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Graph Concatenation for Quantum Codes

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Graphs are closely related to quantum error-correcting codes: every stabilizer code is locally equivalent to a graph code, and every codeword stabilized code can be described by a graph and a classical code. For the construction of good quantum codes of relatively large block length, concatenated quantum codes and their generalizations play an important role. We develop a systematic method for constructing concatenated quantum codes based on "graph concatenation", where graphs representing the inner and outer codes are concatenated via a simple graph operation called "generalized local complementation." Our method applies to both binary and non-binary concatenated quantum codes as well as their generalizations.

I. INTRODUCTION

The discovery of quantum error-correcting codes (QECCs) and the theory of fault-tolerant quantum computation (FTQC) have greatly improved the long-term prospects for quantum communication and computation technology. This general QECC-FTQC framework leads to a remarkable threshold theorem, which indicates that noise likely poses no fundamental barrier to the performance of large-scale quantum computations [1].

Stabilizer codes, a quantum analogue of classical additive codes, are the most important class of QECCs [2, 3]. These codes have dominated the study of QECC-FTQC for the past ten years because of their simple construction based on Abelian groups. The recently introduced codeword stabilized (CWS) quantum codes framework [4–6] provides a unified way of constructing a larger class of quantum codes, both stabilizer and nonadditive codes. Based on the CWS framework, many nonadditive codes which outperform stabilizer codes in terms of coding parameters, have been constructed.

Graphs are closely related to QECCs. It has been shown that every stabilizer code is local Clifford equivalent to a graph code [7, 8]. The basic ingredients of a graph code are a graph and a finite Abelian group from which the code can explicitly be obtained [9]. Every CWS code also has a canonical form, where it can be fully characterized by a graph $\mathcal{G}$ and a classical code $\mathcal{C}$ [4, 5]. So a CWS quantum code $\mathcal{Q}$ is usually denoted by $\mathcal{Q} = (\mathcal{G}, \mathcal{C})$. When the classical code $\mathcal{C}$ is linear, $\mathcal{Q}$ is a graph code; therefore, graph codes, and hence stabilizer codes, are special cases of CWS codes. FIG. 1 demonstrates the relationship between all quantum codes, CWS codes and graph (stabilizer) codes.

For the construction of good QECCs of relatively large block length and good asymptotical performance, concatenated quantum codes and their generalizations play an important role [1, 2, 10, 11]. Combined with the CWS framework, families of good quantum codes, both stabilizer and nonadditive, have been constructed [10, 11]. Concatenated quantum codes also play a central role in FTQC, and the proof of the threshold theorem [1, 12–15]. Given the intimate relations

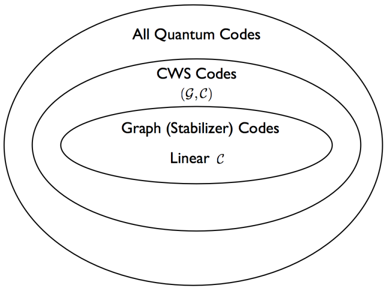

FIG. 1: Quantum codes

between graphs and quantum codes, a question that arises naturally is whether there is a graphical description for concatenated quantum codes and their generalizations. Moreover, if there were such a description, for the case where both the inner and outer codes are CWS codes, the next question is whether the corresponding graph captures the "quantum nature" of the concatenated code.

Previously, some related results on graph codes have been obtained. For instance, concatenation of graph codes may be described graphically by adding some auxiliary vertices. However, it remains unclear what the final graph after removing those auxiliary vertices will look like [16]. The known examples of generalized concatenated codes only provide graphical descriptions in the case where the outer code is of a special form [10, 11]. However, none of these previous works provides a general systematic graphical description for constructing concatenated quantum codes. Lack of such a description seems to indicate that using graphs to describe quantum codes was a very restricted approach. This issue will be addressed in the present work by developing a systematic method for constructing concatenated quantum codes based on a graph operation called "graph concatenation."

To be more precise, we construct the concatenated quantum code

$$\mathcal{Q}_c = \mathcal{Q}_{\text{in}} \sqsubset \mathcal{Q}_{\text{out}}, \tag{1}$$

where the inner code $\mathcal{Q}_{\text{in}} = (\mathcal{G}_{\text{in}}, \mathcal{C}_{\text{in}})$ and the outer code $\mathcal{Q}_{\text{out}} = (\mathcal{G}_{\text{out}}, \mathcal{C}_{\text{out}})$ are both CWS codes. We require $\mathcal{C}_{\text{in}}$ to be linear, but $\mathcal{C}_{\text{out}}$ can be either linear or nonlinear. Since $\mathcal{C}_{\text{in}}$ is linear, $\mathcal{Q}_{\text{in}}$ is a graph (stabilizer) code. We can then denote the parameters of $\mathcal{Q}_{\text{in}}$ by $[[n, k, d]]_p$. For simplicity, throughout the paper we assume that p is a prime number. When $\mathcal{Q}_{\text{in}}$ encodes k qupits, the corresponding outer code $\mathcal{Q}_{\text{out}}$ of length n' must be a subspace of the Hilbert space $\mathcal{H}_{p^k}^{\otimes n'}$, i.e., we can denote the parameters of $\mathcal{Q}_{\text{out}}$ by $((n', K', d'))_{p^k}$. When $\mathcal{C}_{\text{out}}$ is linear, then $\mathcal{Q}_{\text{out}}$ is a graph (stabilizer) code that can also be denoted by $[[n', k', d']]_{p^k}$, where $K' = p^{kk'}$.

We now state our main result.

Main Result: *The concatenated quantum code $\mathcal{Q}_c$ can also be described as a CWS code, i.e.,*

$$\mathcal{Q}_c = (\mathcal{G}_c, \mathcal{C}_c), \tag{2}$$

and $\mathcal{Q}_c$ can be constructed via the following way:

$$\begin{aligned} \mathcal{Q}_c &= \mathcal{Q}_{in} \sqsubset \mathcal{Q}_{out} \\ &= (\mathcal{G}_{in}, \mathcal{C}_{in}) \sqsubset (\mathcal{G}_{out}, \mathcal{C}_{out}) \\ &= (\mathcal{G}_{in} \sqsubset \mathcal{G}_{out}, \mathcal{C}_{in} \sqsubset \mathcal{C}_{out}), \end{aligned} \tag{3}$$

where $\mathcal{C}_c = \mathcal{C}_{in} \sqsubset \mathcal{C}_{out}$ is the classical concatenated code with the inner code $\mathcal{C}_{in}$ and the outer code $\mathcal{C}_{out}$, and the graph concatenation $\mathcal{G}_{in} \sqsubset \mathcal{G}_{out}$ in Eq. (3) gives the graph $\mathcal{G}_c$. And, we show that $\mathcal{G}_c$

can be obtained by concatenating $\mathcal{G}_{in}$ and $\mathcal{G}_{out}$ via a simple graph operation called "generalized local complementation."

The main advantage of constructing concatenated quantum codes via Eq. (3) is that the "quantum part" of this construction is fully characterized by the graph concatenation $\mathcal{G}_{\text{in}} \sqsubset \mathcal{G}_{\text{out}}$. Providing the rule for performing this graph concatenation, the problem of constructing concatenated quantum codes becomes purely classical, i.e., constructing the classical concatenated code $\mathcal{C}_{\text{in}} \sqsubset \mathcal{C}_{\text{out}}$. Despite the restriction that $\mathcal{C}_{\text{in}}$ must be linear, our method of graph concatenation can be applied to very general situations: both binary and non-binary concatenated quantum codes, and their generalizations.

This paper is organized as follows. In Sec. II, we give a simple example which informally demonstrates the rule of graph concatenation via generalized local complementation. In Sec. III, we review definitions of graph states, CWS codes, and graph codes. In Sec. IV, for a simple case that the inner code encodes only a single qupit (i.e., $k = 1$) and the outer code is also a graph code, we provide a description of graph concatenation based on the algebraic structure of stabilizers. We prove our main result in Sec. V. In Sec. VI, we discuss the application of our main result to the situation of the generalized concatenated quantum codes. A final discussion and conclusion is given in Sec. VII.

II. A SIMPLE EXAMPLE AND THE RULE

In this section we give a simple example to demonstrate the idea of our main result given by Eq. (3). We first recall how to describe a CWS code by a graph and a classical code; then we demonstrate how to represent a concatenated quantum code as an encoding graph $\mathcal{G}_c^{\{enc\}}$ with some auxiliary vertices. Finally, we show how to obtain the graph $\mathcal{G}_c$ of the concatenated code $\mathcal{Q}_c = (\mathcal{G}_c, \mathcal{C}_c)$, via "generalized local complementation" and removal of the auxiliary vertices.

A. The graph and the encoding circuit of a CWS code

Let us start by taking the outer code to be a simple $n = 3$ binary CWS quantum code $\mathcal{Q}_{\text{out}} = (\mathcal{G}_{\text{out}}, \mathcal{C}_{\text{out}})$, where $\mathcal{G}_{\text{out}}$ is a triangle given in FIG. 2A. $\mathcal{G}_{\text{out}}$ defines a unique quantum stabilizer state, which we denote $|\psi\rangle_{\mathcal{G}_{\text{out}}}$. $\mathcal{C}_{\text{out}}$ is a classical binary code of length 3, and can be either linear or nonlinear. A basis of the CWS code $\mathcal{Q}_{\text{out}}$ can then be chosen as $\{Z^{\mathbf{c}_{\text{out}}}|\psi\rangle_{\mathcal{G}_{\text{out}}}\}$, for all the codewords $\mathbf{c}_{\text{out}} \in \mathcal{C}_{\text{out}}$ [4].

If $\mathcal{C}_{\text{out}}$ is linear, then $\mathcal{Q}_{\text{out}}$ is a graph (stabilizer) code. The name "graph code" is chosen due to the fact that there is a graphical way to represent the code $\mathcal{Q}_{\text{out}}$, which gives both the information of $\mathcal{G}_{\text{out}}$ and $\mathcal{C}_{\text{out}}$ [9].

To show how to represent $\mathcal{C}_{\text{out}}$ graphically, let us first recall the encoding circuit of $\mathcal{Q}_{\text{out}} = (\mathcal{G}_{\text{out}}, \mathcal{C}_{\text{out}})$. We use the standard quantum circuit notations, for instance as those given in [1]. For a CWS code, in general the encoding can be done by first performing a classical encoder $C_{\text{out}}^{\{enc\}}$ which encodes the classical code $\mathcal{C}_{\text{out}}$ and then a graph encoder $G_{\text{out}}^{\{enc\}}$ which encodes the graph state corresponding to the graph $\mathcal{G}_{\text{out}}$ [4] as shown in the top left circuit of FIG. 3. Here q_1, q_2, q_3 denote qubits $1, 2, 3$ in FIG. 2A, respectively.

Let us now take $\mathcal{C}_{\text{out}} = \{000, 111\}$ which is linear and gives a $[[3, 1, 1]]$ stabilizer code. In this case, the classical encoder $C_{\text{out}}^{\{enc\}}$ which encodes

$$0 \rightarrow 000,\ 1 \rightarrow 111 \tag{4}$$

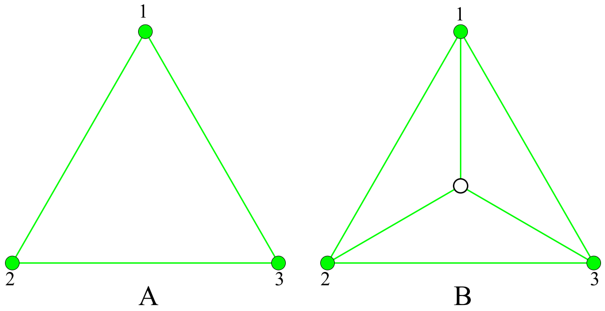

FIG. 2: A $[[3,1,1]]$ graph code.

can be implemented by adding an input qubit q_0 and performing controlled-NOT gates with control qubit q_0 and target qubits q_1, q_2, q_3, followed by measuring q_0 in the Pauli X basis (which can be done by applying a Hadamard gate on q_0 and then measuring in the Pauli Z basis) as shown in the top right circuit of FIG. 3. Throughout the paper we always assume that we get the desired measurement outcome (if not, we just need to perform some local Pauli operations according to the actual measurement outcome).

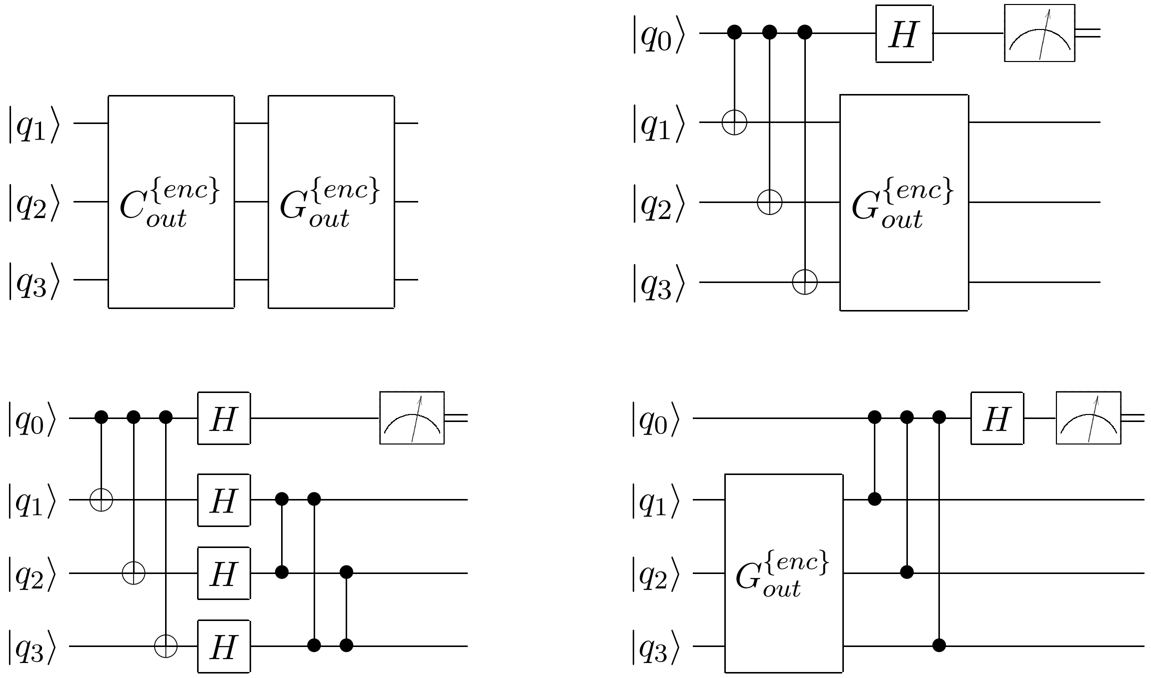

FIG. 3: Encoding circuit for the $[[3,1,1]]$ outer code.

The graph encoder $G_{\text{out}}^{\{enc\}}$ consists of three Hadamard gates on q_1, q_2, q_3 and three controlled-Z gates between them (controlled-Z gates are applied whenever the corresponding vertices are adjacent in graph FIG. 2A), as shown in the bottom left circuit of FIG. 3. Now it is clear that we can "move" the classical encoder $C_{\text{out}}^{\{enc\}}$ to the right of the graph encoder $G_{\text{out}}^{\{enc\}}$ by replacing each controlled-NOT by a controlled-Z, as shown in the bottom right circuit of FIG. 3.

In the following we use the convention to modify the encoding circuit by applying a Hadamard gate on the auxiliary qubit q_0 before applying the classical encoder as shown by the left circuit of

FIG. 5. This modification can be viewed as “a basis change” of the input qubit q_0, i.e., what the “classical encoder” $C_{\text{out}}^{\{enc\}}$ does is then

$$+ \to 000, \ - \to 111, \tag{5}$$

where ± are the labels of the quantum states

$$|\pm\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle \pm |1\rangle \right). \tag{6}$$

This change of basis yields a non-classical encoding circuit, yet we know that it does not make any difference for the quantum code because by this new encoding circuit we obtain the same code space as before. We adopt this convention throughout the paper: for any CWS code $\mathcal{Q} = (\mathcal{G}, \mathcal{C})$, we always assume that the “classical encoder” $C^{\{enc\}}$ maps “classical strings” in the $\{|+\rangle, |-\rangle\}$ basis to “classical codewords” in the $\{|0\rangle, |1\rangle\}$ basis. We will see later that this convention naturally leads to a simple rule for graph concatenation.

Moreover, for any $[[n, k, d]]$ CWS code $\mathcal{Q} = (\mathcal{G}, \mathcal{C})$ with linear $\mathcal{C}$ (i.e., $\mathcal{Q}$ is a graph code), the encoding of $\mathcal{Q}$ can be applied by first performing the graph encoder $G^{\{enc\}}$, and then the classical encoder $C^{\{enc\}}$ as follows: use k input qubits; apply Hadamard on each of the k qubits; replace each controlled-NOT gate performed in the original classical encoder $C^{\{enc\}}$ with a controlled-Z gate; finally measure each of the k auxiliary qubits in the Pauli X basis.

This encoding circuit can be represented graphically: add the k input qubits as k new vertices to the graph $\mathcal{G}$; whenever a controlled-Z is applied in the encoding circuit between an input vertex v and a vertex v' of $\mathcal{G}$, add an edge between them [9]. The corresponding graph representing the graph code $\mathcal{Q} = (\mathcal{G}, \mathcal{C})$ is denoted by $\mathcal{G}^{\mathcal{C}}$.

Therefore, for the outer code $\mathcal{Q}_{\text{out}} = (\mathcal{G}_{\text{out}}, \mathcal{C}_{\text{out}})$ with graph $\mathcal{G}_{\text{out}}$ given in FIG. 2A, where $\mathcal{C}_{\text{out}} = \{000, 111\}$ is linear, we can insert the input qubit q_0 as a new vertex (denoted by 0) to the graph $\mathcal{G}$ (the middle white vertex in FIG. 2B). We then add the edges between 0 and $1, 2, 3$ according to the encoding circuit given by the bottom right circuit in FIG. 3 (see FIG. 2B). This graph is then denoted by $\mathcal{G}_{\text{out}}^{\mathcal{C}_{\text{out}}}$. There are two types of vertices in $\mathcal{G}_{\text{out}}^{\mathcal{C}_{\text{out}}}$: the input vertices (the middle white vertex); and the output vertices (vertices $1, 2, 3$).

B. The encoding graph of a concatenated quantum code

Now we consider the inner code $\mathcal{Q}_{\text{in}} = (\mathcal{G}_{\text{in}}, \mathcal{C}_{\text{in}})$. Notice that due to our restriction for Eq. (3), $\mathcal{C}_{\text{in}}$ must be linear. So $\mathcal{Q}_{\text{in}}$ is a graph code and has a graph representation $\mathcal{G}_{\text{in}}^{\mathcal{C}_{\text{in}}}$. For simplicity we take $\mathcal{Q}_{\text{in}}$ be a $[[2, 1, 1]]$ stabilizer code, which is represented by the graph of FIG. 4A on the vertices $1, 4, 5$. (The subgraph of the vertices $4, 5$ represents $\mathcal{G}_{\text{in}}$, and 1 is the input qubit describing the classical encoder of $\mathcal{C}_{\text{in}}$; hence, $\mathcal{C}_{\text{in}} = \{00, 11\}$.)

To construct the concatenated code $\mathcal{Q}_c = \mathcal{Q}_{\text{in}} \sqsubset \mathcal{Q}_{\text{out}}$, since the outer code has length $n' = 3$, we must take three copies of $\mathcal{G}_{\text{in}}$, to encode qubits $1, 2, 3$ as shown in FIG. 4A. The graphical representation of the concatenation procedure is shown in FIG. 4B. Here, in the outer code, the middle white vertex is encoded into vertices $1, 2, 3$. Then each of these vertices is encoded using the inner code: vertex 1 into vertices $4, 5$; vertex 2 into vertices $6, 7$; and vertex 3 into vertices $8, 9$.

We call this graphical representation of the concatenated code $\mathcal{Q}_c$ with a linear $\mathcal{C}_{\text{out}}$ the encoding graph of $\mathcal{Q}_c$ and denote it by $\mathcal{G}_{\mathcal{Q}_c}^{\mathcal{C}_{\text{out}}\{enc\}}$. We have three types of vertices in $\mathcal{G}_{\mathcal{Q}_c}^{\mathcal{C}_{\text{out}}\{enc\}}$: the input vertices (the middle white vertex in our example); auxiliary vertices which are in the subgraph $\mathcal{G}_{\text{out}}$ (vertices $1, 2, 3$); and output vertices which are in the copies of $\mathcal{G}_{\text{in}}$ (vertices $4, 5, 6, 7, 8, 9$). In general, if $\mathcal{C}_{\text{out}}$ is nonlinear, similarly we can have an encoding graph of $\mathcal{Q}_c$ and denote it by $\mathcal{G}_{\mathcal{Q}_c}^{\{enc\}}$,

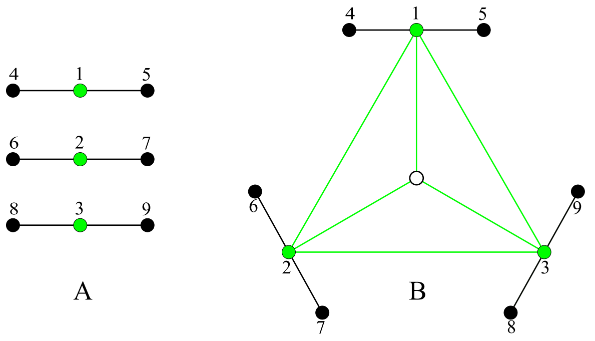

FIG. 4: Concatenated graph code.

which in our example is the subgraph of FIG. 4B without the middle white vertex. Therefore, we only have two types of vertices in $\mathcal{G}_{\mathcal{Q}_c}^{\{enc\}}$: the auxiliary vertices (vertices $1, 2, 3$); and the output vertices(vertices $4, 5, 6, 7, 8, 9$).

The encoding circuit of the concatenated code $\mathcal{Q}_c$ is given by the right circuit in FIG. 5, where $G_{\text{in}}^{\{enc\}}$ denotes the graph encoder for the graph of the inner code $\mathcal{G}_{\text{in}}$. To obtain this encoding circuit, we should recall our convention of adding a Hadamard gate before performing the classical encoder.

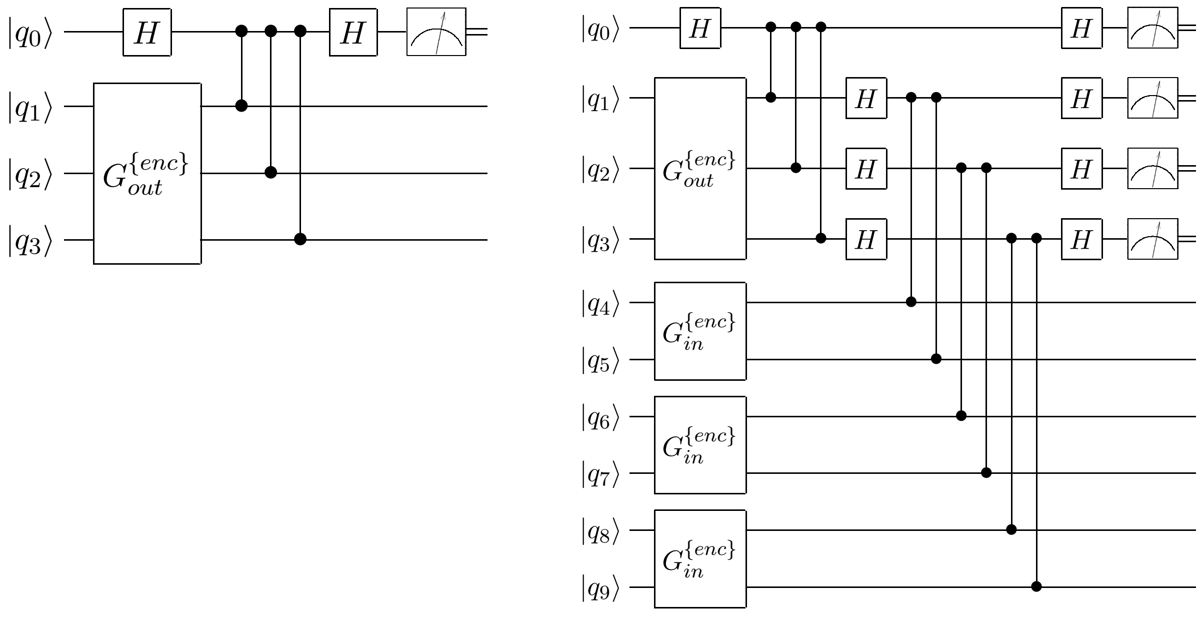

FIG. 5: Encoding circuit for the concatenated code with linear outer code.

In general, if $\mathcal{C}_{\text{out}}$ is nonlinear, the encoding circuit of the concatenated code $\mathcal{Q}_c$ is given by the right circuit in FIG. 6, where $G_{\text{in}}^{\{enc\}}$ denotes the graph encoder for the graph of the inner code $\mathcal{G}_{\text{in}}$. Again, note that we add a Hadamard gate before performing the classical encoder. Also, we need to keep in mind that the "classical encoder" $C_{\text{out}}^{\{enc\}}$ maps "classical strings" in the $\{|+\rangle, |-\rangle\}$ basis to "classical codewords" in the $\{|0\rangle, |1\rangle\}$ basis.

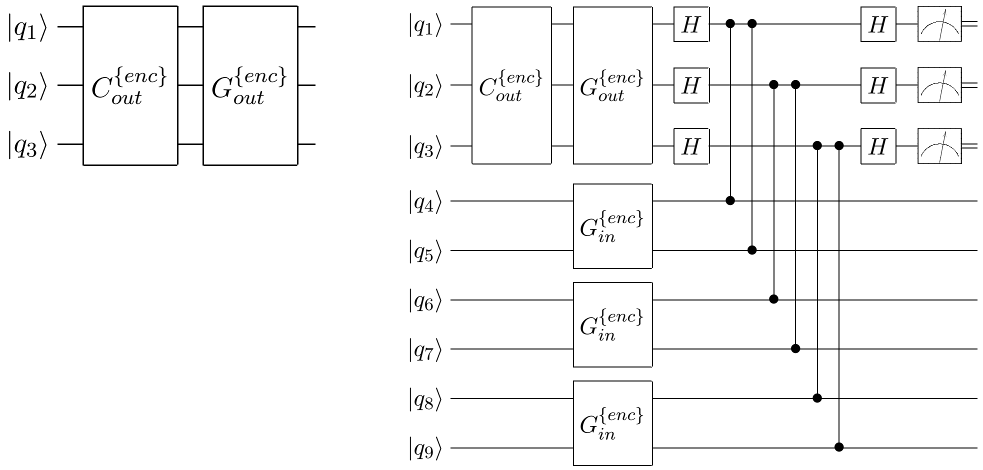

FIG. 6: Encoding circuit for the concatenated code with a general outer code.

C. The rule of the generalized local complementation for graph concatenation

As shown in Sec. II B, given a concatenated quantum code $\mathcal{Q}_c = \mathcal{Q}_{\rm in} \sqsubset \mathcal{Q}_{\rm out}$ with a CWS outer code $\mathcal{Q}_{\rm out} = (\mathcal{G}_{\rm out}, \mathcal{C}_{\rm out})$ and a graph inner code $\mathcal{Q}_{\rm out} = (\mathcal{G}_{\rm out}, \mathcal{C}_{\rm out})$, it is easy to get the encoding graph $\mathcal{G}_{\mathcal{Q}_c}^{\{enc\}}$. We claim (and will show later in Sec. V) that the concatenated quantum code $\mathcal{Q}_c = \mathcal{Q}_{\rm in} \sqsubset \mathcal{Q}_{\rm out}$ can also be described as a CWS code $\mathcal{Q}_c = (\mathcal{G}_c, \mathcal{C}_c)$. Therefore, the real description that we want for the concatenated code $\mathcal{Q}_c$ is a graph $\mathcal{G}_c$ and a classical code $\mathcal{C}_c$ such that $\mathcal{Q}_c = (\mathcal{G}_c, \mathcal{C}_c)$. Also, we want the classical code be given by the "classical concatenation" of the classical code of the inner and outer code, i.e., $\mathcal{C}_c = \mathcal{C}_{\rm in} \sqsubset \mathcal{C}_{\rm out}$, so that the quantum part can be fully taken care of by the graph concatenation $\mathcal{G}_c = \mathcal{G}_{\rm in} \sqsubset \mathcal{G}_{\rm out}$. Furthermore, this graph concatenation should be given by some simple graph operations on the encoding graph $\mathcal{G}_{\mathcal{Q}_c}^{\{enc\}}$, i.e., we want a general rule which gives

$$\mathcal{G}_{\mathcal{Q}_c}^{\{enc\}} \to \mathcal{G}_c, \tag{7}$$

or

$$\mathcal{G}_{\mathcal{Q}_c}^{\mathcal{C}_{\rm out}\{enc\}} \to \mathcal{G}_c^{\mathcal{C}_c}, \tag{8}$$

if the outer code is also a graph code. As discussed in the main result, such a general rule does exist and we call it "generalized local complementation."

In this section, we demonstrate the rule of generalized local complementation for graph concatenation by a simple example, starting from the encoding graph given by FIG. 4B. Keep in mind that we want

$$\mathcal{C}_c = \mathcal{C}_{\rm in} \sqsubset \mathcal{C}_{\rm out} = \{00, 11\} \sqsubset \{000, 111\} = \{00\,00\,00, 11\,11\,11\}. \tag{9}$$

Remark 1 *To obtain the graph $\mathcal{G}_c$ (or $\mathcal{G}_c^{\mathcal{C}_c}$) from FIG. 4B, a naive way is to calculate the stabilizer state $|\psi\rangle$ of the output vertices after performing Pauli X measurements on all the input and the auxiliary vertices in the encoding circuit (given by the right graph of FIG. 6 or the right graph of FIG. 5), and then to represent it as a graph state (or a graph code). Notice that in general it might not be possible to represent the very code as a graph $\mathcal{G}_c$ (or $\mathcal{G}_c^{\mathcal{C}_c}$) does not necessarily exist. Indeed, any stabilizer state is local Clifford equivalent to a graph state (which is not necessarily unique), any any stabilizer code is local Clifford equivalent to a graph code, and any CWS code is* *local Clifford equivalent to a standard form given by a graph and a classical code. However, for a general CWS code, those local Clifford operations transform both the graph and the classical code [5]. Therefore, it is not clear that such a graph $\mathcal{G}_c$ exists such that the classical code is obtained by concatenation, i.e.,* $\mathcal{C}_c = \mathcal{C}_{in} \sqsubset \mathcal{C}_{out}$.

Now we specify the rule of generalized local complementation: given a graph $\mathcal{G}$, for any vertex i, denote the set of its adjacent vertices by $N(i)$. Also let S be a subset of vertices disjoint from $N(i)$. A generalized local complementation on i with respect to S is to replace the bipartite subgraph induced on $N(i) \cup S$ with its complement.

For an example, the generalized local complementation of the graph shown by FIG. 7A on vertex 1 with respect to $S = \{6, 7\}$ results in the graph shown by FIG. 7B. Here $N(1) = \{2, 3, 4\}$. The generalized local complementation replaces the bipartite subgraph of vertices $\{2, 3, 4, 6, 7\}$ and edges $\{(3, 6), (4, 7)\}$ with its complement (i.e. another bipartite subgraph of vertices $\{2, 3, 4, 6, 7\}$ and edges $\{(2, 6), (4, 6), (2, 7), (3, 7)\}$).

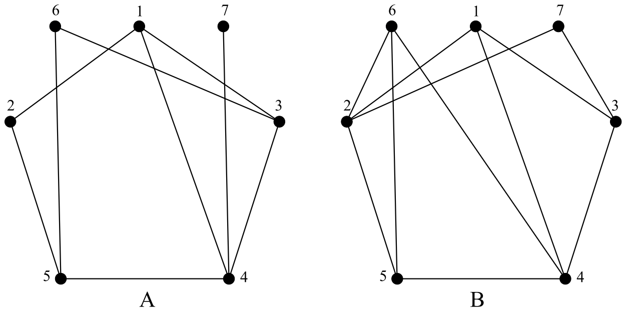

FIG. 7: Generalized local complementation

Now we are ready to specify the rule of graph concatenation in terms of generalized local complementation. (Recall that our goal is to obtain $\mathcal{G}_c$ from $\mathcal{G}_{\mathcal{Q}_c}^{\{enc\}}$, or $\mathcal{G}_c^{\mathcal{C}_c}$ from $\mathcal{G}_{\mathcal{Q}_c}^{\mathcal{C}_{\text{out}}\{enc\}}$.)

Procedure 1 *(Graph Concatenation via Generalized Local Complementation)*

1. *Given the graph* $\mathcal{G}_{\mathcal{Q}_c}^{\{enc\}}$ *(or* $\mathcal{G}_{\mathcal{Q}_c}^{\mathcal{C}_{out}\{enc\}}$*), for each auxiliary vertex* i*, define* S_i *to be the set of all output vertices which are adjacent to* i*.*

2. *For each auxiliary vertex* i*, delete all the edges which connect* i *to vertices in* S_i*.*

3. *For each auxiliary vertex* i*, perform generalized local complementation on* i *with respect to* S_i*. Note that the order in which we apply those generalized local complementations does not matter since the whole procedure on all auxiliary vertices finally gives the same graph.*

4. *Remove all the auxiliary vertices.*

To demonstrate the above rules, let us start from the encoding graph given in FIG. 4B for the concatenated quantum code with the outer code $[[3, 1, 1]]$ given in FIG. 2B and the inner code $[[2, 1, 1]]$ given in FIG. 4A. For convenience we redraw FIG. 4B in FIG. 8A. Now from FIG. 8A we get $S_1 = \{4, 5\}$, $S_2 = \{6, 7\}$, and $S_3 = \{8, 9\}$. Deleting all the edges which connect each auxiliary vertex i and vertices in S_i (for $i = 1, 2, 3$) results in FIG. 8B. Performing local complementation on auxiliary vertex 1 with respect to $S_1 = \{4, 5\}$ leads to FIG. 8C, where we use dashed blue lines to show the edges that we add between output and auxiliary vertices, and solid black lines to show

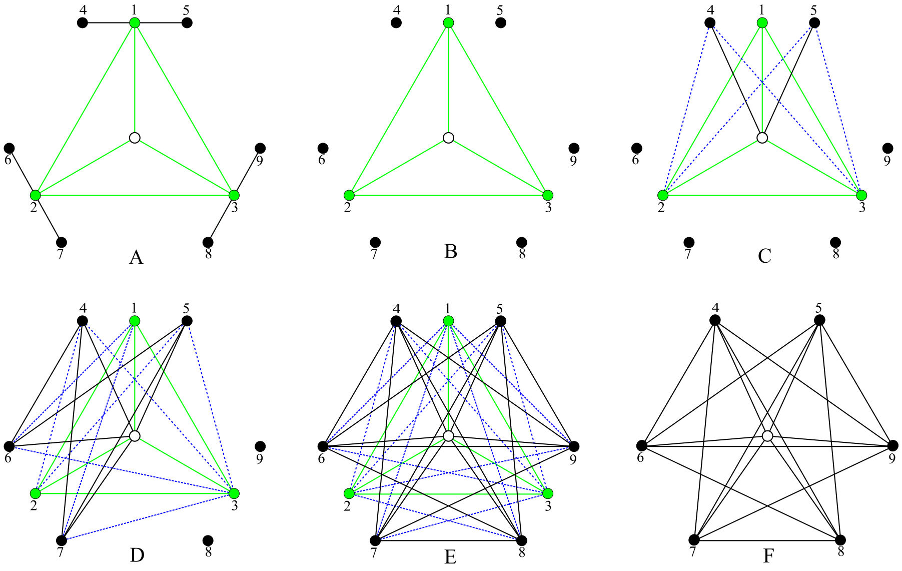

FIG. 8: Generalized local complementation for graph concatenation

other edges. Performing local complementation in FIG. 8C on auxiliary vertex 2 with respect to $S_2 = \{6,7\}$ leads to FIG. 8D, and performing local complementation in FIG. 8D on auxiliary vertex 3 with respect to $S_3 = \{8,9\}$ leads to FIG. 8E. Removing all the auxiliary vertices in FIG. 8E gives FIG. 8F, which is the graph $\mathcal{G}_c^{\mathcal{C}_c}$.

From FIG. 8F one can easily see that the rule for concatenation of the classical codes given in Eq. (9) holds. In general, the outer classical code $\mathcal{C}_{\text{out}}$ is nonlinear, so we do not have the input vertices in the encoding graph of the concatenated code. However, we can still go through the whole procedure of the generalized local complementations on auxiliary vertices to obtain the graph $\mathcal{G}_c$. In our example this procedure is demonstrated by subgraphs of FIG. 8A through FIG. 8F without the middle white vertex.

III. GRAPH STATES, CWS CODES, AND GRAPH CODES

In this section we review the stabilizer formalism to fix our notation especially in the non-binary case; then we define CWS codes, graph codes, and finally describe their encoding circuits.

A. The generalized Pauli group

Let p be a prime number and $\mathbb{F}_p$ be the field of p elements. A qupit is a p-level quantum system whose Hilbert space is represented by the orthonormal basis $\{|r\rangle : r \in \mathbb{F}_p\} = \{|0\rangle, |1\rangle, \ldots, |p-1\rangle\}$. Let $\omega = e^{2\pi i/p}$ be a p-th root of unity. The (generalized) Pauli matrices X and Z are defined as follows.

$$X|r\rangle = |r + 1 \bmod p\rangle, \tag{10}$$

$$Z|r\rangle = \omega^r|r\rangle. \tag{11}$$

It is clear that $X^p = Z^p = I$, and then we can consider the operators X^a and Z^b where $a, b \in \mathbb{F}_p$. We have $Z^b X^a = \omega^{ab} X^a Z^b$; therefore, $X^a Z^b$ and $X^{a'} Z^{b'}$ commute iff $ab' - ba' = 0$ (see e.g. [20] for more details).

The group generated by the (generalized) Pauli matrices X and Z is $\{\omega^c X^a Z^b : \ a, b, c \in \mathbb{F}_p\}$ and is called the (generalized) Pauli group. Notice that, in the binary case ($p = 2$) the Pauli group is generated by Pauli matrices σ_x and σ_z together with iI ($i = \sqrt{-1}$).

Let n be an arbitrary positive integer. For vectors $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ in $\mathbb{F}_p^n$ define

$$X^{\mathbf{a}} = X^{a_1} \otimes \cdots \otimes X^{a_n} \tag{12}$$

and

$$Z^{\mathbf{b}} = Z^{b_1} \otimes \cdots \otimes Z^{b_n}. \tag{13}$$

Again, two Pauli matrices $X^{\mathbf{a}} Z^{\mathbf{b}}$ and $X^{\mathbf{a}'} Z^{\mathbf{b}'}$ commute if and only if $\mathbf{a}\mathbf{b}' - \mathbf{a}'\mathbf{b} = 0$, where $\mathbf{c}\mathbf{d} = c_1 d_1 + \cdots + c_n d_n$ is the usual inner product on $\mathbb{F}_p^n$.

For simplicity, a Pauli operator $X^{\mathbf{a}} Z^{\mathbf{b}}$ is denoted by the vector $(\mathbf{a}\,|\,\mathbf{b})$ of length $2n$. Thus two Pauli operators $X^{\mathbf{a}} Z^{\mathbf{b}}$ and $X^{\mathbf{a}'} Z^{\mathbf{b}'}$ commute iff their corresponding vectors are orthogonal with respect to the "symplectic inner product" defined by

$$(\mathbf{a}\,|\,\mathbf{b}) * (\mathbf{a}'\,|\,\mathbf{b}') = \mathbf{a}\mathbf{b}' - \mathbf{a}'\mathbf{b}. \tag{14}$$

B. Stabilizer states

It is easy to see that for a Pauli matrix $g = \omega^c X^{\mathbf{a}} Z^{\mathbf{b}}$, $g^p = I$. (In the case $p = 2$, the statement might only be true after replacing G by ig in order to get $g^2 = I$. Having this in mind, there is no true difference between the binary and non-binary case in the rest of the paper.) Therefore, the eigenvalues of g are all p-th root of unity. In fact, if $(\mathbf{a}\,|\,\mathbf{b})$ is non-zero, then for any i, ω^i is an eigenvalue of g, and the multiplicity of each of these p eigenvalues is equal to p^{n-1} [20].

Now suppose $g_1 = \omega^{c_1} X^{\mathbf{a^1}} Z^{\mathbf{b^1}}, \dots, g_k = \omega^{c_k} X^{\mathbf{a^k}} Z^{\mathbf{b^k}}$ are k Pauli matrices which pairwise commute and such that the subgroup generated by any $k - 1$ of them does not contain the other one. Additionally, we require that the group generated by $g_1, \dots, g_k$ does not contain a non-trivial multiple of identity. Since $g_1, \dots, g_k$ commute, they can be diagonalized simultaneously.

Lemma 1 *The common eigenspace of all g_i's with eigenvalue* 1 *is a p^{n-k}-dimensional subspace.*

This lemma is a well-known fact in the binary case [1], and a proof for the non-binary case can be found in [20].

Representing the operators $g_1, \dots, g_k$ by the vectors of length $2n$, we obtain the $k \times (2n)$ matrix

$$M = \left(\begin{array}{c|c} \mathbf{a^1} & \mathbf{b^1} \\ \mathbf{a^2} & \mathbf{b^2} \\ \vdots & \vdots \\ \mathbf{a^k} & \mathbf{b^k} \end{array} \right), \tag{15}$$

of rank k (because g_i is not in the subgroup generated by the rest of g_j's). The rows of M are mutually orthogonal with respect to the symplectic inner product (see Eq. (14)).

If we consider n generators, or equivalently an $n \times (2n)$ full-rank self-orthogonal matrix M, Lemma 1 implies that the common eigenspace of all g_i's with eigenvalue 1 is a one-dimensional subspace. Hence there is a unique (up to a scaler) non-zero vector $|\psi\rangle$ such that $g_i|\psi\rangle = |\psi\rangle$. In fact, if we consider the group $\mathcal{S}$ generated by the g_i's, for any $h \in \mathcal{S}$ we have $h|\psi\rangle = |\psi\rangle$. The group $\mathcal{S}$, which is a maximal Abelian subgroup of the Pauli group modulo its center, is called a *stabilizer group*, and the state $|\psi\rangle$ is called a *stabilizer state.*

Notice that for a stabilizer state $|\psi\rangle$, its stabilizer group $\mathcal{S}$ is unique; however, $\{g_1, \dots, g_n\}$ is just some set of generators of $\mathcal{S}$. Suppose $\{h_1, \dots, h_n\}$ is another generating set of $\mathcal{S}$. Then for any i there is $u_{ij} \in \mathbb{F}_p$ such that $h_i = g_1^{u_{i1}} \cdots g_n^{u_{\rm in}}$. As a result, the vector corresponding to h_i is equal to $(u_{i1}, \dots, u_{\rm in})M$.

Lemma 2 *Any set of generators of the stabilizer group $\mathcal{S}$ with k generators can be represented by a matrix UM, where U is an invertible $k \times k$ matrix.*

C. Clifford group

The Clifford group is the normalizer of the Pauli group. In the binary case, it is well-known that the Clifford group is generated by the Hadamard gate, the phase gate, and the controlled-NOT gate [17]. A characterization of the Clifford group in the non-binary case can be found in [18].

Clifford operators are important in the stabilizer formalism because they send any stabilizer state to a stabilizer state. Suppose $|\psi\rangle$ is a stabilizer state with the stabilizer group $\mathcal{S}$. Also, let L be a Clifford operator. For any $g \in \mathcal{S}$ we have $(LgL^\dagger)L|\psi\rangle = L|\psi\rangle$. On the other hand, $LgL^\dagger$ is in $L\mathcal{S}L^\dagger$ which is a subgroup of the Pauli group since L is a Clifford operator. In fact, $L\mathcal{S}L^\dagger$ is a maximal Abelian subgroup of the Pauli group whose corresponding stabilizer state is $L|\psi\rangle$. Therefore, Clifford operators send stabilizer states to stabilizer states.

Based on the characterization of the Clifford group [17, 18], for any two stabilizer states $|\psi\rangle$ and $|\psi'\rangle$ there is a Clifford operator L such that $L|\psi\rangle = |\psi'\rangle$. However, it does not mean that all the stabilizer states are the same in the point of view of quantum coding theory since the operator L may completely change the entanglement of a state. But if we assume that $L = L_1 \otimes \cdots \otimes L_n$ is a local Clifford operator (L is the tensor product of n one-qupit Clifford operators), then the entanglement of $|\psi\rangle$ and $L|\psi\rangle$ are the same. Based on this idea, two stabilizer states are called "local Clifford equivalent" if they are equivalent under the action of the local Clifford group.

For the encoding circuits, we need only two Clifford operators which we describe next. Define the vector

$$|\widehat{r}\rangle = \frac{1}{\sqrt{p}} \sum_{s=0}^{p-1} \omega^{-rs} |s\rangle, \tag{16}$$

for any $r \in \mathbb{F}_p$. $|\widehat{r}\rangle$ is an eigenvector of X, i.e., $X|\widehat{r}\rangle = \omega^r |\widehat{r}\rangle$, and $\{|\widehat{0}\rangle, \dots, |\widehat{p-1}\rangle\}$ is an orthonormal basis. Therefore, the operator

$$H|\widehat{r}\rangle = |r\rangle, \tag{17}$$

which is called the (generalized) Hadamard gate, is unitary. By definition $HXH^\dagger = Z$. Also, it is easy to see that $HZH^\dagger = X^\dagger$. Hence, H is in the Clifford group. Using the above relations the proof of the following lemma is easy.

Lemma 3 *Suppose $|\psi\rangle$ is a stabilizer state whose stabilizer group is represented by the $n \times (2n)$ matrix M. Thus, the matrix representation of the stabilizer state $H_i|\psi\rangle$ (Hadamard gate is applied on the i-th qupit) is obtained from M by exchanging the i-th and $(n+i)$-th columns and then multiplying the i-th column by -1.*

The next operator is a two-qupit gate which is called controlled-Z and is defined by

$$C_z|r\rangle|s\rangle = |r\rangle Z^r|s\rangle = Z^s|r\rangle|s\rangle = \omega^{rs}|r\rangle|s\rangle. \tag{18}$$

We have

$$\begin{aligned} C_z X \otimes I C_z^\dagger &= X \otimes Z, \\ C_z I \otimes X C_z^\dagger &= Z \otimes X, \\ C_z Z \otimes I C_z^\dagger &= Z \otimes I, \\ C_z I \otimes Z C_z^\dagger &= I \otimes Z, \end{aligned} \tag{19}$$

and thus by definition C_z is in the Clifford group.

Lemma 4 *Suppose $|\psi\rangle$ is a stabilizer state whose stabilizer group is represented by the $n \times (2n)$ matrix M. Thus, the matrix representation of the stabilizer state $C_z^{ij}|\psi\rangle$ (the controlled-Z gate is applied on the i-th and j-th qupits) is obtained from M by adding column i to column $n+j$, and column j to column $n+i$.*

D. Graph states

In the following we consider graphs whose edges are labeled by non-zero elements of $\mathbb{F}_p$. Considering the adjacency matrix of a graph $\mathcal{G}$, we can represent it by a symmetric matrix over $\mathbb{F}_p$ with zero diagonal. Suppose G is such a matrix of size $n \times n$. Then $M = (I_n \,|\, G)$ is a full-rank $n \times (2n)$ matrix, and all of its rows are mutually orthogonal with respect to the symplectic inner product; therefore, M represents a stabilizer group which corresponds to a stabilizer state. Such a stabilizer state is called a *graph state*, which we denote by $|\psi\rangle_{\mathcal{G}}$. It is well-known that any stabilizer state is local Clifford equivalent to a graph state [20], so to study the properties of stabilizer states it is sufficient to restrict ourselves to graph states.

Graph states can be generated easily using only Hadamard and controlled-Z gates,

Lemma 5 *The graph state corresponding to the graph with adjacency matrix $G = (g_{ij})$ on n vertices can be generated by the following circuit. Prepare n qupits in the state $|0\rangle$, apply $H^\dagger$ to every one of them, and then for any i, j apply $C_z^{g_{ij}}$ on qupits i and j.*

Proof: The initial state of the n qupits is $|0\rangle \cdots |0\rangle$, which is a stabilizer state with the stabilizer group $\{Z^{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_p^n\}$. This stabilizer group corresponds to the matrix $M_0 = (0_n \,|\, I_n)$. According to Lemma 3, after applying $H^\dagger$ gates the matrix M_0 will be changed to $M_1 = (I_n \,|\, 0)$. Also, by Lemma 4, applying $C_z^{g_{ij}}$ on qupits i and j corresponds to adding columns i and j multiplied by g_{ij} to columns $n+j$ and $n+i$, respectively. Since the first block of M_1 is identity, this operation is the same as to add g_{ij} to the entries ij and ji of the second block. Therefore, at the end we obtain the matrix $M_2 = (I_n \,|\, G)$. □

E. Measurement on graph states

Suppose we have a graph state $|\psi\rangle_{\mathcal{G}}$ which corresponds to the graph $\mathcal{G}$ with adjacency matrix G, and we measure its (say) last qupit in the standard basis and get $|0\rangle$. We claim that the state after the measurement (without the measured qubit) is also a graph state whose corresponding graph is obtained from $\mathcal{G}$ by removing the last vertex. To see this fact precisely notice that since $(I_n \,|\, G)$ represents the stabilizer group of $|\psi\rangle_{\mathcal{G}}$, for any i, we have $X^{\mathbf{e}_i} Z^{\mathbf{g}_i} |\psi\rangle_{\mathcal{G}} = |\psi\rangle_{\mathcal{G}}$, where all coordinates of $\mathbf{e}_i$ are 0 except the i-th one which is 1, and $\mathbf{g}_i$ is the i-th row of G. Let

$$|\psi\rangle_{\mathcal{G}} = \sum_{r=0}^{p-1} \alpha_r |\phi_r\rangle |r\rangle, \tag{20}$$

and for $1 \leq i \leq n-1$ let $\mathbf{g}'_i$ and $\mathbf{e}'_i$ be the vectors of length $n-1$ obtained from $\mathbf{g}_i$ and $\mathbf{e}'_i$, respectively, by deleting the last coordinate. Thus we have

$$|\psi\rangle_{\mathcal{G}} = X^{\mathbf{e}_i} Z^{\mathbf{g}_i} |\psi\rangle_{\mathcal{G}} = \sum_{r=0}^{p-1} \alpha_r Z^{g_{in}} X^{\mathbf{e}'_i} Z^{\mathbf{g}'_i} |\phi_r\rangle |r\rangle = \sum_{r=0}^{p-1} \alpha_r \left(\omega^{r g_{in}} X^{\mathbf{e}'_i} Z^{\mathbf{g}'_i} |\phi_r\rangle \right) |r\rangle. \tag{21}$$

As a result $X^{\mathbf{e}'_i} Z^{\mathbf{g}'_i} |\phi_0\rangle = |\phi_0\rangle$, which means that $|\phi_0\rangle$ is a stabilizer state with the stabilizer group generated by $X^{\mathbf{e}'_i} Z^{\mathbf{g}'_i}$, $1 \leq i \leq n-1$, and the matrix representation of these generators is $(I_{n-1} \,|\, G')$ where G' is the adjacency matrix of the graph obtained from $\mathcal{G}$ by removing its last vertex.

F. CWS codes and graph codes

A CWS code $((n, K, d))_p$ is described by a graph $\mathcal{G}$ with n vertices and edges labeled by $\mathbb{F}_p$, together with a classical code $\mathcal{C}$ which consists of K vectors in $\mathbb{F}_p^n$. Such a code is denoted by $\mathcal{Q} = (\mathcal{G}, \mathcal{C})$ [4–6].

If the classical code $\mathcal{C}$ is linear, then $\mathcal{Q}$ is a graph (stabilizer) code [4–6]. The parameters of such a graph code $\mathcal{Q} = (\mathcal{G}, \mathcal{C})$ are $[[n, k, d]]_p$, where the classical code $\mathcal{C}$ consists of $K = p^k$ vectors in $\mathbb{F}_p^n$ that are indexed by the elements of $\mathbb{F}_p^k$. This $[[n, k, d]]_p$ graph code encodes k qupits into n qupits in the following way. Suppose $|\psi\rangle_{\mathcal{G}}$ is the graph state corresponding to $\mathcal{G}$. To encode a state of the form $H^\dagger \otimes \cdots \otimes H^\dagger |r_1 \dots r_k\rangle$ we first find the classical codeword $\boldsymbol{\alpha} \in \mathcal{C}$ which is indexed by $r_1 \dots r_k$, and then encode $H^\dagger \otimes \cdots \otimes H^\dagger |r_1 \dots r_k\rangle$ into $Z^{\boldsymbol{\alpha}} |\psi\rangle_{\mathcal{G}}$ [21]. Since $\mathcal{C}$ is a linear code, it is a linear subspace of $\mathbb{F}_p^n$. We can then represent $\mathcal{C}$ by k basis vectors $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_k$. In this case, the state $H^\dagger \otimes \cdots \otimes H^\dagger |r_1 \dots r_k\rangle$ is encoded into $Z^{r_1 \boldsymbol{\alpha}_1 + \cdots + r_k \boldsymbol{\alpha}_k} |\psi\rangle_{\mathcal{G}}$.

The encoding circuit of a $[[n, k, d]]$ graph code is simple, as shown in the following procedure.

Procedure 2 *(Encoding circuit for a graph code)*

1. *First generate the graph state $|\psi\rangle_{\mathcal{G}}$ using the circuit described in Lemma 5.*

2. *For any $1 \leq i \leq k$ apply $H^\dagger$ on q_i, where $q_1, \dots, q_k$ are the qupits that we want to encode.*

3. *For any $1 \leq j \leq n$ apply $C_z^{\alpha_{ij}}$ on q_i and the j-th qupit of $|\psi\rangle_{\mathcal{G}}$, where α_{ij} is the j-th coordinate of $\boldsymbol{\alpha}_i$.*

4. *Apply H to $q_1, \dots, q_k$.*

5. *Measure $q_1, \dots, q_k$ in the computational basis.*

For example, the encoding circuit of the graph code with a triangle graph and the classical code $\{000, 111\}$ can be found in the left circuit of FIG. 5. (Notice that in the binary case $H^\dagger = H$.)

In general, for a graph code $\mathcal{Q} = (\mathcal{G}, \mathcal{C})$ the encoding circuit can be represented graphically, and the corresponding graph is denoted by $\mathcal{G}^{\mathcal{C}}$: consider the graph $\mathcal{G}$, for any $1 \leq i \leq k$ add a vertex (input vertices), attach it to the vertices of $\mathcal{G}$ (called the output vertices), and label the edge between this vertex and the j-th vertex of $\mathcal{G}$ by α_{ij}. For example, FIG. 2B gives the graph $\mathcal{G}^{\mathcal{C}}$, where $\mathcal{G}$ is a triangle and $\mathcal{C} = \{000, 111\}$.

Remark 2 *The encoding circuit corresponding to the graph code with graphical representation $\mathcal{G}^{\mathcal{C}}$ is related to the circuit that generates the graph state $|\psi\rangle_{\mathcal{G}^{\mathcal{C}}}$ corresponding to the graph $\mathcal{G}^{\mathcal{C}}$. To see this, notice that the steps 1,2, 3 in Procedure 2 indeed give such a graph encoder.*

To find the logical X and Z operators of an additive graph code we first describe the stabilizer group of the logical $|0\dots0\rangle_L$ state. Notice that

$$|0\dots0\rangle = \frac{1}{\sqrt{p^k}} \sum_{r_1,\dots,r_k} H^\dagger \otimes \cdots \otimes H^\dagger |r_1 \dots r_k\rangle, \tag{22}$$

and then

$$|0\dots0\rangle_L = \frac{1}{\sqrt{p^k}} \sum_{r_1,\dots,r_k} Z^{r_1\boldsymbol{\alpha}_1+\cdots+r_k\boldsymbol{\alpha}_k} |\psi\rangle_{\mathcal{G}}. \tag{23}$$

Therefore, all operators $Z^{\boldsymbol{\alpha}_i}$ are in the stabilizer group of $|0\dots0\rangle_L$, and the logical Z operators are described by the rows of the matrix

$$\left(\begin{array}{c|c} \mathbf{0} & \boldsymbol{\alpha}_1 \\ \vdots & \vdots \\ \mathbf{0} & \boldsymbol{\alpha}_k \end{array}\right). \tag{24}$$

Since the vectors $\boldsymbol{\alpha}_i$ are linearly independent, without loss of generality (by a change of basis for the classical code and reordering the qupits), we may assume that the first block of the second part of this matrix is I_k. So we assume that the matrix $(I_k \;\; A)$, where A is of size $k \times (n-k)$ describes a basis for $\mathcal{C}$, and the logical Z operators are

$$\left(\begin{array}{c|c} 0 & I_k \;\; A \end{array}\right). \tag{25}$$

Assume that

$$G = \begin{pmatrix} G_1 & B \\ B^T & G_2 \end{pmatrix}, \tag{26}$$

where G_1, G_2, and B are of size $k \times k$, $(n-k) \times (n-k)$, and $k \times (n-k)$, respectively. Then the stabilizer group of the state $|\psi\rangle_{\mathcal{G}}$ is represented by

$$\left(\begin{array}{cc|cc} I_k & 0 & G_1 & B \\ 0 & I_{n-k} & B^T & G_2 \end{array}\right). \tag{27}$$

Now note that for any $1 \leq i, j \leq k$, $(Z^{\boldsymbol{\alpha}_i})(X^{\mathbf{e}_j} Z^{\mathbf{g}_j}) = \omega^{\delta_{ij}} (X^{\mathbf{e}_j} Z^{\mathbf{g}_j})(Z^{\boldsymbol{\alpha}_i})$, where δ_{ij} is the Kronecker delta function. On the other hand, the code space is invariant under $X^{\mathbf{e}_j} Z^{\mathbf{g}_j}$. Therefore, the logical

X operators can be described by the matrix

$$\begin{pmatrix} I_k & 0 \,\big|\, G_1 & B \end{pmatrix}. \tag{28}$$

Also, it is not hard to see that the Pauli matrices corresponding to the rows of

$$\begin{pmatrix} -A^T & I_{n-k} \,\big|\, -A^T G_1 + B^T & -A^T B + G_2 \end{pmatrix}, \tag{29}$$

commute with both logical X and logical Z operators. Therefore, the additive graph code $\mathcal{Q}$ is described by the stabilizer group

$$\mathcal{S} = \left(\begin{array}{cc|cc} 0 & 0 & I_k & A \\ -A^T & I_{n-k} & -A^T G_1 + B^T & -A^T B + G_2 \end{array}\right), \tag{30}$$

logical Z operators

$$\mathcal{Z} = \begin{pmatrix} 0 & 0 \,\big|\, I_k & A \end{pmatrix}, \tag{31}$$

and logical X operators

$$\mathcal{X} = \begin{pmatrix} I_k & 0 \,\big|\, G_1 & B \end{pmatrix}. \tag{32}$$

G. Summary of notations

Before going into the detailed proof of the main result, we summarize our notation. Let $\mathcal{Q} = (\mathcal{G}, \mathcal{C})$ be a CWS code. If $\mathcal{C}$ is linear, then $\mathcal{Q}$ is a graph code, where the code has a graphical representation denoted by $\mathcal{G}^{\mathcal{C}}$. The concatenation of two CWS quantum codes $\mathcal{Q}_{\text{in}} = (\mathcal{G}_{\text{in}}, \mathcal{C}_{\text{in}})$ and $\mathcal{Q}_{\text{out}} = (\mathcal{G}_{\text{out}}, \mathcal{C}_{\text{out}})$ is denoted by $\mathcal{Q}_c = \mathcal{Q}_{\text{in}} \sqsubset \mathcal{Q}_{\text{out}}$.

See Table I for the rest of notations.

$\mathcal{C}$	the classical code
C	the generator matrix of the classical code $\mathcal{C}$, if $\mathcal{C}$ is linear
$C^{\{enc\}}$	the encoder of the classical code $\mathcal{C}$
$\mathcal{G}$	the graph corresponding to the graph state $\vert\psi\rangle_{\mathcal{G}}$
G	the adjacency matrix of the graph $\mathcal{G}$ ($G = (g_{ij})$)
$\mathcal{G}^{\mathcal{C}}$	the graph representing the graph code $\mathcal{Q}$, if $\mathcal{C}$ is linear
$G^{\{enc\}}$	the encoding circuit of the graph $\mathcal{G}$
$\mathcal{G}_{\mathcal{Q}_c}^{\{enc\}}$	the encoding graph of the concatenated code $\mathcal{Q}_c$
$\mathcal{G}_{\mathcal{Q}_c}^{\mathcal{C}_{\text{out}}\{enc\}}$	the encoding graph of the concatenated code $\mathcal{Q}_c$, if $\mathcal{C}_{\text{out}}$ is linear

TABLE I: Notations

Most of the notations have already been given in Sec. III, except for $\mathcal{G}_{\mathcal{Q}_c}^{\{enc\}}$ and $\mathcal{G}_{\mathcal{Q}_c}^{\mathcal{C}_{\text{out}}\{enc\}}$, which are discussed in Sec. II B and will be explained in more details in Sec. V.

IV. CONCATENATION OF GRAPH CODES

In this section, we prove our main result in a simple case, where the inner code encodes only a single qupit and the outer code is a graph code. In this situation, we can algebraically obtain the graph and the classical code of the concatenated code using the stabilizer formalism. Although we will prove our main result in the general case in Sec. V, we believe that the proof given in this section is easily accessible to those who are familiar with the stabilizer formalism.

Suppose the inner code $\mathcal{Q}_{\text{in}} = (\mathcal{G}_{\text{in}}, \mathcal{C}_{\text{in}})$ encodes only a single qupit, i.e., $\mathcal{Q}_{\text{in}}$ is an $[[n, 1, d]]_p$ code. Then from the discussion in Sec. III F, it follows that

$$G_{\text{in}} = \begin{pmatrix} 0 & \mathbf{y} \\ \mathbf{y}^T & H' \end{pmatrix}, \tag{33}$$

and

$$\mathcal{C}_{\text{in}} = \{0, (1\ \ \mathbf{b})\}, \tag{34}$$

i.e.,

$$C_{\text{in}} = (1\ \ \mathbf{b}), \tag{35}$$

where both $\mathbf{y}$ and $\mathbf{b}$ are vectors of length $n-1$. (Notice that $\mathcal{Q}_{\text{in}}$ encodes one qupit; thus, C_{in} is one-dimensional.)

Since $\mathcal{Q}_{\text{in}}$ encodes one qupit, the corresponding outer code $\mathcal{Q}_{\text{out}} = (\mathcal{G}_{\text{out}}, \mathcal{C}_{\text{out}})$ is an $((n', K', d'))_p$ code. In this section we assume that $\mathcal{C}_{\text{out}}$ is linear, so $\mathcal{Q}_{\text{out}}$ is a graph code with parameters $[[n', k', d']]_p$, where $K' = p^{k'}$. Then from the discussion in Sec. III F, we have

$$G_{\text{out}} = \begin{pmatrix} G_1 & B \\ B^T & G_2 \end{pmatrix}, \tag{36}$$

and the rows of

$$C_{\text{out}} = (I_{k'}\ \ A) \tag{37}$$

form a basis for $\mathcal{C}_{\text{out}}$.

Thus by Eqs. (30)–(32) the stabilizer group of $\mathcal{Q}_{\text{in}}$ is

$$\mathcal{S}_{\text{in}} = \left(\begin{array}{cc|cc} 0 & 0 & 1 & \mathbf{b} \\ -\mathbf{b}^T & I_{n-1} & \mathbf{y}^T & -\mathbf{b}^T\mathbf{y} + H' \end{array}\right), \tag{38}$$

its logical operator Z is given by

$$\mathcal{Z}_{\text{in}} = \left(\begin{array}{cc|cc} 0 & 0 & 1 & \mathbf{b} \end{array}\right), \tag{39}$$

and its logical operator X is given by

$$\mathcal{X}_{\text{in}} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & \mathbf{y} \end{array}\right). \tag{40}$$

The concatenated code $\mathcal{Q}_c = \mathcal{Q}_{\text{in}} \sqsubset \mathcal{Q}_{\text{out}}$ is a quantum code which encodes k' qupits into nn'

qupits as follows; it first encodes k' qupits into n' qupits using $\mathcal{Q}_{\rm out}$, and then encodes any of the n' qupits into n qupits based on $\mathcal{Q}_{\rm in}$.

The main result of this section is given by the following theorem, which states that the concatenated code $\mathcal{Q}_c = \mathcal{Q}_{\rm in} \sqsubset \mathcal{Q}_{\rm out}$ is also a graph code. The corresponding adjacency matrix of the graph and the generator matrix of the classical code can be computed directly from the adjacency matrices and the generator matrices of the inner and outer codes.

Theorem 1 *Suppose $\mathcal{Q}_{out} = (\mathcal{G}_{out}, \mathcal{C}_{out})$ and $\mathcal{Q}_{in} = (\mathcal{G}_{in}, \mathcal{C}_{in})$ are $[[n', k', d']]_p$ and $[[n, k, d]]_p$ graph codes, respectively, (where $k = 1$) as described by Eqs. (33)–(37). Then the concatenated code $\mathcal{Q}_c = \mathcal{Q}_{in} \sqsubset \mathcal{Q}_{out} = (\mathcal{G}_c, \mathcal{C}_c)$ is a graph code described by the graph $\mathcal{G}_c$ with adjacency matrix*

$$G_c = G_{in} \otimes I_{n'} + \begin{pmatrix} 1 \\ \mathbf{b}^T \end{pmatrix} (1 \;\; \mathbf{b}) \otimes G_{out}, \tag{41}$$

and the classical code with generator matrix

$$C_c = (1 \;\; \mathbf{b}) \otimes (I_{k'} \;\; A), \tag{42}$$

i.e., the classical code is obtained by concatenation as well:

$$\mathcal{C}_c = \mathcal{C}_{in} \sqsubset \mathcal{C}_{out}. \tag{43}$$

Proof: Let us first show that a basis of $\mathcal{C}_c$ is described by Eq. (42). To find $\mathscr{Z}_c$, the logical Z operators of $\mathcal{Q}_c$, we should first consider the logical Z operators of $\mathcal{Q}_{\rm out}$, and then replace any Pauli matrix X and Z of those operators with the logical X and Z operators of $\mathcal{Q}_{\rm in}$. For example, if the logical Z operator acting on the first encoded qupit in $\mathcal{Q}_{\rm out}$ is $Z^{(1,1,0,\ldots,0)}$, the logical Z operator acting on the first qupit in $\mathcal{Q}_c$ is $Z^{((1,\mathbf{b}),(1,\mathbf{b}),0\ldots0)}$ because by Eq. (39) the logical Z operator of $\mathcal{Q}_{\rm in}$ is $Z^{(1,\mathbf{b})}$. Therefore, by changing the order of qubits we can represent this Pauli matrix by the vector $\Big(0 \;\Big|\; (1 \;\; \mathbf{b}) \otimes (1,1,0,\ldots,0)\Big)$, where the zero before the vertical line is actually a zero vector. Now since the logical Z operators of $\mathcal{Q}_{\rm out}$ are represented by rows of Eq. (31), we have

$$\mathscr{Z}_c = \Big(0 \;\Big|\; (1 \;\; \mathbf{b}) \otimes (I_{k'} \;\; A)\Big). \tag{44}$$

Equivalently, $(1 \;\; \mathbf{b}) \otimes (I_{k'} \;\; A)$ is a basis for the linear code $\mathcal{C}_c$.

Analogously, we compute for the logical X operators of $\mathcal{Q}_c$:

$$\begin{aligned} \mathscr{X}_c &= \Big((1 \;\; 0) \otimes (I_{k'} \;\; 0) \;\Big|\; (0 \;\; \mathbf{y}) \otimes (I_{k'} \;\; 0) + (1 \;\; \mathbf{b}) \otimes (G_1 \;\; B)\Big) & (45) \\ &= \Big(I_k \;\; 0 \;\Big|\; G_1 \quad B \quad \mathbf{b} \otimes (G_1 \, B) + \mathbf{y} \otimes (I_{k'} \, 0)\Big). & (46) \end{aligned}$$

It remains to compute the stabilizer group. The first n' rows of $\mathcal{S}_c$ are obtained from rows of $\mathcal{S}_{\rm out}$ (Eq. (30)) by replacing any X and Z with the logical X and Z of the inner code. For the next $(n-1)n'$ rows note that, $g_2, \ldots, g_n$ which are the Pauli matrices corresponding to the rows $2, \ldots, n$ of $\mathcal{S}_{\rm in}$ commute with the logical X and Z of the inner code. In fact, they are in the stabilizer group of the code space (spanned by the states $|0\rangle, \ldots, |p-1\rangle$ states). Now since we replace any of the n' qupits of $\mathcal{Q}_{\rm out}$ with a state in the code space of $\mathcal{Q}_{\rm in}$, each block of n qupits in $\mathcal{Q}_c$ should be stabilized by $g_2, \ldots, g_n$. As a result, $\mathcal{S}_c = (\, M | N \,)$, where

$$M = \begin{pmatrix} (1\ \ 0)\otimes\begin{pmatrix}0 & 0\\ -A^T & I_{n'-k'}\end{pmatrix} \\ (-\mathbf{b}^T\ \ I_{n-1})\otimes I_{n'}\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0\\ -A^T & I_{n'-k'} & 0\\ -\mathbf{b}^T\otimes\begin{pmatrix}I_{k'}\\ 0\end{pmatrix} & -\mathbf{b}^T\otimes\begin{pmatrix}0\\ I_{n'-k'}\end{pmatrix} & I_{(n-1)n'}\end{pmatrix}, \tag{47}$$

and

$$N = \begin{pmatrix} (0\ \ \mathbf{y})\otimes\begin{pmatrix}0 & 0\\ -A^T & I_{n'-k'}\end{pmatrix} + (1\ \ \mathbf{b})\otimes\begin{pmatrix}I_{k'} & A\\ -A^TG_1+B^T & -A^TB+G_2\end{pmatrix} \\ (\mathbf{y}^T\ \ -\mathbf{b}^T\mathbf{y}+H')\otimes I_{n'}\end{pmatrix}. \tag{48}$$

Now to complete the proof of Theorem 1 it is sufficient to show that the stabilizer group, and the logical X and Z operators of the graph code described by Eqs. (41) and (42) are given by Eqs. (44)–(48). We compute these matrices using the construction given by Eqs. (30)–(32).

First of all, the classical part of the code is given by $(I_{k'}\ \ B)$, where $B = (A\ \ \mathbf{b}\otimes(I_{k'}\ \ A))$; therefore, the logical Z operators of the code are the same as Eq. (44).

The block from of matrix G_c of Eq. (41) is given by

$$G_c = \begin{pmatrix}K_1 & W\\ W^T & K_2\end{pmatrix}, \tag{49}$$

where $K_1 = G_1$, $W = (B\ \ \mathbf{y}\otimes(I_{k'}\ \ 0)+\mathbf{b}\otimes(G_1\ \ B))$ and

$$K_2 = \begin{pmatrix} G_2 & \mathbf{y}\otimes(0\ \ I_{n'-k'})+\mathbf{b}\otimes(B^T\ \ G_2)\\ \mathbf{y}^T\otimes\begin{pmatrix}0\\ I_{n'-k'}\end{pmatrix}+\mathbf{b}^T\otimes\begin{pmatrix}B\\ G_2\end{pmatrix} & H'\otimes I_{n'}+\mathbf{b}^T\mathbf{b}\otimes G_2\end{pmatrix}. \tag{50}$$

Hence, the logical X operator of the graph code is

$$\Big(I_{k'}\ \ 0\ \mid\ K_1\ \ W\Big) = \Big(I_{k'}\ \ 0\ \mid\ K_1\quad B\quad \mathbf{y}\otimes(I_{k'}\ \ 0)+\mathbf{b}\otimes(G_1\ \ B)\Big), \tag{51}$$

which is the same as Eq. (45).

The stabilizer group of the graph code is given by

$$\left(\begin{array}{cc|cc} 0 & 0 & I_{k'} & B\\ -B^T & I_{nn'-k'} & -B^TK_1+W^T & -B^TW+K_2\end{array}\right). \tag{52}$$

By Lemma 2 this matrix describes the same group as $\mathcal{S}_c = (\,M\,|\,N\,)$ because we have

$$\mathcal{S}_c = \begin{pmatrix} I_{k'} & 0 & 0\\ 0 & I_{n'-k'} & 0\\ 0 & -\mathbf{b}^T\otimes\begin{pmatrix}0\\ I_{n'-k'}\end{pmatrix} & I_{(n-1)n'}\end{pmatrix}\left(\begin{array}{cc|cc} 0 & 0 & I_{k'} & B\\ -B^T & I_{nn'-k'} & -B^TK_1+W^T & -B^TW+K_2\end{array}\right). \tag{53}$$

□

Notice that from Eq. (41), the adjacency matrix G_c does not depend on the classical code of the outer code ($\mathcal{C}_{\text{out}}$), which indicates that Theorem 1 could also be true even if $\mathcal{C}_{\text{out}}$ is nonlinear (in this case Eq. (42) does no longer apply, but Eq. (43) may still hold). As the stabilizer formalism can no longer be used to handle this case, we need an alternative proof technique which will be presented in the next section. This new technique is based on analyzing the encoding circuit of the concatenated code. It can easily be extended to more general cases, such as nonlinear outer codes, $k > 1$, and even to generalized concatenated quantum codes.

V. GRAPH CONCATENATION BY GENERALIZED LOCAL COMPLEMENTATION

In this section we prove our main result based on analyzing the encoding circuits of the concatenated code. We start with an alternative proof for Theorem 1 in Sec. V A for the simple case that the inner code encodes only a single qupits, and the outer code is a graph code. This proof is based on the rule of "generalized local complementation." Then in Sec. V B, we show that the rule of "generalized local complementation" can be directly applied to the case that the outer code is a general CWS code, which is beyond the result of Theorem 1. In Sec. V C, we discuss the case where the inner code encodes more than one qupit (i.e., $k > 1$); we show that the rule of "generalized local complementation" given in Sec. V A also applies directly to this case, and hence completes the proof of the main result.

A. Alterantive proof for Theorem 1

Recall our main goal: suppose we have two graph codes $\mathcal{Q}_{\text{out}} = (\mathcal{G}_{\text{out}}, \mathcal{C}_{\text{out}})$ and $\mathcal{Q}_{\text{in}} = (\mathcal{Q}_{\text{in}}, \mathcal{C}_{\text{in}})$ given by Eqs. (33)–(37), where $\mathcal{Q}_{\text{in}}$ encodes a single qupit. Let $\mathcal{Q}_c = \mathcal{Q}_{\text{in}} \sqsubset \mathcal{Q}_{\text{out}}$ denote the concatenation of the inner code $\mathcal{Q}_{\text{in}}$ and the outer code $\mathcal{Q}_{\text{out}}$. We would like to show that $\mathcal{Q}_c = (\mathcal{Q}_c, \mathcal{C}_c)$, where $\mathcal{Q}_c$ and $\mathcal{C}_c$ are given in Eqs. (41) and (42), respectively.

In Sec. V A 1, we first specify the encoding circuit of the concatenated code $\mathcal{Q}_c$, then we give the graphical interpretation of this circuit and define the encoding graph $\mathcal{G}_{\mathcal{Q}_c}^{\mathcal{C}_{\text{out}}\{enc\}}$ of the concatenated code $\mathcal{Q}_c$. Then in Sec. V A 2 we define the rule of "generalized local complementation" on a graph; we show how the encoding circuit of the concatenated code $\mathcal{Q}_c$ can be interpreted as generalized local complementation on the encoding graph, and how we can obtain the graph code $\mathcal{G}_c^{\mathcal{C}_c}$ from the encoding graph $\mathcal{G}_{\mathcal{Q}_c}^{\mathcal{C}_{\text{out}}\{enc\}}$; finally we show that $\mathcal{Q}_c$ and $\mathcal{C}_c$ are exactly those given in Eqs. (41) and (42), thereby completing the proof.

1. *Encoding circuit and encoding graph for the concatenated code*

We have already discussed the encoding circuit of a concatenated code in Sec. II B. Here we state it more formally.

Procedure 3 *(Encoding circuit for $\mathcal{Q}_c$ with a graph outer code and an inner code encoding a single qupit)*

1. *Apply the encoding circuit of $\mathcal{Q}_{out}$ that encodes k' qupits into n' qupits which we call $q_1, \ldots, q_{n'}$, as given by Procedure 2.*

2. *Apply n' copies of the circuit that gives the graph state corresponding to $\mathcal{G}_{in}$.*

3. *Apply $H^\dagger$ on all qupits $q_1, \ldots, q_{n'}$.*

4. *Apply the corresponding controlled-Z operators between these qupits and the graph states of $\mathcal{G}_{in}$.*

5. *Apply H on $q_1, \ldots, q_{n'}$.*

6. *Measure $q_1, \ldots, q_{n'}$ in the computational basis.*

For an example, see the right circuit of FIG. 5.

As discussed in Sec. II B, Procedure 3 can be represented by a graph which is denoted by $\mathcal{G}_{\mathcal{Q}_c}^{\mathcal{C}_{\text{out}}\{enc\}}$. This graph is constructed as follows: Step 1 corresponds to the graph $\mathcal{G}_{\text{out}}^{\mathcal{C}_{\text{out}}}$; Step 2 corresponds to adding a copy of the graph $\mathcal{G}_{\text{in}}$ for each vertex q_i of the graph $\mathcal{G}_{\text{out}}$, then we have a graph on $n' + k' + nn'$ vertices; Steps 3, 4, 5 encode the n' qupits of the outer code into n' copies of the inner code, so we just add edges and labels according to controlled-Z gates that are applied between these n' qupits and the graph states of $\mathcal{G}_{\text{in}}$.

For an example of the encoding graph $\mathcal{G}_{\mathcal{Q}_c}^{\mathcal{C}_{\text{out}}\{enc\}}$, see FIG. 4B. (The corresponding encoding circuit is given by the right circuit of FIG. 5.)

2. Graph concatenation via Generalized Local Complementation

We now give a graphical interpretation of Steps 3, 4, 5 given in Procedure 3. Notice that for $1 \leq i \leq n'$ we apply $H^\dagger$ to qupit q_i, then the corresponding controlled-Z operations between q_i and the i-th copy of the graph state $\mathcal{G}_{\text{in}}$, and finally H on q_i. We show that each of these n' steps is equivalent to a generalized local complementation on the graph.

Definition 1 *(Generalized Local Complementation) Suppose $F = (f_{ij})$ is the adjacency matrix of a graph $\mathcal{F}$, i is a vertex of $\mathcal{F}$, and $\mathbf{f}_i$ is the i-th row of F. Also, let $\mathbf{v}$ be a vector whose coordinates are indexed by the vertices of $\mathcal{F}$ such that $\mathbf{v}$ is zero on i and its neighbors, i.e., $v_j = 0$ if $j = i$ or $f_{ij} \neq 0$. Then the generalized local complementation at $(i, \mathbf{v})$ is the operation which sends F to $F + \mathbf{v}^T\mathbf{f}_i + \mathbf{f}_i^T\mathbf{v}$.*

Notice that, since $\mathbf{v}$ is zero on the neighbors of i, for any j and k either $(\mathbf{v}^T\mathbf{f}_i)_{jk}$ or $(\mathbf{f}_i^T\mathbf{v})_{jk}$ is equal to zero.

To get an idea on why we call this operation the generalized local complementation, let us consider the binary case. In this special case $\mathbf{v}$ corresponds to a subset of vertices (j belongs to this set iff $\mathbf{v}_j = 1$). Then this operation is the same as to replace the bipartite graph induced on the neighbors of i and the vertices in $\mathbf{v}$ with its complement. (For an example, see FIG. 7.)

Theorem 2 *(Encoding circuit interpreted as generalized local complementation) Consider a circuit which corresponds to a graph $\mathcal{F}$ with the adjacency matrix F. Let i be a vertex of $\mathcal{F}$ (or equivalently a qupit in the circuit), and let $\mathbf{v}$ be a vector which is zero on i and its neighbors. Suppose we change the circuit by applying $H^\dagger$ on the i-th qupit, $C_z^{v_j}$ (v_j is the j-th coordinate of $\mathbf{v}$) on the qupits i and j, for any j, and then H on the i-th qupit. Then the resulting circuit is equivalent to the graph $\mathcal{F}$ after the generalized local complementation at $(i, \mathbf{v})$.*

Proof: For simplicity assume $i = 1$, and let $\mathbf{f}_1 = (0 \ \mathbf{s})$, where $\mathbf{f}_1$ is the first row of F. Also, let $\mathbf{v} = (0 \ \mathbf{v}')$, and $\mathcal{F}'$ be the graph obtaining from $\mathcal{F}$ by removing its first vertex (and F' its adjacency matrix). Then the stabilizer group corresponding to the circuit is represented by

$$(\, I \,|\, F \,) = \left(\begin{array}{cc|cc} 1 & 0 & 0 & \mathbf{s} \\ 0 & I & \mathbf{s}^T & F' \end{array}\right). \tag{54}$$

Then based on the translation of the action of the Hadamard gate and controlled-Z gate on the stabilizer group (Lemmas 3 and 4), we can compute that stabilizer group after applying those gates as follows:

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & \mathbf{s} \\ 0 & I & \mathbf{s}^T & F' \end{array}\right) \xrightarrow{H^\dagger} \left(\begin{array}{cc|cc} 0 & 0 & -1 & \mathbf{s} \\ \mathbf{s}^T & I & 0 & F' \end{array}\right) \xrightarrow{C_z^{1,\mathbf{v}'}} \left(\begin{array}{cc|cc} 0 & 0 & -1 & \mathbf{s} \\ \mathbf{s}^T & I & \mathbf{v}'^T & F' + \mathbf{s}^T\mathbf{v}' \end{array}\right) \tag{55}$$

$$\xrightarrow{H} \left(\begin{array}{cc|cc} 1 & 0 & 0 & \mathbf{s} \\ -\mathbf{v}'^T & I & \mathbf{s}^T & F' + \mathbf{s}^T\mathbf{v}' \end{array}\right) \tag{56}$$

Now to relate this stabilizer group to a graph, we change the set of generators by multiplying the above matrix by

$$\begin{pmatrix} 1 & 0 \\ \mathbf{v}'^T & 1 \end{pmatrix}, \tag{57}$$

which gives

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & \mathbf{s} \\ 0 & I & \mathbf{s}^T & F' + \mathbf{s}^T\mathbf{v}' + \mathbf{v}'^T\mathbf{s} \end{array}\right). \tag{58}$$

Hence, the adjacency matrix F of the graph is changed to $F + \mathbf{v}^T\mathbf{f}_i + \mathbf{f}_i^T\mathbf{v}$. □

A direct corollary of Theorem 2 is the following:

Corollary 1 *$\mathcal{Q}_c = (\mathcal{G}_c, \mathcal{C}_c)$, and the graph $\mathcal{G}_c^{\mathcal{C}_c}$ can be obtained from the encoding graph $\mathcal{G}_{\mathcal{Q}_c}^{\mathcal{C}_{out}\{enc\}}$ via Procedure 1.*

Notice that Corollary 1 proves our main result in the case that the inner code encodes a single qupit and the outer code is linear.

Also, note that the resulting graph of Corollary 1 is consistent with the one given by Theorem 1 since they both compute the same graph. In other words, the adjacency matrix of the graph $\mathcal{G}_c^{\mathcal{C}_c}$ constructed via Corollary 1 is given by Theorem 1.

Theorem 3 *The graph $\mathcal{G}_c^{\mathcal{C}_c}$ given by Corollary 1 is equal to the graph given by Eqs. (41) and (42).*

Proof: Here we briefly describe a proof only for the binary case, and for the validity of Eq. (41). This proof can simply be captured for the more general setting.

Based on Procedure 1, the graph on which we apply the generalized local complementation operators has the following subgraphs: $\mathcal{G}_{\text{out}}$ with auxiliary vertices $\{1, \dots, n'\}$; and a copy of $\mathcal{G}_{\text{in}}$ with vertex set V_i for each auxiliary vertex $1 \le i \le n'$. Then for each $1 \le i \le n'$ we apply the generalized local complementation on i with respect to $S_i \subseteq V_i$ which is defined based on the classical inner code.

Fact 1 *Eq. (41) describes the unique graph on the vertex set $\bigcup_i V_i$ with the following structure:*

1. *The induced subgraph on V_i, for every i, is isomorphic to $\mathcal{G}_{in}$.*

2. *For every $i \neq j$, there is no edge between vertices in V_i and $V_j \setminus S_j$.*

3. *For every $i \neq j$, there is an edge between vertices $v \in S_i$ and $w \in S_j$ iff i and j are connected in $\mathcal{G}_{out}$.*

Clearly the graph with these properties is unique. Also, it is clear that Eq. (41) represents this unique graph.

Based on this fact, we will show that the graph resulting from Procedure 1 has the above structure.

Fact 2 *During Procedure 1 the changes on the subgraph induced on $\bigcup V_i$ happen only among vertices $v \in S_i$ and $w \in S_j$ for $i \neq j$. As a result, the final graph of Procedure 1 satisfies properties 1 and 2 of Fact 1.*

This is simply because in the generalized local complementations we never touch vertices in $V_i \setminus S_i$. Furthermore,in each step, S_i is disjoint from $N(i)$ (the neighbors of vertex i), so there is no change in the subgraph induced on the vertex set S_i.

Fact 3 *In Procedure 1, suppose we have applied the generalized local complementation on vertices $1, 2, \ldots, l$, for some $1 \leq l \leq n'$. Then for any choice of $v_i \in S_i$, for $1 \leq i \leq l$, the induced subgraph on vertices $\{v_1, \ldots, v_l\} \cup \{l + 1, \ldots, n'\}$ is isomorphic to $\mathcal{G}_{out}$.*

This fact can be proved by a simple induction on l.

Now we can prove the theorem. The resulting graph of Procedure 1 is a graph on the vertex set $\bigcup_i V_i$. According to Fact 2, this graph satisfies properties 1 and 2 of Fact 1. Property 3 of Fact 1 also holds based on Fact 3. Therefore, by the uniqueness of the graph described in Fact 1, we are done. □

We illustrate the graph obtained by generalized local complementation for the code [[25, 1, 9]] which can be obtained by self-concatenation of the code [[5, 1, 3]]. As a graph code, the code [[5, 1, 3]] can be described by a pentagon corresponding to the output nodes and a central input node that is connected to all output nodes. Using auxiliary nodes, the concatenated code [[25, 1, 9]] is show as the left graph in FIG. 9.

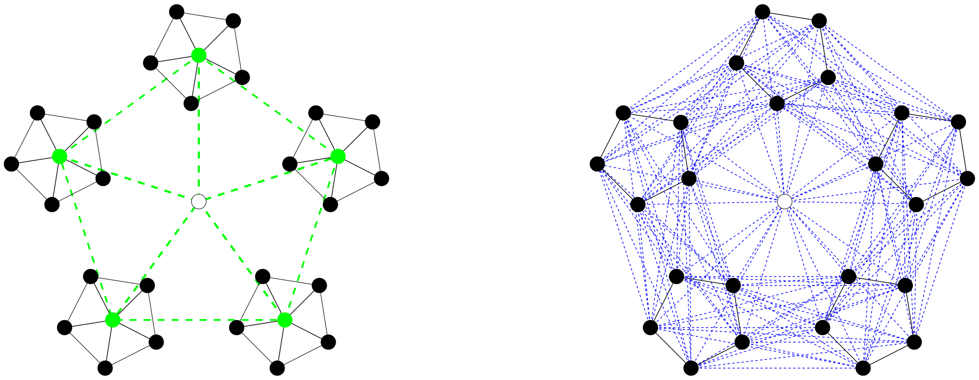

FIG. 9: Self-concatenation of the code [[5, 1, 3]] yielding a code [[25, 1, 9]].

The outer code is given by the large pentagon with green/light dots and dashed lines. The five copies of the inner code correspond to the small pentagons with black dots and solid lines. The final graph is shown on the right of FIG. 9. The five solid black pentagons remain, and any vertex in a small pentagon is connected with blue/dashed lines to any vertex of the neighboring pentagons as well as the central input node.

The situation for the self-concatenation of Steane's code [[7, 1, 3]], which can be realized as a cube, is shown in FIG. 10.

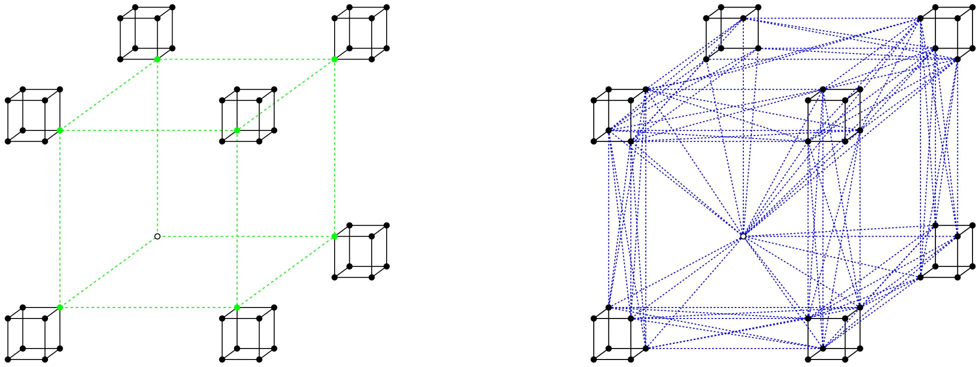

FIG. 10: Self-concatenation of Steane's code $[[7,1,3]]$ yielding a code $[[49,1,9]]$.

B. A general outer code

In this section we consider the case when the outer code is nonadditive. The advantage of Theorem 2 is that it directly applies to this case as well.

Procedure 4 *(Encoding circuit for $\mathcal{Q}_c$ with a general outer code and an inner code encoding a single qupit)*

1. *Apply the encoding circuit of $\mathcal{Q}_{out}$ that encodes K' states into n' qupits which we call $q_1, \ldots, q_{n'}$.*
2. *Apply n' copies of the circuit that gives the graph state corresponding to $\mathcal{G}_{in}$.*
3. *Apply $H^\dagger$ on all qupits $q_1, \ldots, q_{n'}$.*
4. *Apply the corresponding controlled-Z operators between these qupits and the graph states of $\mathcal{G}_{in}$.*
5. *Apply H on $q_1, \ldots, q_{n'}$.*
6. *Measure $q_1, \ldots, q_{n'}$ in the computational basis.*

For an example, see the right circuit of FIG. 6.

Notice that Theorem 2 deals with Steps 3, 4, 5 in Procedure 4, which are exactly the same as Steps 3, 4, 5 as in Procedure 3. Therefore, whether $\mathcal{C}_{\text{out}}$ is linear or not does not actually matter. Consequently, Corollary 1, and thus the main result hold even for nonlinear outer codes.

C. The case $k > 1$

Theorem 2 can also be directly applied to the case when the inner code encodes more than one qupit. Again, to see this we only need to specify the encoding circuit of $\mathcal{Q}_c$.

Procedure 5 *(Encoding circuit for $\mathcal{Q}_c$ with a general outer code and an inner code encoding k qupits)*

1. *Apply the encoding circuit of $\mathcal{Q}_{out}$ that encodes K' states (or kk' qupits if $\mathcal{C}_{out}$ is linear) into kn' qupits which we call $q_1, \ldots, q_{kn'}$.*

2. *Apply n' copies of the circuit that gives the graph state corresponding to $\mathcal{G}_{in}$.*

3. *Apply $H^\dagger$ on all qupits $q_1, \ldots, q_{kn'}$.*

4. *Apply the corresponding controlled-Z operators between these qupits and the graph states of $\mathcal{G}_{in}$.*

5. *Apply H on $q_1, \ldots, q_{kn'}$.*

6. *Measure $q_1, \ldots, q_{kn'}$ in the computational basis.*

Note that Steps 3, 4, 5 remain the same as those given in Procedure 3. Consequently, Corollary 1, and hence our main result hold for the case of $k > 1$.

For an example, the left graph of FIG. 11 is the encoding graph $\mathcal{G}_{\mathcal{Q}_c}^{\mathcal{C}_{\text{out}}\{enc\}}$ of the concatenated code $\mathcal{Q}_c$ with a $[[4,2,2]]_2$ inner code, and a $[[4,2,2]]_{2^2}$ outer code. Note that we decompose the outer code into two copies of a qubit code $[[4,2,2]]_2$. Hence there are $kn' = 2 \times 4 = 8$ auxiliary vertices (green/light vertices) in $\mathcal{G}_{\mathcal{Q}_c}^{\mathcal{C}_{\text{out}}\{enc\}}$. The corresponding encoding circuit is given by the middle circuit in FIG. 11, where "/" on each line indicates that there is a set of qubits, not just one. For instance, the line corresponding to $|q_0\rangle$ represents the 4 input qubits (4 white vertices in the left graph of FIG. 11), the line corresponding to $|q_1\rangle$ represents the 8 auxiliary qubits, and the line corresponding to $|q_2\rangle$ represents the 4 output qubits of a single inner code $\mathcal{Q}_{\text{in}}$. The graph $\mathcal{G}_c^{\mathcal{C}_c}$ of the concatenated code $\mathcal{Q}_c$ can be obtained from the encoding graph $\mathcal{G}_{\mathcal{Q}_c}^{\mathcal{C}_{\text{out}}\{enc\}}$ by applying Corollary 1. The result is shown as the right graph in FIG. 11. The blue/dashed lines are the edges obtained by generalized local complementation.

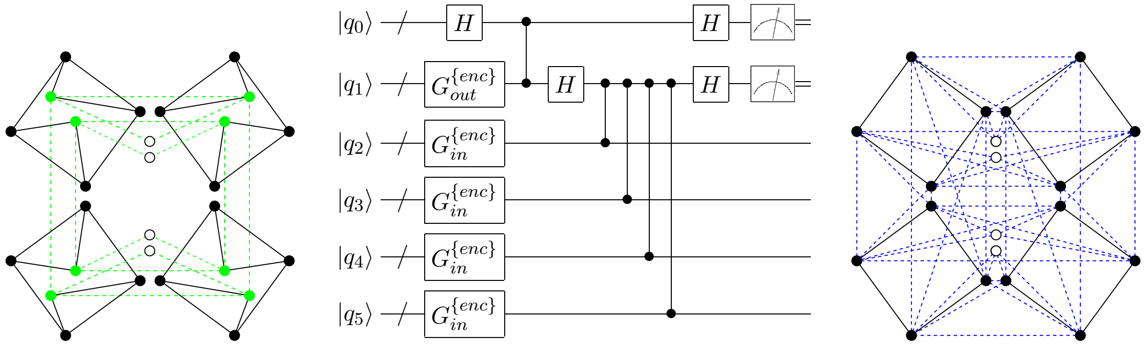

FIG. 11: Graphs and encoding circuit for the concatenated $[[16,4,4]]_2$ code obtained by concatenating an inner code $[[4,2,2]]_2$ with an outer code $[[4,2,2]]_{2^2}$.

VI. GENERALIZED CONCATENATED CODES

In this section, we discuss the application of our main result to the case of generalized concatenated quantum codes (GCQCs). The construction of GCQCs has been recently introduced in [10, 11]. It resulted in many new QECCS, both stabilizer codes and nonadditive codes.

A GCQC is derived from an inner quantum code $\mathcal{Q}_{\text{in}}^{(0)} = ((n, q_1 q_2 \cdots q_r, d_1))_p$, which is first partitioned into q_1 mutually orthogonal subcodes $\mathcal{Q}_{\text{in}\{i_1\}}^{(1)}$ $(0 \le i_1 \le q_1 - 1)$, where each $\mathcal{Q}_{\text{in}\{i_1\}}^{(1)}$ is an $((n, q_2 \cdots q_r, d_2))_p$ code. Then each $\mathcal{Q}_{\text{in}\{i_1\}}^{(1)}$ is partitioned into q_2 mutually orthogonal subcodes $\mathcal{Q}_{\text{in}\{i_1 i_2\}}^{(2)}$ $(0 \le i_2 \le q_2 - 1)$, where each $\mathcal{Q}_{\text{in}\{i_1 i_2\}}^{(2)}$ has parameters $((n, q_3 \cdots q_r, d_3))_p$, and so on.

Finally, each $\mathcal{Q}^{(r-2)}_{\mathrm{in}\{i_1 i_2 \ldots i_{r-2}\}}$ is partitioned into q_{r-1} mutually orthogonal subcodes $\mathcal{Q}^{(r-1)}_{\mathrm{in}\{i_1 i_2 \ldots i_{r-1}\}} = ((n, q_r, d_r))_p$ for $0 \leq i_{r-1} \leq q_{r-1} - 1$. Thus

$$\mathcal{Q}^{(0)}_{\mathrm{in}} = \bigoplus_{i_1=0}^{q_1-1} \mathcal{Q}^{(1)}_{\mathrm{in}\{i_1\}}, \quad \mathcal{Q}^{(1)}_{\mathrm{in}\{i_1\}} = \bigoplus_{i_2=0}^{q_2-1} \mathcal{Q}^{(2)}_{\mathrm{in}\{i_1 i_2\}}, \quad \ldots, \tag{59}$$

and $d_1 \leq d_2 \leq \ldots \leq d_r$. In addition, we take as outer codes a collection of r quantum codes $\mathcal{Q}^{(1)}_{\mathrm{out}}, \ldots, \mathcal{Q}^{(r)}_{\mathrm{out}}$, where $\mathcal{Q}^{(j)}_{\mathrm{out}}$ is an $((n', K'_j, d'_j))_{q_j}$ code over the Hilbert space $\mathcal{H}^{\otimes n'}_{q_j}$.

The generalized concatenated code $\mathcal{Q}_{gc}$ is a quantum code in the Hilbert space $\mathcal{H}^{\otimes nn'}_{q}$ of dimension $K' = K'_1 K'_2 \cdots K'_r$. The detailed construction of $\mathcal{Q}_{gc}$ can be found in [11]. Here we only emphasize that the essence of the "generalization", which is different from the usual concatenated quantum codes, is that the outer code is actually a product of r outer codes, and the inner code is nest-decomposed to specify how those product of outer codes are encoded into each inner code. Therefore, similar to a concatenated code $\mathcal{Q}_c$, a GCQC $\mathcal{Q}_{gc}$ with a graph inner code

$$\mathcal{Q}^{(0)}_{\mathrm{in}} = (\mathcal{G}^{(0)}_{\mathrm{in}}, \mathcal{C}^{(0)}_{\mathrm{in}}) \tag{60}$$

and r CWS outer codes

$$\mathcal{Q}^{(j)}_{\mathrm{out}} = (\mathcal{G}^{(j)}_{\mathrm{out}}, \mathcal{C}^{(j)}_{\mathrm{out}}) \tag{61}$$

naturally has an encoding graph, denoted by $\mathcal{G}^{\{enc\}}_{\mathcal{Q}_{gc}}$, and the corresponding encoding circuit is given by the following procedure.

Procedure 6 *(Encoding circuit for generalized concatenated code $\mathcal{Q}_{gc}$)*

1. *Apply the encoding circuits of $\mathcal{Q}^{(j)}_{out}$ that encodes K'_j states (or $k'_j \log_p q_j$ qupits if $\mathcal{C}^{(j)}_{out}$ is linear) into $n' \log_p q_j$ qupits which we call $q_1, \ldots, q_{n' \log_p q_j}$.*

2. *Apply n' copies of the circuit that gives the graph state corresponding to $\mathcal{G}_{in}$.*

3. *For each $j = 1, \ldots, r$, apply $H^\dagger$ on all qupits $q_1, \ldots, q_{n' \log_p q_j}$.*

4. *Apply the corresponding controlled-Z operators between these qupits and the graph states of $\mathcal{G}_{in}$.*

5. *For each $j = 1, \ldots, r$, apply H on $q_1, \ldots, q_{n' \log_p q_j}$.*

6. *For each $j = 1, \ldots, r$, measure $q_1, \ldots, q_{n' \log_p q_j}$ in the computational basis.*

Notice that Steps 3, 4, 5 remain the same as those given in Procedure 3. Consequently, a similar result as in Corollary 1 holds for constructing GCQCs as well.

Corollary 2 *$\mathcal{Q}_{gc} = (\mathcal{G}_{gc}, \mathcal{C}_{gc})$, where $\mathcal{G}_{gc}$ can be obtained for the encoding graph $\mathcal{G}^{\{enc\}}_{\mathcal{Q}_{gc}}$ via Procedure 1 and $\mathcal{C}_{gc}$ is the classical generalized concatenated code with inner code $\mathcal{C}^{(0)}_{in}$ (with corresponding decomposition given by the decomposition of $\mathcal{Q}^{(0)}_{in}$, see [11] for details) and the outer codes $\mathcal{C}^{(j)}_{out}$ $(j = 1, \ldots, r)$.*

For an example, the left graph of FIG. 12 is the encoding graph $\mathcal{G}_{\mathcal{Q}_{gc}}^{\mathcal{C}_{\text{out}}^{(0)}\{enc\}}$ of the GCQC $\mathcal{Q}_{gc}$ with a $[[4,2,2]]_2$ inner code code that is decomposed into two copies of a code $[[4,1,2]]_2$. There are two different outer codes $[[4,4,1]]_2$ and $[[4,2,2]]_2$. Note that there are $4+2=6$ input vertices (white vertices) and 8 auxiliary vertices (green/light vertices). The corresponding encoding circuit is given by the middle circuit in FIG. 12, where "/" on each line means that the line actually represents a set of qubits. For instance, the line corresponding to $|q_{00}\rangle$ represents the 4 input qubits of the $[[4,4,1]]_2$ outer code, the line corresponding to $|q_{01}\rangle$ represents the 2 input qubits of the $[[4,2,2]]_2$ outer code, the lines corresponding to $|q_{10}\rangle$ and $|q_{11}\rangle$ represents the 4 auxiliary qubits of the $[[4,4,1]]_2$ and the $[[4,2,2]]_2$ outer codes, respectively, and the line corresponding to $|q_2\rangle$ represents the 4 output vertices in a single $\mathcal{Q}_{\text{in}}$. To obtain the graph $\mathcal{G}_{gc}^{\mathcal{C}_{gc}}$ of the concatenated code $\mathcal{Q}_{gc}$ from the encoding graph $\mathcal{G}_{\mathcal{Q}_{gc}}^{\mathcal{C}_{\text{out}}^{(0)}\{enc\}}$ apply Corollary 2. The result is shown as the right graph in FIG. 11.

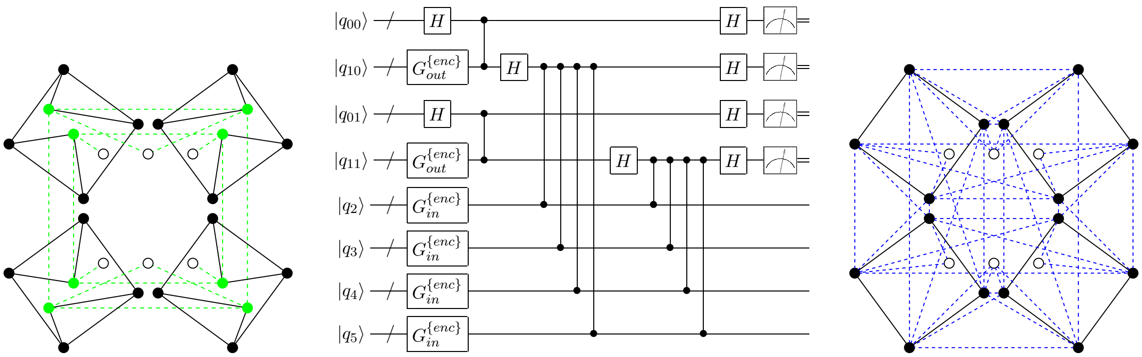

FIG. 12: Graphs and encoding circuit for the generalized concatenated $[[16,6,2]]_2$ code, derived from an inner code $[[4,2,2]]_2$ and outer codes $[[4,2,2]]_2$ and $[[4,4,1]]_2$.

VII. CONCLUSION AND DISCUSSION

In this paper we develop a systematic method for constructing concatenated quantum codes based on "graph concatenation", where graphs representing the inner and outer codes are concatenated via a simple graph operation called "generalized local complementation." The outer code is chosen from a large class of quantum codes, called CWS codes, which includes all the stabilizer codes as well as many good nonadditive codes. The inner code is chosen to be a stabilizer code. Despite the restriction that the inner code must be a stabilizer code, our result applies to very general situations—both binary and nonbinary concatenated quantum codes, and their generalizations.

Our results indicate that graphs indeed capture the "quantum part" of the QECCs. Once the graph part is taken care of, the construction of quantum code is reduced to a pure classical problem. This was essentially the idea of the CWS framework (i.e., the problem of constructing a CWS quantum code is reduced to the problem of finding a classical code with error patterns induced by a given graph). Here we have demonstrated that this idea extends to the construction of (generalized) concatenated quantum codes as well (i.e., to construct (generalized) concatenated quantum codes, given the rule of graph concatenation, one only needs to construct the (generalized) classical concatenated codes). We believe that our results shed light on the further understanding of the role that graphs play in the field of quantum error correction and other related areas in quantum information theory.

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