I. Choice Elements in Multi-sectoral Planning

1. The length of the planning horizon

The choice elements in the multi-sectoral model are qualitatively the same as in the case previously discussed of an aggregative model. They are now only more numerous because of the necessity to interpret some of the variables in the present case as vector variables. We have the following list of choice elements:

(a) the length of the planning horizon, which may still be represented by a single number;

(b) the amounts and composition of the terminal capital stocks;

(c) the rate of growth of consumption during the planning period which must also be interpreted as a vector because we have diverse consumer goods;

(d) the initial levels from which consumption of different items are allowed to rise.

Initial levels of capital stock in the different sectors of the economy are assumed as before as part of the data.

Interrelationships between choice elements constrain our movement in such a fashion that if we specify any three of the elements (a), (b), (c) and (d), the fourth is automatically determined. Once we know elements (a) - (d), the entire timepath of output and capital accumulation (both in the vector sense) is determined as a consequence, over the planning period.

1. Since the time horizon is a single number while the other choice elements are vectors.
The above description suggests that the keynote of this model is consistency and not optimization. Following the specification of three of the choice elements, the objective is to find, if it exists at all, the fourth set which makes them all consistent. In finding this solution the annual levels of output and sectoral allocations of output and investment are determined and, thus, a plan is made in which all the parts are consistent with each other and the choice elements which have been fixed. Investigation of the consequences of alternative time paths of output may be carried out by use of this approach even without introducing optimization as an explicit part of the picture. This can be done by exploring the implications of alternative specifications of three of the four choice elements by means of successive solutions. Alternative rates of growth for the vector of consumption can be investigated or different assumptions can be made about the terminal position desired. For each specification, we can derive the corresponding time pattern of output and accumulation. The policy maker can then decide as to which particular configuration pleases him most.

All such solutions will be characterized by the full use of available capacity or a constant proportion of excess capacity. This raises the important question as to whether these are acceptable features of a planning model. Clearly, a more general formulation involving varying degrees of excess capacity could have solutions which, with a proper specification of the preference function in an optimizing model would dominate the solution corresponding to the no- or constant-excess-capacity case. In other words we may have sets of substitution rates among various desired objectives for which the best procedure is to have some capital idle for some period. To handle such a
general situation, we would need to formulate our planning problems in an explicitly optimizing framework. Since the solution of a proper dynamic programming model is beyond our immediate purpose, the extra possibilities opened up by relaxing the equational structure of the model may be temporarily ignored. However, the test of consistency alone would require that the proper non-negativity conditions are obeyed by all the variables throughout the planning period.

The object of this paper is to demonstrate how a consistent solution can be found following specification of: (a) the length of the planning horizon, (b) the terminal output levels and (c) the intermediate rates of growth of consumption. The discussion starts with a closed economy and with an unchanging set of structural coefficients. It then goes on to consider problems created by the variability of coefficients and, finally, indicates how the model may be adapted to deal with problems of international trade.

2. Description of the initial state

There are a large number of alternative ways of describing an initial state for a particular economy. Naturally, the specific description chosen will correspond to the nature of the model which is going to be set up. Since our primary interest is in the pattern of production in the different sectors of the economy, one natural choice would have been to choose the levels of output in the different sectors in the initial year. But if we are interested in long-term planning, the proper state variables would not be indicated by the initial output levels, but by the full capacity levels of output.
Further, if we assume away excess capacity over the planning period, we can translate if we wish the full capacity levels of output into equivalent estimates of capital stock provided that capital-output ratios are known in advance. However, because of the simplifying assumptions made in this paper, it is a matter of relative indifference if we use capital stock variables or output variables in describing the state of the economy.

3. Description of the terminal state

The terminal stock is described in terms of a set of output levels desired for the terminal year. By the nature of our model, we have what mathematicians call a two-point boundary value problem, which implies that variables describing the initial and end configurations are qualitatively the same. In our case, these variables constitute a vector $X(T)$, having the same dimensionality as $X(0)$. The final bill of goods in the terminal period, in addition to consumption, will provide for government purchase, capital formation and exports. Now, how do we determine this vector $X(T)$?

To be sure, there is an unavoidable element of arbitrariness about this choice. There are various possible types of end configurations that we may want to arrive at. Corresponding to any particular choice, we have the technical problem of translating this description into an equivalent description involving the $X(T)$'s.

Thus, if the consumption vector is to be set as a target, we could make an estimate of the population in the final year of the desired per capita consumption of different products. The expression, "desired per capita consumption", in practice requires to be spelled out in great detail. For one thing,
who determines what is desired and under what conditions, e.g., price-income configurations? If we take into account desires of different individuals, then, we have to bring in income-distribution as an additional important factor. The problem of knowing individuals preference maps may be sidetracked partly by extrapolating from time series data or by utilizing the cross-section information available. However there is an essential circularity in setting the terminal levels of consumption for those items for which consumption is itself dependent on the level of income. If a rigid connection were to be maintained between consumption and income, setting the level of the former would be equivalent to setting the terminal level of income. If the connection is not so rigid, some policy is implied to control the target consumption. In consistency models in which the terminal bill of goods is a consequence of the stipulation of the other three sets of choice elements, this type of problem in setting consumption targets does not exist. In the present "target" version of the consistency model a simple interpretation can be adopted of a planner setting a mix of products which satisfy certain minimums of nutrition, housing and so on.

An estimate of consumption alone is going to be inadequate as a specification of the desired terminal bill of goods, even putting aside government consumption and exports. We are, after all, interested in ensuring growth beyond the terminal year. As a first approximation let us make the provisional assumption that we want consumption of each type to be able to grow at \( r \) per cent per annum beyond the terminal year. This assumption of equiproportional growth as referring to the distant future has a considerable amount of analytical convenience. The levels of output in the different sectors
in the terminal year may be easily calculated once the information on the rate of growth is available, together with the likely values of structural coefficients.

We have the following balance equations:

\[ x_i(t) = \sum_j x_{ij}(T) + \sum_j i_{ij}(T) + c_i(T) \]  

(1)

\[ t = T, x_i(T) = \sum_j a_{ij} x_j(T) + \sum_j b_{ij} x_j(T) + c_i e^{rT} \]  

(2)

Our purpose now is to find a particular solution of this system of differential equations such that the \( x \)'s are growing at the same rate as \( c(t) \). In other words, we want to find a vector \( M \) such that \( x = Me^{rt} \) is a solution of (2). This is easily done by writing

\[ M_1 e^{rt} = \sum_j a_{ij} M_j e^{rt} + \sum_j b_{ij} M_j C e^{rt} + c_i e^{rt} \]  

(3)

or

\[ M_1 = \sum_j a_{ij} M_j + r \sum_j b_{ij} M_j + c_i \]  

(4)

after dividing both sides by \( e^{rt} \);

or

\[ M = AM + r BM + C \]

or

\[ (I - A - rB) M = C \]

Thus, \( M = (I - A - rB)^{-1} C \).  

(5)

\( M \) has meaningful solutions if \( (I - A - rB)^{-1} \) exists and contains non-negative elements. Since \( (A + rB) \) is a non-negative square matrix, this condition will be satisfied if \( \sum_j (a_{ij} + rb_{ij}) < 1 \) for all \( i \). Since \( r \) will normally be a small number, we can expect that in general, condition (5) will be satisfied. In fact, an upper bound on \( r \) may be computed from condition (5).

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2. One obtains the same time-free expression between \( M \) and \( C \) if the analysis proceeds in terms of discrete time units, which can be shown by reworking it in terms of difference equations.
The main advantage of the assumption of equiproportional growth is that we get a vector equation for \( M \), which is entirely independent of \( 't' \). This implies that if \( M \)'s are chosen to satisfy a simple static condition of compatibility as given by equation (5) the growth of the economy at \( r \) per cent per annum is ensured for all \( t \geq T \).

Now, the assumption of equiproportional growth beyond the terminal year is introduced here as a purely auxiliary device. The very nature of the planning problem requires that we must impose some end conditions to ensure a determinate solution. Thus, if the concept of equiproportional growth is held to be economically uninteresting, there is nothing that prevents us from considering non-proportional growth. Naturally, we do not expect the same degree of simplicity to characterize the non-proportional case. In fact, we would not get a simple time-independent restriction on \( M \)'s, because \( e^{rT} \) cannot be cancelled from both sides. This may be shown by means of algebraic reasoning.

However, we can incorporate non-proportionality in growth rates for one time period beyond the terminal data. This ensures the really crucial condition that not all the capital will be eaten up at the terminal date and that some explicit provision is made for growth to continue into the future. All that we actually require is that growth should be continued for one post-terminal period and then new long-term plans may be set up.

Let the demand for various consumption items, in particular, the \( i^{th} \) item be represented by the simple expression \( C_i (1 + r_i)^{t-T} \) for \( t \geq T \).

3. It is more appropriate to use discrete time units in this connection.
Now we want to ensure that at least for \( t = (T, T+1) \) the above equation holds.

For \( t = T \), this is equal to \( C_1 \). At \( t = T+1 \), the demand is \( C_1(l + r_1) \). Hence \( \Delta C_1 = rC_1 \).

Now, we have from the balance equation, \( X_1 = \sum_j a_{ij} x_j + \sum b_{ij} \Delta x_j + C_1 \) for all \( x^i's \).

Then the following incremental relationship must hold:

\[
\Delta x_1 = \sum_j a_{ij} \Delta x_j + \sum b_{ij} \Delta^2 x_j + \Delta C_1
\]

If we ignore \( \Delta^2 x_j \) as being of the second order of smallness, we may write

\[
\Delta x_1 = \sum_j a_{ij} \Delta x_j + \Delta C_1 \quad \text{Thus, } \Delta X = (I-A)^{-1} \Delta C \quad (6)
\]

\( \Delta X \) is then related to \( \Delta X \) by the relation \( \Delta K = b \Delta X \) where \( b \) is the matrix \( b_1 \ldots b_n \). We write in more detail:

\[
\Delta K = b(I-A)^{-1} \Delta C_1 \quad \text{Thus,}
\]

if \( \Delta C_1 \) is prescribed from outside, \( \Delta K_1 \) is known. Once \( K_1 \) is known, we can determine the vector of output requirements at \( t = T \) by using the simple relationship \( X(T) = (I-A)^{-1} (C_1(T) + \Delta K_1(T)) \). In this way, we can incorporate non-proportionality, for the time period \((T, T+1)\).

The above discussion indicates one particular way of describing the terminal state. The advantage of the particular procedures briefly discussed is that specification of a single number 'r' or, a vector \((r)\) denoting growth beyond the terminal year is enough, together with the structural information, to give us a vector of terminal stock requirements. By varying these magnitudes, we may easily trace out the sensitiveness of the timepath of the variables in which we are interested.

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4. This amounts to ignoring the special side effects, if any, which may arise due to the acceleration or deceleration of the rate of growth in the post-terminal period as compared to the terminal period.
4. The model

Having described the initial and terminal conditions in some detail we may now describe the manner in which the model is constructed in order to permit its solution when the intermediate rates of growth of consumption as well as the planning period are specified. This solution will indicate the use of the available resources to produce consumption and capital goods in the initial and subsequent periods. There are two sets of structural coefficients which much be known: current input-output and capital-output ratios.

We consider two subcases:

(i) Equal rates of growth of consumption for \( t < T \).

\[ X(t) = AX(t) + I(t) + C(t). \]

Here, \( A \) is the usual Leontief matrix of current input-output coefficients, \( I(t) \) is the vector of investment goods delivered by sectors, \( C(t) \) is the vector of consumption.

From the balance equation, we get

\[ (I-A) X(t) = I(t) + C(t) \]

or,

\[ (I-A) X(t) = B \dot{X}(t) + C(t) \]

where \( B \) is the matrix of intersectoral capital coefficients, \( \dot{X}(t) \) represents the rate of change of \( X \) at \( t \). Hence,

\[ X(t) = (I-A)^{-1} B \dot{X}(t) + (I-A)^{-1} C(t). \]

Let us now assume that \( C(t) = C e^{\beta t} \) where \( C \) represents the vector of initial consumption levels and \( \beta \) is the common exogenous rate at which consumption of various items is allowed to rise.

The solution to the differential equation written above has two parts. The first is obtained by solving the homogeneous equation:

\[ \dot{X}(t) = \left\{ (I-A)^{-1} B \right\} \dot{X}(t). \]

This has the solution

\[ X(t) = P e^{Nt} \]

where \( N = 1/M \) and \( M = \left\{ (I-A)^{-1} B \right\}. \]
The concept of a matrix exponential is always well defined for a constant matrix; \( P \) is a vector to be determined with the help of initial conditions.

For the non-homogeneous part, we derive the particular solution in the following way.

Put \( X(t) = Qe^{\rho t} \) where \( Q \) is a set of unknowns to be determined and \( \rho \) is the rate of growth of consumption. Substituting \( Qe^{\rho t} \) in the differential equation, we get

\[
Qe^{\rho t} - M\rho Qe^{\rho t} = (I-A)^{-1}Ce^{\rho t}.
\]

\( e^{\rho t} \) cancels out. Thus, we have

\[
Q - M\rho Q = (I-A)^{-1}C.
\]

Since \( \rho \) is just a scalar, we can write the above equation as

\[
Q - \rho M Q = (I-A)^{-1}C.
\]

Hence

\[
(I-\rho M)Q = (I-A)^{-1}C
\]

and

\[
Q = (I-A)^{-1}C/(I-\rho M).
\]

The complete solution for \( X(t) \) may now be written as follows:

\[
X(t) = Pe^{NT} + \left\{ (I-A)^{-1} C/(I-\rho M) \right\} e^{\rho t}
\]

\((I-\rho M)\) has a non-negative inverse because \( \rho M \) is a non-negative square matrix with a maximum characteristic root less than 1.\(^5\) To determine \( P \) and \( C \), we need information regarding initial and terminal states. We have \( n \) initial \( X(0)'s \) and terminal \( X(T)'s \). Thus, we have \( 2n \) equations to determine \( 2n \) unknowns \( P_i's \) and \( C_i's \).

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5. This helps in computing an upper bound on the permissible \( \rho \), i.e., a value of \( \rho \) such that \((I-\rho M)\) has a non-negative inverse.
Once \( X(t) \) is determined, \( K(t) \) is known. Therefore, \( \dot{K}(t) \) the pattern of accumulation is also determined.

(ii) Unequal rates of growth of consumption for \( t < T \)

\( C(t) \) can no longer be written as \( Ce^{\rho t} \) since different components of consumption grow at different rates. The method of solution described above breaks down for this more general case. The use of matrix exponentials, however, enables us to write out the solution for this general case with relative ease.

Our general, non-homogeneous equation is

\[
X(t) = (I-A)^{-1} B \dot{X}(t) + C(t)
\]

where

\[
C(t) = \begin{pmatrix} C_1(t) \\ \vdots \\ C_n(t) \end{pmatrix} = \begin{pmatrix} C_1 \rho_1(t) \\ \vdots \\ C_n \rho_n(t) \end{pmatrix}
\]

or

\[
\dot{x}(t) = \left\{ (I-A)^{-1} B \right\}^{-1} X(t) - \left\{ (I-A)^{-1} B \right\}^{-1} C(t),
\]

which can be written as

\[
\dot{x}(t) = NX(t) + f(t),
\]

where

\[
N = \left\{ (I-A)^{-1} B \right\}^{-1}
\]

and \( f(t) = \left\{ -(I-A)^{-1} B \right\} C(t) \).

Noting

\[
\frac{d}{dt} \left[ e^{-Nt} X(t) \right] = e^{-Nt} \left[ \dot{x}(t) - N X(t) \right] = e^{-Nt} f(t),
\]

Hence

\[
e^{-Nt} X(t) = P + \int_0^t e^{-Nt} f(t) \, dt.
\]

Or

\[
X(t) = e^{Nt} P + \int_0^t e^{-Nt} f(t) \, dt.
\]

Since \( X(t) \) and \( f(t) \) are known functions, the equation for \( X(t) \), with the integrations performed, are functions of \( P \), \( t \) and \( f(t) \). Since \( f(t) \) is, in turn, a function of \( C \)'s and \( P \)'s, \( X(t) \) is a function of \( P \)'s, \( C \)'s, \( Q \)'s and \( t \). \( Q \)'s are given data. \( P \)'s and \( C \)'s are determined with the help of initial and terminal conditions. Thus, for the more general case, the same qualitative results can be obtained, even though the technique of solution differs somewhat.

The presence of gestation lags in the economic system makes a difference equation formulation of the model a more appropriate one for situations where such lags are important in relation to the length of the planning period.

The balance equation \( X(t) = AX(t) + I(t) + C(t) \) would be the same in the difference equation version as in the differential equation form. However, we cannot write \( I(t) = E\dot{X}(t) \), when there is a significant gestation lag in the system. In fact, if the length of the gestation period is assumed to be uniform and equal to \( g \) years then

\[
I(t) = \frac{B}{g}\sqrt{X(t+g)} - X(t) ,
\]

if capacity is assumed to be built up uniformly over the interval '0' to '\( g \)'.

Thus, we have

\[
X(t) = AX(t) + \frac{B}{g}\sqrt{X(t+g)} - X(t) + C(t).
\]

The homogeneous part is now a system of difference equations of order '\( g \)'. With the help of matrix notation, we are treating it as if it were one matrix system of order \( g \). Now, this solution has one dominant root. This may be easily demonstrated. The solution corresponding to the dominant root is written as follows:

\[
X(t) = P \left( \frac{\sqrt{g(I-A) + B/g}}{B} \right)^t 
\]

where \( P \) is to be determined from initial conditions. The \( g \)th root of \( \frac{\sqrt{g(I-A) + B/g}}{B} \) is well defined for \( g \geq 0 \).

\[7\] For this footnote, see bottom of page 13.
We have now to determine a solution for the non-homogeneous part.

We have the following relationship:

\[(1 - A) \mathbf{X} = \frac{B}{g} \sum_{t=1}^{g} \mathbf{X}(t+g) - \mathbf{X}(t) \mathbf{J} + \mathbf{Q} (1 + r)^t \]; \hspace{1cm} (17)

or

\[ (I - A + \frac{B}{g}) \mathbf{X}(t) = \frac{B}{g} \mathbf{X}(t + g) + \mathbf{Q} (1 + r)^t \] \hspace{1cm} (18)

Now try a solution for \( \mathbf{X}(t) = \chi (1 + r)^t \) where \( r \) is the rate at which exogenous consumption is increasing.

Cancelling out \((1 + r)^t\), we get

\[(I - A + \frac{B}{g}) \mathbf{X} = \frac{B}{g} \mathbf{X} (1 + r)^g + \mathbf{Q} ; \hspace{1cm} (19)\]

or

\[ I - A + \frac{B}{g} - \frac{B}{g} (1 + r)^g \mathbf{X} = \mathbf{Q} ; \hspace{1cm} (20)\]

or

\[\mathbf{X} = \frac{\mathbf{Q}}{I - A + \frac{B}{g} - \frac{B}{g} (1 + r)^g} \hspace{1cm} (21)\]

We can write the complete solution as:

\[
\mathbf{X}(t) = P \left[ \frac{g}{g} \left( \frac{I - A + B/g}{B} \right) \right]^t + \frac{\mathbf{Q}}{\left\{I - A + \frac{B}{g} - \frac{B}{g} (1 + r)^g \right\}} (1 + r)^t
\hspace{1cm} (22)
\]

7. (From page 12): Strictly speaking the above procedure yields a valid solution only if we assume a special initial condition to prevail right from the start. Otherwise we are justified in ignoring the other components of the general solution corresponding to the non-dominant roots only if interest is concentrated on the long-run solution. For short periods this procedure breaks down. It is then necessary to postulate \( 'g' \) sets of initial conditions. Together with the open ends pertaining to consumption there will be \((g + 1)\) sets of constants to be determined. Information on the terminal sets of \( \mathbf{X}'s \) together with the information on \( g \) sets of \( \mathbf{X}'s \) pertaining to \( \mathbf{X}(0), \mathbf{X}(-1), \ldots, \mathbf{X}(g-1) \) permit the determination of all the constants of the problem.
For the interesting case, where \( g = 1 \), we get

\[
X(t) = P \frac{(I-A + B)^t}{B} + \frac{Q}{I - A - rB} (1 + r)^t
\]  

(23)

Since we know \( X(0) \) and \( X(T) \), we have 2n equations to determine 2n unknowns \( P \) & \( Q \). Thus, the discussion is complete for the case of gestation lags with an average length equal to \( g \).

If we have different gestation lags in different sectors, then, the formal manipulation of the model will have to be along somewhat different lines, but the essential point remains unchanged.

5. Interpretation of the model

The analysis given in Section 4 shows quite clearly that the one-sector model is capable of being extended to a multi-sector case in a straightforward fashion. The qualitatively interesting point that arises in the multi-sector model is the possibility of non-proportional growth. This has naturally no analogue in the one-sector model. Further, the determination of the terminal state is a tricky business. Apart from that, all we have done is just multiply the number of variables, but the use of matrix exponential keeps the notation.

8. The use of difference equations in multi-sectoral models with lags playing an important part, is discussed in great detail in Chapters IV-VI of The Logic of Investment Planning by S. Chakravarty, North Holland Publishing Company, Amsterdam. The discussion there is primarily from the point of view of numerical extrapolation, a method of solution particularly suited to situations characterized by significant differences in the lengths of sectoral gestation lags.
perspicacions. In fact, it is the use of matrix methods which shows the similarity with the one sector case so clearly.

However, there is a source of complication in the many-sector model which relates not to mathematical consistency of the dynamic model, which is obviously satisfied here. The doubt arises as to whether we would preserve non-negativity of all the variables. Now, an analysis of the solution of the differential equation shows that there exists an initial configuration for which the system possesses a solution which has all non-negative components. Thus, there is no necessary contradiction between the model and the requirement of non-negativity. But the real question is whether for any arbitrary set of initial conditions we shall necessarily preserve non-negativity in all the variables for all future periods.

A closer look at the form of the solution will enable us to clarify the question considerably. We write the solution for \( X(t) \) in the following way:

\[
X(t) = P e^{Ht} + \frac{C'}{I - \phi N} \phi t.
\]  

(23')

This is the solution that corresponds to the continuous case. The matrix \( N \) is actually a composite one and may be written explicitly as follows:

\[
N = \left\{ (I - A)^{-1} B \right\}^{-1}
\]

Now if we ignore for the time being the non-homogeneous part, then \( X(t) \) will preserve non-negativity for arbitrary non-negative initial conditions if \( N \) is a non-negative matrix. In fact, it can be shown to be a sufficient condition.

A necessary and sufficient condition on the matrix \( N \) which ensures non-negativity of solutions for all \( t > 0 \) is the following: if, for any given matrix \( N \), we can find a scalar \( C \) such that the matrix \( CI + N \), where \( I \) is the identity
matrix, is non-negative, then the matrix exponential $e^{Mt}$ is also non-negative. A different way of stating this condition is the following: $e^{Mt}$ is non-negative if and only if the matrix $N = (a_{ij})$ is such that $a_{ij} > 0$ for $i \neq j$. This leaves the sign of the diagonal elements entirely open. As the inverse of a non-negative matrix, $N$ will have mixed signs, but we cannot say offhand if the negative signs apply only to the diagonal elements. The initial conditions are assumed to be non-negative.

In extending the discussion to the non-homogeneous case an interesting possibility arises. In equation (23') if $\rho$ is such that modulus $(\rho M) < 1$, then \( \frac{1}{1-\rho M} \) has a non-negative inverse and has, therefore, a matrix multiplier expansion, e.g., $I + \rho M + (\rho M)^2 + \ldots$ etc. If $C'$ is also non-negative, then $(C'/(I-\rho M))e^{\rho t}$ is a non-negative expression. In other words, the non-homogeneous system has one characteristic positive root $\rho$ and a characteristic vector associated with it which is positive. This positive root is additional to the positive root which the homogeneous system contains. It follows that we may have a value of $\rho$ which is greater in absolute value than any other root and, therefore, dominates all of them. Since the characteristic vector of $\rho$ is non-negative, the system will necessarily move towards the balanced growth expansion represented by the solution corresponding to the non-homogeneous part.

The economic implication of this argument is that if we have final demand expanding at a fast enough rate for all the commodities in question then negative

9. To show this write $e^{At} = e^{(CI + A)t}$. Now $e^{(CI + A)t}$ is non-negative because $CI + A$ is non-negative. $e^{-Clt}$, being a scalar exponential, is necessarily non-negative. Since the product of two non-negative matrices is non-negative, $e^{At}$ is clearly non-negative. See R. Bellman, Introduction to Matrix Analysis.

10. By virtue of Frobenius's assumptions on the matrix $M$. 
output levels cannot arise. This is true because decumulation can arise only if some sectors are contracting while others are expanding. Since contraction is ruled out, the possibility of capital stocks and output levels turning out to be negative is also eliminated. By generalizing the argument to the case where the different sectors are growing at unequal rates, it would be possible to show that the conclusion on non-negativity retains its validity although the derivation of the necessary and sufficient conditions will be more difficult.

From the point of view of preserving non-negativity it would appear to be a good thing if \( \rho \) were made very large. This would increase the probability that \( \rho \) would dominate all other roots. However, given the technology and the initial and terminal conditions, a very high \( \rho \) may not be feasible. Infeasibility of a stipulated \( \rho \) would show itself in rendering \( C' \) negative while the argument of the preceding paragraphs depends on \( C' \) being positive.

In a developing economy there is nothing in the logic of things to prevent certain sectors from declining absolutely, e.g., the case of inferior goods when income expands or the decline in production for export when a primary producing country turns to a concentration on industrialization. However, in the case of countries like India, in which per capita income is quite low, only a very substantial increase in the level of income or a very fine commodity classification would give the inferior good problem any importance. Moreover, for balance of payments purposes, production of primary commodities will have to be expanded for some time to come in most of the less developed countries.

There is also the possibility of changing from any arbitrary set of initial conditions to a special initial configuration which would preserve non-negativity of all output levels. In this case the planning problem is
broken into two phases: (a) the preliminary one of getting to these special proportions and (b) the problem of moving from this special position to the desired terminal situation.

5.1.1. Singularity of the B Matrix

If the B-matrix is singular, then we cannot use the method of solution outlined in previous sections. Instead, we try a different technique of solution. Our fundamental equation is as follows:

\[(I-A) X(t) = B \frac{dx}{dt}\]  \hspace{1cm} (24)

Now, try a solution \(X(t) = e^{\lambda t}\). Substituting this relation in the previous equation, we get

\[(I-A) e^{\lambda t} = \lambda B e^{\lambda t}\]  \hspace{1cm} (24)

Since \(e^{\lambda t}\) may be cancelled out, we thus have the following equation:

\[(I-A) \lambda M = e.\]  \hspace{1cm} (25)

For non-trivial solutions to exist, \(\left| I - A - \lambda B \right| = 0 \).

This gives us the characteristic polynomial in \(\lambda\). This is degree 'n'. We have thus 'n' roots. We get corresponding values of \(X\). These are determined up to a scale factor. Using some normalizing device, we can determine them uniquely.

Our solution for \(X(t)\) can now be written as

\[X(t) = \sum_j C_j M_{ij} e^{\lambda j t}.\]  \hspace{1cm} (26)

The \(C_j\)'s are determined with the help of initial conditions.

The above method of solution does not depend on the non-singularity of the B-matrix.
5.3 The variability of the input coefficients

The entire analysis has proceeded on the assumption of Constancy of input-output and capital-output coefficients. This may be easily criticized as being unduly restrictive. There are some ways in which the limitations of these assumptions may be somewhat relaxed. One partial relaxation would come through making input-output dependence linear rather than proportional. Thus, we write

\[ x_{ij} = a_{ij} x_j + b_{ij} \]

where \( b_{ij} > 0 \), or \( b_{ij} < 0 \). In the first place, relative factor requirement diminishes while, in the other case, it increases. We can consider the first situation as illustrating some aspects of the law of increasing returns, while the second would serve as an example of the operation of the law of diminishing returns. Our equation system is easily adapted to take into account this slightly generalized situation.

There is another way of considering this variability. This method is applicable when we are working out iterative solution to our system of equations.

We reduce for computational purposes the differential equations to the corresponding set of finite difference equations. Then, in working out solutions we use changing sets of coefficients depending on the levels of output already attained. This is a rather effective method of taking to account variability of coefficients, and often corresponds to the way a planner adjusts his production targets through successive approximations.
II. Extension to an Open Economy

The previous discussion has been in terms of a closed economy: there was no demand on resources for production for export and no availability of resources for filling the final bill of goods except from domestic production. No questions arose of import substitution, export drives or balance of payments restrictions. But for an open economy these questions do arise and can and must be taken into account.

Importing a commodity and producing it at home may be regarded as alternative activities for some sectors while non-competitive imports may be treated as a primary factor whose availability is restricted by the foreign exchange constraint. This approach cannot be introduced into the framework of the model itself because analysis of a dynamic model with "built-in" substitution possibilities is beyond our immediate scope. On the other hand, a state model, even with substitution possibilities, would ignore a range of significant factors connected with planning for development over time.

For projecting export demand by different sectors, nothing better is suggested here than the usual projection techniques based on price elasticities. The significance of alternative projections and the use of resources to achieve them can then be investigated by use of the model.

Imports take the place of domestic production but, in a developing economy, it would be a mistake to consider the dependence on imports as constant over time. Yet the degree of dependence is, to some extent, a matter of choice, the consequences of which require analysis. The alternatives from which choices can be made have to be generated outside the model which, as for exports, can then
be used to investigate their implications. The following general approach may be used for this purpose. With an aggregate projection of imports and a forecast of foreign assistance availabilities, the amount and phasing of the foreign exchange restriction can be established. The problem is then to allocate the total among various commodities. If the timing of foreign assistance is flexible, determination of the best timing is an additional problem. An allocation among sectors and over time has to be made on bases exogenous to the formal planning model itself. The solution can then be computed and the results examined and compared with results obtained from an alternative allocation.

The balance equation for an open economy is:

\[ X(t) = AX(t) + B\dot{x}(t) + C(t) + E(t) - M(t), \]

where \( E(t) \) is the vector of exports, \( M(t) \) the vector of competitive imports and all the other symbols have their usual meanings. Non-competitive imports are ignored or included in \( M(t) \). If non-competitive imports are kept outside the model they may be brought into the picture when total export earnings are being compared with the total import requirements. An alternative procedure would be to consider non-competitive imports as constituting "empty sectors". For such empty sectors, \( X(t) = 0 \), implying that domestic availability is zero, and \( E(t) = 0 \) unless re-exports are allowed. They may be used to augment capacity elsewhere in the economy or to supply raw materials or for final consumption.

For sectors 1, ..., \( n \) in which domestic production takes place

\[ X(t) + M(t) = A X(t) + B\dot{x}(t) + C(t) + E(t). \]

If \( M(t) = \lambda X(t) \), where \( \lambda \) is a diagonal matrix \( \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \end{bmatrix} \) indicating the ratio of competitive imports per unit of domestic production,

\[ (I + \lambda - A) X(t) = B\dot{x}(t) + C(t) + E(t). \]
Let $C(t) = C_0e^{ct}$ and $E(t) = E_0e^{dt}$ as a first approximation. This augmented system has the following solution:

$$X(t) = Pe^{At}t + \frac{(I + \lambda - A)^{-1} C_0e^{ct}}{1 - \rho M'} + \frac{(I + \lambda - A)^{-1} E_0e^{dt}}{1 - \lambda M'}$$

where $M' = \frac{1}{M}$ and $M' = \left[(I + \lambda - A)^{-1} B\right]^{-1}$. There are now three sets of constants to determine the $P$'s, $C$'s and $E$'s, but only two sets of equations involving the initial and terminal conditions, the $X(0)$'s and the $X(T)$'s. This leaves, therefore, $n$ degrees of freedom. Any $n$ values can then be assumed on an arbitrary basis. For example, all $C$'s may be chosen arbitrarily or all $E$'s. However, such exclusive choices will often be unrealistic. There are a large number of mixed choices possible consisting partly of $E$'s and $C$'s. The introduction of foreign trade, thus, increases the number of degrees of freedom substantially.

Once $X$'s are known, $M = M' + M''$, where $M'$ and $M''$ stand for competitive and non-competitive imports respectively, may be easily calculated. This total may be compared with the total of export earnings. If there is a deficit on the balance of payments, this may be either met by foreign assistance or by import substitution. The nature and extent of import substitution may be determined from the model by changing $\lambda$'s or by deciding to fill up "empty sectors". The total effectiveness of import substitution in any direction may be determined by computing appropriately changed inverses of $(I + \lambda - A)$ matrices. In this way the main consistency problems that arise in planning in an open economy may be taken into account.