional $(2,0)$ SUSY, which is consisted of a self-dual 2-form $B_{\mu\nu}$ field, five scales and corresponding fermion fields. For a single Type IIB NS fivebrane the massless field is the vector multiplet of six dimensional

(1, 1) SUSY, which is consisted of a six-dimensional gauge field, four scalars and corresponding fermions. Recalling that scales of D-brane world volume theories describe the fluctuation of D-branes in transverse directions, here we should have same explanation. For the Type IIB NS-brane, four scalars match four transverse directions in ten dimensional space-times. But how to explain the Type IIA NS-brane where we have five scalars? In fact, the fifth scalar indicates another hidden direction which will finally lead us to the eleven dimensional M-theory.

The low energy world volume theory of N Type IIB NS-branes is a $5+1$ dimensional $(1,1)$ $U(N)$ SYM theory with gauge coupling constant

$$g_{YM}^2 = l_s^2 \tag{2.17}$$

Comparing with the gauge coupling constant living on the D5-brane, we found the ratio

$$\frac{g_{YM,D5}^2}{g_{YM,NS}^2} = g_s \tag{2.18}$$

The massless spectrum of above low energy theory can be obtained by analyzing the D-string ending on these NS-branes. The low energy world volume theory of N Type IIA NS-branes is more exotic field theory.

Now we turn to M2-branes and M5-branes. The preserved supercharges are given by

$$\epsilon = \Gamma^0\Gamma^1 ... \Gamma^p \epsilon, \tag{2.19}$$

and their tensions are fixed by SUSY to be

$$T_p = \frac{1}{l_{11}^{p+1}}, \qquad p = 2, 5 \tag{2.20}$$

where l_{11} is the eleven dimensional Planck scale. The world volume theories of M2-branes and M5-branes are not very clear at this moment.

To get the Type IIA theory from the M-theory, we need to compactify the M-theory on $R^{1,9} \times S^1$ where the radius of S^1 is R_{10}. The relationship of coupling constants are

$$\frac{R_{10}}{l_{11}^3} = \frac{1}{l_s^2}, \tag{2.21}$$

$$R_{10} = g_s l_s \tag{2.22}$$

Thus the strong coupling limit $g_s \to \infty$ of Type IIA theory is equivalent to un-compactify the M-theory.

To get the Type IIB theory from the M-theory is more involving since Type IIB theory has explicitly $SL(2, Z)$ symmetry which does not showed in M-theory. To realize this symmetry we need to wrap the M-theory on a finite two-torus. In the case of zero vacuum expectation value of the massless RR scalar the two-torus is rectangular with radius R_9, R_{10}. Then the relationship of coupling constants are

$$\frac{R_{10}}{l_{11}^3} = \frac{1}{l_s^2}, \tag{2.23}$$

$$\frac{R_9}{l_{11}^3} = \frac{1}{g_s l_s^2}, \tag{2.24}$$

$$\frac{R_9 R_{10}}{l_{11}^3} = \frac{1}{R_B} \tag{2.25}$$

where R_B is the radius of Type IIB theory wrapping on S^1.

Now let us turn to a new direction: the duality. These branes we introduced above are related to each other by all kinds of duality transformations. They are called the U-duality[38]. However, the ones we familiar with and most used are T-duality and S-duality. By T-duality we can relate Dp-branes in the Type IIA (IIB) theory to D(p+1)-branes or D(p-1)-branes in the Type IIB (IIA) theory dependent on if the wrapped direction is transverse or parallel to them. Under the T-duality, moment modes of the fundamental string in one theory is traded to wrapping modes of the fundamental string in another theory and vice versa. The coupling constant has relationship

$$R_i^A R_i^B = l_s^2, \tag{2.26}$$

$$\frac{g_s^A}{\sqrt{R_i^A}} = \frac{g_s^B}{\sqrt{R_i^B}}. \tag{2.27}$$

For NS fivebranes, the T-duality property is that if the T-dual direction is parallel to the fivebrane, the NS-brane in Type IIA (IIB) will become the NS-brane in Type IIB (IIA), but if the T-dual direction is transverse to the fivebrane, the NS-brane in Type IIA will become the KK-monopole while the Type IIB NS5-brane will turn into the general (p,q)-web.

As we have mentioned, Type IIB string theory has explicitly $SL(2,Z)$ symmetry which includes both T-duality and S-duality. We have discussed the T-duality above. Under the S-duality, the F1-string is exchanged to the D1-brane, the NS5-brane to the D5-brane while the D3-brane is invariant. If we apply both of them arbitrarily, we will have general (p,q) brane webs for dimension one, five and seven.

2.1.2 The brane system and three dimensional field theory

As we have mentioned above, there are a lot of brane systems which are tailed for different problems. Here we will not review all of them, but try just a special case which relates to our concern later. Although it is special, it shows all essential properties of brane setups already. This special case is the NS5-D5-D3 brane system which preserves 8 supercharges at final [10].

In this system, we usually put NS5-branes along X^{12345}, D5-branes along X^{12789} and D3-branes along X^{126}. Using equations (2.7), (2.8) and (2.10) it is easy to see that there are eight supercharges left. The configuration breaks the Lorentz group $SO(1,9)$ to $SO(1,2)\times SO(3)\times SO(3)$, where the $SO(1,2)$ acts on x^{012}, first $SO(3)\equiv SO(3)_V$ acts on $\mathbf{m}=(x^3,x^4,x^5)$ and the second $SO(3)\equiv SO(3)_H$ acts on $\mathbf{w}=(x^7,x^8,x^9)$. The two double covering groups $SU(2)$ of these two $SO(3)$, which have clear geometric picture, will become symmetries of Coulomb and Higgs branches in the corresponding field theories. The way that we can see these symmetries pictorially in the brane setup is a great advantage of this method.

The key part of above system is the D3-brane. More precisely, we will let D3-

branes ending on NS5-branes or D5-branes. Since these five branes have two more extended directions than D3-branes', these five branes are much heavier than D3-branes and can be considered as backgrounds for field theories living on the world volume of D3-branes. Furthermore, since the X^6 direction is finite for ending D3-branes, the effective field theory is three-dimensional gauge theory. Above all observations lead to two important conclusions:

- (a). that the parameters $(z, t, \mathbf{m}, \mathbf{w})$ specifying the positions of five branes will be interpreted as coupling constants in the effective gauge field theory on D3-brane world volume.

- (b). the positions of D3-branes on five branes are dynamical moduli which parameterize the vacua of field theories.

Having above general idea, let us give more details. First let us identify the symmetry of theories. Since there are eight supercharges left, we get $N = 4$ supersymmetric three-dimensional field theory. The R-symmetry $SO(4) = SU(2)_V \times SU(2)_H$ are nothing more, but the two double covering groups of two $SO(3)$ rotations we met above.

Second let us discuss the matter content living on D3-branes. This will be a little of complex and we will do it step by step. First let us consider the infinite D3-brane. In this case, we have $N = 4$ supersymmetric gauge theory in four-dimension and the matter content is the $N = 4$ vector multiplet. An $N = 4$ vector multiplet in four-dimension can be divided into two parts: an $N = 2$ vector multiplet in four-dimension and an $N = 2$ hyper-multiplet in four-dimension. After one dimension reduction we get an $N = 4$ vector multiplet in three-dimension and an $N = 4$ hyper-multiplet in three-dimension. Next we need to know the projection when D3-branes end on five branes. The results are

- (a). If D3-branes end on NS5-branes, the $N = 4$ vector multiplet will be kept while the $N = 4$ hyper-multiplet will be projected out.

- (b). If D3-branes end on D5-branes, the $N = 4$ hyper-multiplet will be kept while the $N = 4$ vector multiplet will be projected out.

- (c). If one side of D3-branes end on D5-branes and another side, on NS5-branes, all massless modes are projected out. The infrared theory has a unique vacuum with a mass gap.

The gauge theory arising when two sides of D3-branes end on NS5-branes is called "electric gauge theory". The gauge coupling constant is

$$\frac{1}{g_{ele}^2} = |t_1 - t_2| \tag{2.28}$$

where t_1, t_2 parametrized the X^6 coordinates of two NS5-branes. The gauge theory arising when two sides of D3-branes ending on D5-branes is called "magnetic gauge theory". The gauge coupling constant is

$$\frac{1}{g_{mag}^2} = |z_1 - z_2| \tag{2.29}$$

where z_1, z_2 parameterize the X^6 coordinates of two D5-branes. We call them this way because after the S-duality, they are exchanged to each other just like the electric charge and magnetic charge are exchanged under the Montonet-Olive duality. To simplify the discussion we will mainly focus on the electric gauge theory.

There is still a little piece we have left. After ending D3-branes on NS5-branes at two sides, we left only the $N = 4$ vector multiplet. To get the $N = 4$ hyper-multiplet coupling to this gauge field, we can do two things. The first way is to put a bunch of D5-branes (k D5-branes) between these two NS5-branes. When D3-branes moving on the NS5-brane along directions X^{345}, D5-branes can touch D3-branes and contribute massless modes, hence k $N = 4$ hyper-multiplets. The second way is to put some D3-branes, for example k again, at another side of one NS-branes (or both). The D3-branes at the two sides of the NS-brane can meet and give massless modes, i.e., k $N = 4$ hyper-multiplets.

Above two methods are in fact equivalent to each other since they are related to each other by the famous "brane transaction", see figure 2-1. This can be explained by the requirement of the conservation of "linking numbers" of five branes. The linking number of an NS5-brane is defined as

$$L_{NS} = \frac{1}{2}(r - l) + (L - R) \tag{2.30}$$

where r, l are the numbers of D5-branes at its right and left sides while L, R are the numbers of D3-branes ending on its left and right sides. The linking number of a D5-brane is defined in a similar way as

$$L_{D5} = \frac{1}{2}(r - l) + (L - R) \tag{2.31}$$

where r, l are the numbers of NS5-branes at its right and left sides.

The last piece we need to finish this subsection is the "s-configuration". This was first proposed in [10] to get the consistent result from the field theory. The basic idea is that if there are more than one D3-brane connecting an NS-brane to a D5-brane, the brane setup is not supersymmetric. This result can be generalized to other dimensions by duality. We will also generalize it to the case in the presence of orientifold planes. More discussions can be found later.

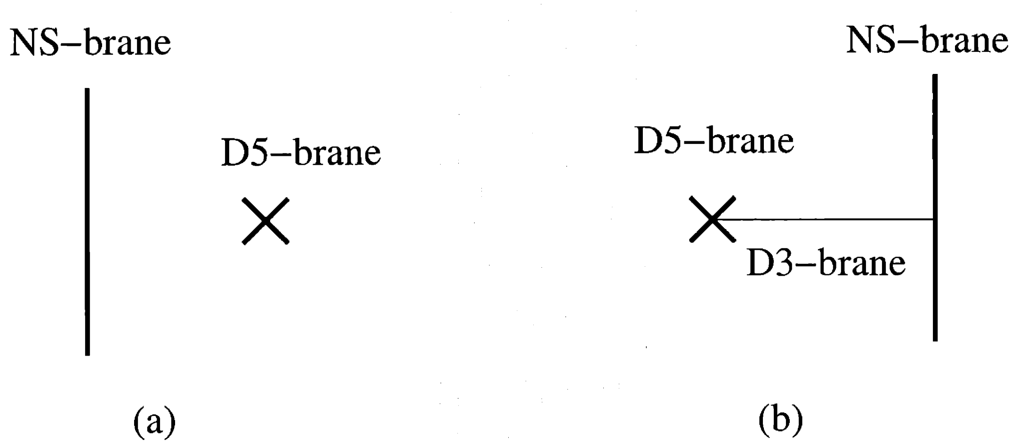


Figure 2-1: One D3-brane is created after the D5-brane crossing the NS-brane.

2.1.3 A simple application: the mirror symmetry

Mirror symmetry of $N = 4$ three-dimensional field theory was first discussed in [39]. The basic idea is that the R-symmetry $SO(4) = SU(2)_L \times SU(2)_R$ make sure that moduli spaces of the Coulomb branch and the Higgs branch are both hyper-Kahler manifolds. Furthermore, moduli spaces and other parameters transform under the group $SU(2)_L \times SU(2)_R$ as

Table 2.2: The symmetries of gauge field theory.

	$SU(2)_L \times SU(2)_R$
Higgs branch	$(1, 3)$
Coulomb branch	$(3, 1)$
gauge coupling	$(1 + 3, 1)$
Mass term	$(3, 1)$
Fayet-Iliopoulos term	$(1, 3)$

The authors of [39] found that, at the nontrivial IR fixed point, there is a duality which relates two different theories by exchanging: (1) $SU(2)_L$ and $SU(2)_R$; (2) The Coulomb branch and the Higgs branch; (3) Mass terms and Fayet-Iliopoulos D-terms. Such a pair is called the "mirror pair". In [39], they showed that $U(1)$ gauge theory with $n+1$ electrons is mirrored to gauge theory given by A_n singularity while $SU(2)$ gauge theory with n quarks is mirrored to gauge theory given by D_n singularity.

Continuing the study of [39], Hanany and Witten [10] used the brane setup to derive the mirror pairs for general situations in much easy way. First there is a correspondence $SU(2)_L \to SU(2)_V$ and $SU(2)_R \to SU(2)_H$ between the field theory and the brane setup. Second, the mirror symmetry in string theory is nothing more but a simple S-duality. Using the S-duality property of NS5-branes, D5-branes and D3-branes, it is easy to identify the mirror pair.

We will use a simple example to demonstrate the method. The example will be $U(2)$ theory with four hyper-multiplets. To find its mirror theory, we do following steps. First we draw the corresponding brane setup in part (a) of Figure 2-2. Second, we go to the Higgs branch by breaking D3-branes among D5-branes and NS5-branes as part (b). Notice the way we break D3-branes makes sure that there is no s-configuration. At third step we make brane transaction in part (c). In last step, we make the S-duality and reach the final mirror theory $U(1) \times U(2) \times U(1)$ with two bi-fundamentals and two $U(2)$ hyper-multiplets as in part (d).

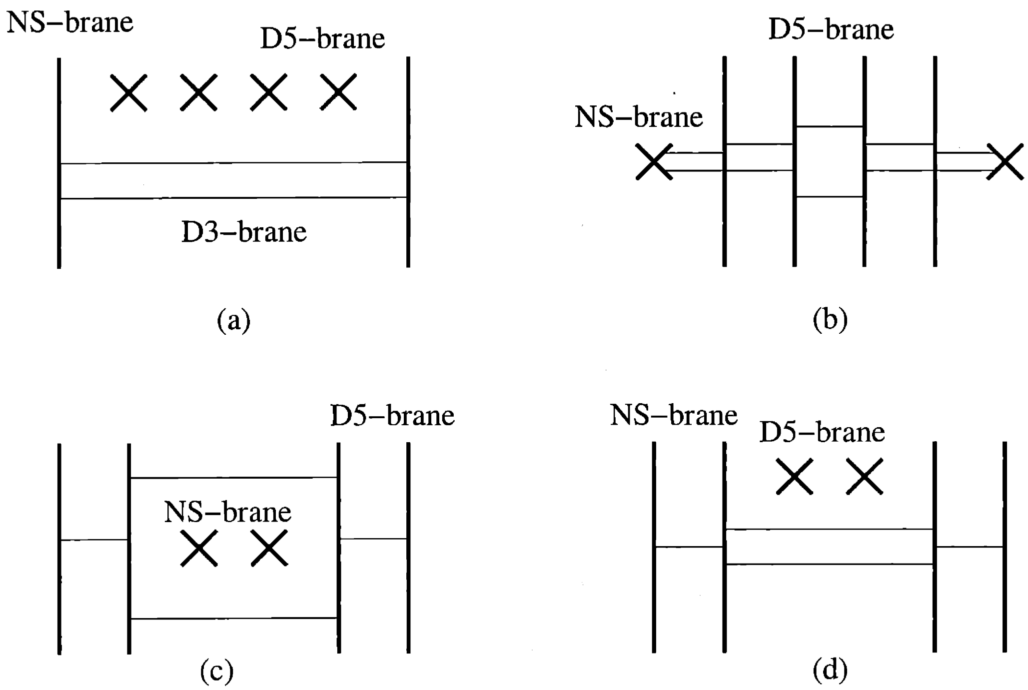


Figure 2-2: The brane method to derive the mirror theory of $U(2)$ theory with four hyper-multiplets.

2.2 Orbifold Resolution by D-branes: The Forward Algorithm

As we have mentioned at the introduction, it is interesting to use D-branes as a probe to study the geometry of backgrounds. This direction was initiated by Douglas and Moore in [40] and continued in [41, 42]. In these papers, it was showed that the effective world volume theory of D-brane probes is a projection of SYM with sixteen supercharges, with Fayet-Iliopoulos terms controlled by twist sector moduli.

For C^2/Γ case, it was observed that the classical gauge theory moduli space is the resolved orbifold. To reach this result, the hyper-Kahler quotient construction of Kronheimer was used [43].

We would like to see if similar thing happens in three dimensions, especially C^3/Γ. In this case, the world volume theory has only four supercharges, i.e., if we use the D3-brane as the probe, we will get $N = 1$ four-dimensional SYM. Since there are less supersymmetries, the manifold would not be hyper-Kahler quotient, but rather singular Kahler quotient. As singularities get resolved in two dimensions, we expect singular Kahler quotient resolved by, for example, blow-up. To show this is true in general will be very difficult. However, for orbifold singularities we have a powerful tool, toric geometry, to help us. Using this tool in [14], it was showed that above expectation is true. The superpotential defines the complex structure and the blow-up, while the D-term defines the periods of the metric (though the full metric depends on both data).

To carry out above idea, the starting point will be the world volume theory of the D-brane probe. When we discuss the singular (complex) three-dimensional space, we usually use D3-branes as the probe. For orbifold singularities, the world volume theory is easy to work out [40, 42, 44, 45, 46, 31, 30]. The result can be summarized into the so called "quiver diagram". Quiver diagram (or Moose diagram) is a easy way to encode the matter content of a given theory, which we will introduce more later. To complete the $N = 1$ theory, we need also to specify the superpotential.

Although it is easy to get world volume theories of orbifold singularities, it is not easy for other singularities. This is one main motivation for our inverse algorithm which will be presented later. The idea is that for toric singularities, it is easy to find the geometry from toric data, then by inverting the line of [14], we may end up at the starting point, i.e., the world volume theory of D3-brane probes on that singularity.

2.2.1 A quick introduction of toric geometry

In this part, we will briefly introduce the toric geometry. There are two books [47, 48] for this topic, but for our purpose we will follow the work of Greene in [49].

There are three ways to see the toric geometry. The first way is by holomorphic quotient. It is a very intuitive way. For example, we are familiar with the expression for complex projective space CP^n

$$CP^n = \frac{C^{n+1} - \{0, 0, ..., 0\}}{C^*} \tag{2.32}$$

A toric variety is just the generalization of above quotient. We remove a point set F_Δ instead of the origin and quotient by a number of C^* actions, the n-dimensional toric variety can be written as

$$V = \frac{C^m - F_\Delta}{(C^*)^p}, \quad m - p = n \tag{2.33}$$

To fix a given toric variety, we need to fix the point set F_Δ and C^* actions (i.e., the action of multiplying nonzero complex number). It was found that these data can be encoded into a certain simple combinatorial data which will be the idea for the third method.

The second way is symplectic quotient. The basic idea is to realize the holomorphic quotient in two steps by writing $C^* = R_+ \times U(1)$. The first step is called "moment map" which fixes the action of R_+. The second step is moduli the $U(1)$ phase factor. Physicist likes the second way because it is explicitly realized by the linear sigma model[50] where the moment map is just the D-term of supersymmetric gauge theory and $U(1)$ phase action is just the $U(1)$ gauge transformation.

The third way which is used a lot in mathematics literatures is to use "fan" Δ. A fan is a collection of *strongly convex rational polyhedral cones* σ_i in the real vector space $N_R = N \otimes_Z R$. Every cone σ_i defines a coordinate patch of V and will be glued together properly. To make the connection to lattice data more clear, we define the dual cone $\check{\sigma}_i$ in M_R as

$$\check{\sigma}_i = \{m \in M_R : \langle m, n \rangle \geq 0, \forall n \in \sigma_i\} \tag{2.34}$$

where $M_R = M \otimes_Z R$ and M is the dual lattice of N, $M = Hom(N, Z)$. Using this we can easily describe the patch. For each dual cone $\check{\sigma}_i$ we choose a finite set of elements $\{m_{i,j} \in M, j = 1, ..., r_i\}$ such that

$$\check{\sigma}_i \cap M = Z_{\geq 0} m_{i,1} + ... + Z_{\geq 0} m_{i,r_i} \tag{2.35}$$

This set has R relations

$$\sum_{j=1}^{r_i} p_{s,j} m_{i,j} = 0, \qquad s = 1, ..., R \tag{2.36}$$

We choose the relations such that every other relation among them can be written as a linear combination of above R's by integer coefficients. Knowing this data we can write the patch as

$$U_{\sigma_i} = \{(u_{i,1}, ..., u_{i,r_i}) \in C^{r_i} | u_{i,1}^{p_{s,1}} u_{i,2}^{p_{s,2}} ... u_{i,r_i}^{p_{s,r_i}} = 1, \quad \forall s\} \tag{2.37}$$

Using same way we can glue two patches together. First we find the relations

$$\sum_{l=1}^{r_i} q_{s,l} m_{i,l} + \sum_{l=1}^{r_j} q'_{s,l} m_{i,l} = 0 \tag{2.38}$$

For every relation we have coordinate transition relation

$$u_{i,1}^{q_{s,1}} u_{i,2}^{q_{s,2}} ... u_{i,r_i}^{q_{s,r_i}} u_{j,1}^{q'_{s,1}} u_{j,2}^{q'_{s,2}} ... u_{j,r_j}^{q'_{s,r_j}} = 1 \tag{2.39}$$

Now we record some results in toric geometry:

- (1). *Let V be a toric variety associated to a fan Δ in N. V is smooth if for each cone σ in the fan we can find an integral basis $\{n_1, ..., n_n\}$ of N and a integer $r \leq n$ such that $\sigma = R_{\geq 0} n_1 + ... + R_{\geq 0} n_r$.*

- (2). To resolve the singularity, we can subdivide the fan. This process is called "blow-up".

- (3). In order to have nothing new to canonical class when resolving the singularity, the singularity must be "canonical Gorenstein singularity".

- (4). *The singularities of the affine toric variety U_σ are canonical Gorenstein singularities if all the points in S lie in an affine hyperplane H in N_R of the form*

$$H = \{x \in N_R | \langle m, x \rangle = 1\}, \tag{2.40}$$

for some $m \in M$ and if there are no lattice points $x \in \sigma \cap N$ with $0 < \langle m, x \rangle < 1$.

2.2.2 The Forward Procedure: Extracting Toric Data From Gauge Theories

We shall here give a brief review of the procedures involved in going from gauge theory data on the D-brane to toric data of the singularity, using primarily the notation and concepts from [14]. In the course thereof special attention will be paid on how toric diagrams, SUSY fields and linear σ-models weave together.

A stack of n D-brane probes on algebraic singularities gives rise to SUSY gauge theories with product gauge groups resulting from the projection of the $U(n)$ theory on the original stack by the geometrical structure of the singularity. For orbifolds $\mathbb{C}^k/\Gamma$, we can use the structure of the finite group Γ to fabricate product $U(n_i)$ gauge groups [40, 42, 44]. For toric singularities, since we have only (Abelian) $U(1)$ toroidal actions, we are so far restricted to product $U(1)$ gauge groups[1]. In physical terms, we have *a single D-brane probe.* Extensive work has been done in [84, 14] to see how the geometrical structure of the variety can be thus probed and how the gauge theory moduli may be encoded. The subclass of toric singularities, namely Abelian orbifolds, has been investigated to great detail [40, 69, 14, 80, 84] and we shall make liberal usage of their properties throughout.

Now let us consider the world-volume theory on the D-brane probe on a toric singularity. Such a theory, as it is a SUSY gauge theory, is characterised by its matter content and interactions. The former is specified by quiver diagrams which in turn give rise to **D-term** equations; the latter is given by a superpotential, whose partial derivatives with respect to the various fields are the so-called **F-term** equations.

[1]Proposals toward generalisations to D-brane stacks have been made [84].

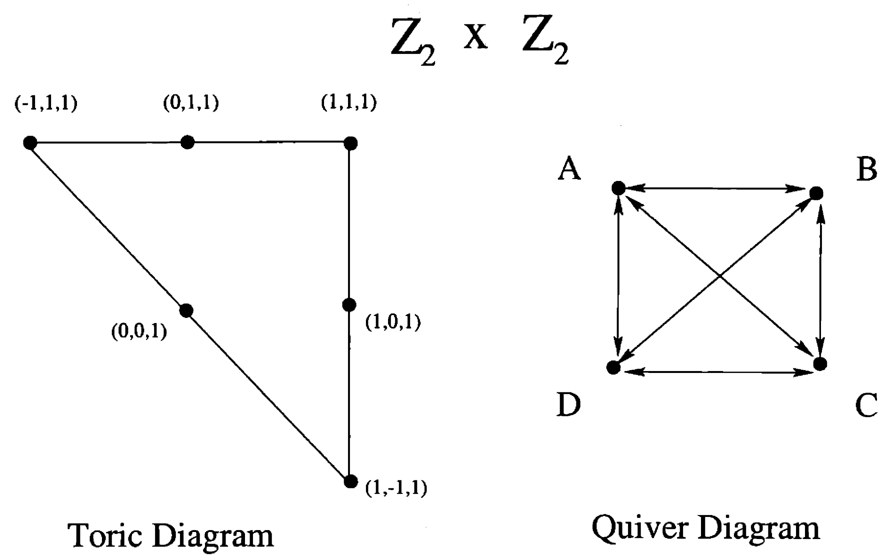


Figure 2-3: The toric diagram for the singularity $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$ and the quiver diagram for the gauge theory living on a D-brane probing it. We have labelled the nodes of the toric diagram by columns of G_t and those of the quiver, with the gauge groups $U(1)_{\{A,B,C,D\}}$.

F and D-flatness subsequently describe the (classical) moduli space of the theory. The basic idea is that the D-term equations together with the FI-parametres, in conjunction with the F-term equations, can be concatenated together into a matrix which gives the vectors forming the dual cone of the toric variety which the D-branes probe. We summarise the algorithm of obtaining the toric data from the gauge theory in the following, and to illuminate our abstraction and notation we will use the simple example of the Abelian orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ as given in Figure 2-3.

1. Quivers and D-Terms:

   (a) The bi-fundamental matter content of the gauge theory can be conveniently encoded into a **quiver diagram** $\mathcal{Q}$, which is simply the (possibly directed) graph whose **adjacency matrix** a_{ij} is precisely the matrix of the bi-fundamentals. In the case of an Abelian orbifold[2] prescribed by the group Γ, this diagram is the McKay Quiver (i.e., for the irreps R_i of Γ, a_{ij} is such that $R\otimes R_i = \oplus_j a_{ij} R_j$ for some fundamental representation R). We

[2]This is true for all orbifolds but of course only Abelian ones have known toric description.

denote the set of nodes as $\mathcal{Q}_0 := \{v\}$ and the set of the edges, $\mathcal{Q}_1 := \{a\}$. We let the number of nodes be r; for Abelian orbifolds, $r = |\Gamma|$ (and for generic orbifolds r is the number of conjugacy classes of Γ). Also, we let the number of edges be m; this number depends on the number of supersymmetries which we have. The adjacency matrix (bi-fundamentals) is thus $r \times r$ and the gauge group is $\prod\limits_{j=1}^{r} SU(w_j)$. For our example of $\mathbb{Z}_2 \times \mathbb{Z}_2$, $r = 4$, indexed as 4 gauge groups $U(1)_A \times U(1)_B \times U(1)_C \times U(1)_D$ corresponding to the 4 nodes, while $m = 4 \times 3 = 12$, corresponding to the 12 arrows in Figure 2-3. The adjacency matrix for the quiver is $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.
Though for such simple examples as Abelian orbifolds and conifolds, brane setups and [44] specify the values of w_j as well as a_{ij} completely[3], there is yet no discussion in the literature of obtaining the matter content and gauge group for generic toric varieties in a direct and systematic manner and a partial purpose of this note is to present a solution thereof.

(b) From the $r \times r$ adjacency matrix, we construct a so-called $r \times m$ incidence matrix d for $\mathcal{Q}$; this matrix is defined as $d_{v,a} := \delta_{v,head(a)} - \delta_{v,tail(a)}$ for $v \in \mathcal{Q}_0$ and $a \in \mathcal{Q}_1$. Because each column of d must contain a 1, a -1 and the rest 0's by definition, one row of d is always redundant; this physically signifies the elimination of an overall trivial $U(1)$ corresponding to the COM motion of the branes. Therefore we delete a row of d to define the matrix Δ of dimensions $(r-1) \times m$; and we could always extract d from Δ by adding a row so as to force each column to sum to zero. This matrix Δ thus contains almost as much information as a_{ij} and once it is specified, the gauge group and matter content are also, with the exception that precise adjoints (those charged under the same gauge group factor and hence correspond to arrows that join a node to itself) are not manifest. For our example the 4×12 matrix d is as follows and Δ is the top 3 rows:

[3]For arbitrary orbifolds, $\sum\limits_j w_i n_i = |\Gamma|$ where n_i are the dimensions of the irreps of Γ; for Abelian case, $n_i = 1$.

$$
d = \left(\begin{array}{c|cccccccccccc}
 & X_{AD} & X_{BC} & X_{CB} & X_{DA} & X_{AB} & X_{BA} & X_{CD} & X_{DC} & X_{AC} & X_{BD} & X_{CA} & X_{DB} \\
A & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\
B & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\
C & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & 0 \\
\hline
D & 1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & -1
\end{array}\right)
$$

(c) The moment maps, arising in the sympletic-quotient language of the toric variety, are simply $\mu := d \cdot |x(a)|^2$ where $x(a)$ are the affine coordinates of the $\mathbb{C}^r$ for the torus $(\mathbb{C}^*)^r$ action. Physically, $x(a)$ are of course the bi-fundamentals in chiral multiplets (in our example they are $X_{ij \in \{A,B,C,D\}}$ as labelled above) and the D-term equations for each $U(1)$ group is [50]

$$D_i = -e^2(\sum_a d_{ia}|x(a)|^2 - \zeta_i)$$

with ζ_i the FI-parameters. In matrix form we have $\Delta \cdot |x(a)|^2 = \vec{\zeta}$ and see that D-flatness gives precisely the moment map. These ζ-parameters will encode the resolution of the toric singularity as we shall shortly see.

2. <u>Monomials and F-Terms:</u>

(a) From the super-potential W of the SUSY gauge theory, one can write the F-Term equation as the system $\frac{\partial}{\partial X_j} W = 0$. The remarkable fact is that we could solve the said system of equations and express the m fields X_i in terms of $r+2$ parameters v_j which can be summarized by a matrix K.

$$X_i = \prod_j v_j^{K_{ij}}, \qquad i = 1, 2, .., m; \quad j = 1, 2, .., r+2 \tag{2.41}$$

This matrix K of dimensions $m \times (r+2)$ is the analogue of Δ in the sense that it encodes the F-terms and superpotential as Δ encodes the D-terms and the matter content. In the language of toric geometry K defines a cone[4] $\mathbf{M}_+$: a non-negative linear combination of m vectors $\vec{K}_i$

[4]We should be careful in this definition. Strictly speaking we have a lattice $\mathbf{M} = \mathbb{Z}^{r+2}$ with its dual lattice $\mathbf{N} \cong \mathbb{Z}^{r+2}$. Now let there be a set of $\mathbb{Z}_+$-independent vectors $\{\vec{k}_i\} \in \mathbf{M}$ and a cone is

in an integral lattice $\mathbb{Z}^{r+2}$.

For our example, the superpotential is

$$W = X_{AC}X_{CD}X_{DA} - X_{AC}X_{CB}X_{BA} + X_{CA}X_{AB}X_{BC} - X_{CA}X_{AD}X_{DC}$$
$$+X_{BD}X_{DC}X_{CB} - X_{BD}X_{DA}X_{AB} - X_{DB}X_{BC}X_{CD},$$

giving us 12 F-term equations and with the manifold of solutions parameterizable by 4 + 2 new fields, whereby giving us the 12×6 matrix (we here show the transpose thereof, thus the horizontal direction corresponds to the original fields X_i and the vertical, v_j):

$K^t =$

	X_{AC}	X_{BD}	X_{CA}	X_{DB}	X_{AB}	X_{BA}	X_{CD}	X_{DC}	X_{AD}	X_{BC}	X_{CB}	X_{DA}
v_1	1	0	0	1	1	0	0	1	0	0	0	0
v_2	0	1	1	0	−1	0	0	−1	0	0	0	0
v_3	0	0	0	0	0	1	0	1	0	1	0	1
v_4	0	0	0	0	1	0	1	0	0	−1	0	−1
v_5	0	0	−1	−1	0	0	0	0	1	1	0	0
v_6	0	0	1	1	0	0	0	0	0	0	1	1

For example, the third column reads $X_{CA} = v_2 v_5^{-1} v_6$, i.e., $X_{AD}X_{CA} = X_{BD}X_{CB}$, which the the F-flatness condition $\frac{\partial W}{\partial X_{DC}=0}$. The details of obtaining W and K from each other are discussed in [14, 84] and Subsection 3.4.

(b) We let T be the space of (integral) vectors dual to K, i.e., $K \cdot T \geq 0$ for all entries; this gives an $(r+2) \times c$ matrix for some positive integer c. Geometrically, this is the definition of a dual cone $\mathbf{N}_+$ composed of vectors $\vec{T}_i$ such that $\vec{K} \cdot \vec{T} \geq 0$. The physical meaning for doing so is that K may have negative entries which may give rise to unwanted singularities and hence we define a new set of c fields p_i (*a priori* we do not know the number c and we present the standard algorithm of finding dual cones in

defined to be generated by these vectors as $\sigma := \{\sum_i a_i \vec{k}_i \mid a_i \in \mathbb{R}_{\geq 0}\}$; Our $\mathbf{M}_+$ should be $\mathbf{M} \cap \sigma$. In much of the literature $\mathbf{M}_+$ is taken to be simply $\mathbf{M}'_+ := \{\sum_i a_i \vec{k}_i \mid a_i \in \mathbb{Z}_{\geq 0}\}$ in which case we must make sure that any lattice point contained in $\mathbf{M}_+$ but not in $\mathbf{M}'_+$ must be counted as an independent generator and be added to the set of generators $\{\vec{k}_i\}$. After including all such points we would have $\mathbf{M}'_+ = \mathbf{M}_+$. Throughout our analyses, our cone defined by K as well the dual cone T will be constituted by such a complete set of generators.

the Appendix). Thus we reduce (2.41) further into

$$v_j = \prod_\alpha p_\alpha^{T_{j\alpha}} \tag{2.42}$$

whereby giving $X_i = \prod_j v_j^{K_{ij}} = \prod_\alpha p_\alpha^{\sum_j K_{ij} T_{j\alpha}}$ with $\sum_j K_{ij} T_{j\alpha} \geq 0$. For our $\mathbb{Z}_2 \times Z_2$ example, $c = 9$ and

$$T_{j\alpha} = \left(\begin{array}{c|ccccccccc} & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 \\ \hline X_{AC} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ X_{BD} & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ X_{BA} & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ X_{CD} & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ X_{AD} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ X_{CB} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array} \right)$$

(c) These new variables p_α are the matter fields in Witten's linear σ-model. How are these fields charged? We have written $r+2$ fields v_j in terms of c fields p_α, and hence need $c-(r+2)$ relations to reduce the independent variables. Such a reduction can be done via the introduction of the new gauge group $U(1)^{c-(r+2)}$ acting on the p_i's so as to give a new set of D-terms. The charges of these fields can be written as $Q_{k\alpha}$. The gauge invariance condition of v_i under $U(1)^{c-(r+2)}$, by (2.42), demands that the $(c-r-2) \times c$ matrix Q is such that $\sum_\alpha T_{j\alpha} Q_{k\alpha} = 0$. This then defines for us our charge matrix Q which is the cokernel of T:

$$TQ^t = (T_{j\alpha})(Q_{k\alpha})^t = 0, \;\; j = 1,..,r+2; \;\; \alpha = 1,..,c; \;\; k = 1,..,(c-r-2)$$

For our example, the charge matrix is $(9-4-2) \times 9$ and one choice is

$$Q_{k\alpha} = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(d) In the linear σ-model language, the F-terms and D-terms can be treated in the same footing, i.e., as the D-terms (moment map) of the new fields p_α; with the crucial difference being that the former must be set exactly to zero[5] while the latter are to be resolved by arbitrary FI-parameters. Therefore in addition to finding the charge matrix Q for the new fields p_α coming from the original F-terms as done above, we must also find the corresponding charge matrix Q_D for the p_i coming from the original D-terms. We can find Q_D in two steps. Firstly, we know the charge matrix for X_i under $U(1)^{r-1}$, which is Δ. By (2.41), we transform the charges to that of the v_j's, by introducing an $(r-1) \times (r+2)$ matrix V so that $V \cdot K^t = \Delta$. To see this, let the charges of v_j be V_{lj} then by (2.41) we have $\Delta_{li} = \sum_j V_{lj} K_{ij} = V \cdot K^t$. A convenient V which does so for our $\mathbb{Z}_2 \times \mathbb{Z}_2$ example is $\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}_{(4-1)\times(4+2)}$. Secondly, we use (2.42) to transform the charges from v_j's to our final variables p_α's, which is done by introducing an $(r+2) \times c$ matrix $U_{j\alpha}$ so that $U \cdot T^t = \mathrm{Id}_{(r+2)\times(r+2)}$. In our example, one choice for U is $U_{j\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}_{(4+2)\times 9}$.

Therefore, combining the two steps, we obtain $Q_D = V \cdot U$ and in our example, $(V \cdot U)_{l\alpha} = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}$.

3. Thus equipped with the information from the two sides: the F-terms and D-terms, and with the two required charge matrices Q and $V \cdot U$ obtained, finally we concatenate them to give a $(c-3) \times c$ matrix Q_t. The transpose of the kernel of Q_t, with (possible repeated columns) gives rise to a matrix G_t. The columns of this resulting G_t then define the vertices of the toric diagram describing the polynomial corresponding to the singularity on which we initially placed our D-

[5] Strictly speaking, we could have an F-term set to a non-zero constant. An example of this situation could be when there is a term $a\phi + \phi\tilde{Q}Q$ in the superpotential for some chargeless field ϕ and charged fields $\tilde{Q}$ and Q. The F-term for ϕ reads $\tilde{Q}Q = -a$ and not 0. However, in our context ϕ behaves like an integration constant and for our purposes, F-terms are set exactly to zero.

branes. Once again for our example, $Q_t = \left(\begin{array}{ccccccccc|c} 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & \zeta_1 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & \zeta_2 \\ -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & \zeta_3 \end{array}\right)$ and $G_t = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. The columns of G_t, up to repetition, are precisely marked in the toric diagram for $\mathbb{Z}_2 \times \mathbb{Z}_2$ in Figure 2-3.

Thus we have gone from the F-terms and the D-terms of the gauge theory to the nodes of the toric diagram. In accordance with [48], G_t gives the algebraic variety whose equation is given by the maximal ideal in the polynomial ring $\mathbb{C}[YZ, XYZ, Z, X^{-1}YZ, XY^{-1}Z, XZ]$ (the exponents (i, j, k) in $X^iY^jZ^k$ are exactly the columns), which is $uvw = s^2$, upon defining $u = (YZ)(XYZ)^2(Z)(XZ)^2; v = (YZ)^2(Z)^2(X^{-1}YZ)^2; w = (Z)^2(XY^{-1}Z)(XZ)^2$ and $s = (YZ)^2(XYZ)(Z)^2(X^{-1}YZ)(XY^{-1}Z)(XZ)^2$; this is precisely $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. In physical terms this equation parametrises the moduli space obtained from the F and D flatness of the gauge theory.

We remark two issues here. In the case of there being no superpotential we could still define K-matrix. In this case, with there being no F-terms, we simply take K to be the identity. This gives T =Id and $Q = 0$. Furthermore U becomes Id and $V = \Delta$, whereby making $Q_t = \Delta$ as expected because all information should now be contained in the D-terms. Moreover, we note that the very reason we can construct a K-matrix is that all of the equations in the F-terms we deal with are in the form $\prod_i X_i^{a_i} = \prod_j X_j^{b_j}$; this holds in general if every field X_i appears twice and precisely twice in the superpotential. More generic situations would so far transcend the limitations of toric techniques.

Schematically, our procedure presented above at length, what it means is as follows: we begin with two pieces of physical data: (1) matrix d from the quiver encoding the gauge groups and D-terms and (2) matrix K encoding the F-term equations. From these we extract the matrix G_t containing the toric data by the flow-chart:

$$
\begin{array}{ccccccc}
\text{Quiver} \to d & \to & \Delta & & & & \\
 & & \downarrow & & & & \\
\text{F-Terms} \to K & \stackrel{V\cdot K^t=\Delta}{\to} & V & & & & \\
\downarrow & & \downarrow & & & & \\
T = \text{Dual}(K) & \stackrel{U\cdot T^t=\text{Id}}{\to} & U & \to & VU & & \\
\downarrow & & & & \downarrow & & \\
Q = [\text{Ker}(T)]^t & & \longrightarrow & & Q_t = \begin{pmatrix} Q \\ VU \end{pmatrix} & \to & G_t = [\text{Ker}(Q_t)]^t
\end{array}
$$

2.3 Witten's Cubic String Field Theory

String field theories have been studied for more than twenty years and become a very broad topic. There are all kinds of string field theories, for example, Witten's bosonic cubic string field theory [15], Witten's supersting field theory[16], Berkovits' superstring field theory [109], Zwiebach's closed string field theory [110] and Witten's boundary string field theory[111]. It will be hard to introduce all of them in this thesis, so we will focus only on the Witten's Cubic String Field Theory (CSFT) which our calculation is based on (for a review, see[112]).

2.3.1 The basic idea

The basic idea in [15] is following. We start with an associative non-commutative algebra B with a Z_2 grading (in fact, in string field theory, it is Z grading) and define three basic operations: **multiplication $\star$, integration $\int$ and derivation Q**, with following properties:

$$
\begin{aligned}
grading(a \star b) &= grading(a) + grading(b), && (2.43)\\
Q(a \star b) &= (Qa) \star b + (-)^a a \star (Qb), && (2.44)\\
Q^2 &= 0, && (2.45)
\end{aligned}
$$

$$\int a \star b = (-)^{ab} \int b \star a, \tag{2.46}$$

$$\int Q(b) = 0, \quad \forall b, \tag{2.47}$$

$$(a \star b) \star c = a \star (b \star c), \tag{2.48}$$

To have an intuition about these requirements, we can compare it with forms in manifold, where Q will be the external derivative d, $\star$ will be the wedge product $\wedge$ and $\int$ will be integration of forms on the manifold. From this comparing, we notice that there is a natural subspace $B_0 \in B$ (*zero* form) which is closed under the multiplication.

Using only above properties, we can define the gauge transformation of $1-form$ as

$$\delta A = Q\Lambda + A \star \Lambda - \Lambda \star A, \tag{2.49}$$

$$A' = e^{-\Lambda}(Q + A)e^{\Lambda} \tag{2.50}$$

where we have written down both small and big gauge transformations. From them, we can construct following quantities

$$F = QA + A \star A, \tag{2.51}$$

$$L = AQA + \frac{2}{3}A^3 \tag{2.52}$$

For Witten's Cubic string field theory, the action is proposed to be

$$I = \int AQA + \frac{2}{3}A^3 \tag{2.53}$$

and the equation of motion is

$$F = QA + A \star A = 0 \tag{2.54}$$

Having the aim in front of us, the task is to find the proper definitions of $Q, \int, \star$ which satisfy these properties (2.43-2.48). To reach this, Witten has used his inspiring geometry picture. Using this picture, the $\star$ product of two strings is defined by gluing the right half string of first string to the left half string of second string, which can be written briefly as

$$(S_L, S_R) \star (T_L, T_R) = (S_L, T_R)\delta(S_R - T_L), \tag{2.55}$$

Using the same gluing idea, the $\int$ can be defined as to glue the left half string to the right half string, i.e.,

$$\int S = \int dS_L dS_R \delta(S_L - S_R) \tag{2.56}$$

The last piece is the derivative operator. For this, there is a natural candidate in string theory, the BRST operator Q_B.

2.3.2 The string field theory in the oscillator modes

Although above geometry picture is very intuitive and attractive, it is hard to make the calculation. Furthermore, to make the theory well defined, we need to deal a lot of subtle points, like the ghost number, delta-interaction. These considerations motived Gross and Jevicki [17, 18] to develop the oscillator mode expression for the string field theory. It is the form most of calculations follow.

To do this, we expand the field X^μ into modes as

$$x(\sigma) = x_0 + \sqrt{2}\sum_{n=1} x_n \cos(n\sigma), \tag{2.57}$$

and define the creation and annihilation operators as

$$x_n = i\sqrt{\frac{\alpha'}{n}}(a_n - a_n^+), \tag{2.58}$$

$$p_n = -i\frac{\partial}{\partial x_n} = \frac{1}{2}\sqrt{\frac{n}{\alpha'}}(a_n + a_n^+), \tag{2.59}$$

$$x_0 = i\sqrt{\frac{\alpha'}{2}}(a_0 - a_0^+), \tag{2.60}$$

$$p_0 = -i\frac{\partial}{\partial x_0} = \sqrt{\frac{1}{2\alpha'}}(a_0 + a_0^+), \tag{2.61}$$

where we put the α' explicitly for later convenience. Do same thing to the ghost fields b, c as

$$c_{\pm}(\sigma) = \sum_{-\infty}^{+\infty} c_n e^{\pm in\sigma} = c(\sigma) \pm i\pi_b(\sigma), \tag{2.62}$$

$$b_{\pm}(\sigma) = \sum_{-\infty}^{+\infty} b_n e^{\pm in\sigma} = \pi_c(\sigma) \pm ib(\sigma), \tag{2.63}$$

where the modes c_n, b_n satisfy

$$\{c_n, b_m\} = \delta_{n,-m}, \tag{2.64}$$

From these, we can write down the BRST operator

$$\begin{aligned} Q_{BRST} = & \sum_{n=1}^{+\infty}[c_{-n}(L_n^x + \frac{1}{2}L_n^{gh}) + (L_{-n}^x + \frac{1}{2}L_{-n}^{gh})c_n] \\ & + c_0(L_0^x + \frac{1}{2}L_0^{gh} - 1), \end{aligned} \tag{2.65}$$

where L_n^x, L_n^{gh} are Virasoro algebra of matter fields and ghost fields respectively.

Now we can give the explicit forms for star-product and integration. Given a string field Ψ as a functional of $x(\sigma), b(\sigma), c(\sigma)$ with ghost number one (recall that the ghost number of c is one and b, -1), we expand the field as

$$\Psi[x(\sigma), b(\sigma), c(\sigma)] = [\phi(x) + A_\mu(x)a_{-1}^\mu + T_{\mu\nu}(x)a_{-1}^\mu a_{-1}^\nu + ...] \left|\Omega\right> \tag{2.66}$$

with

$$\left|\Omega\right> = c_1 \left|0\right>$$

and $\left|0\right>$ is the $SL(2,R)$ invariant vacuum. Then the integration is

$$\int \Psi = \langle I|\Psi\rangle \tag{2.67}$$

where $\langle I|$ is the BPZ-conjugate of identity field $|I\rangle$ [6] and can be expressed as

$$
\begin{aligned}
|I\rangle &= b_+(\frac{1}{2}\pi)b_-(\frac{1}{2}\pi)e^{\sum_{n=1}^{+\infty}[(-)^n c_{-n}b_{-n}-\frac{1}{2}(-)^n a_n^\dagger a_n^\dagger]}c_0c_1\,|0\rangle\,, &(2.68)\\
|I\rangle &= e^E\,|0\rangle\,, &(2.69)\\
E &= -\sum_{n\geq 1}\frac{(-)^n}{2n}\alpha_{-n}\alpha_{-n}+\sum_{n\geq 2}(-)^n c_{-n}b_{-n} &(2.70)\\
&\quad -2c_0\sum_{n\geq 1}(-)^n b_{-2n}-(c_1-c_{-1})\sum_{n\geq 1}(-)^n b_{-2n-1}
\end{aligned}
$$

In above we have written down two forms for the identity field. The first one is coming from [17, 18] and the second is coming from [113]. In fact, it was showed [114] that identity field belongs to a set of wedge states and can be written in the form that only total Virasoro operator L_n acting on the $SL(2,R)$ invariant vacuum. We will come back to this point later.

The star-product is defined by

$$
\langle\Psi_1| = \langle V_3|\,|\Psi_2\rangle\otimes|\Psi_3\rangle \qquad (2.71)
$$

where $\langle\Psi_1|$ is the BPZ-conjugate of $|\Psi_1\rangle$. The matter part of three-string vertex is given by [122, 125]

$$
\begin{aligned}
|V_3^m\rangle &= \int d^{26}p_1d^{26}p_2d^{26}p_3\delta^{26}(p_1+p_2+p_3)e^{-E_m}|0,p>_{123}, &(2.72)\\
E_m &= \frac{1}{2}\sum_{rs;m,n\geq 1}\eta_{\mu\nu}a_m^{(r)\mu\dagger}V_{mn}^{rs}a_n^{(s)\nu\dagger}+\sum_{rs;n\geq 1}\eta_{\mu\nu}p_{(r)}^\mu V_{0n}^{rs}a_n^{(s)\nu\dagger}\\
&\quad +\frac{1}{2}\sum_r\eta_{\mu\nu}p_{(r)}^\mu V_{00}^{rr}p_{(r)}^\nu &(2.73)
\end{aligned}
$$

where $a_m^{(r)\mu\dagger}$ are non-zero oscillator modes with $m\geq 1$, and $|0,p\rangle_{123}=|p_1\rangle\otimes|p_2\rangle\otimes|p_3\rangle$. This expression is in the moment representation and we can translate it into the pure oscillator modes as did in [116, 125]. The ghost part can be found in [135, 175] as

$$
\left|V_3^{gh}\right\rangle = e^{-\sum_{r,s=1}^3\sum_{n\geq 1,m\geq 0}c_{-n}^{(r)}X_{nm}^{rs}b_{-m}^{(s)}}c_0^{(1)}c_1^{(1)}\,|0\rangle_1\otimes
$$

[6] The BPZ-conjugation is defined as $\phi_n\to(-)^{h+n}\phi_{-n}$ on the modes of fields ϕ with the conformal weight h.

$$c_0^{(2)} c_1^{(2)} |0\rangle_2 \otimes c_0^{(3)} c_1^{(3)} |0\rangle_3 \tag{2.74}$$

where X^{rs}_{nm} as well as V^{rs}_{mn} are Neumann coefficients which can be found in [17, 18].

Now we complete the definition of string field theory in the oscillator modes. Using this formula, a lot of numerical works have been carried out to check Sen's three conjectures to very high accuracy.

Before we finish this section, it is worth to mention that except above oscillator mode expression for string field theory, there is another form, i.e., the conformal expression which was initiated in [173]. For a good review, see also [114, 118].

Chapter 3

Mirror symmetry by O3-planes

3.1 Introduction

In [39], Intriligator and Seiberg found a new duality, the so-called "mirror symmetry", between two different $N = 4$ gauge theories in three dimensions. There exists such a mirror duality in three dimensions due to several special properties. First, the $N = 4$ theory has a global R-symmetry $SO(4)$ which can be rewritten as $SU(2)_L \times SU(2)_R$, i.e., as the direct product of two independent $SU(2)$ factors. This is one crucial property for mirror symmetry because one action of the mirror duality is to simply interchange these two $SU(2)$ factors[1]. Under the global R-symmetry, the vector multiplet is in the adjoint of $SU(2)_L$ and is invariant under $SU(2)_R$ while the hypermultiplet is in the adjoint of $SU(2)_R$ and is invariant under $SU(2)_L$ (notice that both multiplets have four scalars if we dualize the gauge field A_μ in three dimensions to a scalar). Furthermore, the mass parameter transforms as $(3, 1)$ of $SU(2)_L \times SU(2)_R$ and the FI-parameter as $(1, 3)$. So after mirror duality, the Coulomb branch and mass parameter of one theory change to the Higgs branch and FI-parameter of the other and vice versa. Such mapping has an immediate application: because the Higgs branch is not renormalized by quantum effects [51], we can get the exact result about

[1]When we discuss the mirror duality of $N = 2$ theory in three dimensions, we must enhance the explicit $U(1)$ global R-symmetry to two $U(1)$'s, i.e., $U(1) \times U(1)$. Otherwise there is no good way to define the mirror theory. For details see [24, 64].

the Coulomb branch of one theory which is corrected by quantum effects by studying the Higgs branch of the mirror theory which can be studied at the classical level. Because of this and other good applications of mirror duality (for details, see [39]), a lot of work [52, 53, 10, 54, 56, 57, 58, 59, 67] has been done in this topic to try and find new mirror pairs.

There are several ways to construct the mirror pairs. The first way is to use the arguments coming from field theory [39, 52]. This method gives a lot of details how fields and parameters map to each other under the mirror duality. However, this method requires a lot of results which are not easy to get in field theory, so it is hard to use it to construct general mirror pairs. The second way is to use M-theory to construct the mirror pairs as done by Porrati and Zaffaroni in [54]. The third way is to use the geometric realization in [55]. The fourth way, which is also the most popular way in the construction of mirror pairs, is given in [10] by using brane setups. The brane setup has the good property of making many quantities in field theory more visible. For example, the R-symmetry $SU(2)_L \times SU(2)_R$ corresponds to rotations in planes X^{345} and X^{789}. The Coulomb branch and Higgs branch become the positions of D3-branes in NS-branes and D5-branes. The mass parameter and FI-parameter also have similar geometric correspondences. These geometric pictures give us some intuition to understand the problem better (for more applications of brane setups, see review [22]). The key observation in [10] is that the mirror duality is just the S-duality in string theory. Using the known property of S-dual transformation of various kinds of branes [10, 53, 56, 57] we can easily find the mirror pairs. In this paper we will follow the last method.

Because we will use the brane setup to find the mirror theory, let us talk more about the general idea [10] of the brane construction. Given a gauge theory with gauge group and some matter contents, first we try to find a proper brane setup which represents the gauge theory (usually it is the Coulomb branch given explicitly in the brane setup). After that, we move to the Higgs branch[2] of the theory by split-

[2]Usually, we can break all gauge symmetries by Higgs mechanism. However, in some cases after Higgsing there are still some massless gauge fields. We call the latter case *"incomplete Higgsing*

ting the D3-branes between NS-branes and D5-branes. Then we make the S-duality transformation (mirror transformation) which changes the NS-brane to D5-brane, D5-brane to NS-brane and D3-brane to itself, while perform the electric-magnetic duality in the world volume theory of D3-branes. When the brane setup involves an orientifold or ON plane, we need to know the S-duality rule for them too. Finally, we read out the corresponding gauge theory given by the S-dual brane setup—it is the mirror theory which we want to find.

In applications, it is straight forward to use the above procedure to give the mirror theory of $U(n)$ gauge theory with some flavors or the product of $U(n)$'s with some bifundamentals because the brane setup of those theories involve only NS5-branes, D5-branes and D3-branes and we know how to deal with them. However, when we try to find the mirror for a gauge group $Sp(k)$ or $SO(n)$, we must use an orientifold plane in the brane setup. Now a problem arises because sometimes we do not know how to read out the gauge theory of the S-dual brane setup of these orientifolds. The orientifolds which are involved in the construction can be divided into two types: the orientifold three plane (O3-plane) and the orientifold five plane (O5-plane). Sen has given an answer about the gauge theory under the ON-projection, which is the S-dual of the $O5^-$ plane plus a physical D5-brane, in [60]. Using this result, we can get the mirror theory for $Sp(k)$ [56, 57] by using the orientifold five plane in the initial brane setup. For $SO(k)$, if we insist on using the orientifold five plane again in the brane setup, we must know what is the gauge theory under the ON^+ projection which is the S-dual of $O5^+$ plane. It is still an open problem to read it out.

In the above paragraph, we mention that there is a difficulty to use orientifold five-plane to construct the mirror theory of $SO(n)$ gauge group. However, for constructing the $Sp(k)$ or $SO(n)$ gauge theory we can use an O3-plane instead of the O5-plane. Because under S-duality the O3-plane changes into another O3-plane, we know how to read out the gauge theory (unlike the O5-plane which becomes ON plane under S-duality). Motivated by this observation, in this paper we use O3-planes to investigate the mirror theory of $SO(n)$ and $Sp(k)$ gauge groups. In particular, we get the mirror theory for $SO(n)$ gauge group which is a completely new result. Furthermore, our proposal for the construction of the mirror theory predicts a nontrivial strong coupling limit of field theories with eight supercharges.

The contents of the paper are as follows. In section 2, we discuss some basic facts on Op-planes which will set the stage for calculating the mirrors. These include the four kinds of O3-planes and the s-configuration involving 1/2NS-brane and 1/2D5-brane. In section 3 we discuss the splitting of physical D5-branes on O3-planes. It is a crucial ingredient in our construction of mirror theory. By S-duality, we get the rules for how a physical NS-brane can split into two 1/2NS-branes or conversely how two 1/2NS-branes can combine into a physical NS-brane. The latter predicts a nontrivial transition of strongly coupled field theories. After these preparations, we give the mirror theory of a single gauge group with some flavors: $Sp(k)$ in section 4, $Sp'(k)$ in section[3] 5, $SO(2k)$ in section 6 and $SO(2k+1)$ in section 7. In sections eight and nine we generalize the mirror construction to products of two gauge groups: $Sp(k)\times SO(2m)$ in section 8 and $Sp'(k)\times SO(2m+1)$ in section 9. Finally, we give conclusions in section 10.

3.2 Some facts concerning O3-planes

In this section, we summarize some facts about the O3-plane which will be useful for the mirror construction later.

3.2.1 The four kinds of O3-planes

There are four kinds of O3-planes which we will meet in this paper (for a more detailed discussion, see [61]): $O3^+, O3^-, \widetilde{O3^+}, \widetilde{O3^-}$. However, before entering the specific discussion of $O3$-planes let us start from general Op-planes. When $p \leq 5$, there exist four kinds of orientifolds $Op^+, Op^-, \widetilde{Op^+}, \widetilde{Op^-}$. Among these four we are very familiar with $Op^+, Op^-, \widetilde{Op^-}$. They can be described perturbatively as the fixed planes of the orientifold projection Ω which acts on the world sheet as well as the

[3]There are two ways to get $Sp(k)$ gauge group: by $O3^+$-plane or $\widetilde{O3^+}$-plane. We denote the theory given by $O3^+$-plane as $Sp(k)$ and the theory given by $\widetilde{O3^+}$-plane as $Sp'(k)$.

Chan-Paton factors. By different choices of the action Ω on the Chan-Paton factors we get two kinds of projections which we denote as $\pm$ projection. In the $+$ case, we can put only an even number of 1/2Dp-branes and the corresponding plane is the Op^+ plane. In the $-$ case, we can put an even or odd number of 1/2Dp-branes and the corresponding plane is Op^- for even number of 1/2Dp-branes and $\widetilde{Op}^-$ for odd number of 1/2Dp-branes. For $\widetilde{Op}^-$, because there is an odd number of 1/2Dp-branes, one 1/2Dp-brane must be stuck on the orientifold plane so that sometimes we consider the $\widetilde{Op}^-$ as the bound state of the Op^- and the 1/2Dp-brane (for more detailed discussion, the reader is referred to [2]). The $\widetilde{Op}^+$ is more complicated and is discussed in detail by Witten in [61]. In that paper, Witten observes $O3$-planes from a more unified point of view, namely discrete torsion (he deals with $O3$-planes. However the discussion can be easily generalized to other Op-planes). We can distinguish Op-planes by two Z_2 charges (b, c) with the definition $b = \int_{RP^2} B_{NS}$ and $c = \int_{RP^{5-p}} C^{5-p}$ (the (b, c) is defined under modular two and the discussion presented here comes from lecture [62] already given by one of the authors at ITP, Santa Barbara; see also [63]). The second charge c exists only for $p \leq 5$. For $p > 5$, it can not be defined and we are left only with two types of Op-planes (it is a little mysterious that $\widetilde{Op}^-$ does not exist for $p > 5$, some arguments can be found in [62, 63]). We summarize the properties of these four Op-planes according the discrete torsions (b, c) in Table 3.2.1 (where S-duality is applied only to $p = 3$).

Table 3.1: The summary of the properties of the four Op-planes. The charge is in units of physical Dp-brane.

(b, c)	notation	charge	Gauge group	(b,c) after S-duality($p = 3$ only)
$(0,0)$	O^-	-2^{p-5}	$SO(2n)$	(0,0) O^-
$(0,1)$	$\widetilde{O}^-$	$\frac{1}{2} - 2^{p-5}$	$SO(2n+1)$	(1,0) O^+
$(1,0)$	O^+	2^{p-5}	$Sp(n)$	(0,1) $\widetilde{O}^-$
$(1,1)$	$\widetilde{O}^+$	2^{p-5}	$Sp'(n)$	(1,1) $\widetilde{O}^+$

These four kinds of O-planes are not unrelated to each other and in fact change to each other when they pass through the 1/2NS-brane or 1/2D-brane [29, 57, 63]. The change is shown in Figure 3-1: when $Op^-(\widetilde{Op}^-)$ passes through the 1/2NS-brane, it changes to $Op^+(\widetilde{Op}^+)$ and vice versa; when $Op^-(Op^+)$ passes through the 1/2D(p+2)

-brane, it changes to $\widetilde{Op^-}(\widetilde{Op^+})$ and vice versa.

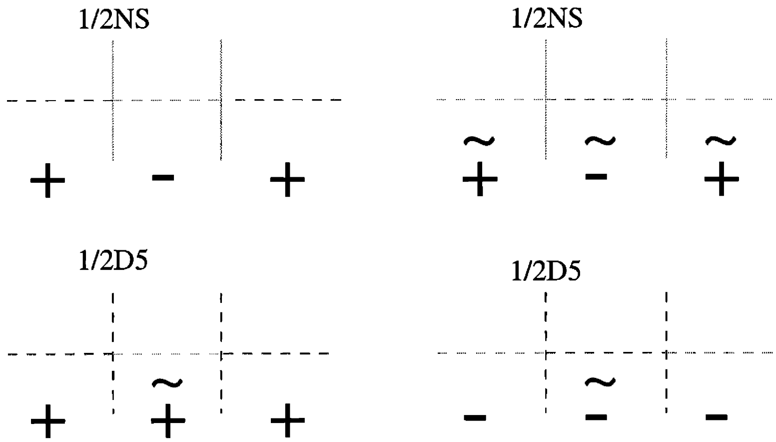


Figure 3-1: The change of four kinds of O3-planes as they cross 1/2NS-branes and 1/2D5-branes. In our brane setup, D3-brane and $O3$-plane will extend along X^{0126}, D5-brane, X^{012789} and NS-brane, X^{012345}. Henceforth, we use cyan (if the reader uses colored postscript rendering) lines to denote the 1/2NS-brane, blue lines to denote the 1/2D5-brane, dotted horizontal (red) lines to denote the $O3^+$-plane, dotted horizontal (green) lines to denote the $O3^-$-plane, dotted horizontal (yellow) lines to denote the $\widetilde{O3^+}$ and finally dotted horizontal (pink) lines to denote the $\widetilde{O3^-}$. Furthermore, for simplicity, we use $-, +, \tilde{-}, \tilde{+}$ to denote $O3^-, O3^+, \widetilde{O3^-}, \widetilde{O3^+}$ respectively.

After the discussion of general Op-planes, we focus on $O3$-planes which will be used throughout this paper. For $O3$-planes, the charge of $O3^-$ is $-1/4$ while the charges of $O3^+, \widetilde{O3^-}, \widetilde{O3^+}$ are $1/4$. The fact that the charges for the latter three $O3$-planes are identical is not a coincidence and they are related to each other by the $SL(2, Z)$ duality symmetry in Type IIB. In particular, under S-duality $O3^+$ and $\widetilde{O3^-}$ transform to each other while $\widetilde{O3^+}$ transforms to itself. $O3^-$ transforms to itself also under S-duality because it is the only O3-plane with $-1/4$ charge. One immediate application of the above S-duality property is that the change of O3-planes crossing the 1/2NS-brane is exactly S-dual to the change of O3-planes crossing the 1/2D5-brane. So our rule is consistent. The above discussions will be useful later in the study of mirror symmetry.

3.2.2 The supersymmetric configuration

In the procedures involved in mirror transformations, we need to break the D3-branes between the NS-brane and D5-brane to avoid the so called s-rule [10]. Furthermore, to read out the mirror theory from the brane setup it is convenient to move a 1/2NS-brane along the X^6 direction (our notations and conventions for the brane setups for all kinds of branes is given in the caption of Figure 3-1.) to pass through the 1/2D5-brane such that the D3-branes ending on the 1/2NS-branes are annihilated in order to keep the linking number between 1/2NS-brane and 1/2D5-brane invariant. All these actions require the understanding of supersymmetric configurations in the presence of O3-planes. We summarize these results in this subsection. The tool in our discussion of s-configuration is still the conservation of linking number between 1/2NS-brane and 1/2D5-brane. The formula of linking number for 1/2NS-brane and 1/2D5-brane [10] is

$$\begin{aligned} L_{NS} &= \tfrac{1}{2}(R_{D5} - L_{D5}) + (L_{D3} - R_{D3}) \\ L_{D5} &= \tfrac{1}{2}(R_{NS} - L_{NS}) + (L_{D3} - R_{D3}) \end{aligned} \tag{3.1}$$

where R_{D5} (L_{D5}) is the D5-charge to the right (left) of NS-brane (1/2D5-brane has 1/2 charge) and similar definition to others. Because we have four kinds of O3-planes we will have four kinds of supersymmetric configurations including one 1/2-NS brane and one 1/2-D5 brane. These four different cases are:

$$\begin{array}{ll} (1) & O3^- \;\; (1/2D5 - 1/2NS) \; or \; (1/2NS - 1/2D5) \;\; \widetilde{O3^+}, \\ (2) & O3^+ \;\; (1/2D5 - 1/2NS) \; or \; (1/2NS - 1/2D5) \;\; \widetilde{O3^-}, \\ (3) & \widetilde{O3^-} \;\; (1/2D5 - 1/2NS) \; or \; (1/2NS - 1/2D5) \;\; O3^+, \\ (4) & \widetilde{O3^+} \;\; (1/2D5 - 1/2NS) \; or \; (1/2NS - 1/2D5) \;\; O3^-, \end{array} \tag{3.2}$$

where the configuration $O3^-$ $(1/2D5 - 1/2NS)$ or $(1/2NS - 1/2D5)$ $\widetilde{O3^+}$ means that the $O3^-$ plane is at the left, $\widetilde{O3^+}$ at the right. In the middle we put 1/2NS-brane and 1/2D5-brane according to the order 1/2D5-1/2NS from left to right (see part (a) of Figure 3-2) or 1/2NS-1/2D5 (see part (b) of Figure 3-2).

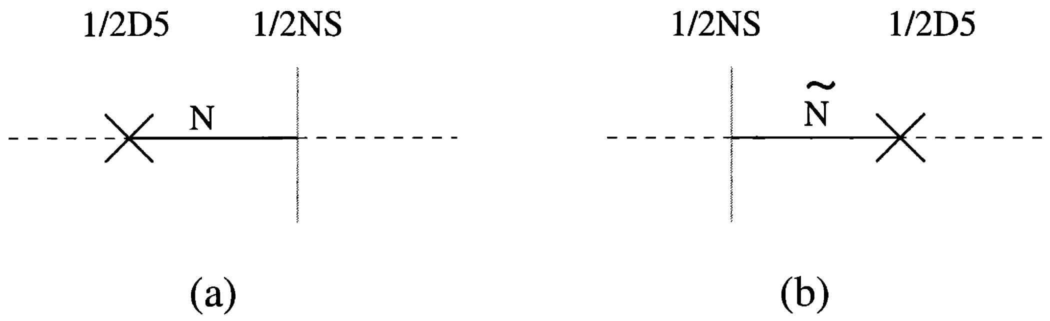


Figure 3-2: Starting from any one (left or right figure) , we move the 1/2NS-brane along X^6 direction to pass 1/2D5-brane and get the other (right or left). To allow such a process, we must conserve the linking number with the condition that $N, \tilde{N} \geq 0$. In this figure and henceforth, when NS-brane and D5-brane show at the same time in the figure with proper two-dimensional coordinates (for example, here $X = X^6, Y = X^5$), for clarification we use a line to denote an extended brane in these coordinates and use a cross to denote a point-like brane.

The general pattern for the above four supersymmetric configurations is shown in Figure 3-2, where we assume the number of connected D3-branes (in physical units) from the left 1/2D5-brane to the right 1/2NS-brane is N and from the left 1/2NS-brane to the right 1/2 D5-brane is $\widetilde{N}$. So a configuration to be supersymmetric is equivalent to the solution of $N, \tilde{N} \geq 0$ such that they conserve the linking number after crossing.

The first case: $O3^- - - - \widetilde{O3}^+$

In this case we start from the brane setup (a) of Figure 3-2 with $O3^-$ plane at the left, $\widetilde{O3}^-$ plane in the middle and $\widetilde{O3}^+$ plane at the right. The linking numbers are $L_{1/2D5} = \frac{1}{2}(\frac{1}{2}-0)+[(-\frac{1}{4})-(\frac{1}{4}+N)] = -N-\frac{1}{4}$ and $L_{1/2NS} = \frac{1}{2}(0-\frac{1}{2})+[(\frac{1}{4}+N)-(\frac{1}{4})] = N-\frac{1}{4}$. Now we move the 1/2D5 along X^6 direction to pass through 1/2NS and get the (b) of Figure 3-2 with $O3^-$ plane at the left, $O3^+$ plane in the middle and $\widetilde{O3}^+$ plane at the right. For the latter we have linking numbers as $L_{1/2D5} = \frac{1}{2}(0-\frac{1}{2}) + [(\tilde{N}+\frac{1}{4})-(\frac{1}{4})] = \tilde{N}-\frac{1}{4}$ and $L_{1/2NS} = \frac{1}{2}(\frac{1}{2}-0)+[(-\frac{1}{4})-(\tilde{N}+\frac{1}{4})] = -\tilde{N}-\frac{1}{4}$. Comparing these two linking numbers we get

$$N = -\tilde{N} \tag{3.3}$$

It is a highly constraining equation. For the supersymmetric configuration, the only solution is $N = \tilde{N} = 0$. This means that when we break the D3-brane to go to the Higgs branch, we **can not** put D3-brane between 1/2NS-brane and 1/2D5-brane in this orientifold configuration.

The second case: $O3^+ --- \widetilde{O3}^-$

Starting from brane setup (a) of Figure 3-2 with $O3^+$ at the left, $\widetilde{O3}^+$ in the middle and $\widetilde{O3}^-$ at the right, we find the linking numbers as $L_{1/2D5} = \frac{1}{2}(\frac{1}{2}-0)+[(\frac{1}{4})-(\frac{1}{4}+N)] = -N+\frac{1}{4}$ and $L_{1/2NS} = \frac{1}{2}(0-\frac{1}{2})+[(\frac{1}{4}+N)-(\frac{1}{4})] = N-\frac{1}{4}$. Again by moving the 1/2D5-brane to pass through 1/2NS-brane we get the brane setup as (b) with the middle O3-plane changed from $\widetilde{O3}^+$ in (a) to $O3^-$ in (b) (the left and right O3-plane are invariant under the motion). The linking numbers for the latter are $L_{1/2D5} = \frac{1}{2}(0-\frac{1}{2})+[(\tilde{N}-\frac{1}{4})-(\frac{1}{4})] = \tilde{N}-\frac{3}{4}$ and $L_{1/2NS} = \frac{1}{2}(\frac{1}{2}-0)+[(\frac{1}{4})-(\tilde{N}-\frac{1}{4})] = -\tilde{N}+\frac{3}{4}$. From these relations we find the equation

$$-N+1 = \tilde{N}. \tag{3.4}$$

So for a consistent supersymmetric configuration there are three solutions: $(N,\tilde{N}) = (0,1); (\frac{1}{2},\frac{1}{2}); (1,0)$.

The third case: $\widetilde{O3}^- --- O3^+$

For the third case, we start from the brane setup (a) with $\widetilde{O3}^-$ at the left, $O3^-$ in the middle and $O3^+$ at the right. The linking numbers are $L_{1/2D5} = \frac{1}{2}(\frac{1}{2}-0)+[(\frac{1}{4})-(-\frac{1}{4}+N)] = -N+\frac{3}{4}$ and $L_{1/2NS} = \frac{1}{2}(0-\frac{1}{2})+[(-\frac{1}{4}+N)-(\frac{1}{4})] = N-\frac{3}{4}$. Now we move the 1/2D5-brane to pass through 1/2NS-brane and get the brane setup (b) with $\widetilde{O3}^+$ in the middle. The linking numbers become $L_{1/2D5} = \frac{1}{2}(0-\frac{1}{2})+[(\tilde{N}+\frac{1}{4})-(\frac{1}{4})] = \tilde{N}-\frac{1}{4}$ and $L_{1/2NS} = \frac{1}{2}(\frac{1}{2}-0)+[(\frac{1}{4})-(\tilde{N}+\frac{1}{4})] = -\tilde{N}+\frac{1}{4}$. By comparing these relations we have

$$-N+1 = \tilde{N}. \tag{3.5}$$

So again there are three solutions: $(N,\tilde{N}) = (0,1); (\frac{1}{2},\frac{1}{2}); (1,0)$.

The fourth case: $\widetilde{O3}^+ - - - O3^-$

For the last case we start from the brane setup (a) with $\widetilde{O3}^+$ at the left, $O3^+$ in the middle and $O3^-$ at the right. The linking numbers are $L_{1/2D5} = \frac{1}{2}(\frac{1}{2}-0)+[(\frac{1}{4})-(\frac{1}{4}+N)] = -N+\frac{1}{4}$ and $L_{1/2NS} = \frac{1}{2}(0-\frac{1}{2})+[(\frac{1}{4}+N)-(-\frac{1}{4})] = N+\frac{1}{4}$. Now we move the 1/2D5-brane to pass through 1/2NS-brane and get the brane setup (b) with $\widetilde{O3}^-$ in the middle. The linking numbers change to $L_{1/2D5} = \frac{1}{2}(0-\frac{1}{2})+[(\tilde{N}+\frac{1}{4})-(-\frac{1}{4})] = \tilde{N}+\frac{1}{4}$ and $L_{1/2NS} = \frac{1}{2}(\frac{1}{2}-0)+[(\frac{1}{4})-(\tilde{N}+\frac{1}{4})] = -\tilde{N}+\frac{1}{4}$. From these relations we have

$$-N = \tilde{N}. \tag{3.6}$$

The only solution is $(N, \tilde{N}) = (0, 0)$ as in the first case.

Let us summarize the results in the last four subsections. When the charge of $O3$-planes at the two sides are the same (case two and case three), the condition is $N + \tilde{N} = 1$, so there is annihilation or creation of D3-branes in crossing. When the charge of $O3$-planes at the two sides are different (case one and case four), the condition is $N = \tilde{N} = 0$, so there can not be any D3-branes between the 1/2NS-brane and 1/2D5-brane.

3.3 The splitting of the physical brane

To construct the mirror theory by brane setups, we can follow the procedure given in the introduction [10]. However, in the presence of the O3-plane, we need one new input: how to *split* the physical D5-brane into two 1/2D5-branes on the O3-plane. Initially, the physical D5-brane can be placed off the O3-plane in pairs of 1/2D5-branes (see Figure 3-3). We can move the pair of 1/2D5-branes to touch the O3-plane. After touching the O3-plane, in principle every 1/2D5-brane can move freely on the O3-plane. We call such an independent motion of the 1/2D5-brane as "*splitting*" of the physical D5-brane. We want to emphasize that the splitting of a physical D5-brane into two 1/2D5-branes is a nontrivial dynamical process in string theory and can be applied to many situations. Here we need the splitting because in the mirror theory, the gauge theory is given by D3-branes ending on 1/2NS-branes which are the S-dual of 1/2D5-branes in the original theory. In this paper, we give only a preliminary discussion. We found some novel results: sometimes there is a creation of one physical D3-brane between these two 1/2D5-branes; sometimes there is an annihilation and sometimes, no creation and no annihilation. We found these results by matching the Higgs branch moduli of the Sp or SO theory with the correct dimension of moduli space.

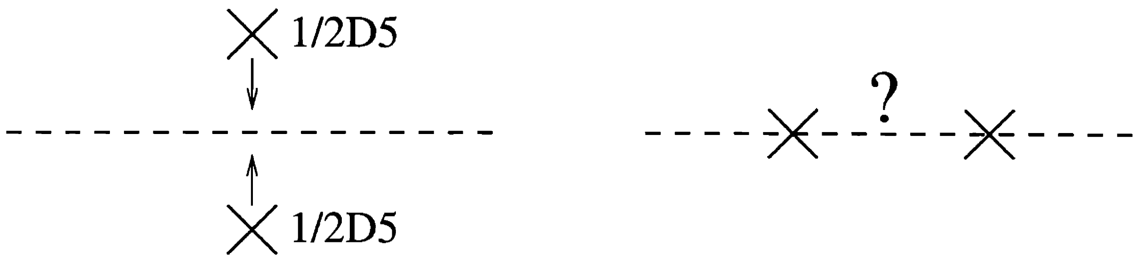


Figure 3-3: Splitting of a D5-brane on the O3-planes. The left figure shows that a pair of 1/2D5-branes moving to touch the O3-plane. The right figure shows that when they touch the O3-plane they can split. The ? in the middle of these two 1/2D5-branes means there is nontrivial dynamics dependent on different situations.

3.3.1 The splitting of D5-branes without ending D3-branes

Before going to the general situation let us discuss the splitting of D5-branes which do not have any D3-branes ending on them. First we discuss the case where there is only one physical D5-brane and O3-plane (see Figure 3-3). Before splitting, every 1/2D5-brane has linking number *zero*. After splitting, there can be N physical D3-branes between these two 1/2D5-branes (to keep supersymmetry, there can not be anti-D3-branes between them; furthermore, because here we do not have any D3-branes initially, there can not be annihilation either). Let us calculate the linking number after splitting:

$O3\ before\ splitting$	ΔL_{left}	ΔL_{right}
$O3^+$	$-N$	N
$\widetilde{O3}^+$	$-N$	N
$O3^-$	$-\frac{1}{2}-N$	$\frac{1}{2}+N$
$\widetilde{O3}^-$	$\frac{1}{2}-N$	$-\frac{1}{2}+N$

(3.7)

In BPS states, we have the tension of D5-branes proportional to their charge (linking number). To have the minimum tension configuration, it is natural to have $N = 0$ for the first three cases. However, for the last case, $N = 0$ and $N = 1$ are equally favorable just from the point view of tension. We will fix the ambiguity in the next paragraph. However, before we end this paragraph, we want to emphasize that no matter what case it is, the total change in linking number is always

$$\Delta L = 0 \qquad or \qquad \Delta L = \pm\frac{1}{2} \ .$$

We can fix the ambiguity for the last case by considering Higgsing. Starting from the $SO(3)$ gauge theory with one flavor, we can Higgs it to $SO(2)$ with one singlet (there are $3 - 1 = 2$ gauge fields which get mass, so we leave only $3 \times 1 - 2 = 1$ singlet). In part (a) of Figure 3-4 we assume $N = 0$ in the splitting process and go to the Higgs branch. By moving 1/2D5-branes outside we find the final theory is $SO(2)$ without singlets in part (b). This means that our assumption is wrong. Choosing the other assumption $N = 1$ in part (c), by moving 1/2D5-branes outside we get the final theory is $SO(2)$ with a singlet in part (d) which is exactly what we expect from the field theory. This shows that, for matching the correct moduli dimensions of Higgs branch, in last case of (3.7) there should be a D3-brane created in the splitting.

The discussion of the splitting of physical D5-branes becomes more complex if there are more than one D5-brane to be split. The complexity manifests in the last two cases in (3.7) because in these cases there is a change of linking number ($\Delta L = \pm\frac{1}{2}$) for every 1/2D5-brane. Before splitting, we have, for example, $2n$ 1/2D5-branes with linking number *zero*. After splitting, we have n 1/2D5-branes with linking number $\frac{1}{2}$

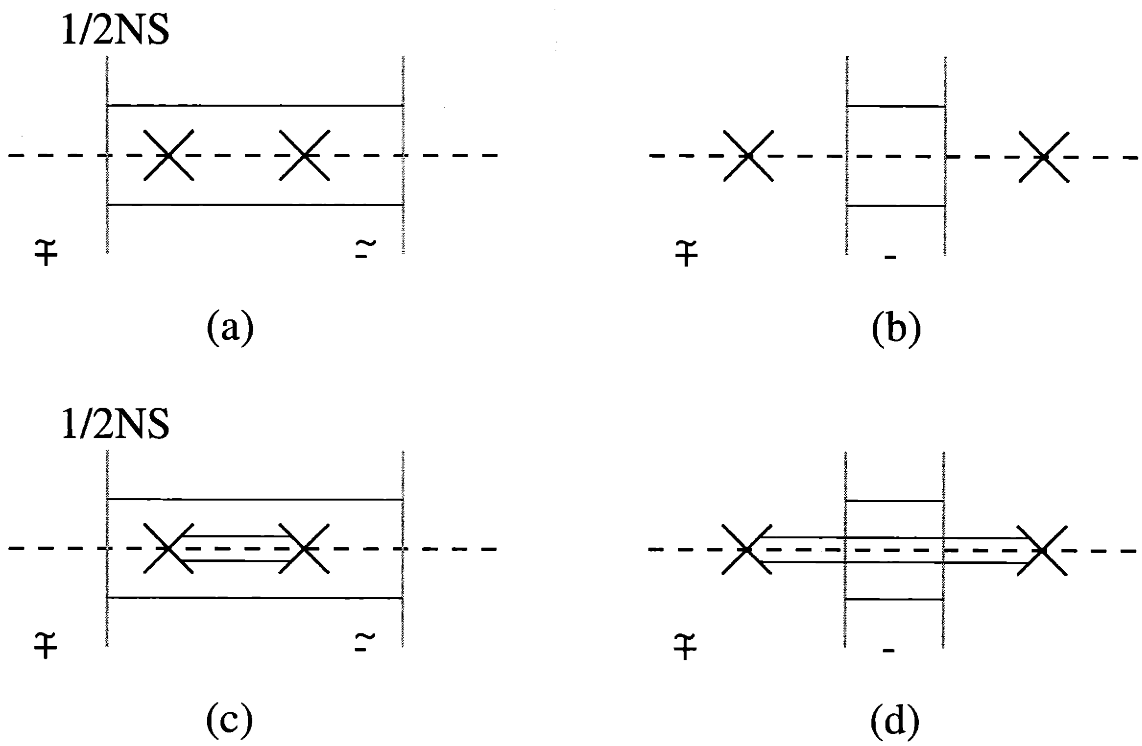


Figure 3-4: The Higgs branch of $SO(3)$ with one flavor. (a) We assume when the splitting, there is no D3-brane generated. (b) By moving 1/2D5-branes outside, we get $SO(2)$ without singlet. (c) We assume when the splitting, there is a D3-brane generated. (d) By moving 1/2D5-branes outside, we get $SO(2)$ with one singlet given by D3-brane ending on 1/2D5-brane.

and n, with linking number $-\frac{1}{2}$. The different order of linking number gives different physical content, i.e., the order determines when there should be D3-branes created and when there are no D3-branes created.

To illustrate our idea, let us see Figure 3-5. After splitting one D5-brane according to the analysis in the last paragraph, we continue to split the second D5-brane. However, in this case, we have two choices. In the first choice, the second D5-brane is far away from the first D5-brane in X^6 direction like part (a). So locally the splitting should be the same as the first D5-brane and we get part (b). Notice that the order of linking number of 1/2D5-branes is $-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}$. In the second choice, the second D5-brane is in the middle of the pair of first 1/2D5-branes as part (c). Naively, the second D5-brane will see the $O3^-$-plane (in fact, D5-brane will see more) so the splitting looks like to go as part (d) with the order of linking number $-\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}$. However, part (d) is not consistent with the Higgs branch of $SO(3)$ with two flavors.

Furthermore, because there are eight supercharges, the different positions of D5-branes should not effect the physics. So we argue that from part (c) we should get part (b) too. In part (c), the second D5-brane sees not only the $O3^-$-plane, but also the one created D3-branes, 1/2D5-branes at left with $\Delta = -\frac{1}{2}$ and 1/2D5-branes at right with $\Delta = +\frac{1}{2}$. This more complete information determines that the second D5-brane will split to part (b).

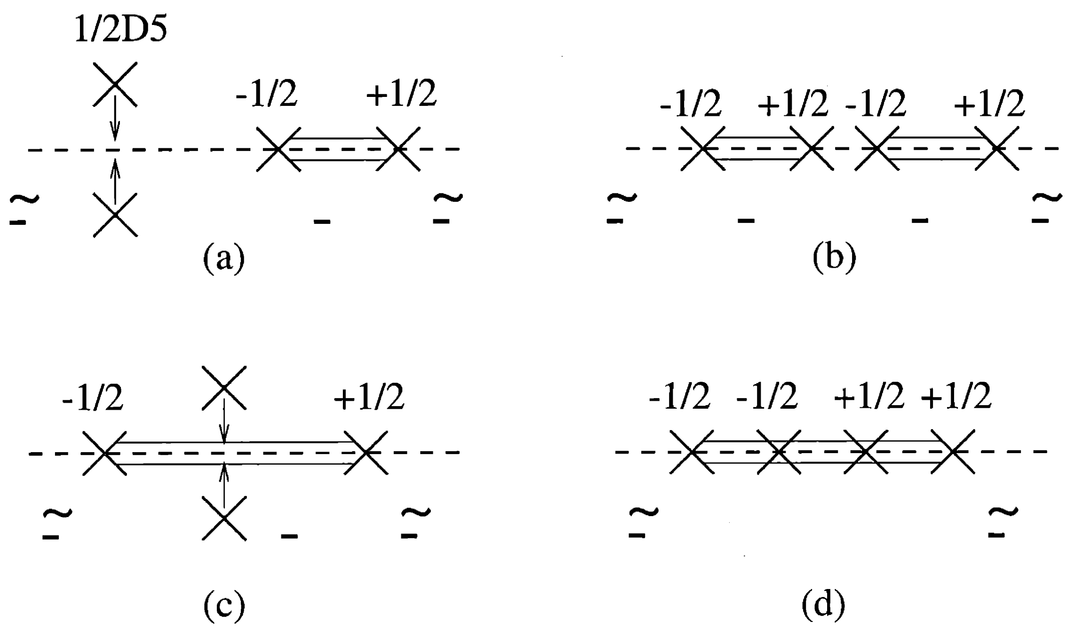


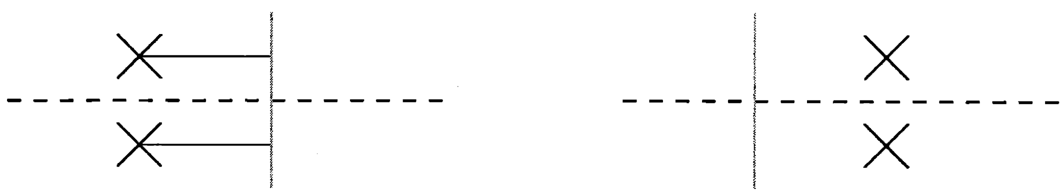
Figure 3-5: The splitting of the second D5-brane. (a) second D5-brane is far away from the first D5-brane. (b) the splitting of second D5-brane from configuration in part (a). (c). second D5-brane is in the middle of first D5-brane. (d) the naive splitting of second D5-brane which turns out to be wrong.

From the above observation, we propose that the correct order of linking number should be $-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, ..., -\frac{1}{2}, +\frac{1}{2}$ (notice the alternating fashion of $-\frac{1}{2}$ and $+\frac{1}{2}$). We make such a suggestion because it is the only correct order which can produce the consistent Higgs pattern for $SO(K)$ gauge group with N flavors. It will be very interesting if we can derive such a rule from string theory. Furthermore, this proposal will give very interesting predictions which we will discuss later.

Let us pause a moment to summarize the results we have obtained above. Without the D3-brane ending on D5-branes, (1) the change of linking number of 1/2D5-branes is $\Delta L = 0$ for $O3^+, \widetilde{O3}^+$ and $\Delta L = \pm\frac{1}{2}$ for $O3^-, \widetilde{O3}^-$; (2) for the splitting of a bunch of D5-branes, the order of linking number is $-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, ..., -\frac{1}{2}, +\frac{1}{2}$.

3.3.2 The splitting of D5-branes with ending D3-branes

After the discussion of the splitting of D5-branes without D3-branes ending on them, we consider the case that there are N D3-branes ending on them. The results for this latter case can be derived from the results in the last subsection. For example, let us discuss the case of one D5-brane with one ending D3-brane in Figure 3-6. We can add one 1/2NS-brane such that the D3-brane ending on it as part (a). Then we can move D5-brane to the right of 1/2NS-brane and annihilate the D3-brane as part (b). Now the part (b) is the case we discussed in the last subsection. We can split the physical D5-brane and move two 1/2D5-branes to left of 1/2NS-brane by using the result in section 2. By this loop, we finally get the splitting of D5-brane with one ending D3-brane. For more D3-branes ending on D5-branes we can add more 1/2NS-brane and repeat the above procedure.



(a) (b)

Figure 3-6: (a) One D3-brane ends on a physical D5-brane. We can add a 1/2NS-brane at the right. It should not affect the discussion. (b) By moving D5-brane to right of 1/2NS-brane we annihilate the ending D3-brane.

Although the above trick solves our problem completely, it is too tedious and we need a more direct way to see it. Notice that the change of the linking number of 1/2D5-branes happens only at splitting. So we can use the changing of linking number as the rule to determine the splitting of D5-brane. In general there will be N_L D3-branes ending on D5-brane from the left and N_R D3-branes, from right. The rule depends only on the absolute difference between N_L, N_R, i.e., $N = |N_L - N_R|$. We summarize the rule in Table 3.3.2.

Table 3.2: The rules of splitting of D5-brane, where $N = |N_L - N_R|$ is the difference of D3-branes ending on D5-brane from the left and the right.

O3-plane	$N = even$	$N = odd$
$O3^+$	$\Delta L = 0$	$\Delta L = \pm\frac{1}{2}$ in order of $-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, ...$
$\widetilde{O3}^+$	$\Delta L = 0$	$\Delta L = \pm\frac{1}{2}$ in order of $-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, ...$
$O3^-$	$\Delta L = \pm\frac{1}{2}$ in order of $-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, ...$	$\Delta L = 0$
$\widetilde{O3}^-$	$\Delta L = \pm\frac{1}{2}$ in order of $-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, ...$	$\Delta L = 0$

Table 3.3: The rules of splitting of NS-brane, where $N = |N_L - N_R|$ is the difference of D3-branes ending on NS-brane from the left and the right.

O3-plane	$N = even$	$N = odd$
$\widetilde{O3}^-$	$\Delta L = 0$	$\Delta L = \pm\frac{1}{2}$ in order of $-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, ...$
$\widetilde{O3}^+$	$\Delta L = 0$	$\Delta L = \pm\frac{1}{2}$ in order of $-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, ...$
$O3^-$	$\Delta L = \pm\frac{1}{2}$ in order of $-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, ...$	$\Delta L = 0$
$O3^+$	$\Delta L = \pm\frac{1}{2}$ in order of $-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, ...$	$\Delta L = 0$

3.3.3 The splitting of NS-branes and novel predictions of field theory in the strong coupling limit

Making S-duality, we can get the rules of splitting physical NS-branes into 1/2NS-branes on O3-plane as Table 3.3.3.

From Table 3.3.3, we get two predictions of $N = 4$ three dimensional field theory in the strong coupling limit (see Figure 3-7). In the first case (part (a) of Figure 3-7), the field theory is $SO(2k) \times Sp(k) \times SO(2k)$ with two bifundamentals. From the brane setup in part (a), we see that, by reversing the process of the splitting of the NS-brane, we can move two middle 1/2NS-branes to meet together and leave $O3^-$-plane. In field theory, moving two middle 1/2NS-branes together corresponds to the strong coupling limit of $Sp(2k)$ gauge theory, and moving NS-brane off the $O3^-$-plane corresponds to turning on *"FI-parameters"* [4]. So our brane configuration predicts that, at the strong coupling limit of $Sp(2k)$ and the turning of FI-parameter, the original theory $SO(2k) \times Sp(k) \times SO(2k)$ with two bifundamentals will flow to $SO(2k)$ without any flavor. The second case is given in part (b) of Figure 3-7. By the similar arguments, we predict that at the strong coupling limit of $SO(2k+2)$ and the turning of FI-parameter, the field theory $Sp(k) \times SO(2k+2) \times Sp(k)$ with two

[4]In fact, it is a hidden *"FI-parameters"*. We will discuss it more in section 4.3

bifundamentals will flow to $Sp(k)$ without any flavor. It will be interesting to check these two predictions from the field theory point of view.

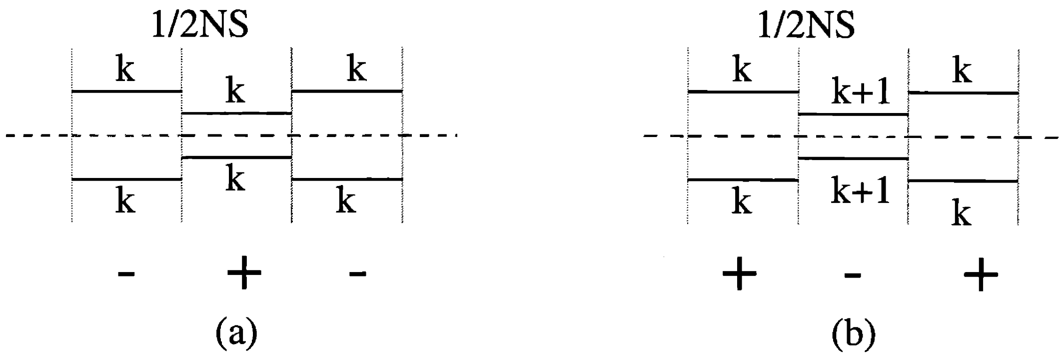


Figure 3-7: (a) $SO(2k) \times Sp(k) \times SO(2k)$ gauge theory with two bifundamentals. (b) $Sp(k) \times SO(2k+2) \times Sp(k)$ gauge theory with two bifundamentals.

3.4 The mirror of $Sp(k)$ gauge theory

Now we start to construct mirror pairs using the above knowledge. First let us discuss $Sp(k)$ gauge theory with N fundamental flavors. In this case, the brane setup is as follows: we put $2k$ 1/2D3-branes, i.e., branes and their images under the O-plane (extended in X^{0126}) ending on two 1/2NS branes (extended in X^{012345}) along X^6 while $2N$ 1/2D5-branes (extended in X^{012789}) are put in the middle (see (a) of figure 3-8). Then $O3$-planes from the left to right read as $O3^-, O3^+, O3^-$. This $O3$-plane configuration reminds us of the special s-configuration discussed in the last section. Because in the presence of $O3$-planes the s-configuration is a little different from the known s-rule in [10], we will demonstrate the detailed steps for the mirror construction for $Sp(1)$ with three flavors. Thereafter we quickly go to the general $Sp(k)$ case.

3.4.1 $Sp(1)$ with the 3 flavors

For the $Sp(1)$ gauge theory with 3 flavors we have the following information about the moduli space of the Higgs branch and the Coulomb branch as well as the FI-parameters and mass parameters:

$$\begin{aligned} d_v &= 1, \\ d_H &= 3\times 2 - 3 = 3, \\ \#m &= 3, \\ \#\zeta &= 0 \end{aligned} \tag{3.8}$$

After the mirror map, we should have a mirror theory which has $d_v = 3, d_H = 1, \#\zeta = 3, \#m = 0$, i.e. the Coulomb branch and the Higgs branch are interchanged while the mass parameters and FI-parameters are exchanged [39]. However, when we count these parameters, sometimes we meet nontrivial situations, such as the "hidden FI-term" explained in [56]. We will see later that these "hidden parameters" arise in our construction and will discuss them in more detail later.

The details of the mirror construction are given in Figure 3-8. Let us go step by step. Part (a) is just the brane setup for $Sp(1)$ with three fundamental flavors. By moving the physical D5-brane to touch the orientifold $O3^+$ plane, i.e., setting the masses to zero, we can split them into 1/2D5-branes as in part (b). Now we go to the Higgs branch by splitting the D3-branes between those 1/2NS-branes and 1/2D5-branes. However, from (3.3) and (3.6), we must split these D3-branes as given by part (c). The crucial point is that there is no D3-brane connected between the 1/2NS-brane and its nearest 1/2D5-brane because it is prohibited by the supersymmetric configuration discussed in section 2. Now we can use the rules (3.3) and (3.6) to move the left 1/2NS-brane crossing the neighboring right 1/2D5-brane and the right 1/2NS-brane crossing the neighboring left 1/2D5-brane. The result is given by part (d). Notice that in such a process, no D3-brane is created or annihilated. Applying (3.4) and (3.5) to move the 1/2NS-brane across 1/2D5-brane, we reach part (e). In this process, the physical D3-brane which connects the 1/2NS-brane and 1/2D5-brane is annihilated. Now we can apply the mirror transformation to give the result shown in part (f). However, it is a little hard to read out the final gauge theory because of the $O3^-$ and $\widetilde{O3}^-$ projections in the same interval. We can get rid of this ambiguity by applying (3.3) and (3.6) again to reach the result in part (g).

Now we have the brane setup for the mirror theory in part (g) of figure 3-8. We can read out the theory directly from the brane setup according the standard rule: For $2k$ 1/2D3-brane stretching between two 1/2NS-branes with $O3^-, \widetilde{O3^-}, O3^+, \widetilde{O3^+}$ planes we get $SO(2k), SO(2k+1), Sp(k), Sp'(k)$ gauge groups respectively. For one 1/2D5-brane between two 1/2NS-branes it contributes one fundamental half-hypermultiplet for that gauge group. For one physical D5-brane between two 1/2NS-branes it contributes one fundamental hypermultiplet for the gauge group. For two gauge groups which have a common 1/2NS-brane there is a bifundamental (in the presence of $O3$-plane, such bifundamental is, more exactly, half-hypermultiplet). Applying the above rules we immediately get the mirror theory as $SO(2) \times Sp(1) \times SO(2)$ with two bi-fundamentals and one fundamental for $Sp(1)$. Here we want to emphasize that in general we get only half fundamental hypermultiplets coming from the 1/2D5-brane. The unusual point for this explicit example is that the two 1/2D5-branes are in the same interval such that they can combine together and leave the orientifold (see section 3). Now let us calculate the moduli spaces and parameters to see if they are really mirror to each other. For the mirror theory in the part(h) of Figure 3-8, it is easy to get the dimensions of moduli spaces as $d_v = 1+1+1 = 3$ and $d_H = (2 \times 2 + 2 \times 2)/2 + 1 \times 2 - (1+1+3) = 6-5 = 1$, so we see the results match when comparing to 3.8. However, when we turn to calculate the mass parameters and FI-parameters, a mismatch occurs. In the mirror theory, we have two bifundamentals and one fundamental. For the two bifundamentals we do not know how to turn the mass parameters so we get the $\#m = 1$. Because there are no $U(1)$ factors in the mirror theory, it seems that we should get $\#\zeta = 0$. Now comparing with the original theory, we find a mismatch in the mass parameters and FI-parameters. The solution of the above mismatch is given by the concept of "hidden FI-term" which we will discuss later [56].

3.4.2 Another method to go to the Higgs branch

In the above procedure, we split D5-branes first, then went to the Higgs branch by splitting the D3-branes. However, we can go to the Higgs branch in another way by splitting the D3-brane first on the physical D5-brane and then splitting the D5-brane on the O3-plane. The procedure of this second method is drawn in Figure 3-9. In part (a) , we keep the D5-branes off the O3-plane and split the D3-branes to go the Higgs branch (such splitting is very familiar to us already, see [10]). By moving the physical D5-brane to cross the 1/2NS-brane, we can get rid of the D3-brane ending one D5-brane and 1/2NS-brane. The result is shown in part (b). Now we move the D5-brane to the O3-plane and split them. For consistency with the first method in the last subsection we must require the splitting of D5-brane with one D3-brane ending on it as the rule given in section 3.2. In fact, as we discussed above, we find all rules in section 3.2 in this way. It is easy to check that in this example we should get the same result as part (e) of Figure 3-8.

3.4.3 The "hidden FI-term"

We have met the mismatch of mass and FI parameters in the above mirror pair. It is time for us to talk more about it in this subsection. In fact, such a mismatch of mass and FI parameters in mirror pair is not new to us. Kapustin found this problem in [56]. In that paper, he considers the mirror of $Sp(k)$ with an antisymmetric tensor and n fundamental flavors. He found that when $n = 2, 3$ the quivers of the mirror theory are in fact affine A_1 for $n = 2$ and affine A_3 for $n = 3$. However, it is a well-known fact that a gauge theory given by an affine A_n quiver has one mass parameter. On the other hand, classically the original $Sp(k)$ theory does not have any FI parameters. Kapustin suggests the concept of **"hidden FI term"** to resolve the conflict. Such a term arises as the deformation in the infrared limit and has the same quantum number as a FI-term. Because it is a quantum effect, these deformations need not have a Lagrangian description in the ultraviolet. To count the number of hidden FI deformation we simply count the mass parameters in the mirror theory. Now applying Kapustin's explanation to our example, we find there is one "hidden FI-term" for the original theory and three "hidden FI-terms" for the mirror theory. This result is consistent with Kapustin's result. Notice that for $k = 1$ the antisymmetric tensor of $Sp(k)$ does not exist, so his theory is in exact agreement with our original theory and we both find one “hidden FI-term”. Hereafter we do not discuss the matching of the mass and FI parameters anymore, but we will mention the case when there exists a “hidden FI-term” for the original theory.

The appearance of the *“hidden FI term”* indicates another important aspect of the possible enhanced hidden global symmetry. In [39], the authors observed that the fixed point can have global symmetries which are manifest in one description but hidden in another (i.e., can be seen only quantum mechanically). For example, the $U(1)$ with two flavors is a self-mirror theory. On a classical level we have $SU(2) \times U(1)$ global symmetry, where $SU(2)$ is the flavor symmetry and $U(1)$ is the global symmetry connecting one FI-parameter (FI-parameter can be considered as a component in the background vector supermultiplet of $U(1)$). However, at the fixed point, the $U(1)$ global symmetry is enhanced to $SU(2)$. This enhanced symmetry can be easily seen in the brane setup of the mirror theory because in this special case ($U(1)$ with two flavors), the two D5-branes (the S-dual of two NS-branes in original symmetry) meet in same interval. This is another advantage of brane setup because we can see a lot of nontrivial phenomena pictorially. In later sections, when we find the case where there is a *“hidden FI term”*, we will also discuss the enhanced global symmetry.

There is another interesting aspect which is worth mentioning. If our construction is right, it seems that we have two different theories which are mirror to the same one because in [56, 57] we can construct the mirror of $Sp(k)$ gauge theory by using the $O5^-$ plane. This is also met by Kapustin in [56]. He noticed that two theories, (1) the $Sp(k)$ gauge theory with an antisymmetric tensor plus two or three fundamental and (2) the $U(k)$ gauge theory with an adjoint plus two or four fundamental flavors, are mirror to the same affine A_1 or A_3 quiver theory. Because mirror symmetry is a property in the infrared limit of gauge theory, such a non-uniqueness is allowed. Actually the brane picture provides a definition of the theory beyond the infrared limit and the non-uniqueness can be seen in nature by having two different brane representations of the same field theory.

3.4.4 $Sp(2)$ with 6 flavors

With the experience of $Sp(1)$ gauge theory, we can deal with the $Sp(2)$ with 6 flavors very quickly. The moduli spaces for the original theory have $d_v = 2$ and $d_H = 6\times 4 - 10 = 14$ (as mentioned above, in the following discussion we do not discuss the issue of mass parameters and FI-parameters). The steps for getting the mirror theory is in Figure 3-10. From the brane setup (d) in Figure 3-10 we read out that the mirror theory as $SO(2)\times Sp(1)\times SO(4)\times Sp(2)\times SO(5)\times Sp(2)\times SO(4)\times Sp(1)\times SO(2)$ with 8 bifundamentals and two fundamental half-hypermultiplets one for each $Sp(2)$ gauge theory. By an easy calculation, we can check the moduli spaces as: $d_v = 4\times 1+5\times 2 = 14$ and $d_H = (2\times 4+2\times 8+2\times 16+2\times 20+2\times 4)/2-(2+2\times 3+2\times 6+3\times 10) = 52-50 = 2$.

3.4.5 The general case

Now we discuss the general case, i.e., $Sp(k)$ with N fundamental flavors (to get the complete Higgsing, we have to assume that $N \geq 2k$). The moduli space has $d_v = k$ and $d_H = 2kN - k(2k+1)$. The steps for getting the mirror theory are shown in Figure 3-11. From it we can read out that the mirror theory are $SO(2)\times Sp(1)\times SO(4)\times Sp(2)\cdots\times Sp(k-1)\times SO(2k)\times(Sp(k)\times SO(2k+1))^{n-2k-1}\times Sp(k)\times SO(2k)\times Sp(k-1)\cdots\times Sp(1)\times SO(2)$ with bifundamentals and one fundamental half-hypermultiplet for each the first and the last $Sp(k)$ gauge groups. For clarity, the corresponding quiver diagram of the above mirror theory is also drawn in part (c) of this figure. Now we can calculate the moduli spaces of the mirror as

$$\begin{aligned}
d_v &= 4\sum_{n=1}^{k-1} n + (2N-4k+1)k = 2Nk - k(2k+1), \\
d_H &= [\frac{1}{2}\times 2\sum_{n=1}^{k-1}((2n)^2 + 2n(2n+2)) + \frac{1}{2}(2(2k)^2 \\
&+ (2N-4k-2)2k(2k+1)) + 2k] - [2\sum_{n=1}^{k-1}(n(2n-1)+n(2n+1)) \\
&+ (2N-4k-1)k(2k+1) + 2k(2k-1)] \\
&= k. \qquad (3.9)
\end{aligned}$$

As mentioned above, for general N, k we get only half-hypermultiplets coming from the 1/2D5-branes in the mirror theory. However, there are two degenerate cases where one fundamental hypermultiplet does exist instead of two half-hypermultiplets. The first case is when $N = 2k$. In this case, we do not need to move the 1/2NS brane further from part (a) to part (b) in Figure 3-11. Instead, we can make the mirror transformation directly from part(a). In the mirror theory, we get only one $SO(2k)$ gauge group but with one flavor for this $SO(2k)$. As explained above, such a flavor hints a "hidden FI-term" in the original theory. The second case is when $N = 2k+1$, where we get only one $Sp(k)$ gauge group in mirror theory, but also with one flavor of the $Sp(k)$ which also suggests a "hidden FI-term" in the original theory. For $k = 1$, the two cases where a "hidden FI-term" shows is given in [56]. For $k \geq 2$ it is a new result.

As we mentioned in section 4.3, in the case where a "hidden FI-term" shows we should consider the possible enhancement of global symmetry. In general the theory has global $SO(2N)$ flavor symmetry. When $N = 2k$, the global symmetry will be enhanced to $SO(2N) \times Sp(1)$. The factor $Sp(1)$ can be seen from the mirror theory, where two 1/2D5-branes meet and give one flavor to the $SO(2k)$ gauge group (notice the flavor symmetry for $Sp(k)$ gauge groups is $SO(2N)$, for $Sp'(k)$ gauge groups, $SO(2N+1)$, for $SO(2k)$ gauge groups, $Sp(N)$ and for $SO(2k+1)$ gauge groups, $Sp'(N)$). When $N = 2k+1$, the global symmetry will be enhanced to $SO(2N) \times SO(2)$ because in this case, the one extra flavor in mirror theory belongs to the $Sp(k)$ gauge group.

3.5 The mirror of $Sp'(k)$ gauge theory

We know that the $O3^+$ and $\widetilde{O3}^+$ projections both give $Sp(k)$ gauge theory. To distinguish them, we denote the gauge theory given by $O3^+$ projection as $Sp(k)$ and that by $\widetilde{O3}^+$ as $Sp'(k)$. After the discussion of the mirror of $Sp(k)$ gauge group in the last section, we now address the $Sp'(k)$ case in this section. The brane setup of Sp' is just to replace the $O3^{\pm}$ in $Sp(k)$ by $\widetilde{O3}^{\pm}$ (for example, see figure 3-12). However, by such a replacement, the theory becomes $Sp(k)$ gauge theory with n flavors plus two half-hypermultiplets contributed from the $\widetilde{O3}^-$ at the two sides (notice that $\widetilde{O3}^-$ can be considered as $O3^-$ plus a 1/2D3-brane). We will start the discussion also from a simple example, then go to the general case. Furthermore, we will compare the mirror of $Sp(k)$ and $Sp'(k)$ and show that in fact they give the same mirror theory.

3.5.1 $Sp'(1)$ with 3 fundamental flavors

In this example, the theory is $Sp(1)$ gauge group with three hypermultiplets and two half-hypermultiplets. The moduli space has $d_v = 1$ and $d_H = 3 \times 2 + 2 \times 2/2 - 3 = 5$. The steps for finding the mirror theory are drawn in Figure 3-12. First we go to the Higgs branch. Now equations (3.4) and (3.5) allow us the break the D3-branes between the 1/2NS-branes and the neighboring 1/2D5-brane as part (a). Form part (a) we move 1/2NS-brane inside to pass one 1/2D5-brane and get part (b). In this step, the 1/2NS-branes get rid of the D3-brane ending on them already. However, this brane setup does not readily give the correct mirror theory and we need go to the next step, i.e., moving 1/2NS-brane one step further inside as in part (c). Finally, we make the S-duality transformation and get the mirror theory in part (d). The mirror theory is $SO(2) \times Sp(1) \times SO(3) \times Sp(1) \times SO(2)$ with four bifundamentals and two half-hypermultiplets one for each $Sp(1)$ gauge group. We can check the moduli spaces of the mirror theory as having $d_v = 5$ and $d_H = (2\times 4 + 2\times 6 + 2\times 2)/2 - (2 + 3\times 3) = 12 - 11 = 1$.

3.5.2 The general case

Now we discuss the general case, i.e., $Sp'(k)$ with n hypermultiplets and two half-hypermultiplets. The moduli spaces have $d_v = k$ and $d_H = 2nk + 2k - k(2k+1) = 2nk - k(2k-1)$. The main steps to get the mirror theory are in Figure 3-13. We can read out the mirror theory from the quiver diagram in part(b) and check the moduli space as having

$$\begin{aligned}
d_v &= 4\sum_{t=1}^{k} t + k(2n-4k-1) = 2nk - k(2k-1), \\
d_H &= [2\sum_{t=1}^{k-1}((2t)^2 + 2t(2t+2)) + 2(2k)^2 + (2n-4k)2k(2k+1) \\
&+ 2(2k)]/2 - [2\sum_{t=1}^{k-1}(t(2t-1) + t(2t+1)) \\
&+ 2k(2k-1) + (2n-4k+1)k(2k+1)] \\
&= k
\end{aligned} \tag{3.10}$$

As the case of Sp gauge group when $n = 2k$, the mirror theory has only one $Sp(k)$ gauge group and the two half-hypermultiplets combine together to give one flavor for $Sp(k)$. It means that we have a "hidden FI-term" in the original theory. However, it is not the end of the story. By careful observation, we find that when $n = 2k-1$, the mirror theory has only one $SO(2k)$ gauge group and two half-hypermultiplets also combine together to give one flavor for $SO(2k)$ (this happens because in this case, we do not need move 1/2NS-brane one further step as we did from part (b) to part (c) in Figure 3-12). So we get a "hidden FI-term" in this case also. This is not expected initially because it seems that for $n = 2k-1$ we can not get the complete Higgs branch, but this is not true. By studying the part (a) of Figure 3-12, we find that for $n = 1$ in $Sp'(1)$ we indeed get complete Higgsing. Furthermore by the discussion in the next subsection we will see more clearly the reason why $n = 2k-1$ gives a "hidden FI-term".

Now let us discuss the global symmetry. The results are very similar to those at the end of section 4. In the general case we have global $SO(2N+1)$ flavor symmetry[5]. When $N = 2k-1$, the global symmetry goes to $SO(2N+1) \times Sp(1)$. When $N = 2k$, the global symmetry goes to $SO(2N+1) \times SO(2)$.

[5]From the discussion in the next subsection, the mirrors of single $Sp'(k)$ with N flavors and single $Sp(k)$ with $N+1$ flavors are identical. In the latter case, the flavor symmetry is $SO(2N+2)$, but in the former case, we see only an obvious $SO(2N+1)$ flavor symmetry. However, in current situation of product gauge theories the argument of section 5.3 can not be applied directly. There is true distinguishing between $Sp(k)$ and $Sp'(k)$ gauge theories

3.5.3 Comparing the mirror of $Sp(k)$ and $Sp'(k)$

In the above, we have discussed the mirror of two kinds of Sp gauge groups, i.e., $Sp(k)$ and $Sp'(k)$. We want to ask ourselves whether there is any relation between the mirrors of these two Sp gauge groups? By checking the two quivers in Figure 3-11 and Figure 3-13, we find that these two quivers are exactly the same, except that N flavors in $Sp'(k)$ should correspond to $N+1$ flavors in $Sp(k)$. This is reasonable because for $Sp'(k)$ with N flavors there are two half-hypermultiplets which give the same degrees of freedom as one flavor. However, in principle there is a difference between one flavor and two half-hypermultiplets: for the former we can involve one mass parameter, but for the latter there is no such mass parameter. We will show, in the case of $Sp'(k)$, that the two half-hypermultiplets do combine to give one flavor with the mass parameter. To see this, we move one 1/2D5-brane from infinity at each side to pass the 1/2NS-brane. By using the s-configuration in section 2, we get the brane setup of $Sp(k)$ with an additional flavor. The whole discussion is shown in figure 3-14. Furthermore, it is easily to show that the two cases where a "hidden FI-parameter" shows in $Sp(k)$ and $Sp'(k)$ exactly match each other.

3.6 The mirror of $SO(2k)$ gauge theory

After the discussion of the mirror theories for $Sp(k)$ gauge groups, we now discuss $SO(2k)$. There are no known results for the mirror of $SO(2k)$ gauge groups and it is the main motivation of this paper to calculate it using the $O3$ plane. As in the last two sections, we first present the simple case of $SO(2)$ with three flavors, then give the general results for $SO(2k)$ with N flavors.

3.6.1 $SO(2)$ with 3 flavors

For $SO(2)$ gauge theory with three flavors, the moduli spaces have $d_v = 1$ and $d_H = 3 \times 2 - 1 = 5$. The steps for the mirror transformation are given in Figure 3-15. In part (a) , we break the D3-branes by preserving the supersymmetric configurations, then use (3.4) and (3.5) to move the 1/2NS-brane passing the 1/2D5-brane to get part (b). Unlike the Sp case, part (b) is already convenient for the mirror transformation, so we can make S-duality directly and get part (c) . From the brane setup in part (c) we read out the mirror theory to be $Sp(1) \times SO(2) \times Sp(1) \times SO(2) \times Sp(1)$ with four bifundamentals and two half-hypermultiplets for the leftmost $Sp(1)$ and two half-hypermultiplets for the rightmost $Sp(1)$. Here we want to emphasize that in the two half-hypermultiplets for the leftmost $Sp(1)$, one comes from the 1/2D5-brane and the other from the $\widetilde{O3}^-$ projection (same for the rightmost $Sp(1)$). That the half-hypermultiplets come from different sources is a general phenomenon in $SO(2k)$. However, for our simple example, we can combine these two half-hypermultiplets together by moving the 1/2D5-brane in part (c) to go part (d). Now we have one flavor of $Sp(1)$ given by one physical D3-brane stuck between the 1/2NS-brane and the 1/2D5-brane. We need to emphasize that because the physical D3-brane is stuck between the 1/2NS-brane and the 1/2D5-brane, it does not contribute to the mass parameter. It will be interesting to compare it with the discussion in section 4.3, where we find that the two half-hypermultiplets of $Sp'(k)$ can combine to give a flavor with free mass parameter. Finally, we calculate the dimension of the moduli spaces of mirror theory as $d_v = 5$ and $d_H = (4\times4/2+2\times2)-(2+3\times3) = 12-11 = 1$.

3.6.2 An exotic example: $SO(2)$ with 2 flavors

In this subsection, we discuss the mirror of $SO(2)$ with two flavors. This theory will show one nontrivial phenomenon. The moduli are $d_v = 1$ and $d_H = 2 \times 2 - 1 = 3$. According to the standard procedure introduced in the last subsection we get the Higgs branch as part (a) in Figure 3-16 and the mirror theory in part (b). The dimensions of moduli spaces of the mirror theory in part (b) are $d_v = 1 + 1 + 1 = 3$ and $d_H = 2 \times 4/2 + 4 \times 2/2 - (3 + 3 + 1) = 9 - 8 = 1$.

However it seems we can get another possible Higgs branch in part (c) by moving the 1/2NS-brane one further step inside from part (a). If these two 1/2NS-branes do not meet together, the brane setup is not convenient to perform S-duality to get the mirror theory and we must go back to part (a). But in this special example, these two 1/2NS-branes do meet together. Now if these two 1/2NS-brane can combine to leave the $O3^+$ plane, we do get another mirror theory like part (d). Let us assume it is correct first and calculate the moduli spaces. In the part (d), the mirror theory is $Sp(1) \times SO(3) \times Sp(1)$ with two bifundamentals, two half-hypermultiplets for the two $Sp(1)$ and one fundamental for $SO(3)$, so the moduli are $d_v = 3$ and $d_H = 2 \times 6/2 + 2 \times 2/2 + 3 - (3 + 3 + 3) = 11 - 9 = 2$. Therefore the results do not match. There is another inconsistent result because in the mirror theory of part(d) we get one "hidden FI-term" which does not exist in the mirror theory of part (b).

What is the resolution for the above inconsistency? Notice the combination of two 1/2NS-branes on the $O3^+$ is S-dual to the combination of two 1/2D5-branes on the $\widetilde{O3}^-$. We have discussed this configuration in section 3.1, where we showed, only when there is an extra physical D3-brane between these two 1/2D5-branes (1/2NS-branes) can they combine and leave the O3-plane. So the conclusion is that the two 1/2NS-branes in part (c) can not combine and leave the O3-plane. We are left with only one correct mirror theory in part (b).

3.6.3 The general $SO(2k)$ with N flavors

With the experience of the $SO(2)$ case, we can now work on the general $SO(2k)$ with N flavors. The moduli for this theory are $d_v = k$ and $d_H = 2kN - k(2k-1)$. The steps for the mirror theory are given in Figure 3-17. Again, we first break the D3-branes according to the supersymmetric configuration, then move the 1/2NS-branes inside to go to part(a). The brane setup in part(a) can be considered as the brane setup of S-duality just by exchanging the roles of the 1/2NS-brane and the 1/2D5-brane and putting in a proper O3-plane. For clarity, we draw the quiver diagram of the mirror theory in part(b). Let us check the result again by calculating the moduli of the mirror theory as

$$\begin{aligned}
d_v &= 4\sum_{n=1}^{k-1} n + k(2N-4k+3) = 2kN - k(2k-1), \\
d_H &= [\tfrac{1}{2}\times 2\sum_{n=1}^{2k-2}(n+1)(n+2) + \tfrac{1}{2}4k^2(2N-4k+2) \\
&+ \tfrac{1}{2}(2k+2k+2+2)] - [4\sum_{n=1}^{k-1} n(2n+1) \\
&+ k(2k+1)(N-2k+2) + k(2k-1)(N-2k+1)] \\
&= k.
\end{aligned} \tag{3.11}$$

By checking part (a) in Figure 3-17, we find that there is a "hidden FI-parameter" in the original theory when $N = 2k-1$ because two 1/2NS-branes will meet in same interval of $\widetilde{O3}^+$ plane. For general N, k, the global symmetry is an $Sp(N)$ flavor symmetry, but in the case $N = 2k-1$ it is enhanced to $Sp(N)\times SO(3)$. We want to point out that there is only one case where *"hidden FI-parameters"* show in $SO(2k)$ while for $Sp(k)$ and $Sp'(k)$ there are two cases. This difference can be seen very clearly in part (c) of Figure 3-16. In that case two 1/2NS-branes do meet in same interval, but they can not combine and leave $O3^+$-plane. So there is no *"hidden FI-parameters"*.

3.7 The mirror of $SO(2k+1)$ gauge theory

In this section, we discuss the mirror theories of $SO(2k+1)$ to complete our study of single gauge groups. We first present the simple example of $SO(3)$ with two flavors, then give the general results for $SO(2k+1)$ with N flavors.

3.7.1 $SO(3)$ with 2 flavors

For $SO(3)$ with two flavors, the dimensions of moduli space are $d_v = 1$ and $d_H = 2\times 3 - 3 = 3$. The steps to get the mirror theory are shown in Figure 3-18. In part (a) we split the physical D5-branes into the 1/2D5-branes according the rules given in section two. In such a process we see the generation of two physical D3-branes which is necessary to account for the correct Higgs branch. In part (b) we split the D3-brane between the 1/2D5-branes and 1/2NS-branes to go to the Higgs branch. Notice that there is no D3-branes connecting 1/2NS-brane and the nearest 1/2D5-brane which is required by s-rule. In part (c) we move the 1/2NS-branes inside to get rid of the D3-branes ending on them. Now we can make S-duality to give the mirror theory in part (d). However, in our example, there is a special property: two 1/2D5-branes can combine together and leave the $\widetilde{O3^+}$-plane to give one flavor.

Now we can read out the mirror theory as $SO(3) \times Sp(1) \times SO(3)$ with two bifundamentals and one flavor for $Sp(1)$. Let us calculate the dimension of moduli space. For the Coulomb branch, we have $d_v = 1+1+1 = 3$ which matches the Higgs branch of the original theory. For the Higgs branch, naively we should have $d_H = [\frac{1}{2}(6+6)+2]-[3+3+3] = -1$. However, the dimension can never be negative. The negative result means that our naive calculation is wrong. The reason is that in our naive calculation we assumed that there is complete Higgsing. However, in our example, there is no complete Higgsing in the mirror theory. After Higgsing, we still keep two $SO(2)$ gauge groups which give the correct $d_H = [8]-[9-2] = 1$ and match the Coulomb branch in the original theory. Furthermore, in our example, we have one flavor in the mirror theory which means that there is a "hidden FI-term" in the original theory.

3.7.2 The general case: $SO(2k+1)$ with N flavors

Now let us discuss the mirror of $SO(2k+1)$ with N flavors. The dimensions of moduli spaces are $d_v = k$ and $d_H = (2k+1)N - k(2k+1)$. The steps to get the mirror theory are given in Figure 3-19. In part (a), we give the brane setup of the Higgs branch. In fact, we can consider it as well as the brane setup of the mirror theory by just changing the role of the vertical line and cross line (in Higgs branch, vertical lines denote 1/2D5-branes and cross lines, 1/2NS-branes; in the mirror theory, vertical lines denote 1/2NS-branes and cross lines, 1/2D5-branes). For convenience, we give the quiver diagram of the mirror theory in part (b).

Let us calculate the dimensions of the moduli spaces of the mirror theory to see if they match the dimensions of the moduli spaces of the original theory. The calculations are given as

$$
\begin{aligned}
d_v &= [\sum_{i=1}^{k-1} 4i] + k(N-2k+3) + (k+1)(N-2k) \\
&= (2k+1)N - k(2k+1) \\
d_H &= 2\sum_{i=1}^{k-1}[\frac{(2i+1)2i}{2} + \frac{2i(2i+3)}{2} - 2i(2i+1)] \\
&+ (2N-4k)\frac{2k(2k+2)}{2} - (N-2k+1)k(2k+1) \\
&- (N-2k)(k+1)(2(k+1)-1) + 2\frac{2k}{2} + 2\frac{2k(2k+1)}{2} - 2k(2k+1) + N \\
&= [2k(k-1)] + [-(N-2k) - k(2k+1)] + [2k] + [N] \\
&= k
\end{aligned} \tag{3.12}
$$

Notice that we add N when we calculate d_H because after the Higgsing, the mirror theory still keep N $SO(2)$ gauge group. Furthermore, from the part (a) in Figure 3-19 we see when $N = 2k$, two 1/2-branes can combine together and leave the orientifold plane. This means that when $N = 2k$ there is a *"hidden FI-term"* in the original theory. This also means that in the special case, the original theory has an enhanced global $Sp'(N) \times SO(3)$ symmetry instead of $Sp'(N)$ flavor symmetry in general.

3.7.3 Comparing the mirrors of $SO(2k)$ and $SO(2k+1)$

At the end of this section, let us compare the mirror theories of $SO(2k)$ and $SO(2k+1)$. First we can start from the $SO(2k+1)$ with $N+1$ flavors to go to $SO(2k)$ with N flavors by Higgsing one flavor. At the other side, by comparing the quivers in Figure 3-17 and Figure 3-19, it is obvious that if we change the $SO(d)$ gauge group in Figure 3-19 to $SO(d-2)$ while keeping the $Sp(d/2)$ gauge group we get exactly the quiver in Figure 3-17. In particular, the two $SO(3)$ gauge group in Figure 3-19 go to $SO(1)$ and disappear as a gauge group but add two half-hypermultiplets to two $Sp(1)$ at the two ends of quiver in Figure 3-17. This pattern can also be found if we higgs $SO(2k)$ with N flavors to $SO(2k-1)$ with $N-1$ flavors. In the latter case, we change the $Sp(d/2)$ gauge group in Figure 3-17 to $Sp(d/2-1)$ gauge group while keeping $SO(d)$ gauge group. After such a change, the quiver in Figure 3-17 becomes exactly the quiver in Figure 3-19 (the two nodes at the ends in Figure 3-17 disappear). Notice that the Higgsing in the original theory should correspond to the reduction of the Coulomb branch in the mirror theory. The change of gauge group is exactly the required reduction of the Coulomb branch in the mirror theory.

The above pattern passes another consistency check. Notice that for $SO(2k+1)$ gauge theory with $N+1$ flavors, it has an enhanced $SO(3)$ global symmetry when $N+1=2k$. After Higgsing, we get $SO(2k)$ with N flavors. For the latter, it has an enhanced $SO(3)$ global symmetry exactly when $N=2k-1$. We see such hidden global symmetry is not broken by the Higgs mechanism as it should be.

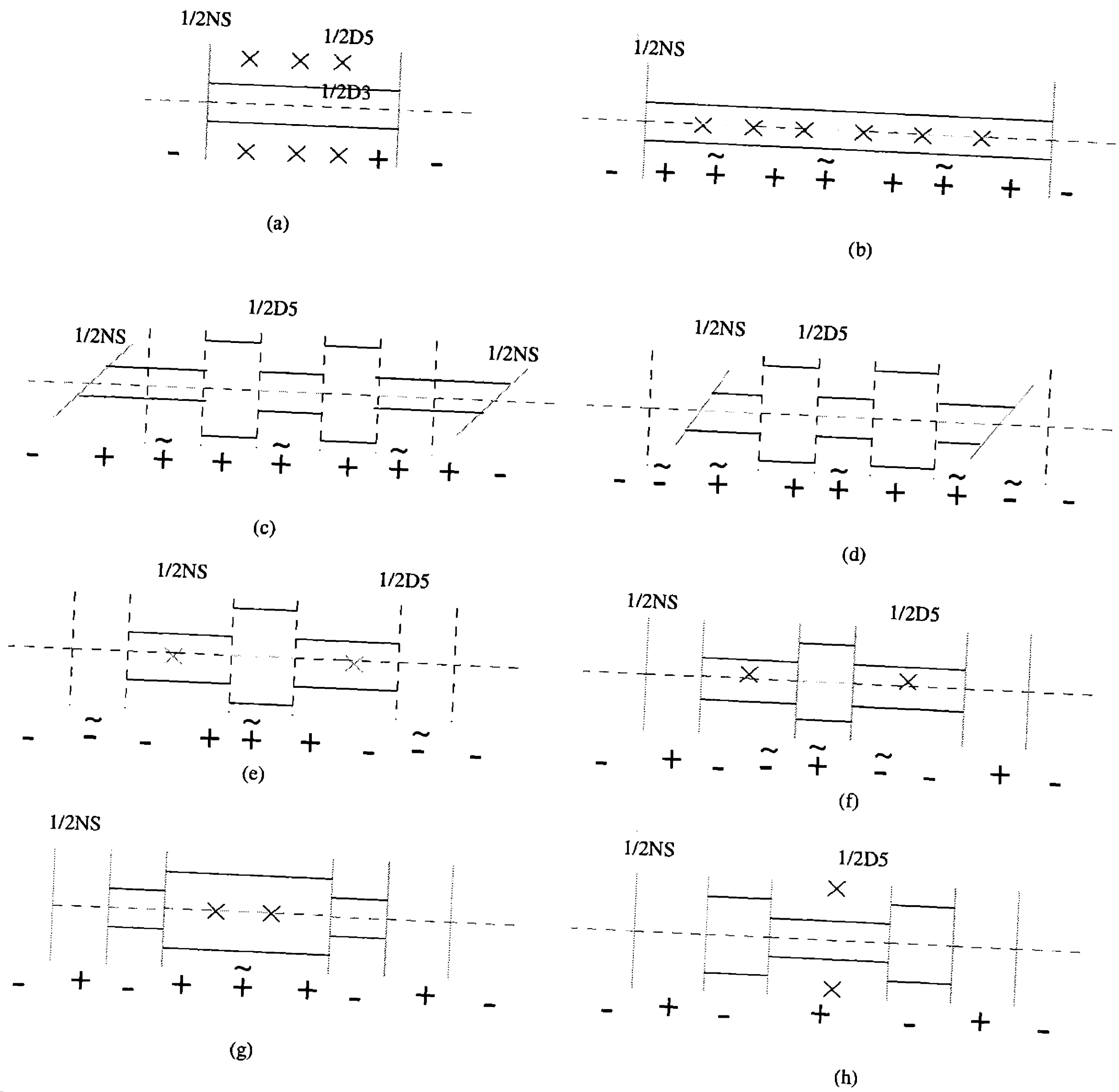


Figure 3-8: The detailed steps for getting the mirror of $Sp(1)$ with three fundamental flavors. (a) The brane setup. (b) Splitting of the physical D5-branes. (c) The Higgs branch obtained by splitting the D3-brane. Notice the special splitting of these D3-branes. (d) Using the result of supersymmetric configuration we can move 1/2NS-brane one step inside. (e) Using again the rule of supersymmetric configuration we move the 1/2NS-brane one further step inside. In this step, the D3-brane ending on the 1/2NS-brane is annihilated. (f) S-dual of part (e). (g) However, we can not read out the final gauge theory from the brane setup in (f). For avoiding the ambiguity, we can move 1/2NS-brane one further step inside. (h) A special property of our example is that we can combine two 1/2D5-branes in part (g) together and leave the $O3^+$ plane.

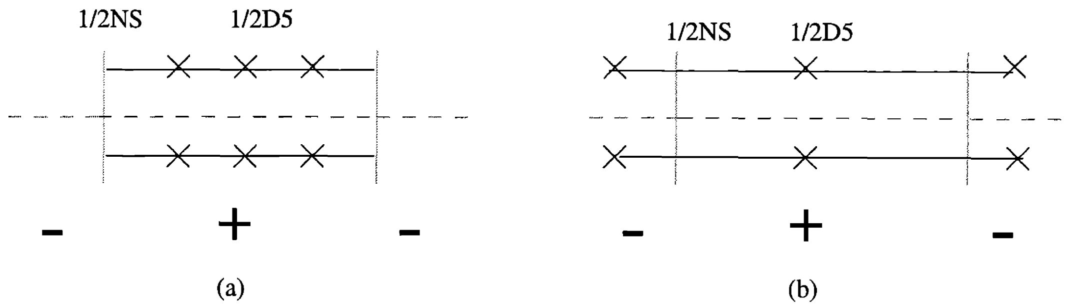


Figure 3-9: The other method to go to the Higgs branch: (a) The incomplete Higgs branch when D5-branes are off the O3-plane. (b) By moving 1/2NS-brane one step inside, we get rid of the D3-brane ending on 1/2NS-brane.

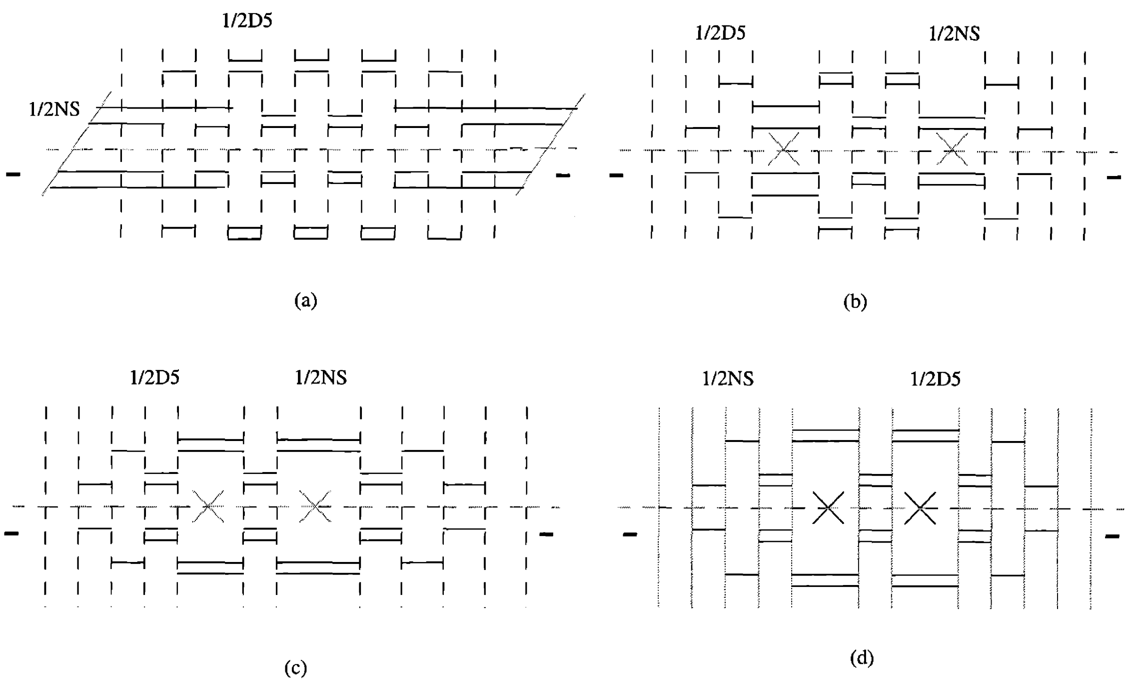


Figure 3-10: (a) The Higgs branch of $Sp(2)$ with 6 flavors. Notice how we split the D3-branes according the supersymmetric configuration. (b) By moving 1/2NS-brane across the 1/2D5-brane, we get rid of the D3-brane ending on 1/2NS-brane. (c) However, to read out the correct mirror theory, we need to move the 1/2NS-brane one step further inside. (d) By S-duality of part (c) we get the brane setup of the mirror theory.

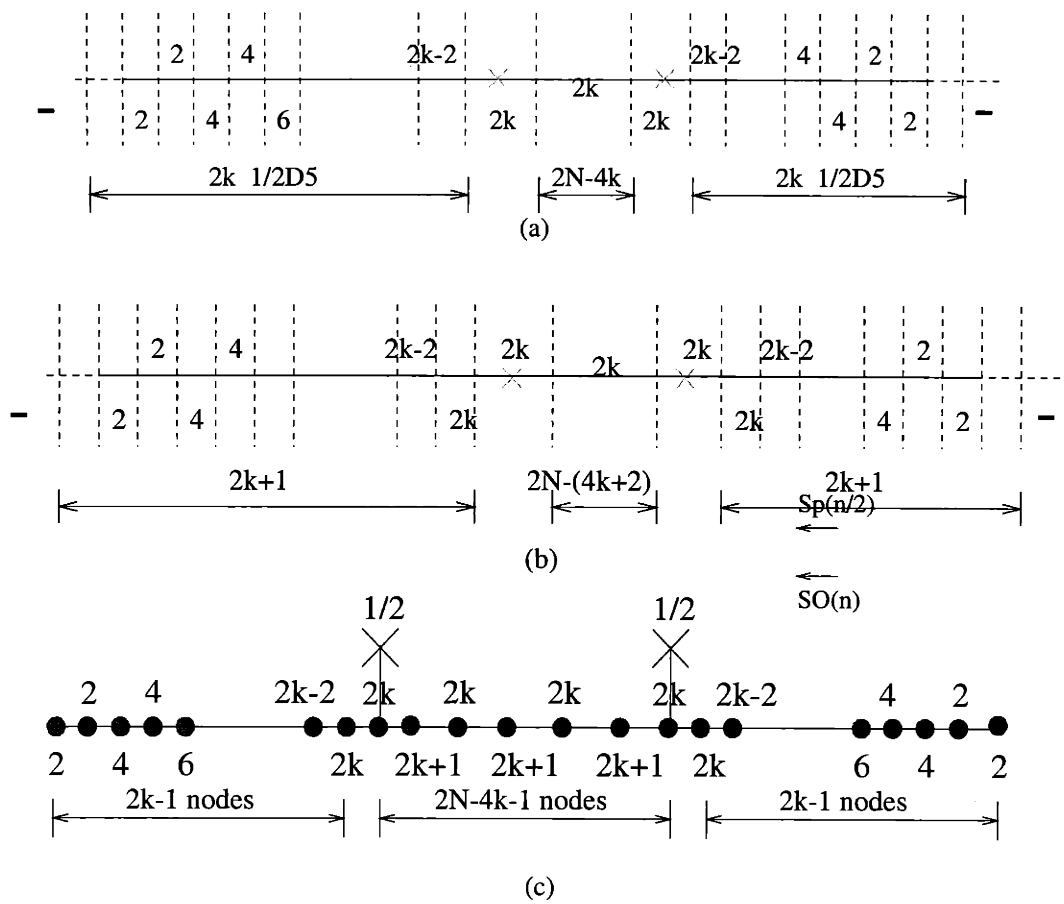


Figure 3-11: The mirror of $Sp(k)$ gauge theory with N flavors. Because of the complexity, in this figure we do not keep track of the change of O3-plane anymore and use dotted horizontal black line to express all O3-planes. However, we do keep track of the intervals which give the Sp or SO group in the final mirror theory by using the number above the O3-plane to denote the Sp group and below to denote SO group. (a) The Higgs branch of $Sp(k)$ with N flavors. Notice that the pattern of the number of 1/2-D3 branes between two nearby 1/2D5-branes is, from left to right, $0, 2, 2, 4, 4, 6, ...2k-2, 2k^{2N-4k+1}, 2k-2, 2k-2, ..., 4, 4, 2, 2, 0$. (b) To read out the mirror theory in general we need to move the 1/2NS-brane one step further inside. However, we can consider (b) as the brane setup of the mirror theory too by just thinking of the dotted vertical line as 1/2NS brane and the cross as 1/2D5-brane. (c) For convenience, we draw the quiver diagram. We use red dots for SO groups and blue dots for Sp groups. We also write the number above for an Sp group and under for an SO group.

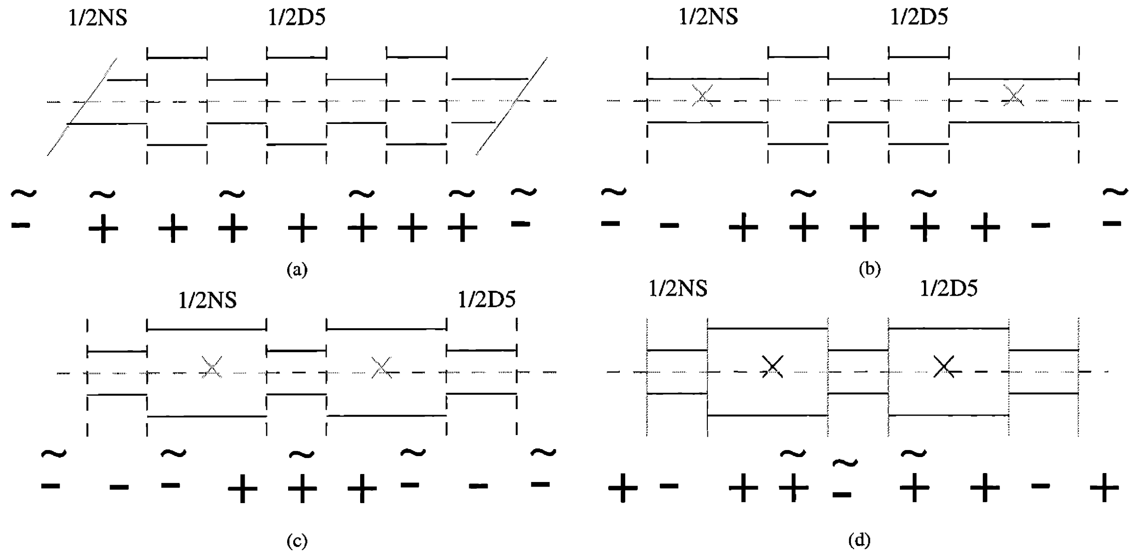


Figure 3-12: (a) The Higgs branch of $Sp'(1)$ with three flavors. Notice how we split D3-branes to satisfy the supersymmetric configuration. (b) (c) Using the rule of supersymmetric configuration, we reach the brane setup which is good for the mirror transformation. (d) The brane setup of the mirror theory.

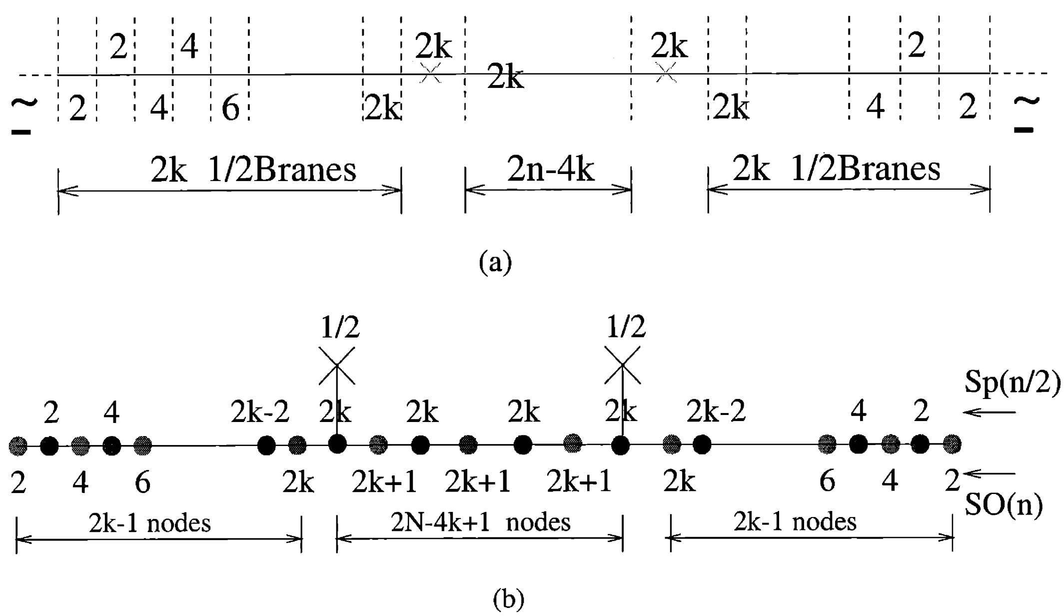


Figure 3-13: (a) The Higgs branch of $Sp'(k)$ with N flavors after moving the two 1/2NS-branes inside. As before, the numbers above and below mean the number of 1/2D3-branes which connect two neighboring 1/2D5-branes. We can also consider it as the brane setup of the mirror theory just by considering the vertical line as 1/2NS-brane instead of 1/2D5-brane and the crosses as 1/2D5-branes instead of 1/2NS-branes. (b) The quiver diagram of the mirror theory. The numbers written above the (blue) node denote the Sp gauge group and the numbers written under the (red) node denote the SO gauge group.

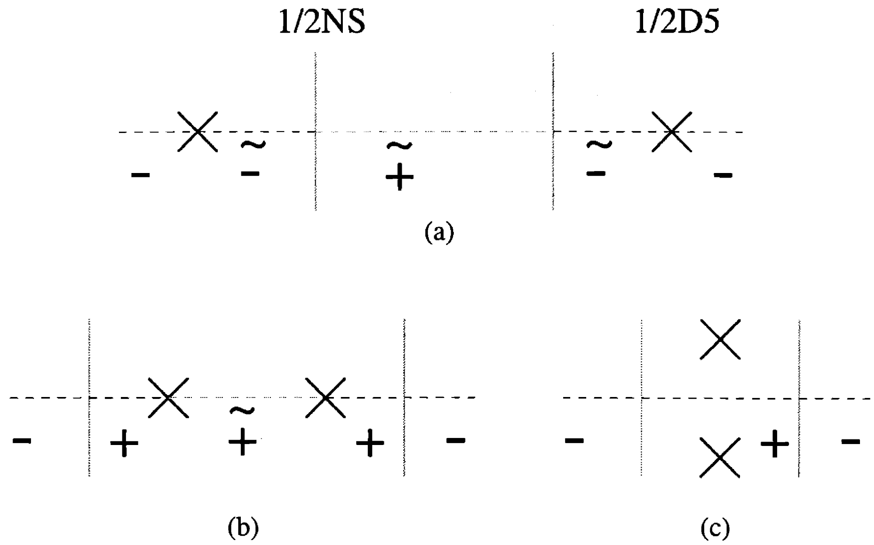


Figure 3-14: (a) We move one 1/2D5-brane from the left and right infinity. (b) By using the s-configuration in section 2, we change the position of 1/2D5-branes inside. (c) By combining the two 1/2D5-brane we get one physical D5-brane which can be moved off the $O3^+$-plane. From it we see that we change $Sp'(k)$ to $Sp(k)$ with one additional flavor.

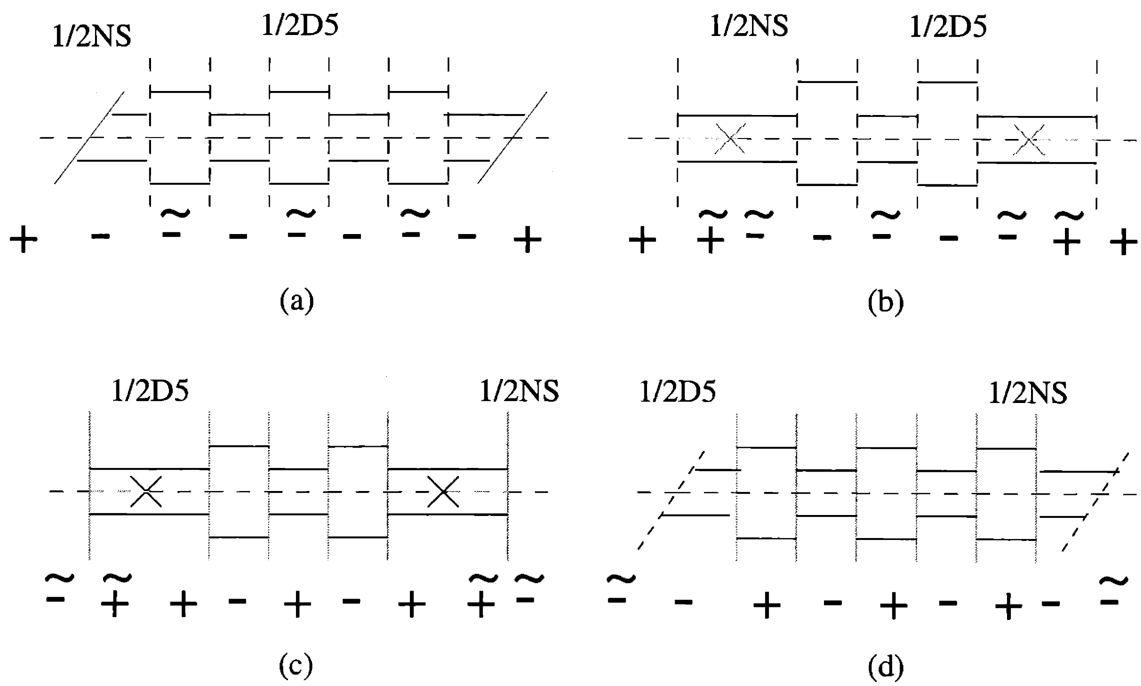


Figure 3-15: (a) The Higgs branch of $SO(2)$ with three flavors. Notice how we split the D5-brane to satisfy the supersymmetric configurations. (b) Using the rule of supersymmetric configuration, we move the 1/2NS-brane one step to reach the brane setup which is convenient for the mirror transformation. (c) The brane setup of the mirror theory. (d) By moving the 1/2D5-brane one step outside, we combine the two half-hypermultiplets into one hypermultiplet.

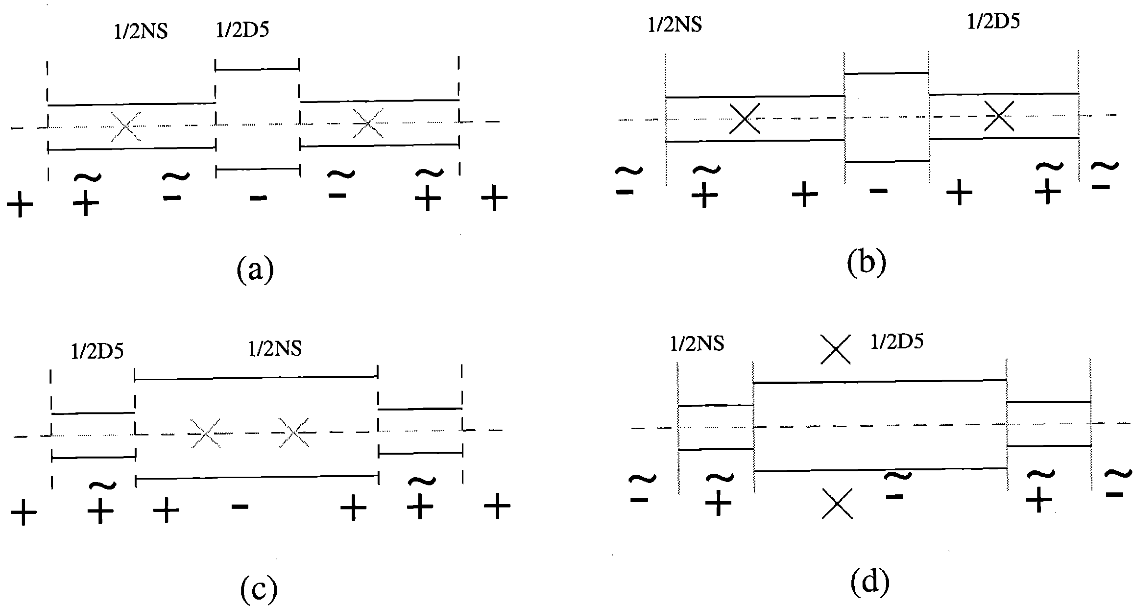


Figure 3-16: (a) The Higgs branch of $SO(2)$ with two flavors. (b) By S-duality, we get the mirror theory as $Sp(1) \times SO(2) \times Sp(1)$ with two bifundamentals and four half-hypermultiplets for the two $Sp(1)$ gauge theories. (c) However, for this special case, it seems we can get another mirror theory by moving the 1/2NS-brane one further step inside from part (a) to part (c). In our case, now two 1/2NS-branes are in same interval. If they can combine together and leave the $O3^+$ plane, we can make the S-duality to get part (d). (d) The mirror theory got from part (c) is $Sp(1) \times SO(3) \times Sp(1)$ with two bifundamentals , two half-hypermultiplets for two $Sp(1)$ and one fundamental for $SO(3)$.

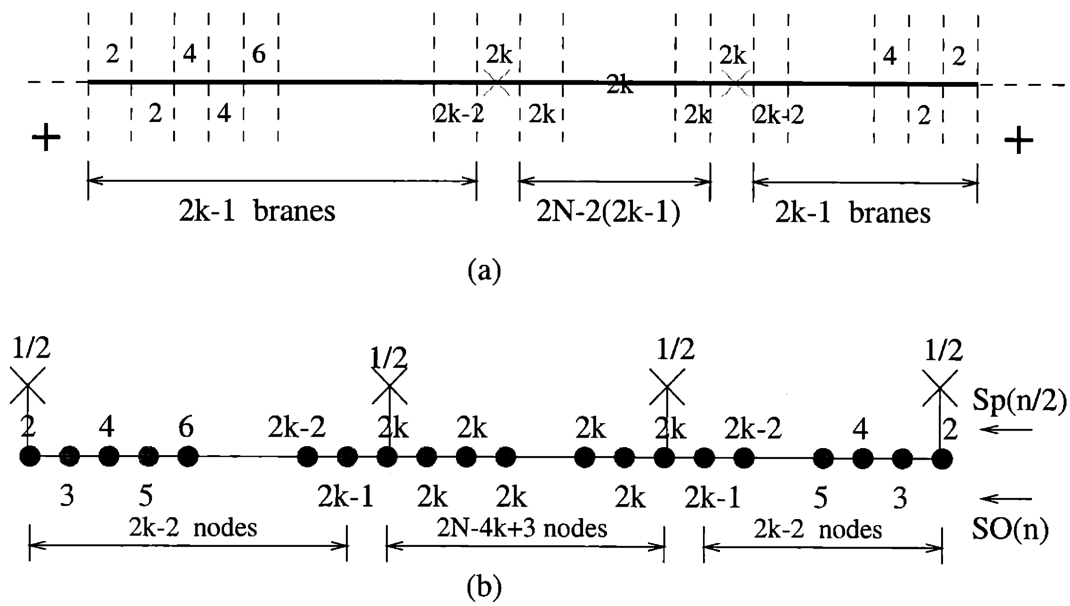


Figure 3-17: (a) The Higgs branch of $SO(2k)$ with N flavors in the setup of the D5-brane splitting. The numbers in the interval denote the number of 1/2D3-branes connecting the two neighboring 1/2D5-branes. (b) The quiver diagram of the mirror theory of $SO(2k)$ with N flavors. Notice that the index above the node means $Sp(n/2)$ and index below the node means $SO(n)$. The 1/2 means the half-fundamental.

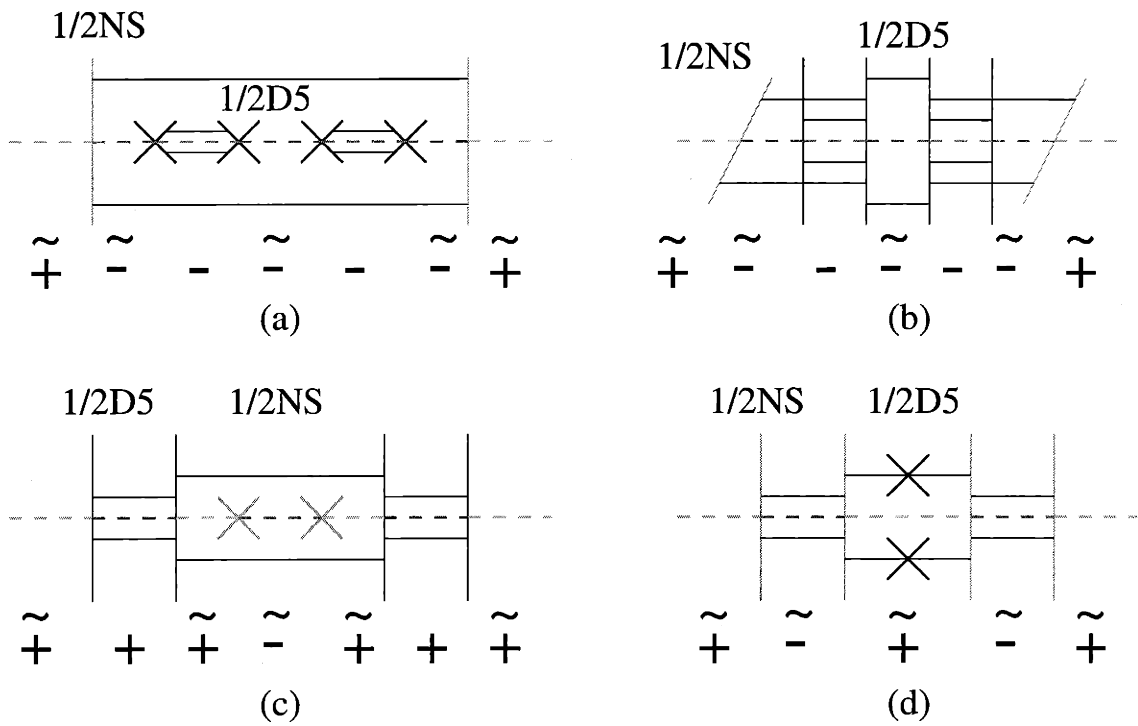


Figure 3-18: The mirror of $SO(3)$ with two flavors. (a) Splitting of D5-branes according the rules given above. Notice the generation of D3-branes between 1/2D5-branes. (b) The Higgs branch of $SO(3)$ theory. (c) By moving 1/2NS-branes inside we get rid of D3-brane ending on 1/2NS-branes and ready to go to the mirror theory. (d) The mirror theory. However, here we combine two 1/2D5-branes to give one physical D5-brane.

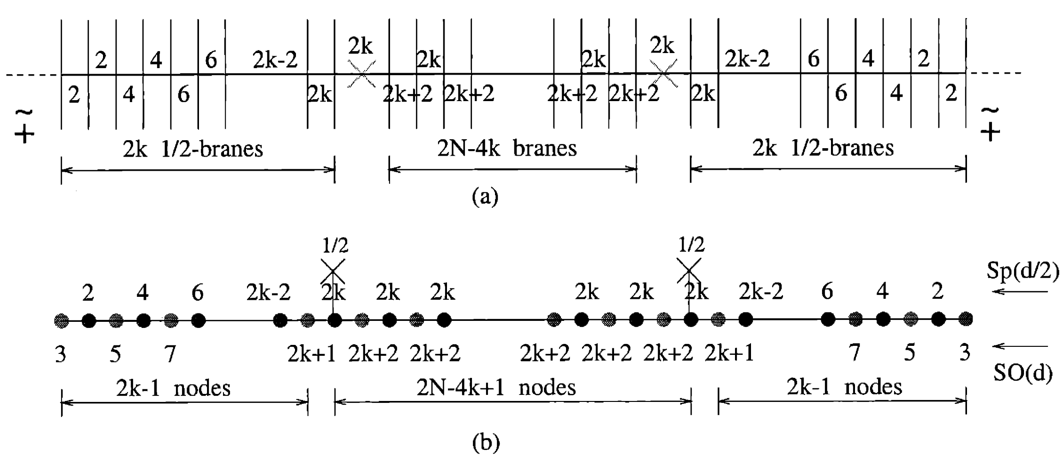


Figure 3-19: The mirror of $SO(2k+1)$ with N flavors. (a) The Higgs branch of the original theory or the Coulomb branch of the mirror theory. (b) The quiver diagram of the mirror theory.

3.8 The mirror of $Sp(k) \times SO(2m)$

We have discussed the mirror for a single Sp or SO group above. In this section, we generalize the above construction to the case of the product of Sp and SO gauge groups. Because after crossing the 1/2NS-brane $O3^{\pm}(\widetilde{O3}^{\pm})$ change to $O3^{\mp}(\widetilde{O3}^{\mp})$ and vise versa, we get two series of products $SO(2n_1) \times Sp(k_1) \times SO(2n_2) \times Sp(k_2)$.. and $SO(2n_1+1) \times Sp'(k_1) \times SO(2n_2+1) \times Sp'(k_2)$... In this section, we discuss the first series and leave the second series to next section. For simplicity, we will discuss only the product of two gauge groups, i.e., $Sp(k) \times SO(2m)$ (the case of more product groups can be directly generalized). For this simple case, we still have two choices, the so called "elliptic model" [25] (X^6 direction is compactified) , or the "non-elliptic model" (X^6 direction is not compactified). We discuss these two models one by one.

3.8.1 The non-elliptic model

For the non-elliptic model, there are N fundamentals for $SO(2m)$, H fundamentals for $Sp(k)$ and one bifundamental (for simplicity we assume that N, H are sufficiently large. For N, H too small, there are a lot of special cases which need to be discussed individually and are tedious without providing too much new insight). The moduli are $d_v = m+k$ and $d_H = 2mN+2kH+2mk-m(2m-1)-k(2k+1)$. In constructing the mirror theory, we need to study three cases: $m > k$, $m = k$ and $m < k$. Let us start with the case of $m > k$. The mirror theory is given in Figure 3-20. When we go to the Higgs branch, we can connect the D3-branes at the two sides of middle 1/2NS-brane. Because $m > k$, we can connect only k D3-branes such that they end on the left and the right 1/2NS-branes. There are still $m-k$ D3-branes ending on the middle 1/2NS-brane from the right. To get rid of those D3-branes, we must move the middle 1/2NS-brane to the right. The final Higgs branch after such a motion is given in part (a) of Figure 3-20 and the quiver diagram of the mirror, in part (b). The moduli of the mirror can be calculated as

$$d_v = [\sum_{p=1}^{p=k-1} 2p] + (2H-2k+1)k + [2\sum_{p=k+1}^{m-1} p]$$

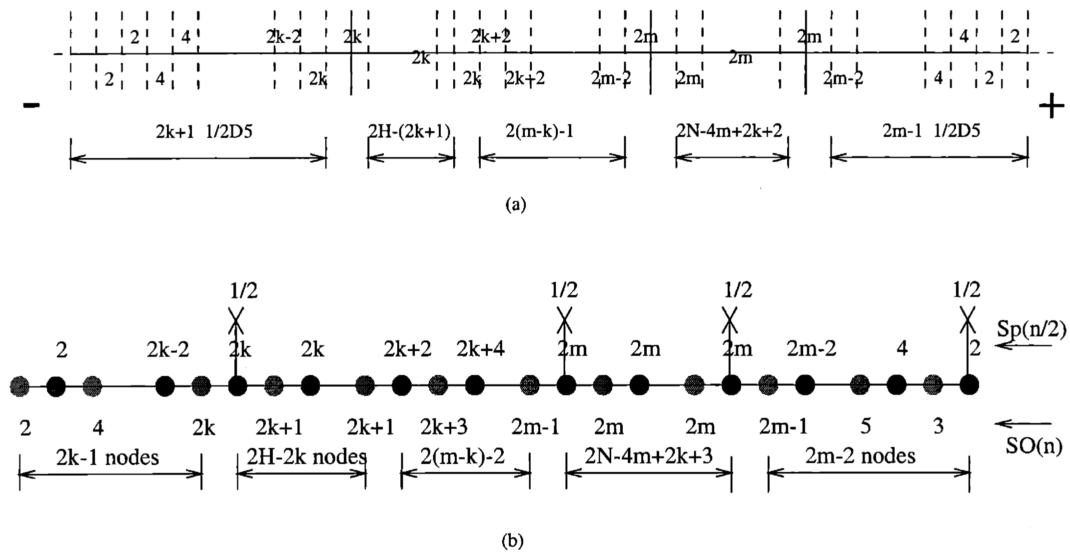


Figure 3-20: (a) The Higgs branch of $Sp(k) \times SO(2m)$ with N fundamentals for $SO(2m)$, H fundamentals for $Sp(k)$ and one bifundamental in the case of $m > k$. (b) The quiver diagram of the mirror theory of part(a). Notice that the index n above the node denotes $Sp(n/2)$ and index n below the node denotes $SO(n)$. The 1/2 means half-hypermultiplets.

$$
\begin{aligned}
& + \quad (2N-4m+2k+3)m + [2 \sum_{p=1}^{p=m-1} p] \\
& = \quad 2mN + 2kH + 2mk - m(2m-1) - k(2k+1), \\
d_H & = \quad [\sum_{i=1}^{i=k-1} (2i \times 2i + 2i \times (2i+2))/2 - i(2i-1) - i(2i+1)] \\
& + \quad [4k^2/2 + (2H-2k-1)2k(2k+1)/2 \\
& - \quad (H-k)2k(2k+1) - k(2k-1)] \\
& + \quad [\sum_{i=1}^{m-k-1} (2k+2i-1)(2k+2i)/2 + (2k+2i)(2k+2i+1)/2 \\
& - \quad 2(k+i)(2k+2i+1)] \\
& + \quad [2m(2m-1)/2 + 4m^2(2N-4m+2k+2)/2 \\
& \qquad -(N-2m+k+1)(m(2m-1)+m(2m+1)) - m(2m+1)] \\
& + \quad [\sum_{i=1}^{m-1} 2i(2i+1)/2 + (2i+1)(2i+2)/2 - 2i(2i+1)] \\
& + \quad [2k/2 + 2m/2 + 2/2 + 2m/2] \\
& = \quad [k(k-1)] + [-2k^2] + [(-k-m)(m-k-1)] + [-2m] \\
& + \quad [m^2-1] + [k+2m+1]
\end{aligned}
$$

$$= \quad k + m \tag{3.13}$$

From part (a) of Figure 3-20, we see that when $2N - 4m + 2k + 2 = 0$, the two 1/2NS-branes meet together which indicates a "hidden FI-parameter" in the original theory.

After the discussion of the $m > k$ case, we go to the $m = k$ case. Here, by connecting the D3-branes between the two sides of the middle 1/2NS-brane, we get the Higgs branch looking like part (a) in Figure 3-21. From the quiver diagram part

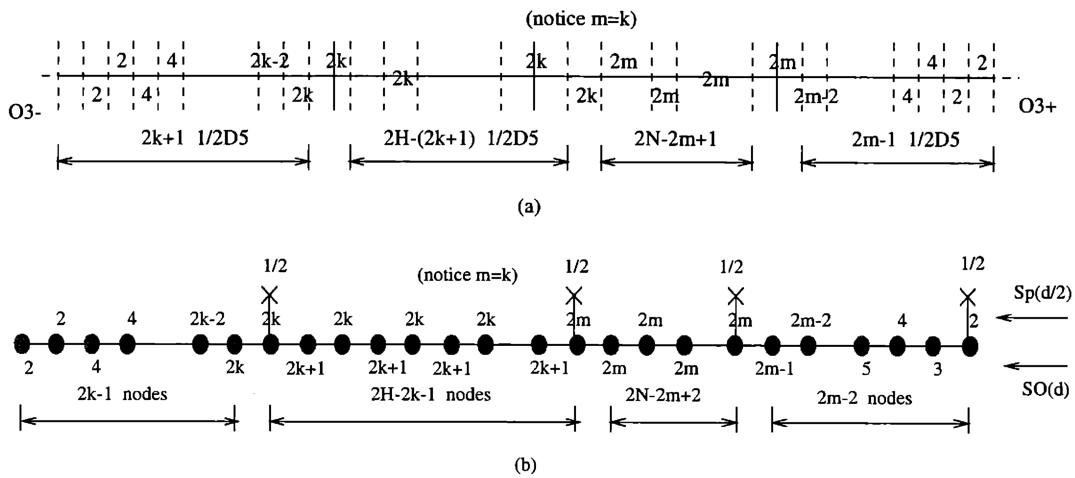


Figure 3-21: (a) The Higgs branch of $Sp(k) \times SO(2m)$ with N fundamentals for $SO(2m)$, H fundamentals for $Sp(k)$ and one bifundamental in the case of $m = k$. (b) The quiver diagram of the mirror theory of part(a). Notice that the index n above the node denotes $Sp(n/2)$ and index n below the node denotes $SO(n)$. The 1/2 means the half-hypermultiplet.

(b) we recalculate the moduli space as:

$$\begin{aligned}
d_v &= [\sum_{p=1}^{p=k-1} 2p] + (2H - 2k - 1 + 2N - 2m + 2 + 1)k + [2 \sum_{p=k+1}^{m-1} p] \\
&= 2kN + 2kH - 2k^2 \\
&= 2mN + 2kH + 2mk - m(2m-1) - k(2k+1) \quad when \;\; m = k, \\
d_H &= [\sum_{i=1}^{i=k-1} (2i \times 2i + 2i \times (2i+2))/2 - i(2i-1) - i(2i+1)] \\
&+ [4k^2/2 + (2H - 2k - 2)2k(2k+1)/2 \qquad (3.14)
\end{aligned}$$

$$
\begin{aligned}
&- \quad (2H-2k-1)k(2k+1) - k(2k-1)] \\
&+ \quad [(2N-2m+2)4m^2/2 - (N-m+1)(m(2m+1)+m(2m-1))] \\
&+ \quad [\sum_{i=1}^{m-1} 2i(2i+1)/2 + (2i+1)(2i+2)/2 - 2i(2i+1)] \\
&+ \quad [2k/2 + 2m/2 + 2/2 + 2m/2] \\
&= \quad [k(k-1)] + [-2k^2] + [0] + [m^2-1] + [k+2m+1] \\
&= \quad k+m. \qquad (3.15)
\end{aligned}
$$

From the figure again, when $2H-2k=0$ or $2N-2m+2=0$, the two 1/2NS-branes meet together to give a "hidden FI-term" in the original theory.

Now we are left with only one case, i.e., $m < k$. In this last case, to get rid of the D3-branes, the middle 1/2NS brane should move to the left direction. The result is shown in Figure 3-22.

The moduli of the mirror theory are

$$
\begin{aligned}
d_v &= [\sum_{p=1}^{p=k-1} 2p] + (2H-4k+2m+1)k + [2\sum_{p=m+1}^{k-1} p] \\
&+ (2N-2m+3)m + [2\sum_{p=1}^{p=m-1} p] \\
&= 2mN + 2kH + 2mk - m(2m-1) - k(2k+1), \\
d_H &= [\sum_{i=1}^{i=k-1} (2i\times 2i + 2i\times(2i+2))/2 - i(2i-1) - i(2i+1)] \\
&+ [4k^2/2 + (2H-4k+2m-2)2k(2k+1)/2 + 4k^2/2 \\
&\quad -(2H-4k+2m-1)k(2k+1) - 2k(2k-1)] \\
&+ [\sum_{i=1}^{k-m-1} 2(m+i)2(m+i)/2 + 2(m+i)2(m+i+1)/2 \\
&\quad -(m+i)(2(m+i)-1) - (m+i)(2(m+i)+1)] \\
&+ [2m(2m+2)/2 + 4m^2(2N-2m+2)/2 \\
&\quad -(N-m+1)(m(2m-1)+m(2m+1)) - m(2m+1)] \\
&+ [\sum_{i=1}^{m-1} 2i(2i+1)/2 + (2i+1)(2i+2)/2 - 2i(2i+1)] \\
&+ [2k/2 + 2k/2 + 2/2 + 2m/2]
\end{aligned}
$$

$$
\begin{aligned}
&= [k(k-1)] + [-2k^2 + k] + [(k+m)(k-m-1)] \\
&+ [m] + [m^2 - 1] + [2k + m + 1] \\
&= k + m
\end{aligned} \tag{3.16}
$$

There is also a possible "hidden FI-term" in original theory when $2H-4k+2m-2=0$ which can be explicitly seen in part(a) in Figure 3-22.

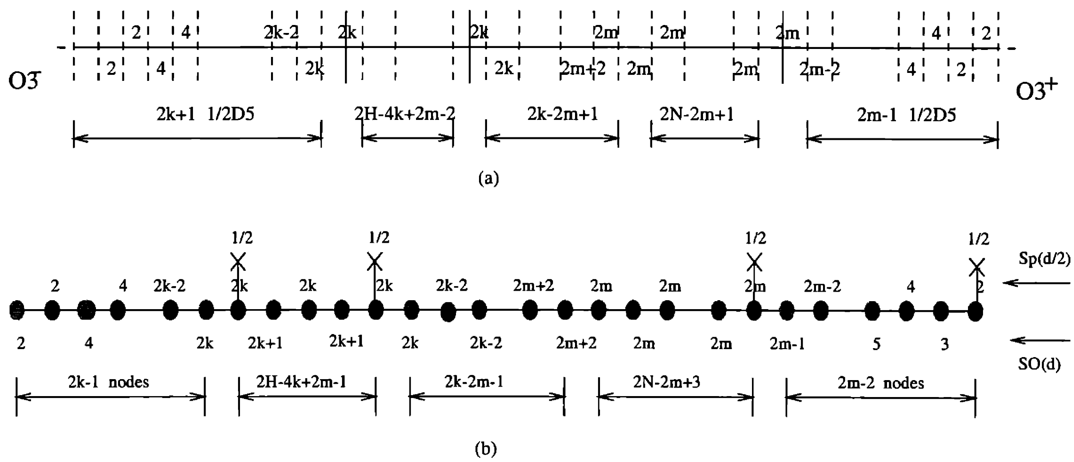


Figure 3-22: (a) The Higgs branch of $Sp(k) \times SO(2m)$ with N fundamentals for $SO(2m)$, H fundamentals for $Sp(k)$ and one bifundamental in the case of $m < k$. (b) The quiver diagram of the mirror theory of part(a). Notice that the index n above the node denotes $Sp(n/2)$ and index n below the node denotes $SO(n)$. The 1/2 denotes the half-fundamental.

3.8.2 The elliptic model

In the elliptic model, the X^6 direction is compactified such that for consistency, we must have an even number of 1/2NS-branes and an even number of gauge groups where half of them are Sp gauge groups and the other half, SO gauge groups. We discuss the case of only two gauge groups, i.e., $Sp(k) \times SO(2m)$ with H fundamentals for $Sp(k)$, N fundamentals for $SO(2m)$ and two bifundamentals. The moduli for this theory are $d_v = k + m$ and $d_H = 2mN + 2kH + 4mk - m(2m-1) - k(2k+1)$. The mirror theory for the elliptic model is similar to the non-elliptic model. The only difference is that in the non-elliptic model we can connect the D3-branes only at the middle 1/2NS-brane, but here in the elliptic model we can connect the D3-branes to all 1/2NS-branes (here two 1/2NS-branes). Again we divide into three cases to discuss. The simple one is the case $m = k$. In this case, we can connect all D3-branes such that no D3-brane is left to end on the 1/2NS-branes. The Higgs branch and the quiver of the mirror are given in parts (a) (b) of Figure 3-23 and the moduli are:

$$\begin{aligned} d_v &= (2H + 2N)k \\ &= 2mN + 2kH + 4mk - m(2m-1) - k(2k+1) \quad when \quad m = k, \\ d_H &= [(2H-2)2k(2k+1)/2 - (H-1)2k(2k+1) - k(2k+1)] \\ &+ [(2N+2)4k^2/2 - N(k(2k+1) + k(2k-1)) - k(2k-1)] \\ &+ [2k/2 + 2k/2] \\ &= [-k(2k+1)] + [2k^2 + k] + [2k] = k + m \end{aligned} \tag{3.17}$$

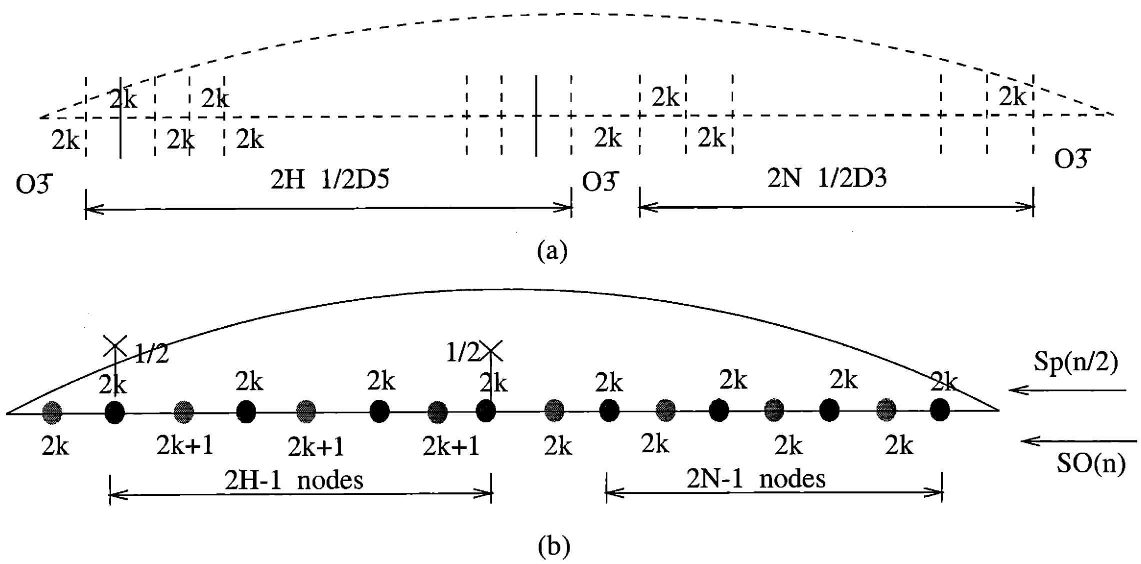


Figure 3-23: (a) The Higgs branch of elliptic $Sp(k) \times SO(2m)$ with N fundamentals for $SO(2m)$, H fundamentals for $Sp(k)$ and two bifundamentals in the case of $m = k$. The number here denotes how many 1/2D3-branes are connected to neighboring 1/2D5-branes. (b) The quiver diagram of the mirror theory of part(a). Notice that the index n above the node denotes $Sp(n/2)$ and index n below the node denotes $SO(n)$. The 1/2 denotes the half hypermultiplets.

There is still one case where the "hidden FI-term" appears, namely when $H = 1$. In this case, two 1/2NS-branes in part(a) of Figure 3-23 meet together.

Now we move to the case of $k > m$. The Higgs branch looks like the superposition of the Higgs branch of the $m = k$ case together with that of a single $Sp(k-m)$ gauge theory. We give the result in Figure 3-24. To check the result, we calculate the moduli as

$$
\begin{aligned}
d_v &= [4\textstyle\sum_{i=1}^{i=k-m-1}(m+i)] + k(2H-4(k-m)+1) + m(2N+3) \\
&= 2mN + 2kH + 4mk - m(2m-1) - k(2k+1), \\
d_H &= [2\textstyle\sum_{i=1}^{i=k-m-1} 2(m+i)2(m+i)/2 + 2(m+i)2(m+i+1)/2 \\
&\quad -(m+i)(2(m+i)-1) - (m+i)(2(m+i)+1)] \\
&+ [2\times 4k^2/2 + (2H-4(k-m)-2)2k(2k+1)/2 \\
&\quad -(2H-4(k-m)-1)k(2k+1) - 2k(2k-1)] \\
&+ [(2N+2)4m^2/2 + 2\times 2m(2m+2)/2 \\
&- (N+1)(m(2m+1)+m(2m-1)) \\
&- m(2m+1)] + [2\times 2k/2] \\
&= [2(k^2-m^2-k-m)] + [-2k^2+k] + [2m^2+3m] + [2k] \\
&= k+m.
\end{aligned}
\tag{3.18}
$$

When $2H-4(k-m)-2=0$, two 1/2NS-branes will meet together in part(a) of Figure 3-24. This is the condition that a "hidden FI-term" exists.

Now the remainder case is $m > k$. In this case, the Higgs branch looks like the superposition of two Higgs branches: that of the $m = k$ case and that of a single $SO(2(m-k))$ theory. The result can be found in Figure 3-25. The moduli of the mirror theory are

$$
\begin{aligned}
d_v &= [4\sum_{i=1}^{i=m-k-1}(k+i)] + k(2H+1) + m(2N-4(m-k)+3) \\
&= 2mN + 2kH + 4mk - m(2m-1) - k(2k+1), \\
d_H &= [2\sum_{i=1}^{i=m-k-1} 2(k+i)(2k+2i+1)/2 + (2k+2i+1)(2k+2i+2)/2 \\
&\quad -2(k+i)(2k+2i+1)] \\
&+ [(2N-4(m-k)+2)4m^2/2 - (N-2(m-k)+1) \\
&\quad (m(2m+1)+m(2m-1)) - m(2m+1)] \\
&+ [2h2k(2k+1)/2 + 2(2k+1)(2k+2)/2 - (2H+1)k(2k+1)]
\end{aligned}
$$

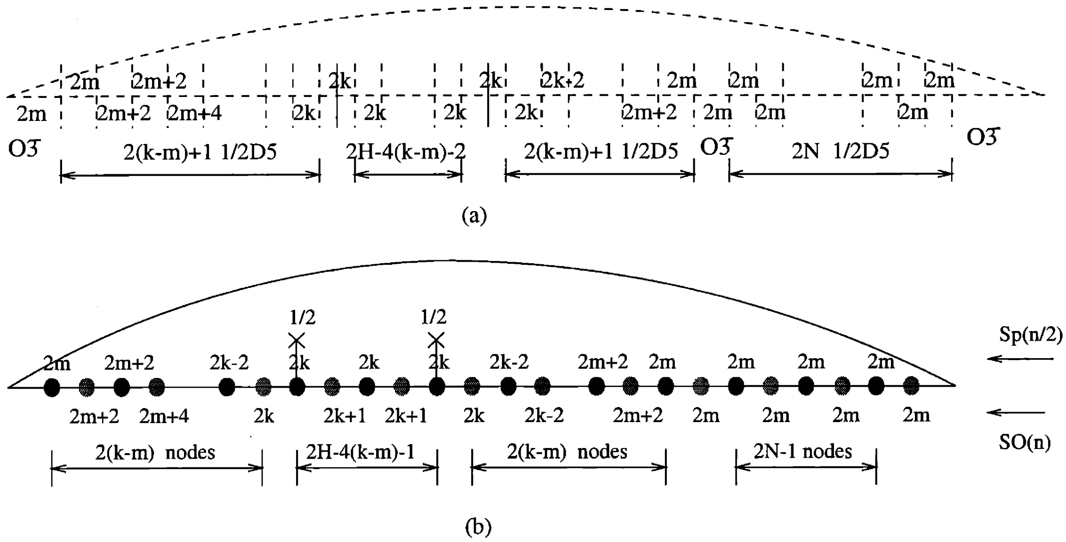


Figure 3-24: (a) The Higgs branch of elliptic $Sp(k) \times SO(2m)$ with N fundamentals for $SO(2m)$, H fundamentals for $Sp(k)$ and two bifundamentals in the case of $m < k$. The number here denotes how many 1/2D3-branes are connected to neighboring 1/2D5-branes. (b). The quiver diagram of the mirror theory of part(a). Notice that the index n above the node denotes $Sp(n/2)$ and index n below the node denotes $SO(n)$. The 1/2 denotes the half hypermultiplets.

$$
\begin{aligned}
&+ \quad [2 \times 2m/2] \\
&= \quad [2m^2 - 2(k+1)^2] + [-2m^2 - m] + [2k^2 + 5k + 2] + [2k] \\
&= \quad k + m
\end{aligned}
\tag{3.19}
$$

When $2N - 4(m-k) + 2 = 0$, there is a "hidden FI-term" in the original theory.

3.9 The mirror of $Sp'(k) \times SO(2m+1)$

For completion, we give one more example: the mirror theory of $Sp'(k) \times SO(2m+1)$. We assume that there are H flavors for $Sp'(k)$ gauge theory and N flavors for $SO(2m+1)$ gauge theory. Besides, there are one or two bifundamentals and half-hypermultiplet for $Sp'(k)$ depend on different situations. Again we divide our discussion into two parts: non-elliptic model and elliptic model.

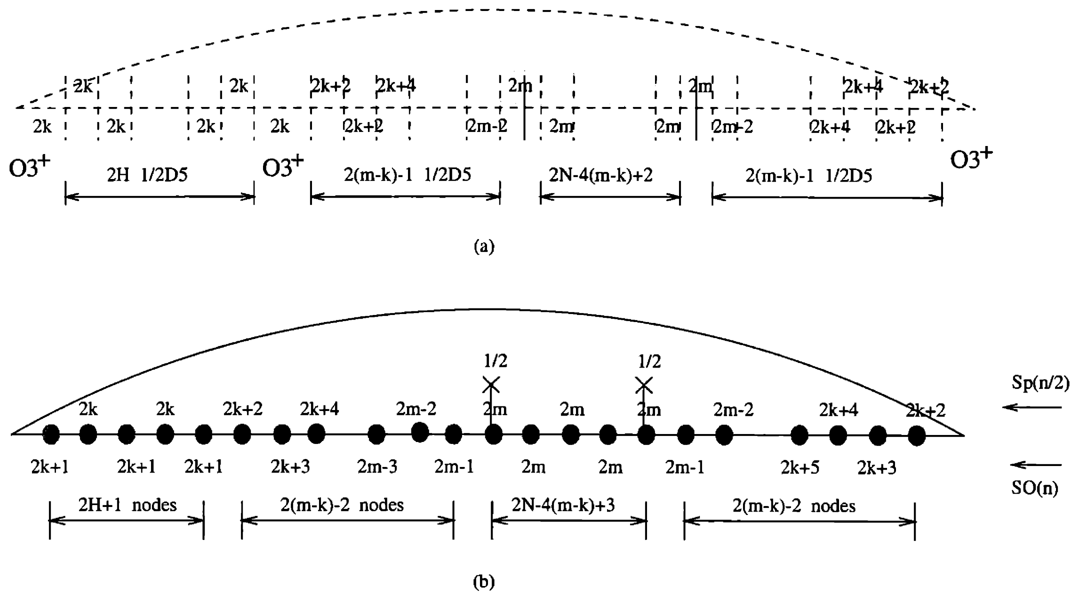


Figure 3-25: (a).The Higgs branch of elliptic $Sp(k) \times SO(2m)$ with N fundamentals for $SO(2m)$, H fundamentals for $Sp(k)$ and two bifundamentals in the case of $m > k$. The number here means how many 1/2D3-branes are connected to neighboring 1/2D5-branes. (b). The quiver diagram of the mirror theory of part(a). Notice that the index n above the node denotes $Sp(n/2)$ and index n below the node denotes $SO(n)$. The 1/2 denotes the half hypermultiplets.

3.9.1 The non-elliptic model

Let us start from the non-elliptic model. In this case, the dimensions of moduli spaces are $d_v = k+m$ and $d_H = 2kH+(2m+1)N+k(2m+1)+k-k(2k+1)-m(2m+1) = 2kH+(2m+1)N-2k^2+k-2m^2-m+2km$ (here again, for simplicity we assume N, H are sufficiently large to avoid special cases). The mirror theory depends on whether $m > k$, $m = k$ or $m < k$. We first give the mirror of the case $m = k$ because in this particular case, we can combine the D3-branes at the two sides of middle 1/2NS-brane such that there is no D3-branes ending on the middle 1/2NS-brane anymore. The mirror theory is given in Figure 3-26.

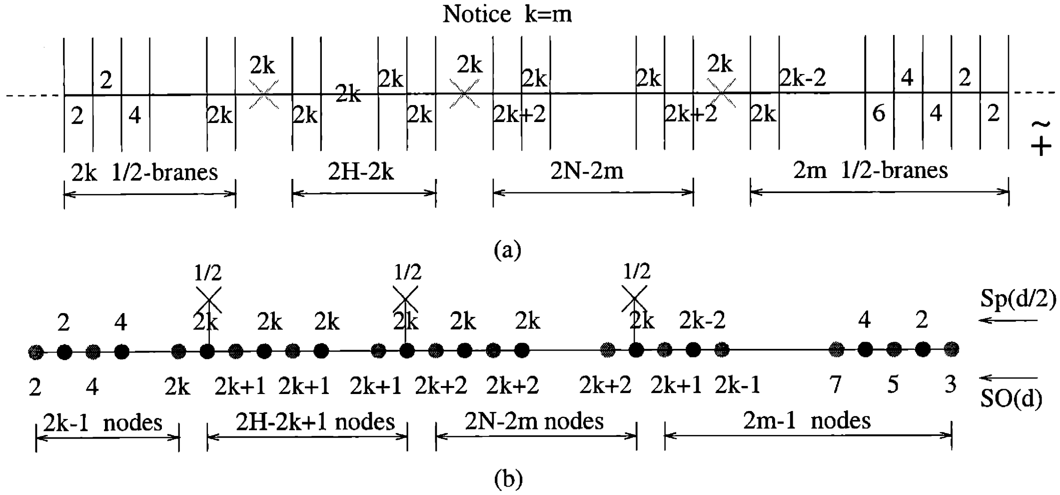


Figure 3-26: The mirror of $Sp'(k) \times SO(2m+1)$ with H flavors for $Sp'(k)$, N flavors for $SO(2m+1)$, a half-hypermultiplet for $Sp'(k)$ and one bifundamental in case of $m=k$. (a) The Higgs branch of original theory or the Coulomb branch of the mirror theory. (b) The quiver diagram of mirror theory.

Let us check it by calculating the dimensions of moduli spaces of the mirror theory:

$$\begin{aligned}
d_v &= 2\sum_{i=1}^{k-1} i + (2H-2k+2)k + (N-k)(k+k+1) + k + 2\sum_{i=1}^{k-1} i \\
&= 2kH + (2k+1)N - 2k^2 \\
d_H &= \sum_{i=1}^{k-1}[\tfrac{2i2i}{2} + \tfrac{2i(2i+2)}{2} - i(2i-1) - i(2i+1)] \\
&+ \sum_{i=1}^{k-1}[\tfrac{(2i+1)2i}{2} + \tfrac{2i(2i+3)}{2} - 2i(2i+1)] \\
&+ \tfrac{2k(2k+1)}{2}(2H-2k) - (H-k)2k(2k+1) \\
&+ \tfrac{2k(2k+2)}{2}(2N_2k) - (N-k)(k(2k+1) + (k+1)(2k+1)) \\
&+ \tfrac{2k2k}{2} - k(2k+1) - k(2k-1) + \tfrac{2k(2k+1)}{2} - k(2k+1) \\
&+ 3\tfrac{2k}{2} + N \\
&= [k^2-k] + [k^2-k] + [0] + [-(N-k)] + [-2k^2] + [3k+N] \\
&= 2k,
\end{aligned} \tag{3.20}$$

where when we calculate the d_H we add the term N to account for the remaining H $SO(2)$ gauge groups after Higgsing (this happens for latter examples so we will not mention it every time). When $2H-2m=0$ there is a *"hidden FI-term"* in the original theory.

Now we go to the case that $k > m$. In this case, after connecting the D3-branes at the two sides of the middle 1/2NS-brane, we still have $k-m$ D3-brane ending on the middle 1/2NS-brane from the left. The mirror theory is given in Figure 3-27. The

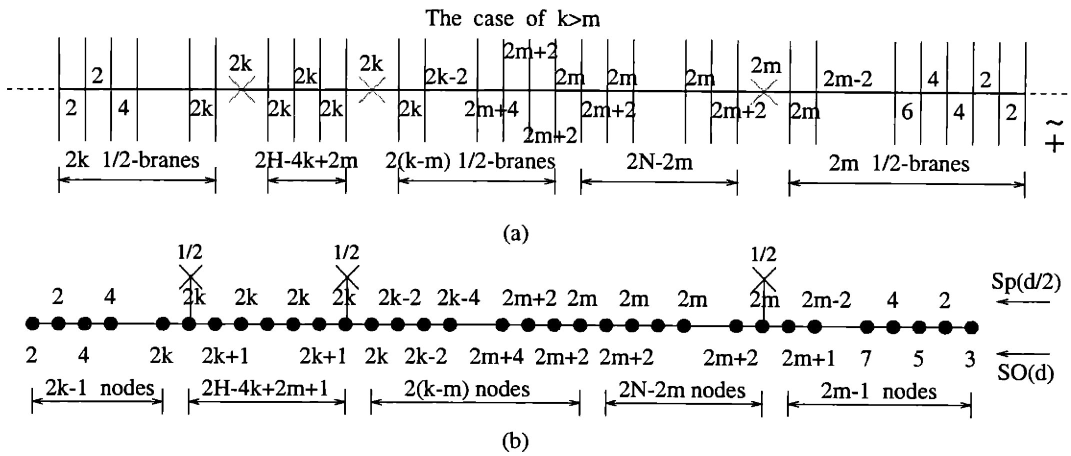


Figure 3-27: The mirror of $Sp'(k) \times SO(2m+1)$ with H flavors for $Sp'(k)$, N flavors for $SO(2m+1)$, a half-hypermultiplet for $Sp'(k)$ and one bifundamental in case of $k > m$. (a) The Higgs branch of original theory or the Coulomb branch of the mirror theory. (b) The quiver diagram of mirror theory.

dimensions of the moduli space of the mirror theory are

$$
\begin{aligned}
d_v &= [2\sum_{i=1}^{k-1} i] + [2\sum_{i=1}^{m-1} i] + (2H-4k+2m+3)k \\
&+ [2\sum_{i=1}^{k-m-1}(m+i)] + [(N-m+2)m + (N-m)(m+1)] \\
&= [k^2-k] + [m^2-m] + [2kH - 4k^2 + 2km + 3k] \\
&+ [k^2 - m^2 - k - m] + [(2m+1)N - 2m^2 + m] \\
&= 2kH + (2m+1)N - 2k^2 - 2m^2 + k - m + 2km \\
d_H &= \sum_{i=1}^{k-1}[\frac{2i2i}{2} + \frac{2i(2i+2)}{2} - i(2i-1) - i(2i+1)] \\
&+ 2\frac{2k2k}{2} + \frac{2k(2k+1)}{2}(2H-4k+2m) - (2H-4k+2m+1)k(2k+1) \\
&\quad -2k(2k-1) + \sum_{i=1}^{k-m-1}[\frac{(2(m+i))^2}{2} + \frac{2(m+i)2(m+i+1)}{2} \\
&\quad -(m+i)(2(m+i)-1) - (m+i)(2(m+i)+1)] \\
&+ \frac{2m(2m+2)}{2}(2N-2m+1) - (N-m+1)m(2m+1) \\
&\quad -(N-m)m(2(m+1)-1) + \sum_{i=1}^{m-1}[\frac{(2i+1)2i}{2} + \frac{2i(2i+3)}{2} - 2i(2i+1)]
\end{aligned}
$$

$$
\begin{aligned}
+ & \frac{2m(2m+1)}{2} - m(2m+1) + 2\frac{2k}{2} + \frac{2m}{2} + N \\
= & [k^2 - k] + [-2k^2 + k] + [k^2 - m^2 - k - m] \\
& +[-N + 2m] + [m^2 - m] + [N + 2k + m] \\
= & k + m. \qquad (3.21)
\end{aligned}
$$

After the discussion of above two cases, we go to the last case: $k < m$. In this case, because $k < m$, after the combination of D3-branes at the two sides of middle 1/2NS-brane, we still leave $m - k$ D3-brane ending on it from the right. The mirror theory is given in Figure 3-28. Let us calculate the dimensions of moduli spaces:

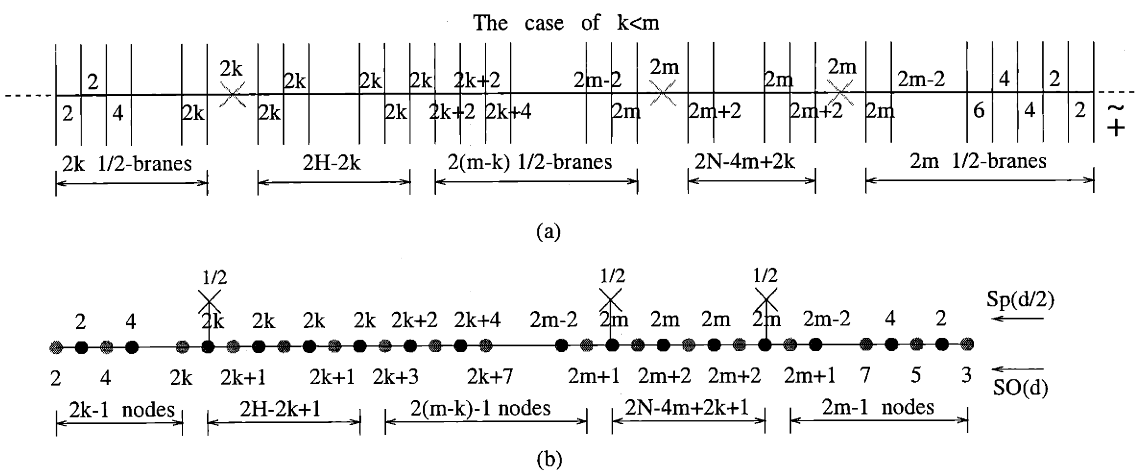


Figure 3-28: The mirror of $Sp'(k) \times SO(2m+1)$ with H flavors for $Sp'(k)$, N flavors for $SO(2m+1)$, a half-hypermultiplet for $Sp'(k)$ and one bifundamental in case of $k < m$. (a) The Higgs branch of the original theory or the Coulomb branch of the mirror theory. (b) The quiver diagram of the mirror theory.

$$
\begin{aligned}
d_v &= [2\textstyle\sum_{i=1}^{k-1} i] + [2\sum_{i=1}^{m-1} i] + (2H-2k+2)k \\
&+ 2\textstyle\sum_{i=1}^{m-k-1}(k+i) + m(N-2m+k+3) + (m+1)(N-2m+k) \\
&= [k^2-k] + [m^2-m] + [2kH - 2k^2 + 2k] + [m^2 - k^2 - k - m] \\
&+ [(2m+1)N - 4m^2 + m + k + 2km] \\
&= 2kH + (2m+1)N + 2km - 2k^2 - 2m^2 - m + k \\
d_H &= \textstyle\sum_{i=1}^{k-1}[\frac{2i2i}{2} + \frac{2i(2i+2)}{2} - i(2i-1) - i(2i+1)] \\
&+ \textstyle\sum_{i=1}^{m-1}[\frac{(2i+1)2i}{2} + \frac{2i(2i+3)}{2} - 2i(2i+1)] \\
&+ \tfrac{(2k)^2}{2} + \tfrac{2k(2k+1)}{2}(2H-2k) - k(2k-1) - (2H-2k+1)k(2k+1) \qquad (3.22) \\
&+ \textstyle\sum_{i=1}^{m-k-1}[\frac{(2k+2i+1)2(k+i)}{2} + \frac{2(k+i)(2k+2i+3)}{2} - 2(k+i)(2(k+i)+1)] \\
&+ [2\tfrac{2m(2m+1)}{2} + \tfrac{2m(2m+2)}{2}(2N-4m+2k) \\
&\quad -(N-2m+k+3)m(2m+1) - (N-2m+k)(m+1)(2m+1)] \\
&+ \tfrac{2k(2k+3)}{2} + k + 2m + N \\
&= [k^2-k] + [m^2-m] + [-2k^2] + [m^2 - k^2 - k - m] \\
&+ [-N - 2m^2 + m - k] + [2k^2 + 4k + 2m + N] \\
&= m + k.
\end{aligned}
$$

3.9.2 The elliptic model

In this section, we discuss the mirror theory of $Sp'(k) \times SO(2m+1)$ in the elliptic model. Now because X^6 is compact, the matter contents are H flavors for $Sp'(k)$, N flavors for $SO(2m+1)$ and two bifundamentals. The dimensions of the moduli spaces are $d_v = k+m$ and $d_H = 2kH + (2m+1)N + 2k(2m+1) - k(2k+1) - m(2m+1) = 2kH + (2m+1)N + 4km - 2k^2 - 2m^2 - m + k$. Again, our investigation will be divided into three cases $k = m$, $k > m$ and $k < m$.

Let us start from the case $k = m$. In this case, because we can combine all D3-branes at the two sides of 1/2NS-branes, it makes the mirror theory very simple as shown in Figure 3-29. Let us check the dimensions of moduli spaces:

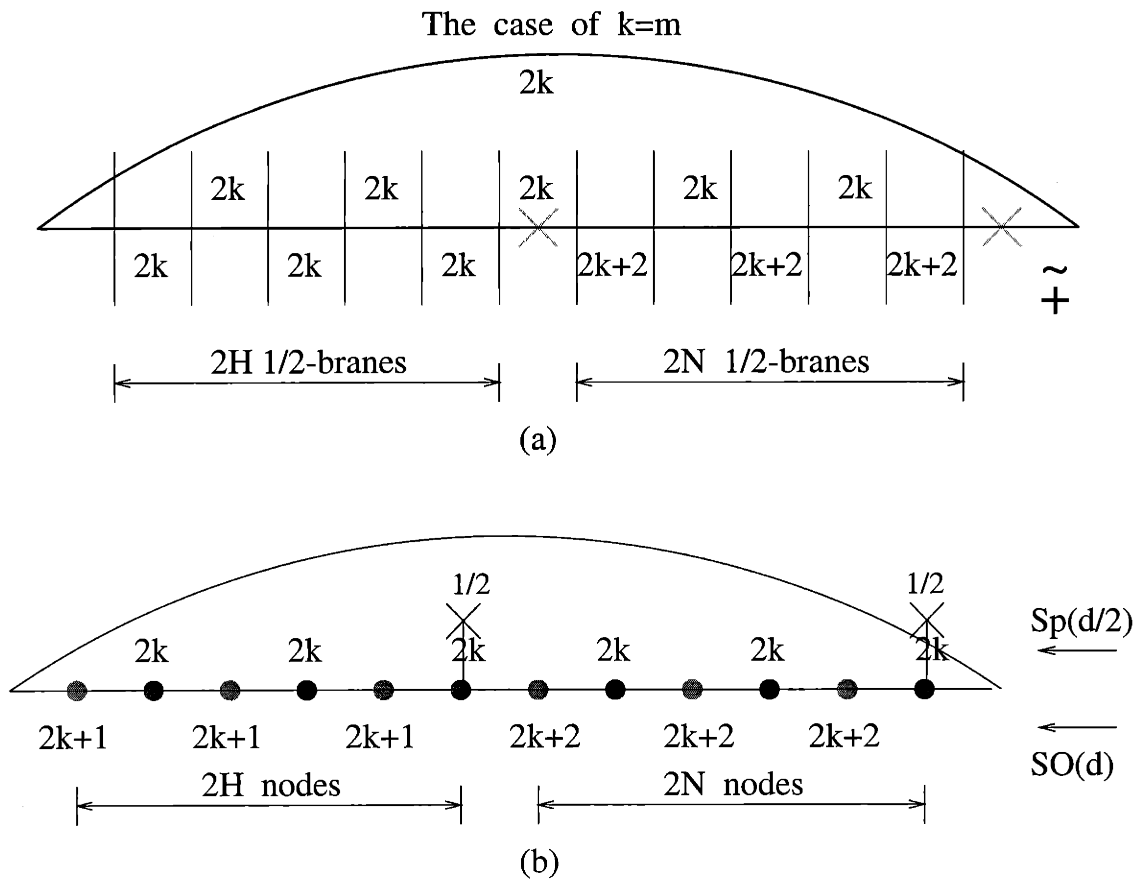


Figure 3-29: The mirror of $Sp'(k) \times SO(2m+1)$ with H flavors for $Sp'(k)$, N flavors for $SO(2m+1)$ and two bifundamentals in case of $k = m$. (a) The Higgs branch of the original theory or the Coulomb branch of the mirror theory. (b) The quiver diagram of the mirror theory.

$$\begin{aligned} d_v &= 2kH + Nk + N(k+1) = 2kH + (2k+1)N \\ d_H &= [\tfrac{2k(2k+1)}{2}2H - 2Hk(2k+1)] + [N] + [2\tfrac{2k}{2}] \\ &+ [\tfrac{2k(2k+2)}{2}2N - Nk(2k+1) - N(k+1)(2k+1)] \\ &= 2k. \end{aligned} \tag{3.23}$$

Now we go to the case of $k > m$. In this case, After combining the D3-branes, we still leave $k - m$ D3-branes in the interval of $\widetilde{O3}^+$-plane. The mirror theory is given in Figure 3-30. The dimensions of moduli spaces are

$$\begin{aligned} d_v &= 2\sum_{i=1}^{k-m-1} 2(m+i) + k(2H - 4k + 4m + 3) + m(N+1) + (m+1)N \\ &= [2k^2 - 2m^2 - 2k - 2m] + [2kH + (2m+1)N + 4km - 4k^2 + 3k + m] \end{aligned}$$

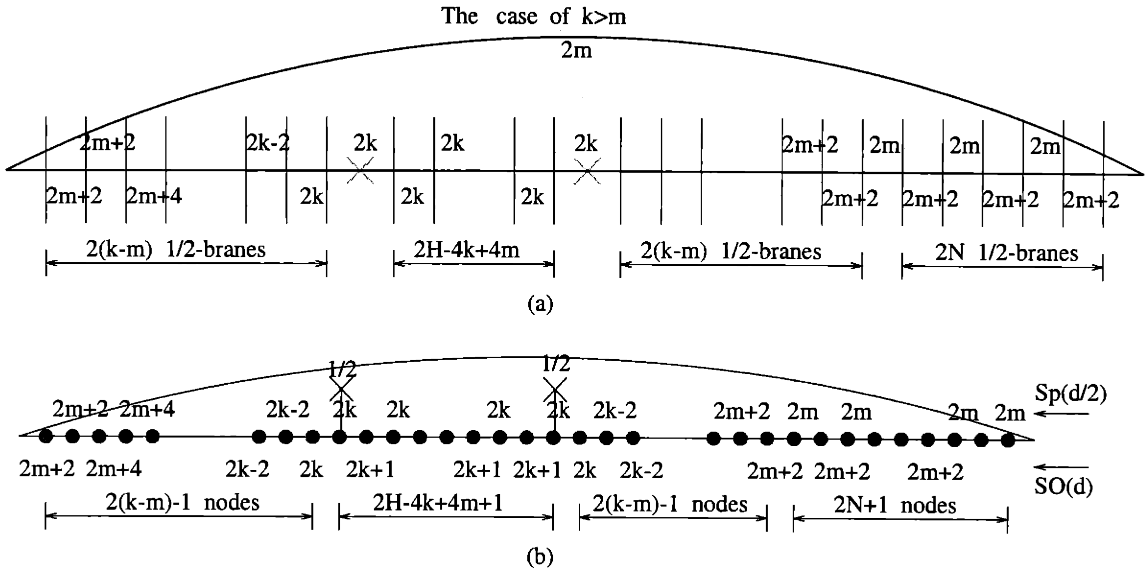


Figure 3-30: The mirror of $Sp'(k) \times SO(2m+1)$ with H flavors for $Sp'(k)$, N flavors for $SO(2m+1)$ and two bifundamentals in case of $k > m$. (a) The Higgs branch of the original theory or the Coulomb branch of the mirror theory. (b) The quiver diagram of the mirror theory.

$$
\begin{aligned}
&= 2kH + (2m+1)N + 4km - 2k^2 - 2m^2 + k - m \\
d_H &= 2\sum_{i=1}^{k-m-1}\Big[\frac{(2(m+i))^2}{2} + \frac{2(m+i)2(m+i+1)}{2} \\
&\quad -(m+i)(2m+2i-1) - (m+i)(2m+2i+1)\Big] \\
&+ 2\frac{(2k)^2}{2} + \frac{2k(2k+1)}{2}(2H-4k+4m) - (2H-4k+4m+1)k(2k+1) \\
&\quad -2k(2k-1) + 2k + N \\
&= [2k^2 - 2m^2 - 2k - 2m] + [-2k^2 + k] + [-N + 2m^2 + 3m] + [N + 2k] \\
&= k + m \qquad (3.24)
\end{aligned}
$$

We are left only one more example, i.e., the case of $k < m$. For this case, after the combination, we still have $m - k$ D3-branes in the interval of $\widetilde{O3}^-$-plane. The mirror theory is given in Figure 3-31. The dimensions of moduli spaces are

$$
\begin{aligned}
d_v &= 2\sum_{i=1}^{m-k-1} 2(k+i) + k(2H+1) + m(N - 2m + 2k + 3) \\
&\quad +(m+1)(N - 2m + 2k) \\
&= [2m^2 - 2k^2 - 2k - 2m] + [2kH + (2m+1)N + 4km + m + 3k]
\end{aligned}
$$

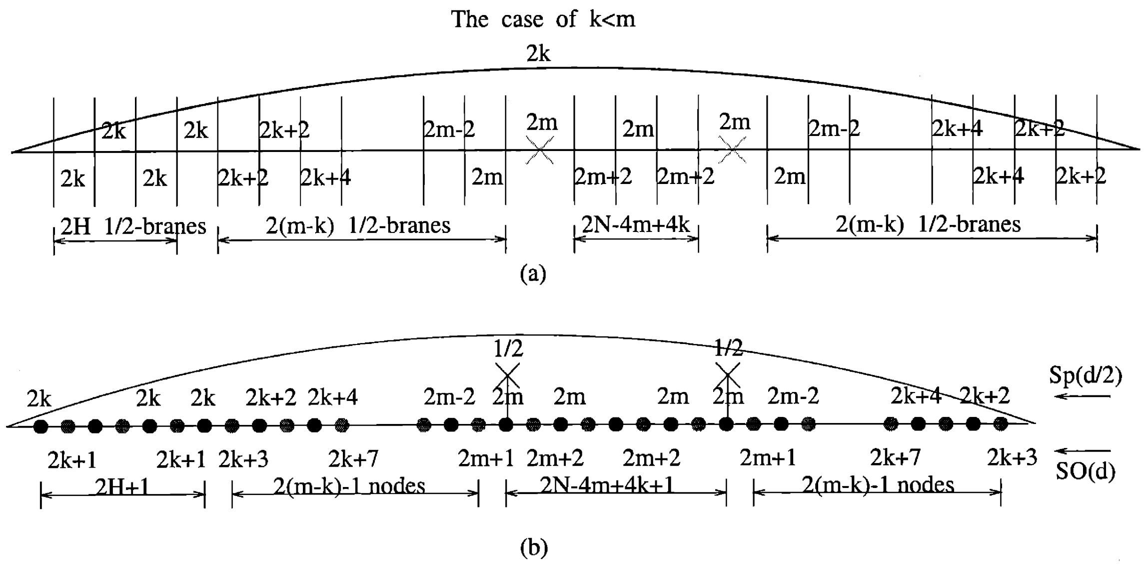


Figure 3-31: The mirror of $Sp'(k) \times SO(2m+1)$ with H flavors for $Sp'(k)$, N flavors for $SO(2m+1)$ and two bifundamentals in case of $k < m$. (a) The Higgs branch of the original theory or the Coulomb branch of the mirror theory. (b) The quiver diagram of the mirror theory.

$$
\begin{aligned}
&= 2kH + (2m+1)N - 2m^2 - 2k^2 - m + k \\
d_h &= 2\sum_{i=1}^{m-k-1}[\frac{(2k+2i)(2k+2i+1)}{2} + \frac{(2k+2i)(2k+2i+3)}{2} \\
&\quad -2(k+i)(2k+2i+1)] + [2\frac{2m(2m+1)}{2} + \frac{2m(2m+2)}{2}(2n-4m+4k) \\
&\quad -(N-2m+2k+3)m(2m+1) - (N-2m+2k)(m+1)(2m+1)] \\
&+ \frac{2k(2k+1)}{2}(2H) - (2H+1)k(2k+1) + 2\frac{2k(2k+3)}{2} \\
&+ 2m + N \\
&= [2m^2 - 2k^2 - 2k - 2m] + [-N - 2m^2 - 2k + m] + [2k^2 + 5k] + [2m + N] \\
&= k + m. \qquad (3.25)
\end{aligned}
$$

3.10 Conclusion

In this paper, we give the mirror theories of $Sp(k)$ and $SO(n)$ gauge theories. In particular, for the first time the mirror of $SO(n)$ gauge theory is given. In the construction of the mirror, we have made an assumption about the splitting of D5-

branes on O3-planes in the *brane-plane* system[6]. We want to emphasize that because the splitting of D5-brane on O3-plane is a nontrivial dynamical process and we do not fully understand it at this moment, we can not really prove our assumption by calculation. However, although our discussions in this paper indicate that our assumption is consistent, the other independent checks are favorable. This gives one direction of further work as to prove our observation.

Furthermore, as we discussed in section three, our rules observed in this paper about the splitting of physical brane predict some nontrivial strong coupling limit of a particular field theory. It will be very interesting to use the Seiberg-Witten curve [25, 26, 65] to show whether it is true.

There is another direction to pursue our investigation. By rotating one of the 1/2NS-branes [27, 28, 66, 24] we break the $N = 4$ theory in three dimensions to an $N = 2$ theory. Then we can discuss the mirror of $N = 2$ in three dimension as we have done in this paper. However, because there is less supersymmetry in the $N = 2$ case, things become more complex (for a detailed explanation of new features in $N = 2$, see [24]). Indeed, we can even break the supersymmetry further to discuss the mirror symmetry in the $N = 1$ case [68].

[6]For more details about the brane-plane system see [63]

Chapter 4

D-Brane Gauge Theories from Toric Singularities and Toric Duality

4.1 Introduction

The study of D-branes as probes of geometry and topology of space-time has by now been of wide practice (cf. e.g. [49]). In particular, the analysis of the moduli space of gauge theories, their matter content, superpotential and β-function, as world-volume theories of D-branes sitting at geometrical singularities is still a widely pursued topic. Since the pioneering work in [40], where the moduli and matter content of D-branes probing ALE spaces had been extensively investigated, much work ensued. The primary focus on (Abelian) orbifold singularities of the type $\mathbb{C}^2/\mathbb{Z}_n$ was quickly generalised using McKay's Correspondence, to arbitrary (non-Abelian) orbifold singularities $\mathbb{C}^2/(\Gamma \subset SU(2))$, i.e., to arbitrary ALE spaces, in [42].

Several directions followed. With the realisation [70, 71] that these singularities provide various horizons, [40, 42] was quickly generalised to a treatment for arbitrary finite subgroups $\Gamma \subset SU(N)$, i.e., to generic Gorenstein singularities, by [44]. The case of $SU(3)$ was then promptly studied in [45, 46, 72] using this technique and a generalised McKay-type Correspondence was proposed in [45, 73]. Meanwhile, via T-duality transformations, certain orbifold singularities can be mapped to type II brane-setups in the fashion of [10]. The relevant gauge theory data on the world volume can thereby be conveniently read from configurations of NS-branes, D-brane stacks as well as orientifold planes. For $\mathbb{C}^2$ orbifolds, the A and D series have been thus treated [10, 56], whereas for $\mathbb{C}^3$ orbifolds, the Abelian case of $\mathbb{Z}_k \times \mathbb{Z}_{k'}$ has been solved by the brane box models [30, 31]. First examples of non-Abelian $\mathbb{C}^3$ orbifolds have been addressed in [74, 75] as well as recent works in [76, 77].

Thus rests the status of orbifold theories. What we note in particular is that once we specify the properties of the orbifold in terms of the algebraic properties of the finite group, the gauge theory information is easily extracted. Of course, orbifolds are a small subclass of algebro-geometric singularities. This is where we move on to **toric varieties**. Inspired by the linear σ-model approach of [50], which provides a rich structure of the moduli space, especially in connexion with various geometrical phases of the theory, the programme of utilising toric methods to study the behaviour of the gauge theory on D-branes which live on and hence resolve certain singularities was initiated in [14]. In this light, toric methods provide a powerful tool for studying the moduli space of the gauge theory. In treating the F-flatness and D-flatness conditions for the SUSY vacuum in conjunction, these methods show how branches of the moduli space and hence phases of the theory may be parametrised by the algebraic equations of the toric variety. Recent developments in "brane diamonds," as an extension of the brane box rules, have been providing great insight to such a wider class of toric singularities, especially the generalised conifold, via blown-up versions of the standard brane setups [78]. Indeed, with toric techniques much information could be extracted as we can actually analytically describe patches of the moduli space.

Now Abelian orbifolds have toric descriptions and the above methodolgy is thus immediately applicable thereto. While bearing in mind that though non-Abelian orbifolds have no toric descriptions, a single physical D-brane has been placed on various general toric singularities. Partial resolutions of $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, such as the conifold and the suspended pinched point have been investigated in [79, 80] and brane setups giving the field theory contents are constructed by [81, 83, 82]. Groundwork for the next family, coming from the toric orbifold $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$, such as the del Pezzo surfaces and the zeroth Hirzebruch, has been laid in [84]. Essentially, given the gauge theory data on the D-brane world volume, the procedure of transforming this information (F and D terms) into toric data which parametrises the classical moduli space is by now well-established.

One task is therefore immediately apparent to us: how do we proceed in the reverse direction, i.e., *when we probe a toric singularity with a D-brane, how do we know the gauge theory on its world-volume?* We recall that in the case of orbifold theories, [44] devised a general method to extract the gauge theory data (matter content, superpotential etc.) from the geometry data (the characters of the finite group Γ), and *vice versa* given the geometry, brane-setups for example, conveniently allow us to read out the gauge theory data. The same is not true for toric singularities, and the second half of the above bi-directional convenience, namely, a general method which allows us to treat the inverse problem of extracting gauge theory data from toric data is yet pending, or at least not in circulation.

The reason for this shortcoming is, as we shall see later, that the problem is highly non-unique. It is thus the purpose of this writing to address this inverse problem: given the geometry data in terms of a toric diagram, how does one read out (at least one) gauge theory data in terms of the matter content and superpotential? We here present precisely this algorithm which takes the matrices encoding the singularity to the matrices encoding a good gauge theory of the D-brane which probes the said singularity.

The structure of this chapter is as follows. In Subsection 2.1, we demonstrate how to extract the matter content and F-terms from the charge matrix of the toric singularity. In Subsection 2.2, we exemplify our algorithm with the well-known suspended pinched point before presenting in detail in Subsection 2.3, the general algorithm of how to obtain the gauge theory information from the toric data by the method of partial resolutions. In Subsection 2.4, we show how to integrate back to obtain the actual superpotential once the F-flatness equations are extracted from the toric data. Section 3 is then devoted to the illustration of our algorithm by tabulating the D-terms and F-terms of D-brane world volume theory on the toric del Pezzo surfaces and Hirzebruch zero. We finally discuss in Section 4, the non-uniqueness of the inverse problem and provide, through the studying of two types of ambiguities, ample examples of rather different gauge theories flowing to the same toric data. Discussions and future prospects are dealt with in Section 5.

4.2 The Inverse Procedure: Extracting Gauge Theory Information from Toric Data

As outlined above we see that wherever possible, the gauge theory of a D-brane probe on certain singularities such as Abelian orbifolds, conifolds, etc., can be conveniently encoded into the matrix Q_t which essentially concatenates the information contained in the D-terms and F-terms of the original gauge theory. The cokernel of this matrix is then a list of vectors which prescribes the toric diagram corresponding to the singularity. It is natural to question ourselves whether the converse could be done, i.e., whether given an arbitrary singularity which affords a toric description, we could obtain the gauge theory living on the D-brane which probes the said singularity. This is the inverse problem we projected to solve in the introduction.

4.2.1 Quiver Diagrams and F-terms from Toric Diagrams

Our result must be two-fold: first, we must be able to extract the D-terms, or in other words the quiver diagram which then gives the gauge group and matter content; second, we must extract the F-terms, which we can subsequently integrate back to give the superpotential. These two pieces of data then suffice to specify the gauge theory. Essentially we wish to trace the arrows in the above flow-chart from G_t back to Δ and K. The general methodology seems straightforward:

1. Read the column-vectors describing the nodes of the given toric diagram, repeat the appropriate columns to obtain G_t and then set $Q_t = \text{Coker}(G_t)$;

2. Separate the D-term $(V \cdot U)$ and F-term (Q_t) portions from Q_t;

3. From the definition of Q, we obtain[1] $T = \ker(Q)$.

4. Farka's Theorem [48] guarantees that the dual of a convex polytope remains convex whence we could invert and have $K = \text{Dual}(T^t)$; Moreover the duality theorem gives that $\text{Dual}(\text{Dual}(K)) = K$, thereby facilitating the inverse procedure.

5. Definitions $U \cdot T^t = \text{Id}$ and $V \cdot K^t = \Delta \Rightarrow (V \cdot U) \cdot (T^t \cdot K^t) = \Delta$.

We see therefore that once the appropriate Q_t has been found, the relations

$$K = \text{Dual}(T^t) \qquad \Delta = (V \cdot U) \cdot (T^t \cdot K^t) \tag{4.1}$$

retrieve our desired K and Δ. The only setback of course is that the appropriate Q_t is NOT usually found. Two ambiguities are immediately apparent to us: (A) In step 1 above, there is really no way to know a priori which of the vectors we should repeat when writing into the G_t matrix; (B) In step 2, to separate the D-terms and the F-terms, i.e., which rows constitute Q and which constitute $V \cdot U$ within Q_t, seems arbitrary. We shall in the last section discuss these ambiguities in more detail and actually perceive it to be a matter of interest. Meanwhile, in light thereof, we must find an alternative, to find a canonical method which avoids such ambiguities and gives us a consistent gauge theory which has such well-behaved properties as having only bi-fundamentals etc.; this is where we appeal to partial resolutions.

Another reason for this canonical method is compelling. The astute reader may question as to how could we guarantee, in our mathematical excursion of performing the inverse procedure, that the gauge theory we obtain at the end of the day is one that still lives on the world-volume of a D-brane probe? Indeed, if we naïvely traced back the arrows in the flow-chart, bearing in mind the said ambiguities, we have no *a fortiori* guarantee that we have a brane theory at all. However, the method via partial

[1] As mentioned before we must ensure that such a T be chosen with a complete set of $\mathbb{Z}_+$-independent generators;

resolution of Abelian orbifolds (which are themselves toric) does give us assurance. When we are careful in tuning the FI-parametres so as to stay inside cone-partitions of the space of these parametres (and avoid flop transitions) we do still have the resulting theory being physical [84]. Essentially this means that with prudence we tune the FI-parametres in the allowed domains from a parent orbifold theory, thereby giving a subsector theory which still lives on the D-brane probe and is well-behaved. Such tuning we shall practice in the following.

The virtues of this appeal to resolutions are thus twofold: not only do we avoid ambiguities, we are further endowed with physical theories. Let us thereby present this canonical mathod.

4.2.2 A Canonical Method: Partial Resolutions of Abelian Orbifolds

Our programme is standard [84]: theories on the Abelian orbifold singularity of the form $\mathbb{C}^k/\Gamma$ for $\Gamma(k,n) = \mathbb{Z}_n \times \mathbb{Z}_n \times ... \mathbb{Z}_n$ ($k-1$ times) are well studied. The complete information (and in particular the full Q_t matrix) for $\Gamma(k,n)$ is well known: $k=2$ is the elliptic model, $k=3$, the Brane Box, etc. In the toric context, $k=2$ has been analysed in great detail by [40], $k=3, n=2$ in e.g. [81, 83, 82], $k=3, n=3$ in [84]. Now we know that given any toric diagram of dimension k, we can embed it into such a $\Gamma(k,n)$-orbifold for some sufficiently large n; and we choose the smallest such n which suffices. This embedding is always possible because the toric diagram for the latter is the k-simplex of length n enclosing lattice points and any toric diagram, being a collection of lattice points, can be obtained therefrom via deletions of a subset of points. This procedure is known torically as **partial resolutions** of $\Gamma(k,n)$. The crux of our algorithm is that the deletions in the toric diagram corresponds to the turning-on of the FI-parametres, and which in turn induces a method to determine a Q_t matrix for our original singularity from that of $\Gamma(n,k)$.

We shall first turn to an illustrative example of the suspended pinched point singularity (SPP) and then move on to discuss generalities. The SPP and conifold

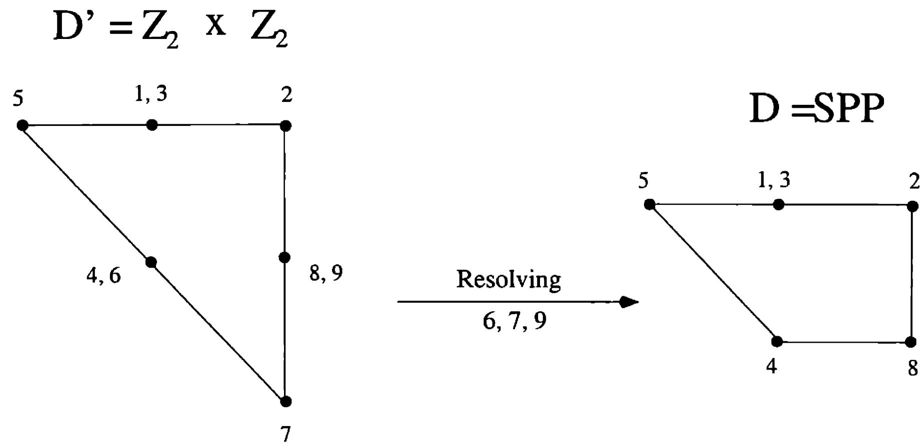


Figure 4-1: The toric diagram showing the resolution of the $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ singularity to the suspended pinch point (SPP). The numbers i at the nodes refer to the i-th column of the matrix G_t and physically correspond to the fields p_i in the linear σ-model.

as resolutions of $\Gamma(3,2) = \mathbb{Z}_2 \times \mathbb{Z}_2$ have been extensively studied in [83]. The SPP, given by $xy = zw^2$, can be obtained from the $\Gamma(3,2)$ orbifold, $xyz = w^2$, by a single $\mathbb{P}^1$ blow-up. This is shown torically in Figure 4-1. Without further ado let us demonstrate our procedure.

1. Embedding into $\mathbb{Z}_2 \times \mathbb{Z}_2$: Given the toric diagram D of SPP, we recognise that it can be embedded minimally into the diagram D' of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Now information on D' is readily at hand [83], as presented in the previous section. Let us recapitulate:

$$Q'_t := \left(\begin{array}{cccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & \zeta_1 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & \zeta_2 \\ -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & \zeta_3 \end{array}\right),$$

and

$$G'_t := \mathrm{coker}(Q'_t) = \left(\begin{array}{ccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right),$$

which is drawn in Figure 2-3. The fact that the last row of G_t has the same number (i.e., these three-vectors are all co-planar) ensures that D' is Calabi-Yau [49]. Incidentally, it would be very helpful for one to catalogue the list of Q_t matrices of $\Gamma(3,n)$ for $n = 2, 3...$ which would suffice for all local toric singularities of Calabi-Yau threefolds.

In the above definition of Q'_t we have included an extra column $(0,0,0,\zeta_1,\zeta_2,\zeta_3)$ so as to specify that the first three rows of Q'_t are F-terms (and hence exactly zero) while the last three rows are D-terms (and hence resolved by FI-parametres $\zeta_{1,2,3}$). We adhere to the notation in [83] and label the columns (linear σ-model fields) as $p_1...p_9$; this is shown in Figure 4-1.

2. Determining the Fields to Resolve by Tuning ζ: We note that if we turn on a single FI-parametre we would arrive at the SPP; this is the resolution of D' to D. The subtlety is that one may need to eliminate more than merely the 7th column as there is more than one field attributed to each node in the toric diagram and eliminating column 7 some other columns corresponding to the adjacent nodes (namely out of 4,6,8 and 9) may also be eliminated. We need a judicious choice of ζ for a consistent blowup. To do so we must solve for fields $p_{1,..,9}$ and tune the ζ-parametres such that at least p_7 acquires non-zero VEV (and whereby resolved). Recalling that the D-term equations are actually linear equations in the modulus-squared of the fields, we shall henceforth define $x_i := |p_i|^2$ and consider linear-systems therein. Therefore we perform Gaussian row-reduction on Q' and solve all fields in terms of x_7 to give: $\vec{x} = \{x_1, x_2, x_1 + \zeta_2 + \zeta_3, \frac{2\,x_1 - x_2 + x_7 - \zeta_1 + \zeta_2}{2}, 2\,x_1 - x_2 + \zeta_2 + \zeta_3, \frac{2\,x_1 - x_2 + x_7 + \zeta_1 + \zeta_2 + 2\,\zeta_3}{2}, x_7, \frac{x_2 + x_7 - \zeta_1 - \zeta_2}{2}, \frac{x_2 + x_7 + \zeta_1 + \zeta_2}{2}\}$.

The nodes far away from p_7 are clearly unaffected by the resolution, thus the fields corresponding thereto continue to have zero VEV. This means we solve the above set of solutions $\vec{x}$ once again, setting $x_{5,1,3,2} = 0$, with $\zeta_{1,2,3}$ being the variables, giving upon back substitution, $\vec{x} = \{0, 0, 0, \frac{x_7 - \zeta_1 - \zeta_3}{2}, 0, \frac{x_7 + \zeta_1 + \zeta_3}{2}, x_7, \frac{x_7 - \zeta_1 + \zeta_3}{2}, \frac{x_7 + \zeta_1 - \zeta_3}{2}\}$. Now we have an arbitrary choice and we set $\zeta_3 = 0$ and $x_7 = \zeta_1$ to make p_4 and p_8 have zero VEV. This makes $p_{6,7,9}$ our candidate for fields to be resolved and seems perfectly reasonable observing Figure 4-1. The constraint on our choice is that all solutions must be ≥ 0 (since the x_i's are VEV-squared).

3. Solving for G_t: We are now clear what the resolution requires of us: in order to remove node p_7 from D' to give the SPP, we must also resolve 6, 7 and 9. Therefore we immediately obtain G_t by directly removing the said columns from G'_t:

$$G_t := \text{coker}(Q_t) = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_8 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

the columns of which give the toric diagram D of the SPP, as shown in Figure 4-1.

4. Solving for Q_t: Now we must perform linear combination on the rows of Q'_t to obtain Q_t so as to force columns 6, 7 and 9 zero. The following constraints must be born in mind. Because G_t has 6 columns and 3 rows and is in the null space of Q_t, which itself must have $9-3$ columns (having eliminated $p_{6,7,9}$), we must have $6-3=3$ rows for Q_t. Also, the row containing ζ_1 must be eliminated as this is precisely our resolution chosen above (we recall that the FI-parametres are such that $\zeta_{2,3}=0$ and are hence unresolved, while $\zeta_1>0$ and must be removed from the D-terms for SPP).

   We systematically proceed. Let there be variables $\{a_{i=1,..,6}\}$ so that $y := \sum_i a_i \text{row}_i(Q'_t)$ is a row of Q_t. Then (a) the 6th, 7th and 9th columns of y must be set to 0 and moreover (b) with these columns removed y must be in the nullspace spanned by the rows of G_t. We note of course that since Q'_t was in the nullspace of G'_t initially, that the operation of row-combinations is closed within a nullspace, and that the columns to be set to 0 in Q'_t to give Q_t are precisely those removed in G'_t to give G_t, condition (a) automatically implies (b). This condition (a) translates to the equations $\{a_1+a_6=0, -a_1+a_2-a_6=0, -a_2+a_4=0\}$ which afford the solution $a_1=-a_6; a_2=a_4=0$. The fact that

$a_4 = 0$ is comforting, because it eliminates the row containing ζ_1. We choose $a_1 = 1$. Furthermore we must keep row 5 as ζ_2 is yet unresolved (thereby setting $a_5 = 1$). This already gives two of the 3 anticipated rows of Q_t: row$_5$ and row$_1$ - row$_6$. The remaining row must corresponds to an F-term since we have exhausted the D-terms, this we choose to be the only remaining variable: $a_3 = 1$. Consequently, we arrive at the matrix

$$Q_t = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_8 & \\ 1 & -1 & 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 & \zeta_2 \\ -1 & 0 & 0 & -1 & 1 & 1 & \zeta_3 \end{pmatrix}.$$

5. Obtaining K and Δ Matrices: The hard work is now done. We now recognise from Q_t that $Q = (1, -1, 1, 0, -1, 0)$, giving

$$T_{j\alpha} := \ker(Q) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \qquad K^t := \mathrm{Dual}(T^t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Subsequently we obtain

$$T^t \cdot K^t = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which we do observe indeed to have every entry positive semi-definite. Furthermore we recognise from Q_t that

$$V \cdot U = \begin{pmatrix} -1 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 1 & 1 \end{pmatrix},$$

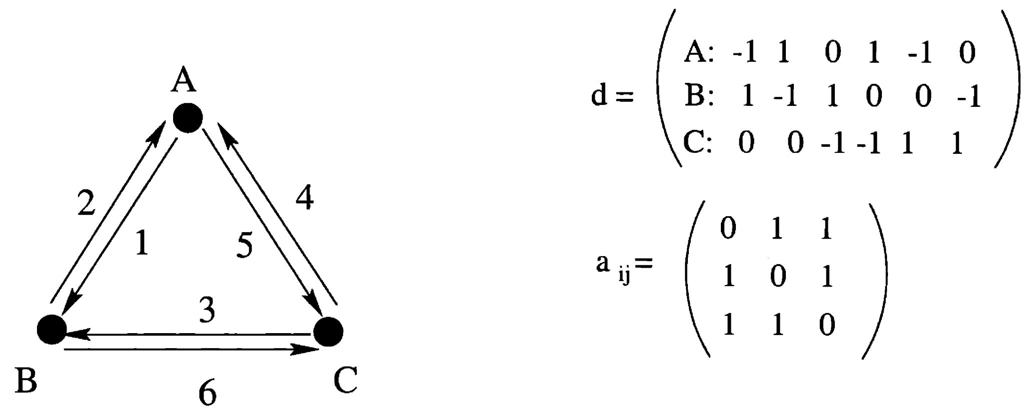


Figure 4-2: The quiver diagram showing the matter content of a D-brane probing the SPP singularity. We have not marked in the chargeless field ϕ (what in a non-Abelian theory would become an adjoint) because thus far the toric techniques do not yet know how to handle such adjoints.

whence we obtain at last, using (4.1),

$$\Delta = \begin{pmatrix} -1 & 1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 & 0 & -1 \end{pmatrix} \quad \Rightarrow \quad d = \left(\begin{array}{c|cccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \\ U(1)_A & -1 & 1 & 0 & 1 & -1 & 0 \\ U(1)_B & 1 & -1 & 1 & 0 & 0 & -1 \\ \hline U(1)_C & 0 & 0 & -1 & -1 & 1 & 1 \end{array} \right),$$

giving us the quiver diagram (included in Figure 4-2 for reference), matter content and gauge group of a D-brane probe on SPP in agreement with [83]. We shall show in the ensuing sections that the superpotential we extract has similar accordance.

4.2.3 The General Algorithm for the Inverse Problem

Having indulged ourselves in this illustrative example of the SPP, we proceed to outline the general methodology of obtaining the gauge theory data from the toric diagram.

1. Embedding into $\mathbb{C}^k/(\mathbb{Z}_n)^{k-1}$: We are given a toric diagram D describing an algebraic variety of complex dimension k (usually we are concerned with local Calabi-Yau singularities of $k = 2, 3$ so that branes living thereon give $\mathcal{N} = 2, 1$ gauge theories). We immediately observe that D could always be embedded

into D', the toric diagram of the orbifold $\mathbb{C}^k/(\mathbb{Z}_n)^{k-1}$ for some sufficiently large integer n. The matrices Q'_t and G'_t for D' are standard. Moreover we know that the matrix G_t for our original variety D must be a submatrix of G'_t. Equipped with Q'_t and G'_t our task is to obtain Q_t; and as an additional check we could verify that Q_t is indeed in the nullspace of G_t.

2. Determining the Fields to Resolve by Tuning ζ: Q'_t is a $k \times a$ matrix[2] (because D' and D are dimension k) for some a; G'_t, being its nullspace, is thus $(a-k) \times a$. D is a partial resolution of D'. In the SPP example above, we performed a single resolution by turning on one FI-parametre, generically however, we could turn on as many ζ's as the embedding permits. Therefore we let G_t be $(a-k) \times (a-b)$ for some b which depends on the number of resolutions. Subsequently the Q_t we need is $(k-b) \times (a-b)$.

   Now b is determined directly by examining D' and D; it is precisely the number of fields p associated to those nodes in D' we wish to eliminate to arrive at D. Exactly which b columns are to be eliminated is determined thus: we perform Gaussian row-reduction on Q'_t so as to solve the k linear-equations in a variables $x_i := |p_i|^2$, with F-terms set to 0 and D-terms to FI-parametres. The a variables are then expressed in terms of the ζ_i's and the set B of x_i's corresponding to the nodes which we definitely know will disappear as we resolve $D' \to D$. The subtlety is that in eliminating B, some other fields may also acquire non zero VEV and be eliminated; mathematically this means that $\text{Order}(B) < b$.

   Now we make a judicious choice of which fields will remain and set them to zero and impose this further on the solution $x_{i=1,..,a} = x_i(\zeta_j; B)$ from above until $\text{Order}(B) = b$, i.e., until we have found all the fields we need to eliminate. We know this occurs and that our choice was correct when all $x_i \geq 0$ with those equaling 0 corresponding to fields we do not wish to eliminate as can be observed from the toric diagram. If not, we modify our initial choice and repeat until satisfaction. This procedure then determines the b columns which we wish

[2] We henceforth understand that there is an extra column of zeroes and ζ's.

to eliminate from Q'_t.

3. Solving for G_t and Q_t: Knowing the fields to eliminate, we must thus perform linear combinations on the k rows of Q'_t to obtain the $k-b$ rows of Q_t based upon the two constraints that (1) the b columns must be all reduced to zero (and thus the nodes can be removed) and that (2) the $k-b$ rows (with b columns removed) are in the nullspace of G_t. As mentioned in our SPP example, condition (1) guarantees (2) automatically.

   In other words, we need to solve for k variables $\{x_{i=1,..,k}\}$ such that

$$\sum_{i=1}^{k} x_i \; (Q'_t)_{ij} = 0 \qquad \text{for } \; j = p_1, p_2, ...p_b \in B. \tag{4.2}$$

   Moreover, we immediately obtain G_t by eliminating the b columns from G'_t. Indeed, as discussed earlier, (4.2) implies that $\sum\limits_{i=1}^{k} \sum\limits_{j \neq p_{1...b}} x_i \; (Q'_t)_{ij} \; (G_t)_{mj} = 0$ for $m = 1, ..., a-k$ and hence guarantees that the Q_t we obtain is in the nullspace of G_t.

   We could phrase equation (4.2) for x_i in matrix notation and directly evaluate

$$Q_t = NullSpace(W)^t \cdot \tilde{Q'_t} \tag{4.3}$$

   where $\tilde{Q'_t}$ is Q'_t with the appropriate columns $(p_{1...b})$ removed and W is the matrix constructed from the deleted columns.

4. Obtaining the K Matrix (F-term): Having obtained the $(k-b) \times (a-b)$ matrix Q_t for the original variety D, we proceed with ease. Reading from the extraneous column of FI-parametres, we recognise matrices Q (corresponding to the rows that have zero in the extraneous column) and $V \cdot U$ (corresponding to those with combinations of the unresolved ζ's in the last column). We let $V \cdot U$ be $c \times (a-b)$ whereby making Q of dimension $(k-b-c) \times (a-b)$. The number c is easily read from the embedding of D into D' as the number of unresolved FI-parametres.

   From Q, we compute the kernel T, a matrix of dimensions $(a-b)-(k-b-$

$c) \times (a-b) = (a-k+c) \times (a-b)$ as well as the matrix K^t of dimensions $(a-k+c) \times d$ describing the dual cone to that spanned by the columns of T. The integer d is uniquely determined from the dimensions of T in accordance with the algorithm of finding dual cones presented in the Appendix. From these two matrices we compute $T^t \cdot K^t$, of dimension $(a-b) \times d$.

5. Obtaining the Δ Matrix (D-term): Finally, we use (4.1) to compute $(V \cdot U) \cdot (T^t \cdot K^t)$, arriving at our desired matrix Δ of dimensions $c \times d$, the incidence matrix of our quiver diagram. The number of gauge groups we have is therefore $c+1$ and the number of bi-fundamentals, d.

   Of course one may dispute that finding the kernel T of Q is highly non-unique as any basis change in the null-space would give an equally valid T. This is indeed so. However we note that it is really the combination $T^t \cdot K^t$ that we need. This is a dot-product in disguise, and by the very definition of the dual cone, this combination remains invariant under basis changes. Therefore this step of obtaining the quiver Δ from the charge matrix Q_t is a unique procedure.

4.2.4 Obtaining the Superpotential

Having noticed that the matter content can be conveniently obtained, we proceed to address the interactions, i.e., the F-terms, which require a little more care. The matrix K which our algorithm extracts encodes the F-term equations and must at least be such that they could be integrated back to a single function: the *superpotential.*

Reading the possible F-flatness equations from K is *ipso facto* straight-forward. The subtlety exists in how to find the right candidate among many different linear relations. As mentioned earlier, K has dimensions $m \times (r-2)$ with m corresponding to the fields that will finally manifest in the superpotential, $r-2$, the fields that solve them according to (2.41) and (2.42); of course, $m \geq r-2$. Therefore we have $r-2$ vectors in $\mathbb{Z}^m$, giving generically $m-r+2$ linear relations among them. Say we have $\text{row}_1 + \text{row}_3 - \text{row}_7 = 0$, then we simply write down $X_1 X_3 = X_7$ as one of the candidate F-terms. In general, a relation $\sum\limits_i a_i K_{ij} = 0$ with $a_i \in \mathbb{Z}$ implies an F-term

$\prod_i X_i^{a_i} = 1$ in accordance with (2.41). Of course, to find all the linear relations, we simply find the $\mathbb{Z}$-nullspace of K^t of dimension $m - r + 2$.

Here a great ambiguity exists, as in our previous calculations of nullspaces: any linear combinations therewithin may suffice to give a new relation as a candidate F-term[3]. Thus educated guesses are called for in order to find the set of linear relations which may be most conveniently integrated back into the superpotential. Ideally, we wish this back-integration procedure to involve no extraneous fields (i.e., integration constants[4]) other than the m fields which appear in the K-matrix. Indeed, as we shall see, this wish may not always be granted and sometimes we must include new fields. In this case, the whole moduli space of the gauge theory will be larger than the one encoded by our toric data and the new fields parametrise new branches of the moduli in the theory.

Let us return to the SPP example to enlighten ourselves before generalising. We recall from subsection 3.2, that $K =$

	X_1	X_2	X_3	X_4	X_5	X_6
v_1	1	0	0	0	0	0
v_2	0	0	1	0	1	0
v_3	0	1	0	0	0	0
v_4	0	0	1	1	0	0
v_5	0	0	0	1	0	1

from which we can read out only one relation $X_3X_6 - X_4X_5 = 0$ using the rule described in the paragraph above. Of course there can be only one relation because the nullspace of K^t is of dimension $6 - 5 = 1$.

Next we must calculate the charge under the gauge groups which this term carries. We must ensure that the superpotential, being a term in a Lagrangian, be a gauge invariant, i.e., carries no overall charge under Δ. From

$d =$

	X_1	X_2	X_3	X_4	X_5	X_6
$U(1)_A$	-1	1	0	1	-1	0
$U(1)_B$	1	-1	1	0	0	-1
$U(1)_C$	0	0	-1	-1	1	1

we find the charge of X_3X_6 to be $(q_A, q_B, q_C) = (0+0, 1+(-1), (-1)+1) = (0,0,0)$;

[3]Indeed each linear relation gives a possible candidate and we seek the correct ones. For the sake of clarity we shall call candidates "relations" and reserve the term "F-term" for a successful candidate.

[4]By constants we really mean functions since we are dealing with systems of partial differential equations.

of course by our very construction, X_4X_5 has the same charge. Now we have two choices: (a) to try to write the superpotential using only the six fields; or (b) to include some new field ϕ which also has charge $(0,0,0)$. For (a) we can try the ansatz $W = X_1X_2(X_3X_6 - X_4X_5)$ which does give our F-term upon partial derivative with respect to X_1 or X_2. However, we would also have a new F-term $X_1X_2X_3 = 0$ by $\frac{\partial}{\partial X_6}$, which is inconsistent with our K since columns 1, 2 and 3 certainly do not add to 0.

This leaves us with option (b), i.e., $W = \phi(X_3X_6 - X_4X_5)$ say. In this case, when $\phi = 0$ we not only obtain our F-term, we need not even correct the matter content Δ. This branch of the moduli space is that of our original theory. However, when $\phi \neq 0$, we must have $X_3 = X_4 = X_5 = X_6 = 0$. Now the D-terms read $|X_1|^2 - |X_2|^2 = -\zeta_1 = \zeta_2$, so the moduli space is: $\{\phi \in \mathbb{C}, X_1 \in \mathbb{C}\}$ such that $\zeta_1 + \zeta_2 = 0$ for otherwise there would be no moduli at all. We see that we obtain another branch of moduli space. As remarked before, this is a general phenomenon when we include new fields: the whole moduli space will be larger than the one encoded by the toric data. As a check, we see that our example is exactly that given in [83], after the identification with their notation, $Y_{12} \to X_6, X_{24} \to X_3, Z_{23} \to X_1, Z_{32} \to X_2, Y_{34} \to X_4, X_{13} \to X_5, Z_{41} \to \phi$ and $(X_1X_2 - \phi) \to \phi$. We note that if we were studying a non-Abelian extension to the toric theory, as by brane setups (e.g. [83]) or by stacks of probes (in progress from [84]), the chargeless field ϕ would manifest as an adjoint field thereby modifying our quiver diagram. Of course since the study of toric methods in physics is so far restricted to product $U(1)$ gauge groups, such complexities do not arise. To avoid confusion we shall henceforth mark only the bi-fundamentals in our quiver diagrams but will write the chargeless fields explicit in the superpotential.

Our agreement with the results of [83] is very reassuring. It gives an excellent example demonstrating that our canonical resolution technique and the inverse algorithm do indeed, in response to what was posited earlier, give a theory living on a D-brane probing the SPP (T-dual to the setup in [83]). However, there is a subtle point we would like to mention. There exists an ambiguity in writing the superpotential when the chargeless field ϕ is involved. Our algorithm gives $W = \phi(X_3X_6 - X_4X_6)$

while [83] gives $W = (X_1X_2 - \phi)(X_3X_6 - X_4X_6)$. Even though they have identical moduli, it is the latter which is used for the brane setup. Indeed, the toric methods by definition (in defining Δ from a_{ij}) do not handle chargeless fields and hence we have ambiguities. Fortunately our later examples will not involve such fields.

The above example of the SPP was a naïve one as we need only to accommodate a single F-term. We move on to a more complicated example. Suppose we are now given

$$d = \begin{pmatrix} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \\ A & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\ B & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ C & 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ D & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \text{ and } K = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

We shall see in the next section, that these arise for the del Pezzo 1 surface. Now the nullspace of K has dimension $10-6=4$, we could obtain a host of relations from various linear combinations in this space. One relation is obvious: $X_2X_7 - X_3X_6 = 0$. The charge it carries is $(q_A, q_B, q_C, q_D) = (0+0, -1+0, 0+1, 1+(-1)) = (0, -1, 1, 0)$ which cancels that of X_9. Hence $X_9(X_2X_7 - X_3X_6)$ could be a term in W. Now $\frac{\partial}{\partial X_2}$ thereof gives X_7X_9 and from K we see that $X_7X_9 - X_1X_5X_{10} = 0$, therefore, $-X_1X_2X_5X_{10}$ could be another term in W. We repeat this procedure, generating new terms as we proceed and introducing new fields where necessary. We are fortunate that in this case we can actually reproduce all F-terms without recourse to artificial insertions of new fields: $W = X_2X_7X_9 - X_3X_6X_9 - X_4X_8X_7 - X_1X_2X_5X_{10} + X_3X_4X_{10} + X_1X_5X_6X_8$.

Enlightened by these examples, let us return to some remarks upon generalities. Making all the exponents of the fields positive, the F-terms can then be written as

$$\prod_i X_i^{a_i} = \prod_j X_j^{b_j}, \tag{4.4}$$

with $a_i, b_j \in \mathbb{Z}^+$. Indeed if we were to have another field X_k such that $k \notin \{i\}, \{j\}$ then the term $X_k \left(\prod\limits_i X_i^{a_i} - \prod\limits_j X_j^{b_j} \right)$, on the condition that X_k appears only this once, must be an additive term in the superpotential W. This is because the F-flatness condition $\frac{\partial W}{\partial X_k} = 0$ implies (4.4) immediately. Of course judicious observations are called for to (A) find appropriate relations (4.4) and (B) find X_k among our m fields. Indeed (B) may not even be possible and new fields may be forced to be introduced, whereby making the moduli space of the gauge theory larger than that encodable by the toric data.

In addition, we must ensure that each term in W be chargeless under the product gauge groups. What this means for us is that for each of the terms $X_k\left(\prod_i X_i^{a_i} - \prod_j X_j^{b_j}\right)$ we must have $\text{Charge}_s(X_k) + \sum_i a_i \text{Charge}_s(X_i) = 0$ for $s = 1,..,r$ indexing through our r gauge group factors (we note that by our very construction, for each gauge group, the charges for $\prod_i X_i^{a_i}$ and for $\prod_j X_j^{b_j}$ are equal). If X_k in fact cannot be found among our m fields, it must be introduced as a new field ϕ with appropriate charge. Therefore with each such relation (4.4) read from K, we iteratively perform this said procedure, checking $\Delta_{sk}+\sum_i a_i\Delta_{si} = 0$ at each step, until a satisfactory superpotential is reached. The right choices throughout demands constant vigilance and astuteness.

4.3 An Illustrative Example: the Toric del Pezzo Surfaces

As the $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ resolutions were studied in great detail in [83], we shall use the data from [84] to demonstrate the algorithm of finding the gauge theory from toric diagrams extensively presented in the previous section.

The toric diagram of the dual cone of the (parent) quotient singularity $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ as well as those of its resolution to the three toric del Pezzo surface are presented in Figure 4-3.

<u>del Pezzo 1:</u> Let us commence our analysis with the first toric del Pezzo surface[5]. From its toric diagram, we see that the minimal $\mathbb{Z}_n \times \mathbb{Z}_n$ toric diagram into which it embeds is $n = 3$. As a reference, the toric diagram for $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is given in

[5]Now some may identify the toric diagram of del Pezzo 1 as given by nodes (using the notation in Figure 4-3) $(1,-1,1)$, $(2,-1,0)$, $(-1,1,1)$, $(0,0,1)$ and $(-1,0,2)$ instead of the one we have chosen in the convention of [84], with nodes $(0,-1,2)$, $(0,0,1)$, $(-1,1,1)$, $(1,0,0)$ and $(0,1,0)$. But of course these two G_t matrices describe the same algebraic variety. The former corresponds to $\text{Spec}\left(\mathbb{C}[XY^{-1}Z, X^2Y^{-1}, X^{-1}YZ, Z, X^{-1}Z^2]\right)$ while the latter corresponds to $\text{Spec}\left(\mathbb{C}[Y^{-1}Z^2, Z, X^{-1}YZ, X, Y]\right)$. The observation that $(X^2Y^{-1}) = (X)(X^{-1}YZ)^{-1}(Z)$, $(XY^{-1}Z) = (X)(Y)^{-1}(Z)$ and $(X^{-1}Z^2) = (Y^{-1}Z^2)(Y)(X^{-1})$ for the generators of the polynomial ring gives the equivalence. In other words, there is an $SL(5,\mathbb{Z})$ transformation between the 5 nodes of the two toric diagrams.

Figure 4-3 and the quiver diagram, given later in the convenient brane-box form, in Figure 4-4. Luckily, the matrices Q_t' and G_t' for this Abelian quotient is given in [84]. Adding the extra column of FI-parametres we present these matrices below[6]:

$$G_t' = \left(\begin{array}{cccccccccccccccccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} & p_{19} & p_{20} & p_{21} & p_{22} & p_{23} & p_{24} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 3 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \end{array}\right. \cdots\cdots$$

$$\cdots\cdots \left.\begin{array}{cccccccccccccccccc} p_{25} & p_{26} & p_{27} & p_{28} & p_{29} & p_{30} & p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} & p_{37} & p_{38} & p_{39} & p_{40} & p_{41} & p_{42} \\ 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

[6]In [84], a canonical ordering was used; for our purposes we need not belabour this point and use their Q'_{total} as Q_t'. This is perfectly legitimate as long as we label the columns carefully, which we have done.

and

$$
Q'_t = \begin{pmatrix}
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} & p_{19} & p_{20} & p_{21} & p_{22} & p_{23} & p_{24} & \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_1 \\
0 & 1 & -1 & -1 & 2 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_2 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_3 \\
1 & -2 & 1 & 0 & -2 & 2 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_4 \\
-1 & 1 & -1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \zeta_6 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_7 \\
0 & 1 & -1 & 0 & 2 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \zeta_8
\end{pmatrix}
$$

p_{25}	p_{26}	p_{27}	p_{28}	p_{29}	p_{30}	p_{31}	p_{32}	p_{33}	p_{34}	p_{35}	p_{36}	p_{37}	p_{38}	p_{39}	p_{40}	p_{41}	p_{42}	
−1	0	0	0	0	0	0	0	0	1	−1	0	0	0	1	−1	−1	1	0
−1	0	0	0	0	0	0	0	1	0	−1	0	0	0	1	−1	−1	1	0
−1	0	0	0	0	0	0	−1	2	0	−1	0	0	0	2	−2	−2	2	0
−1	0	0	0	0	0	0	−1	1	1	−1	0	0	0	2	−1	−2	1	0
−2	0	0	0	0	0	0	−1	2	1	−2	0	0	0	3	−2	−2	2	0
−2	0	0	0	0	0	0	0	2	0	−1	0	0	0	1	−1	−1	1	0
−2	0	0	0	0	0	0	0	1	1	−1	0	0	0	1	−1	−1	1	0
−1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	−1	0	1	0
−1	0	0	0	0	0	0	0	0	1	−1	0	0	0	1	−1	0	1	0
−1	0	0	0	0	0	0	−1	1	0	−1	0	0	0	2	−1	−1	2	0
−1	0	0	0	0	0	0	−1	2	−1	0	0	0	0	1	−1	−1	2	0
0	0	0	0	0	0	0	−1	1	0	0	0	0	0	1	−1	−1	1	0
0	0	0	0	0	0	0	−1	0	0	0	0	0	0	1	0	−1	1	0
1	0	0	0	0	0	0	0	−1	0	0	0	0	0	−1	1	1	−1	0
1	0	0	0	0	0	0	0	−1	−1	1	0	0	0	−1	2	0	−1	0
1	0	0	0	0	0	0	−1	0	−1	1	0	0	0	−1	1	0	0	0
1	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	−1	1	0	0	0	−1	1	0	0	0
0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	1	0	0	0
−1	0	0	0	0	0	0	0	0	1	−1	0	0	0	1	−1	0	0	0
−1	0	0	0	0	0	0	0	1	0	−1	0	0	0	1	0	−1	0	0
−1	0	0	0	0	0	0	0	1	−1	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	−1	0	0	−1	0	0	0	1	−1	0	1	0
0	0	1	0	0	0	0	−1	1	−1	0	0	0	0	0	−1	0	1	0
0	0	0	1	0	0	0	0	−1	1	−1	0	0	0	0	−1	1	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	−1	−1	1	0	0
0	0	0	0	0	1	0	−1	0	1	−1	0	0	0	1	−1	0	0	0
0	0	0	0	0	0	1	−1	1	0	−1	0	0	0	1	−1	−1	1	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	−1	−1	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	−1	1	0	−1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	−1	0	1	−1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_4
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_5
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_6
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_7
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_8

According to our algorithm, we must perform Gaussian row-reduction on Q'_t to solve for 42 variables x_i. When this is done we find that we can in fact express all variables in terms of 3 x_i's together with the 8 FI-parametres ζ_i. We choose these three x_i's to be $x_{10,29,36}$ corresponding to the 3 outer vertices which we know must be resolved in going from $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ to del Pezzo 1.

Next we select the fields which must be kept and set them to zero in order to determine the range for ζ_i. Bearing in mind the toric diagrams from Figure 4-3, these fields we judiciously select to be: $p_{13,8,37,38}$. Setting $x_{13,8,37,38} = 0$ gives us the solution $\{\zeta_6 = 0; x_{29} = \zeta_7 = \zeta_3 = \zeta_1 - \zeta_5; x_{10} = \zeta_4 + \zeta_5 + \zeta_3; x_{36} = \zeta_7 - \zeta_8\}$, which upon back-substitution to the solutions x_i we obtained from Q'_t, gives zero for $x_{13,8,37,38}$

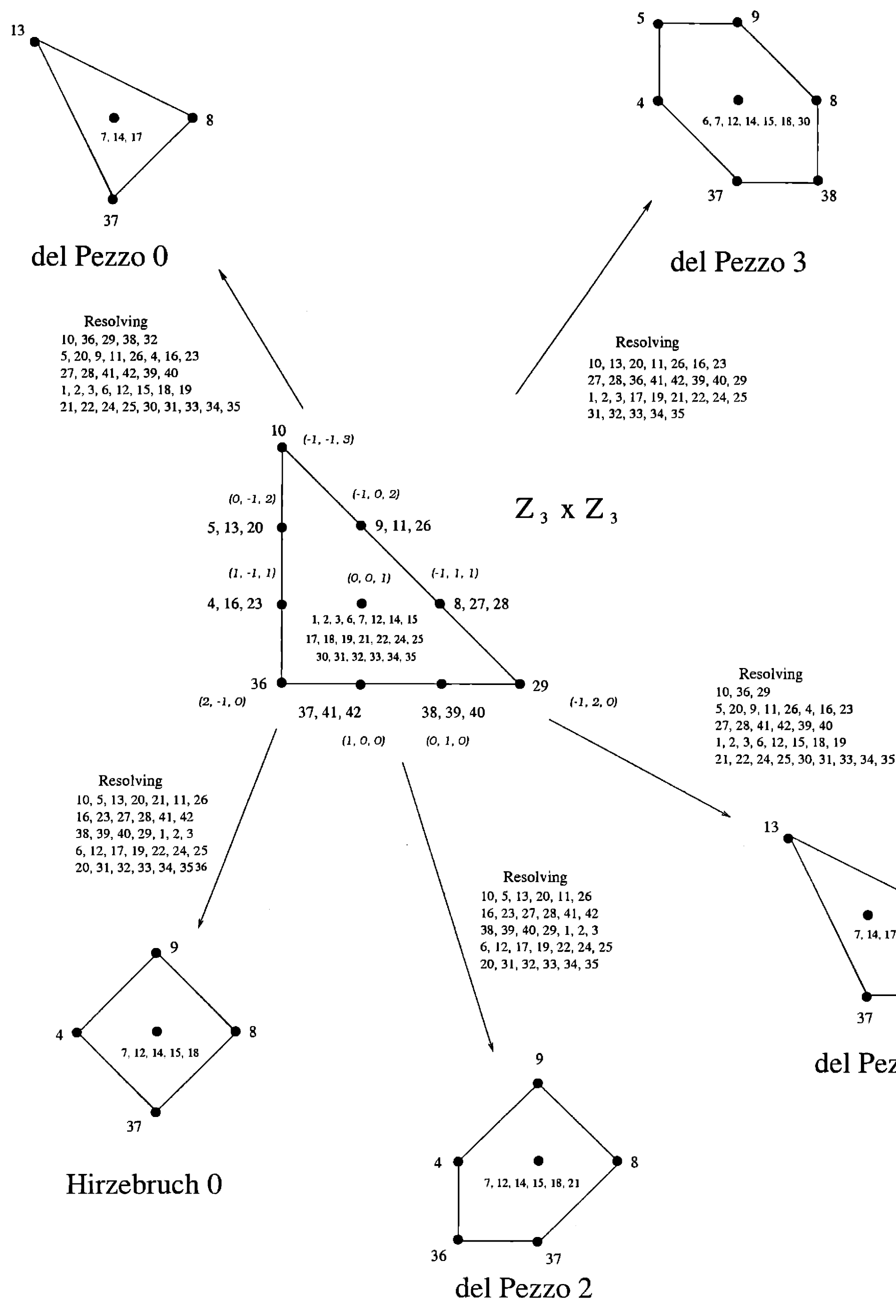


Figure 4-3: The resolution of the Gorenstein singularity $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ to the three toric del Pezzo surfaces as well as the zeroth Hirzebruch surface. We have labelled explicitly which columns (linear σ-model fields) are to be associated to each node in the toric diagrams and especially which columns are to be eliminated (fields acquiring non-zero VEV) in the various resolutions. Also, we have labelled the nodes of the parent toric diagram with the coordinates as given in the matrix G_t for $\mathbb{C}^3/(\mathbb{Z}_3\times\mathbb{Z}_3)$.

(which we have chosen by construction) as well as $x_{7,14,17,32}$; for all others we obtain positive values. This means precisely that all the other fields are to be eliminated and these 8 columns { 13, 8, 37, 38, 7,14,17,32 } are to be kept while the remaining 42-8=34 are to be eliminated from Q'_t upon row-reduction to give Q_t. In other words, we have found our set B to be {1,2,3,4,5,6,9,10,11,12,15,16,18,19,20,21,22,23,24,25, 26,27,28,29,30,31,33,34,35,36,39,40,41,42} and thus according to (4.3) we immediately obtain

$$Q_t = \begin{pmatrix} p_7 & p_8 & p_{13} & p_{14} & p_{17} & p_{32} & p_{37} & p_{38} & \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \zeta_2+\zeta_8 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & \zeta_6 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \zeta_1+\zeta_3+\zeta_5 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We note of course that 5 out of the 8 FI-parametres have been eliminated automatically; this is to be expected since in resolving $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ to del Pezzo 1, we remove precisely 5 nodes. Obtaining the D-terms and F-terms is now straightforward. Using (4.1) and re-inserting the last row we obtain the D-term equations (incidence matrix) to be

$$d = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

From this matrix we immediately observe that there are 4 gauge groups, i.e., $U(1)^4$ with 10 matter fields X_i which we have labelled in the matrix above. In an equivalent notation we rewrite d as the adjacency matrix of the quiver diagram (see Figure 4-4) for the gauge theory:

$$a_{ij} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 1 & 2 & 0 & 0 \end{pmatrix}.$$

The K-matrix we obtain to be:

$$K^t = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

which indicates that the original 10 fields X_i can be expressed in terms of 6. This was actually addressed in the previous section and we rewrite that pleasant superpotential here:

$$W = X_2X_7X_9 - X_3X_6X_9 - X_4X_8X_7 - X_1X_2X_5X_{10} + X_3X_4X_{10} + X_1X_5X_6X_8.$$

<u>del Pezzo 2:</u> Having obtained the gauge theory for del Pezzo 1, we now repeat the above analysis for del Pezzo 2. Now we have the FI-parametres restricted as $\{p_{36} = \zeta_2 = 0; \zeta_3 = \zeta_4; x_{29} = \zeta_4 + \zeta_6; x_{10} = \zeta_1 + \zeta_4\}$, making the set to be eliminated as $B = \{$ 1, 2, 3, 5, 6, 10, 11, 13, 16, 17, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 38, 39, 40, 41, 42 $\}$. Whence, we obtain

$$Q_t = \begin{pmatrix} p_4 & p_7 & p_8 & p_9 & p_{12} & p_{14} & p_{15} & p_{18} & p_{21} & p_{36} & p_{37} & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \zeta_4+\zeta_6+\zeta_8 \\ 1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \zeta_7 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \zeta_1+\zeta_3+\zeta_5 \\ -1 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & \zeta_2 \\ 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and observe that 4 D-terms have been resolved, as 4 nodes have been eliminated from $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. From this we easily extract (see Figure 4-4)

$$
d = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} & X_{13} \\ -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix};
$$

moreover, we integrate the F-term matrices

$$
K^t = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} & X_{13} \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

to obtain the superpotential

$$
\begin{aligned} W = X_2X_9X_{11} - X_9X_3X_{10} - X_4X_8X_{11} - X_1X_2X_7X_{13} + X_{13}X_3X_6 \\ -X_5X_{12}X_6 + X_1X_5X_8X_{10} + X_4X_7X_{12}. \end{aligned}
$$

<u>del Pezzo 3:</u> Finally, we shall proceed to treat del Pezzo 3. Here we have the range of the FI-parametres to be $\{\zeta_1 = \zeta_6 = \zeta_6 = 0; x_{29} = \zeta_3 = -\zeta_5; x_{10} = \zeta_4; \zeta_2 = x_{36}; \zeta_8 = -\zeta_2 - \zeta_{10}\}$, which gives the set B as {1, 2, 3, 10, 11, 13, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 36, 39, 40, 41, 42}, and thus according to (4.3) we immediately obtain

$$
Q_t = \begin{pmatrix} p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{12} & p_{14} & p_{15} & p_{18} & p_{30} & p_{37} & p_{38} & \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \zeta_2+\zeta_4+\zeta_8 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \zeta_7 \\ -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & \zeta_6 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \zeta_3+\zeta_5 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

We note indeed that 3 out of the 8 FI-parametres have been automatically resolved, as we have removed 3 nodes from the toric diagram for $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. The matter content (see Figure 4-4) is encoded in

$$d = \left(\begin{array}{cccccccccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} & X_{13} & X_{14} \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{array}\right),$$

and from the F-terms

$$K^t = \left(\begin{array}{cccccccccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} & X_{13} & X_{14} \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

we integrate to obtain the superpotential

$$\begin{aligned} W = & X_3X_8X_{13} - X_8X_9X_{11} - X_5X_6X_{13} - X_1X_3X_4X_{10}X_{12} \\ & +X_7X_9X_{12} + X_1X_2X_5X_{10}X_{11} + X_4X_6X_{14} - X_2X_7X_{14}. \end{aligned}$$

Note that we have a quintic term in W; this is an interesting interaction indeed.

del Pezzo 0: Before proceeding further, let us attempt one more example, viz., the degenerate case of the del Pezzo 0 as shown in Figure 4-3. This time we note that the ranges for the FI-parametres are $\{\zeta_5 = -x_{29} + \zeta_6 - A; \zeta_6 = x_{29} - B; x_{29} = B + C; \zeta_8 = -x_{36} + B; x_{36} = B + C + D; x_{10} = A + E\}$ for some positive A, B, C, D and E, that $B = \{$ 1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 15, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42 $\}$ and whence the charge matrix is

$$Q_t = \begin{pmatrix} p_7 & p_8 & p_{13} & p_{14} & p_{17} & p_{37} & \\ 1 & 0 & 0 & 0 & -1 & 0 & \zeta_2+\zeta_6+\zeta_8 \\ -1 & 0 & 0 & 1 & 0 & 0 & \zeta_1+\zeta_3+\zeta_5 \\ -1 & 1 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}.$$

We extract the matter content (see Figure 4-4) as $d = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 \\ -1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}$, and the F-terms as $K^t = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$, and from the latter we integrate to obtain the superpotential

$$W = X_1X_4X_9 - X_4X_5X_7 - X_2X_3X_9 - X_1X_6X_8 + X_2X_5X_8 + X_3X_6X_7.$$

Of course we immediately recognise the matter content (which gives a triangular quiver which we shall summarise below in Figure 4-4) as well as the superpotential from equations (4.7-4.14) of [14]; it is simply the theory on the Abelian orbifold $\mathbb{C}^3/\mathbb{Z}_3$ with action $(\alpha \in \mathbb{Z}_3) : (z_1, z_2, z_3) \to (e^{\frac{2\pi i}{3}} z_1, e^{\frac{2\pi i}{3}} z_2, e^{\frac{2\pi i}{3}} z_3)$. Is our del Pezzo 0 then $\mathbb{C}^3/\mathbb{Z}_3$? We could easily check from the G_t matrix (which we recall is obtained from G'_t of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ by eliminating the columns corresponding to the set B):

$$G_t = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 0 \end{pmatrix}.$$

These columns (up to repeat) correspond to monomials $Z, X^{-1}YZ, Y^{-1}Z^2, X$ in the polynomial ring $\mathbb{C}[X, Y, Z]$. Therefore we need to find the spectrum (set of maximal ideals) of the ring $\mathbb{C}[Z, X^{-1}YZ, Y^{-1}Z^2, X]$, which is given by the minimal polynomial relation: $(X^{-1}YZ) \cdot (Y^{-1}Z^2) \cdot X = (Z)^3$. This means, upon defining $p = X^{-1}YZ; q = Y^{-1}Z^2; r = X$ and $s = Z$, our del Pezzo 0 is described by $pqr = s^3$ as an algebraic variety in $\mathbb{C}^4(p, q, r, s)$, which is precisely $\mathbb{C}^3/\mathbb{Z}_3$. Therefore we have actually come through a full circle in resolving $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ to $\mathbb{C}^3/\mathbb{Z}_3$ and the validity of our algorithm survives this consistency check beautifully. Moreover, since we know that our gauge theory is exactly the one which lives on a D-brane probe on $\mathbb{C}^3/\mathbb{Z}_3$, this gives a good check for physicality: that our careful tuning of FI-parametres via canonical partial resolutions does give a physical D-brane theory at the end. We tabulate the matter content a_{ij} and the superpotential W for the del Pezzo surfaces below, and the quiver diagrams, in Figure 4-4.

	del Pezzo 1	**del Pezzo 2**	**del Pezzo 3**
$a_{ij} =$	$\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 1 & 2 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$
$W =$	$X_2X_7X_9 - X_3X_6X_9$ $-X_4X_8X_7 - X_1X_2X_5X_{10}$ $+X_3X_4X_{10} + X_1X_5X_6X_8$	$X_2X_9X_{11} - X_9X_3X_{10}$ $-X_4X_8X_{11} - X_1X_2X_7X_{13}$ $+X_{13}X_3X_6 - X_5X_{12}X_6$ $+X_1X_5X_8X_{10} + X_4X_7X_{12}$	$X_3X_8X_{13} - X_8X_9X_{11}$ $-X_5X_6X_{13} - X_1X_3X_4X_{10}X_{12}$ $+X_7X_9X_{12} + X_1X_2X_5X_{10}X_{11}$ $+X_4X_6X_{14} - X_2X_7X_{14}$

	del Pezzo 0 $\cong \mathbb{C}^3/\mathbb{Z}_3$	**Hirzebruch 0** $\cong \mathbb{P}^1 \times \mathbb{P}^1 := F_0 = E_1$
Matter a_{ij}	$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$
Superpotential W	$X_1X_4X_9 - X_4X_5X_7$ $-X_2X_3X_9 - X_1X_6X_8$ $+X_2X_5X_8 + X_3X_6X_7$	$X_1X_8X_{10} - X_3X_7X_{10}$ $-X_2X_8X_9 - X_1X_6X_{12}$ $+X_3X_6X_{11} + X_4X_7X_9$ $+X_2X_5X_{12} - X_4X_5X_{11}$

Upon comparing Figure 4-3 and Figure 4-4, we notice that as we go from del Pezzo 0 to 3, the number of points in the toric diagram increases from 4 to 7, and the number of gauge groups (nodes in the quiver) increases from 3 to 6. This is consistent with the observation for $\mathcal{N} = 1$ theories that the number of gauge groups equals the number of perimetre points (e.g., for del Pezzo 1, the four nodes 13, 8, 37 and 38) in the toric diagram. Moreover, as discussed in [36], the rank of the global symmetry group (E_i for del Pezzo i) which must exist for these theories equals the number of perimetre point minus 3; it would be an intereting check indeed to see how such a symmetry manifests itself in the quivers and superpotentials.

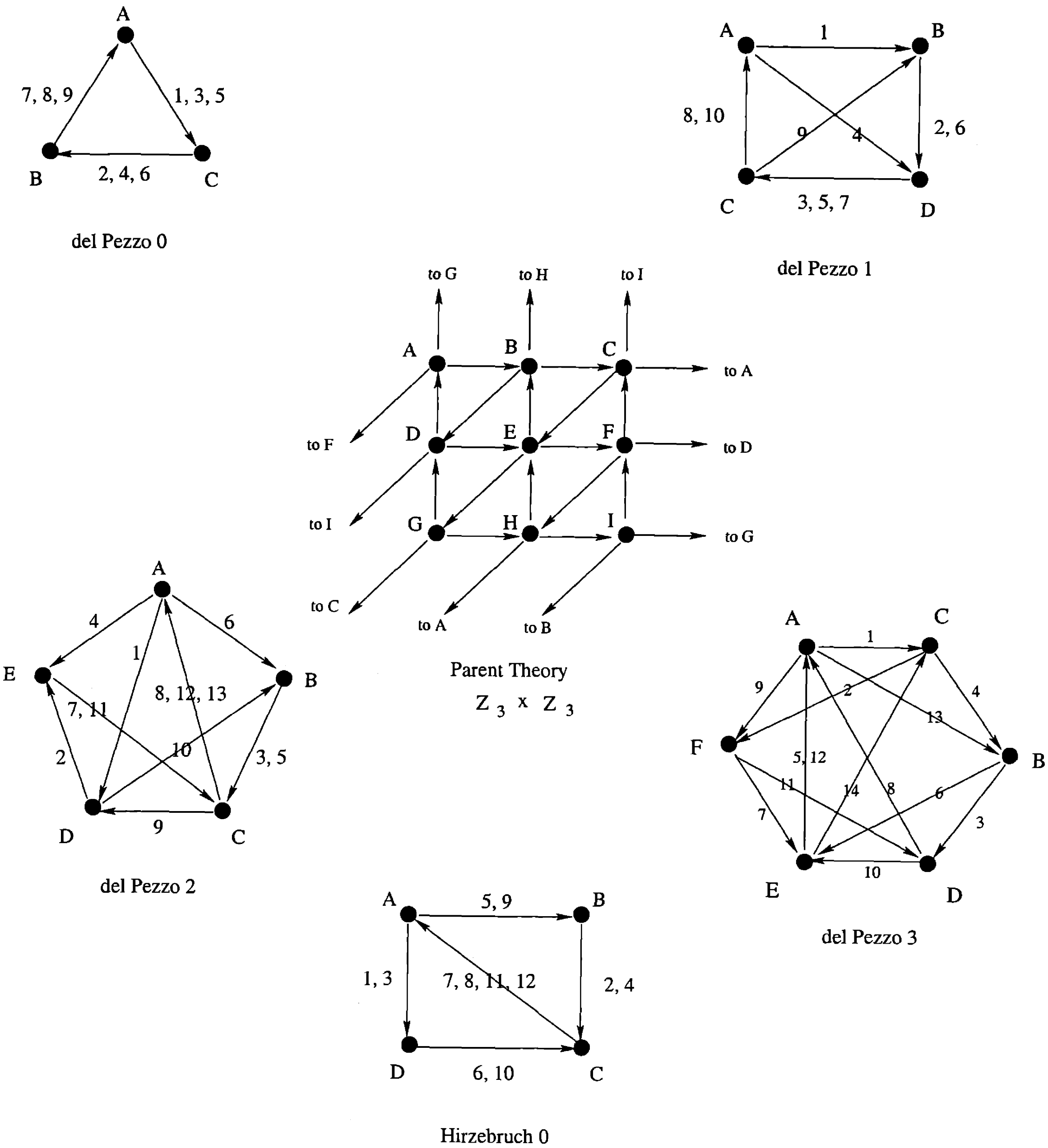


Figure 4-4: The quiver diagrams for the matter content of the brane world-volume gauge theory on the 4 toric del Pezzo singularities as well as the zeroth Hirzebruch surface. We have specifically labelled the $U(1)$ gauge groups (A, B, ..) and the bi-fundamentals (1, 2, ..) in accordance with our conventions in presenting the various matrices Q_t, Δ and K. As a reference we have also included the quiver for the parent $\mathbb{Z}_3 \times \mathbb{Z}_3$ theory.

Hirzebruch 0: Let us indulge ourselves with one more example, namely the 0th Hirzebruch surface, or simply $\mathbb{P}^1 \times \mathbb{P}^1 := F_0 := E_1$. The toric diagram is drawn in Figure 4-3. Now the FI-parametres are $\{\zeta_4 = -x_{29} - x_{36} - \zeta_5 - \zeta_8 - A; \zeta_5 = -A - B; \zeta_7 = x_{10} + x_{29} + x_{36} + \zeta_8 - C; \zeta_8 = -x_{10} - x_{29} - x_{36} + D; D = A + B; C = A + B; A = x_{10} - E; x_{10} = E + F; x_{29} = B + G\}$ for positive A, B, C, D, E, F and G. Moreover, $B = \{$ 1, 2, 3, 5, 6, 10, 11, 13, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42 $\}$. We note that this can be obtained directly by partial resolution of fields 21 and 36 from del Pezzo 2 as is consistent with Figure 4-3. Therefrom we obtain the charge matrix

$$Q_t = \begin{pmatrix} p_4 & p_7 & p_8 & p_9 & p_{12} & p_{14} & p_{15} & p_{18} & p_{37} & \\ -1 & 2 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & \zeta_2 + \zeta_4 + \zeta_6 + \zeta_8 \\ 1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & \zeta_7 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \zeta_1 + \zeta_3 + \zeta_5 \\ 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

from which we have the matter content

$$d = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} \\ -1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

the quiver for which is presented in Figure 4-4. The F-terms are

$$K^t = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

from which we obtain

$$\begin{aligned} W &= X_1X_8X_{10} - X_3X_7X_{10} - X_2X_8X_9 - X_1X_6X_{12} + X_3X_6X_{11} \\ &\quad + X_4X_7X_9 + X_2X_5X_{12} - X_4X_5X_{11}, \end{aligned}$$

a perfectly acceptable superpotential with only cubic interactions. We include these results with our table above.

4.4 Uniqueness?

In our foregoing discussion we have constructed in detail an algorithm which calculates the matter content encoded by Δ and superpotential encoded in K, given the toric diagram of the singularity which the D-branes probe. As abovementioned, though this algorithm gives one solution for the quiver and the K-matrix once the matrix Q_t is determined, the general inverse process of going from toric data to gauge theory data, is highly **non-unique** and a classification of all possible theories having the same toric description would be interesting[7]. Indeed, by the very structure of our algorithm, in immediately appealing to the partial resolution of gauge theories on $\mathbb{Z}_n \times \mathbb{Z}_n$ orbifolds which are well-studied, we have granted ourselves enough extraneous information to determine a unique Q_t and hence the ability to proceed with ease (this was the very reason for our devising the algorithm).

However, generically we do not have any such luxury. At the end of subsection 3.1, we have already mentioned two types of ambiguities in the inverse problem. Let us refresh our minds. They were (A) the **F-D ambiguity** which is the inability to decide, simply by observing the toric diagram, which rows of the charge matrix Q_t are D-terms and which are F-terms and (B) the **repetition ambiguity** which is the inability to decide which columns of G_t to repeat once having read the vectors from the toric diagram. Other ambiguities exist, such as in each time when we compute nullspaces, but we shall here discuss to how ambiguities (A) and (B) manifest themselves and provide examples of vastly different gauge theories having the same toric description. There is another point which we wish to emphasise: as mentioned at the end of subsection 3.1, the resolution method guarantees, upon careful tuning of the FI-parametres, that the resulting gauge theory does originate from the world-volume of a D-brane probe. Now of course, by taking liberties with experimentation

[7]We thank R. Plesser for pointing this issue out to us.

of these ambiguities we are no longer protected by physicality and in general the theories no longer live on the D-brane. It would be a truly interesting exercise to check which of these different theories do.

F-D Ambiguity: First, we demonstrate type (A) by returning to our old friend the SPP whose charge matrix we had earlier presented. Now we write the same matrix without specifying the FI-parametres:

$$Q_t = \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

We could apply the last steps of our algorithm to this matrix as follows.

(a) If we treat the first row as Q (the F-terms) and the second and third as $V \cdot U$ (the D-terms) we obtain the gauge theory as discussed in subsection 3.3 and in [83].

(b) If we treat the second row as Q and first with the third as $V \cdot U$, we obtain $d = \begin{pmatrix} -1 & 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}$ which is an exotic theory indeed with a field (p_5) charged under three gauge groups.

Let us digress a moment to address the stringency of the requirements upon matter contents. By the very nature of finite group representations, any orbifold theory must give rise to only adjoints and bi-fundamentals because its matter content is encodable by an adjacency matrix due to tensors of representations of finite groups. The corresponding incidence matrix d, has (a) only 0 and ± 1 entries specifying the particular bi-fundamentals and (b) has each column containing precisely one 1, one -1 and with the remaining entries 0. However more exotic matter contents could arise from more generic toric singularities, such as fields charged under 3 or more gauge group factors; these would then have d matrices with conditions (a) and (b) relaxed[8]. Such exotic quivers (if we could even call them quivers still) would give interesting enrichment to

[8]Note that we still require that each column sums to 0 so as to be able to factor out an overall $U(1)$.

those well-classified families.

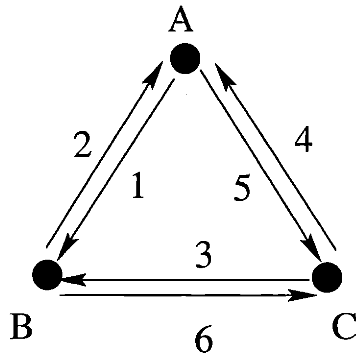
Moreover we must check the anomaly cancellation conditions. These could be rather involved; even though for $U(1)$ theories they are a little simpler, we still need to check *trace anomalies* and *cubic anomalies.* In a trace-anomaly-free theory, for each node in the quiver, the number of incoming arrows must equal the number of outgoing (this is true for a $U(1)$ theory which is what toric varieties provide; for a discussion on this see e.g. [45]). In matrix language this means that each row of d must sum to 0.

Now for a theory with only bi-fundamental matter with ± 1 charges, since $(\pm 1)^3 = \pm 1$, the cubic is equal to the trace anamaly; therefore for these theories we need only check the above row-condition for d. For more exotic matter content, which we shall meet later, we do need to perform an independent cubic-anomaly check.

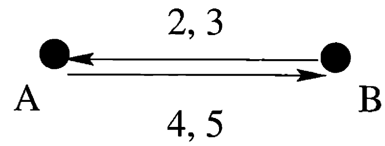
Now for the above d, the second row does not sum to zero and whence we do unfortunately have a problematic anomalous theory. Let us push on to see whether we have better luck in the following.

(c) Treating row 3 as the F-terms and the other two as the D-terms gives $d = \begin{pmatrix} 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}$ which has the same anomaly problem as the one above.

(d) Now let rows 1 and 2 as the F-terms and the 3rd, as the D-terms, we obtain $d = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & -1 & -1 & 1 & 1 \end{pmatrix}$, which is a perfectly reasonable matter content. Integrating $K = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$ gives the superpotential $W = \phi(X_1X_2X_5 - X_3X_4)$ for some field ϕ of charge $(0,0)$ (which could be an adjoint for example; note however that we can not use X_1 even though it has charge $(0,0)$ for otherwise the F-terms would be altered). This theory is perfectly legitimate. We compare the quiver diagrams of theories (a) (which we recall from Figure 4-2) and this present example in Figure 4-5. As a check, let us define the gauge invariant



Theory (a)



Theory (d)

Figure 4-5: The vastly different matter contents of theories (a) and (d), both anomaly free and flow to the toric diagram of the suspended pinched point in the IR.

quantities: $a = X_2X_4$, $b = X_2X_5$, $c = X_3X_4$, $d = X_3X_5$ and $e = X_1$. Then we have the algebraic relations $ad = bc$ and $eb = c$, from which we immediately obtain $ad = eb^2$, precisely the equation for the SPP.

(e) As a permutation on the above, treating rows 1 and 3 as the F-terms gives a theory equivalent thereto.

(f) Furthermore, we could let rows 2 and 3 be Q giving us $d = \begin{pmatrix} 0 & 1 & -1 & -1 & -1 \\ 0 & -1 & 1 & 1 & 1 \end{pmatrix}$, but this again gives an anomalous matter content.

(g) Finally, though we cannot treat all rows as F-terms, we can however treat all of them as D-terms in which Q_t is simply Δ as remarked at the end of Section 2 before the flow chart. In this case we have the matter content $d = \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$ which clearly is both trace-anomaly free (each row adds to zero) and cubic-anomaly-free (the cube-sum of the each row is also zero). The superpotential, by our very choice, is of course zero. Thus we have a perfectly legitimate theory without superpotential but with an exotic field (the first column) charged under 4 gauge groups.

We see therefore, from our list of examples above, that for the simple case of the SPP we have 3 rather different theories (a,d,g) with contrasting matter content and superpotential which share the same toric description.

Repetition Ambiguity: As a further illustration, let us give one example of type (B) ambiguity. First let us eliminate all repetitive columns from the G_t of SPP, giving us:

$$G_t = \begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

which is perfectly allowed and consistent with Figure 4-1. Of course many more possibilities for repeats are allowed and we could redo the following analyses for each of them. As the nullspace of our present choice of G_t, we find Q_t, and we choose, in light of the foregoing discussion, the first row to represent the D-term:

$$Q_t = \begin{pmatrix} -1 & 1 & -1 & 0 & 1 & \zeta \\ 1 & -2 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Thus equipped, we immediately retrieve, using our algorithm,

$$d = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \\ 1 & -1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 1 & 0 \end{pmatrix} \quad K^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We see that d passes our anomaly test, with the same bi-fundamental matter content as theory (d). The superpotential can be read easily from K (since there is only one relation) as $W = \phi(X_5^2 - X_3X_4)$. As a check, let us define the gauge invariant quantities: $a = X_1X_2$, $b = X_1X_4$, $c = X_3X_2$, $d = X_3X_4$ and $e = X_5$. These have among themselves the algebraic relations $ad = bc$ and $e^2 = d$, from which we immediately obtain $bc = ae^2$, the equation for the SPP. Hence we have yet another interesting anomaly free theory, which together with our theories (a), (d) and (g) above, shares the toric description of the SPP.

Finally, let us indulge in one more demonstration. Now let us treat both rows of our Q_t as D-terms, whereby giving a theory with no superpotential and the exotic matter content $d = \begin{pmatrix} -1 & 1 & -1 & 0 & 1 \\ 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & -1 \end{pmatrix}$ with a field (column 2) charged under 3 gauge groups.

Indeed though the rows sum to 0 and trace-anomaly is avoided, the cube-sum of the second row gives $1^3 + 1^3 + (-2)^3 = -6$ and we do have a cubic anomaly.

In summary, we have an interesting phenomenon indeed! Taking so immediate an advantage of the ambiguities in the above has already produced quite a few examples of vastly different gauge theories flowing in the IR to the same universality class by having their moduli spaces identical. The vigilant reader may raise two issues. First, as mentioned earlier, one may take the pains to check whether these theories do indeed live on a D-brane. Necessary conditions such as that the theories may be obtained from an $\mathcal{N} = 4$ theory must be satisfied. Second, the matching of moduli spaces may not seem so strong since they are on a classical level. However, since we are dealing with product $U(1)$ gauge groups (which is what toric geometry is capable to dealing with so far), the classical moduli receive no quantum corrections[9]. Therefore the matching of the moduli for these various theories do persist to the quantum regime, which hints at some kind of "duality" in the field theory. We shall call such a duality **toric duality**. It would be interesting to investigate how, with non-Abelian versions of the theory (either by brane setups or stacks of D-brane probes), this toric duality may be extended.

4.5 Conclusions and Prospects

The study of resolution of toric singularities by D-branes is by now standard. In the concatenation of the F-terms and D-terms from the world volume gauge theory of a single D-brane at the singularity, the moduli space could be captured by the algebraic data of the toric variety. However, unlike the orbifold theories, the inverse problem where specifying the structure of the singularity specifies the physical theory has not yet been addressed in detail.

We recognise that in contrast with D-brane probing orbifolds, where knowing the group structure and its space-time action uniquely dictates the matter content and superpotential, such flexibility is not shared by generic toric varieties due to the highly

[9] We thank K. Intriligator for pointing this out.

non-unique nature of the inverse problem. It has been the purpose and main content of the current writing to device an **algorithm** which constructs the matter content (the incidence matrix d) and the interaction (the F-term matrix K) of a well-behaved gauge theory given the toric diagram D of the singularity at hand.

By embedding D into the Abelian orbifold $\mathbb{C}^k/(\mathbb{Z}_n)^{k-1}$ and performing the standard partial resolution techniques, we have investigated how the induced action upon the charge matrices corresponding to the toric data of the latter gives us a convenient charge matrix for D and have constructed a programmatic methodology to extract the matter content and superpotential of *one* D-brane world volume gauge theory probing D. The theory we construct, having its origin from an orbifold, is nicely behaved in that it is anomaly free, with bi-fundamentals only and well-defined superpotentials. As illustrations we have tabulated the results for all the toric del Pezzo surfaces and the zeroth Hirzebruch surface.

Directions of further work are immediately clear to us. From the patterns emerging from del Pezzo surfaces 0 to 3, we could speculate the physics of higher (non-toric) del Pezzo cases. For example, we expect del Pezzo n to have $n + 3$ gauge groups. Moreover, we could attempt to fathom how our resolution techniques translate as Higgsing in brane setups, perhaps with recourse to diamonds, and realise the various theories on toric varieties as brane configurations.

Indeed, as mentioned, the inverse problem is highly non-unique; we could presumably attempt to classify all the different theories sharing the same toric singularity as their moduli space. In light of this, we have addressed two types of ambiguity: that in having multiple fields assigned to the same node in the toric diagram and that of distinguishing the F-terms and D-terms in the charge matrix. In particular we have turned this ambiguity to a matter of interest and have shown, using our algorithm, how vastly different theories, some with quite exotic matter content, may have the same toric description. This commonality would correspond to a duality wherein different gauge theories flow to the same universality class in the IR. We call this phenomenon **toric duality**. It would be interesting indeed how this duality may manifest itself as motions of branes in the corresponding setups. Without further ado however, let us pause here awhile and leave such investigations to forthcoming work.

Appendix: Finding the Dual Cone

Let us be given a convex polytope C, with the edges specifying the faces of which given by the matrix M whose columns are the vectors corresponding to these edges. Our task is to find the dual cone $\tilde{C}$ of C, or more precisely the matrix N such that

$$N^t \cdot M \geq 0 \qquad \text{for all entries.}$$

There is a standard algorithm, given in [48]. Let M be $n \times p$, i.e., there are p n-dimensional vectors spanning C. We note of course that $p \geq n$ for convexity. Out of the p vectors, we choose $n-1$. This gives us an $n \times (n-1)$ matrix of co-rank 1, whence we can extract a 1-dimensional null-space (as indeed the initial p vectors are all linearly independent) described by a single vector u.

Next we check the dot product of u with the remaining $p-(n-1)$ vectors. If all the dot products are positive we keep u, and if all are negative, we keep $-u$, otherwise we discard it.

We then select another $n-1$ vectors and repeat the above until all combinations are exhausted. The set of vectors we have kept, u's or $-u$'s then form the columns of N and span the dual cone $\tilde{C}$.

We note that this is a very computationally intensive algorithm, the number of steps of which depends on $\left(\begin{array}{c} p \\ n-1 \end{array} \right)$ which grows exponentially.

A subtle point to remark. In light of what we discussed in a footnote in the paper on the difference between $\mathbf{M}_+ = \mathbf{M} \cap \sigma$ and $\mathbf{M}'_+$, here we have computed the dual of σ. We must ensure that $\mathbb{Z}_+$-independent lattice points inside the cones be not missed.

Chapter 5

Phase Structure of D-brane Gauge Theories and Toric Duality

5.1 Introduction

The methods of toric geometry have been a crucial tool to the understanding of many fundamental aspects of string theory on Calabi-Yau manifolds (cf. e.g. [49]). In particular, the connexions between toric singularities and the manufacturing of various gauge theories as D-brane world-volume theories have been intimate.

Such connexions have been motivated by a myriad of sources. As far back as 1993, Witten [50] had shown, via the so-called gauged linear sigma model, that the Fayet-Illiopoulos parametre r in the D-term of an $\mathcal{N}=2$ supersymmetric field theory with $U(1)$ gauge groups can be tuned as an order-parametre which extrapolates between the Landau-Ginzburg and Calabi-Yau phases of the theory, whereby giving a precise viewpoint to the LG/CY-correspondence. What this means in the context of Abelian gauge theories is that whereas for $r \ll 0$, we have a Landau-Ginzberg description of the theory, by taking $r \gg 0$, the space of classical vacua obtained from D- and F-flatness is described by a Calabi-Yau manifold, and in particular a toric variety.

With the advent of D-brane technologies, vast amount of work has been done to study the dynamics of world-volume theories on D-branes probing various geometries. Notably, in [40], D-branes have been used to probe Abelian singularities of the form

C^2/Z_n. Methods of studying the moduli space of the SUSY theories describable by quiver diagrams have been developed by the recognition of the Kronheimer-Nakajima ALE instanton construction, especially the moment maps used therein [42].

Much work followed [71, 70, 44]. A key advance was made in [14], where, exemplifying with Abelian C^3 orbifolds, a detailed method was developed for capturing the various phases of the moduli space of the quiver gauge theories as toric varieties. In another vein, the huge factory built after the brane-setup approach to gauge theories [10] has been continuing to elucidate the T-dual picture of branes probing singularities (e.g. [30, 31, 45]). Brane setups for toric resolutions of $Z_2 \times Z_2$, including the famous conifold, were addressed in [83, 79]. The general question of how to construct the quiver gauge theory for an arbitrary toric singularity was still pertinent. With the AdS/CFT correspondence emerging [71, 70], the pressing need for the question arises again: given a toric singularity, how does one determine the quiver gauge theory having the former as its moduli space?

The answer lies in "Partial Resolution of Abelian Orbifolds" and was introduced and exemplified for the toric resolutions of the $Z_3 \times Z_3$ orbifold [14, 84]. The method was subsequently presented in an algorithmic and computationally feasible fashion in [85] and was applied to a host of examples in [86].

One short-coming about the inverse procedure of going from the toric data to the gauge theory data is that it is highly non-unique and in general, unless one starts by partially resolving an orbifold singularity, one would not be guaranteed with a physical world-volume theory at all! Though the non-uniqueness was harnessed in [85] to construct families of quiver gauge theories with the same toric moduli space, a phenomenon which was dubbed "toric duality," the physicality issue remains to be fully tackled.

The purpose of this writing is to analyse toric duality within the confinement of the canonical method of partial resolutions. Now we are always guaranteed with a world-volume theory at the end and this physicality is of great assurance to us. We find indeed that with the restriction of physical theories, toric duality is still very much at work and one can construct D-brane quiver theories that flow to the same moduli space.

We begin in §2 with a seeming paradox which initially motivated our work and which *ab initio* appeared to present a challenge to the canonical method. In §3 we resolve the paradox by introducing the well-known mathematical fact of toric isomorphisms. Then in §4, we present a detailed analysis, painstakingly tracing through each step of the inverse procedure to see how much degree of freedom one is allowed as one proceeds with the algorithm. We consequently arrive at a method of extracting torically dual theories which are all physical; to these we refer as "phases." As applications of these ideas in §5 we re-analyse the examples in [85], viz., the toric del Pezzo surfaces as well as the zeroth Hirzebruch surface and find the various phases of the quiver gauge theories with them as moduli spaces. Finally in §6 we end with conclusions and future prospects.

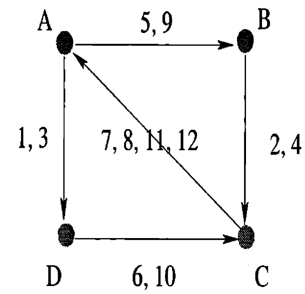
5.2 A Seeming Paradox

In [85] we noticed the emergence of the phenomenon of "Toric Duality" wherein the moduli space of vast numbers of gauge theories could be parametrised by the *same* toric variety. Of course, as we mentioned there, one needs to check extensively whether these theories are all physical in the sense that they are world-volume theories of some D-brane probing the toric singularity.

Here we shall discuss an issue of more immediate concern to the physical probe theory. We recall that using the method of *partial resolutions of Abelian orbifolds* [85, 14, 84, 83], we could always extract a canonical theory on the D-brane probing the singularity of interest.

However, a discrepancy of results seems to have risen between [85] and [71] on the precise world-volume theory of a D-brane probe sitting on the zeroth Hirzebruch surface; let us compare and contrast the two results here.

- Results from [85]: The matter contents of the theory are given by (on the left we present the quiver diagram and on the right, the incidence matrix that encodes the quiver):



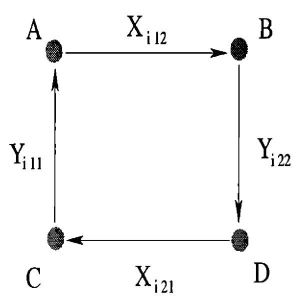
$d =$

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}
A	−1	0	−1	0	−1	0	1	1	−1	0	1	1
B	0	−1	0	−1	1	0	0	0	1	0	0	0
C	0	1	0	1	0	1	−1	−1	0	1	−1	−1
D	1	0	1	0	0	−1	0	0	0	−1	0	0

and the superpotential is given by

$$\begin{aligned} W &= X_1X_8X_{10} - X_3X_7X_{10} - X_2X_8X_9 - X_1X_6X_{12} + X_3X_6X_{11} \\ &\quad + X_4X_7X_9 + X_2X_5X_{12} - X_4X_5X_{11}. \end{aligned} \tag{5.1}$$

- Results from [71]: The matter contents of the theory are given by (for $i = 1, 2$):



$d =$

	$X_{i\ 12}$	$X_{i\ 21}$	$Y_{i\ 11}$	$Y_{i\ 22}$
A	−1	0	1	0
B	1	0	0	−1
C	0	1	−1	0
D	0	−1	0	1

and the superpotential is given by

$$W = \epsilon^{ij}\epsilon^{kl} X_{i\ 12} Y_{k\ 22} X_{j\ 21} Y_{l\ 11}. \tag{5.2}$$

Indeed, even though both these theories have arisen from the canonical partial resolutions technique and hence are world volume theories of a brane probing a Hirzebruch singularity, we see clearly that they differ vastly in both matter content and superpotential! Which is the "correct" physical theory?

In response to this seeming paradox, let us refer to Figure 5-1. Case 1 of course

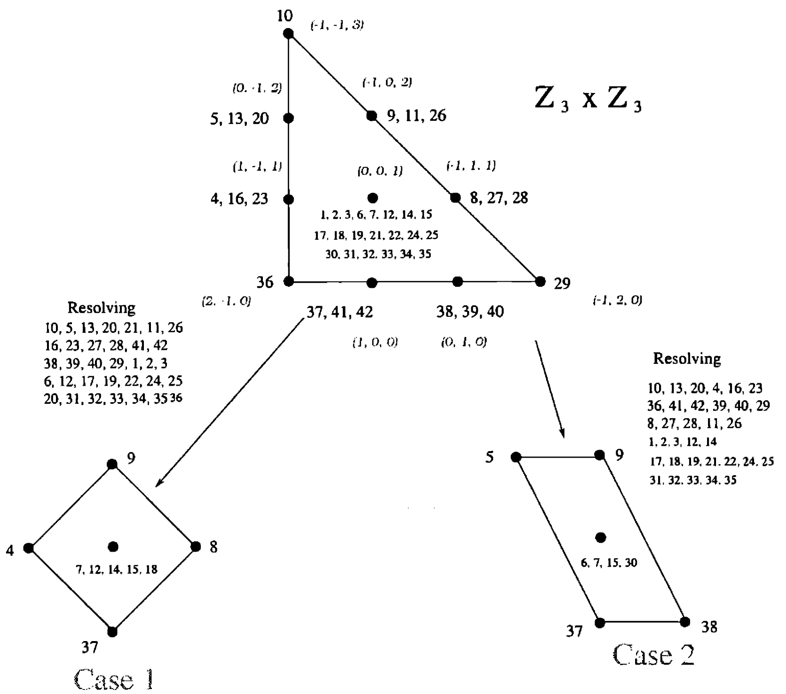


Figure 5-1: Two alternative resolutions of $C^2/Z_3 \times Z_3$ to the Hirzebruch surface F_0: Case 1 from [85] and Case 2 from [71].

was what had been analysed in [85] (q.v. ibid.) and presented in (5.1); let us now consider case 2. Using the canonical algorithm of [84, 85], we obtain the matter content (we have labelled the fields and gauge groups with some foresight)

$$d_{ia} = \left(\begin{array}{c|cccccccc} & X_1 & X_1' & X_2' & Y_1 & Y_2 & Y_1' & Y_2 & Y_2' \\ \hline D & 0 & 1 & 1 & 0 & 0 & -1 & 0 & -1 \\ A & -1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ B & 1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ C & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 1 \end{array}\right)$$

and the dual cone matrix

$$K_{ij}^T = \begin{pmatrix} & X_1 & X_1' & X_2' & Y_1 & Y_2 & Y_1' & X_2 & Y_2' \\ \hline p_1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ p_2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ p_3 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ p_4 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ p_5 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ p_6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

which translates to the F-term equations

$$X_1 Y_2' = p_1 p_3 p_6 = Y_1' X_2; \quad X_1' Y_2 = p_2 p_4 p_5 = Y_1 X_2'.$$

What we see of course, is that with the field redefinition $X_i \leftrightarrow X_{i\ 12}, X_i' \leftrightarrow Y_{i\ 22}, Y_i \leftrightarrow Y_{i\ 11}$ and $Y_i' \leftrightarrow X_{i\ 21}$ for $i = 1, 2$, the above results are in exact agreement with the results from [71] as presented in (5.2).

This is actually of no surprise to us because upon closer inspection of Figure 5-1, we see that the toric diagram for Cases 1 and 2 respectively has the coordinate points

$$G1_t = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 \end{pmatrix} \qquad G2_t = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 1 \end{pmatrix}.$$

Now since the algebraic equation of the toric variety is given by [48]

$$V(G_t) = Spec_{Max}\left(\mathbb{C}[X_i^{G_t^\vee \cap Z^3}]\right),$$

we have checked that, using a reduced Gröbner polynomial basis algorithm to compute the variety [88], the equations are identical up to redefinition of variables.

Therefore we see that the two toric diagrams in Cases 1 and 2 of Figure 5-1 both describe the zeroth Hirzebruch surface as they have the same equations (embedding into C^9). Yet due to the particular choice of the diagram, we end up with strikingly different gauge theories on the D-brane probe despite the identification of the moduli space in the IR. This is indeed a curiously strong version of "toric duality."

Bearing the above in mind, in this paper, we will analyse the degrees of freedom in the Inverse Algorithm expounded upon in [85], i.e., for a given toric singularity, how many different physical gauge theories (phase structures), resulting from various partial resolutions can one have for a D-brane probing such a singularity? To answer this question, first in §2 we present the concept of toric isomorphism and give the conditions for different toric data to correspond to the same toric variety. Then in §3 we follow the Forward Algorithm and give the freedom at each step from a given set of gauge theory data all the way to the output of the toric data. Knowing these freedoms, we can identify the sources that may give rise to different gauge theories in the Inverse Algorithm starting from a prescribed toric data. In section 4, we apply the above results and analyse the different phases for the partial resolutions of the $Z_3 \times Z_3$ orbifold singularity, in particular, we found that there are two inequivalent phases of gauge theories respectively for the zeroth Hirzebruch surface and the second del Pezzo surface. Finally, in section 5, we give discussions for further investigation.

5.3 Toric Isomorphisms

Extending this observation to generic toric singularities, we expect classes of inequivalent toric diagrams corresponding to the same variety to give rise to inequivalent gauge theories on the D-brane probing the said singularity. An immediate question is naturally posed: "is there a classification of these different theories and is there a transformation among them?"

To answer this question we resort to the following result. Given M-lattice cones σ and σ', let the linear span of σ be $\text{lin}\sigma = \mathbb{R}^n$ and that of σ' be $\mathbb{R}^m$. Now each cone gives rise to a semigroup which is the intersection of the dual cone $\sigma^\vee$ with the dual lattice M, i.e., $S_\sigma := \sigma^\vee \cap M$ (likewise for σ'). Finally the toric variety is given as the maximal spectrum of the polynomial ring of C adjoint the semigroup, i.e., $X_\sigma := Spec_{Max}\left(C[S_\sigma]\right)$.

DEFINITION 5.3.1 *We have these types of isomorphisms:*

1. *We call σ and σ'* cone isomorphic, *denoted $\sigma \cong_{cone} \sigma'$, if $n = m$ and there is a unimodular transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ with $L(\sigma) = \sigma'$;*

2. *we call S_σ and $S_{\sigma'}$* monomial isomorphic, *denoted $S_\sigma \cong_{mon} S_{\sigma'}$, if there exists mutually inverse monomial homomorphisms between the two semigroups.*

Thus equipped, we are endowed with the following

THEOREM 5.3.1 *([90], VI.2.11) The following conditions are equivalent:*

$$(a) \quad \sigma \cong_{cone} \sigma' \Leftrightarrow (b) \quad S_\sigma \cong_{mon} S_{\sigma'} \Leftrightarrow (c) \quad X_\sigma \cong X_{\sigma'}$$

What this theorem means for us is simply that, for the n-dimensional toric variety, an $SL(n; Z)$ transformation[1] on the original lattice cone amounts to merely coördinate transformations on the polynomial ring and results in the same toric variety. This, is precisely what we want: different toric diagrams giving the same variety.

The necessity and sufficiency of the condition in Theorem 5.3.1 is important. Let us think of one example to illustrate. Let a cone be defined by (e_1, e_2), we know this corresponds to C^2. Now if we apply the transformation

$$(e_1, e_2) \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = (2e_1 - e_2, e_2),$$

which corresponds to the variety $xy = z^2$, i.e., C^2/Z_2, which of course is not isomorphic to C^2. The reason for this is obvious: the matrix we have chosen is certainly not unimodular.

---

[1]Strictly speaking, by unimodular we mean $GL(n; Z)$ matrices with determinant ± 1; we shall denote these loosely by $SL(n; Z)$.

5.4 Freedom and Ambiguity in the Algorithm

In this section, we wish to step back and address the issue in fuller generality. Recall that the procedure of obtaining the moduli space encoded as toric data once given the gauge theory data in terms of product $U(1)$ gauge groups, D-terms from matter contents and F-terms from the superpotential, has been well developed [71, 14]. Such was called the **forward algorithm** in [85]. On the other hand the **reverse algorithm** of obtaining the gauge theory data from the toric data has been discussed extensively in [84, 85].

It was pointed in [85] that both the forward and reverse algorithm are highly non-unique, a property which could actually be harnessed to provide large classes of gauge theories having the same IR moduli space. In light of this so-named "toric duality" it would be instructive for us to investigate how much freedom do we have at each step in the algorithm. We will call two data related by such a freedom *equivalent* to each other. Thence further we could see how freedoms at every step accumulate and appear in the final toric data. Modulo such equivalences we believe that the data should be uniquely determinable.

5.4.1 The Forward Algorithm

We begin with the forward algorithm of extracting toric data from gauge data. A brief review is at hand. To specify the gauge theory, we require three pieces of information: the number of $U(1)$ gauge fields, the charges of matter fields and the superpotential. The first two are summarised by the so-called charge matrix d_{li} where $l = 1, 2, ..., L$ with L the number of $U(1)$ gauge fields and $i = 1, 2, ..., I$ with I the number of matter fields. When using the forward algorithm to find the vacuum manifold (as a toric variety), we need to solve the D-term and F-term flatness equations. The D-terms are given by d_{li} matrix while the F-terms are encoded in a matrix K_{ij} with $i, 1, 2, ..., I$ and $j = 1, 2, ..., J$ where J is the number of independent parameters needed to solve the F-terms. By gauge data then we mean the matrices d (also called the incidence matrix) and the K (essentially the dual cone); the forward algorithm takes these as input. Subsequently we trace a flow-chart:

$$
\begin{array}{ccccccc}
\text{D-Terms} \rightarrow d & \rightarrow & \Delta & & & & \\
 & & \downarrow & & & & \\
\text{F-Terms} \rightarrow K & \stackrel{V \cdot K^T = \Delta}{\rightarrow} & V & & & & \\
\downarrow & & \downarrow & & & & \\
T = \text{Dual}(K) & \stackrel{U \cdot T^T = \text{Id}}{\rightarrow} & U & \rightarrow & VU & & \\
\downarrow & & & & \downarrow & & \\
Q = [\text{Ker}(T)]^T & & \longrightarrow & & Q_t = \begin{pmatrix} Q \\ VU \end{pmatrix} & \rightarrow & G_t = [\text{Ker}(Q_t)]^T
\end{array}
$$

arriving at a final matrix G_t whose columns are the vectors which prescribe the nodes of the toric diagram.

What we wish to investigate below is how much procedural freedom we have at each arrow so as to ascertain the non-trivial toric dual theories. Hence, if A_1 is the matrix whither one arrives from a certain arrow, then we would like to find the most general transformation taking A_1 to another solution A_2 which would give rise to an identical theory. It is to this transformation that we shall refer as "freedom" at the particular step.

Superpotential: the matrices K and T

The solution of F-term equations gives rise to a dual cone $K_1 = K_{ij}$ defined by I vectors in Z^J. Of course, we can choose different parametres to solve the F-terms and arrive at another dual cone K_2. Then, K_1 and K_2, being integral cones, are equivalent if they are unimodularly related, i.e., $K_2^T = A \cdot K_1^T$ for $A \in GL(J, Z)$ such that $\det(A) = \pm 1$. Furthermore, the order of the I vectors in Z^J clearly does not matter, so we can permute them by a matrix S_I in the symmetric group $\mathcal{S}_I$. Thus far we have two freedoms, multiplication by A and S:

$$
K_2^T = A \cdot K_1^T \cdot S_I, \tag{5.3}
$$

and $K_{1,2}$ should give equivalent theories.

Now, from K_{ij} we can find its dual matrix $T_{j\alpha}$ (defining the cone T) where $\alpha = 1, 2, ..., c$ and c is the number of vectors of the cone T in Z^J, as constrained by

$$K \cdot T \geq 0 \tag{5.4}$$

and such that T also spans an integral cone. Notice that finding dual cones, as given in a algorithm in [48], is actually unique up to permutation of the defining vectors. Now considering the freedom of K_{ij} as in (5.3), let T_2 be the dual of K_2 and T_1 that of K_1, we have $K_2 \cdot T_2 = S_I^T \cdot K_1 \cdot A^T \cdot T_2 \geq 0$, which means that

$$T_1 = A^T \cdot T_2 \cdot S_c. \tag{5.5}$$

Note that here S_c is the permutation of the c vectors of the cone T in and not that of the dual cone in (5.3).

The Charge Matrix Q

The next step is to find the charge matrix $Q_{k\alpha}$ where $\alpha = 1, 2, ..., c$ and $k = 1, 2, ..., c - J$. This matrix is defined by

$$T \cdot Q^T = 0. \tag{5.6}$$

In the same spirit as the above discussion, from (5.5) we have $T_1 \cdot Q_1^T = A^T \cdot T_2 \cdot S_c \cdot Q_1^T = 0$. Because A^T is a invertible matrix, this has a solution when and only when $T_2 \cdot S_c \cdot Q_1^T = 0$. Of course this is equivalent to $T_2 \cdot S_c \cdot Q_1^T \cdot B_{kk'} = 0$ for some invertible $(c - J) \times (c - J)$ matrix $B_{kk'}$. So the freedom for matrix Q is

$$Q_2^T = S_c \cdot Q_1^T \cdot B. \tag{5.7}$$

We emphasize a difference from (5.4); there we required both matrices K and T to be integer where here (5.6) does not possess such a constraint. Thus the only condition for the matrix B is its invertibility.

Matter Content: the Matrices d, $\widetilde{V}$ and U

Now we move onto the D-term and the integral d_{li} matrix. The D-term equations are $d \cdot |X|^2 = 0$ for matter fields X. Obviously, any transformation on d by an invertible matrix $C_{L\times L}$ does not change the D-terms. Furthermore, any permutation S_I of the order the fields X, so long as it is consistent with the S_I in (5.3), is also game. In other words, we have the freedom:

$$d_2 = C \cdot d_1 \cdot S_I. \tag{5.8}$$

We recall that a matrix V is then determined from Δ, which is d with a row deleted due to the centre of mass degree of freedom. However, to not to spoil the above freedom enjoyed by matrix d in (5.8), we will make a slight amendment and define the matrix $\widetilde{V}_{lj}$ by

$$\widetilde{V} \cdot K^T = d. \tag{5.9}$$

Therefore, whereas in [14, 85] where $V \cdot K^T = \Delta$ was defined, we generalise V to $\widetilde{V}$ by (5.9). One obvious way to obtain $\widetilde{V}$ from V is to add one row such that the sum of every column is zero. However, there is a caveat: when there exists a vector h such that

$$h \cdot K^T = 0,$$

we have the freedom to add h to any row of $\widetilde{V}$. Thus finding the freedom of $\widetilde{V}_{lj}$ is a little more involved. From (5.3) we have $d_2 = \widetilde{V}_2 \cdot K_2^T = \widetilde{V}_2 \cdot A \cdot K_1^T \cdot S_I$ and $d_2 = C \cdot d_1 \cdot S_I = C \cdot \widetilde{V}_1 \cdot K_1^T \cdot S_I$. Because S_I is an invertible square matrix, we have $(\widetilde{V}_2 \cdot A - C \cdot \widetilde{V}_1) \cdot K_1^T = 0$, which means $\widetilde{V}_2 \cdot A - C \cdot \widetilde{V}_1 = CH_{K_1}$ for a matrix H constructed by having the aforementioned vectors h as its columns. When K^T has maximal rank, H is zero and this is in fact the more frequently encountered situation. However, when K^T is not maximal rank, so as to give non-trivial solutions of h, we have that $\widetilde{V}_1$ and $\widetilde{V}_2$ are equivalent if

$$\widetilde{V}_2 = C \cdot (\widetilde{V}_1 + H_{K_1}) \cdot A^{-1}. \tag{5.10}$$

Moving on to the matrix $U_{j\alpha}$ defined by

$$U \cdot T^T = \mathbb{I}_{jj'}, \tag{5.11}$$

we have from (5.5) $\mathbb{I}_{jj'} = U_1 \cdot T_1^T = U_1 \cdot S_c^T \cdot T_2^T \cdot A$, whence $A^{-1} = U_1 \cdot S_c^T \cdot T_2^T$ and $\mathbb{I} = A \cdot U_1 \cdot S_c^T \cdot T_2^T$. This gives $(A \cdot U_1 \cdot S_c^T - U_2) \cdot T_2^T = 0$ which has a solution $A \cdot U_1 \cdot S_c^T - U_2 = H_{T_2}$ where $H_{T_2} \cdot T_2^T = 0$ is precisely as defined in analogy of the H above. Therefore the freedom on U is subsequently

$$U_2 = A \cdot (U_1 - H_{T_1}) \cdot S_c^T, \tag{5.12}$$

where $H_{T_1} = A^{-1} H_{T_2} (S_c^T)^{-1}$ and $H_{T_1} \cdot T_1^T = (A^{-1} H_{T_2} (S_c^T)^{-1})(S_c^T \cdot T_2^T \cdot A) = 0$. Finally using (5.10) and (5.12), we have

$$(\tilde{V}_2 \cdot U_2) = C \cdot (\tilde{V}_1 + H_{K_1}) \cdot A^{-1} \cdot A \cdot (U_1 - H_{T_1}) \cdot S_c^T = C \cdot (\tilde{V}_1 + H_{K_1})(U_1 - H_{T_1}) \cdot S_c^T, \tag{5.13}$$

determining the freedom of the relevant combination $(\tilde{V} \cdot U)$.

Let us pause for an important observation that in most cases $H_{K_1} = 0$, as we shall see in the examples later. From (5.6), which propounds the existence of a non-trivial nullspace for T, we see that one can indeed obtain a non-trivial H_{T_1} in terms of the combinations of the rows of the charge matrix Q, whereby simplifying (5.13) to

$$(\tilde{V}_2 \cdot U_2) = C \cdot (\tilde{V}_1 \cdot U_1 + H_{VU_1}) \cdot S_c^T, \tag{5.14}$$

where every row of H_{VU_1} is linear combination of rows of Q_1 and the sum of its columns is zero.

Toric Data: the Matrices Q_t and G_t

At last we come to $\tilde{Q}_t$, which is given by adjoining Q and $\tilde{V} \cdot U$. The freedom is of course, by combining all of our results above,

$$(\tilde{Q}_t)_2 = \begin{pmatrix} Q_2 \\ \tilde{V}_2 \cdot U_2 \end{pmatrix} = \begin{pmatrix} B^T \cdot Q_1 \cdot S_c^T \\ C \cdot (\tilde{V}_1 \cdot U_1 + H_{VU_1}) \cdot S_c^T \end{pmatrix} = \begin{pmatrix} B^T \cdot Q_1 \\ C \cdot (\tilde{V}_1 \cdot U_1 + H_{VU_1}) \end{pmatrix} \cdot S_c^T \tag{5.15}$$

Now $\tilde{Q}_t$ determines the nodes of the toric diagram $(G_t)_{p\alpha}$ $(p = 1, 2, .., (c-(L-1)-J)$ and $\alpha = 1, 2, ..., c)$ by

$$Q_t \cdot G_t^T = 0; \tag{5.16}$$

The columns of G_t then describes the toric diagram of the algebraic variety for the vacuum moduli space and is the output of the algorithm. From (5.16) and (5.15) we find that if $(\tilde{Q}_t)_1 \cdot (G_t)_1^T = 0$, i.e., $Q_1 \cdot (G_t)_1^T = 0$ and $\tilde{V}_1 \cdot U_1 \cdot (G_t)_1^T = 0$, we automatically have the freedom $(\tilde{Q}_t)_2 \cdot (S_c^T)^{-1} \cdot (\tilde{G}_t)_1^T = 0$. This means that at most we can have

$$(G_t)_2^T = (S_c^T)^{-1} \cdot (G_t)_1^T \cdot D, \tag{5.17}$$

where D is a $GL(c - (L-1) - J, Z)$ matrix with $\det(D) = \pm 1$ which is exactly the unimodular freedom for toric data as given by Theorem 5.3.1.

One immediate remark follows. From (5.16) we obtain the nullspace of Q_t in Z^c. It seems that we can choose an arbitrary basis so that D is a $GL(c - (L-1) - J, Z)$ matrix with the only condition that $\det(D) \neq 0$. However, this is not stringent enough: in fact, when we find cokernel G_t, we need to find the *integer basis* for the null space, i.e., we need to find the basis such that any integer null vector can be decomposed into a linear combination of the columns of G_t. If we insist upon such a choice, the only remaining freedom[2] is that $\det(D) = \pm 1$, viz, unimodularity.

[2] We would like to express our gratitude to M. Douglas for clarifying this point to us.

5.4.2 Freedom and Ambiguity in the Reverse Algorithm

Having analysed the equivalence conditions in last subsection, culminating in (5.15) and (5.17), we now proceed in the opposite direction and address the ambiguities in the reverse algorithm.

The Toric Data: G_t

We note that the G_t matrix produced by the forward algorithm is not minimal in the sense that certain columns are repeated, which after deletion, constitute the toric diagram. Therefore, in our reverse algorithm, we shall first encounter such an ambiguity in deciding which columns to repeat when constructing G_t from the nodes of the toric diagram. This so-called *repetition ambiguity* was discussed in [85] and different choices of repetition may indeed give rise to different gauge theories. It was pointed out (*loc. cit.*) that arbitrary repetition of the columns certainly does not guarantee physicality. By physicality we mean that the gauge theory arrived at the end of the day should be *physical* in the sense of still being a D-brane world-volume theory. What we shall focus here however, is the inherent symmetry in the toric diagram, given by (5.17), that gives rise to the same theory. This is so that we could find truly inequivalent *physical* gauge theories not related by such a transformation as (5.17).

The Charge Matrix: from G_t to Q_t

From (5.16) we can solve for Q_t. However, for a given G_t, in principle we can have two solutions $(Q_t)_1$ and $(Q_t)_2$ related by

$$(Q_t)_2 = P(Q_t)_1, \tag{5.18}$$

where P is a $p \times p$ matrix with p the number of rows of Q_t. Notice that the set of such transformations P is much larger than the counterpart in the forward algorithm given in (5.15). This is a second source of ambiguity in the reverse algorithm. More explicitly, we have the freedom to arbitrarily divide the Q_t into two parts, viz., the D-term part $\widetilde{V}U$ and the F-term part Q. Indeed one may find a matrix P such that $(Q_t)_1$ and $(Q_t)_2$ satisfy (5.18) but not matrices B and C in order to satisfy (5.15). Hence different choices of Q_t and different division therefrom into D and F-term parts give rise to different gauge theories. This is what we called *FD Ambiguity* in [85]. Again, arbitrary division of the rows of Q_t was pointed out to not to ensure physicality. As with the discussion on the repetition ambiguity above, what we shall pin down is the freedom due to the linear algebra and not the choice of division.

The Dual Cone and Superpotential: from Q to K

The nullspace of Q is the matrix T. The issue is the same as discussed at the paragraph following (5.17) and one can uniquely determine T by imposing that its columns give an integral span of the nullspace. Going further from T to its dual K, this is again a unique procedure (while integrating back from K to obtain the superpotential is certainly not). In summary then, these two steps give no sources for ambiguity.

The Matter Content: from $\widetilde{V}U$ to d matrix

The d matrix can be directly calculated as [85]

$$d = (\widetilde{V}U) \cdot T^T \cdot K^T. \tag{5.19}$$

Substituting the freedoms in (5.3), (5.5) and (5.13) we obtain

$$\begin{aligned} d_2 &= (\widetilde{V}_2 \cdot U_2) \cdot T_2^T \cdot K_2^T = C \cdot [(\widetilde{V}_1 \cdot U_1) + H_{VU_1}] \cdot S_c^T \cdot (S_c^T)^{-1} \cdot T_1^T \cdot A^{-1} \cdot A \cdot K_1^T \cdot S_I \\ &= C \cdot (\widetilde{V}_1 \cdot U_1) \cdot T_1^T \cdot K_1^T \cdot S_I + C \cdot H_{VU_1} \cdot T_1^T \cdot K_1^T \cdot S_I = C \cdot d_1 \cdot S_I, \end{aligned}$$

which is exactly formula (5.8). This means that the matter matrices are equivalent up to a transformation and there is no source for extra ambiguity.

5.5 Application: Phases of $Z_3 \times Z_3$ Resolutions

In [85] we developed an algorithmic outlook to the Inverse Procedure and applied it to the construction of gauge theories on the toric singularities which are partial resolutions of $Z_3 \times Z_3$. The non-uniqueness of the method allowed one to obtain many different gauge theories starting from the same toric variety, theories to which we referred as being toric duals. The non-uniqueness mainly comes from three sources: (i) the repetition of the vectors in the toric data G_t (Repetition Ambiguity), (ii) the different choice of the null space basis of Q_t and (iii) the different divisions of the rows of Q_t (F-D Ambiguity). Many of the possible choices in the above will generate unphysical gauge theories, i.e., not world-volume theories of D-brane probes. We have yet to catalogue the exact conditions which guarantee physicality.

However, *Partial Resolution* of Abelian orbifolds, which stays within subsectors of the latter theory, does indeed constrain the theory to be physical. To these physical theories we shall refer as **phases** of the partial resolution. As discussed in [85] any k-dimensional toric diagram can be embedded into Z_n^{k-1} for sufficiently large n, one obvious starting point to obtain different phases of a D-brane gauge theory is to try various values of n. We leave some relevances of general n to the Appendix. However, because the algorithm of finding dual cones becomes prohibitively computationally intensive even for $n \geq 4$, this approach may not be immediately fruitful.

Yet armed with Theorem 5.3.1 we have an alternative. We can certainly find all possible unimodular transformations of the given toric diagram which still embeds into the same Z_n^{k-1} and then perform the inverse algorithm on these various *a fortiori* equivalent toric data and observe what physical theories we obtain at the end of the day. In our two examples in §1, we have essentially done so; in those cases we found that two inequivalent gauge theory data corresponded to two unimodularly equivalent toric data for the examples of Z_5-orbifold and the zeroth Hirzebruch surface F_0.

The strategy lays itself before us. Let us illustrate with the same examples as was analysed in [85], namely the partial resolutions of $C^3/(Z_3 \times Z_3)$, i.e., F_0 and the toric del Pezzo surfaces $dP_{0,1,2,3}$. We need to (i) find all $SL(3; Z)$ transformations of the toric diagram G_t of these five singularities that still remain as sub-diagrams of that of $Z_3 \times Z_3$ and then perform the inverse algorithm; therefrom, we must (ii) select theories not related by any of the freedoms we have discussed above and summarised in (5.15).

5.5.1 Unimodular Transformations within $Z_3 \times Z_3$

We first remind the reader of the G_t matrix of $Z_3 \times Z_3$ given in Figure 5-1, its columns are given by vectors: $(0,0,1)$, $(1,-1,1)$, $(0,-1,2)$, $(-1,1,1)$, $(-1,0,2)$, $(-1,-1,3)$, $(1,-1,1)$, $(-1,2,0)$, $(1,0,0)$, $(0,1,0)$. Step (i) of our above strategy can be immediately performed. Given the toric data of one of the resolutions G'_t with x columns, we select x from the above 10 columns of G_t and check whether any $SL(3;Z)$ transformation relates any permutation thereof unimodularly to G'_t. We shall at the end find that there are three different cases for F_0, five for dP^0, twelve for dP_1, nine for dP_2 and only one for dP_3. The (unrepeated) G_t matrices are as follows:

$(F_0)_1$	$(0,0,1),(1,-1,1),(-1,1,1),(-1,0,2),(1,0,0)$
$(F_0)_2$	$(0,0,1),(0,-1,2),(0,1,0),(-1,0,2),(1,0,0)$
$(F_0)_3$	$(0,0,1),(1,-1,1),(-1,1,1),(0,-1,2),(0,1,0)$
$(dP_0)_1$	$(0,0,1),(1,0,0),(0,-1,2),(-1,1,1)$
$(dP_0)_2$	$(0,0,1),(1,0,0),(-1,-1,3),(0,1,0)$
$(dP_0)_3$	$(0,0,1),(-1,2,0),(1,-1,1),(0,-1,2)$
$(dP_0)_4$	$(0,0,1),(0,1,0),(1,-1,1),(-1,0,2)$
$(dP_0)_5$	$(0,0,1),(2,-1,0),(-1,1,1),(-1,0,2)$
$(dP_1)_1$	$(1,0,0),(0,1,0),(-1,1,1),(0,-1,2),(0,0,1)$
$(dP_1)_2$	$(-1,-1,3),(0,-1,2),(1,0,0),(0,1,0),(0,0,1)$
$(dP_1)_3$	$(0,-1,2),(1,-1,1),(1,0,0),(-1,1,1),(0,0,1)$
$(dP_1)_4$	$(0,-1,2),(1,-1,1),(0,1,0),(-1,2,0),(0,0,1)$
$(dP_1)_5$	$(0,-1,2),(1,-1,1),(0,1,0),(-1,0,2),(0,0,1)$
$(dP_1)_6$	$(0,-1,2),(1,-1,1),(-1,2,0),(-1,1,1),(0,0,1)$
$(dP_1)_7$	$(0,-1,2),(1,0,0),(-1,1,1),(-1,0,2),(0,0,1)$
$(dP_1)_8$	$(1,-1,1),(2,-1,0),(-1,1,1),(-1,0,2),(0,0,1)$
$(dP_1)_9$	$(1,-1,1),(1,0,0),(0,1,0),(-1,0,2),(0,0,1)$
$(dP_1)_{10}$	$(1,-1,1),(0,1,0),(-1,1,1),(-1,0,2),(0,0,1)$
$(dP_1)_{11}$	$(2,-1,0),(1,0,0),(-1,1,1),(-1,0,2),(0,0,1)$
$(dP_1)_{12}$	$(-1,-1,3),(1,0,0),(0,1,0),(-1,0,2),(0,0,1)$
$(dP_2)_1$	$(2,-1,0),(1,-1,1),(-1,0,2),(-1,1,1),(1,0,0),(0,0,1)$
$(dP_2)_2$	$(-1,-1,3),(0,-1,2),(1,0,0),(0,1,0),(-1,0,2),(0,0,1)$
$(dP_2)_3$	$(0,-1,2),(1,-1,1),(1,0,0),(0,1,0),(-1,1,1),(0,0,1)$
$(dP_2)_4$	$(0,-1,2),(1,-1,1),(1,0,0),(0,1,0),(-1,0,2),(0,0,1)$
$(dP_2)_5$	$(0,-1,2),(1,-1,1),(1,0,0),(-1,1,1),(-1,0,2),(0,0,1)$
$(dP_2)_6$	$(0,-1,2),(1,-1,1),(0,1,0),(-1,2,0),(-1,1,1),(0,0,1)$
$(dP_2)_7$	$(0,-1,2),(1,-1,1),(0,1,0),(-1,1,1),(-1,0,2),(0,0,1)$
$(dP_2)_8$	$(0,-1,2),(1,0,0),(0,1,0),(-1,1,1),(-1,0,2),(0,0,1)$
$(dP_2)_9$	$(1,-1,1),(1,0,0),(0,1,0),(-1,1,1),(-1,0,2),(0,0,1)$
dP_3	$(0,-1,2),(1,-1,1),(1,0,0),(0,1,0),(-1,1,1),(-1,0,2),(0,0,1)$

The reader is referred to Figure 5-2 to Figure 5-6 for the toric diagrams of the data above. The vigilant would of course recognise $(F_0)_1$ to be Case 1 and $(F_0)_2$ as Case 2 of Figure 5-1 as discussed in §2 and furthermore $(dP_{0,1,2,3})_1$ to be the cases addressed in [85].

5.5.2 Phases of Theories

The Inverse Algorithm can then be readily applied to the above toric data; of the various unimodularly equivalent toric diagrams of the del Pezzo surfaces and the zeroth Hirzebruch, the details of which fields remain massless at each node (in the notation of [85]) are also presented in those figures immediately referred to above.

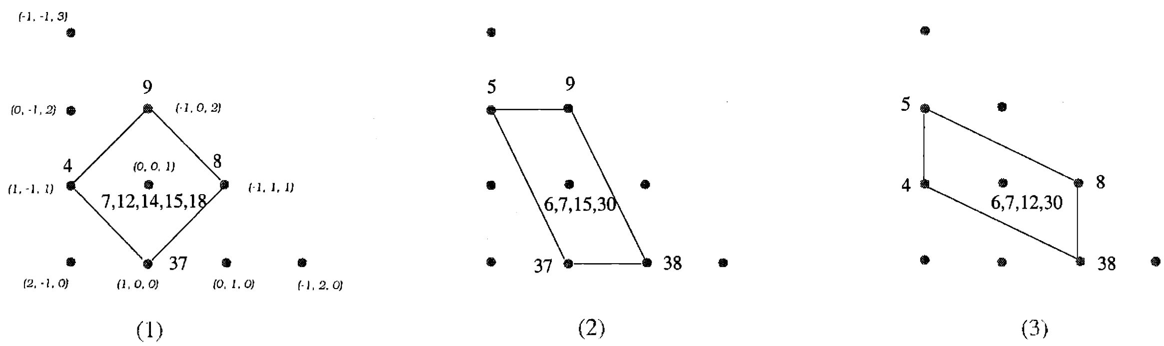


Figure 5-2: The 3 equivalent representations of the toric diagram of the zeroth Hirzebruch surface as a resolution of $Z_3 \times Z_3$. We see that (2) and (3) are related by a reflection about the 45^o line (a symmetry inherent in the parent $Z_3 \times Z_3$ theory) and we have the two giving equivalent gauge theories as expected.

Subsequently, we arrive at a number of D-brane gauge theories; among them, all five cases for dP^0 are equivalent (which is in complete consistency with the fact that dP^0 is simply C^3/Z_3 and there is only one nontrivial theory for this orbifold, corresponding to the decomposition $\mathbf{3} \to 1+1+1$). For dP_1, all twelve cases give back to same gauge theory (q.v. Figure 5 of [85]). For F_0, the three cases give two inequivalent gauge theories as given in §2. Finally for dP_2, the nine cases again

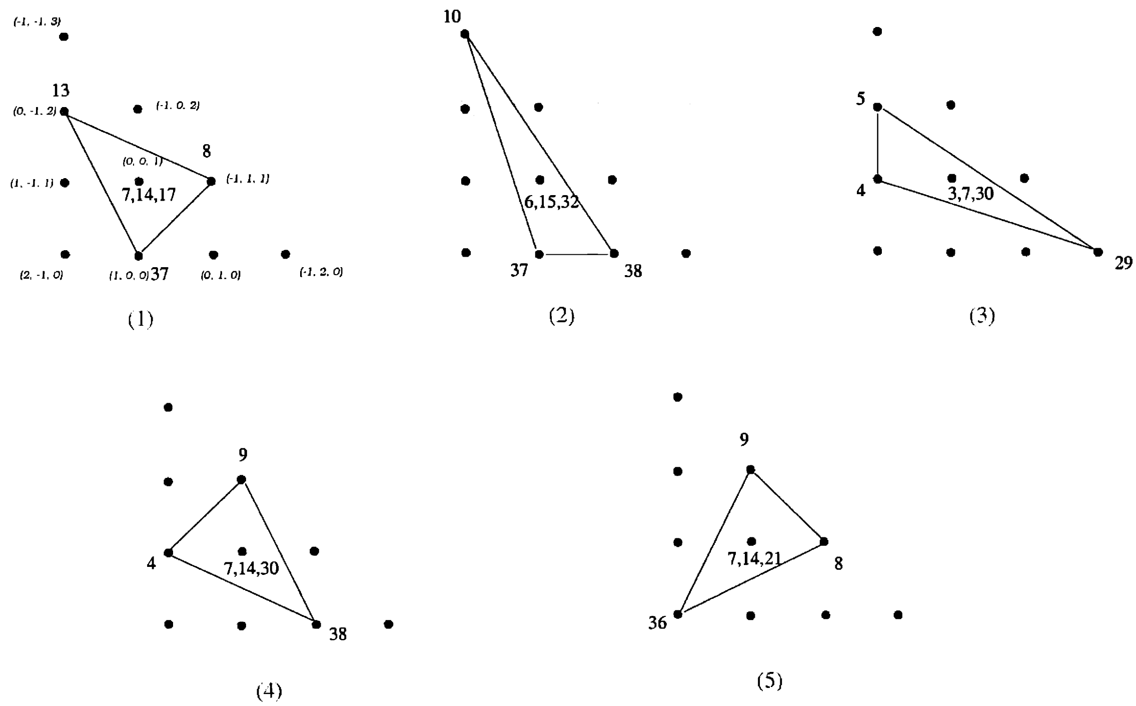


Figure 5-3: The 5 equivalent representations of the toric diagram of the zeroth del Pezzo surface as a resolution of $Z_3 \times Z_3$. Again (1) and (4) (respectively (2) and (3)) are related by the 45^o reflection, and hence give equivalent theories. In fact further analysis shows that all 5 are equivalent.

give two different theories. For reference we tabulate the D-term matrix d and F-term matrix K^T below. If more than 1 theory are equivalent, then we select one representative from the list, the matrices for the rest are given by transformations (5.3) and (5.8).

Sing	Matter Content d	Superpotential

$(F_0)_1$

	1	2	3	4	5	6	7	8	9	10	11	12
A	$\bar{1}$	0	$\bar{1}$	0	$\bar{1}$	0	1	1	$\bar{1}$	0	1	1
B	0	$\bar{1}$	0	$\bar{1}$	1	0	0	0	1	0	0	0
C	0	1	0	1	0	1	$\bar{1}$	$\bar{1}$	0	1	$\bar{1}$	$\bar{1}$
D	1	0	1	0	0	$\bar{1}$	0	0	0	$\bar{1}$	0	0

Superpotential: $X_1X_8X_{10} - X_3X_7X_{10} - X_2X_8X_9 - X_1X_6X_{12} + X_3X_6X_{11} + X_4X_7X_9 + X_2X_5X_{12} - X_4X_5X_{11}$

$(F_0)_{2,3}$

	X_{112}	Y_{122}	Y_{222}	Y_{111}	Y_{211}	X_{121}	X_{212}	X_{221}
A	$\bar{1}$	0	0	1	1	0	$\bar{1}$	0
B	1	$\bar{1}$	$\bar{1}$	0	0	0	1	0
C	0	0	0	$\bar{1}$	$\bar{1}$	1	0	1
D	0	1	1	0	0	$\bar{1}$	0	$\bar{1}$

Superpotential: $\epsilon^{ij}\epsilon^{kl}X_{i\ 12}Y_{k\ 22}X_{j\ 21}Y_{l\ 11}$

(dP_0)

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
A	$\bar{1}$	0	$\bar{1}$	0	$\bar{1}$	0	1	1	1
B	0	1	0	1	0	1	$\bar{1}$	$\bar{1}$	$\bar{1}$
C	1	$\bar{1}$	1	$\bar{1}$	1	$\bar{1}$	0	0	0

Superpotential: $X_1X_4X_9 - X_4X_5X_7 - X_2X_3X_9 - X_1X_6X_8 + X_2X_5X_8 + X_3X_6X_7$

(dP_1)

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
A	$\bar{1}$	0	0	$\bar{1}$	0	0	0	1	0	1
B	1	$\bar{1}$	0	0	0	$\bar{1}$	0	0	1	0
C	0	0	1	0	1	0	1	$\bar{1}$	$\bar{1}$	$\bar{1}$
D	0	1	$\bar{1}$	1	$\bar{1}$	1	$\bar{1}$	0	0	0

Superpotential: $X_2X_7X_9 - X_3X_6X_9 - X_4X_8X_7 - X_1X_2X_5X_{10} + X_3X_4X_{10} + X_1X_5X_6X_8$

$(dP_2)_1$

	1	2	3	4	5	6	7	8	9	10	11	12	13
A	$\bar{1}$	0	0	$\bar{1}$	0	$\bar{1}$	0	1	0	0	0	1	1
B	0	0	$\bar{1}$	0	$\bar{1}$	1	0	0	0	1	0	0	0
C	0	0	1	0	1	0	1	$\bar{1}$	$\bar{1}$	0	1	$\bar{1}$	$\bar{1}$
D	1	$\bar{1}$	0	0	0	0	0	0	1	$\bar{1}$	0	0	0
E	0	1	0	1	0	0	$\bar{1}$	0	0	0	$\bar{1}$	0	0

Superpotential: $X_2X_9X_{11} - X_9X_3X_{10} - X_4X_8X_{11} - X_1X_2X_7X_{13} + X_{13}X_3X_6 - X_5X_{12}X_6 + X_1X_5X_8X_{10} + X_4X_7X_{12}$

$(dP_2)_2$

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}
A	$\bar{1}$	0	$\bar{1}$	0	0	0	1	0	0	0	1
B	1	$\bar{1}$	0	0	$\bar{1}$	0	0	0	1	0	0
C	0	0	1	$\bar{1}$	0	1	0	0	$\bar{1}$	0	0
D	0	0	0	0	0	$\bar{1}$	$\bar{1}$	1	0	1	0
E	0	1	0	1	1	0	0	$\bar{1}$	0	$\bar{1}$	$\bar{1}$

Superpotential: $X_5X_8X_6X_9 + X_1X_2X_{10}X_7 + X_{11}X_3X_4 - X_4X_{10}X_6 - X_2X_8X_7X_3X_9 - X_{11}X_1X_5$

$(dP_3)_1$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A	$\bar{1}$	0	0	0	1	0	0	1	$\bar{1}$	0	0	1	$\bar{1}$	0
B	0	0	$\bar{1}$	1	0	$\bar{1}$	0	0	0	0	0	0	1	0
C	1	$\bar{1}$	0	$\bar{1}$	0	0	0	0	0	0	0	0	0	1
D	0	0	1	0	0	0	0	$\bar{1}$	0	$\bar{1}$	1	0	0	0
E	0	0	0	0	$\bar{1}$	1	1	0	0	1	0	$\bar{1}$	0	$\bar{1}$
F	0	1	0	0	0	0	$\bar{1}$	0	1	0	$\bar{1}$	0	0	0

Superpotential: $X_3X_8X_{13} - X_8X_9X_{11} - X_5X_6X_{13} - X_1X_3X_4X_{10}X_{12} + X_7X_9X_{12} + X_1X_2X_5X_{10}X_{11} + X_4X_6X_{14} - X_2X_7X_{14}$

The matter content for these above theories are represented as quiver diagrams in Figure 5-7 (multi-valence arrows are labelled with a number) and the superpotentials, in the table below.

In all of the above discussions, we have restricted ourselves to the cases of $U(1)$ gauge groups, i.e., with only a single brane probe; this is because such is the only case to which the toric technique can be applied. However, after we obtain the matter contents and superpotential for $U(1)$ gauge groups, we should have some idea for multi-brane probes. One obvious generalization is to replace the $U(1)$ with $SU(N)$

gauge groups directly. For the matter content, the generalization is not so easy. A field with charge $(1,-1)$ under gauge groups $U(1)_A \times U(1)_B$ and zero for others generalised to a bifundamental $(N,\bar{N})$ of $SU(N)_A \times SU(N)_B$. However, for higher charges, e.g., charge 2, we simply do not know what should be the generalization in the multi-brane case (for a discussion on generalised quivers cf. e.g. [89]). Furthermore, a field with zero charge under all $U(1)$ groups, generalises to an adjoint of one $SU(N)$ gauge group in the multi-brane case, though we do not know which one.

The generalization of the superpotential is also not so straight-forward. For example, there is a quartic term in the conifold with nonabelian gauge group [83, 79], but it disappears when we go to the $U(1)$ case. The same phenomenon can happen when treating the generic toric singularity.

For the examples we give in this paper however, we do not see any obvious obstruction in the matter contents and superpotential; they seem to be special enough to be trivially generalized to the multi-brane case; they are all charge ± 1 under no more than 2 groups. We simply replace $U(1)$ with $SU(N)$ and $(1,-1)$ fields with bifundamentals while keeping the superpotential invariant. Generalisations to multi-brane stack have also been discussed in [84].

5.6 Discussions and Prospects

It is well-known that in the study of the world-volume gauge theory living on a D-brane probing an orbifold singularity C^3/Γ, different choices of decomposition into irreducibles of the space-time action of Γ lead to different matter content and interaction in the gauge theory and henceforth different moduli spaces (as different algebraic varieties). This strong relation between the decomposition and algebraic variety has been shown explicitly for Abelian orbifolds in [92]. It seems that there is only one gauge theory for each given singularity.

A chief motivation and purpose of this paper is the realisation that the above strong statement can not be generalised to arbitrary (non-orbifold) singularities and in particular toric singularities. It is possible that there are several gauge theories on the D-brane probing the same singularity. The moduli space of these inequivalent theories are indeed by construction the same, as dictated by the geometry of the singularity.

In analogy to the freedom of decomposition into irreps of the group action in the orbifold case, there too exists a freedom in toric singularities: any toric diagram is defined only up to a unimodular transformation (Theorem 5.3.1). We harness this toric isomorphism as a tool to create inequivalent gauge theories which live on the D-brane probe and which, by construction, flow to the same (toric) moduli space in the IR.

Indeed, these theories constitute another sub-class of examples of *toric duality* as proposed in [85]. A key point to note is that unlike the general case of the duality (such as F-D ambiguities and repetition ambiguities as discussed therein) of which we have hitherto little control, these particular theories are all physical (i.e., guaranteed to be world-volume theories) by virtue of their being obtainable from the canonical method of partial resolution of Abelian orbifolds. We therefore refer to them as *phases* of partial resolution.

As a further tool, we have re-examined the Forward and Inverse Algorithms developed in [84, 85, 14] of extracting the gauge theory data and toric moduli space data from each other. In particular we have taken the pains to show what *degree of freedom* can one have at each step of the Algorithm. This will serve to discriminate whether or not two theories are physically equivalent given their respective matrices at each step.

Thus equipped, we have re-studied the partial resolutions of the Abelian orbifold $C^3/(Z_3 \times Z_3)$, namely the 4 toric del Pezzo surfaces $dP_{0,1,2,3}$ and the zeroth Hirzebruch surface F_0. We performed all possible $SL(3;Z)$ transformation of these toric diagrams which are up to permutation still embeddable in $Z_3 \times Z_3$ and subsequently initiated the Inverse Algorithm therewith. We found at the end of the day, in addition to the physical theories for these examples presented in [85], an additional one for both F_0 and dP_2. Further embedding can of course be done, viz., into $Z_n \times Z_n$ for $n > 3$; it is expected that more phases would arise for these computationally prohibitive cases, for example for dP_3.

A clear goal awaits us: because for the generic (non-orbifold) toric singularity there is no concrete concept corresponding to the different decomposition of group action, we do not know at this moment how to classify the phases of toric duality. We certainly wish, given a toric singularity, to know (a) how many inequivalent gauge theory are there and (b) what are the corresponding matter contents and superpotential. It will be a very interesting direction for further investigation.

Many related questions also arise. For example, by the AdS/CFT correspondence, we need to understand how to describe these different gauge theories on the supergravity side while the underline geometry is same. Furthermore the dP^2 theory can be described in the brane setup by (p, q)-5 brane webs [36], so we want to ask how to understand these different phases in such brane setups. Understanding these will help us to get the gauge theory in higher del Pezzo surface singularities.

Another very pertinent issue is to clarify the meaning of "toric duality." So far it is merely an equivalence of moduli spaces of gauge theories in the IR. It would be very nice if we could make this statement stronger. For example, could we find the explicit mappings between gauge invariant operators of various toric-dual theories? Indeed, we believe that the study of toric duality and its phase structure is worth further pursuit.

5.7 Appendix: Gauge Theory Data for $Z_n \times Z_n$

For future reference we include here the gauge theory data for the $Z_n \times Z_n$ orbifold, so that, as mentioned in [85], any 3-dimensional toric singularity may exist as a partial resolution thereof.

We have $3n^2$ fields denoted as X_{ij}, Y_{ij}, Z_{ij} and choose the decomposition $\mathbf{3} \rightarrow (1,0) + (0,1) + (-1,-1)$. The matter content (and thus the d matrix) is well-known from standard brane box constructions, hence we here focus on the superpotential

[91] (and thus the K matrix):

$$X_{ij}Y_{i(j+1)}Z_{(i+1)(j+1)} - Y_{ij}X_{(i+1)j}Z_{(i+1)(j+1)},$$

from which the F-terms are

$$\begin{array}{lrcl} \frac{\partial W}{\partial X_{ij}}: & Y_{i(j+1)}Z_{(i+1)(j+1)} & = & Z_{i(j+1)}Y_{(i-1)j} \\ \frac{\partial W}{\partial Y_{ij}}: & Z_{(i+1)j}X_{i(j-1)} & = & X_{(i+1)j}Z_{(i+1)(j+1)} \\ \frac{\partial W}{\partial Z_{(i+1)(j+1)}}: & X_{ij}Y_{i(j+1)} & = & Y_{ij}X_{(i+1)j}. \end{array} \tag{5.20}$$

Now let us solve (5.20). First we have $Y_{i(j+1)} = Y_{ij}X_{(i+1)j}/X_{ij}$. Thus if we take Y_{i0} and X_{ij} as the independent variables, we have

$$Y_{i(j+1)} = \frac{\prod_{l=0}^{j} X_{(i+1)l}}{\prod_{l=0}^{j} X_{il}} Y_{i0}. \tag{5.21}$$

There is of course the periodicity which gives

$$Y_{in} = Y_{i0} \Longrightarrow \prod_{l=0}^{n-1} X_{(i+1)l} = \prod_{l=0}^{n-1} X_{il}. \tag{5.22}$$

Next we use X_{ij} to solve the Z_{ij} as $Z_{i(j+1)} = Z_{ij}X_{(i-1)(j-1)}/X_{ij}$, whence

$$Z_{i(j+1)} = \frac{\prod_{l=0}^{j} X_{(i-1)(l-1)}}{\prod_{l=0}^{j} X_{il}} Z_{i0}. \tag{5.23}$$

As above,

$$Z_{in} = Z_{i0} \Longrightarrow \prod_{l=0}^{n-1} X_{(i-1)(l-1)} = \prod_{l=0}^{n-1} X_{il}. \tag{5.24}$$

Putting the solution of Y, Z into the first equation of (5.20) we get

$$\frac{\prod_{l=0}^{j} X_{(i+1)l}}{\prod_{l=0}^{j} X_{il}} Y_{i0} \frac{\prod_{l=0}^{j} X_{(i)(l-1)}}{\prod_{l=0}^{j} X_{(i+1)l}} Z_{(i+1)0} = \frac{\prod_{l=0}^{j} X_{(i-1)(l-1)}}{\prod_{l=0}^{j} X_{il}} Z_{i0} \frac{\prod_{l=0}^{j-1} X_{il}}{\prod_{l=0}^{j-1} X_{(i-1)l}} Y_{(i-1)0},$$

which can be simplified as $Y_{i0}Z_{(i+1)0}X_{i(n-1)} = Z_{i0}Y_{(i-1)0}X_{(i-1)(n-1)}$, or $X_{i(n-1)} =$

$X_{(i-1)(n-1)}\frac{Y_{(i-1)0}}{Y_{i0}}\frac{Z_{i0}}{Z_{(i+1)0}}$. From this we solve

$$X_{i(n-1)} = X_{0(n-1)}\prod_{l=0}^{i-1}\frac{Y_{l0}}{Y_{(l+1)0}}\frac{Z_{(l+1)0}}{Z_{(l+2)0}}. \tag{5.25}$$

The periodicity gives

$$\prod_{l=0}^{n-1}\frac{Y_{l0}}{Y_{(l+1)0}}\frac{Z_{(l+1)0}}{Z_{(l+2)0}} = 1. \tag{5.26}$$

Now we have the independent variables Y_{i0} Z_{i0} and X_{ij} for $j \neq n-1$ and $X_{0(n-1)}$, plus three constraints (5.22) (5.24) (5.26). In fact, considering the periodic condition for X, (5.22) is equivalent to (5.24). Furthermore considering the periodic conditions for Z_{i0} and Y_{i0}, (5.26) is trivial. So we have only one constraint. Putting the expression (5.25) into (5.22) we get $\prod_{l=0}^{n-2} X_{(i+1)l}\frac{Y_{i0}}{Y_{(i+1)0}}\frac{Z_{(i+1)0}}{Z_{(i+2)0}} = \prod_{l=0}^{n-2} X_{il} \Rightarrow \prod_{l=0}^{n-2} X_{(i+1)l}\frac{1}{Y_{(i+1)0}Z_{(i+2)0}} = \prod_{l=0}^{n-2} X_{il}\frac{1}{Y_{i0}Z_{(i+1)0}}$.

From this we can solve the $X_{i(n-1)}$ for $i \neq 0$ as

$$X_{i(n-2)} = (\prod_{l=0}^{n-2} X_{0l})\frac{Y_{i0}Z_{(i+1)0}}{Y_{00}Z_{10}}(\prod_{l=0}^{n-2} X_{il})^{-1}. \tag{5.27}$$

The periodic condition does not give new constraints.

Now we have finished solving the F-term and can summarise the results into the K-matrix. We use the following independent variables: Z_{i0}, Y_{i0} for $i = 0, 1, ..., n-1$; X_{ij} for $i = 0, 1, ..., n-1$ $j = 0, 1, ..., n-3$ and $X_{0(n-2)}$ $X_{0(n-1)}$, so the total number of variables is $2n + n(n-2) + 2 = n^2 + 2$. This is usually too large to calculate. For example, even when $n = 4$, the K matrix is 48×18. The standard method to find the dual cone T from K needs to analyse some $48!/(17!31!)$ vectors, which is computationally prohibitive.

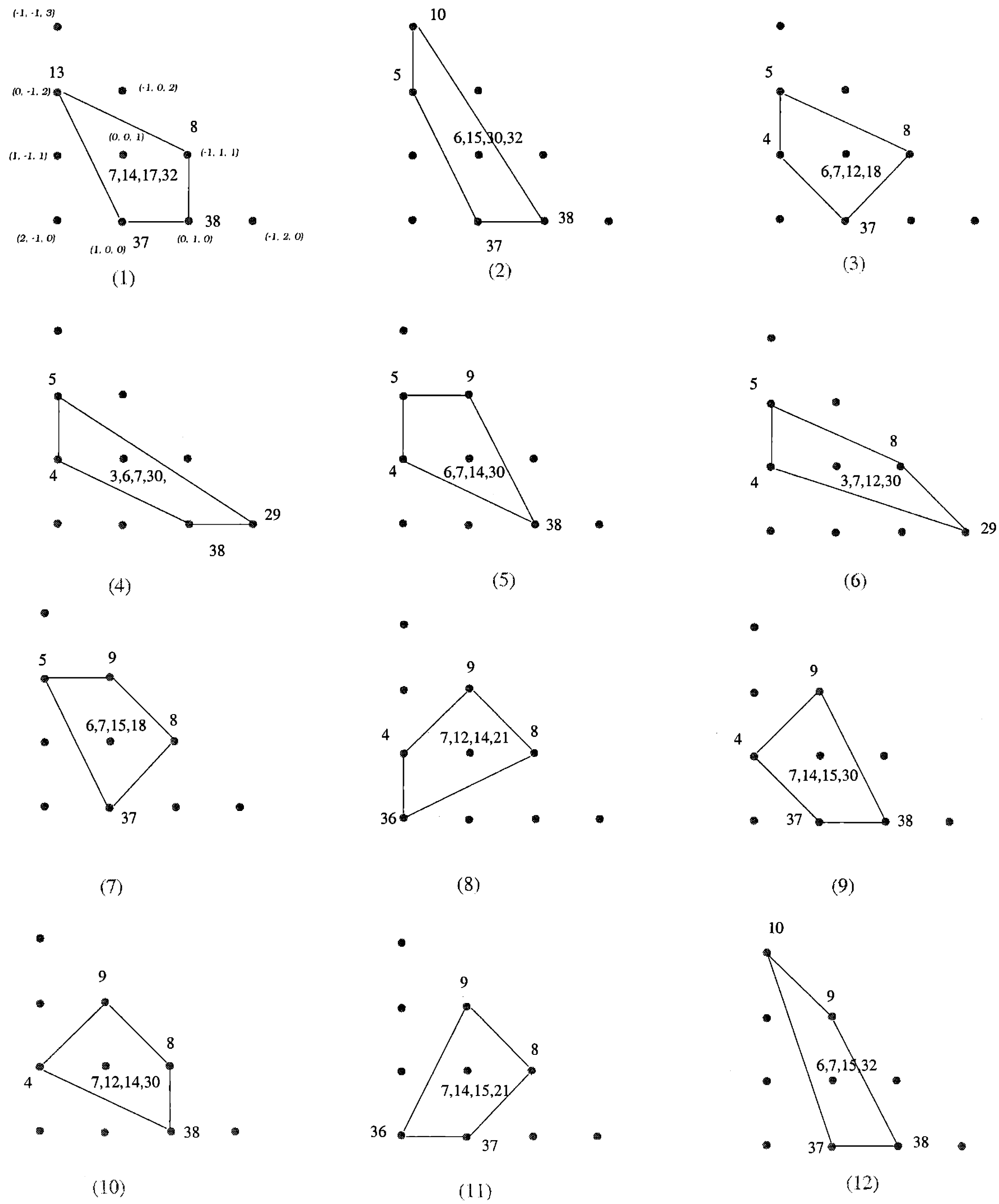


Figure 5-4: The 12 equivalent representations of the toric diagram of the first del Pezzo surface as a resolution of $Z_3 \times Z_3$. The pairs (1,5); (2,4); (3,9); (6,12); (7,10) and (8,11) are each reflected by the 45^o line and give mutually equivalent gauge theories indeed. Further analysis shows that all 12 are equivalent.

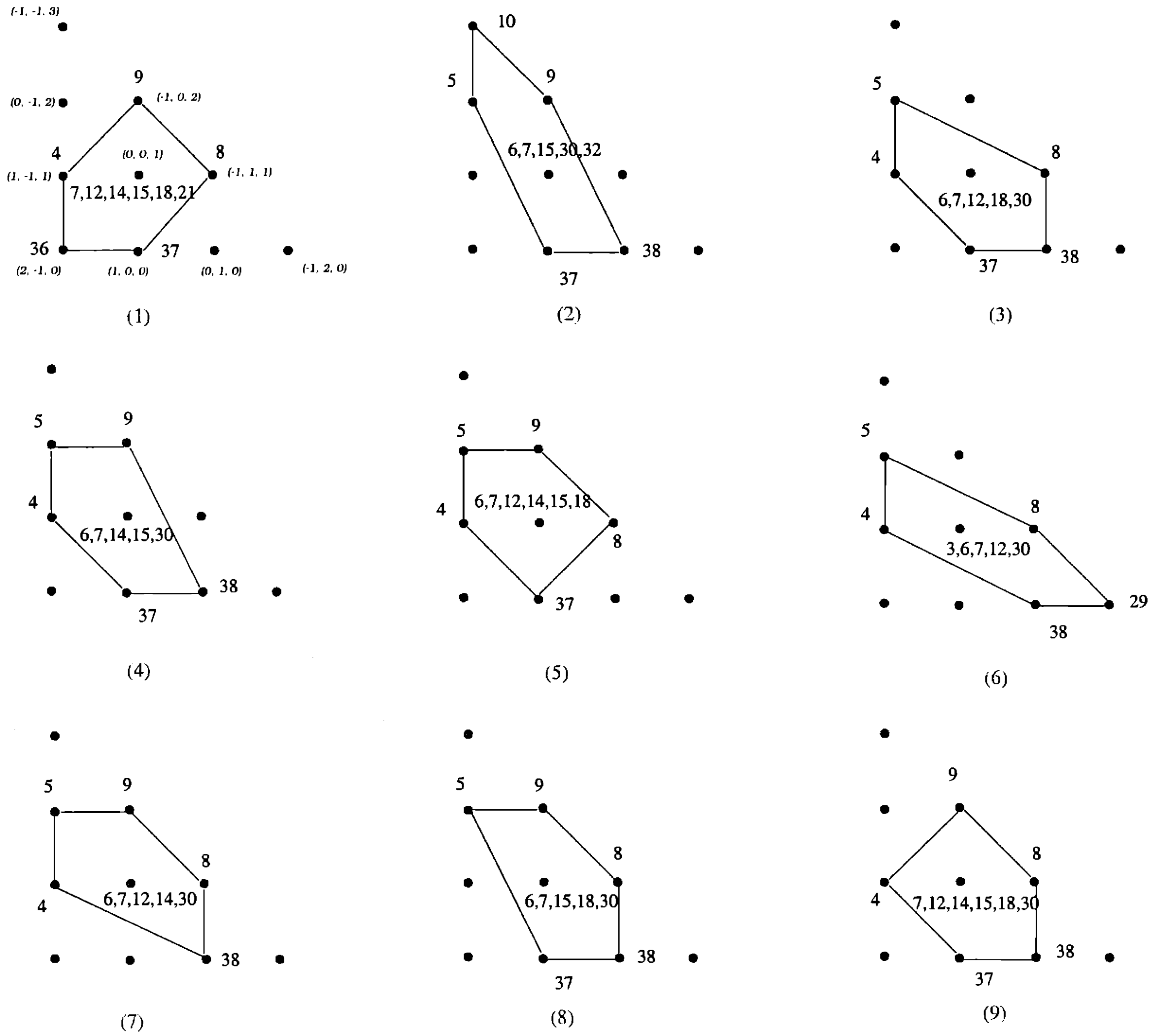
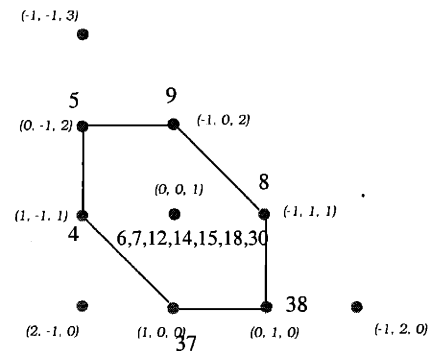


Figure 5-5: The 9 equivalent representations of the toric diagram of the second del Pezzo surface as a resolution of $Z_3 \times Z_3$. The pairs (2,6); (3,4); (5,9) and (7,8) are related by 45^o reflection while (1) is self-reflexive and are hence give pairwise equivalent theories. Further analysis shows that there are two phases given respectively by (1,5,9) and (2,3,4,6,7,8).



(1)

Figure 5-6: The unique representations of the toric diagram of the third del Pezzo surface as a resolution of $Z_3 \times Z_3$.

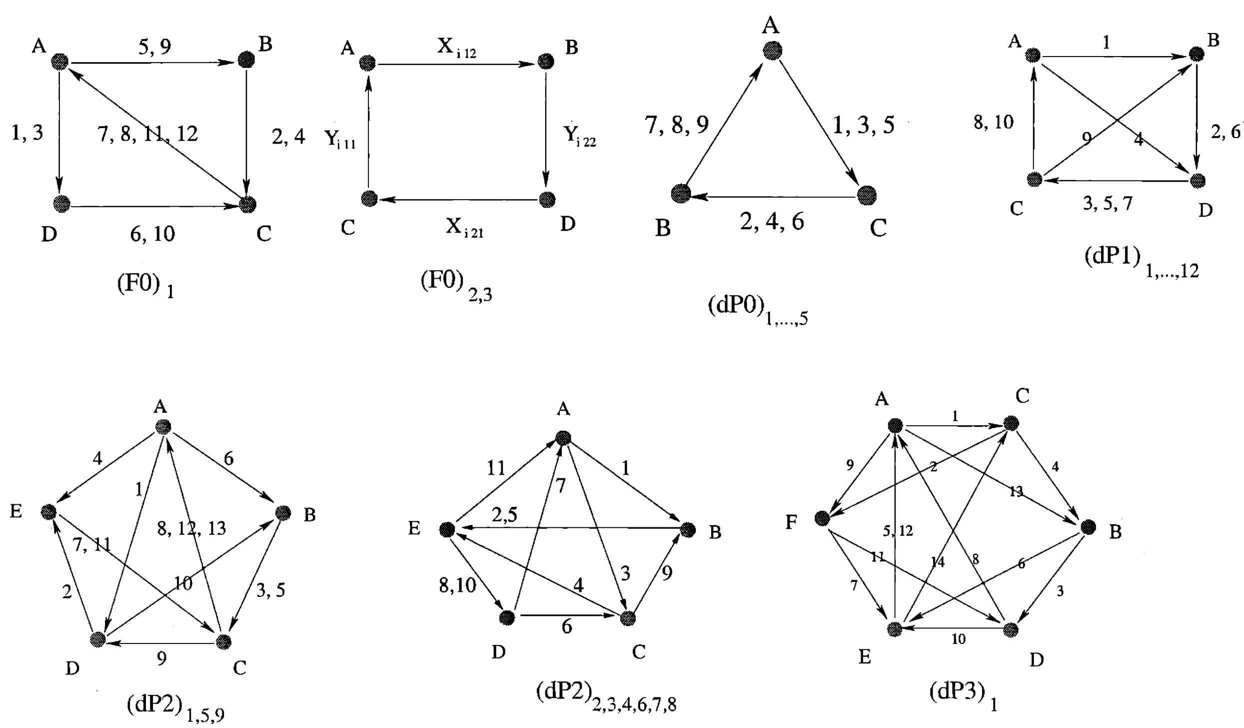


Figure 5-7: The quiver diagrams for the various phases of the gauge theory for the del Pezzo surfaces and the zeroth Hirzebruch surface.

Chapter 6

Toric Duality as Seiberg Duality and Brane Diamonds

6.1 Introduction

Witten's gauge linear sigma approach [50] to $\mathcal{N} = 2$ super-conformal theories has provided deep insight not only to the study of the phases of the field theory but also to the understanding of the mathematics of Geometric Invariant Theory quotients in toric geometry. Thereafter, the method was readily applied to the study of the $\mathcal{N} = 1$ supersymmetric gauge theories on D-branes at singularities [14, 71, 83, 84]. Indeed the classical moduli space of the gauge theory corresponds precisely to the spacetime which the D-brane probes transversely. In light of this therefore, toric geometry has been widely used in the study of the moduli space of vacua of the gauge theory living on D-brane probes.

The method of encoding the gauge theory data into the moduli data, or more specifically, the F-term and D-term information into the toric diagram of the algebraic variety describing the moduli space, has been well-established [14, 71]. The reverse, of determining the SUSY gauge theory data in terms of a given toric singularity upon which the D-brane probes, has also been addressed using the method partial resolutions of abelian quotient singularities. Namely, a general non-orbifold singularity is regarded as a partial resolution of a worse, but orbifold, singularity.

This "Inverse Procedure" was formalised into a linear optimisation algorithm, easily implementable on computer, by [85], and was subsequently checked extensively in [86].

One feature of the Inverse Algorithm is its non-uniqueness, viz., that for a given toric singularity, one could in theory construct countless gauge theories. This means that there are classes of gauge theories which have identical toric moduli space in the IR. Such a salient feature was dubbed in [85] as **toric duality**. Indeed in a follow-up work, [93] attempted to analyse this duality in detail, concentrating in particular on a method of fabricating dual theories which are physical, in the sense that they can be realised as world-volume theories on D-branes. Henceforth, we shall adhere to this more restricted meaning of toric duality.

Because the details of this method will be clear in later examples we shall not delve into the specifics here, nor shall we devote too much space reviewing the algorithm. Let us highlight the key points. The gauge theory data of D-branes probing Abelian orbifolds is well-known (see e.g. the appendix of [93]); also any toric diagram can be embedded into that of such an orbifold (in particular any toric local Calabi-Yau threefold D can be embedded into $C^3/(Z_n \times Z_n)$ for sufficiently large n. We can then obtain the subsector of orbifold theory that corresponds the gauge theory constructed for D. This is the method of "Partial Resolution."

A key point of [93] was the application of the well-known mathematical fact that the toric diagram D of any toric variety has an inherent ambiguity in its definition: namely any unimodular transformation on the lattice on which D is defined must leave D invariant. In other words, for threefolds defined in the standard lattice Z^3, any $SL(3;C)$ transformation on the vector endpoints of the defining toric diagram gives the same toric variety. Their embedding into the diagram of a fixed Abelian orbifold on the other hand, certainly is different. Ergo, the gauge theory data one obtains in general are vastly different, even though per constructio, they have the same toric moduli space.

What then is this "toric duality"? How clearly it is defined mathematically and yet how illusive it is as a physical phenomenon. The purpose of the present writing is to make the first leap toward answering this question. In particular, we shall show, using brane setups, and especially brane diamonds, that known cases for toric duality are actually interesting realisations of Seiberg Duality. Therefore the mathematical equivalence of moduli spaces for different quiver gauge theories is related to a real physical equivalence of the gauge theories in the far infrared.

The paper is organised as follows. In Section 2, we begin with an illustrative example of two torically dual cases of a generalised conifold. These are well-known to be Seiberg dual theories as seen from brane setups. Thereby we are motivated to conjecture in Section 3 that toric duality is Seiberg duality. We proceed to check this proposal in Section 4 with all the known cases of torically dual theories and have successfully shown that the phases of the partial resolutions of $C^3/(Z_3 \times Z_3)$ constructed in [85] are indeed Seiberg dual from a field theory analysis. Then in Section 6 we re-analyse these examples from the perspective of brane diamond configurations and once again obtain strong support of the statement. From rules used in the diamond dualisation, we extracted a so-called "quiver duality" which explicits Seiberg duality as a transformation on the matter adjacency matrices. Using these rules we are able to extract more phases of theories not yet obtained from the Inverse Algorithm. In a more geometrical vein, in Section 7, we remark the connection between Seiberg duality and Picard-Lefschetz and point out cases where the two phenomena may differ. Finally we finish with conclusions and prospects in Section 8.

While this manuscript is about to be released, we became aware of the nice work [108], which discusses similar issues.

6.2 An Illustrative Example

We begin with an illustrative example that will demonstrate how Seiberg Duality is realised as toric duality.

6.2.1 The Brane Setup

The example is the well-known generalized conifold described as the hypersurface $xy = z^2w^2$ in C^4, and which can be obtained as a Z_2 quotient of the famous conifold $xy = zw$ by the action $z \to -z, w \to -w$. The gauge theory on the D-brane sitting at such a singularity can be established by orbifolding the conifold gauge theory in [87], as in [99]. Also, it can be derived by another method alternative to the Inverse Algorithm, namely performing a T-duality to a brane setup with NS-branes and D4-branes [99, 100]. Therefore this theory serves as an excellent check on our methods.

The setup involves stretching D4 branes (spanning 01236) between 2 pairs of NS and NS′ branes (spanning 012345 and 012389, respectively), with x^6 parameterizing a circle. These configurations are analogous to those in [27]. There are in fact two inequivalent brane setups (a) and (b) (see Figure 6-1), differing in the way the NS- and NS′-branes are ordered in the circle coordinate. Using standard rules [10, 27],

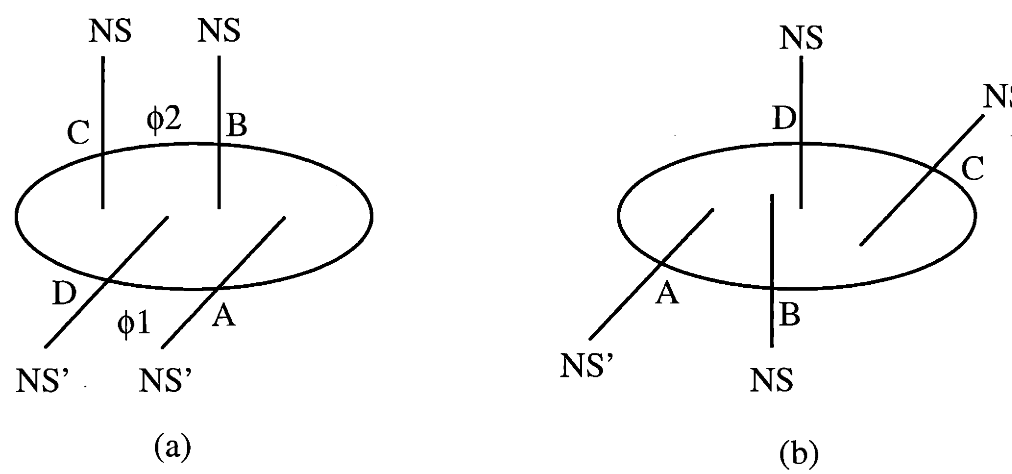


Figure 6-1: The two possible brane setups for the generalized conifold $xy = z^2w^2$. They are related to each other passing one NS-brane through an NS'-brane. A_i, B_i, C_i, D_i $i = 1, 2$ are bifundamentals while ϕ_1, ϕ_2 are two adjoint fields.

we see from the figure that there are 4 product gauge groups (in the Abelian case, it is simply $U(1)^4$. As for the matter content, theory (a) has 8 bi-fundamental chiral multiplets A_i, B_i, C_i, D_i $i = 1, 2$ (with charge $(+1, -1)$ and $(-1, +1)$ with respect to adjacent $U(1)$ factors) and 2 adjoint chiral multiplets $\phi_{1,2}$ as indicated. On the other hand (b) has only 8 bi-fundamentals, with charges as above. The superpotentials are respectively [66, 99]

$$(a) \quad W_a = \; -A_1A_2B_1B_2 + B_1B_2\phi_2 - C_1C_2\phi_2 + C_1C_2D_1D_2 - D_1D_2\phi_1 + A_1A_2\phi_1,$$

$$(b) \quad W_b = \; A_1A_2B_1B_2 - B_1B_2C_1C_2 + C_1C_2D_1D_2 - D_1D_2A_1A_2$$

With some foresight, for comparison with the results later, we rewrite them as

$$W_a = (B_1B_2 - C_1C_2)(\phi_2 - A_1A_2) + (A_1A_2 - D_1D_2)(\phi_1 - C_1C_2) \qquad (6.1)$$

$$W_b = (A_1A_2 - C_1C_2)(B_1B_2 - D_1D_2) \qquad (6.2)$$

6.2.2 Partial Resolution

Let us see whether we can reproduce these field theories with the Inverse Algorithm. The toric diagram for $xy = z^2w^2$ is given in the very left of Figure 6-2. Of course, the hypersurface is three complex-dimensional so there is actually an undrawn apex for the toric diagram, and each of the nodes is in fact a three-vector in Z^3. Indeed the fact that it is locally Calabi-Yau that guarantees all the nodes to be coplanar. The next step is the realisation that it can be embedded into the well-known toric diagram for the Abelian orbifold $C^3/(Z_3 \times Z_3)$ consisting of 10 lattice points. The reader is referred to [85, 93] for the actual coördinates of the points, a detail which, though crucial, we shall not belabour here.

The important point is that there are six ways to embed our toric diagram into the orbifold one, all related by $SL(3;C)$ transformations. This is indicated in parts (a)-(f) of Figure 6-2. We emphasise that these six diagrams, drawn in red, are *equivalent* descriptions of $xy = z^2w^2$ by virtue of their being unimodularly related; therefore they are all candidates for toric duality.

Now we use our Inverse Algorithm, by partially resolving $C^3/(Z_3 \times Z_3)$, to obtain the gauge theory data for the D-brane probing $xy = z^2w^2$. In summary, after exploring the six possible partial resolutions, we find that cases (a) and (b) give identical results, while (c,d,e,f) give the same result which is inequivalent from (a,b). There-

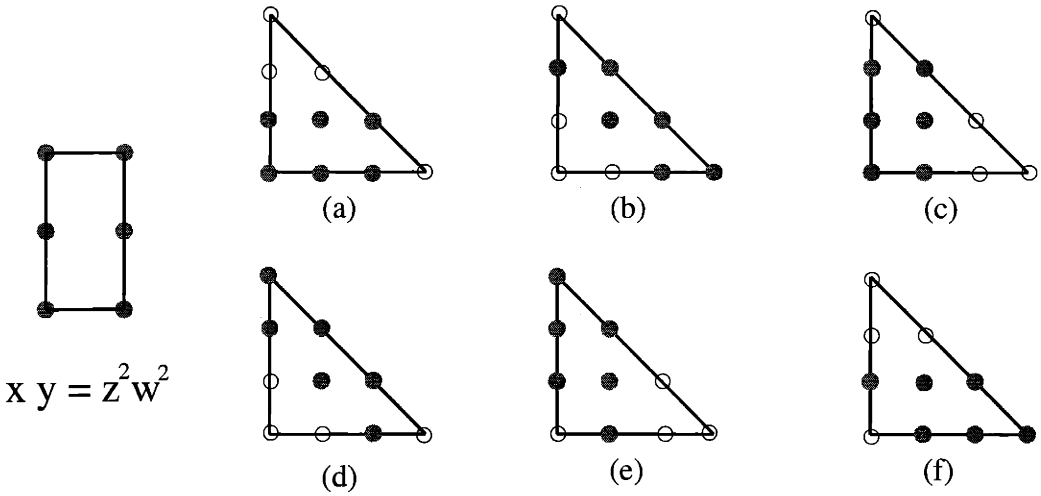


Figure 6-2: The standard toric diagram for the generalized conifold $xy = uv = z^2$ (far left). To the right are six $SL(3;C)$ transformations (a)-(f) thereof (drawn in red) and hence are equivalent toric diagrams for the variety. We embed these six diagrams into the Abelian orbifold $C^3/(Z_3 \times Z_3)$ in order to perform partial resolution and thus the gauge theory data.

fore we conclude that cases (a) and (c) are inequivalent torically dual theories for $xy = z^2w^2$. In the following we detail the data for these two contrasting cases. We refer the reader to [85, 93] for details and notation.

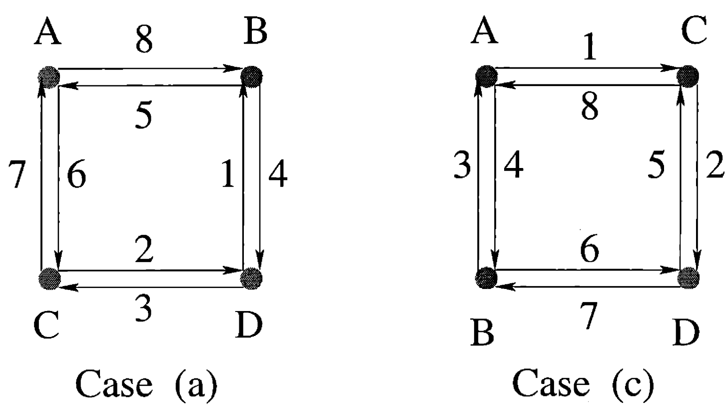


Figure 6-3: The quiver diagram encoding the matter content of Cases (a) and (c) of Figure 6-2.

6.2.3 Case (a) from Partial Resolution

For case (a), the matter content is encoded the d-matrix which indicates the charges of the 8 bi-fundamentals under the 4 gauge groups. This is the incidence matrix for the quiver diagram drawn in part (a) of Figure 6-3.

$$\begin{pmatrix} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\ U(1)_A & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ U(1)_B & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ U(1)_C & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 \\ U(1)_D & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

On the other hand, the F-terms are encoded in the K-matrix

$$\begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

From K we get two relations $X_5X_8 = X_6X_7$ and $X_1X_4 = X_2X_3$ (these are the relations one must impose on the quiver to obtain the final variety; equivalently, they correspond to the F-term constraints arising from the superpotential). Notice that here each term is chargeless under all 4 gauge groups, so when we integrate back to get the superpotential, we should multiply by chargeless quantities also[1].

The relations must come from the F-flatness $\frac{\partial}{\partial X_i}W = 0$ and thus we can use these relations to integrate back to the superpotential W. However we meet some

[1]In more general situations the left- and right-hand sides may not be singlets, but transform in the same gauge representation.

ambiguities [2]. In principle we can have two different choices:

$$\begin{aligned} (i) \qquad W_1 = &\ (X_5X_8 - X_6X_7)(X_1X_4 - X_2X_3) \\ (ii) \qquad W_2 = &\ \psi_1(X_5X_8 - X_6X_7) + \psi_2(X_1X_4 - X_2X_3) \end{aligned}$$

where for now ψ_i are simply chargeless fields.

We shall evoke physical arguments to determine which is correct. Expanding (i) gives $W_1 = X_5X_8X_1X_4 - X_6X_7X_1X_4 - X_5X_8X_2X_3 + X_6X_7X_2X_3$. Notice the term $X_6X_7X_1X_4$: there is no common gauge group under which there four fields are charged, i.e. these 4 arrows (q. v. Figure 6-3) do not intersect at a single node. This makes (i) very unnatural and exclude it.

Case (ii) does not have the above problem and indeed all four fields X_5, X_8, X_6, X_7 are charged under the $U(1)_A$ gauge group, so considering ψ_1 to be an adjoint of $U(1)_A$, we do obtain a physically meaningful interaction. Similarly ψ_2 will be the adjoint of $U(1)_D$, interacting with X_1, X_4, X_2, X_3.

However, we are not finish yet. From Figure 6-3 we see that X_5, X_8, X_1, X_4 are all charged under $U(1)_B$, while X_6, X_7, X_2, X_3 are all charged under $U(1)_C$. From a physical point of view, there should be some interaction terms between these fields. Possibilities are $X_5X_8X_1X_4$ and $X_6X_7X_2X_3$. To add these terms into W_2 is very easy, we simply perform the following replacement:[3] $\psi_1 \longrightarrow \psi_1 - X_1X_4, \qquad \psi_2 \longrightarrow \psi_2 - X_6X_7$. Putting everything together, we finally obtain that Case (a) has matter content as described in Figure 6-3 and the superpotential

$$W = (\psi_1 - X_1X_4)(X_5X_8 - X_6X_7) + (\psi_2 - X_6X_7)(X_1X_4 - X_2X_3) \qquad (6.3)$$

This is precisely the theory (a) from the brane setup in the last section! Comparing

[2]The ambiguities arise because in the abelian case (toric language) the adjoints are chargeless. In fact, no ambiguity arises if one performs the Higgsing associated to the partial resolution in the non-abelian case. We have performed this exercise in cases (a) and (c), and verified the result obtained by the different argument offered in the text.

[3]Here we choose the sign purposefully for later convenience. However, we do need, for the cancellation of the unnatural interaction term $X_1X_4X_6X_7$, that they both have the same sign.

(6.3) with (6.1), we see that they are exact same under the following redefinition of variables:

$$B_1, B_2 \Longleftrightarrow X_5, X_8 \qquad C_1, C_2 \Longleftrightarrow X_6, X_7 \qquad D_1, D_2 \Longleftrightarrow X_2, X_3$$
$$A_1, A_2 \Longleftrightarrow X_1, X_4 \qquad \phi_2 \Longleftrightarrow \psi_1 \qquad \phi_1 \Longleftrightarrow \psi_2$$

In conclusion, case (a) of our Inverse Algorithm reproduces the results of case (a) of the brane setup.

6.2.4 Case (c) from Partial Resolution

For case (c), the matter content is given by the quiver in Figure 6-3, which has the charge matrix d equal to

$$\begin{pmatrix} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\ U(1)_A & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ U(1)_B & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ U(1)_C & 1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 \\ U(1)_D & 0 & 1 & 0 & 0 & -1 & 1 & -1 & 0 \end{pmatrix}$$

This is precisely the matter content of case (b) of the brane setup. The F-terms are given by

$$K = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

From it we can read out the relations $X_1X_8 = X_6X_7$ and $X_2X_5 = X_3X_4$. Again there are two ways to write down the superpotential

$$(i) \qquad W_1 = (X_1X_8 - X_6X_7)(X_3X_4 - X_2X_5)$$

$$(ii) \qquad W_2 = \psi_1(X_1X_8 - X_6X_7) + \psi_2(X_3X_4 - X_2X_5)$$

In this case, because X_1, X_8, X_6, X_7 are not charged under any common gauge group, it is impossible to include any adjoint field ψ to give a physically meaningful interaction and so (ii) is unnatural. We are left the superpotential W_1. Indeed, comparing with (6.2), we see they are identical under the redefinitions

$$\begin{array}{ll} A_1, A_2 \Longleftrightarrow X_1, X_8 & \qquad B_1, B_2 \Longleftrightarrow X_3, X_4 \\ C_1, C_2 \Longleftrightarrow X_6, X_7 & \qquad D_1, D_2 \Longleftrightarrow X_2, X_5 \end{array}$$

Therefore we have reproduced case (b) of the brane setup.

What have we achieved? We have shown that toric duality due to inequivalent embeddings of unimodularly related toric diagrams for the generalized conifold $xy = z^2w^2$ gives two inequivalent physical world-volume theories on the D-brane probe, exemplified by cases (a) and (c). On the other hand, there are two T-dual brane setups for this singularity, also giving two inequivalent field theories (a) and (b). Upon comparison, case (a) (resp. (c)) from the Inverse Algorithm beautifully corresponds to case (a) (resp. (b)) from the brane setup. Somehow, a seemingly harmless trick in mathematics relates inequivalent brane setups. In fact we can say much more.

6.3 Seiberg Duality versus Toric Duality

As follows from [27], the two theories from the brane setups are actually related by Seiberg Duality [94], as pointed out in [99] (see also [81, 101]. Let us first review the main features of this famous duality, for unitary gauge groups.

Seiberg duality is a non-trivial infrared equivalence of $\mathcal{N} = 1$ supersymmetric field theories, which are different in the ultraviolet, but flow the the same interacting fixed point in the infrared. In particular, the very low energy features of the different theories, like their moduli space, chiral ring, global symmetries, agree for Seiberg dual theories. Given that toric dual theories, by definition, have identical moduli spaces, etc , it is natural to propose a connection between both phenomena.

The prototypical example of Seiberg duality is $\mathcal{N} = 1$ $SU(N_c)$ gauge theory with N_f vector-like fundamental flavours, and no superpotential. The global chiral symmetry is $SU(N_f)_L \times SU(N_f)_R$, so the matter content quantum numbers are

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$
Q	$\square$	$\square$	1
Q'	$\overline{\square}$	1	$\overline{\square}$

In the conformal window, $3N_c/2 \leq N_f \leq 3N_c$, the theory flows to an interacting infrared fixed point. The dual theory, flowing to the same fixed point is given $N = 1$ $SU(N_f - N_c)$ gauge theory with N_f fundamental flavours, namely

	$SU(N_f - N_c)$	$SU(N_f)_L$	$SU(N_f)_R$
q	$\square$	$\overline{\square}$	1
q'	$\overline{\square}$	1	$\square$
M	1	$\square$	$\overline{\square}$

and superpotential $W = Mqq'$. From the matching of chiral rings, the 'mesons' M can be thought of as composites QQ' of the original quarks.

It is well established [27], that in an $\mathcal{N} = 1$ (IIA) brane setup for the four dimensional theory such as Figure 6-1, Seiberg duality is realised as the crossing of 2 non-parallel NS-NS′ branes. In other words, as pointed out in [99], cases (a) and (b) are in fact a Seiberg dual pair. Therefore it seems that the results from the previous section suggest that toric duality is a guise of Seiberg duality, for theories with moduli space admitting a toric descriptions. It is therefore the intent of the remainder of this paper to examine and support

CONJECTURE 6.3.1 *Toric duality is Seiberg duality for $\mathcal{N} = 1$ theories with toric moduli spaces.*

6.4 Partial Resolutions of $C^3/(Z_3 \times Z_3)$ and Seiberg duality

Let us proceed to check more examples. So far the other known examples of torically dual theories are from various partial resolutions of $C^3/(Z_3 \times Z_3)$. In particular it was found in [93] that the (complex) cones over the zeroth Hirzebruch surface as well as the second del Pezzo surface each has two toric dual pairs. We remind the reader of these theories.

6.4.1 Hirzebruch Zero

There are two torically dual theories for the cone over the zeroth Hirzebruch surface F_0. The toric and quiver diagrams are given in Figure 6-4, the matter content and interactions are

		Matter Content d												Superpotential
		1	2	3	4	5	6	7	8	9	10	11	12	
I	A	−1	0	−1	0	−1	0	1	1	−1	0	1	1	$X_1X_8X_{10} - X_3X_7X_{10} - X_2X_8X_9$
	B	0	−1	0	−1	1	0	0	0	1	0	0	0	$-X_1X_6X_{12} + X_3X_6X_{11} + X_4X_7X_9$
	C	0	1	0	1	0	1	−1	−1	0	1	−1	−1	$+X_2X_5X_{12} - X_4X_5X_{11}$
	D	1	0	1	0	0	−1	0	0	0	−1	0	0	
		X_{112}	Y_{122}	Y_{222}	Y_{111}	Y_{211}	X_{121}	X_{212}	X_{221}					
II	A	−1	0	0	1	1	0	−1	0					
	B	1	−1	−1	0	0	0	1	0					$\epsilon^{ij}\epsilon^{kl}X_{i\ 12}Y_{k\ 22}X_{j\ 21}Y_{l\ 11}$
	C	0	0	0	−1	−1	1	0	1					
	D	0	1	1	0	0	−1	0	−1					

(6.4)

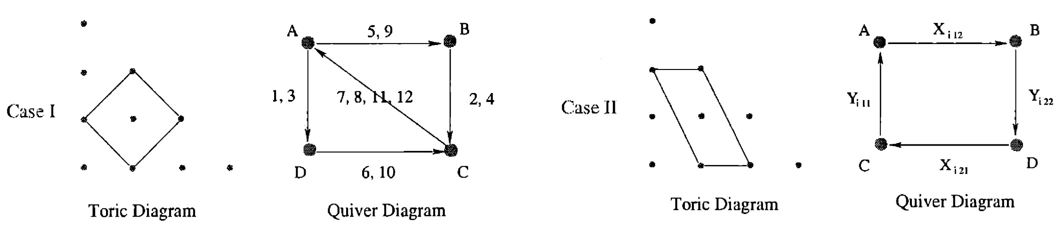


Figure 6-4: The quiver and toric diagrams of the 2 torically dual theories corresponding to the cone over the zeroth Hirzebruch surface F_0.

Let us use the field theory rules from Section 3 on Seiberg Duality to examine these two cases in detail. The charges of the matter content for case II, upon promotion from $U(1)$ to $SU(N)$ [4] (for instance, following the partial resolution in the non-abelian case, as in [71, 83]), can be re-written as (redefining fields $(X_i, Y_i, Z_i, W_i) := (X_{i\ 12}, Y_{i\ 22}, X_{i\ 21}, Y_{i\ 11})$ with $i = 1, 2$ and gauge groups $(a, b, c, d) := (A, C, B, D)$ for convenience):

	$SU(N)_a$	$SU(N)_b$	$SU(N)_c$	$SU(N)_d$
X_i	$\square$	$\overline{\square}$		
Y_i		$\square$	$\overline{\square}$	
Z_i			$\square$	$\overline{\square}$
W_i	$\overline{\square}$			$\square$

The superpotential is then

$$W_{II} = X_1Y_1Z_2W_2 - X_1Y_2Z_2W_1 - X_2Y_1Z_1W_2 + X_2Y_2Z_1W_1.$$

Let us dualise with respect to the a gauge group. This is a $SU(N)$ theory with $N_c = N$ and $N_f = 2N$ (as there are two X_i's). The chiral symmetry is however broken from $SU(2N)_L \times SU(2N)_R$ to $SU(N)_L \times SU(N)_R$, which moreover is gauged as $SU(N)_b \times SU(N)_d$. Ignoring the superpotential W_{II}, the dual theory would be:

	$SU(N)_{a'}$	$SU(N)_b$	$SU(N)_c$	$SU(N)_d$
q_i	$\overline{\square}$	$\square$		
Y_i		$\square$	$\overline{\square}$	
Z_i			$\square$	$\overline{\square}$
q'_i	$\square$			$\overline{\square}$
M_{ij}		$\overline{\square}$		$\square$

(6.5)

We note that there are M_{ij} giving 4 bi-fundamentals for bd. They arise from the Seiberg mesons in the bi-fundamental of the enhanced chiral symmetry $SU(2N) \times SU(2N)$, once decomposed with respect to the unbroken chiral symmetry group. The

[4] Concerning the $U(1)$ factors, these are in fact generically absent, since they are anomalous in the original $Z_3 \times Z_3$ singularity, and the Green-Schwarz mechanism canceling their anomaly makes them massive [102] (see [103, 40, 104] for an analogous 6d phenomenon). However, there is a well-defined sense in which one can use the abelian case to study the toric moduli space [71].

superpotential is

$$W' = M_{11}q_1q_1' - M_{12}q_2q_1' - M_{21}q_1q_2' + M_{22}q_2q_2'.$$

The choice of signs in W' will be explained shortly.

Of course, W_{II} is not zero and so give rise to a deformation in the original theory, analogous to those studied in e.g. [66]. In the dual theory, this deformation simply corresponds to W_{II} rewritten in terms of mesons, which can be thought of as composites of the original quarks, i.e., $M_{ij} = W_iX_j$. Therefore we have

$$W_{II} = M_{21}Y_1Z_2 - M_{11}Y_2Z_2 - M_{22}Y_1Z_1 + M_{12}Y_2Z_1$$

which is written in the new variables. The rule for the signs is that e.g. the field M_{21} appears with positive sign in W_{II}, hence it should appear with negative sign in W', and analogously for others. Putting them together we get the superpotential of the dual theory

$$\begin{aligned} W_{II}^{dual} &= W_{II} + W' = M_{11}q_1q_1' - M_{12}q_2q_1' \\ &\quad -M_{21}q_1q_2' + M_{22}q_2q_2' + M_{21}Y_1Z_2 - M_{11}Y_2Z_2 - M_{22}Y_1Z_1 + M_{12}Y_2Z_1 \end{aligned} \tag{6.6}$$

Upon the field redefinitions

$$\begin{array}{cccc} M_{11} \to X_7 & M_{12} \to X_8 & M_{21} \to X_{11} & M_{22} \to X_{12} \\ q_1 \to X_4 & q_2 \to X_2 & q_{1'} \to X_9 & q_{2'} \to X_5 \\ Y_1 \to X_6 & Y_2 \to X_{10} & Z_1 \to X_1 & Z_2 \to X_3 \end{array}$$

we have the field content (6.5) and superpotential (6.6) matching precisely with case I in (6.4). We conclude therefore that the two torically dual cases I and II obtained from partial resolutions are indeed Seiberg duals!

6.4.2 del Pezzo 2

Encouraged by the results above, let us proceed with the cone over the second del Pezzo surface, which also have 2 torically dual theories. The toric and quiver diagrams are given in Figure 6-5.

	Matter Content d														Superpotential
		1	2	3	4	5	6	7	8	9	10	11	12	13	
I	A	-1	0	0	-1	0	-1	0	1	0	0	0	1	1	$Y_2Y_9Y_{11} - Y_9Y_3Y_{10} - Y_4Y_8Y_{11}$
	B	0	0	-1	0	-1	1	0	0	0	1	0	0	0	$-Y_1Y_2Y_7Y_{13} + Y_{13}Y_3Y_6 - Y_5Y_{12}Y_6$
	C	0	0	1	0	1	0	1	-1	-1	0	1	-1	-1	$+Y_1Y_5Y_8Y_{10} + Y_4Y_7Y_{12}$
	D	1	-1	0	0	0	0	0	0	1	-1	0	0	0	
	E	0	1	0	1	0	0	-1	0	0	0	-1	0	0	
		1	2	3	4	5	6	7	8	9	10	11			
II	A	-1	0	-1	0	0	0	1	0	0	0	1			$X_5X_8X_6X_9 + X_1X_2X_{10}X_7 + X_{11}X_3X_4$
	B	1	-1	0	0	-1	0	0	0	1	0	0			$-X_4X_{10}X_6 - X_2X_8X_7X_3X_9 - X_{11}X_1X_5$
	C	0	0	1	-1	0	1	0	0	-1	0	0			
	D	0	0	0	0	0	-1	-1	1	0	1	0			
	E	0	1	0	1	1	0	0	-1	0	-1	-1			

(6.7)

Again we start with Case II. Working analogously, upon dualisation on node D

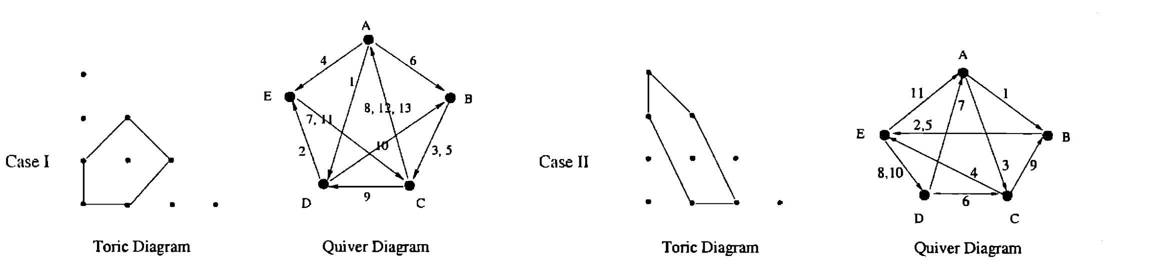


Figure 6-5: The quiver and toric diagrams of the 2 torically dual theories corresponding to the cone over the second del Pezzo surface.

neglecting the superpotential, the matter content of II undergoes the following change:

	A	B	C	D	E
X_1	$\overline{\square}$	$\square$			
X_2		$\overline{\square}$			$\square$
X_5		$\overline{\square}$			$\square$
X_3	$\overline{\square}$		$\square$		
X_4			$\overline{\square}$		$\square$
X_9		$\square$	$\overline{\square}$		
X_{11}	$\square$				$\overline{\square}$
X_6			$\square$	$\overline{\square}$	
X_7	$\square$			$\overline{\square}$	
X_8				$\square$	$\overline{\square}$
X_{10}				$\square$	$\overline{\square}$

dual on D $\Longrightarrow$

	A	B	C	D	E
X_6	$\overline{\square}$	$\square$			
X_5		$\overline{\square}$			$\square$
X_3		$\overline{\square}$			$\square$
X_1	$\overline{\square}$		$\square$		
X_4			$\overline{\square}$		$\square$
X_{10}		$\square$	$\overline{\square}$		
X_{13}	$\square$				$\overline{\square}$
$\widetilde{X}_6$			$\overline{\square}$	$\square$	
$\widetilde{X}_7$	$\overline{\square}$			$\square$	
$\widetilde{X}_8$				$\overline{\square}$	$\square$
$\widetilde{X}_{10}$				$\overline{\square}$	$\square$
$M_{EA,1}$	$\square$				$\overline{\square}$
$M_{EA,2}$	$\square$				$\overline{\square}$
$M_{EC,1}$			$\square$		$\overline{\square}$
$M_{EC,2}$			$\square$		$\overline{\square}$

(6.8)

Let us explain the notations in (6.8). Before Seiberg duality we have 11 fields $X_{1,\ldots,11}$. After the dualisation on gauge group D, the we obtain dual quarks (corresponding to bi-fundamentals conjugate to the original quark X_6, X_7, X_8, X_{10}) which we denote $\widetilde{X}_6, \widetilde{X}_7, \widetilde{X}_8, \widetilde{X}_{10}$. Furthermore we have added meson fields $M_{EA,1}, M_{EA,2}, M_{EC,1}, M_{EC,2}$, which are Seiberg mesons decomposed with respect to the unbroken chiral symmetry group.

As before, one should incorporate the interactions as a deformation of this duality. Naïvely we have 15 fields in the dual theory, but as we will show below, the resulting superpotential provides a mass term for the fields X_4 and $M_{EC,2}$, which transform in conjugate representations. Integrating them out, we will be left with 13 fields, the number of fields in Case I. In fact, with the mapping

dual of II	X_1	X_2	X_5	X_3	X_4	X_9	X_{11}	$\widetilde{X}_6$	$\widetilde{X}_7$	$\widetilde{X}_8$	$\widetilde{X}_{10}$
Case I	Y_6	Y_5	Y_3	Y_1	massive	Y_{10}	Y_{13}	Y_2	Y_4	Y_{11}	Y_7

and

dual of II	$M_{EA,1}$	$M_{EA,2}$	$M_{EC,1}$	$M_{EC,2}$
Case I	Y_8	Y_{12}	Y_9	massive

we conclude that the matter content of the Case II dualised on gauge group D is identical to Case I!

Let us finally check the superpotentials, and also verify the claim that X_4 and $M_{EC,2}$ become massive. Rewriting the superpotential of II from (6.7) in terms of the dual variables (matching the mesons as composites $M_{EA,1} = X_8X_7$, $M_{EA,2} = X_{10}X_7$, $M_{EC,1} = X_8X_6$, $M_{EC,2} = X_{10}X_6$), we have

$$\begin{aligned} W_{II} &= X_5M_{EC,1}X_9 + X_1X_2M_{EA,2} + X_{11}X_3X_4 \\ &\quad -X_4M_{EC,2} - X_2M_{EA,1}X_3X_9 - X_{11}X_1X_5. \end{aligned}$$

As is with the previous subsection, to the above we must add the meson interaction terms coming from Seiberg duality, namely

$$W_{meson} = M_{EA,1}\widetilde{X}_7\widetilde{X}_8 - M_{EA,2}\widetilde{X}_7\widetilde{X}_{10} - M_{EC,1}\widetilde{X}_6\widetilde{X}_8 + M_{EC,2}\widetilde{X}_6\widetilde{X}_{10},$$

(notice again the choice of sign in W_{meson}). Adding this two together we have

$$\begin{aligned} W_{II}^{dual} &= X_5M_{EC,1}X_9 + X_1X_2M_{EA,2} + X_{11}X_3X_4 \\ &\quad -X_4M_{EC,2} - X_2M_{EA,1}X_3X_9 - X_{11}X_1X_5 \\ &\quad +M_{EA,1}\widetilde{X}_7\widetilde{X}_8 - M_{EA,2}\widetilde{X}_7\widetilde{X}_{10} - M_{EC,1}\widetilde{X}_6\widetilde{X}_8 + M_{EC,2}\widetilde{X}_6\widetilde{X}_{10}. \end{aligned}$$

Now it is very clear that both X_4 and $M_{EC,2}$ are massive and should be integrated out:

$$X_4 = \widetilde{X}_6\widetilde{X}_{10}, \qquad M_{EC,2} = X_{11}X_3.$$

Upon substitution we finally have

$$\begin{aligned} W_{II}^{dual} &= X_5M_{EC,1}X_9 + X_1X_2M_{EA,2} + X_{11}X_3\widetilde{X}_6\widetilde{X}_{10} - X_2M_{EA,1}X_3X_9 \\ &\quad -X_{11}X_1X_5 + M_{EA,1}\widetilde{X}_7\widetilde{X}_8 - M_{EA,2}\widetilde{X}_7\widetilde{X}_{10} - M_{EC,1}\widetilde{X}_6\widetilde{X}_8, \end{aligned}$$

which with the replacement rules given above we obtain

$$\begin{aligned} W_{II}^{dual} &= Y_3Y_9Y_{10} + Y_6Y_5Y_{12} + Y_{13}Y_1Y_2Y_7 - Y_5Y_1Y_{10}Y_8 \\ &\quad -Y_{13}Y_6Y_3 + Y_8Y_4Y_{11} - Y_{12}Y_4Y_7 - Y_9Y_2Y_{11}. \end{aligned}$$

This we instantly recognise, by referring to (6.7), as the superpotential of Case I.

In conclusion therefore, with the matching of matter content and superpotential, the two torically dual cases I and II of the cone over the second del Pezzo surface are also Seiberg duals.

6.5 Brane Diamonds and Seiberg Duality

Having seen the above arguments from field theory, let us support that toric duality is Seiberg duality from yet another perspective, namely, through brane setups. The use of this T-dual picture for D3-branes at singularities will turn out to be quite helpful in showing that toric duality reproduces Seiberg duality.

What we have learnt from the examples where a brane interval picture is available (i.e. NS- and D4-branes in the manner of [10]) is that the standard Seiberg duality by brane crossing reproduces the different gauge theories obtained from toric arguments (different partial resolutions of a given singularity). Notice that the brane crossing corresponds, under T-duality, to a change of the B field in the singularity picture, rather than a change in the singularity geometry [99, 81]. Hence, the two theories arise on the world-volume of D-branes probing the same singularity.

Unfortunately, brane intervals are rather limited, in that they can be used to study Seiberg duality for generalized conifold singularities, $xy = w^k w^l$. Although this is a large class of models, not many examples arise in the partial resolutions of $C^3/(Z_3 \times Z_3)$. Hence the relation to toric duality from partial resolutions cannot be checked for most examples.

Therefore it would be useful to find other singularities for which a nice T-dual brane picture is available. Nice in the sense that there is a motivated proposal to realize Seiberg duality in the corresponding brane setup. A good candidate for such a brane setup is **brane diamonds**, studied in [78].

Reference [30] (see also [91, 31]) introduced brane box configurations of intersecting NS- and NS'-branes (spanning 012345 and 012367, respectively), with D5-branes (spanning 012346) suspended among them. Brane diamonds [78] generalized (and refined) this setup by considering situations where the NS- and the NS'-branes recombine and span a smooth holomorphic curve in the 4567 directions, in whose holes D5-branes can be suspended as soap bubbles. Typical brane diamond pictures are as in figures in the remainder of the paper.

Brane diamonds are related by T-duality along 46 to a large set of D-branes at singularities. With the set of rules to read off the matter content and interactions in [78], they provide a useful pictorial representation of these D-brane gauge field theories. In particular, they correspond to singularities obtained as the abelian orbifolds of the conifold studied in Section 5 of [99], and partial resolutions thereof. Concerning this last point, brane diamond configurations admit two kinds of deformations: motions of diamond walls in the directions 57, and motions of diamond walls in the directions 46. The former T-dualize to geometric sizes of the collapse cycles, hence trigger partial resolutions of the singularity (notice that when a diamond wall moves in 57, the suspended D5-branes snap back and two gauge factors recombine, leading to a Higgs mechanism, triggered by FI terms). The later do not modify the T-dual singularity geometry, and correspond to changes in the B-fields in the collapsed cycles.

The last statement motivates the proposal made in [78] for Seiberg duality in this setup. It corresponds to closing a diamond, while keeping it in the 46 plane, and reopening it with the opposite orientation. The orientation of a diamond determines the chiral multiplets and interactions arising from the picture. The effect of this is shown in fig 7 of [78]: The rules are

1. When the orientation of a diamond is flipped, the arrows going in or out of it change orientation;

2. one has to include/remove additional arrows to ensure a good 'arrow flow' (ultimately connected to anomalies, and to Seiberg mesons)

3. Interactions correspond to closed loops of arrows in the brane diamond picture.

4. In addition to these rules, and based in our experience with Seiberg duality, we propose that when in the final picture some mesons appear in gauge representations conjugate to some of the original field, the conjugate pair gets massive.

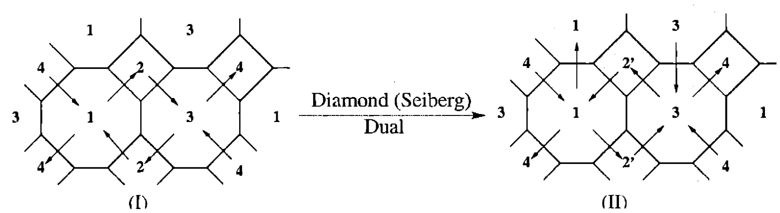


Figure 6-6: Seiberg duality from the brane diamond construction for the generalized conifold $xy = z^2w^2$. Part (I) corresponds to the brane interval picture with alternating ordering of NS- and NS$'$-branes, whereas part (II) matches the other ordering.

These rules reproduce Seiberg duality by brane crossing in cases where a brane interval picture exists. In fact, one can reproduce our previous discussion of the $xy = z^2w^2$ in this language, as shown in figure Figure 6-6. Notice that in analogy with the brane interval case the diamond transition proposed to reproduce Seiberg duality does not involve changes in the T-dual singularity geometry, hence ensuring that the two gauge theories will have the same moduli space.

Let us re-examine our aforementioned examples.

6.5.1 Brane diamonds for D3-branes at the cone over F_0

Now let us show that diamond Seiberg duality indeed relates the two gauge theories arising on D3-branes at the singularity which is a complex cone over F_0. The toric diagram of F_0 is similar to that of the conifold, only that it has an additional point (ray) in the middle of the square. Hence, it can be obtained from the conifold diagram by simply refining the lattice (by a vector $(1/2, 1/2)$ if the conifold lattice is generated by $(1,0)$, $(0,1)$). This implies [92]) that the space can be obtained as a Z_2 quotient of the conifold, specifically modding $xy = zw$ by the action that flips all coordinates.

Performing two T-dualities in the conifold one reaches the brane diamond picture described in [78] (fig. 5), which is composed by two-diamond cell with sides identified, see Part (I) of Figure 6-7. However, we are interested not in the conifold but on a

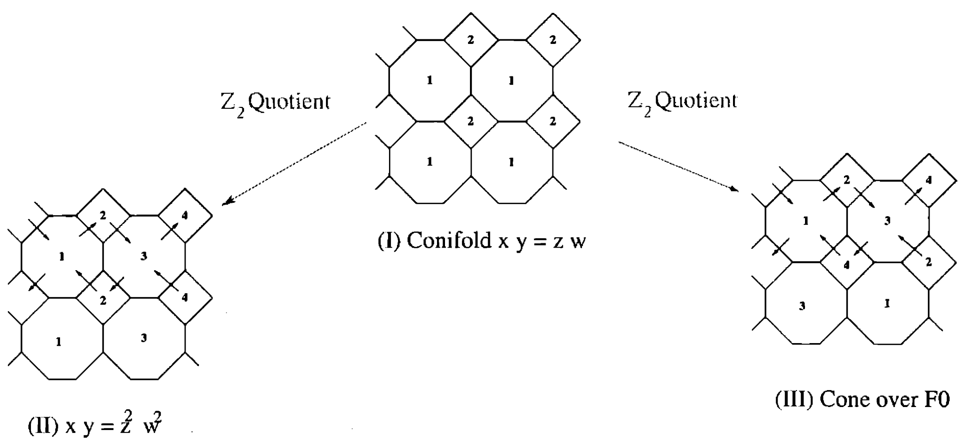


Figure 6-7: (I) Brane diamond for the conifold. Identifications in the infinite periodic array of boxes leads to a two-diamond unit cell, whose sides are identified in the obvious manner. From (I) we have 2 types of Z_2 quotients: (II) Brane diamond for the Z_2 quotient of the conifold $xy = z^2w^2$, which is a case of the so-called generalised conifold. The identifications of sides are trivial, not tilting. The final spectrum is the familiar non-chiral spectrum for a brane interval with two NS and two NS' branes (in the alternate configuration); (III) Brane diamond for the Z_2 quotient of the conifold yielding the complex cone over F_0. The identifications of sides are shifted, a fact related to the specific 'tilted' refinement of the toric lattice.

Z_2 quotient thereof. Quotienting a singularity amounts to including more diamonds in the unit cell, i.e. picking a larger unit cell in the periodic array. There are two possible ways to do so, corresponding to two different Z_2 quotients of the conifold.

One corresponds to the generalized conifold $xy = z^2w^2$ encountered above, and whose diamond picture is given in Part (II) of Figure 6-7 for completeness. The second possibility is shown in Part (III) of Figure 6-7 and does correspond to the T-dual of the complex cone over F_0, so we shall henceforth concentrate on this case. Notice that the identifications of sides of the unit cell are shifted. The final spectrum agrees with the quiver before eq (2.2) in [85]. Moreover, following [78], these fields have quartic interactions, associated to squares in the diamond picture, with signs given by the orientation of the arrow flow. They match the ones in case II in (6.4).

Now let us perform the diamond duality in the box labeled 2. Following the diamond duality rules above, we obtain the result shown in Figure 6-8. Careful comparison with the spectrum and interactions of case I in (6.4), and also with the Seiberg dual computed in Section 4.1 shows that the new diamond picture reproduces the toric dual / Seiberg dual of the initial one. Hence, brane diamond configurations provide a new geometric picture for this duality.

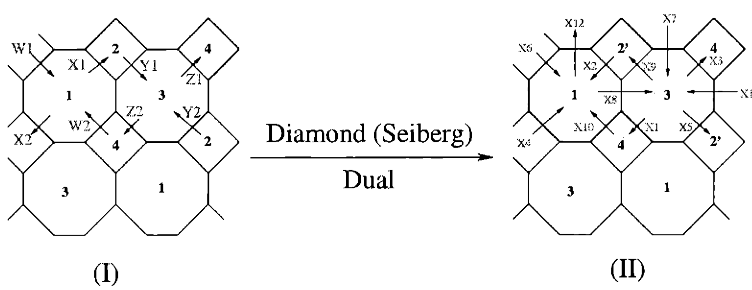


Figure 6-8: Brane diamond for the two cases of the cone over F_0. (I) is as in Figure 6-7 and (II) is the result after the diamond duality. The resulting spectrum and interactions are those of the toric dual (and also Seiberg dual) of the initial theory (I).

6.5.2 Brane diamonds for D3-branes at the cone over dP_2

The toric diagram for dP_2 shows it cannot be constructed as a quotient of the conifold. However, it is a partial resolution of the orbifolded conifold described as $xy = v^2$, $uv = z^2$ in C^5 (we refer the reader to Figure 6-9. This is a $Z_2 \times Z_2$ quotient of the conifold whose brane diamond, shown in Part (I) of Figure 6-10, contains 8 diamonds in its unit cell. Partial resolutions in the brane diamond language correspond to partial

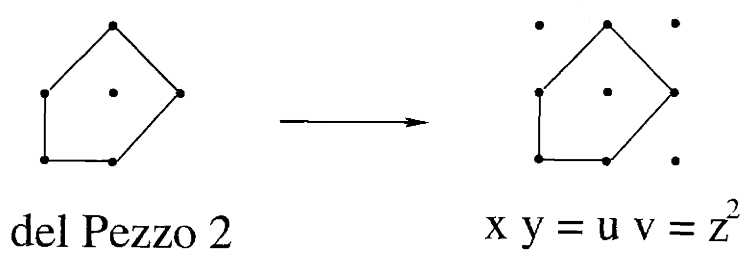


Figure 6-9: Embedding the toric diagram of dP2 into the orbifolded conifold described as $xy = v^2$, $uv = z^2$.

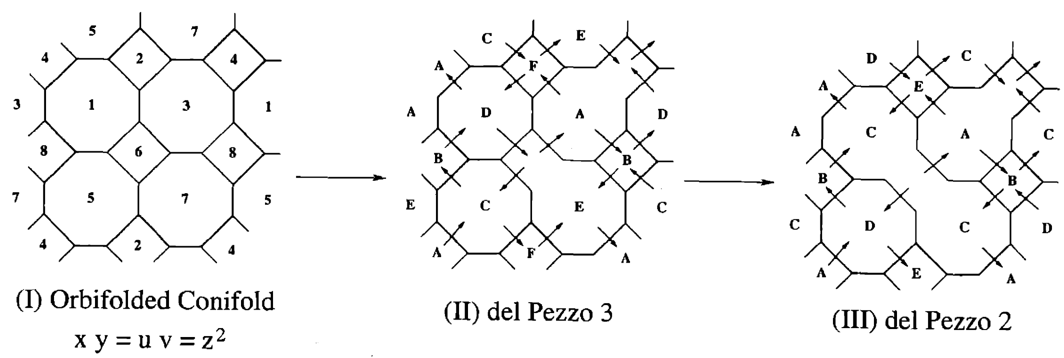


Figure 6-10: (I) Brane diamond for a $Z_2 \times Z_2$ orbifold of the conifold, namely $xy = z^2; uv = z^2$. From this we can partial resolve to (II) the cone over $dP3$ and thenceforth again to (III) the cone over dP_2, which we shall discuss in the context of Seiberg duality.

Higgsing, namely recombination of certain diamonds. As usual, the difficult part is to identify which diamond recombination corresponds to which partial resolution. A systematic way proceed would be[5]:

1. Pick a diamond recombination;

2. Compute the final gauge theory;

3. Compute its moduli space, which should be the partially resolved singularity.

[5]As an aside, let us remark that the use of brane diamonds to follow partial resolutions of singularities may provide an alternative to the standard method of partial resolutions of orbifold singularities [71, 85]. The existence of a brane picture for partial resolutions of orbifolded conifolds may turn out to be a useful advantage in this respect.

However, instead of being systematic, we prefer a shortcut and simply match the spectrum of recombined diamond pictures with known results of partial resolutions. In order to check we pick the right resolutions, it is useful to discuss the brane diamond picture for some intermediate step in the resolution to dP_2. A good intermediate point, for which the field theory spectrum is known is the complex cone over dP_3.

By trial and error matching, the diamond recombination which reproduces the world-volume spectrum for D3-branes at the cone over dP_3 (see [85, 93]), is shown in Part (II) of Figure 6-10. Performing a further resolution, chosen so as to match known results, one reaches the brane diamond picture for D3-branes on the cone over dP_2, shown in Part (III) of Figure 6-10. More specifically, the spectrum and interactions in the brane diamond configuration agrees with those of case I in (6.7).

This brane box diamond, obtained in a somewhat roundabout way, is our starting point to discuss possible dual realizations. In fact, recall that there is a toric dual field theory for dP_2, given as case II in (6.7). After some inspection, the desired effect is obtained by applying diamond Seiberg duality to the diamond labeled B. The corresponding process and the resulting diamond picture are shown in Figure 6-11. Two comments are in order: notice that in applying diamond duality using the rules above, some vector-like pairs of fields have to be removed from the final picture; in fact one can check by field theory Seiberg duality that the superpotential makes them massive. Second, notice that in this case we are applying duality in the direction opposite to that followed in the field theory analysis in Section 4.2; it is not difficult to check that the field theory analysis works in this direction as well, namely the dual of the dual is the original theory. Therefore this new example provides again a geometrical realization of Seiberg duality, and allows to connect it with Toric Duality.

We conclude this Section with some remarks. The brane diamond picture presumably provides other Seiberg dual pairs by picking different gauge factors. All such models should have the same singularities as moduli space, and should be toric duals in a broad sense, even though all such toric duals may not be obtainable by partial resolutions of $C^3/(Z_3 \times Z_3)$. From this viewpoint we learn that Seiberg duality can provide us with new field theories and toric duals beyond the reach of present computational tools. This is further explored in Section 7.

A second comment along the same lines is that Seiberg duality on nodes for which $N_f \neq 2N_c$ will lead to dual theories where some gauge factors have different rank. Taking the theory back to the 'abelian' case, some gauge factors turn out to be non-abelian. Hence, in these cases, even though Seiberg duality ensures the final theory has the same singularity as moduli space, the computation of the corresponding symplectic quotient is beyond the standard tools of toric geometry. Therefore, Seiberg duality can provide ('non-toric') gauge theories with toric moduli space.

6.6 A Quiver Duality from Seiberg Duality

If we are not too concerned with the superpotential, when we make the Seiberg duality transformation, we can obtain the matter content very easily at the level of the quiver diagram. What we obtain are rules for a so-called "quiver duality" which is a rephrasing of the Seiberg duality transformations in field (brane diamond) theory in the language of quivers. Denote $(N_c)_i$ the number of colors at the i^{th} node, and a_{ij} the number of arrows from the node i to the j (the adjacency matrix) The rules on the quiver to obtain Seiberg dual theories are

1. Pick the dualisation node i_0. Define the following sets of nodes: I_{in} := nodes having arrows going into i_0; I_{out} := those having arrow coming from i_0 and I_{no} := those unconnected with i_0. The node i_0 should not be included in this classification.

2. Change the rank of the node i_0 from N_c to $N_f - N_c$ where N_f is the number of vector-like flavours, $N_f = \sum\limits_{i \in I_{in}} a_{i,i_0} = \sum\limits_{i \in I_{out}} a_{i_0,i}$

3. Reverse all arrows going in or out of i_0, therefore

$$a_{ij}^{dual} = a_{ji} \qquad \text{if either } i, j = i_0$$

4. Only arrows linking I_{in} to I_{out} will be changed and all others remain unaffected.

5. For every pair of nodes A, B, $A \in I_{out}$ and $B \in I_{in}$, change the number of arrows a_{AB} to

$$a_{AB}^{dual} = a_{AB} - a_{i_0 A} a_{B i_0} \qquad \text{for } A \in I_{out}, \quad B \in I_{in}.$$

If this quantity is negative, we simply take it to mean $-a^{dual}$ arrow go from B to A.

These rules follow from applying Seiberg duality at the field theory level, and therefore are consistent with anomaly cancellation. In particular, notice the for any node $i \in I_{in}$, we have replaced $a_{i,i_0} N_c$ fundamental chiral multiplets by $-a_{i,i_0}(N_f - N_c) + \sum_{j \in I_{out}} a_{i,i0} a_{i_0,j}$ which equals $-a_{i,i_0}(N_f - N_c) + a_{i,i0} N_f = a_{i,i_0} N_c$, and ensures anomaly cancellation in the final theory. Similarly for nodes $j \in I_{out}$.

It is straightforward to apply these rules to the quivers in the by now familiar examples in previous sections.

In general, we can choose an arbitrary node to perform the above Seiberg duality rules. However, not every node is suitable for a toric description. The reason is that, if we start from a quiver whose every node has the same rank N, after the transformation it is possible that this no longer holds. We of course wish so because due to the very definition of the C^* action for toric varieties, toric descriptions are possible iff all nodes are $U(1)$, or in the non-Abelian version, $SU(N)$. If for instance we choose to Seiberg dualize a node with $3N$ flavours, the dual node will have rank $3N - N = 2N$ while the others will remain with rank N, and our description would no longer be toric. For this reason we must choose nodes with only $2N_f$ flavors, if we are to remain within toric descriptions.

One natural question arises: if we Seiberg-dualise every possible allowed node, how many different theories will we get? Moreover how many of these are torically dual? Let we re-analyse the examples we have thus far encountered.

6.6.1 Hirzebruch Zero

Starting from case (II) of F_0 (recall Figure 6.4) all of four nodes are qualified to yield toric Seiberg duals (they each have 2 incoming and 2 outgoing arrows and hence $N_f = 2N$). Dualising any one will give to case (I) of F_0. On the other hand, from (I) of F_0, we see that only nodes B, D are qualified to be dualized. Choosing either, we get back to the case (II) of F_0. In another word, cases (I) and (II) are closed under the Seiberg-duality transformation. In fact, this is a very strong evidence that there are only two toric phases for F_0 no matter how we embed the diagram into higher $Z_k \times Z_k$ singularities. This also solves the old question [85, 93] that the Inverse Algorithm does not in principle tell us how many phases we could have. Now by the closeness of Seiberg-duality transformations, we do have a way to calculate the number of possible phases. Notice, on the other hand, the existence of non-toric phases.

6.6.2 del Pezzo 0,1,2

Continuing our above calculation to del Pezzo singularities, we see that for dP_0 no node is qualified, so there is only one toric phase which is consistent with the standard result [93] as a resolution $\mathcal{O}_{\mathbf{P}^2}(-1) \rightarrow C^3/Z_3$. For dP_1, nodes A, B are qualified (all notations coming from [93]), but the dualization gives back to same theory, so it too has only one phase.

For our example dP_2 studied earlier (recall Figure 6.7), there are four points A, B, C, D which are qualified in case (II). Nodes A, C give back to case (II) while nodes B, D give rise to case (I) of dP_2. On the other hand, for case (I), three nodes B, D, E are qualified. Here nodes B, E give case (II) while node D give case (I). In other words, cases (I) and (II) are also closed under the Seiberg-duality transformation, so we conclude that there too are only two phases for dP_2, as presented earlier.

6.6.3 The Four Phases of dP_3

Things become more complex when we discuss the phases of dP_3. As we remarked before, due to the running-time limitations of the Inverse Algorithm, only one phase was obtained in [93]. However, one may expect this case to have more than just one phase, and in fact a recent paper has given another phase [98]. Here, using the closeness argument we give evidence that there are four (toric) phases for dP_3. We will give only one phase in detail. Others are similarly obtained. Starting from case (I) given in [93] and dualizing node B, (we refer the reader to Figure 6-12) we get the charge (incidence) matrix d as

	q_1	q_2	q_1'	q_2'	X_1	X_2	X_7	X_9	X_{10}	X_{11}	M_1	X_{14}	M_2	X_8	M_1'	X_5	X_{12}	M_2'
A	0	1	0	0	−1	0	0	−1	0	0	0	0	0	1	−1	1	1	−1
B	−1	−1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C	1	0	0	0	1	−1	0	0	0	0	−1	1	−1	0	0	0	0	0
D	0	0	−1	0	0	0	0	0	−1	1	1	0	0	−1	1	0	0	0
E	0	0	0	−1	0	0	1	0	1	0	0	−1	1	0	0	−1	−1	1
F	0	0	0	0	0	1	−1	1	0	−1	0	0	0	0	0	0	0	0

where

$$M_1 = X_4X_3, \quad M_2 = X_4X_6, \quad M_1' = X_{13}X_3, \quad M_2' = X_{13}X_6$$

are the added mesons. Notice that X_{14} and M_2 have opposite charge. In fact, both are massive and will be integrate out. Same for pairs (X_8, M_1') and (X_5, M_2').

Let us derive the superpotential. Before dual transformation, the superpotential is [85]

$$\begin{aligned} W_I &= X_3X_8X_{13} - X_8X_9X_{11} - X_5X_6X_{13} - X_1X_3X_4X_{10}X_{12} \\ &\quad X_7X_9X_{12} + X_4X_6X_{14} + X_1X_2X_5X_{10}X_{11} - X_2X_7X_{14} \end{aligned}$$

After dualization, superpotential is rewritten as

$$\begin{aligned} W' &= M_1'X_8 - X_8X_9X_{11} - X_5M_2' - X_1M_1X_{10}X_{12} \\ &\quad X_7X_9X_{12} + M_2X_{14} + X_1X_2X_5X_{10}X_{11} - X_2X_7X_{14}. \end{aligned}$$

It is very clear that fields $X_8, M_1', X_5, M_2', X_{14}, M_2$ are all massive. Furthermore, we need to add the meson part

$$W_{meson} = M_1 q_1' q_1 - M_2 q_1 q_2' - M_1' q_1' q_2 + M_2' q_2' q_2$$

where we determine the sign as follows: since the term $M_1' X_8$ in W' in positive, we need term $M_1' q_1' q_2$ to be negative. After integration all massive fields, we get the superpotential as

$$\begin{aligned} W_{II} &= -q_1' q_2 X_9 X_{11} - X_1 M_1 X_{10} X_{12} + X_7 X_9 X_{12} \\ &\quad + X_1 X_2 q_2' q_2 X_{10} X_{11} - X_2 X_7 q_1 q_2' + M_1 q_1' q_1. \end{aligned}$$

The charge matrix now becomes

$$\begin{pmatrix} & q_1 & q_2 & q_1' & q_2' & X_1 & X_2 & X_7 & X_9 & X_{10} & X_{11} & M_1 & X_{12} \\ A & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ B & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ D & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ E & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ F & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 0 \end{pmatrix}$$

This is in precise agreement with [98]; very re-assuring indeed!

Without further ado let us present the remaining cases. The charge matrix for the third one (dualising node C of (I)) is

$$\begin{pmatrix} & q_1 & q_1' & q_2' & q_2 & X_5 & X_{12} & X_3 & X_8 & X_9 & M_1 & X_{10} & X_{11} & X_{13} & M_2 \\ A & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 & -1 & 0 & 0 & -1 & -1 \\ B & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ C & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ D & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ E & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ F & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \end{pmatrix}$$

with superpotential

$$\begin{aligned} W_{III} &= X_3X_8X_{13} - X_8X_9X_{11} - X_5q_2q_2'X_{13} - M_2X_3X_{10}X_{12} \\ &\quad +q_2q_1'X_9x_{12} + M_1X_5X_{10}X_{11} - M_1q_1q_1' + M_2q_1q_2'. \end{aligned}$$

Finally the fourth case (dualising node E of (III)) has the charge matrix

$$\begin{pmatrix} & q_1 & W_1 & W_2 & q_1' & q_2' & X_3 & X_8 & W_1' & W_2' & X_9 & M_1 & X_{11} & X_{13} & M_2 & p_1 & p_1' & p_2' & p_2 \\ A & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 \\ B & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ C & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ D & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ F & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with superpotential

$$\begin{aligned} W_{IV} &= X_3X_8X_{13} - X_8X_9X_{11} - W_1q_2'X_{13} - M_2X_3W_2' + q_1'X_9W_2 + M_1W_1'X_{11} \\ &\quad -M_1q_1q_1' + M_2q_1q_2' + W_1p_1p_1' - W_2p_1p_2' - W_1'p_2p_1' + W_2'p_2p_2' \end{aligned}$$

6.7 Conclusions

In [85, 93] a mysterious duality between classes of gauge theories on D-branes probing toric singularities was observed. Such a Toric Duality identifies the infrared moduli space of very different theories which are candidates for the world-volume theory on D3-branes at threefold singularities. On the other hand, [99, 81] have recognised certain brane-moves for brane configurations of certain toric singularities as Seiberg duality.

In this paper we take a unified view to the above. Indeed we have provided a physical interpretation for toric duality. The fact that the gauge theories share by definition the same moduli space motivates the proposal that they are indeed physically equivalent in the infrared. In fact, we have shown in detail that toric dual gauge theories are connected by Seiberg duality.

This task has been facilitated by the use of T-dual configurations of NS and D-branes, in particular brane intervals and brane diamonds [78]. These constructions show that the Seiberg duality corresponds in the singularity picture to a change of B-fields in the collapsed cycles. Hence, the specific gauge theory arising on D3-branes at a given singularity, depends not only on the geometry of the singularity, but also on the B-field data. Seiberg duality and brane diamonds provide us with the tools to move around this more difficult piece of the singular moduli space, and probe different phases.

This viewpoint is nicely connected with that in [85, 93], where toric duals were obtained as different partial resolutions of a given orbifold singularity, $C^3/(Z_3 \times Z_3)$, leading to equivalent geometries (with toric diagrams equivalent up to unimodular transformations). Specifically, the original orbifold singularity has a specific assignments of B-fields on its collapsed cycles. Different partial resolutions amount to choosing a subset of such cycles, and blowing up the rest. Hence, in general different partial resolutions leading to the same geometric singularity end up with different assignments of B-fields. This explains why different gauge theories, related by Seiberg duality, arise by different partial resolutions.

In particular we have examined in detail the toric dual theories for the generalised conifold $xy = z^2w^2$, the partial resolutions of $C^3/(Z_3 \times Z_3)$ exemplified by the complex cones over the zeroth Hirzebruch surface as well as the second del Pezzo surface. We have shown how these theories are equivalent under the above scheme by explicitly having

1. unimodularly equivalent toric data;

2. the matter content and superpotential related by Seiberg duality;

3. the T-dual brane setups related by brane-crossing and diamond duality.

The point d'appui of this work is to show that the above three phenomena are the same.

As a nice bonus, the physical understanding of toric duality has allowed us to construct new toric duals in cases where the partial resolution technique provided only one phase. Indeed the exponential running-time of the Inverse Algorithm currently prohibits larger embeddings and partial resolutions. Our new perspective greatly facilitates the calculation of new phases. As an example we have constructed three new phases for the cone over del Pezzo three one of which is in reassuring agreement with a recent work [98] obtained from completely different methods.

Another important direction is to understand the physical meaning of Picard-Lefschetz transformations. As we have pointed out in Section 7, PL transformation and Seiberg duality are really two different concepts even though they coincide for certain restricted classes of theories. We have provided examples of two theories which are related by one but not the other. Indeed we must pause to question ourselves. For those which are Seiberg dual but not PL related, what geometrical action does correspond to the field theory transformation. On the other hand, perhaps more importantly, for those related to each other by PL transformation but not by Seiberg duality, what kind of duality is realized in the dynamics of field theory? Does there exists a new kind of dynamical duality not yet uncovered??

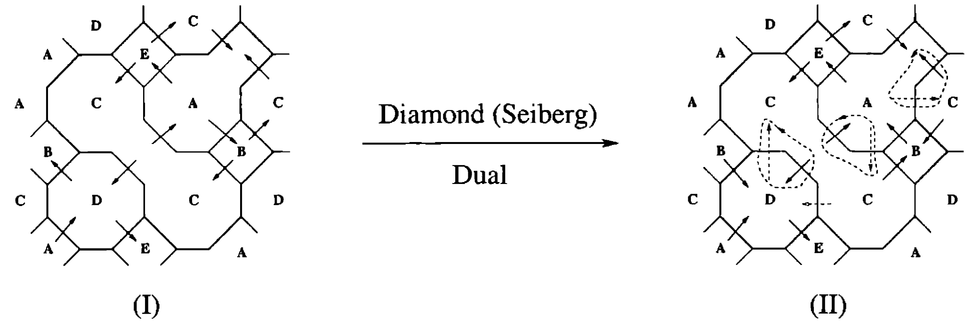


Figure 6-11: The brane diamond setup for the Seiberg dual configurations of the cone over dP_2. (I) is as in Figure 6-10 and (II) is the results after Seiberg (diamond) duality and gives the spectrum for the toric dual theory. The added meson fields are drawn in dashed blue lines. Notice that applying the diamond dual rules carelessly one gets some additional vectorlike pairs, shown in the picture within dotted lines. Such multiplets presumably get massive in the Seiberg dualization, hence we do not consider them in the quiver.

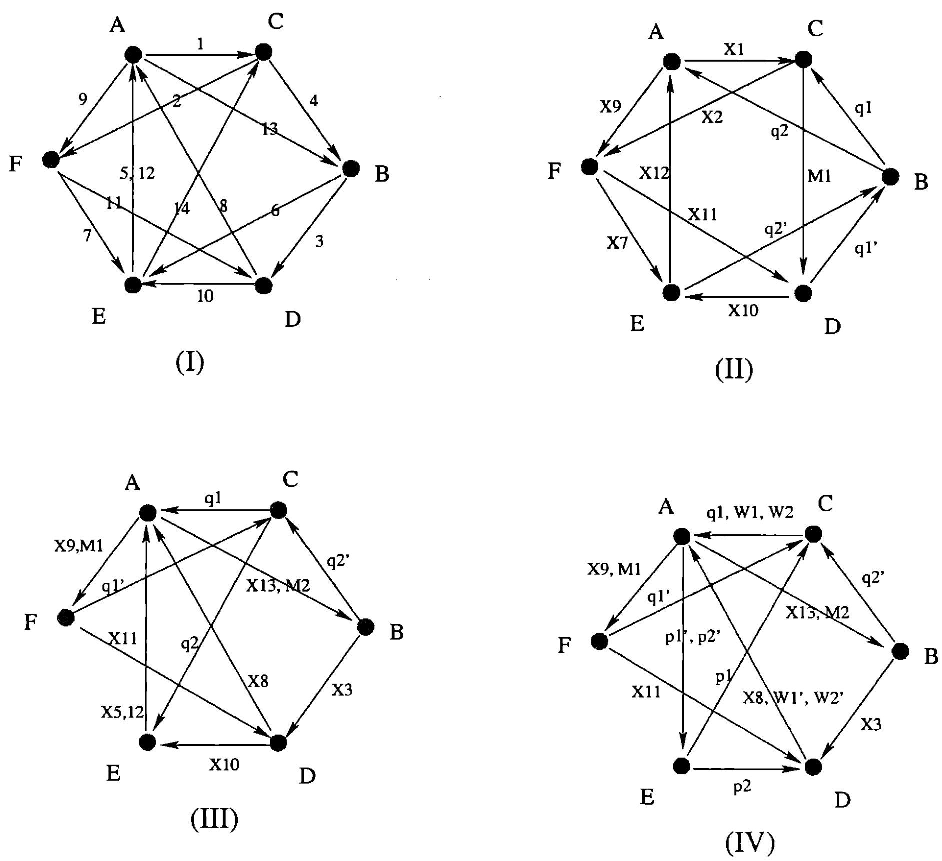


Figure 6-12: The four Seiberg dual phases of the cone over dP_3.

Chapter 7

The Identity String Field and the Tachyon Vacuum

7.1 Introduction

Regarding the fate of the tachyon in various systems such as brane-antibrane pairs in Type II theories as well as the D25-brane in the bosonic string theory, Sen proposed his famous three conjectures in [20, 126]. These state that (i) The difference in energy between the perturbative and the tachyon vacuum exactly cancels the tension of the corresponding D-brane system; (ii) After the tachyon condenses, all open string degrees of freedom disappear, leaving us with the closed string vacuum; and (iii) Non-trivial field configurations correspond to lower-dimensional D-branes.

Because tachyon condensation is an off-shell process[1], we must use the formalism of string field theory. Both Witten's cubic open string field theory [15] and his background independent open string field theory [15, 128, 129, 130] seem to be good candidates. Indeed, in the last two years, there has been a host of works aimed to understand Sen's three conjectures by using the above two string field theories as well as the non-linear sigma-model (Born-Infeld action) [131]. Thus far, Sen's first and third conjectures have been shown to be true to a very high level of accuracy ([132]

[1]For some early works concerning tachyon condensation please consult [127].

- [149]); they have also been proven analytically in Boundary String Field Theory ([150] - [152]). The second conjecture however, is still puzzling.

Let us clarify the meaning of this conjecture. From a physical point of view, after the tachyon condenses to the vacuum, the corresponding D-brane system disappears and there is no place for open strings to end on. Therefore at least all perturbative conventional open string excitations (of ghost number 1) should decouple from the theory. There has been a lot of work to check this statement, for example ([153]-[164]). In particular, using level truncation, [165] verifies that the scalar excitations at even levels (the Q closed scalar fields) are also Q-exact to very high accuracy.

However, as proposed in [115, 116] there is a little stronger version for the second conjecture. There, Rastelli, Sen and Zwiebach suggest that after a field redefinition, the new BRST operator may be taken[2] to be simply c_0, or more generally a linear combination of operators of the form $(c_n + (-)^n c_{-n})$. For such a new BRST operator, not only should the conventional excitations of ghost number 1 disappear, but more precisely the full cohomology of any ghost number of the new BRST operator around the tachyon vacuum vanishes[3]. Hence these authors propose that Sen's second conjecture should hold in such a stronger level. In fact, Sen's second conjecture suggests also that around the tachyon vacuum, there should be only closed string dynamics. However, we will not touch upon the issue of closed strings in our paper and leave the reader to the references [162, ?, 168, 172].

Considering the standing of the second conjecture, it is the aim of this paper to address to what degree does it hold, i.e., whether the cohomology of Q_{Ψ_0} is trivial only for ghost number 1 fields or for fields of any ghost number. We will give evidence which shows that the second conjecture holds in the strong sense, and is hence consistent with the proposal in [115, 116].

Our discussion relies heavily upon the existence of a string field $\mathcal{I}$ of ghost number

[2]The first String Field Theory action with pure ghost kinetic operator was written down in [167].

[3]An evidence for the triviality of a subset of the discrete ghost number one cohomology was presented recently in [166] which complemented [165].

0 which is the identity of the $\star$-product. It satisfies

$$\mathcal{I} \star \psi = \psi \star \mathcal{I} = \psi$$

for any state[4] ψ. The state $\mathcal{I}$ was first constructed in the oscillator basis in [17, 18]. Then a recent work [114] gave a recursive way of constructing the identity in the (background independent) total-Virasoro basis which shows its universal property in string field theory. As a by-product of our analysis, we have found a new and elegant analytic expression for $\mathcal{I}$ without recourse to the complicated recursions.

Ignoring anomalies, the fact that Q_{Ψ_0} is a derivation of the $\star$-algebra implies that $\mathcal{I}$ is Q_{Ψ_0} closed and the problem is to determine whether it is also Q_{Ψ_0} exact, i.e., if there exists a ghost number -1 field A, such that $\mathcal{I} = Q_{\Psi_0}A$. If so, then for an arbitrary Q_{Ψ_0} closed state ϕ we would have

$$\begin{aligned} Q_{\Psi_0}(A \star \phi) &= (Q_{\Psi_0}A) \star \phi - A \star (Q_{\Psi_0}\phi) \\ &= \mathcal{I} \star \phi \\ &= \phi, \end{aligned}$$

where in the second step, we used the fact that ϕ is Q_{Ψ_0}-closed, and in the last step, that $\mathcal{I}$ acts as the identity on ϕ. This means that any Q_{Ψ_0}-closed field ϕ is also Q_{Ψ_0}-exact, in other words, the entire cohomology of Q_{Ψ_0} is trivial.

Therefore we have translated the problem of the triviality of the cohomology of Q_{Ψ_0} into the issue of the exactness of the identity $\mathcal{I}$. In this paper, we will use the level truncation method to show that the state A indeed exists for the tachyon vacuum Ψ_0 up to an accuracy of 3.2%.

The paper is structured as follows. In Section 2, we explain the above idea of the exactness of $\mathcal{I}$ in detail. In Section 3, we use two different methods to find the state

[4]There are some mysteries regarding of the identity. For example, in [114] the authors showed that this identity string field is subject to anomalies, with consequences that $\mathcal{I}$ may be the identity of the $\star$-algebra only on a subspace of the whole Hilbert space. In the following, we will first assume that $\mathcal{I}$ behaves well on the whole Hilbert space, and postpone some discussions thereupon to Section 4.

A: one without gauge fixing and the other, in the Feynman-Siegel gauge. They give the results up to an accuracy of 2.4% and 3.2% respectively. In Section 4, we discuss the behaviour of $\mathcal{I}$ under level truncation and perform a few consistency checks on our approximations. Finally, in Section 5 we make some concluding remarks and address some further problems and directions.

A few words on nomenclature before we proceed. By $|0\rangle$ we mean the $SL(2,\mathbb{R})$-invariant vacuum and $|\Omega\rangle := c_1 |0\rangle$. We consider $|\Omega\rangle$ to be level 0 and hence $|0\rangle$ is level 1. Furthermore, in this paper we expand our fields in the universal basis (matter Virasoro and ghost oscillator modes).

7.2 The Proposal

To reflect the trivial cohomology of the BRST operator at the stable vacuum, Rastelli, Sen and Zwiebach [116] proposed that after a field redefinition, the new BRST operator Q_{new} may be taken to be simply c_0, or more generally a linear combination of operators of the form $(c_n + (-)^n c_{-n})$. For such operators, there is an important fact: there is an operator A such that

$$\{A, Q_{new}\} = I,$$

where I is the identity operator. For example, if $Q_{new} = c_n + (-)^n c_{-n}$, we can choose $A = \frac{1}{2}(b_{-n} + (-)^n b_n)$ because $\{\frac{1}{2}(b_{-n} + (-)^n b_n), Q_{new}\} = \{\frac{1}{2}(b_{-n} + (-)^n b_n), c_n + (-)^n c_{-n}\} = 1$. Therefore, if the state Φ is closed, i.e., $Q_{new}\Phi = 0$, then we have

$$\begin{aligned} \Phi &= \{A, Q_{new}\}\Phi = AQ_{new}\Phi + Q_{new}A\Phi \\ &= Q_{new}(A\Phi) \end{aligned} \tag{7.1}$$

which means that Φ is also exact. Thus the existence of such an A guarantees that the cohomology of Q_{new} is trivial.

In fact the converse is true. Given a Q_{new} which has vanishing cohomology we can always construct an A such that $\{A, Q_{new}\} = I$. Suppose that we denote the string

Hilbert space at ghost level g by V_g. Define the subspace V_g^C as the set of all closed elements of V_g. We can then pick a complement, V_g^N, to this subspace[5] satisfying $V_g = V_g^C \oplus V_g^N$. Note that it consist of vectors which are not killed by Q_{new}. This subspace V_g^N, is not gauge invariant but any specific choice will do. The important point is that because we have assumed that Q_{new} has no cohomology, the restriction of Q_{new} to V_g^N given by

$$Q_{new}\Big|_{V_g^N} : V_g^N \to V_{g+1}^C,$$

has no kernel and is surjective[6] on V_{g+1}^C. Thus it has an inverse which we denote A

$$A\Big|_{V_{g+1}^C} \equiv Q_{new}^{-1}\Big|_{V_{g+1}^C} : V_{g+1}^C \to V_g^N.$$

This insures that on the space V_g^C, $\{A, Q_{new}\} = I$ holds since if Φ is Q_{new}-closed,

$$\{A, Q_{new}\}\Phi = AQ_{new}\Phi + Q_{new}A\Phi = Q_{new}Q_{new}^{-1}\Phi = \Phi.$$

The above discussion only defines the action of A on V_g^C, what remains is to define its action on the complement V_g^N. Here there is quite a bit of freedom since one can choose any map that takes V_g^N into V_{g-1}^C. Assuming this, we have that for $\Phi \in V_g^N$,

$$\{A, Q_{new}\}\Phi = AQ_{new}\Phi + Q_{new}A\Phi = Q_{new}^{-1}Q_{new}\Phi + Q_{new}^2\chi = \Phi,$$

where by assumption $A\Phi$ is Q_{new}-closed (because it is in V_{g-1}^C) and thus equals $Q_{new}\chi$ for some $\chi \in V_{g-2}^N$. In general one can insist that A satisfies more properties. For example if we set $A|_{V_g^N} = 0$ we get that $A^2 = 0$. We summarize the above discussion as

PROPOSITION 7.2.1 *The cohomology of Q_{new} is trivial iff there exists an operator A such that $\{A, Q_{new}\} = I$.*

---

[5]More precisely, the space V_g could be split as $V_g = V_g^C \oplus (V_g/V_g^C)$ where (V_g/V_g^C) is a vector space of equivalence classes under the addition of exact states. V_g^N should be considered as a space of representative elements in (V_g/V_g^C).

[6]As remarked in the previous footnote, if we use (V_g/V_g^C) instead of V_g^N, the mapping is an isomorphism of vector spaces.

The basic hypothesis of this paper is that not only does such an operator A exist for Q_{Ψ_0}, but also for special choices of A, *the action of* A *can be expressed as the left multiplication by the ghost number* -1 *string field* which we denote as $A\star$. Thus we are now interested in satisfying the equation $\{A\star, Q_{new}\} = I$. Writing this out explicitly we have

$$\begin{aligned}
\{A\star, Q_{new}\}\Phi &= A\star(Q_{new}\Phi) + Q_{new}(A\star\Phi) \\
&= A\star Q_{new}(\Phi) + (Q_{new}A)\star\Phi - A\star(Q_{new}\Phi) \\
&= (Q_{new}A)\star\Phi.
\end{aligned}$$

In order for the last line to equal Φ for all Φ we need that

$$Q_{new}A = \mathcal{I}, \tag{7.2}$$

where $\mathcal{I}$ is the identity of the $\star$-algebra.

For the case of interest, we wish to study the physics around the minimum of the tachyon potential. We recall that for a state Φ, the new BRST operator around the solution ψ of the EOM is given by

$$Q_\psi\Phi = Q_B(\Phi) + \psi\star(\Phi) - (-)^\Phi(\Phi)\star\psi. \tag{7.3}$$

Using this expression for the BRST operator we can rewrite the basic equation (7.2) as $Q_\psi A = Q_B(A) + \psi\star(A) + (A)\star\psi = \mathcal{I}$. For general vacua ψ, such a string field A will not exist. For example in the perturbative vacuum, $\psi = 0$, Q_ψ is simply Q_B. It is easy to show here that there is no solution for A because the Q_B action preserves levels while $\mathcal{I}$ has a component at level one (namely $|0\rangle$), but the minimum level of a ghost number -1 state A is 3. Indeed, for a more general solution $\psi \neq 0$ (such as the tachyon vacuum), the star product will not preserve the level and so it may be possible to find A. Our endeavor will be to use the level truncation scheme to find A

for the tachyon vacuum Ψ_0, i.e., to find a solution A to the equation

$$Q_{\Psi_0} A = \mathcal{I}. \tag{7.4}$$

Note that this equation is invariant under

$$A \to A + Q_{\Psi_0} B$$

for some B of ghost number -2, thereby giving A a gauge freedom. This is an important property to which we shall turn in the next section.

Having expounded upon the properties of A, our next task is clear. In the following section, we show that for the tachyon vacuum Ψ_0, we can find the state A satisfying (7.4) in the approximation of the level truncation scheme.

7.3 Finding The State A

Let us now solve (7.4) by level truncation. To do so, let us proceed in two ways. We recall from the previous section that A is well-defined up to the gauge transformation $A \to A + Q_{\Psi_0} B$ where B is a state of ghost number -2. Because in the level truncation scheme, this gauge invariance is broken, we first try to find the best fit results without fixing the gauge of A. The fitting procedure is analogous to that used in [166] and we shall not delve too much into the details. We shall see below that at level 9, the result is accurate to 2.4%. However, when we check the behaviour of the numerical coefficients of A as we increase the accuracy from level 3 to 9, we found that they do not seem to converge. We shall explain this phenomenon as the consequence of the gauge freedom in the definition of A; we shall then redo the fitting in the Feynman-Siegel gauge. With this second method, we shall find that the coefficients do converge and the best fit at level 9 is to 3.2% accuracy. These results support strongly the existence of a state A in (7.4) and hence the statement that the cohomology around the tachyon vacuum is indeed trivial. In the following subsections let us present our methods and results in detail.

7.3.1 The Fitting without Gauge Fixing A

To solve the condition (7.4), we first need an explicit expression of the identity $\mathcal{I}$. Such an expression has been presented in [17] and [114], differing by a mere normalization factor $-4i$. In this paper, we will follow the conventions of [114] which has[7]

$$
\begin{aligned}
|\mathcal{I}\rangle &= e^{L_{-2}-\frac{1}{2}L_{-4}+\frac{1}{2}L_{-6}-\frac{7}{12}L_{-8}+\frac{2}{3}L_{-10}+\dots}\,|0\rangle \qquad (7.5)\\
&= |0\rangle + L_{-2}\,|0\rangle + \frac{1}{2}(L_{-2}^2 - L_{-4})\,|0\rangle \\
&+ \left(\frac{1}{6}L_{-2}^3 - \frac{1}{4}L_{-2}L_{-4} - \frac{1}{4}L_{-4}L_{-2} + \frac{1}{2}L_{-6}\right)|0\rangle \\
&+ \left(\frac{1}{24}L_{-2}^4 + \frac{1}{4}(L_{-2}L_{-6} + L_{-6}L_{-2}) + \frac{1}{8}L_{-4}^2 - \frac{7}{12}L_{-8}\right. \\
&\left. -\frac{1}{12}(L_{-2}^2L_{-4} + L_{-2}L_{-4}L_{-2} + L_{-4}L_{-2}^2)\right)|0\rangle \qquad (7.6)
\end{aligned}
$$

where $L_n = L_n^m + L_n^g$, the sum of the ghost (L_n^g) and matter (L_n^m) parts, is the total Virasoro operator. For later usage we have expanded the exponential up to level 9. Furthermore, we split L_n into matter and ghost parts and expand the latter into b_n, c_n operators as

$L_m^g := \sum\limits_{n=-\infty}^{\infty} (2m-n) : b_n c_{m-n} : -\delta_{m,0}$. In other words, we write the states in the so-called "Universal Basis" [114].

As a by-product, we have found an elegant expression for $\mathcal{I}$ which avoids the recursions[8] needed to generate the coefficients in the exponent. In fact, one can show that only L_{-m} for m being a power of 2 survive in the final expression, thus significantly reducing the complexity of the computation of level-truncation for $\mathcal{I}$:

$$
\begin{aligned}
|\mathcal{I}\rangle &= \left(\prod_{n=2}^{\infty} \exp\left\{-\frac{2}{2^n}L_{-2^n}\right\}\right) e^{L_{-2}}\,|0\rangle \\
&= \dots \exp(-\frac{2}{2^3}L_{-2^3})\exp(-\frac{2}{2^2}L_{-2^2})\exp(L_{-2})\,|0\rangle\,, \qquad (7.7)
\end{aligned}
$$

[7]With the normalization $\langle c_1, c_1, c_1\rangle = 3$ that we are using, we should scale this expression by a factor of $K^3/3$, where $K = 3\sqrt{3}/4$. However, as the normalization of the identity will not change our analysis, we will use this right normalization only in Section 4, where we are dealing with expressions like $\mathcal{I} \star \Phi$.

[8]Indeed the expression given in [18] has no recursion either, however their oscillator expansion is not normal-ordered due to ghost insertions at the string mid-point.

where we emphasize that the Virasoro's of higher index stack to the left *ad infinitum*. We leave the proof of this fact to the Appendix.

It is worth noticing that in the expansion of $\mathcal{I}$ only odd levels have nonzero coefficients. This means that we can constrain the solution A of (7.4), if it exists, to have only odd levels in its expansion. The reason for this is as follows. Equation (7.4) states that $Q_B A + \Psi_0 \star A + A \star \Psi_0 = \mathcal{I}$, moreover we recall that (cf. e.g. Appendix A.4 of [166]) the coefficient $k_{\ell,i}$ in the expansion of the star product $x \star y = \sum\limits_{\ell,i} k_{\ell,i}\psi_{\ell,i}$ is $k_{\ell,i} = \langle \tilde{\psi}_{\ell,i}, x, y\rangle$ for the orthogonal basis $\tilde{\psi}$ to ψ. Now the triple correlator has the symmetry $\langle x, y, z\rangle = (-)^{1+g(x)g(y)+\ell(x)+\ell(y)+\ell(z)}\langle x, z, y\rangle$, where $g(x)$ and $\ell(x)$ are the ghost number and level of the field x respectively. Whence, one can see that the even levels of $\Psi_0 \star A + A \star \Psi_0$ will be zero because the tachyon vacuum Ψ_0 has only even levels and A is constrained to odd levels. Furthermore, $Q_B = \sum\limits_n c_n L^m_{-n} + \frac{1}{2}(m-n) : c_m c_n b_{-m-n} : -c_0$ preserves level. Therefore, in order that both the left and right hand sides of (7.4) have only odd levels, A must also have only odd level fields.

Now the procedure is clear. We expand A into odd levels of ghost number -1 with coefficients as parameters and calculate $Q_{\Psi_0}A$. Indeed as with [166], all the states will be written as Euclidean vectors whose basis is prescribed by the fields at a given level; the components of the vectors are thus the expansion coefficients in each level. Then we compare $Q_{\Psi_0}A$ with $\mathcal{I}$ up to the same level and determine the coefficients of A by minimizing the quantity

$$\epsilon = \frac{|Q_{\Psi_0}A - \mathcal{I}|}{|\mathcal{I}|},$$

which we of course wish to be as close to zero as possible. We refer to this as the *"fitting of the coefficients"*. The norm $|.|$ is the Euclidean norm (for our basis, see the Appendix) . As observed in [165], different normalizations do not significantly change the values from the fitting procedure, so for simplicity we use the Euclidean norm to define the above measure of proximity ϵ. The minimum level of the ghost number -1 field A is 3, so we start our fitting from this level and continue to up to level 9 (higher levels will become computationally prohibitive).

First we list the number of components of odd levels for the fields A and $\mathcal{I}$ up to given levels:

	level 3	level 5	level 7	level 9
Number of Components of A	1	4	14	43
Number of Components of $\mathcal{I}$	4	14	43	118

From this table, we see that at level 3, we have only one parameter to fit 4 components. At level 5, we have 4 parameters to fit 14 components. As the level is increased the number of components to be fitted increases faster that the number of free parameters. Therefore it is not a trivial fitting at all.

A up to level 3

At level 3 the identity is:

$$\begin{aligned}\mathcal{I}_3 &= |0\rangle + L_{-2}\,|0\rangle \\ &= |0\rangle - b_{-3}c_1\,|0\rangle - 2b_{-2}c_0\,|0\rangle + L^m_{-2}\,|0\rangle\end{aligned}$$

and we find the best fit of A (recall that at level 3 we have only 1 degree of freedom) to be

$$A_3 = 1.12237\, b_{-2}\,|0\rangle\,,$$

with an ϵ of 17.1%.

A up to level 5

Continuing to level 5, we have

$$\begin{aligned}\mathcal{I}_5 &= |0\rangle + L_{-2}\,|0\rangle + \tfrac{1}{2}(L^2_{-2} - L_{-4})\,|0\rangle \\ &= |0\rangle - b_{-3}c_1\,|0\rangle - 2b_{-2}c_0\,|0\rangle + L^m_{-2}\,|0\rangle + b_{-5}c_1\,|0\rangle - b_{-2}c_{-2}\,|0\rangle \\ &\quad + b_{-3}c_{-1}\,|0\rangle + 2b_{-3}b_{-2}c_0c_1\,|0\rangle + 2b_{-4}c_0\,|0\rangle - \tfrac{1}{2}L^m_{-4}\,|0\rangle \\ &\quad - b_{-3}c_1L^m_{-2}\,|0\rangle - 2b_{-2}c_0L^m_{-2}\,|0\rangle + \tfrac{1}{2}L^m_{-2}L^m_{-2}\,|0\rangle\,.\end{aligned}$$

To this level we have determined the best-fit A to be

$$A_5 = 1.01893\, b_{-2}\,|0\rangle + 0.50921\, b_{-3}b_{-2}c_1\,|0\rangle - 0.518516\, b_{-4}\,|0\rangle + 0.504193\, b_{-2}L^m_{-2}\,|0\rangle\,,$$

with an ϵ of 11.8%.

The detailed data of the field A to levels 7 and 9 are given in table 7.9 of the Appendix. Here we just summarize the results of the best-fit measure ϵ:

	level 3	level 5	level 7	level 9
$\epsilon = \frac{\lvert Q_{\Psi_0}A-\mathcal{I}\rvert}{\lvert\mathcal{I}\rvert}$	0.171484	0.117676	0.0453748	0.0243515

This indicates that up to an accuracy of 2.4% at level 9, there exists an A that satisfies (7.4); moreover the accuracy clearly gets better with increasing levels. This is truly an encouraging result.

7.3.2 The Stability of Fitting

There is a problem however. Looking carefully at the coefficients of A given in the table 7.9, especially the fitting coefficients between levels 7 and 9, we see that these two groups of data have a large difference. Naively it means that our solution for A does not converge as we increase level. How do we solve this puzzle?

We recall that A is well-defined only up to the gauge freedom

$$A \longrightarrow A + Q_{\Psi_0}B.$$

It means that the solutions of (7.4) should consist of a family of gauge equivalent A. However, because $Q^2_{\Psi_0} \neq 0$ under the level truncation approximation, the family (or the moduli space) is broken into isolated pieces. Similar phenomena were found in [165] where the momentum-dependent closed states were given by points instead of a continuous family. Using this fact, our explanation is that the fitting of levels 7 and 9 are related by $Q_{\Psi_0}B$ for some field B of ghost number -2. To show this, we solve a new $\tilde{A}$ up to level 9 that minimizes

$$\frac{|(\tilde{A})_7 - A_7|}{|A_7|} + \frac{|Q_{\Psi_0}\tilde{A} - \mathcal{I}_9|}{|\mathcal{I}_9|}$$

where A_7 is the known fitting data at level seven, $\mathcal{I}_9$ is the identity up to level nine and $(\tilde{A})_7$ refers to the first 14 components (i.e., the components up to level seven) of the level 9 expansion of $\tilde{A}$. By minimizing this above quantity, we balance the stability of fitting from level 7 to 9. The data is given in the last column of 7.9. Though having gained stability, the fitting for level 9 is a little worse, with ϵ increasing from 2.44% to 3.56%.

The next thing is to check whether $\tilde{A} - A_9$ is an exact state $Q_{\Psi_0}B$. We find that this is indeed true and we find a state B such that

$$\frac{|(\tilde{A} - A_9) - Q_{\Psi_0}B|}{|\tilde{A} - A_9|} = 0.28\%.$$

7.3.3 Fitting A in the Feynman-Siegel Gauge

Alternatively, by gauge-fixing, we can also avoid the instability of the fit. If we require the state A to be in the Feynman-Siegel gauge, A will not have the gauge freedom anymore and the fitting result should converge as we do not have isolated points in the gauge moduli space to jump to. We have done so and do find much greater stability of the coefficients.

Notice that in the Feynman-Siegel gauge, A has the same field bases in levels 3 and 5, so the fitting at these two levels is the same as in Subsections 3.1.1 and 3.1.2. However, in this gauge it has one parameter less at level 7 and 5 less in level 9. Performing the fit with these parameters we have reached an accuracy of $\epsilon = 4.8\%$ at level 7 and $\epsilon = 3.2\%$ at level 9, which is still a good result. The details are presented in Table 7.10 in the Appendix.

7.4 Some Subtleties of the Identity

As pointed out in the Introduction, there are some mysterious and anomalous features of the identity $\mathcal{I}$. For example, $\mathcal{I}$ is not a normalizable state [169], moreover, c_0, contrary to expectation, does not annihilate $\mathcal{I}$ even though it is a derivation [114]. We shall show in the following that with a slight modification of the level truncation scheme, this unnormalizability does not effect the results and furthermore that in our approximation $Q_{\Psi_0}\mathcal{I}$ indeed vanishes as it must for consistency.

Let us first show how problems may arise in a naive attempt at level truncation. Consider the quantity $\mathcal{I}_\ell \star |\Omega\rangle - |\Omega\rangle$, where $\mathcal{I}_\ell$ denotes the identity truncated to level ℓ and $|\Omega\rangle := c_1 |0\rangle$. We of course expect this to approach 0 as we increase ℓ. Using the methods of the previous section, we shall define the measure of proximity

$$\eta \equiv \frac{|\mathcal{I}_\ell \star |\Omega\rangle - |\Omega\rangle\,|}{|\,|\Omega\rangle\,|} = |\mathcal{I}_\ell \star |\Omega\rangle - |\Omega\rangle\,|,$$

where $|.|$ is our usual norm. We list η to levels 3, 5, 7, and 9 in the following Table:

level ℓ	3	5	7	9
$\eta = \lvert\mathcal{I}_\ell \star \lvert\Omega\rangle - \lvert\Omega\rangle\,\rvert$	2.06852	2.87917	3.56054	3.9452

Our η obviously does not converge to zero, hence star products involving $\mathcal{I}$ do not converge in the usual sense of level truncation. It is however not yet necessary to despair, as weak convergence will come to our rescue[9].

Indeed, instead of truncating the result to level ℓ, let us use a slightly different scheme. We truncate $\mathcal{I}_\ell \star |\Omega\rangle$ to a fixed level $m < \ell$ and observe how the coefficients of the fields up to level m converge as we increase ℓ. In the following table we list the values of the coefficients coeff(x) of the basis for $m = 2$ (i.e., fields x of level 0, 1 and

[9]We thank B. Zwiebach for this suggestion.

2) for the expression $\mathcal{I}_\ell \star |\Omega\rangle$.

$\mathcal{I}_\ell \star \|\Omega\rangle$	coeff($\|\Omega\rangle$)	coeff($b_{-1}c_0 \|\Omega\rangle$)	coeff($b_{-1}c_{-1} \|\Omega\rangle$)	coeff($b_{-2}c_0 \|\Omega\rangle$)	coeff($L^m_{-2} \|\Omega\rangle$)
$\ell = 3$	0.6875	0.505181	-0.905093	-0.930556	0.465278
$\ell = 5$	1.16898	-0.278874	0.38846	0.520748	-0.260374
$\ell = 7$	0.911094	0.16252	-0.197833	-0.296607	0.148304
$\ell = 9$	1.05767	-0.0971502	0.0902728	0.163579	-0.0817895

We see that the $|\Omega\rangle$ component converges to 1 while the others converge to 0, as was hoped. We note however that this (oscillating) convergence is rather slow and we thus expect slow weak convergence for other calculations involving the identity.

Having shown that as $\ell \to \infty$ we get a weak convergence $\mathcal{I}_\ell \star |\Omega\rangle \to |\Omega\rangle$, we now consider $Q_{\Psi_0}\mathcal{I}_\ell$ as $\ell \to \infty$, which should tend to zero. Since Q_B preserves level and $Q_B\mathcal{I} = 0$, we have that $Q_B\mathcal{I} = 0$ in the level expansion; thus $Q_{\Psi_0}\mathcal{I} = \Psi_0 \star \mathcal{I} - \mathcal{I} \star \Psi_0$, which should converge to zero.

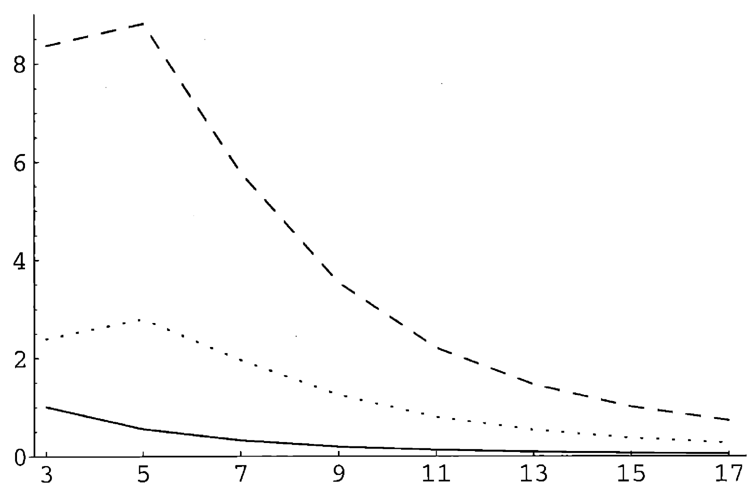


Figure 7-1: A plot of $q_{0,1}(\ell)$ (solid curve), $q_{2,1}(\ell)$ (dotted curve) and $q_{2,3}(\ell)$ (dashed curve) as functions of the level ℓ of the identity. ℓ goes from 3 to 17.

As the expression $Q_{\Psi_0}\mathcal{I}$ is linear in every component of Ψ_0, that $\mathcal{I}$ is Q_{Ψ_0}-closed will be established if we can show that for each component ϕ in Ψ_0, $\phi\star\mathcal{I}-\mathcal{I}\star\phi \equiv [\phi\star, \mathcal{I}]$ converges to zero as the level of $\mathcal{I}$ is increased. We plot in Fig.7-1, the absolute values of the coefficient of $c_0 |0\rangle$ in the expressions $[(c_1 |0\rangle)\star, \mathcal{I}_\ell]$, $[(c_{-1} |0\rangle)\star, \mathcal{I}_\ell]$ and

$\left[(L^m_{-2}c_1)\left|0\right\rangle \star, \mathcal{I}_\ell\right]$, which we denote by $q_{0,1}(\ell)$, $q_{2,1}(\ell)$ and $q_{2,3}(\ell)$ respectively. It seems clear that the coefficients do converge to zero.

The weak convergence we have shown above can be interpreted in a more abstract setting. Let us examine the quantity $|\mathcal{I}_\ell \star \Phi - \Phi|$. It was shown in [170] that the $\star$-algebra of the open bosonic string field theory is a C^*-algebra. A well-known theorem dictates that any C^*-algebra M (with or without unit) has a so-called *approximate identity* which is a set of operators $\{\mathcal{I}_i\}$ in M indexed by i satisfying (i) $||\mathcal{I}_i|| \leq 1$ for every i and (ii) $||\mathcal{I}_i x - x|| \to 0$ and $||x\mathcal{I}_i - x|| \to 0$ for all $x \in M$ with respect to the (Banach) norm $||.||$ of M (cf. e.g. [171]).

The level ℓ in our level truncation scheme is suggestive of an index for $\mathcal{I}$. Furthermore the weak convergence we have found in this section is analogous to property (ii) of the theorem (being of course a little cavalier about the distinction of the Banach norm of the C^*-algebra with the Euclidean norm used here). Barring this subtlety, it is highly suggestive that our $\mathcal{I}_\ell$ is an approximate identity of the $\star$-algebra indexed by level ℓ.

7.5 Conclusion and Discussions

According to a strong version of Sen's Second Conjecture, there should be an absence of any open string states around the perturbatively stable tachyon vacuum Ψ_0. This disappearance of all states, not merely the physical ones of ghost number 1, means that the cohomology of the new BRST operator Q_{Ψ_0} should be completely trivial near the vacuum. It is the key observation of this paper that this statement of triviality is implied by the existence of a ghost number -1 field A satisfying

$$Q_{\Psi_0} A = Q_B A + \Psi_0 \star A + A \star \Psi_0 = \mathcal{I}.$$

That is to say that if the identity of the $\star$-algebra $\mathcal{I}$ is a Q_{Ψ_0} exact state, then the cohomology of Q_{Ψ_0} would be trivial.

The level truncation scheme was subsequently applied to check our proposal. We have found that such a state A exists up to an accuracy of 3.2% at level 9. Although these numerical results give a strong support to the proposal for the existence of A and hence the triviality of Q_{Ψ_0}-cohomology near the vacuum, an analytic expression for A would be most welcome. However, to obtain such an analytic form of A, it seems that we would require the analytic expression for the vacuum Ψ_0, bringing us back to an old problem. It is perhaps possible that by choosing different gauges other than the Feynman-Siegel gauge we may find such a solution.

Our solution A satifies $\{A, Q\} = I$. It would be nice to see whether we can choose A cleverly to make $A \star A = 0$ (our Feynman-Siegel gauge fitting may not satisfy this equation). We are interested with this case because for the proposal of $Q_B = c_n + (-)^n c_{-n}$ made in [115, 116], one could find that $A = \frac{1}{2}(b_{-n} + (-)^n b_n)$ which does satisfy $A^2 = 0$. It would be interesting to mimic this nilpotency within the $\star$-algebra. Furthermore, it would be fascinating to see if we can make a field redefinition to reduce A to a simple operator such as b_0, and at the same time reduce Q_{Ψ_0} to a new BRST operator as suggested in [116], for example, c_0.

Last but not least, an interesting question is about the identity $\mathcal{I}$. In this paper we have given an elegant analytic expression for $\mathcal{I}$ which avoids the usage of complicated recursion relations. Furthermore, we have suggested that though the $\star$-algebra of OSFT may be a non-unital C^*-algebra, $\mathcal{I}$ still may serve as a so-called approximate identity. However, as we discussed before, anomalies related to the identity in the String Field Theory make the calculation in level truncation converge very slowly. It will be useful to understand more about $\mathcal{I}$.

7.6 Appendix

7.6.1 The Perturbatively Stable Vacuum Solution at Level $(M, 3M)$

We tabulate the coefficient of the expansion of the stable vacuum solution Ψ_0 at various levels and interaction [135].

$gh = 1$ field basis	level $(2, 6)$	level $(4, 12)$	level $(6, 18)$	level $(8, 24)$
$\|\Omega\rangle$	0.3976548947184288	0.4007200390749924	0.4003790755638671	0.39973608190423154
$b_{-1}c_{-1}\|\Omega\rangle$	−0.1389738152295008	−0.1502869559917484	−0.15477497270540513	−0.15712091953765914
$L^m_{-2}\|\Omega\rangle$	0.0408931493261807	0.04159452148973691	0.04175525359702033	0.041806849347695574
$b_{-1}c_{-3}\|\Omega\rangle$		0.041073385934010505	0.041936906548529496	0.042358626301118626
$b_{-2}c_{-2}\|\Omega\rangle$		0.02419174563180113	0.02489022878379843	0.025301843897808124
$b_{-3}c_{-1}\|\Omega\rangle$		0.013691128644670262	0.013978968849509828	0.014119542100372846
$L^m_{-4}\|\Omega\rangle$		−0.003741923212578628	−0.0037331617302832193	−0.0037279001402683682
$b_{-1}c_{-1}L^m_{-2}\|\Omega\rangle$		0.005013189182427192	0.005410660944694899	0.005620705137023851
$L^m_{-2}L^m_{-2}\|\Omega\rangle$		−0.00043064009114185083	−0.0004545462255696699	−0.0004654022166127481
$b_{-1}c_{-5}\|\Omega\rangle$			−0.02193107815206234	−0.022161386573208323
$b_{-2}c_{-4}\|\Omega\rangle$			−0.013702048066242712	−0.01385275004340868
$b_{-3}c_{-3}\|\Omega\rangle$			−0.00834273227278023	−0.008359650003474304
$b_{-4}c_{-2}\|\Omega\rangle$			−0.0068510240331213544	−0.0069263750217043295
$b_{-5}c_{-1}\|\Omega\rangle$			−0.004386215630412471	−0.0044322773146416965
$b_{-2}b_{-1}c_{-2}c_{-1}\|\Omega\rangle$			−0.005651485281802872	−0.00580655453652034
$L^m_{-6}\|\Omega\rangle$			0.0010658398347450269	0.0010617366766707361
$b_{-1}c_{-1}L^m_{-4}\|\Omega\rangle$			−0.0008498595740547494	−0.0008732233330861659
$b_{-1}c_{-2}L^m_{-3}\|\Omega\rangle$			−0.000046769138331183204	−0.000052284121618944255
$b_{-2}c_{-1}L^m_{-3}\|\Omega\rangle$			−0.000023384569165591568	−0.000026142060809472097
$L^m_{-3}L^m_{-3}\|\Omega\rangle$			$4.479437511126653 \times 10^{-6}$	$5.080488681869039 \times 10^{-6}$
$b_{-1}c_{-3}L^m_{-2}\|\Omega\rangle$			−0.002457790374962076	−0.002528657337188949
$b_{-2}c_{-2}L^m_{-2}\|\Omega\rangle$			−0.0020680241879350277	−0.002125416342663475
$b_{-3}c_{-1}L^m_{-2}\|\Omega\rangle$			−0.0008192634583206926	−0.0008428857790629816
$L^m_{-4}L^m_{-2}\|\Omega\rangle$			0.00022330350231085353	0.00022500193649010967
$b_{-1}c_{-1}L^m_{-2}L^m_{-2}\|\Omega\rangle$			−0.00011131535311028013	−0.00012817322136544294
$L^m_{-2}L^m_{-2}L^m_{-2}\|\Omega\rangle$			$-7.241008154399294 \times 10^{-6}$	$-6.240064701718801 \times 10^{-6}$

continued...

$gh = 1$ field basis	level $(2,6)$	level $(4,12)$	level $(6,18)$	level $(8,24)$
$b_{-1}c_{-7}\ket{\Omega}$				0.014312021693536028
$b_{-2}c_{-6}\ket{\Omega}$				0.009158200585940239
$b_{-3}c_{-5}\ket{\Omega}$				0.005674268936470511
$b_{-4}c_{-4}\ket{\Omega}$				0.004838957768226669
$b_{-5}c_{-3}\ket{\Omega}$				0.0034045613618823045
$b_{-6}c_{-2}\ket{\Omega}$				0.0030527335286467446
$b_{-2}b_{-1}c_{-3}c_{-2}\ket{\Omega}$				−0.0035422558218537676
$b_{-7}c_{-1}\ket{\Omega}$				0.0020445745276480116
$b_{-2}b_{-1}c_{-4}c_{-1}\ket{\Omega}$				0.0037527555019998804
$b_{-3}b_{-1}c_{-3}c_{-1}\ket{\Omega}$				0.0004302428004449616
$b_{-3}b_{-2}c_{-2}c_{-1}\ket{\Omega}$				−0.0011807519406179202
$b_{-4}b_{-1}c_{-2}c_{-1}\ket{\Omega}$				0.0018763777509999383
$L^m_{-8}\ket{\Omega}$				−0.00041801038699211334
$b_{-1}c_{-1}L^m_{-6}\ket{\Omega}$				0.00029329813765991303
$b_{-1}c_{-2}L^m_{-5}\ket{\Omega}$				$6.281489731737461 \times 10^{-6}$
$b_{-2}c_{-1}L^m_{-5}\ket{\Omega}$				$3.140744865868727 \times 10^{-6}$
$b_{-1}c_{-3}L^m_{-4}\ket{\Omega}$				0.000500528172313894
$b_{-2}c_{-2}L^m_{-4}\ket{\Omega}$				0.00030379159554779373
$b_{-3}c_{-1}L^m_{-4}\ket{\Omega}$				0.00016684272410463048
$L^m_{-4}L^m_{-4}\ket{\Omega}$				−0.000021999720024591806
$b_{-1}c_{-4}L^m_{-3}\ket{\Omega}$				0.00003496149452657495
$b_{-2}c_{-3}L^m_{-3}\ket{\Omega}$				$-3.2753561169368668 \times 10^{-6}$
$b_{-3}c_{-2}L^m_{-3}\ket{\Omega}$				$-2.1835707446245427 \times 10^{-6}$
$b_{-4}c_{-1}L^m_{-3}\ket{\Omega}$				$8.74037363164371 \times 10^{-6}$
$L^m_{-5}L^m_{-3}\ket{\Omega}$				$-1.3196771313891132 \times 10^{-6}$
$b_{-1}c_{-1}L^m_{-3}L^m_{-3}\ket{\Omega}$				$1.2594432286572633 \times 10^{-6}$
$b_{-1}c_{-5}L^m_{-2}\ket{\Omega}$				0.001534533432927412
$b_{-2}c_{-4}L^m_{-2}\ket{\Omega}$				0.0013556709245221895
$b_{-3}c_{-3}L^m_{-2}\ket{\Omega}$				0.0006166063072874846
$b_{-4}c_{-2}L^m_{-2}\ket{\Omega}$				0.0006778354622610939
$b_{-5}c_{-1}L^m_{-2}\ket{\Omega}$				0.00030690668658548353
$b_{-2}b_{-1}c_{-2}c_{-1}L^m_{-2}\ket{\Omega}$				0.0005782814358972997
$L^m_{-6}L^m_{-2}\ket{\Omega}$				−0.00007624602726052426
$b_{-1}c_{-1}L^m_{-4}L^m_{-2}\ket{\Omega}$				0.00006375616369006518
$b_{-1}c_{-2}L^m_{-3}L^m_{-2}\ket{\Omega}$				$5.9626436110722614 \times 10^{-6}$
$b_{-2}c_{-1}L^m_{-3}L^m_{-2}\ket{\Omega}$				$2.9813218055361256 \times 10^{-6}$
$L^m_{-3}L^m_{-3}L^m_{-2}\ket{\Omega}$				$-5.422796727699355 \times 10^{-7}$
$b_{-1}c_{-3}L^m_{-2}L^m_{-2}\ket{\Omega}$				0.00004728162691342103
$b_{-2}c_{-2}L^m_{-2}L^m_{-2}\ket{\Omega}$				0.00010011937816215435
$b_{-3}c_{-1}L^m_{-2}L^m_{-2}\ket{\Omega}$				0.000015760542304474034
$L^m_{-4}L^m_{-2}L^m_{-2}\ket{\Omega}$				$-4.371565449219928 \times 10^{-6}$
$b_{-1}c_{-1}L^m_{-2}L^m_{-2}L^m_{-2}\ket{\Omega}$				$-3.759766768481099 \times 10^{-7}$
$L^m_{-2}L^m_{-2}L^m_{-2}L^m_{-2}\ket{\Omega}$				$7.259081254041818 \times 10^{-7}$

(7.8)

7.6.2 Fitting of the Parameters of A

A up to Level 9 without Gauge Fixing

As A is of ghost number -1 and has only odd levels, we here tabulate such field basis at levels 3, 5, 7 and 9. The best-fit numbers are the coefficients of A obtained by best-fit via minimizing $\epsilon = \frac{|Q_{\Psi_0}A - \mathcal{I}|}{|\mathcal{I}|}$. The stable fit at level 9 is constructed so as to control the convergence behaviour of the coefficients.

Field Basis	level 3 fit	level 5 fit	level 7 fit	level 9 fit	stable level 9 fit
$b_{-2}\|0\rangle$	1.12237	1.01893	0.948316	1.25995	0.931864
$b_{-3}b_{-2}c_1\|0\rangle$		0.50921	0.37306	0.660674	0.401547
$b_{-4}\|0\rangle$		−0.518516	−0.753272	−0.25828	−0.753004
$b_{-2}L^m_{-2}\|0\rangle$		0.504193	0.50695	0.400769	0.496562
$b_{-4}b_{-3}c_1\|0\rangle$			0.698601	−0.10683	0.691255
$b_{-5}b_{-2}c_1\|0\rangle$			0.893251	−1.8453	0.888407
$b_{-6}\|0\rangle$			−0.531323	1.40819	−0.541737
$-b_{-3}b_{-2}c_{-1}\|0\rangle$			−1.87167	3.14822	−1.86475
$-b_{-4}b_{-2}c_0\|0\rangle$			−2.54254	3.2966	−2.54625
$b_{-2}L^m_{-4}\|0\rangle$			0.264611	−0.750856	0.255304
$b_{-3}L^m_{-3}\|0\rangle$			0.00193005	−0.0539165	−0.0191971
$b_{-3}b_{-2}c_1L^m_{-2}\|0\rangle$			0.358002	0.301463	0.338645
$b_{-4}L^m_{-2}\|0\rangle$			−0.724095	0.163428	−0.744985
$b_{-2}L^m_{-2}L^m_{-2}\|0\rangle$			0.166002	0.180328	0.169096
$b_{-5}b_{-4}c_1\|0\rangle$				0.0796036	0.273844
$b_{-6}b_{-3}c_1\|0\rangle$				−1.09893	−0.107261
$b_{-7}b_{-2}c_1\|0\rangle$				0.847731	0.195816
$b_{-8}\|0\rangle$				−0.313743	−0.277211
$b_{-3}c_{-3}b_{-2}\|0\rangle$				−19.0376	−4.11409
$-b_{-4}b_{-2}c_{-2}\|0\rangle$				−0.147445	−0.626872
$-b_{-4}b_{-3}c_{-1}\|0\rangle$				1.80597	−0.0745503
$-b_{-5}b_{-2}c_{-1}\|0\rangle$				−0.172462	−0.356920
$b_{-4}b_{-3}b_{-2}c_0c_1\|0\rangle$				1.05994	−0.102556
$-b_{-5}b_{-3}c_0\|0\rangle$				1.48397	−0.319450
$-b_{-6}b_{-2}c_0\|0\rangle$				−0.784562	0.0949989
$b_{-2}L^m_{-6}\|0\rangle$				0.103719	−0.00879977
$b_{-3}L^m_{-5}\|0\rangle$				−0.530976	−0.0537990
$b_{-3}b_{-2}c_1L^m_{-4}\|0\rangle$				0.428303	0.0633010
$b_{-4}L^m_{-4}\|0\rangle$				0.114766	0.111182
$b_{-4}b_{-2}c_1L^m_{-3}\|0\rangle$				0.687831	0.200100
$b_{-5}L^m_{-3}\|0\rangle$				−0.165379	−0.134011
$-b_{-3}b_{-2}c_0L^m_{-3}\|0\rangle$				−2.72288	−0.722198
$b_{-2}L^m_{-3}L^m_{-3}\|0\rangle$				0.3427	0.0910701
$b_{-4}b_{-3}c_1L^m_{-2}\|0\rangle$				−0.01845	0.304266
$b_{-5}b_{-2}c_1L^m_{-2}\|0\rangle$				−0.628564	−0.137309
$b_{-6}L^m_{-2}\|0\rangle$				0.39923	0.195490
$-b_{-3}b_{-2}c_{-1}L^m_{-2}\|0\rangle$				−0.537685	−0.289167
$-b_{-4}b_{-2}c_0L^m_{-2}\|0\rangle$				0.951973	−0.288878
$b_{-2}L^m_{-4}L^m_{-2}\|0\rangle$				−0.237783	−0.0856879
$b_{-3}L^m_{-3}L^m_{-2}\|0\rangle$				−0.332135	−0.0868470
$b_{-3}b_{-2}c_1L^m_{-2}L^m_{-2}\|0\rangle$				0.128844	0.126029
$b_{-4}L^m_{-2}L^m_{-2}\|0\rangle$				−0.00185911	−0.160345
$b_{-2}L^m_{-2}L^m_{-2}L^m_{-2}\|0\rangle$				0.0403381	0.0402361
$\epsilon = \|Q_{\Psi_0}A - \mathcal{I}\|/\|\mathcal{I}\|$	0.171484	0.117676	0.0453748	0.0243515	0.0356226

(7.9)

Fitting A in the Feynman-Siegel gauge

As A enjoys the gauge freedom $A \to A + Q_{\Psi_0}B$, we can fix it to be in the Feynman-Siegel gauge. This is another way to control the convergence behaviour of the coefficients.

fields	level 3 fit	level 5 fit	level 7 fit	level 9 fit
$b_{-2}\|0\rangle$	1.12237	1.01893	1.12465	1.05322
$b_{-3}b_{-2}c_1\|0\rangle$		0.50921	0.467	0.500266
$b_{-4}\|0\rangle$		−0.518516	−0.503772	−0.53228
$b_{-2}L^m_{-2}\|0\rangle$		0.504193	0.476325	0.504269
$b_{-4}b_{-3}c_1\|0\rangle$			0.333428	0.326986
$b_{-5}b_{-2}c_1\|0\rangle$			−0.330557	−0.328381
$b_{-6}\|0\rangle$			0.346811	0.331188
$-b_{-3}b_{-2}c_{-1}\|0\rangle$			0.325862	0.327997
$-b_{-4}b_{-2}c_0\|0\rangle$			0	0
$b_{-2}L^m_{-4}\|0\rangle$			−0.166799	−0.164306
$b_{-3}L^m_{-3}\|0\rangle$			0.00133026	0.000334022
$b_{-3}b_{-2}c_1L^m_{-2}\|0\rangle$			0.341592	0.328637
$b_{-4}L^m_{-2}\|0\rangle$			−0.332864	−0.327326
$b_{-2}L^m_{-2}L^m_{-2}\|0\rangle$			0.1686	0.165931
$b_{-5}b_{-4}c_1\|0\rangle$				0.245489
$b_{-6}b_{-3}c_1\|0\rangle$				−0.253014
$b_{-7}b_{-2}c_1\|0\rangle$				0.250149
$b_{-8}\|0\rangle$				−0.257672
$b_{-3}c_{-3}b_{-2}\|0\rangle$				0.249999
$-b_{-4}b_{-2}c_{-2}\|0\rangle$				−0.256812
$-b_{-4}b_{-3}c_{-1}\|0\rangle$				0.246526
$-b_{-5}b_{-2}c_{-1}\|0\rangle$				−0.25213
$b_{-4}b_{-3}b_{-2}c_0c_1\|0\rangle$				0
$-b_{-5}b_{-3}c_0\|0\rangle$				0
$-b_{-6}b_{-2}c_0\|0\rangle$				0
$b_{-2}L^m_{-6}\|0\rangle$				0.00104113
$b_{-3}L^m_{-5}\|0\rangle$				0.0000151443
$b_{-3}b_{-2}c_1L^m_{-4}\|0\rangle$				−0.126025
$b_{-4}L^m_{-4}\|0\rangle$				0.12448
$b_{-4}b_{-2}c_1L^m_{-3}\|0\rangle$				−0.0004548
$b_{-5}L^m_{-3}\|0\rangle$				−0.000819122
$-b_{-3}b_{-2}c_0L^m_{-3}\|0\rangle$				0
$b_{-2}L^m_{-3}L^m_{-3}\|0\rangle$				0.0000905036
$b_{-4}b_{-3}c_1L^m_{-2}\|0\rangle$				0.250728
$b_{-5}b_{-2}c_1L^m_{-2}\|0\rangle$				−0.251499
$b_{-6}L^m_{-2}\|0\rangle$				0.250865
$-b_{-3}b_{-2}c_{-1}L^m_{-2}\|0\rangle$				0.249179
$-b_{-4}b_{-2}c_0L^m_{-2}\|0\rangle$				0
$b_{-2}L^m_{-4}L^m_{-2}\|0\rangle$				−0.123363
$b_{-3}L^m_{-3}L^m_{-2}\|0\rangle$				0.000457948
$b_{-3}b_{-2}c_1L^m_{-2}L^m_{-2}\|0\rangle$				0.126358
$b_{-4}L^m_{-2}L^m_{-2}\|0\rangle$				−0.125248
$b_{-2}L^m_{-2}L^m_{-2}L^m_{-2}\|0\rangle$				0.0406385
$\epsilon = \|Q_{\Psi_0}A - \mathcal{I}\|/\|\mathcal{I}\|$	0.171484	0.117676	0.0480658	0.0320384

(7.10)

Expansion of $\mathcal{I}$ up to level 9

Immediately below the field basis at ghost number 0 and levels 1, 3, 5, 7 and 9 is given the coefficient of the expansion of $\mathcal{I}$.

$\|0\rangle$	$b_{-3}c_1\|0\rangle$	$-b_{-2}c_0\|0\rangle$	$L^m_{-2}\|0\rangle$	$b_{-5}c_1\|0\rangle$
1	−1	2	1	1
$-b_{-2}c_{-2}\|0\rangle$	$-b_{-3}c_{-1}\|0\rangle$	$b_{-3}b_{-2}c_0c_1\|0\rangle$	$-b_{-4}c_0\|0\rangle$	$L^m_{-4}\|0\rangle$
1	−1	2	−2	$-\frac{1}{2}$
$b_{-2}c_1L^m_{-3}\|0\rangle$	$b_{-3}c_1L^m_{-2}\|0\rangle$	$-b_{-2}c_0L^m_{-2}\|0\rangle$	$L^m_{-2}L^m_{-2}\|0\rangle$	$b_{-7}c_1\|0\rangle$
0	−1	2	$\frac{1}{2}$	−1
$-b_{-2}c_{-4}\|0\rangle$	$-b_{-3}c_{-3}\|0\rangle$	$b_{-3}b_{-2}c_{-2}c_1\|0\rangle$	$-b_{-4}c_{-2}\|0\rangle$	$b_{-4}b_{-2}c_{-1}c_1\|0\rangle$
0	−1	1	0	0
$-b_{-5}c_{-1}\|0\rangle$	$b_{-4}b_{-3}c_0c_1\|0\rangle$	$b_{-5}b_{-2}c_0c_1\|0\rangle$	$-b_{-6}c_0\|0\rangle$	$b_{-3}b_{-2}c_{-1}c_0\|0\rangle$
1	2	−2	2	2
$L^m_{-6}\|0\rangle$	$b_{-2}c_1L^m_{-5}\|0\rangle$	$b_{-3}c_1L^m_{-4}\|0\rangle$	$-b_{-2}c_0L^m_{-4}\|0\rangle$	$b_{-4}c_1L^m_{-3}\|0\rangle$
0	0	1/2	−1	0
$-b_{-2}c_{-1}L^m_{-3}\|0\rangle$	$-b_{-3}c_0L^m_{-3}\|0\rangle$	$L^m_{-3}L^m_{-3}\|0\rangle$	$b_{-5}c_1L^m_{-2}\|0\rangle$	$-b_{-2}c_{-2}L^m_{-2}\|0\rangle$
0	0	0	1	1
$-b_{-3}c_{-1}L^m_{-2}\|0\rangle$	$b_{-3}b_{-2}c_0c_1L^m_{-2}\|0\rangle$	$-b_{-4}c_0L^m_{-2}\|0\rangle$	$L^m_{-4}L^m_{-2}\|0\rangle$	$b_{-2}c_1L^m_{-3}L^m_{-2}\|0\rangle$
−1	2	−2	−1/2	0
$b_{-3}c_1L^m_{-2}L^m_{-2}\|0\rangle$	$-b_{-2}c_0L^m_{-2}L^m_{-2}\|0\rangle$	$L^m_{-2}L^m_{-2}L^m_{-2}\|0\rangle$	$b_{-9}c_1\|0\rangle$	$-b_{-2}c_{-6}\|0\rangle$
−1/2	1	1/6	1	0
$-b_{-3}c_{-5}\|0\rangle$	$b_{-3}b_{-2}c_{-4}c_1\|0\rangle$	$-b_{-4}c_{-4}\|0\rangle$	$b_{-4}b_{-2}c_{-3}c_1\|0\rangle$	$-b_{-5}c_{-3}\|0\rangle$
0	0	1	0	0
$b_{-4}b_{-3}c_{-2}c_1\|0\rangle$	$b_{-5}b_{-2}c_{-2}c_1\|0\rangle$	$-b_{-6}c_{-2}\|0\rangle$	$b_{-5}b_{-3}c_{-1}c_1\|0\rangle$	$b_{-6}b_{-2}c_{-1}c_0\|0\rangle$
0	−1	0	0	0
$-b_{-7}c_{-1}\|0\rangle$	$b_{-3}b_{-2}c_{-2}c_{-1}\|0\rangle$	$b_{-5}b_{-4}c_0c_1\|0\rangle$	$b_{-6}b_{-3}c_0c_1\|0\rangle$	$b_{-7}b_{-2}c_0c_1\|0\rangle$
−1	−1	2	−2	2
$-b_{-8}c_0\|0\rangle$	$b_{-3}b_{-2}c_{-3}c_0\|0\rangle$	$b_{-4}b_{-2}c_{-2}c_0\|0\rangle$	$b_{-4}b_{-3}c_{-1}c_0\|0\rangle$	$b_{-5}b_{-2}c_{-1}c_0\|0\rangle$
−2	2	−2	2	−2
$L^m_{-8}\|0\rangle$	$b_{-2}c_1L^m_{-7}\|0\rangle$	$b_{-3}c_1L^m_{-6}\|0\rangle$	$-b_{-2}c_0L^m_{-6}\|0\rangle$	$b_{-4}c_1L^m_{-5}\|0\rangle$
−1/4	0	0	0	0
$-b_{-2}c_{-1}L^m_{-5}\|0\rangle$	$-b_{-3}c_0L^m_{-5}\|0\rangle$	$b_{-5}c_1L^m_{-4}\|0\rangle$	$-b_{-2}c_{-2}L^m_{-4}\|0\rangle$	$-b_{-3}c_{-1}L^m_{-4}\|0\rangle$
0	0	−1/2	−1/2	1/2
$b_{-3}b_{-2}c_0c_1L^m_{-4}\|0\rangle$	$-b_{-4}c_0L^m_{-4}\|0\rangle$	$L^m_{-4}L^m_{-4}\|0\rangle$	$b_{-6}c_1L^m_{-3}\|0\rangle$	$-b_{-2}c_{-3}L^m_{-3}\|0\rangle$
−1	1	1/8	0	0
$-b_{-3}c_{-2}L^m_{-3}\|0\rangle$	$b_{-3}b_{-2}c_{-1}c_1L^m_{-3}\|0\rangle$	$-b_{-4}c_{-1}L^m_{-3}\|0\rangle$	$b_{-4}b_{-2}c_0c_1L^m_{-3}\|0\rangle$	$-b_{-5}c_0L^m_{-3}\|0\rangle$
0	0	0	0	0
$L^m_{-5}L^m_{-3}\|0\rangle$	$b_{-2}c_1L^m_{-4}L^m_{-3}\|0\rangle$	$b_{-3}c_1L^m_{-3}L^m_{-3}\|0\rangle$	$-b_{-2}c_0(L^m_{-3})^2\|0\rangle$	$b_{-7}c_1L^m_{-2}\|0\rangle$
0	0	0	0	−1
$-b_{-2}c_{-4}L^m_{-2}\|0\rangle$	$-b_{-3}c_{-3}L^m_{-2}\|0\rangle$	$b_{-3}b_{-2}c_{-2}c_1L^m_{-2}\|0\rangle$	$-b_{-4}c_{-2}L^m_{-2}\|0\rangle$	$b_{-4}b_{-2}c_{-1}c_1L^m_{-2}\|0\rangle$
0	−1	1	0	0
$-b_{-5}c_{-1}L^m_{-2}\|0\rangle$	$b_{-4}b_{-3}c_0c_1L^m_{-2}\|0\rangle$	$b_{-5}b_{-2}c_0c_1L^m_{-2}\|0\rangle$	$-b_{-6}c_0L^m_{-2}\|0\rangle$	$b_{-3}b_{-2}c_{-1}c_0L^m_{-2}\|0\rangle$
1	2	−2	2	2
$L^m_{-6}L^m_{-2}\|0\rangle$	$b_{-2}c_1L^m_{-5}L^m_{-2}\|0\rangle$	$b_{-3}c_1L^m_{-4}L^m_{-2}\|0\rangle$	$-b_{-2}c_0L^m_{-4}L^m_{-2}\|0\rangle$	$b_{-4}c_1L^m_{-3}L^m_{-2}\|0\rangle$
0	0	1/2	−1	0
$-b_{-2}c_{-1}L^m_{-3}L^m_{-2}\|0\rangle$	$-b_{-3}c_0L^m_{-3}L^m_{-2}\|0\rangle$	$(L^m_{-3})^2L^m_{-2}\|0\rangle$	$b_{-5}c_1(L^m_{-2})^2\|0\rangle$	$-b_{-2}c_{-2}(L^m_{-2})^2\|0\rangle$
0	0	0	1/2	1/2
$-b_{-3}c_{-1}(L^m_{-2})^2\|0\rangle$	$b_{-3}b_{-2}c_0c_1(L^m_{-2})^2\|0\rangle$	$-b_{-4}c_0(L^m_{-2})^2\|0\rangle$	$L^m_{-4}(L^m_{-2})^2\|0\rangle$	$b_{-2}c_1L^m_{-3}(L^m_{-2})^2\|0\rangle$
−1/2	1	−1	−1/4	0
$b_{-3}c_1(L^m_{-2})^3\|0\rangle$	$-b_{-2}c_0(L^m_{-2})^3\|0\rangle$	$(L^m_{-2})^4\|0\rangle$		
−1/6	1/3	1/24		

(7.11)

7.6.3 The Proof for the Simplified Expression for the Identity

In this section we wish to present the proof for the analytic expression for the identity as given in (7.7). We remind the reader of the expression:

$$
\begin{aligned}
|\mathcal{I}\rangle &= \left(\prod_{n=2}^{\infty} \exp\left\{-\frac{2}{2^n} L_{-2^n}\right\}\right) e^{L_{-2}} |0\rangle \\
&= \ldots \exp(-\frac{2}{2^3} L_{-2^3}) \exp(-\frac{2}{2^2} L_{-2^2}) \exp(L_{-2}) |0\rangle ,
\end{aligned}
\tag{7.12}
$$

or its BPZ conjugate form[10]

$$
\langle \mathcal{I}| = \langle 0| U_h U_{f_2} U_{f_3} U_{f_4} \ldots , \tag{7.13}
$$

where $U_{f_n} = e^{-\frac{2}{2^n} L_{2^n}}$ for $n \geq 2$ and $U_h = e^{L_2}$. In [114], the identity is given by $\langle \mathcal{I}| = \langle 0| U_{f_{\mathcal{I}}}$ where $U_{f_{\mathcal{I}}}$ is the operator corresponding to the function

$$
f_{\mathcal{I}}(z) = \frac{z}{1 - z^2}.
$$

Using the composition law $U_{g_1} U_{g_2} = U_{g_1 \circ g_2}$, what we need is to prove

$$
U_h U_{f_2} U_{f_3} U_{f_4} \ldots = U_{h \circ f_2 \circ f_3 \circ \ldots} = U_{f_{\mathcal{I}}}
$$

which is equivalent to proving

$$
\lim_{k \to \infty} h \circ f_2 \circ \ldots \circ f_k(z) = f_{\mathcal{I}}(z) = \frac{z}{1 - z^2}. \tag{7.14}
$$

[10]Please notice that, besides the replacement $L_n \to (-)^n L_{-n}$, the orders under BPZ-conjugation are also reversed. This is because we use L_n instead of the oscillators α_m, whose orders do not get reversed under BPZ.

For the operator $U_f = e^{aL_n}$, the corresponding function f is given by [173]

$$f(z) = \exp\left\{az^{n+1}\partial_z\right\} z = \frac{z}{(1 - anz^n)^{1/n}} ,$$

so we have

$$\begin{aligned} h(z) &= \tfrac{z}{(1-2z^2)^{1/2}} \\ f_n(z) &= \tfrac{z}{(1+2z^{2^n})^{1/2^n}}. \end{aligned}$$

A useful property of the f_n is that $f_n(z) = (g(z^{2^n}))^{1/2^n}$ where

$$g(z) := \frac{z}{1+2z} = \frac{1}{2+1/z}.$$

Before writing down the general form, first let us do an example:

$$\begin{aligned} f_2 \circ f_3 \circ f_4(z) &= f_2 \circ f_3[(g[z^{2^4}])^{1/2^4}] \\ &= f_2[(g[((g[z^{2^4}])^{1/2^4})^{2^3}])^{1/2^3}] \\ &= f_2[(g[g^{1/2}[z^{2^4}]])^{1/2^3}] \\ &= (g[((g[g^{1/2}[z^{2^4}]])^{1/2^3})^{2^2}])^{1/2^2} \\ &= (g[g^{1/2}[g^{1/2}[z^{2^4}]]])^{1/2^2} \\ &= (g^{1/2}[g^{1/2}[g^{1/2}[z^{2^4}]]])^{1/2}. \end{aligned}$$

Now it is easy to see that the general form is

$$h \circ f_2 \circ f_3 \circ \ldots \circ f_{k+1}(z) = h \circ (\underbrace{g^{\frac{1}{2}} \circ \ldots \circ g^{\frac{1}{2}}}_{k}(z^{2^{k+1}}))^{\frac{1}{2}}.$$

Thus equation (7.14) is equivalent to showing that

$$\lim_{k\to\infty} \underbrace{g^{\frac{1}{2}} \circ \ldots \circ g^{\frac{1}{2}}}_{k}(z^{2^{k+1}}) = (h^{-1}(f(z)))^2 = \frac{z^2}{1+z^4}.$$

The left hand side can be written as

$$(\underbrace{(2+(2+\ldots+(2}_{k}+1/z^{2^{k+1}})^{\frac{1}{2}}\ldots)^{\frac{1}{2}})^{\frac{1}{2}})^{-1} = z^2((2z^{2^2}+(2z^{2^3}+\ldots(2z^{2^{k+1}}+1)^{\frac{1}{2}}\ldots)^{\frac{1}{2}})^{\frac{1}{2}})^{-1}.$$

Thus (7.14) reduces to the verification of the equation

$$\lim_{k\to\infty}(2z^{2^2}+(2z^{2^3}+\ldots(2z^{2^{k+1}}+1)^{\frac{1}{2}}\ldots)^{\frac{1}{2}})^{\frac{1}{2}} = 1+z^{2^2}.$$

This can be done as follows. Consider first squaring both sides of the above equation and canceling $2z^{2^2}$ from the two sides, we get

$$\lim_{k\to\infty}(2z^{2^3}+\ldots(2z^{2^{k+1}}+1)^{\frac{1}{2}}\ldots)^{\frac{1}{2}})^{\frac{1}{2}} = 1+z^{2^3}.$$

Repeating the above operation k times, the left hand side gives 1 while the right hand side gives $1+z^{2^{k+2}}$. Thus as long as $z<1$, we get that the left and right hand sides do converge to each other as $k\to\infty$.

The list of works

In this part, we will list all works we have done in this periods.

- B. Feng, Y. H. He and N. Moeller, "Zeeman spectroscopy of the star algebra," arXiv:hep-th/0203175.
- B. Feng, Y. H. He and N. Moeller, "The spectrum of the Neumann matrix with zero modes," arXiv:hep-th/0202176.
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- B. Feng, Y. H. He, A. Karch and A. M. Uranga, "Orientifold dual for stuck NS5 branes," JHEP **0106**, 065 (2001) [arXiv:hep-th/0103177].
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- B. Feng, A. Hanany, Y. H. He and A. M. Uranga, "Toric duality as Seiberg duality and brane diamonds," JHEP **0112**, 035 (2001) [arXiv:hep-th/0109063].
- B. Feng, A. Hanany and Y. H. He, "Phase structure of D-brane gauge theories and toric duality," JHEP **0108**, 040 (2001) [arXiv:hep-th/0104259].

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- B. Feng, A. Hanany and Y. H. He, "The Z(k) x D(k') brane box model," JHEP **9909**, 011 (1999) [arXiv:hep-th/9906031].

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