# Stochastic Stability Properties of a Singularly Perturbed Chemical Langevin Equation

Narmada Herath and Domitilla Del Vecchio

Massachusetts Institute of Technology

March 18, 2014

## Introduction

In this report, we consider the model reduction of a set of singularly perturbed chemical Langevin equations, according to Theorem 3.1 in [1].

## Mathematical Notation and Terminology

We use $||\cdot||$ to denote the euclidean norm and $|\cdot|$ to denote the absolute value.

A continuous function $\alpha : [0, a) \to [0, \infty)$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. [2]

A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$, is said to belong to class $\mathcal{KL}$ if, for each fixed $s$, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to $r$ and, for each fixed $r$, the mapping $\beta(r, s)$ is decreasing with respect to $s$ and $\beta(r, s) \to \infty$ as $s \to \infty$. [2]

## System Model

We consider a set of singularly perturbed chemical Langevin equations (1) - (4) with the singular perturbation parameter $0 < \epsilon \ll 1$, $x = (x_1, x_2)'$, $z = (z_1, z_2)'$, being the state variables :

$$\epsilon \dot{z_1} = u - \delta_1 z_1 + \sqrt{\epsilon u}\ \Gamma_1 - \sqrt{\epsilon \delta_1 z_1}\ \Gamma_2, \tag{1}$$

$$\dot{x_1} = \beta_1 z_1 - \delta_1(x_1 - z_2) + \sqrt{\beta_1 x_1}\ \Gamma_3 - \sqrt{\delta_1(x_1 - z_2)}\ \Gamma_4, \tag{2}$$

$$\epsilon \dot{z_2} = b(x_1 - z_2) - a\delta_1 z_2 + \sqrt{\epsilon b(x_1 - z_2)}\ \Gamma_5 - \sqrt{a\epsilon\delta_1 z_2}\ \Gamma_6, \tag{3}$$

$$\dot{x_2} = \beta_2 z_2 - \delta_2 x_2 + \sqrt{\beta_2 z_2}\ \Gamma_7 - \sqrt{\delta_2 x_2}\ \Gamma_8. \tag{4}$$

The equations (1) - (4), can be reduced to a system with $\epsilon = 0$ according to Theorem 3.1 in [1]. The assumptions, given by (A1) - (A4), and the results of Theorem 3.1 in [1] are as follows.

Consider a set of singularly perturbed nonlinear Itô differential equations

$$\begin{aligned} \epsilon dz &= f_1(x,z,\theta,\epsilon)dt + g_1(x,z,\theta,\epsilon)dW_1, \\ dx &= f_2(x,z,\theta,\epsilon)dt + g_2(x,z,\theta,\epsilon)dW_2, \end{aligned} \tag{5}$$

where $z \in \mathbb{R}^q, x \in \mathbb{R}^n$ and $\theta \in \mathbb{R}^m$ is an input that is absolutely continuous, and $\epsilon$ is a small positive constant (singular perturbation parameter).

**A1** : The equation $f_1(x,z,\theta,0) = 0$ admits a unique solution $z_s = h(x,\theta)$ which further satisfies $g_1(x,z,\theta,0) = 0$. Moreover, the function $h : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^p$ as well as its first and second derivatives are locally Lipschitz.

**A2** : The reduced slow subsystem given by

$$dx = f_2(x, h(x,\theta), \theta, 0)dt + g_2(x, h(x,\theta), \theta, 0)dW_2,$$

is SISS with respect to input $\theta \in \mathbb{R}^m$, i.e., $\forall \nu > 0$, there exist a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and class $\mathcal{K}$ functions $\gamma_1(\cdot)$, such that

$$P\left\{ ||x(t)|| < \beta(||x_0||, t) + \gamma_1\left(\sup_{0\le s\le t} ||\theta(s)||\right)\right\} \ge 1 - \nu, \qquad \forall t \ge 0, \forall x_0 \in \mathbb{R}^2 \setminus \{0\}.$$

**A3** : The reduced fast subsystem defined by

$$dy = f_1(x, h(x,\theta) + y, \theta, 0)d\tau + \tilde{g}_1(x, h(x,\theta) + y, \theta, 0)d\tilde{W}_1,$$

where $\tilde{W}$ is a standard Wiener process on the fast time scale, $\tilde{g} = \lim_{\epsilon\to 0} g_1/\sqrt{\epsilon}$ is assumed to be locally Lipschitz, and $x, \theta$ are to be viewed as constants (on the fast time scale), is SISS with respect to state $x \in \mathbb{R}^n$ and input $\theta \in \mathbb{R}^m$, i.e., $\forall \nu > 0$, there exist a class $\mathcal{KL}$ function $\beta_y(\cdot,\cdot)$ and class $\mathcal{K}$ functions $\gamma_2^x(\cdot), \gamma_2(\cdot)$, such that

$$P\left\{ ||y(t)|| < \beta(||y_0||, t) + \gamma_2^x\left(\sup_{0\le s\le t} ||x(s)||\right) + \gamma_2\left(\sup_{0\le s\le t} ||\theta(s)||\right)\right\} \ge 1 - \nu, \ \ \forall t \ge 0,$$
$$\forall z_0 \in \mathbb{R}^2 \setminus \{0\}.$$

**A4** : There exist class $\mathcal{K}_\infty$ function $p_1$, a class $\mathcal{K}$ function $\gamma_1^{\tilde{y}}$, a non-increasing function $b : \mathbb{R}_+ \times \mathbb{R}_+ \to (0,1]$ and positive constants $\delta_x, \delta$ such that

$$2\gamma_1^{\tilde{y}} \circ (I + p_1) \circ \frac{\gamma_2^x}{b(\delta_x, \delta)}(s) \le s.$$

Then, given $\nu > 0$, there exist class $\mathcal{KL}$ functions $\delta_1, \delta_2$, class $\mathcal{K}$ functions $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_1^\epsilon, \tilde{\gamma}_2^\epsilon$, and a positive real number satisfying $\max\{||x_0||, ||y_0||, ||\theta||, ||\dot{\theta}||\} \le \delta$ and $0 < \epsilon < \epsilon^*$, the following relationship holds:

$$P\Big\{||x(t)|| < \delta_1(||(x_0,y_0)||,t) + \tilde{\gamma}_1\left(\sup_{0\le s\le t}||\theta(s)||\right) + \tilde{\gamma}_1^{\epsilon}(\epsilon),$$
$$\text{and } ||y(t)|| < \delta_2(||(x_0,y_0)||,t/\epsilon) + \tilde{\gamma}_2^{\epsilon}(\epsilon) + \tilde{\gamma}_2\left(\sup_{0\le s\le t}||\theta(s)||\right)\Big\} \ge 1-v, \ \ \forall t \ge 0.$$

Using these results, we can see that $||y(t)||$ is bounded in probability and the bound decreases as $\epsilon \to 0$. Therefore, as $\epsilon$ becomes smaller, $h(x,\theta)$ becomes a better approximation of $z$.

To apply Theorem 3.1, we write the system (1) - (4) in the form of a set of It$\hat{o}$ differential equations, using the relation $dW/dt = \Gamma$ where $dW$ is Wiener increment. Since $\Gamma_i$ are independent identical Gaussian white noise processes, these dynamics are given by

$$\epsilon dz_1 = (u - \delta_1 z_1)dt + \sqrt{\epsilon u + \epsilon\delta_1 z_1}\ dW_1, \tag{6}$$
$$dx_1 = (\beta_1 z_1 - \delta_1(x_1 - z_2))dt + \sqrt{\beta_1 x_1 + \delta_1(x_1 - z_2)}\ dW_2, \tag{7}$$
$$\epsilon dz_2 = (b(x_1 - z_2) - a\delta_1 z_2)dt + \sqrt{\epsilon b(x_1 - z_2) + a\epsilon\delta_1 z_2}\ dW_3, \tag{8}$$
$$dx_2 = (\beta_2 z_2 - \delta_2 x_2)dt + \sqrt{\beta_2 z_2 + \delta_2 x_2}\ dW_4, \tag{9}$$

and correspond to system (5), with $f_1(x,z,\theta,\epsilon) = (u - \delta_1 z_1, b(x_1 - z_2) - a\delta_1 z_2)'$, $f_2(x,z,\theta,\epsilon) = (\beta_1 z_1 - \delta_1(x_1 - z_2), \beta_2 z_2 - \delta_2 x_2)'$, $g_1(x,z,\theta,\epsilon) = (\sqrt{\epsilon u + \epsilon\delta_1 z_1}, \sqrt{\epsilon b(x_1 - z_2) + a\epsilon\delta_1 z_2})'$ and $g_2(x,z,\theta,\epsilon) = (\sqrt{\beta_1 x_1 + \delta_1(x_1 - z_2)}, \sqrt{\beta_2 z_2 + \delta_2 x_2})'$.

In the following sections, we demonstrate that each of the assumptions (A1) - (A4) are satisfied for the system (6) - (9).

## Verification of A1

When $\epsilon = 0$, the equation $\begin{bmatrix} u - \delta_1 z_1 \\ b(x_1 - z_2) - a\delta_1 z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ admits the unique solution,

$$z_s = \begin{bmatrix} \frac{u}{\delta_1} \\ \frac{bx_1}{b + a\delta_1} \end{bmatrix} = \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix},$$

and with $\epsilon = 0$, we have $\begin{bmatrix} \sqrt{\epsilon u}\ \Gamma_1 - \sqrt{\epsilon\delta_1 z_1}\ \Gamma_2 \\ \sqrt{\epsilon b(x_1 - z_2)}\ \Gamma_5 - \sqrt{a\epsilon\delta_1 z_2}\ \Gamma_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Therefore, A1 is satisfied.

## Verification of A2

We can obtain the reduced slow subsystem by substituting $h_1(x)$ and $h_2(x)$ in (7) and (9) as follows:

$$
\begin{aligned}
dx_1 &= \left(\frac{\beta_1 u}{\delta_1} - \delta_1\left(x_1 - \frac{bx_1}{b+a\delta_1}\right)\right)dt + \sqrt{\frac{\beta_1 u}{\delta_1} + \delta_1\left(x_1 - \frac{bx_1}{b+a\delta_1}\right)}\, dW_2,\\
dx_2 &= \left(\frac{\beta_2 bx_1}{b+a\delta_1} - \delta_2 x_2\right)dt + \sqrt{\frac{\beta_2 bx_1}{b+a\delta_1} + \delta_2 x_2}\, dW_4.
\end{aligned}
$$

Simplifying further, these dynamics are given by

$$
\begin{aligned}
dx_1 &= \left(\frac{\beta_1 u}{\delta_1} - \left(\frac{a\delta_1^2}{b+\delta_1}\right)x_1\right)dt + \sqrt{\frac{\beta_1 u}{\delta_1} + \left(\frac{a\delta_1^2}{b+\delta_1}\right)x_1}\, dW_2,\\
dx_2 &= \left(\frac{\beta_2 bx_1}{b+a\delta_1} - \delta_2 x_2\right)dt + \sqrt{\frac{\beta_2 bx_1}{b+a\delta_1} + \delta_2 x_2}\, dW_4.
\end{aligned}
$$

Let

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & -\left(\frac{a\delta_1^2}{b+\delta_1}\right) \\ \frac{\beta_2 b}{b+a\delta_1} & -\delta_2 \end{bmatrix} = \begin{bmatrix} 0 & -\gamma \\ k_2 & -\delta_2 \end{bmatrix},\\
g(x) &= \begin{bmatrix} \frac{\beta_1 u}{\delta_1} + \sqrt{\frac{\beta_1 u}{\delta_1} + \left(\frac{a\delta_1^2}{b+\delta_1}\right)x_1} \\ \sqrt{\frac{\beta_2 bx_1}{b+a\delta_1} + \delta_2 x_2} \end{bmatrix} = \begin{bmatrix} k_1 + \sqrt{k_1 + \gamma x_1} \\ \sqrt{k_2 x_1 + \delta_2 x_2} \end{bmatrix}.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
dx &= Axdt + \begin{bmatrix} \frac{\beta_1 u}{\delta_1} + \sqrt{\frac{\beta_1 u}{\delta_1} + \left(\frac{a\delta_1^2}{b+\delta_1}\right)x_1} \\ \sqrt{\frac{\beta_2 bx_1}{b+a\delta_1} + \delta_2 x_2} \end{bmatrix}\begin{bmatrix} dW_2 \\ dW_4 \end{bmatrix},\\
dx &= Axdt + g(x)dW_x. \qquad (10)
\end{aligned}
$$

$$(11)$$

where $dW_x = (dW_2, dW_4)'$. To prove that the system described by equation (10) is SISS with respect to an input $\theta \in \mathbb{R}^2$ that will be defined at the end of this section, we proceed by using a change of coordinates such that $v = P^{-1}x$, with $A = PDP^{-1}$ where $D$ is a diagonal matrix. Specifically, we have that

$$
D = \begin{bmatrix} -\frac{\delta_2}{2} - \frac{\sqrt{\delta_2^2 - 4\gamma k_2}}{2} & 0 \\ 0 & -\frac{\delta_2}{2} + \frac{\sqrt{\delta_2^2 - 4\gamma k_2}}{2} \end{bmatrix} = \begin{bmatrix} -D_1 & 0 \\ 0 & -D_2 \end{bmatrix},
$$

$$P = \begin{bmatrix} \frac{\delta_2}{k_2} - \frac{1}{k_2}\left[\frac{\delta_2}{2} + \frac{\sqrt{\delta_2^2-4\gamma k_2}}{2}\right] & \frac{\delta_2}{k_2} - \frac{1}{k_2}\left[\frac{\delta_2}{2} - \frac{\sqrt{\delta_2^2-4\gamma k_2}}{2}\right] \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} P_1 & P_2 \\ 1 & 1 \end{bmatrix},$$

$$P^{-1} = \begin{bmatrix} \frac{-k_2}{\sqrt{\delta^2-4\gamma k_2}} & \frac{\delta_2^2+\sqrt{\delta^2-4\gamma k_2}}{2\sqrt{\delta^2-4\gamma k_2}} \\ \frac{k_2}{\sqrt{\delta^2-4\gamma k_2}} & \frac{-\delta_2^2+\sqrt{\delta^2-4\gamma k_2}}{2\sqrt{\delta^2-4\gamma k_2}} \end{bmatrix} = \begin{bmatrix} F_1 & F_3 \\ F_2 & F_4 \end{bmatrix}.$$

Then,

$$\begin{aligned} x = Pv &= \begin{bmatrix} P_1 & P_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} P_1v_1 + P_2v_2 \\ v_1 + v_2 \end{bmatrix}, \\ dv &= Dzdt + P^{-1}g(x)dW_x, \\ dv &= \left(\begin{bmatrix} -D_1 & 0 \\ 0 & -D_2 \end{bmatrix} v + \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}\begin{bmatrix} k_1 \\ 0 \end{bmatrix}\right) dt + \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}\begin{bmatrix} \sqrt{k_1+\gamma x_2} \\ \sqrt{k_2x_1+\delta_2x_2} \end{bmatrix}\begin{bmatrix} dW_2 \\ dW_4 \end{bmatrix}. \end{aligned} \tag{12}$$

We use Proposition 2.3 in [1] to prove that the slow subsystem defined by equation (12) is SISS with respect to an appropriate input $\theta$ to be defined later. Consider the Lyapunov function $V = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$. Then,

$$\begin{aligned} LV = &\begin{bmatrix} v_1 & v_2 \end{bmatrix}\left(Dv + \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}\begin{bmatrix} k_1 \\ 0 \end{bmatrix}\right) \\ &+ \frac{1}{2}Tr\left\{\begin{bmatrix} \sqrt{k_1+\gamma x_1} & \sqrt{k_2x_1+\delta_2x_2} \end{bmatrix}\begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}\begin{bmatrix} F_1 & F_3 \\ F_2 & F_4 \end{bmatrix}\begin{bmatrix} \sqrt{k_1+\gamma x_1} \\ \sqrt{k_2x_1+\delta_2x_2} \end{bmatrix}\right\}, \\ &= -D_1v_1^2 - D_2v_2^2 + F_1k_1v_1 + F_3k_1v_2 \\ &\qquad + \frac{1}{2}((F_1^2+F_2^2)(k_1+\gamma x_1) + 2(F_1F_3+F_2F_4)\sqrt{k_2x_1+\delta_2x_2}\sqrt{k_1+\gamma x_1}) \\ &\qquad + \frac{1}{2}((F_3^2+F_4^2)(k_2x_1+\delta_2x_2)), \\ &= -D_1v_1^2 - D_2v_2^2 + F_1k_1v_1 + F_3k_1v_2 \\ &\qquad + \frac{1}{2}((F_1^2+F_2^2)(k_1+\gamma x_1)) + (F_1F_3+F_2F_4)\sqrt{k_2x_1+\delta_2x_2}\sqrt{k_1+\gamma x_1} \\ &\qquad + \frac{1}{2}((F_3^2+F_4^2)(k_2x_1+\delta_2x_2)), \\ &= -D_1v_1^2 - D_2v_2^2 + F_1k_1v_1 + F_3k_1v_2 + \frac{1}{2}((F_1^2+F_2^2)(k_1+\gamma(P_1v_1+P_2v_2))) \\ &\qquad + (F_1F_3+F_2F_4)\sqrt{k_2(P_1v_1+P_2v_2)+\delta_2(v_1+v_2)}\sqrt{k_1+\gamma(P_1v_1+P_2v_2)} \\ &\qquad + \frac{1}{2}((F_3^2+F_4^2)(k_2(P_1v_1+P_2v_2)+\delta_2(v_1+v_2))). \end{aligned}$$

Using that

$$\begin{aligned} &\sqrt{k_2(P_1v_1+P_2v_2)+\delta_2(v_1+v_2)}\sqrt{k_1+\gamma(P_1v_1+P_2v_2)} \\ &\qquad \leq \left(\frac{k_2(P_1v_1+P_2v_2)+\delta_2(v_1+v_2)}{2} + \frac{k_1+\gamma(P_1v_1+P_2v_2)}{2}\right), \end{aligned}$$

we can write

$$
\begin{aligned}
LV \leq & -D_1 v_1^2 - D_2 v_2^2 + F_1 k_1 v_1 + F_3 k_1 v_2 + \frac{1}{2}((F_1^2 + F_2^2)(k_1 + \gamma(P_1 v_1 + P_2 v_2))) \\
& + (F_1 F_3 + F_2 F_4)\left(\frac{k_2(P_1 v_1 + P_2 v_2) + \delta_2(v_1 + v_2)}{2} + \frac{k_1 + \gamma(P_1 v_1 + P_2 v_2)}{2}\right) \\
& + \frac{1}{2}((F_3^2 + F_4^2)(k_2(P_1 v_1 + P_2 v_2) + \delta_2(v_1 + v_2))) \\
\leq & -D_1 v_1^2 - D_2 v_2^2 + \\
& \left(F_1 k_1 + \frac{1}{2}\left((F_1^2 + F_2^2)\gamma P_1 + (F_1 F_3 + F_2 F_4)(k_2 P_1 + \delta_2 + \gamma P_1) + (F_3^2 + F_4^2)(k_2 P_1 + \delta_2)\right)\right) v_1 \\
& + \left(F_3 k_1 + \frac{1}{2}((F_1^2 + F_2^2)\gamma P_2 + (F_1 F_3 + F_2 F_4)(k_2 P_2 + \delta_2 + \gamma P_2) + (F_3^2 + F_4^2)(k_2 P_2 + \delta_2))\right) v_2 \\
& + \frac{1}{2}((F_1^2 + F_2^2)k_1 + (F_1 F_3 + F_2 F_4)k_1).
\end{aligned}
$$

Let

$$
\begin{aligned}
E_1 &= \left(F_1 k_1 + \frac{1}{2}\left((F_1^2 + F_2^2)\gamma P_1 + (F_1 F_3 + F_2 F_4)(k_2 P_1 + \delta_2 + \gamma P_1) + (F_3^2 + F_4^2)(k_2 P_1 + \delta_2)\right)\right), \\
E_2 &= \left(F_3 k_1 + \frac{1}{2}((F_1^2 + F_2^2)\gamma P_2 + (F_1 F_3 + F_2 F_4)(k_2 P_2 + \delta_2 + \gamma P_2) + (F_3^2 + F_4^2)(k_2 P_2 + \delta_2))\right), \\
E_3 &= \frac{1}{2}((F_1^2 + F_2^2)k_1 + (F_1 F_3 + F_2 F_4)k_1).
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
LV &\leq -D_1 v_1^2 - D_2 v_2^2 + E_1 v_1 + E_2 v_2 + E_3, \\
LV &\leq -\frac{D_1}{2} v_1^2 - \frac{D_1}{2}\left(\left(v_1 - \frac{E_1}{D_1}\right)^2 - \left(\frac{E_1}{D_1}\right)^2\right) - \frac{D_2}{2} v_2^2 - \frac{D_2}{2}\left(\left(v_2 - \frac{E_2}{D_2}\right)^2 - \left(\frac{E_2}{D_2}\right)^2\right) + E_3, \\
LV &\leq -\frac{D_1}{2} v_1^2 - \frac{D_1}{2}\left(v_1 - \frac{E_1}{D_1}\right)^2 + \left(\frac{E_1^2}{2D_1}\right) - \frac{D_2}{2} v_2^2 - \frac{D_2}{2}\left(v_2 - \frac{E_2}{D_2}\right)^2 + \left(\frac{E_2^2}{2D_2}\right) + E_3.
\end{aligned}
$$

We can show that $LV \leq -\eta_1(v_1^2 + v_2^2)$, with $\eta_1 = (\min \frac{D_1}{2}, \frac{D_2}{2})/2$, if

$$
\left(\frac{D_1}{2} - \eta_1\right) v_1^2 + \left(\frac{D_2}{2} - \eta_1\right) v_2^2 \geq \left|\left(\frac{E_1^2}{2D_1}\right) + \left(\frac{E_2^2}{2D_2}\right) + E_3\right|, \tag{13}
$$

i.e.,

$$
v_1^2 \geq \frac{\left|\left(\frac{E_1^2}{2D_1}\right) + \left(\frac{E_2^2}{2D_2}\right)\right|}{\left(\frac{D_1}{2} - \eta_1\right)} \qquad \text{and} \qquad v_2^2 \geq \frac{|E_3|}{\left(\frac{D_2}{2} - \eta_1\right)}.
$$

Then,

$$\sqrt{v_1^2+v_2^2} \geq \sqrt{\frac{\left|\left(\frac{E_1^2}{2D_1}\right)+\left(\frac{E_2^2}{2D_2}\right)\right|}{\left(\frac{D_1}{2}-\eta_1\right)} + \frac{|E_3|}{\left(\frac{D_2}{2}-\eta_1\right)}}. \tag{14}$$

Therefore, for an input $\theta = \left(\frac{\left|\left(\frac{E_1^2}{2D_1}\right)+\left(\frac{E_2^2}{2D_2}\right)\right|}{\left(\frac{D_1}{2}-\eta_1\right)} + \frac{|E_3|}{\left(\frac{D_2}{2}-\eta_1\right)}, \frac{\left|\frac{\delta_1}{8}+u\right|}{\left(\frac{\delta_1}{2}-\eta_2\right)}\right)'$, we have that $LV \leq -\eta_1(v_1^2+v_2^2)$, for $||v|| \geq \sqrt{||\theta||}$, and therefore, according to Proposition 2.3 the system is SISS with input $\theta$.

Thus, $\forall \nu > 0$, there exist a class $KL$ function $\beta(\cdot,\cdot)$ and class $K$ functions $\gamma_1(\cdot)$, such that

$$P\left\{||v(t)|| < \beta(||v_0||,t) + \gamma_1\left(\sup_{0\leq s\leq t}||\theta(s)||\right)\right\} \geq 1-\nu, \qquad \forall t\geq 0, \forall v_0 \in \mathbb{R}^2 \setminus \{0\}.$$

As $||v||^2 = x^T(P^{-1})^T P^{-1} x$, where $(P^{-1})^T P^{-1}$ is a positive definite matrix, we have that

$$\lambda_{min}((P^{-1})^T P^{-1})||x||^2 \leq ||v||^2 \leq \lambda_{max}((P^{-1})^T P^{-1})||x||^2,$$

where $\lambda_{max}$ and $\lambda_{min}$ are the maximum and minimum eigenvalues of $((P^{-1})^T P^{-1})$, respectively.

Therefore, according to the definition of SISS, we obtain that the slow subsystem in the original coordinates is SISS, i.e., $\forall \nu > 0$, there exist a class $KL$ function $\beta_x(\cdot,\cdot)$ and class $K$ function $\gamma_{1x}(\cdot)$, such that

$$P\left\{||x(t)|| < \beta_x(||x_0||,t) + \gamma_{1x}\left(\sup_{0\leq s\leq t}||\theta(s)||\right)\right\} \geq 1-\nu, \qquad \forall t\geq 0, \forall x_0 \in \mathbb{R}^2 \setminus \{0\}.$$

where $\beta_x(\cdot,\cdot) = \frac{\beta(\cdot,\cdot)}{\sqrt{\lambda_{min}((P^{-1})^T P^{-1})}}$ and $\gamma_{1x}(\cdot) = \frac{\gamma_1(\cdot)}{\sqrt{\lambda_{min}((P^{-1})^T P^{-1})}}$.

This satisfies A2.

## Verification of A3

To obtain the reduced fast system, we define $y_1 = z_1 - h_1(x)$ and $y_2 = z_2 - h_2(x)$. Then the fast subsystem is given by

$$\begin{aligned}
dy_1 &= (u - \delta_1(y_1 + h_1(x)))d\tau + \sqrt{u + \delta_1(y_1 + h_1(x))}\; dW_{y_1}, \\
dy_2 &= (bx_1 - b(y_2 + h_2(x)))d\tau - a\delta_1(y_2 + h_2(x)) + \sqrt{bx_1 - b(y_2 + h_2(x)) + a\delta_1(y_2 + h_2(x))}\; dW_{y_2}.
\end{aligned}$$

Simplifying further, we obtain

$$dy_1 = -\delta_1 y_1 d\tau + \sqrt{2u + \delta_1 y_1}\ dW_{y_1}, \tag{15}$$

$$dy_2 = -(b + a\delta_1)y_2 d\tau + \sqrt{\left(\frac{2a\delta_1 b x_1}{b + a\delta_1}\right) - (b - a\delta_1)y_2}\ dW_{y_2}. \tag{16}$$

Proposition 2.3 in [1] can be used to prove that the reduced fast subsystem defined by the equation (15) - (16) is SISS with respect to the input $\theta$ and slow variable $x$, by considering an input $\theta_x = (\theta, x)'$, with $\theta$ defined in the previous section. Consider the Lyapunov function $V(y) = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2$. Then,

$$\begin{aligned}
LV &= \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} -\delta_1 y_1 \\ -(b + a\delta_1)y_2 \end{bmatrix} \\
&\quad + \frac{1}{2}Tr\left\{ \begin{bmatrix} \sqrt{2u + \delta_1 y_1} & \sqrt{\left(\frac{2a\delta_1 b x_1}{b+a\delta_1}\right) - (b - a\delta_1)y_2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2u + \delta_1 y_1} \\ \sqrt{\left(\frac{2a\delta_1 b x_1}{b+a\delta_1}\right) - (b - a\delta_1)y_2} \end{bmatrix} \right\}, \\
LV &= -\delta_1 y_1^2 - (b + a\delta_1)y_2^2 + \frac{1}{2}\left[2u + \delta_1 y_1 + \left(\frac{2a\delta_1 b x_1}{b + a\delta_1}\right) - (b - a\delta_1)y_2\right] \\
LV &= -\delta_1 y_1^2 - (b + a\delta_1)y_2^2 + u + \frac{\delta_1 y_1}{2} + \frac{a\delta_1 b x_1}{b + a\delta_1} - (b - a\delta_1)y_2 \\
LV &= -\frac{\delta_1}{2}y_1^2 - \frac{\delta_1}{2}(y_1^2 - y_1) - \frac{(b + a\delta_1)}{2}y_2^2 - \frac{(b + a\delta_1)}{2}\left(y_2^2 + \frac{(b - a\delta_1)}{(b + a\delta_1)}y_2\right) + u + \frac{a\delta_1 b x_1}{b + a\delta_1}, \\
LV &= -\frac{\delta_1}{2}y_1^2 - \frac{\delta_1}{2}\left(\left(y_1 - \frac{1}{2}\right)^2 - \frac{1}{4}\right) - \frac{(b + a\delta_1)}{2}y_2^2 \\
&\quad - \frac{(b + a\delta_1)}{2}\left(\left(y_2 + \frac{(b - a\delta_1)}{2(b + a\delta_1)}\right)^2 - \left(\frac{(b - a\delta_1)}{2(b + a\delta_1)}\right)^2\right) + u + \frac{a\delta_1 b x_1}{b + a\delta_1}, \\
LV &= -\frac{\delta_1}{2}y_1^2 - \frac{\delta_1}{2}\left(y_1 - \frac{1}{2}\right)^2 + \frac{\delta_1}{8} - \frac{(b + a\delta_1)}{2}y_2^2 \\
&\quad - \frac{(b + a\delta_1)}{2}\left(y_2 + \frac{(b - a\delta_1)}{2(b + a\delta_1)}\right)^2 - \left(\frac{(b - a\delta_1)}{2(b + a\delta_1)}\right)^2 + u + \frac{a\delta_1 b x_1}{b + a\delta_1}.
\end{aligned}$$

We can show that $LV \leq -\eta_2(y_1^2 + y_2^2)$ with $\eta_2 = (\min(\frac{\delta_1}{2}, \frac{(b+a\delta_1)}{2}))/2$ if

$$\left(\frac{\delta_1}{2} - \eta_2\right)y_1^2 + \left(\frac{(b + a\delta_1)}{2} - \eta_2\right)y_2^2 \geq \left|\frac{\delta_1}{8} + u + \frac{a\delta_1 b x_1}{b + a\delta_1}\right|,$$

i.e.,

$$\left(\frac{\delta_1}{2} - \eta_2\right)y_1^2 \geq \left|\frac{\delta_1}{8} + u\right| \quad \text{and} \quad \left(\frac{(b + a\delta_1)}{2} - \eta_2\right)y_2^2 \geq \left|\frac{a\delta_1 b x_1}{b + a\delta_1}\right|.$$

Then,

$$y_1^2 + y_2^2 \geq \frac{\left|\frac{\delta_1}{8} + u\right|}{\left(\frac{\delta_1}{2} - \eta_2\right)} + \frac{\left|\frac{a\delta_1 b x_1}{b+a\delta_1}\right|}{\left(\frac{(b+a\delta_1)}{2} - \eta_2\right)}.$$

This condition will be satisfied if

$$\sqrt{y_1^2 + y_2^2} \geq \max\left(\sqrt{\frac{2\left|\frac{\delta_1}{8} + u\right|}{\left(\frac{\delta_1}{2} - \eta_2\right)}}, \sqrt{\frac{2\left|\frac{a\delta_1 b x_1}{b+a\delta_1}\right|}{\left(\frac{(b+a\delta_1)}{2} - \eta_2\right)}}\right),$$

$$\sqrt{y_1^2 + y_2^2} \geq \max\left(\sqrt{2}\sqrt{\frac{\left|\frac{\delta_1}{8} + u\right|}{\left(\frac{\delta_1}{2} - \eta_2\right)}}, \sqrt{\frac{2\left|\frac{a\delta_1 b}{b+a\delta_1}\right||x_1|}{\left(\frac{(b+a\delta_1)}{2} - \eta_2\right)}}\right). \quad (17)$$

Given the input $\theta = \left(\frac{\left|\left(\frac{E_1^2}{2D_1}\right)+\left(\frac{E_2^2}{2D_2}\right)\right|}{\left(\frac{D_1}{2}-\eta_1\right)} + \frac{|E_3|}{\left(\frac{D_2}{2}-\eta_1\right)}, \frac{\left|\frac{\delta_1}{8}+u\right|}{\left(\frac{\delta_1}{2}-\eta_2\right)}\right)'$, the state variable $x = (x_1, x_2)'$ and $\theta_x = (\theta, x)'$, then a sufficient condition for satisfying (17) is given by

$$\sqrt{y_1^2 + y_2^2} \geq \max\left(\sqrt{2}\sqrt{||\theta_x||}, \sqrt{\frac{2\left|\frac{a\delta_1 b}{b+a\delta_1}\right|||\theta_x||}{\left(\frac{(b+a\delta_1)}{2} - \eta_2\right)}}\right)$$

which can be written as

$$\sqrt{y_1^2 + y_2^2} \geq \rho(||\theta_x||)$$

where $\rho(||s||) = \max\left(\sqrt{2}\sqrt{||s||}, \sqrt{\frac{2\left|\frac{a\delta_1 b}{b+a\delta_1}\right|||s||}{\left(\frac{(b+a\delta_1)}{2}-\eta_2\right)}}\right)$.

Then, applying Proposition 2.3, the reduced fast subsystem is SISS with respect to the input $\theta$ and slow variable $x$, i.e, $\forall \nu > 0$, there exist a class $\mathcal{KL}$ function $\beta_y(\cdot,\cdot)$ and class $\mathcal{K}$ functions $\gamma_2^x(\cdot), \gamma_2(\cdot)$, such that

$$P\left\{||y(t)|| < \beta(||y_0||, t) + \gamma_2^x\left(\sup_{0\leq s\leq t} ||x(s)||\right) + \gamma_2\left(\sup_{0\leq s\leq t} ||\theta(s)||\right)\right\} \geq 1 - \nu, \ \ \forall t \geq 0, \forall y_0 \in \mathbb{R}^2 \setminus \{0\}.$$

Therefore, A3 is satisfied.

### Verification of A4

We can also show that there exist a class $K_\infty$ function $p_1$, a class $K$ function $\gamma_1^{\tilde{y}}$, a non-increasing function $b : \mathbb{R}_+ \times \mathbb{R}_+ \to (0, 1]$ and positive constants $\delta_x, \delta$ such that

$$2\gamma_1^{\tilde{y}} \circ (I + p_1) \circ \frac{\gamma_2^x}{b(\delta_x, \delta)}(s) \le s,$$

by defining the functions

$$\begin{aligned} \gamma_1^{\tilde{y}}(s) &= \frac{1}{2}s, \\ p_1(s) &= s, \\ \gamma_2^x(s) &= \frac{s}{4}, \\ b(\delta_x, \delta) &= 1. \end{aligned}$$

This satisfies A4.

### Conclusion

Applying Theorem 3.1 in [1], we can see that the bound on $||y(t)||$ decreases as $\epsilon \to 0$. Therefore, $z$ is better approximated by $h(x, \theta)$ as $\epsilon$ becomes smaller.

## References


[1] C. Tang and T. Basar.“Stochastic stability of singularly perturbed nonlinear systems.” Decision and Control, 2001. Proceedings of the 40th IEEE Conference on. Vol. 1. IEEE, 2001.

[2] H. K Khalil. Nonlinear systems. Vol. 3. Upper Saddle River: Prentice hall, 2002.