NOTES ON OPTIMAL ALLOCATION OF INVESTMENT IN ALTERNATIVE PROCESSES

Richard S. Eckaus

Center for International Studies
Massachusetts Institute of Technology
Cambridge, Massachusetts
ERRATA

Page 2, line 3: for "and" read "end".

Page 4, paragraph 2, line 7: for "effects on" read "effects of".

" " line 9: for "problems" read "problem".

Page 7, paragraph 2, line 2: for "has" read "had".

" " paragraph 3, line 4: for "fully" read "full".

Page 8, line 3: for "not" read "no".

" " paragraph 3, line 3: for "( )" read "(t-1)".

Page 14, line 3: for "scarce" read "abundant".

Page 15, paragraph 2, line 1: "in process 1" should read "in process 2".

Page 19, paragraph 2, line 1: for "allocation" read "allocating".

Page 28, line 1: for "sufficient" read "sufficient 1"

" " equation (37): the right hand side of equation (37) should read "α".

Page 17, equation (22): the right hand side of the equation reads:

\[
\frac{P_1}{a_{12}} \left( t-1 \right) \left( 1 - \frac{a_{22}}{a_{12}} \right)
\]

Page 19, line 3: should read: "Since \( \frac{1(t)}{a_{12}} < \frac{1(t)}{a_{11}} \),"

Page 29, equation (38): reads,

\[
m < 1 + \frac{1}{a_{13} - \beta} \quad \text{or} \quad m < 2 + \frac{1 - \frac{a_{13}}{a_{12}}}{\frac{1}{a_{13}} - \beta}
\]
NOTES ON OPTIMAL ALLOCATION OF INVESTMENT
IN ALTERNATIVE PROCESSES

I. Introduction.

The previous papers on factor proportions problems were unashed statics. However, there can be no doubt of the desirability of a dynamic analysis to match the dynamic character of economic development. Moreover it has sometimes been suggested that deductions from static analysis cannot be valid for a dynamic phenomenon like economic development and thus that dynamics is not only desirable but absolutely necessary.

This paper, in the by-now-established manner, will take a small step rather than a big one toward a dynamic analysis. It is concerned with the problem of optimal choice between alternative processes at successive stages in economic growth. As usual a number of simple models will be analyzed. Each presents a somewhat different problem and each, it is hoped, adds something to the understanding of economic growth or provides a stepping-stone to an analysis which will. Of the issues raised here only a few are fully resolved, but I believe that the work has proceeded far enough to benefit from criticism.

The static rule for choosing among alternative processes to achieve maximum output is well-known and can be summarized as "Economize the scarce factors." The factor proportions analysis did not add anything to this rule but showed sources of possible conflict between its application to maximize output and the goal of full employment. The effects of certain types of "market imperfections" in distorting the application of the rule were also
analyzed.

Application of the static rule to overpopulated and under-developed areas would, ceteribus paribus, lead one to suggest the use of processes of production at the labor-intensive and of the scale of factor proportions rather than the more capital intensive projects. This recommendation often arouses uneasiness and criticisms, however, based on more or less explicit dynamic considerations. Some of the criticisms of this type are the following:

Emphasis on labor-intensive processes would commit a country to low labor productivity techniques when what it really wants is to raise labor productivity. Even if surplus labor exists which would make it most economical in the near future to distribute scarce capital in labor-intensive processes, this situation will not persist. A country should not build capital which will become obsolete quickly in the sense that the capital requires factors in proportions which will no longer prevail after a few years. Moreover, if capital invested in the labor intensive processes wears out much more quickly than if invested in more capital intensive processes, the former may turn out to be the more expensive in terms of capital resources.

Questions of the above type involve dynamic factors such as the rates of growth of productive resources, the length of the planning period and the rate of depreciation of capital. The theoretical models used in this paper are intended to highlight such influences and, therefore, have been simplified in other respects.
Section II is designed solely to illustrate the meaning of optimal choice among alternative processes when capital is a non-specific resource and any investment yields an infinite flow of capital services at a constant level. After this discussion the problem of allocation of investment is analyzed in Section III under the assumption that capital invested in any process has no alternative uses. Finally, in Section IV the assumption is dropped that capital is infinitely durable and the problem of choice is investigated assuming a simple form of depreciation.

II. Choice Between Processes when Capital is Non-Specific.

In this and the following sections the analysis will be confined to a single good economy. There are two factors of production: capital and labor. The supply of labor is determined exogenously; labor may be considered to have its own rate of growth which, at first, will be set at zero; i.e., the supply of labor will be a constant. It will also be assumed to be completely mobile between processes. Consumption is endogenously determined and in the first examples will also be considered to be constant. New investment is the residual when consumption is subtracted from the national product.

It will be assumed that there are a number of alternative processes for the production of national product and thus a number of different ways of investing. All the different possible investments have the same period of gestation, however. Moreover, capital is assumed never to be "sunk" but always able to be shifted from one process to another.

Finally, it is necessary to specify for the capital model the patterns of the investment inputs and of the flow of capital services.
resulting. It will be assumed that the investment period is one unit long after which capital services flow at a constant level for an infinite period. That is a type of "point input-period output" model will be used in which the period is infinite. This capital model is represented in Figure 1.

Figure 1.

```
+-------------------+---------------------+---------------------+---------------------+
|                   | Level of initial investment | Level of flow of capital services |
|                   | t                      | t+1                  | t+2                  |
+-------------------+---------------------+---------------------+---------------------+
```

The capital coefficient relates the size of the initial investment to the resulting goods output.

Starting from some initial resource endowment the problem now is to decide how to allocate resources optimally among alternative processes. The assumptions are such as to permit an easy answer. The time dimension is not essential. All capital has the same pattern of investment and flow of capital services; since there is no depreciation, the flow of capital services is at a constant level. Therefore, the relative effects on any particular amount of investment in any process can be judged by comparing the results of the first period of output. The problems of choice has thus been made similar to a static problem since the maximum rate of growth will be produced by using resources to produce maximum output in each period and thus creating the largest surplus over consumption for investment. The static rule for maximizing output is then still applicable: The process chosen in each period should economize the scarce resources.
A numerical example may help in seeing through this case quickly and will introduce similar examples to be constructed in the following section. Figure 2 illustrates a group of alternative processes \( P_1, P_2, P_3, P_4 \), and the initial factor endowment \( E_1 \). The numbers in parentheses indicate the number of units of capital and labor necessary for the production of one unit of output, i.e. the technical coefficients of production for each process.

![Figure 2](image-url)

In constructing Table 1 we have used process 1 to start with in period 1. It can be seen that since even in this most labor intensive process labor is redundant, ten of the nineteen original units of labor are originally unemployed. Two units of output are produced so that with consumption set at 1.0 investment is 1.0. No other single process or combination of processes could be used to produce
a higher output in period 1. For example, in processes 2, 3 and 4 labor would have a higher average productivity but total output from use of these processes would have been only \( \frac{h}{3}, \frac{h}{5} \) or \( \frac{h}{6} \) as compared to 2 units from process 1. Having been lucky the first time we play process 1 for three more periods.

Table 1.

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Periods 2 and 3 are like period 1: even with the additional capital labor is still redundant and no other process would produce more output, investment or faster growth than process 1.

In period 4 we find, however, that capital has grown so fast that now production in process 1 is limited by a shortage of labor; the amount of labor available, 19 units, is less than the amount required to use all the capital available, 21.875 units. In fact, the shortage of labor forces us to leave some of our capital unused. This is a suspicious situation; it seems, and is, inefficient to leave this produced resource idle. Since we can reconsider our decisions we
may try other uses of the investment resources available during period 4. In Table 1 at period 4' are the results of putting all the capital available, old and new, in process 2. This gives even worse results in terms of output than the previous try; any other concentration of capital in processes 3 or 4 would produce even less output.

Suppose instead of having shifted all the capital to process 2 in period 4 only the new investment has been directed into that process. Then we would have the situation shown in line 4'' of Table 1. This does better in terms of output than either of the previous two allocations for period 4. Following the reasoning of static analysis, however, it becomes obvious that the optimal use of processes in period 4 would be one which fully utilizes all the resources available. Since capital as well as labor is completely mobile such an arrangement is possible and is shown on Table 1 as line 4''''; output from the full employment output is higher than from any alternative use of resources. All further choices between processes are like those just analysed so that it is not necessary to push this example further.

Once period 4 is reached, full employment of resources becomes possible and optimum use of processes after that period should always imply full employment. As between alternative arrangements of resources both of which imply fully employment, the one yielding the highest total output is, of course, optimal. In this model there is no advantage in using capital intensive processes prior to the time at which they are suited to the factor proportions available. Premature emphasis on capital intensive techniques would only involve lower income and investment throughout time.
It is possible to be more explicitly dynamic in the analysis of this model but only by using some mathematics. Since the life of capital is infinite and there is not depreciation \( y_d(t) \), gross output of period \( t \), is equal to \( y_N(t) \), net output of period \( t \). When only one process \( j \) is in use \( y_N(t) \) is equal to the output of that process \( y_{Nj}(t) \). In turn

\[
y_{Nj}(t) = \frac{x_{1j}(t)}{a_{1j}},
\]

where \( x_{1j} \) is the scarce factor and \( a_{1j} \) the technical coefficient.

The resource which is the scarce factor is determined, of course, by a comparison of \( \frac{x_{1j}(t)}{a_{1j}} \) and \( \frac{x_{2j}(t)}{a_{2j}} \). The process to be used is determined by comparison of the maximum outputs of each process with the given resources. This process of choice is analyzed rigorously in the next section.

The amount of capital available in period \( t \) is

\[
x_1(t) = x_1(t-1) + i(t-1),
\]

where \( i(t-1) \), investment of period \( (t-1) \) is

\[
i(t-1) = y_N(t-1) - c(t-1).
\]

\( c(t) \) is consumption of period \( t \). Using equations (1), (2) and (3) the following difference equation is obtained for net income when only one process is being used:

\[
y_N(t) = y_N(t-1) + \frac{1}{a_{1j}} y_N(t-1) - \frac{1}{a_{1j}} c(t-1).
\]

If consumption is held at a constant level \( c \), as in the numerical example, the solution to equation (4) is

\[
y_N(t) = y_N(o) - c \left( 1 + \frac{1}{a_{1j}} \right)^t + c,
\]

where \( y_N(o) \) is the net income produced by the original factor endowment.
If \( y(\cdot) \) income from the original factor endowment is not greater than \( c \) the original level of consumption, there is no chance of growth; simply put, growth requires investment which requires saving. If there is some original saving, growth will ensue as equation (5) is always unstable. \( a_{1j} \) is always positive which means that \( (1 + \frac{1}{a_{1j}}) \) is always greater than unity and, thus, in successive periods \( (1 + 1/a_{1j})^t \) will grow without limit. The rate of growth of net income as \( t \) becomes large will tend to \( 1/a_{1j} \).

If consumption is made a linear function of income,
\[
C(t) = \alpha + \beta y(t) (t - 1)
\]
the income equation (4) changes so that the new stability conditions are
\[
\text{(7a)} \quad \beta < 1 \quad ; \\
\text{(7b)} \quad \alpha_{1j} < \frac{1}{\beta} .
\]
Thus, output will grow in this new case only if at least one of the conditions (7) does not hold.

When two processes are used simultaneously the difference equation describing the model's growth becomes somewhat more complicated than the simple system just presented. The stability conditions are also somewhat different.

The simplicity of the model and of the criteria for optimal choice results, of course, from the drastic assumptions. Capital is a homogeneous factor with infinite life which means that investment also results in a permanent addition to capital resources and potential output. This simplifies the problem of choice into finding the processes which will maximize these increments of investment and output.

If the assumption of a labor force of a constant size were replaced by ascribing to it some given growth rate the analysis would become more involved but the essentials of the problem of choice would
remain unchanged. If, in addition, per capita consumption were made a function of income the essence of the problem of choice would not change but the model could now demonstrate the effects of increasing population on growth.  

III. Choice between Processes when Capital is a Specific Factor.

In the preceding section, it was assumed that capital as well as labor could be shifted from process to process at will, that is, that both factors were non specific. Although this was only one of the many "unrealistic" assumptions, it is one whose modification will, I believe, bring the model much closer to an operational framework. Therefore, in this section it will be assumed that capital once invested in a particular process is "sunk" and would have no productivity in any other process. The other basic assumptions of the previous one good, two-factor model will be maintained: (1) the labor supply is exogenously determined and, at first, will be held constant and labor, itself, is a completely mobile factor; (2) capital invested in any process has an infinite life as in Figure 1; (3) consumption is endogenously determined but will also, at first, be held constant; (4) there are a number of alternative processes available and some "original endowment" of factors, which to illustrate the full range of choice, will imply redundant labor.

Again a numerical example is enlightening. The processes and factor endowment illustrated in Figure 2 can also be used for this case. Now, however, we know that whatever capital is invested in process 1

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2 Again this is a subject for further analysis.
will eventually stand idle, because as the capital stock grows the factor endowment will become suitable for the more capital intensive processes. Although the most labor-intensive process would yield the highest output at the early stages, would we not gain these early increases at the expense of later output? Capital in this system has an infinite life and would we not be committing ourselves for eternity to processes which we know will not be optimal in several "years" when the factor endowment has changed? Why not compromise a little on present output and "build ahead" by using processes embodying factor proportions suitable for some future date?

Of course, upon reflection we know that building capital in a particular form does not create an indefinite commitment to the use of the capital; capital can be scrapped or abandoned with some disposal cost which may be negative or positive, but which we shall assume for the present to be zero. On the other hand, the procedure of building capital in processes with factor proportions suitable for the existing endowment and then scrapping the capital when it is "outgrown" will only be justified if the alternative of "building ahead" yields a lower rate of growth.

In Table 2 are shown the results for ten periods of investing in each period in the process which is most suitable for the prevailing factor proportions. The amount of redundant labor or capital when it appears is shown in parentheses in the sector to which it belongs. At first, only labor is redundant due to a general scarcity of capital. Then, however, optimal use of resources requires that capital in process 1 must be left idle as available labor is drawn into other sectors. Eventually this also happens in process 2 as labor is drawn into process 3, and so on. In effect, capital
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becomes obsolete in the labor-intensive processes. However, under the present assumptions it does pay to build capital using labor-intensive processes when labor is relatively scarce and it becomes optimal to leave this capital idle when factor proportions change.

Compare Table 2 with Table 3 which represents one of many possible ways of "building ahead" of existing factor proportions. Total output and new investment in every period is less in Table 3 as compared to Table 2. Of course, there are no evidences of outmoded process 1 capital dotting the landscape as a result of the procedure of Table 3 but obsolete and unused capital capacity would emerge in process 2 after period 11. In the progression described in Table 2, there is more capital investment of all sorts and more obsolete capital than in Table 3 but there is also more output. Unless there is a disposal cost for the obsolete capital these examples, as one would expect, show that optimal choice requires a progressive shift in techniques from less to more capital intensive processes, at least in this model.

The numerical examples are good representations of a more general analysis as can be shown. Suppose there are three alternative processes $P_1$, $P_2$ and $P_3$ as shown in Figure 3.

The relations between the technical coefficients are:

(8) $a_{11} < a_{12} < a_{13}$

and

(9) $a_{21} > a_{22} > a_{23}$

![Figure 3](image-url)
The original factor endowment $x_1(t), x_2(t)$ is chosen to illustrate the problem of choice between processes when labor is originally a redundant factor, as well as when full employment of labor is possible; that is, it is assumed that

\[
\frac{x_1(t)}{x_2(t)} < \frac{a_{11}}{a_{21}}
\]

It will also be assumed that the original capital endowment is not "sunk" in any process so that its allocation is also one of the problems to be solved. Rearrangement of (10) in the form

\[
\frac{x_1(t)}{a_{11}} < \frac{x_2(t)}{a_{21}}
\]

shows that $x_1$, capital, is the limiting factor in process 1.

Using the relations in (8) gives

\[
\frac{x_1(t)}{a_{11}} > \frac{x_1(t)}{a_{12}}
\]

i.e., that a given amount of $x_1$ produces more in process 1 than in process 2. Noting that, likewise

\[
\frac{x_2(t)}{a_{22}} > \frac{x_2(t)}{a_{21}}
\]

and, thus, by combining (11), (12) and (13):

\[
\frac{x_2(t)}{a_{22}} > \frac{x_1(t)}{a_{12}}
\]

This says that $x_1$ is also limiting in process 1. Since $x_1$ is limiting in both processes 1 and 2 and a given amount will produce more in process 1 than in process 2, output would be maximized by putting all $x_1$ in process 1.

Once having been invested in process 1, however, capital is sunk. As long as the increment of investment to capital stock
this fact does not change the inequality (10); i.e. if
\[
(15) \quad \frac{x_{11}(t-1) + i(t)}{x_2(t)} < \frac{a_{11}}{a_{21}}
\]
the preceding reasoning does not change. It shows also that "building ahead" of existing factor proportions in this situation does not pay as any such alternative procedure can be compared with the optimal procedure and found not to be as productive. This, then, is a general demonstration of the choice of processes indicated in periods 1, 2 and 3 in Table 2.

When the increment to investment is such that (15) does not hold, a new problem of choice arises. Reversal of the inequality sign in (15) would mean that not all the additional capital could be absorbed in process 1. The limited supply of labor would be a bottleneck. In addition to the reversal of the inequality sign of (15), we shall also assume that the second inequality of (9) holds, to cover all possible cases.
\[
(16) \quad \frac{a_{11}}{a_{21}} < \frac{x_{11}(t-1) + i(t)}{x_2(t)} < \frac{a_{12}}{a_{22}}
\]

It is convenient to consider the potential use of the additional capital as falling into two possibilities: (1) all of the increment could be put into process 1; (2) the increment of investment can be divided between processes 1 and 2 in various proportions. As a particular case of the last possibility all of the increment could be put into process 2. The procedure which will be followed is to compare the income produced from each alternative to determine which is optimal.

First of all, if the entire investment \( i(t) \) is put into process 1 there will be insufficient labor so that labor will be the limiting factor and some of the capital will be redundant. This emerges
clearly by rewriting the first inequality of (16) as

\[ \frac{x_2(t)}{a_{12}} < \frac{x_{11}(t) + i(t-1)}{a_{11}}. \]

Thus, the income produced by following this alternative, denoted \( y_1(t) \), is

\[ y_1(t) = y_1(t) = \frac{x_2(t)}{a_{21}}. \]

On the other hand, the additional investment may be diverted in whole or in part to process 2. The size of the increment of capital determines the optimum allocation. The simplest case occurs when the increment to capital is so large that if placed in process 2 with enough labor for its full utilization, labor would become scarce and the limiting factor in process 1. Output in this case would be the sum of the outputs from the separate processes. If \( P_1 \) of \( i(t-1) \) went to process 2, output from this process would be

\[ y_2(t) = \frac{P_1 \cdot i(t-1)}{a_{12}} \]

and would use \( x_{22}(t) = y_2(t) \cdot a_{22} = \alpha_{x_2} \cdot \frac{P_1 \cdot i(t-1)}{a_{12}} \)

amount of labor leaving \( x_{21}(t) = x_2(t) - x_{22}(t) \) labor in process 1. We assume that the labor supply is now limiting in this process so that

\[ y_2(t) = \frac{1}{a_{12}} \left[ x_{22}(t) \cdot a_{22} - \frac{P_1 \cdot i(t-1)}{a_{12}} \right]. \]

The total output from this alternative, \( y_{II}(t) \), would therefore be

\[ y_{II}(t) = \frac{1}{a_{12}} \left[ x_2(t) - \frac{a_{22} \cdot P_1 \cdot i(t-1)}{a_{12}} \right] + \frac{P_1 \cdot i(t-1)}{a_{12}}. \]

Comparing \( y_{II}(t) \) with \( y_1(t) \) we find

\[ y_{II}(t) = y_1(t) = \frac{P_1 \cdot i(t-1)}{a_{12}} \left( 1 - \frac{a_{22}}{a_{12}} \right). \]
Since $a_{22} < a_{12}$, $y_{II}(t) - y_{I}(t)$ will always be positive indicating that the second alternative would be preferable to the first.

It may be shown further that if the new investment when placed in process 2 is large enough to make labor limiting in process 1 then it is optimal to place all the capital increment in process 2. The output resulting from placing all the increment in process 2 will be called $y_{22}$. It can then be shown that

$$y_{II}^i(t) - y_{II}^o(t) = (1-p_1) \left( 1 - \frac{a_{21}}{a_{11}} \right)$$

clearly the difference will be largest when $p_1 = 0$. This is the reasoning for the allocations in period 5 and after in Table 2.

One other case remains to be considered: the optimal allocation when the increment to capital is such that although satisfying (16), the increment is not so large that when placed in process 2 labor becomes limiting in process 1. The investment of period 3 in Table II is in this category. Withdrawing a little capital from process 2 would, in this case, allow the employment of more labor in process 1 than had been displaced in process 2. In this case, then it is possible to make an allocation which will imply full employment of both factors and such an allocation would, in fact, be optimal.

The output resulting from full employment of both factors, $y_{FE}(t)$, is

$$y_{FE}(t) = y_1(t) + y_2(t) = \frac{(a_{21} - a_{22}) x_1(t) + (a_{12} - a_{11}) x_2(t)}{a_{12} a_{21} - a_{22} a_{11}}$$

It can be shown that this output is larger than that which can be achieved from any other allocation of $i(t-1)$ by a virtual allocation of all $i(t-1)$ to process 2; capital would then be limiting both in process 1 and 2. A small withdrawal of capital from process 2 would
I decrease its output by \( \frac{\Delta i(t)}{a_{12}} \), and release labor in the amount \( \frac{\Delta i(t)}{a_{12}} \). Putting this increment of capital into process 1 would increase output by \( \frac{\Delta i(t)}{a_{12}} \). Since \( \frac{\Delta i(t)}{a_{12}} \), this would represent an increase in output, and explain the allocation in period 4 in Table 2.

The procedure followed in Table 2 in allocation new increments of capital is, then, perfectly general, at least within the limited framework of assumptions used for the present analysis. Moreover, it is possible to withdraw some of the most limiting assumptions without changing the nature of the conclusions. As in the preceding section, in which it was assumed that capital is completely non-specific, the assumptions of a constant level of consumption and working force are not essential to the demonstration. If the laboring force had its own rate of growth both the arithmetic and algebra would be more complicated but the conclusions would be the same. Likewise, if consumption were dependent on income, rather than assumed to be constant, there would be no fundamental changes in the method of optimal allocation.

The time period used in the model is the period of gestation of investment. If this period should change, the magnitude of the technical coefficients would also have to change. It should also be noticed that the process chosen for new investment should require factor proportions suitable to those which will exist at the end of the period of gestation, not those existing when the new investment is undertaken.

The assumption of a series of fixed coefficient processes is, also, not essential I believe. The processes actually used
can be regarded as points on a curve of continuous factor substitution without changing the essential logic.

The dynamic equations of growth will change, however, as the assumptions about the working force and consumption are varied. When only one process is in use and the labor force is fixed in size, the dynamic analysis of the preceding section will also serve for this case; it does not matter that capital in the single process is "sunk". If two processes are used and capital is a specific factor a new analysis is necessary.

Total income $y(t)$ is now the sum of income produced in each process, $y_1(t)$ and $y_2(t)$. Since we allocate factors to make capital limiting in the more capital-intensive process and labor limiting in the more labor-intensive process we have

\[
(25) \quad y_N(t) = \frac{x_{21}(t)}{a_{21}} + \frac{x_{12}(t)}{a_{12}}
\]

The capital increment in each year is allocated fully to process 2 and all capital in process 2 is fully employed so that

\[
(26) \quad \frac{x_{12}(t)}{a_{12}} = \frac{x_{12}(t-1) + y_N(t-1) - c}{a_{12}}
\]

The labor residual is limiting in process 1; thus

\[
(27) \quad \frac{x_{21}(t)}{a_{21}} = \frac{x_2(t) - a_{22}y_2(t)}{a_{21}}
\]

These and similar equations for period $t-1$ permit the derivation of the following income equation for $y(t)$ solely in terms of $y(t-1)$

\[
(28) \quad y(t) + \frac{a_{22} - 2a_{21}}{a_{12} a_{21}} y(t-1) = -c
\]

Since equation (28) requires for stability that

\[
(29) \quad -\left(\frac{a_{22} - 2a_{21}}{a_{12} a_{21}}\right) < c \quad \text{or} \quad a_{21} < \frac{a_{22}}{2}
\]
equation (28) will always produce unstable movements of income. The stability conditions would change if the above assumptions about consumption and the working force were changed and, of course, the entire analysis rests on the assumption that capital has an infinite life.

It is tempting to generalize the findings of this section. For example, if capital equipment did possess an infinite life there would arise, I believe, conflicts between investment decisions which were optimal from an overall point of view and those which would be taken by private entrepreneurs.

In the analysis just completed, it has been demonstrated that investing in processes which use factors in proportions suitable to those currently prevailing will maximize the output which can be achieved in any given time, or minimize the time necessary to achieve a particular output target. If capital once invested in a particular process cannot be shifted to any other process, this rule implies that it will be optimal to scrap capital even if it has been used for only a few years if it requires factors in proportions no longer existing. Only under the special conditions of perfect competition, when private entrepreneurs are assumed to have perfect foresight, might individual decisions produce the social optimum.

Suppose individual businessmen expected some future shift in factor proportions which would affect relative factor prices and, thus, the costs involved in using different processes. Since their equipment would have a permanent life, the businessmen in trying to
maximize discounted future net income, or some average of future incomes, would choose a process using factor proportions suitable to some future period rather than the current period. Only through perfect foresight could private decisions create supply and demand conditions which would justify the use of the processes which would produce the optimum rate of growth.

In an economy with a slow rate of change in relative factor endowment "building ahead" of the current factor proportions will mean a longer period of less than optimal output than in an economy in which the relative factor proportions change rapidly. A rapid rate of change in relative factor endowment would mean that factor proportions would "catch up" quickly to the proportions required in the existing equipment. On the other hand, expectations of frequent change in relative factor costs might also lead to a greater amount of "building ahead" as compared to an economy with slow or infrequent changes in relative factor endowment. But in the latter type of system a "mistake" in creating capital with unsuitable factor proportions would persist longer as a mistake.

The above generalisations depend, however, on an analysis in which it was assumed that capital has an infinite life. Eliminating this assumption will lead to major modifications in the previous results.

IV. Choice between Processes in which Capital has Different Degrees of Durability.

The analysis which follows is far from complete but has, I believe, been carried far enough to be suggestive. The problem of choice to be faced now is one in which the alternative processes not only involve different intensities of factor use but different life expectancies for capital.
There are, of course, a number of alternative methods of capital depreciation. The capital model now assumed and depicted in Figure 4 is the "one-horse shay" model. Capital, once created, delivers services at a constant rate for a given period, \( m \), which varies from process to process; it then break down and is completely depreciated at the end of this period. This method of depreciation is used not because it is the most accurate of commonly employed technique but for its simplicity.

It will also be assumed here that capital is a specific factor. Since this section will only take up the comparison of the effects of the use of one process at a time, it would be possible to assume that capital is non-specific but has a different rate of wear and, therefore, a different life span when used in the different processes. The labor force and consumption will again be held constant.

It is helpful to begin again with a numerical example; the data of Figure 2 will be used with the additional assumption that capital in process 1 lasts two years and in process 2 lasts four years.\(^1\) Following our earlier procedure and only using process 1 at first because in the original factor endowment labor is redundant,  

\(^1\) The labor endowment is increased to give a longer run using only a single process.
the income progression of Table 4 is obtained.

Table 4

| t  | \( x_{11} \) | \( x_{12} \) | \( x_{13} \) | \( x_{14} \) | \( x_1 \) | \( x_{21} \) | \( x_{22} \) | \( x_{23} \) | \( x_{24} \) | \( y_1 \) | \( y_2 \) | \( y_{16} \) | \( y_{26} \) | \( y_{36} \) | \( y_{46} \) | \( y_0 \) | \( C \) | \( I_0 \) | \( d \) | \( I_N \) | \( y_N \) |
|----|-------------|-------------|-------------|-------------|--------|-------------|-------------|-------------|-------------|--------|--------|-------------|-------------|-------------|-------------|--------|--------|--------|-------------|
| 1  | 4           | 4           | 10          |            | 50     | 2           |             |             |             | 2      | 1      | 0           |             |             |             |        |        | 1      | 2           |
| 2  | 5           | 5           | 12.5        |            | 50     | 2.5         |             |             |             | 2.5    | 1      | 1.5          |             |             |             |        |        | 1.5    | 2.5         |
| 3  | 6.5         | 6.5         | 16.25       |            | 50     | 3.25        |             |             |             | 3.25   | 1      | 2.25         |             |             |             |        |        | 1.25   | 2.25        |
| 4  | 7.75        | 7.75        | 19.375      |            | 50     | 3.875       |             |             |             | 3.875  | 1      | 2.875        |             |             |             |        |        | 1.375  | 2.375       |
| 5  | 9.125       | 9.125       | 22.8        |            | 50     | 4.56        |             |             |             | 4.56   | 1      | 3.56         |             |             |             |        |        | 1.31   | 2.31        |
| 6  | 10.43       | 10.43       | 26.10       |            | 50     | 5.22        |             |             |             | 5.22   | 1      | 4.22         |             |             |             |        |        | 1.35   | 2.34        |
| 7  | 11.78       | 11.78       | 29.45       |            | 50     | 5.89        |             |             |             | 5.89   | 1      | 4.89         |             |             |             |        |        | 1.33   | 2.33        |
| 8  | 13.11       | 13.11       | 32.75       |            | 50     | 6.55        |             |             |             | 6.55   | 1      | 5.55         |             |             |             |        |        | 1.33   | 2.33        |

Since capital does wear out, it is necessary to keep gross and net items distinguished, which is done here by use of the subscripts \( G \) and \( N \); \( d \) is annual depreciation which is the total (i.e. gross) investment of two period previous.

Comparison of Table 1 and Table 4 shows some of the effects of the depreciation of capital. The behaviour of net income in Table 1 is quite different from that in Table 4. When depreciation is subtracted, net income not only grows more slowly at first but also flattens out and stops growing by period 7, while in Table 1 growth of net income is continuous if not steady.

Table 5 shows the results of starting with and continuing to use process 2 in which capital lasts for four years.
Table 5

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In Table 5 net income at first grows more slowly than in Table 4. But unlike Table 4, net income in Table 5 continues to grow. By period 10 it has outstripped the level of income produced by process 1 alone.\(^1\)

\(^1\)It will be noticed that in both Tables 4 and 5 the original capital endowment is never depreciated. This stems from an oversight in planning the exposition rather than from an economically useful assumption. Tables 4 and 5 are constructed so as to be comparable to the previous arithmetic examples. But when the previous tables were constructed, since depreciation of capital was not involved, the original factor endowments and consumption were set at levels which would obviate growth when depreciation was introduced. Rather than redoing the previous arithmetic, the expedient was adopted of not depreciating the original capital endowment.
The arithmetical examples of Tables 4 and 5 are sufficient to demonstrate important differences between models in which there is capital depreciation and those in which capital has an infinite life span. Under the assumption that capital has an infinite life and produces a constant stream of services, any investment represents a permanent addition to resources whose choice can be evaluated in purely static terms by comparing alternative outputs for a single following year. When capital depreciates the rule derived from statics of economizing the scarce factor does not carry over intact for dynamic decisions. Following the rule may even prohibit growth, as in Table 4.

Choice between processes now requires a specifically dynamic analysis. One approach to such an analysis will be partially demonstrated here.

Gross output from process \( j \), in which it is assumed that capital \( x_1 \) is the limiting factor is

\[
(30) \quad y_{jG}(t) = \frac{x_{1j}(t)}{a_{1j}}
\]

while net income is

\[
(31) \quad y_{jN}(t) = y_{jG}(t) - d_j(t)
\]

where \( d_j(t) \) is the depreciation of capital in process \( i \) in period \( t \). The "one horse shay" depreciation technique means that

\[
(32) \quad d_j(t) = i_g(t - m)
\]

\( i_g(t - m) \) being the gross investment of \( m \) periods previous to \( t \), and \( m \) the time required for capital to wear out. Finally,

\[
(33) \quad i_{jG}(t) = y_{jG}(t) - o(t) = i_{jN}(t) + d_j(t)
\]
Combining equations (30), (31), (32) and (33) yields the income equation

\[ y^N(t) = \frac{1}{a_{1j}} \sum_{n=0}^{m} y^N(t-n) + \frac{1}{a_{1j}} \sum_{n=0}^{m} c(t-n) \]

Unless we specify \( a_{1j} \) and \( m \) the general solution of equation (34) is likely to be unenlightening or impossible for \( m \) greater than 2.

However, once having specified these parameters it is possible, although perhaps tedious, to find the roots of the characteristic equation of (34) and thus write the "dynamic part" (i.e. the homogeneous solution) of the equation's complete solution. It would again be tedious but not impossible to complete the solution (by finding the particular solution, adjoining this to the homogeneous solution and imposing the initial conditions). Having the complete solution for each of the alternative processes, it is then possible to compare and choose between the time paths which investment in each process would generate.

However, it is not necessary to solve (34) generally in order to distinguish between processes in which investment would not lead to economic growth and processes in which investment would lead to a cumulative development. This distinction may be achieved through application of stability conditions to equation (34). It may also be difficult to work out these stability conditions except for the simplest cases but they are, I think, worth investigation.

Suppose that consumption is maintained at a constant level, \( c \), from period to period. Equation (34) could then be written

\[ y^N(t) = \frac{1}{a_{1j}} \sum_{n=0}^{m} y^N(t-n) + \frac{1}{a_{1j}} \sum_{n=0}^{m} c(t-n) \]
There is only one necessary and sufficient condition for stability for equations like (35), which is

\[(36) \quad 1 - \sum_{a_{1j}} \frac{1}{a_{1j}} > 0 \quad \text{or} \quad 1 - \frac{m}{a_{1j}} > 0 \quad \text{or} \quad a_{1j} > m\]

If condition (36) does not hold, investment in the particular process will lead to continuous economic change.

Condition (36) confirms the impression gained from the arithmetic example in Table 4: investment may or may not produce economic growth depending on its capital intensity and its length of life. Moreover, by looking at condition (36) we can appreciate why the models were so uniformly unstable in which it was assumed that capital had an infinite life.

Condition (36) also provides the logic for our intuitive feeling that highly capital intensive projects will pay off in underdeveloped areas only if they have a very long life and that investment projects with a very short life are advisable only if they use relatively little capital.

If consumption is assumed to be a simple linear function of income as in equation (6), the income equation becomes

\[(37) \quad y_N(t) - \frac{1}{a_{1j}} y_N(t - 1) - \frac{L}{a_{1j}} - \beta [y_N(t - 2) + \ldots + y_N(t-m)] - \gamma y_N(t-m+1) = \]

If \( \frac{1}{a_{1j}} > \beta \), the stability condition derived by Solow can still be applied and leads to the form

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1 I have not been able to locate a copy of the issue of Econometrica in which a relevant article by R. Solow appeared and thus am not absolutely sure that (36) is both necessary and sufficient. I believe that the appropriate Solow article is "On the Structure of Linear Models," Econometrica, Vol. 20, Jan., 1952, pp. 29 - 46.
This condition leads to considerations similar, though not identical, to (36), about what the relation must be between $a_{ij}$ and $m$ for growth and, in addition, helps to indicate the role of consumption. For any given $a_{ij}$ and $m$, relatively large values of the marginal propensity to consume, $\beta$, will be associated with stable income behaviour as compared to the smaller values of $\beta$ which would help make growth possible.1

V. Summary.

The issues raised in this paper are far from settled by the analysis which is contained here. Yet the discussion does, I believe, help in the appreciation of the proper relationships between the use of processes of varying capital intensity, economic growth and alternative capital models. Some of the more simple relations are analyzed explicitly here and this analysis, in turn, suggests the significance of alternative formulations of capital models, depreciation techniques and market structure.

First of all, drastic assumptions were made about the mobility and growth of the working force and the behaviour of consumption in order to focus the analysis sharply on the problem of allocation of investment among alternative processes.2

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1 If $\frac{1}{a_{ij}} \cdot \beta$, it is less easy to write down meaningful general stability conditions and this case is not analyzed here.

2 Labor is, of course, not fully mobile and resource inouts and a "gestation period" are necessary for its shifting and training. As a result, labor has some of the characteristics of capital and should be analyzed in the same way. The implications of the analysis of this paper are, therefore, not confined to capital.
The general principle that countries at different stages of development should each use processes which require factors in proportions suitable to their different relative factor endowments is derived from economic statics. The validity of the principle as a guide to investment in a dynamic system has been shown to depend on the length of life of the capital equipment, as well as the capital output ratio. More generally, the analysis suggests the significance for optimal choice between alternative processes of the time-aspects of capital investment: the time pattern of investment during the period of gestation and the time pattern of the resulting flow of capital services. Only a simple type of variation in the time pattern of the flow of capital services has actually been investigated, however, and further analysis would be profitable, I believe.

When capital services flow from an initial investment at a constant level for an infinite period, the dynamic problem of choice reduces to a static problem. And in this case optimizing means allocating investment to the different processes to produce the maximum output in each period.

However, when the life of capital is finite, even though the flow of capital services is assumed to be at a constant level for the life of the capital, the problem of choice becomes more complicated. As an example of the difficulties involved, suppose there are two processes with different capital and labor coefficients but the same life span. One cannot say that it will always be optimal to choose the process using factors in proportions most similar to those actually available. Depending on the relation between the capital-output ratio and the life span of the capital, continued investment in a particular process might or might not lead to continuous economic
growth. Thus, another possible source of labor redundancy in underdeveloped areas may arise if the relatively labor-intensive processes which are available also have relatively short life spans.

Only a beginning has been made, however, and more thought is required along a number of lines.

The macro analysis can be pushed further to consider the problem of choice between different patterns of investment inputs and capital service flows. I am not convinced, however, that the method of attack used above will produce the most useful results. Unless the difference equations involved will reduce to simple forms in ways which I do not presently see, all that can be produced on a general level, I believe, are valid but relatively unenlightening stability conditions for large matrices. Further investigation and trials are required to clarify this though.

The macro analysis must also be related more closely to micro-economic behaviour. This would help in identifying the effects of market imperfections and the kind of market structure which will lead to the optimal investment patterns. Although the conclusions of Section II must be qualified by those of Section III, it is still possible that a necessary consequence of the optimal investment pattern would be a rate of obsolescence of capital more rapid than would be permitted by private, individual decision making.

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1 "Obsolescent" is used here to describe investment using factors in such proportions as to involve costs higher than those which would result from alternative processes.