Stochastic Matched Field Processing for Localization and Nulling of Acoustic Sources

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Abstract

Underwater acoustic source localization can be performed using Matched Field Processing (MFP), which correlates a received pressure field with one generated by an acoustic propagation model. Successful localization is contingent upon accurate signal and environmental input parameters to the model. Incorrect modeling parameters can lead to bias in the source location estimation. Some parameters, notably the sound velocity profile (SVP), vary both spatially and temporally, and are thus difficult to treat as deterministic quantities.

This thesis seeks to improve the quality of source localization by proposing a new method of MFP which treats the SVP as a random quantity, and consequently, the modeled pressure field as same. Rather than rely on a single replica vector to correlate against the received pressure field, this method correlates multiple orthonormal replica vectors. Each correlation is weighted and summed to yield a detection statistic. A Maximum-Likelihood nonrandom parameter estimation method is then applied to determine the source location, with and without the presence of colored Gaussian noise caused by discrete interferers. Shallow water localization results are shown using both simulated and real data from the 1998 Santa Barbara Channel Experiment.

Thesis Supervisor: Arthur B. Baggeroer
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Chapter 1

Introduction

1.1 Matched Field Processing

Matched Field Processing (MFP) is a technique for acoustic source localization. It allows one to simultaneously and passively determine the range, bearing and depth of a sound source in the water. MFP was first applied to underwater acoustics by Bucker[1] in 1975. It exploits the multipath and multimode propagation of sound through water, requiring an array of acoustic receivers, and the ability to model the acoustic energy propagating from a source to the receiver array. Accurate modeling requires knowledge of many input variables, including the presumed location of the acoustic source, the speed of sound as a function of depth and range, the depth and composition of the bottom sediment, and the precise spatial location of each element on the receiver array. The modeled signal is compared, or “matched,” with the actual received signal. One iterates through many candidate source locations, comparing each modeled signal with the original received signal. The candidate source location corresponding to the modeled signal with the best received signal match is then chosen as the “true” estimate of the source location.

MFP works very well when the inputs to the modeled signal are accurate [2]. If the model inputs are incorrect, the quality of the source location estimate degrades, and errors appear[3]. One of the inputs which affects bias error is the speed of sound as a function of depth, otherwise known as the sound velocity profile (SVP).

This thesis proposes a new method of Matched Field Processing which mitigates errors in the modeled sound velocity profile, by treating the speed of sound in water as a random quantity. This causes the modeled pressure field to also be treated as a random quantity. One characterizes the modeled pressure field by using a second order covariance matrix, instead of a first order pressure vector. Random signal detection and estimation theory is then used to determine the spatial location of the acoustic source. The end result is a significant reduction in localization error, especially as one increases range in shallow water.

This thesis proposes a new method of nulling acoustic interferers. In some situations, the presence of
a spatially separated, uncorrelated acoustic source in addition to the desired source can corrupt its location estimate. One can reject all acoustic information generated by the interfering source by applying a filter on both the received and modeled pressure field, creating a spatial null. This document treats the acoustic energy radiated by the interferer as a random quantity, allowing one to apply a whitening filter to the data, creating a robust null, and allowing for localization of the desired source in the presence of environmental uncertainty.

1.2 Roadmap

This thesis assumes the reader has a firm, undergraduate level understanding of Linear Algebra and Probability Theory. Strang[4] and Drake[5] provide excellent references for their respective topics. Additionally, knowledge of detection and estimation theory is helpful; one can refer to Kay[6] and Van Trees[7, 8]. Knowledge of underwater acoustic propagation, including normal mode and parabolic equation methods, is also assumed; see Jensen et al.[9] for background information. Finally, familiarity with array signal processing is beneficial to understanding some of the concepts presented. Johnson, et al. [10] provide a solid foundation for study in their text.

This arrangement of information in this document is straightforward. Chapter 2 provides background information on stochastic propagation and Matched Field Processing, in the form of a literature review. Chapter 3 applies random signal detection and estimation to the estimation of the incident angle of signal arrival using a linearly spaced one-dimensional array. Chapter 4 applies random signal detection and estimation to stochastic MFP, showing results from both simulations and experimental data. Appendices A and B provide additional information on stochastic propagation, while Appendix C reviews detection and estimation theory.

1.3 Symbols and Conventions

This document attempts to be consistent with notation throughout. Scalar variables, such as $x$, are represented in italics. Vector quantities ($\mathbf{r}$) are set in bold. Matrices are represented by capital letters in sans-serif font ($\mathbf{U}$). The determinant of a matrix is expressed using vertical lines ($\det \mathbf{A} = |\mathbf{A}|$). The complex conjugate of a variable is shown by an asterix ($x^*$), while the transposition of a vector or matrix is given by a raised $T$, ($\mathbf{r}^T$). The Hermitian (conjugate transpose) of a vector or matrix is shown by the dagger symbol, ($\mathbf{U}^\dagger$). A covariance matrix based on information in the vector $\mathbf{r}$ is represented by the ($\mathbf{R}_r$) symbol. Throughout this document, explanations for variables are given in the text immediately following their first introduction in an equation.
Chapter 2

Stochastic Propagation and Localization

Review

This chapter is a review of the current literature concerning stochastic acoustic propagation and source localization using stochastic Matched Field Processing. This chapter and its associated Appendices are intended to be review and motivate study in Stochastic MFP. Those who are familiar with these topics can proceed to the next chapter, where stochastic detection and estimation are applied to angle-of-arrival estimation using a linear array.

2.1 Stochastic Propagation

There has been substantial interest in underwater acoustic propagation through random media. Attention has been focused on deriving analytical expressions which describe the Probability Density Function (PDF) of a received pressure field, given the PDF of certain random propagation parameters.

Due to the complexity of the acoustic propagation problem, the task of deriving such a pressure PDF is nontrivial. A derived distribution approach using the Parabolic Equation (PE) method is included in Appendix A, if only to illustrate the computational complexity involved with determining the PDF of the pressure field.

Flatté wrote an excellent review article which outlined past efforts to model Wave Propagation in Random Media (WPRM)[11]. The study of WPRM was recorded as early as the writings of Newton, pertaining to observation of celestial bodies. Newton noticed how the turbulent nature of the air causes stars to twinkle in the night sky. His solution to the problem was to observe stars from the top of a mountain, removing most of the atmospheric turbulence. More contemporary solutions have further removed the turbulent propagation medium from the equation, by taking observations from outside the atmosphere on the Hubble Space Telescope.
Later researchers sought to build models of the dynamic nature of the propagation medium. Understanding the effect of turbulence-induced perturbations of the propagation path was essential to countering its negative effects. WPRM models were first applied to electromagnetic wave propagation for astronomy, then modified for underwater acoustics. In the case of a nonrandom medium, the path of propagation through the atmosphere is assumed to be a straight line. With underwater acoustic propagation, one must take into account the non-constant speed of sound, especially as a function of depth.

With this in mind, several different approaches have been taken to derive analytical expressions for the Mutual Coherence Function (MCF), $\Gamma$, or the vertical covariance of the pressure field at a receiver array, as a function of the vertical covariance of the sound velocity profile (SVP) or (equivalently) the index of refraction. The approaches can be differentiated based on the underlying propagation model which is employed.

The first approaches took advantage of deep water environments which require no bottom interaction. The propagation of energy is assumed to be entirely through the water, with no reflections from the top of the ocean or the bottom sediment layers. Dashen, Flatté and Reynolds[12] developed an analytical expression to describe the MCF. They assumed the propagation of sound could be modeled through the use of geometric ray tracing. They characterized the variability of the propagation environment with two parameters: $\Phi$ and $\Lambda$. $\Phi$ is a strength parameter, representing the RMS phase fluctuation of a ray for a fixed ray path. $\Lambda$ characterizes the diffraction of the propagation medium. Using these two parameters, one could qualitatively characterize the propagation medium. For low values of $\Phi$ and $\Lambda$, the propagation medium is said to be unsaturated. Here, wave propagation is adequately described by geometric optics; one need not consider stochastic propagation to accurately model energy propagating between source and receiver. High values of $\Phi$ and $\Lambda$ characterize a saturated propagation medium. Here, a single geometric ray path fractures into several uncorrelated micro-rays, whose travel times are close to the original ray. One must take into account these fractures while modeling the acoustic pressure field. Between these two extremes is the partially saturated region, where micro rays are created by small scale fluctuations in the medium, but they are still correlated with one another by the large scale fluctuations. A companion paper [13] by the authors showed good experimental correlation between the predicted MCF and experimental measurements, given acoustic travel paths which were entirely refractive. Bates and Bates also verified the path-integral formulation in a later experiment[14]. The path-integral formulation has been used to predict stochastic propagation more recently by Colosi during the Acoustic Thermometry of the Ocean Climate (ATOC) experiment. Unfortunately, predictions of the pulse time spread were off by nearly two orders of magnitude[15]. This was due to the misapplication of Flatté’s narrowband theory to the wideband acoustic signals employed during ATOC.

Current interest has been in shallow water acoustics, where bottom interaction is present. The path integral method using rays assumes wholly refractive propagation, so it cannot be used for low frequency, shallow water propagation. Thus, the PE method proposed by TatarskiI was chosen for additional study. A summary of his derivation can be found in Appendix B. Uscinski, Macaskill, and their colleagues have performed considerable research in the area over the past 30 years. Figure 2-1 diagrams their major contributions. Their
work started by first incorporating an arbitrary mean sound velocity profile into the propagation equation[16]. Not satisfied with propagating only the first and second moments of the pressure field, the authors went on to derive analytical expressions for the fourth moment of the acoustic field intensity. This was used to approximate the overall PDF of the pressure field envelope. Their attention gradually became more focused on using stochastic propagation for acoustic source localization.

Tatarskii(1969) propagation of mean and second moments through random media

Tappert(1977) small angle PE wave equation applied to ocean acoustics

Macaskill and Uscinski(1981) Propagation of mean and second moments with arbitrary deterministic SVP

Macaskill and Ewart(1984) Comparison of analytical intensity second moments with experimental data

Uscinski(1982) Analytical derivation of fourth moment of Intensity function

Ewart and Percival(1986) Derived two parameter model for Intensity PDF

Ewart(1986) Compared analytical Intensity PDF to experimental data

Macaskill and Ewart(1994) Comparison of fourth moment Intensity with experimental data

Ritcey, Gordon and Ewart(1996) Complex pressure envelope expressed in terms of the fourth moment Intensity

Jackson and Ewart(1994) First application of fourth moment intensity propagation to MFP

Song, Ritcey and Ewart(1998) Range estimation from fourth moment intensity propagation

Figure 2-1: Roadmap of Stochastic Propagation by Uscinski, Macaskill, Ewart, and colleagues.

In 1984, Macaskill and Ewart[17] simulated wave propagation through a random medium. Using Monté-Carlo simulation techniques, they sought to verify previous analytical results on second moment estimation given by Uscinski, and wished to use the predicted higher order moments. Their interest was restricted to plane waves propagating in a stochastic medium the behavior of which obeyed Gaussian statistics. Later that year, Macaskill and Ewart published a journal article[18] which compared the measured and theoretical normalized moments of the intensity Probability Density Function. Several theoretical models were used: two from Furutsu[19, 20], the $K$ distribution, and a newer model developed by the authors. With the exception of the $K$ distribution, all models required the second and third moments of the propagating signal as input.

In a later paper, Ewart[21] compared the analytical expressions for the PDF of the wave field intensity with simulated results. He assumed plane wave propagation in a medium, whose randomness was described by a temporally and spatially stationary correlation function. In 1996 Macaskill and Ewart[22] again compared
experimental acoustic moment data to numerical evaluations of analytical fourth moment expressions of the complex pressure field envelope. Ritcey et al. [23] combined the intensity PDF results of Ewart with a simplified ocean model to obtain a joint PDF for the complex pressure field envelope.

Jackson and Ewart [24] later applied stochastic propagation to deep water Matched Field Processing. Here, internal waves were assumed to create random perturbations in the environment, necessitating the use of stochastic propagation techniques. The authors focused on perturbations of the index of refraction which were both vertically stationary and non-stationary. Using the analytical expressions derived by Uscinski and his colleagues, one could quickly calculate the covariance of the received pressure field if vertically stationary indices of refraction were assumed. The authors realized vertically stationary indices of refraction were not representative of realistic environmental perturbations; they suggested using Monté-Carlo methods to simulate acoustic propagation from source to receiver, estimating the resulting pressure covariance matrix.

Song et al. [25] applied Ewart's earlier PDF estimation work to source localization. Song measured a received acoustic pressure field, estimating its probability density function, and using Ewart's model for the PDF, solved for the range to the source.

In summary, Uscinski, Macaskill, and their colleagues sought to extend TatarskiI's original work on stochastic propagation through random media. Their objective was to find an analytical expression to describe the pressure field at a receiver array, given stochastic propagation parameters. Their work was based on the PE method of acoustic propagation, and could correctly calculate the effect of acoustic pressure interaction with the bottom.

Unfortunately, the authors' approach assumed a vertically stationary refractive index covariance matrix, something which is not representative of shallow water ocean acoustic environments. Because of this, their analytical expressions could not be used to process experimental data. Until such an analytical expression is derived, accurate stochastic propagation must be estimated through the use of Monté-Carlo simulations.

This section is only a sample of currently available stochastic propagation methods. Many other authors have taken different modeling approaches. For example, Baer, et al. [26] developed a PE based computer model for deep water simulations. A well documented normal-mode based approach, again for deep water, was proposed by Gorodetskaya et al. [27]

### 2.2 Constrained Matched Field Processing

Contemporary passive acoustic underwater source localization relies on accurate modeling of acoustic propagation from source to receiver. Matched Field Processing (MFP) is one localization method [1]. It simulates the pressure field from a candidate source location to an acoustic receiver, and correlates the simulated field with the actual received pressure field. The point of highest correlation between the simulated field and the received field yields the true location of the acoustic source.

Accurate modeling of acoustic propagation is essential to successful localization. Most acoustic models
treat the ocean as a variable waveguide, with energy reflected from the ocean surface and sediment or rock layers of the bottom. The acoustic pressure field is heavily affected by the speed of sound in the water column. Sound speed is influenced by temperature, pressure, and salinity of the water, all of which vary with depth and range.

From a practical standpoint it is difficult to know the exact speed of sound at all locations between the source and receiver. Measurements of the sound speed are taken at sea by a CTD (Conductivity, Temperature and Depth sensor), the outputs of which are fed into empirically derived formulas for determining the speed of sound. Such on-site measurements are costly and valid only for a limited geographic and temporal range.

It is assumed that one source of input error to the acoustic model is the speed of sound in the water column. Other errors can be introduced by supplying incorrect bottom sediment information, or using a model not designed for the particular propagation environment. It is important to process acoustic data in a way which minimizes the effects of environmental uncertainty.

### 2.2.1 MFP processors

The correlation step of MFP generally requires a comparison between the received and modeled acoustic field. A quadratic correlator is often used, with the modeled field compared to an estimated receiver covariance matrix, $\Gamma_r$.

$$ R_n(f) = \int_0^T w(t)q(t + \nu n T) e^{-j2\pi f t} dt \quad (2.1) $$
\[
\Gamma_r(f) = E[R(f)R^\dagger(f)] \approx \frac{1}{N} \sum_{n=1}^{N} R_n(f)R_n^\dagger(f),
\] (2.2)

where \( t \) is time, \( q(t) \) is the received signal at a single hydrophone, \( w(t) \) is a temporal weight vector (such as a Hanning window), \( v \) is an overlap parameter, \( 0 < v < 1 \), \( n \) is the current data snapshot number, and \( N \) is the total number of data snapshots used to create the covariance matrix. In the narrowband case, the covariance matrix is the expected value of the outer product of the received data vector. The covariance matrix expresses the phase relationships between elements in a receiver array. This correlation operation,

\[
y(f, r, z) = s^\dagger(f, r, z)\Gamma_r(f)s(f, r, z),
\] (2.3)

multiplies the covariance matrix with the simulated pressure field vector, \( s(r, z, f) \) to produce an ambiguity surface, \( y \). In reality, \( \Gamma_r \) is estimated by summing the outer products of windowed, Fourier-transformed data (see Equation 2.2). This processor is called the Conventional matched field processor. Although simple in its design, resulting ambiguity surfaces suffer from interference sidelobes, which can obscure the target. Throughout this document, the frequency \( (f) \) is assumed to be a constant, so it will be dropped from further notation.

### 2.2.2 Constraint processors

One way of reducing the sidelobes is to use an adaptive processor. This places the correlation operation in the context of a beamformer, with array weights, \( w \) instead of a simulated pressure field, \( s \),

\[
y(r, z) = w^\dagger(r, z)\Gamma_r w(r, z).
\] (2.4)

One scans through candidate depths \( (z) \) and ranges \( (r) \), applying the beamformer to each “look” direction. The beamformer output is arranged in an ambiguity surface; the maxima of the surface indicate target locations. The general idea behind a constraint processor is to minimize the beamformer output, \( y \), subject to a given constraint. Sidelobes can be reduced or eliminated by this approach, as they contribute energy to the total output. The constraint on the weight vector can be expressed as:

\[
Cw = c.
\] (2.5)

Using Lagrange multipliers to solve for this generalized constraint beamformer yields the weight vector[10]

\[
w = \Gamma_r^{-1}C^\dagger (C\Gamma_r^{-1}C^\dagger)^{-1}c,
\] (2.6)

with the beamformer output given by substituting Equation 2.6 into Equation 2.4, yielding
\[ y = c^\dagger (C \Gamma_r^{-1} C^\dagger)^{-1} c. \]  

(2.7)

One of the simplest constraints which can be applied is that of the minimum variance (MV) beamformer[28, 29]. It has only one constraint: the output of the beamformer in the “look” direction must be normalized at one. Mathematically, this can be expressed by substituting the Hermitianed replica vector, \( s^\dagger \), for \( C \), and setting \( c \) to 1. The resulting weight and output vectors are

\[
\begin{align*}
w_{MV} &= \Gamma_r^{-1} s (s^\dagger \Gamma_r^{-1} s) \\
y_{MV} &= (s^\dagger \Gamma_r^{-1} s)^{-1}
\end{align*}
\]  

(2.8) (2.9)

The single point constraint allows many degrees of freedom for the weight vector. This reduces unwanted sidelobes, providing a very sharp ambiguity surface free from interference. Unfortunately, this level of precision requires extremely accurate modeling of the replica vector, \( s \), as well as a well conditioned covariance matrix \( \Gamma_r \). For stationary sources, the rank of \( \Gamma_r \) can be improved by increasing the number of data snapshots used. Alternatively, diagonal loading or other form of white noise augmentation[30] can be used to artificially increase the covariance matrix rank.

Provided the covariance matrix is well conditioned, another side effect of the Maximum Likelihood Method (MLM) beamformer is an extremely narrow main lobe. While this is a feature which can be used to resolve closely spaced sources, one drawback is the requirement for a finely sampled grid of candidate source positions. Additionally, the accuracy of the position of this main lobe is determined by the environmental data supplied to the propagation model. The MLM beamformer is extremely sensitive to errors in the environmental model.

To increase the width of the main lobe in the MLM processor, while retaining the depressed sidelobes, one can modify the constraints of the MLM processor. Rather than employ a single constraint which causes the main lobe to have a unity gain at its peak, one can add constraints which force the shape of the main lobe to match that which would be generated by a conventional beamformer. These constraints, using data from candidate locations closest to the current candidate location, are arranged columnwise to form a constraint matrix,

\[
S(r, z) = [s(r, z)] s(r + \Delta r, z) s(r - \Delta r, z) s(r, z + \Delta z) s(r, z - \Delta z)].
\]  

(2.10)

then applied to the constraint equation,

\[
S^\dagger(r, z) w(r, z) = d(r, z)
\]  

(2.11)

The constraint vector,
\[ \mathbf{d}(r, z) = S'(r, z)s(r, z) \quad (2.12) \]

corresponds to the value of the main lobe response of a conventional beam, evaluated at one candidate cell away from the source position. This raises the number of constraints from 1 in the MLM processor, to 5 for the Multiple Constraint Method (MCM) processor. Solving for the weight vector in the same manner as the MLM processor, using Lagrange multipliers, one obtains a weight vector,

\[ \mathbf{w}_{mcm}(r, z) = \mathbf{F} r^{-1} \mathbf{S}_r(\mathbf{r}, z) (\mathbf{S}_r(\mathbf{r}, z)\mathbf{F}^{-1} \mathbf{S}_r(\mathbf{r}, z))^{-1} \mathbf{d}(r, z) \quad (2.13) \]

which can be substituted into Equation 2.3 to solve for the ambiguity surface,

\[ \gamma_{mcm} = \frac{\mathbf{d}^\dagger(r, z) (\mathbf{S}_r(\mathbf{r}, z)\mathbf{F}^{-1} \mathbf{S}_r(\mathbf{r}, z))^{-1} \mathbf{d}(r, z)}{\mathbf{d}(r, z) (\mathbf{S}_r(\mathbf{r}, z)\mathbf{S}_r(\mathbf{r}, z))^{-1} \mathbf{d}(r, z)}. \quad (2.14) \]

The Multiple Constraint processor represents an improvement over the MLM processor due to its wider main lobe\[31\]. This wider main lobe reduces the sensitivity of the processor to environmental mismatch, but only as a side effect; it is not designed for environmental robustness.

### 2.3 Stochastic Matched Field Processing

Errors in the replica model are more difficult to compensate. Two possible sources of error are using a model poorly suited for the environment, and supplying incorrect environmental model parameters to the propagation code. Examples of using poorly suited models include using a model suited for range independent calculations, when the environment is actually range-dependent (such as bathymetric variations); and using a model which cannot model shear wave propagation in an acoustic layer (such as ice, or certain types of ocean bottoms), when such environments dominate the acoustic propagation model. Environmental parameters which may be incorrectly supplied to the propagation model include speed of sound in the water column, incorrect bottom properties, and bathymetric errors. These errors are compounded when some environmental parameters are temporally dependent, such as the speed of sound in the water column.

Thus far, there have been two approaches to MFP using stochastic acoustic propagation. The first, termed “Minimum Variance, Environmental Perturbation Constraint (MV-EPC)” by Krolik, uses environmental variability as a constraint to derive an optimal weight vector. The second, termed “Optimum Uncertain Field Processing” by Richardson and Nolte, correlates a weighted set of multiple realizations of the random environment to determine the location of an acoustic source.
2.3.1 Environmental Perturbation Constraint

Krolik[32] developed a constrained beamformer which was tolerant to errors in the environmental model, by introducing perturbations into the model inputs, and processing the results based on the most dominant pressure field. Using his approach, the pressure field at the receiver is simulated with these perturbations, and the result is stacked columnwise in a pressure field matrix, $P$. A singular value decomposition of the matrix is taken, with only the first $J$ singular value/vectors kept,

$$P \simeq HA^\dagger.$$  

(2.15)

The value of $J$ is typically based on the singular values which make up $\Lambda$, where $\lambda_j \geq \zeta \lambda_1$ and $\zeta \ll 1$. (for example, $\zeta = 0.1$). This allows one to include a sufficient number of singular vectors to characterize the variability of the environment, while allowing enough free singular vectors, (assuming $J < N$, the number of receiver elements), to successfully null out sidelobes.

The constraint equation can be rewritten in terms of these singular value pairings. Substituting $P^\dagger$ for $C$ in Equation 2.5, one obtains

$$P^\dagger w = c.$$  

(2.16)

Substituting $P$ with its decomposition (Equation 2.15), and isolating $H$ yields

$$H^\dagger w = \Lambda^{-1} \Gamma^\dagger c$$  

(2.17)

The next step is to determine the appropriate value for $c$, the original constraint. Rather than using a constant, as with the MLM processor, or a spatial constraint used in Schmidt’s MCM[33] processor, Krolik maximized the overall gain of the beamformer output in the look direction; he selected the singular vector which corresponds to the maximum singular value of the pressure field matrix, $P$,

$$c = h_1,$$  

(2.18)

where $h_1$ is the first column of the left singular matrix, $H$. $h_1$ represents the normalized pressure field components which dominate the ensemble of pressure fields that make up $P$, the pressure field matrix. By selecting the dominant singular vector, the probability of a correct match with the actual received pressure field increases.

Substituting Equation 2.18 into the right side of Equation 2.17 yields $H^\dagger w = e_1$, where $e_1 = [1 \ 0 \ 0 \ \cdots \ 0]^T$. Solving for the weight vector and ambiguity surface (using Equations 2.6 and 2.7) gives

$$w_{MV-EPC} = \Gamma^{-1} (H^\dagger \Gamma^{-1} H)^{-1} e_1$$  

(2.19)
\[ y_{MV-EPC}(r, z) = e_1^* \left( H^H \Gamma_r^{-1} H \right)^{-1} e_1. \] (2.20)

This processor was modified and employed by several of Krolik's colleagues for specific applications[34]. A similar environmentally tolerant constraint approach was derived by Preisig[35, 36]. Both Krolik and Preisig assumed a random propagation environment when calculating the optimal weight, or replica vector to correlate against a covariance matrix of received data. Their criteria for calculating the optimal weight vector differed. Unfortunately, by limiting their processing to a single weight vector, both authors neglected to consider additional information present in other (possibly orthogonal) weight vectors.

### 2.3.2 Optimum Uncertain Field Processing (OUFP)

OUFP[37] assumes the source location can be represented by a probability density function, \( p(a|R) \), where \( a \) is a random parameter vector indicating the spatial location (range and depth) of an acoustic source. This can be expressed as an a posteriori probability using

\[
 p_{a|r}(a|R) = \frac{p_a(a)}{p_r(R)} p_{r|a}(R|A). \tag{2.21}
\]

Assuming a signal model

\[ r = s(a, \Psi) + n \tag{2.22} \]

with the source signal \( s \) parameterized by its spatial location \( a \), and environmental parameters of the propagation medium, \( \Psi \). Noise \( n \) is assumed Gaussian and uncorrelated. The complex received signal has a random component; its magnitude has a Rayleigh distribution, and phase is uniformly distributed from 0 to 2\( \pi \). Thus, the expression for the a posteriori PDF reduces to

\[
 p_{a|r}(a|R) = C p_a(a) \int_\Psi (E + 1)^{-1} \exp \left( -\frac{|R|^2}{E + 1} \right) p_{\Psi|a}(\Psi|a) d\Psi \tag{2.23}
\]

with \( C \) a normalization constant, and \( E \) and \( R \) given by

\[
 E = F s^H(a, \Psi) \Gamma_n^{-1} s(a, \Psi) \quad R = F s^H(a, \Psi) \Gamma_n^{-1} p \tag{2.24}
\]

where \( F \) is the average signal to noise ratio at the source, multiplied by the observation time. \( \Gamma_n \) is the noise spatial covariance matrix, and \( s(a, \Psi) \) is the predicted acoustic transfer function of a source at location \( a \) and propagation environment parameterized by \( \Psi \). \( p \) is the observed pressure field at the receiver, normalized by observation time.

What is interesting about the OUF processor is its use of multiple "replica vectors" or simulated pressure fields, \( s(a, \Psi) \) to determine the probability a source is present at location \( a \). By integrating across the range
of $\Psi$, the uncertain parameters of propagation, the processor can take into account all possible environmental variations present in the acoustic waveguide. This is in contrast to other MF processors, which seek to obtain a single optimal weight or replica vector to correlate against a received signal covariance matrix. OUFP takes an ensemble of possible replica vectors, weighs them according to their probability of occurrence in the given propagation medium, and sums the result to obtain a probability of localization for a given source location $a$. The correct source location would correspond to the value of $a$ where the probability of localization was maximized.

Shorey and his colleagues[38] realized the evaluation of the integral in Equation 2.23 would be computationally tedious, especially as the number of uncertain parameters in $\Psi$ increased. The authors sought to reduce computational complexity by first assuming the probability distribution of the random parameters was uniform. Such an assumption would be valid if only a limited amount of information were available about the environmental parameters (e.g., their extreme minima and maxima.) Another computational savings which the authors employed was to replace the integral by a summation,

$$\lambda(r, a) = \frac{1}{M} \sum_{i=1}^{M} \left[ E(a, \Psi_i) + 1 \right]^{-1} \exp \left[ -\frac{|R(r, a, \Psi_i)|^2}{E(a, \Psi_i) + 1} \right]$$

and to choose evaluation points, $\Psi_i$ randomly. This Monté-Carlo approach allowed one to obtain an accurate source localization estimate with far fewer realizations of $\Psi$. As one did not need to run through as many simulations of the acoustic propagation environment, significant computational savings were realized.

Shorey and Nolte[39] applied this technique to source localization in the presence of environmental uncertainties. They applied the OUFP processor to long range acoustic simulations which contained mesoscale eddies (cyclonic currents). The authors demonstrated through simulation the sensitivity of MFP to the presence of such eddies, as well as the quantities which parameterized the disturbances. The performance of more conventional methods of MFP; those which did not take into account the environmental variability of the eddies degraded, but the OUFP processor showed a remarkable ability to localize sources in the presence of such environmental uncertainties.

The authors extended[40] the narrowband OUFP processor to process multiple narrowband tones simultaneously. They assumed an acoustic source would radiate random signals which were uncorrelated across frequency. Thus, one could form a multitonal probability density function of the received signal by multiplying together the corresponding narrowband PDFs,

$$p(r|a, \Psi) = \prod_f p(r_f|a, \Psi).$$

The authors applied the multitonal OUFP processor to both simulations and data obtained from the Hudson Canyon [41] experiment, conducted off the coast of New Jersey, USA. This range independent shallow water localization exercise provided an excellent opportunity to verify the performance of the OUFP processor using real data. The authors found the multitonal OUFP processor had a significantly higher probability of correct
localization (PCL) than an equivalent multitonal conventional MF processor. This was to be expected, since
the OUF processor took environmental uncertainty into account, whereas the conventional processor assumes
a deterministic propagation environment.

In their zeal to reduce the computational complexity of the Monte-Carlo OUF processor, Harrison et
al.\[42\] proposed several steps to improve the OUF processor. The first was to approximate the exponential in
Equation 2.25 by a geometric series, retaining only the first term. The new estimator can be reduced to

$$p_{a|r}(a|R) \approx \sum_{i=1}^{M} |s^T(a, \Psi_i)R|^2. \quad (2.27)$$

The approximation of the exponential is valid when the perturbations produced by the random propagation
parameters $\Psi$ are small. Using this processor, one still assumed the distribution of random parameters was
uniform across their domains. Taking the expected value of Equation 2.27 with respect to the simulated
propagation transfer function, $s$, one obtains

$$p_{a|r}(a|R) \approx R^T \Gamma_s(a) R \quad \text{with} \quad \Gamma_s(a) \approx \frac{1}{M} \sum_{i=1}^{M} s(a, \Psi_i)s(a, \Psi_i)^T. \quad (2.28)$$

The signal covariance matrix $\Gamma_s(a)$ was constructed from the outer products of the simulated propagation
vectors, $s$. If the signal perturbations caused by the random parameters $\Psi$ were small, the rank of $\Gamma_s$ would
be low. Thus, the authors realized additional computational savings by decomposing the signal covariance
matrix

$$\Gamma_s = U_s \Sigma_s U_s^T \quad (2.29)$$

into a diagonal matrix of its singular values, $\Sigma_s$ and a unitary matrix of corresponding singular vectors, $U_s$.
Eliminating the lower order singular values results in a reduced rank covariance matrix, expressed as

$$\Gamma_s \simeq U_1 \Sigma_1 U_1^T. \quad (2.30)$$

where $\Sigma_1$ and $U_1$ represent the reduced rank singular values and vectors, respectively. Such a reduction
in rank allows a corresponding reduction in computational effort. Harrison called this revised processor
\textit{FASTMAP}, acknowledging the tradeoff between approximation of the a posteriori exponential with reduced
computational complexity.

Equation 2.28 can be thought of as a combination of exponentially weighted conventional ambiguity
surfaces. Each surface is a correlation between a modeled pressure vector and the received data vector. As
the modeled environmental parameters become closer to the actual propagation parameters, the result of
each correlation approaches unity. Accurate modeling of the continuum of propagation vectors requires a
large number of realizations to be included in the estimate of $p(a|r)$. Harrison\[43\] proposed weighting the
combination of vectors with an exponent greater than unity. This gave simulated pressure vectors which had a
correlation close to unity, meaning, the parameters used were more “correct,” more weight during summation, allowing for one to reduce the number of realizations, and thus, the computational complexity. The resulting estimator would take the form

\[ p_{a|R}(a|R) \approx \sum_{m=1}^{M} \left| s^\dagger(a, \Psi_i)R \right|^2 \]  

with \( p \) the power weight, generally taken to be greater than 10. As one increases \( p \) to infinity, the correlation vector \( s \) which most closely approximates the received data vector \( R \) dominates. The other terms of the summation can be neglected, obtaining the expression,

\[ p_{a|R}(a|R) = \arg \max_{\Psi_i} \left| s^\dagger(a, \Psi_i)R \right|^2. \]  

Harrison called Equation 2.32 the \( L_\infty \) estimator. One implements the estimator by simulating propagation from source position \( a \) to the receiver, using \( M \) different realizations of the environmental parameters \( \Psi \). The simulated pressure vector \( s(a, \Psi_i) \) with the best correlation to the received vector \( R \) is chosen. This procedure is repeated for each trial source location, \( a \), and the acoustic source location corresponded to where \( p_{a|R}(a|R) \) is maximized. Harrison went on to show analytically that localization error was a function of the relative phases between elements of \( R \) and \( s \), and that such error could be predicted by comparing simulated and received wavenumber gradients.

### 2.4 Summary

This chapter has sought to review the current literature present in stochastic propagation and stochastic MFP. Currently, for realistic ocean environments, no analytical methods exist which express the vertical covariance of the complex pressure field as a function of the vertical covariance of the index of refraction, i.e. sound velocity profile. One is instead limited to Monté-Carlo estimation methods.

For Stochastic MFP, both the MV-EPC and OUFP-derived methods of stochastic source localization have advantages, and both have shown good results in the face of environmental uncertainty. Unfortunately, these methods have some drawbacks. MV-EPC seeks to find the single, best weight vector to describe stochastic propagation. Its optimum criteria may not be the best in situations where the environmental variability is high, or its characterization is flawed. OUFP attempts to redress this drawback by using multiple realizations of random signal vectors, weighting and summing the result. In principle, this allows one to successfully localize targets in areas of high uncertainty. However, its implementation to date has been hindered by a uniform restriction on the PDF of the environmental parameters.
Chapter 3

Random Signal Detection and Estimation Using a One-Dimensional Linearly-Spaced Array

In many scenarios involving underwater acoustics, one would like to determine whether a random signal is present or absent in received data, as well as its incident Angle of Arrival (AOA). This chapter applies detection of random signals in noise to underwater acoustics, specifically, to the presence or absence of random acoustic signals incident on a one dimensional, linearly spaced array. Detection performance is evaluated for both white and colored Gaussian noise. Then, detection theory is extended to estimation theory, and a Maximum-Likelihood estimator is used to find the incident Angle of Arrival for a random signal.

One is motivated to study the AOA problem because it allows one to examine the effect of stochastic signal detection and parameter estimation. Bearing estimation is one of the simpler problems in underwater acoustics; general performance metrics shown in this chapter can be applied to more complicated acoustic processing, including range and depth estimation using Matched Field Processing. The use of whitening filter to null out a random, interfering signal is shown in this chapter; its application can also be extended to MFP.

3.1 Random Signal Model

Assume one is given a zero mean, plane wave \( s(t) \) propagating through space in the direction of unit vector \( \mathbf{a} \). This wave is received by an acoustic element located in space at \( z \). The received signal can be characterized in several different ways. If \( s(t) \) is a zero-mean Gaussian random process, its statistics can be described completely by the space-time correlation function,

\[
R_s(t_1, t_2; z_1, z_2) = E[s(t_1, z_1)s(t_2, z_2)^*], \quad (3.1)
\]
where \( t_1 \) and \( t_2 \) are different points in time, and \( z_1 \) and \( z_2 \) correspond to the spatial locations of two different receiver elements. If the process is Wide Sense Stationary (WSS) in time, one can index the correlation function by the differences in time \( \tau = t_1 - t_2 \), instead of discrete measurement points. If the process is spatially WSS, the spatial indexing can be reduced to \( \Delta z = z_1 - z_2 \). Such a temporally and spatially WSS process would be described by its covariance function, \( R_s(\tau, \Delta z) \).

The Fourier Transform of the space-time correlation function yields the spectral covariance function,

\[
S_s(\omega, \Delta z) = \int_{-\infty}^{\infty} R_s(\tau, \Delta z) e^{-j\omega \tau} d\tau, \tag{3.2}
\]

which maps time to frequency. As the Fourier Transform represents only a change of basis, a Gaussian random process which is completely described by \( R_s \) will be completely characterized by \( S_s \) as well.

Taking the Fourier transform of the Spectral Covariance Function yields the Frequency Wavenumber Function,

\[
P_s(\omega, k) = \int_{-\infty}^{\infty} S_s(\omega, \Delta z) e^{-jk\Delta z} d(\Delta z), \tag{3.3}
\]

which describes the received signal in the wavenumber, rather than the spatial domain. This allows one to consider the effect of array beam patterns which are not present in the spatial domain. It also allows one to consider the effect of both real (when the wavenumber is real) and evanescent (when the wavenumber is imaginary) signals. As with the Space-Time Correlation Function, the Fourier Transform is only a change in basis. Thus, a stationary Gaussian process can be described completely by its Frequency Wavenumber Function. Figure 3-2 summarizes the relationships between statistical characterizations of Gaussian random processes discussed here.

One can design a random signal \( s \) by specifying its Frequency Wavenumber Function, \( P_s(\omega, k) \). Consider,
Space-Time Correlation Function
\[ R_s(\tau, \Delta z) \]

Spectral Covariance Function
\[ S_s(\omega, \Delta z) \]

Wavenumber-Time Correlation Function
\[ F_s(\tau, k) \]

Frequency Wavenumber Function
\[ P_s(\omega, k) \]

Figure 3-2: Relationships between Space-Time Correlation Function, Spectral Covariance Function, and Frequency-Wavenumber Function.

\[ P_s(\omega, k; \Delta k) = \frac{2\rho \Delta k}{(\Delta k)^2 + (k - k_0)^2} \] (3.4)

with \( \omega \) and \( k \) specifying the frequency and wavenumber, and \( \Delta k \) the 3 dB beamwidth of the signal on the \( k \) axis. Figure 3-3 shows \( P_s(\omega, k) \) for several values of \( \Delta k \), with \( k_0 = 0 \) and \( \rho = 1 \). Note the distribution of the signal about \( k_0 \) is dependent on \( \Delta k \); one can vary the angle of arrival spread by changing \( \Delta k \).

It is useful to attach a physical significance to \( \Delta k \). The performance of any detector can be related to the angular resolution of the receiving array. Linear arrays with equal element spacing and uniform weighting have a wavenumber resolution of approximately the 3 dB beamwidth of the main response lobe. One can use the wavenumber response function of a conventional, uniformly weighted array to solve for the 3 dB beamwidth, \( \Delta k_c \),

\[ W(\omega, k_c) = \frac{\text{sinc} \left( \frac{\Delta k_c L}{2} \right)}{\text{sinc} \left( \frac{\Delta k_c \Delta z}{2} \right)} = \frac{1}{2} \] (3.5)

with \( L \) the length of the array, and \( \Delta z \) its inter-element spacing. Normalizing \( \Delta k \) in Equation 3.4 by the 3 dB...
beamwidth of $\mathcal{W}(\omega, k)$ allows one to specify a signal spread, $\Delta k = \Delta k_s \times \Delta k_c$, as a function of the array resolution,

$$P_s(\omega, k; \Delta k_s, \Delta k_c) = \frac{2\rho \Delta k_s \Delta k_c}{(\Delta k_s \Delta k_c)^2 + (k - k_0)^2}.$$  \hspace{1cm} (3.6)

Here, $\Delta k_c$ is the 3 dB beamwidth, with units of $m^{-1}$, and $\Delta k_s$ is a dimensionless quantity which specifies the relative wavenumber spread of the random signal.

Given the Frequency Wavenumber function for a signal, one can analytically derive an expression for its Spectral Covariance, simply by taking its inverse Fourier transform. The Spectral Covariance of Equation 3.6 can be evaluated,

$$S_s(\omega, \Delta z) = \int_{-\infty}^{\infty} P(\omega, k) e^{-j k \Delta z} dk = \rho e^{j k_0 \Delta z} \exp[-\Delta k_s \Delta k_c |\Delta z|],$$ \hspace{1cm} (3.7)

and converted to a signal covariance matrix by substituting $\Delta z = z_i - z_j$,

$$[\Gamma_s]_{ij} = \rho e^{j k_0 (z_i - z_j)} \exp[-\Delta k_s \Delta k_c |z_i - z_j|]$$ \hspace{1cm} (3.8)
One needs to know only a Gaussian random process' mean and covariance to determine its probability density function (PDF). Equation 3.8 gives the covariance matrix for the zero mean signal under study. As the signal is complex Gaussian, its PDF is

$$p_s(S) = \frac{1}{\pi^N |\Gamma_s|^2} \exp \left[-S^1\Gamma_s^{-1}S\right].$$

(3.9)

### 3.2 Signal Dimensionality

Decomposition of the Hermitian matrix $\Gamma_s$ yields a unitary matrix of orthonormal singular vectors $U_s$ and a diagonal matrix of singular values, $\Sigma_s$, whose entries range from $\sigma^2_{s_1}$ to $\sigma^2_{s_N}$. For a signal with no angular spread ($\Delta k_s = 0$), the covariance matrix $\Gamma_s$ reduces to a rank of 1. As $\Delta k_s$ increases, the source signal spreads in $k$, increasing the rank of $\Gamma_s$.

Figure 3-4 illustrates the distribution across singular values for a 30 element linear array, with $\lambda/2$ separation between elements. The top plot shows the degenerate case, with $\Delta k_s = 0$. All energy of $\Gamma_s$ is contained in the first singular vector, whose value is $N = 30$, the number of elements in the array. As $\Delta k$ increases, the signal energy spreads across eigenvalues, while the total energy, $|\Gamma_s| = \text{Tr}[\Sigma_s] = \sum_{i=1}^{N} \sigma^2_{s_i} = N$, remains constant. Figure 3-5 illustrates the eigenvalue distribution (in dB) over a range of $\rho$, while Figure 3-6 shows the eigenvalue distribution over a range of $\Delta k_s$. The sum of the eigenvalues for any given $\Delta k$ is $N \rho^2$, the number of elements in the array, multiplied by the square of the signal strength, $\rho$.

One can also quantify signal dimensionality by examining the singular values of the signal covariance matrix. Summation of the squared singular values, normalized by their sum squared,

$$DOF = \frac{\left[\sum_{i=1}^{N} \sigma^2_{s_i}\right]^2}{\sum_{i=1}^{N} \left[\sigma^2_{s_i}\right]^2}$$

(3.10)

yields the number of Degrees of Freedom (DOF), as derived by Brillinger[44]. Figure 3.2 plots the DOF as a function of signal spread, $\Delta k_s$. The DOF is not affected by signal strength, $\rho$, or incident wavenumber, $k_0$. When the signal spread is narrow, with $\Delta k_s$ at -20 dB, the DOF is 1, equivalent to a Rank-1 signal covariance matrix. As $\Delta k_s$ increases, the DOF rises, especially as the signal spread exceeds the main lobe width, at $\Delta k_s = 0$ dB.
Figure 3-4: Singular Values of $\Gamma_s$.

Top: Degenerate case, when $\Delta k_s = 0$. Only the first singular value is non-zero. Bottom: Random Signal Example, with the random signal spread equal to the array resolution, $\Delta k_s = 1.0$. 
Figure 3-5: Singular Values of $\Gamma_s$, sweeping through $\rho$. 
Figure 3-6: Singular Values of $\Gamma_s$, sweeping through $\Delta k_s$. 
Figure 3-7: Degrees of Freedom for $\Gamma_s$, sweeping through $\rho$ and $\Delta k_s$.

As one increases the signal spread ($\Delta k_s$), the number of degrees of freedom occupied by the signal increases substantially.
3.3 Detection of Random Signals in Noise

Detection in noise requires apriori knowledge of the statistics of the signal. The Gaussian random signal in Equation 3.8, incident on a linear acoustic array has four properties; its frequency, $(\omega)$; incident wavenumber, $k_0$; wavenumber spread, $\Delta k_z$; and signal amplitude, $\rho$. Attention is restricted to narrowband (single frequency) Gaussian random signals. Establishing the presence of such a signal in uncorrelated Gaussian noise is straightforward. Drawing from Van Trees[7, 8] and the review of Detection and Estimation theory in Appendix C, one can construct a binary hypothesis test,

$$ r = s + n \text{ if signal } s \text{ is present (Hypothesis } H_1) $$

$$ r = n \text{ if } s \text{ is absent (Hypothesis } H_0) $$

and establish a mathematical operation on the $N$ dimensional vector $r$ to reduce the problem to a one dimensional detection statistic, $l$. Equation 3.8 and its associated PDF (Equation 3.9) completely characterize the random signal under study. Its PDF can be inserted into a Likelihood Ratio Test (LRT) to enable one to determine if the random signal $s$ is present,

$$ l = \ln \Lambda (R') = \sum_{i=1}^{N} \left( \frac{\sigma^2}{\sigma^2_n} \right) \frac{r'_i r'_i}{\sigma^2_n + \sigma^2_i} - \sum_{i=1}^{N} \ln \left( 1 + \frac{\sigma^2_n}{\sigma^2_i} \right) \in H_0 \ln \gamma = \gamma^* \tag{3.11} $$

where $R' = U_s^\dagger R$, with $R$ the received signal vector, and $U_s$ the matrix of the singular vectors of $\Gamma_s$. $r_i$ is the $i$th element of $R'$, and $\sigma^2_i$ is the $i$th singular value of $\Gamma_s$. Multiplication of the received vector by the unitary matrix $U_s$ represents a change of basis to that of the anticipated received signal. Uncorrelated Gaussian random noise is represented by its variance, $\sigma^2_n$.

3.4 Random Detector Performance Metrics

Consider an example which uses the signal represented in the bottom plot of Figure 3-4, and a background noise level of 0 dB. The first four singular values are above the noise floor, but all thirty contribute to constructing the detection statistic, $l$. One can derive and expression for the PDF of $l$, $p_l|H_0(l|H_0)$, given Hypothesis $H_0$, or $p_l|H_1(l|H_1)$, given Hypothesis $H_1$. Equation 3.11 suggests $l$ will have a $\chi^2$ distribution, due to the presence of the squared Gaussian variable, $|r'_i|^2$. As one increases the number of sensors, $N$, the distribution of $l$ would approach Gaussian, due to the Central Limit Theorem. An analytical expression for $p_l(L)$ is derived in Appendix C, Section C.12. One can also use numerical simulations are used to evaluate $p_l(L)$.

A histogram plot of $l$ under both Hypothesis is shown in Figure 3-8. Using these parameters, the PDFs of both hypothesis overlap. Despite this, one can still choose a threshold to minimize detection error. As indicated in Appendix C, neither PDF from $H_0$ nor $H_1$ are Gaussian, but that of $H_0$ approaches Gaussian through the Central Limit Theorem.
Performance of a detector can be expressed using two metrics: the output signal-to-noise ratio, $d^2$, and the Receiver Operating Characteristic (ROC) Curve. The output SNR is a measure of the normalized distance between the means of the detection statistic PDFs, $P_{l|H_0}(L|H_0)$ and $P_{l|H_1}(L|H_1)$.

$$d^2 = \frac{(E[l|H_1] - E[l|H_0])^2}{\sigma_{l|H_0}^2}.$$  \hspace{1cm} (3.12)

Figure 3-9 illustrates the Output SNR for the given random signal. The signal strength $\rho$ and spread $\Delta k_s$ parameters are varied from -20 to +20 dB, and the Output SNR is shown as a series of contours. One can infer from the plot that for a given $\rho$, output SNR decreases as $\Delta k_s$ increases, and for a given $\Delta k_s$, output SNR increases with signal strength, $\rho$. The ambient noise level is assumed to be $\sigma_n^2 = 0$ dB. The values of $d^2$ are independent of $k_0$, the incident wavenumber.

The variable $\gamma_s$ in Equation 3.11 is the detection threshold. If $l$ is less than $\gamma_s$, then Hypothesis $H_0$ is declared true and the signal said to be absent. Conversely, if $l$ is greater than $\gamma_s$, Hypothesis $H_1$ is taken to be true and the signal is assumed present. There may be cases where the detector fails; this is especially true when the PDFs of $l$ for the two Hypotheses overlap, as in Figure 3-8. The two error types are: (1) False Alarm, when the output of the detector is above $\gamma_s$, but there is no signal present, and (2) Missed Detection, when the detector output $l < \gamma$, but the signal is actually present. These errors can be estimated for a given detector, and stated in terms of their probabilities: $P_F$, the probability of a False Alarm, and $P_M$, the probability of a missed detection. These metrics are defined by integrating the PDFs of the detection statistics,
Deflection Coefficient \( (d^2) \), in dB

Figure 3-9: Output SNR, \( 10 \log_{10} d^2 \), with respect to signal strength \( (\rho) \) and spread \( (\Delta k_s) \). Output SNR rises with signal strength \( (\rho) \), but decreases as the signal spread \( (\Delta k_s) \)

\[
P_F = \int_{-\infty}^{\gamma^*} p_{|H_0}(L|H_0) dL \quad P_M = \int_{\gamma}^{\infty} p_{|H_1}(L|H_1) dL
\]  \hspace{1cm} (3.13)

Figure 3-10 illustrates \( P_F \) and \( P_M \) when both \( \rho \) and \( \Delta k_s \) are 0 dB. One can observe that \( P_F \) decreases as \( \gamma^* \) increases, but at the expense of an increased \( P_M \). These plots were generated from Monté-Carlo simulations of the detection statistic, \( l \). If the simulated signal were real, one could employ the Chernoff bound (see Appendix C, as well as Shapiro[45]) to obtain an analytical approximation for the curves.

Both \( P_F \) and \( P_M \) can be combined into one plot to show the Receiver Operating Characteristic (ROC) curve of the detector. Figures 3-11 and 3-12 illustrate this, with \( P_F \) plotted on the horizontal axis, and \( P_D \), the Probability of Detection \( (1 - P_M) \) on the vertical axis. These curves show the trade-offs of setting a particular threshold on the detector. One can force the detector to have a particular \( P_D \) value, but is then constrained to a corresponding value of \( P_F \). Ideally, one would like the curve to extend as close to the upper left corner as possible.

Figure 3-11 illustrates the ROC when \( \Delta k_s \) is held constant at 0 dB. Background noise is also held constant at \( \sigma_n^2 = 0 \) dB. One can observe that increased signal strength \( (\rho) \) improves the performance of the detector. Figure 3-12 shows ROC curves when \( \rho \) is held constant at \(-10 \) dB. Recall the sum of the singular
An appropriate threshold value which would minimize both $P_F$ and $P_M$ would be at the point where the two curves cross, at $\gamma = 0.5$.

vectors is constant, regardless of $\Delta k_s$. Increasing the signal spread has a slightly positive effect on detector performance, with other parameters held constant.
Figure 3-11: Receiver Operating Characteristic curve for varying $\rho$, when $\Delta k_s = 0$ dB
As one increases signal strength ($\rho$), the performance of the detector improves.
Figure 3-12: Receiver Operating Characteristic curve for varying $\Delta k_s$, when $\rho = -10$ dB. Detector performance is only marginally improved through a decrease in the signal spread.
3.5 Signal Detection in the Presence of Colored Noise

Noise is defined as any unwanted signal which interferes with detection. One cannot depend on the absence of noise when formulating a signal detection problem; instead, signal detectors must be designed to function in the presence of noise. Thus far, only uncorrelated Gaussian noise has been assumed. In the context of a linear underwater array, this type of noise could be from sensor electronics noise, flow noise around the sensor, or very widely distributed sources. Conversely, other sources of noise can be spatially concentrated; these cause the noise covariance matrix to contain off-diagonal terms. Strong interferers can mask the presence of the signal one is trying to detect. One must formulate a signal detector to deal with the presence of correlated noise.

In Appendix C, one approach for signal detection in colored Gaussian noise is considered: using a whitening filter to rotate the basis of the colored noise, making it uncorrelated. One needs only to have characterized the colored noise by its covariance matrix, $\Gamma_{bn}$ in order to build a whitening filter,

$$H_w = U_b \Sigma_{bn}^{-1/2}$$

where $U_b$ is the unitary singular vector matrix and $\Sigma_{bn}^{-1/2}$ is the diagonal singular value matrix of the noise covariance matrix. One would need to apply the whitening filter to both the received signal and the signal covariance matrix,

$$r_w = H_w^T r \quad \Gamma_{aw} = H_w^T \Gamma_a H_w,$$

and use the results in the random signal detector (Equation 3.11). By converting the correlated noise to uncorrelated noise, one can use the previously derived detection methods and performance metrics.

Consider correlated noise from a random interferer, with the same frequency wavenumber function as the random signal in Equation 3.6, but fixing the normalized incident wavenumber at $k_{0i} = 0.5$, its signal spread parameter $\Delta k_s$ to be 0 dB, and its strength $\rho_i = 10$ dB. One can then calculate the Output SNR for a random detector, in the presence of both the interferer and 0 dB of ambient background noise.

Figure 3-13 shows the output SNR (in dB) plotted as a function of incident wavenumber, $k$, and signal strength, $\rho$. The signal spread, $\Delta k_s$ is held constant at 0 dB. The parameters of the interferer are fixed. The numerical results show the output of the signal whitening process. When $\rho$ is held constant, the output SNR dips near the spatial location of the interferer. This trend is particularly acute when the signal strength $\rho$ falls below 0 dB. Figure 3-14 shows the Output SNR as a function of signal spread, $\Delta k_s$, and normalized wavenumber, $k$. Again, the parameters of the interference signal are held constant, as is the desired signal amplitude, $\rho = 0$ dB. The output SNR shows little variability with respect to signal spread, but exhibits a slight improvement near the spatial location of the interference signal.

This section has illustrated the performance metrics of a linearly spaced one dimensional underwater acoustics array in the detection of incident random signals. A random signal model was provided, which furnished signal statistics to the detector. ROC curves illustrated the output SNR and detection thresholds.
Required to achieve low error rates. Generally, spatially distributed signals had better detector performance than spatially focused signals. Detection in the presence of colored noise was shown to be difficult, especially when the interference noise was spatially distributed.

The next section extends the signal detection problem to one of parameter estimation. In this section, the objective was only to determine if a signal were present or absent at a given bearing. The next section outlines the steps needed to perform detection when the incident angle is not known.
Figure 3-14: Output SNR of random signal detector with interferer, with respect to $\Delta k_a$ and $k$. An interferer with both strength and signal spread of 0 dB is present at $k = 0.5$. Output SNR is reduced in the presence of the interferer, regardless of the spread of the desired signal ($\Delta k_a$).
3.6 Incident Wavenumber Estimation

Previous sections of this chapter were devoted entirely towards signal detection: the determination if a signal were present or absent in both white and colored Gaussian noise. Once a signal has been detected, the next task is to estimate the parameters of a signal. Attention here is focused on the deterministic incident wavenumber of the signal. Assuming plane-wave propagation using a linear array, one can estimate the Angle of Arrival (AOA) of the incoming signal (See Figure 3-1).

Several different nonrandom parameter estimators exist. The Maximum Likelihood (ML) estimator is an extension of the random signal detector shown in the last section. Recall from Appendix C the zero mean random signal log-likelihood statistic from complex data,

\[ l(a) = \sum_{i=1}^{N} \left( \frac{\sigma_{n}^2(a)}{\sigma_n^2} \right) \frac{u_{a_i}(a) \hat{\Gamma}_r u_{a_i}(a)}{\sigma_n^2(a) + \sigma_n^2} - \sum_{i=1}^{N} \ln \left( 1 + \frac{\sigma_{n}^2(a)}{\sigma_n^2} \right), \]  

(3.16)

where \( \sigma_n^2 \) is the variance of the uncorrelated Gaussian noise, \( u_{a_i}(a) \) is the \( i^{th} \) singular vector and \( \sigma_{n}^2 \) is the \( i^{th} \) singular value of the signal covariance matrix \( \Gamma_{a}(a) \). The estimated received signal covariance matrix \( \hat{\Gamma}_r \) is a function of the received signal, \( \hat{\Gamma}_r = \mathcal{E}[RR^t] \). The calligraphic \( \mathcal{E} \) indicates the average is over multiple snapshots of the received signal. One evaluates this expression across the domain of \( a \), the nonrandom parameter, and chooses the value of \( a \) which maximizes \( l \).

For AOA estimation, the parameter of interest is the incident wavenumber, \( k_0 \), which can range from \(-2\pi/\lambda\) to \(2\pi/\lambda\). One can estimate the value of \( k_0 \) by selecting the maximum of the normalized detection statistic, \( l(k) \). To illustrate this, Figures 3-15 and 3-16 plot the normalized detection statistic, \( l(k) \),

\[ l(k) = \frac{\left[ \prod_{i=1}^{N} \left( 1 + \frac{\sigma_{n}^2(k)}{\sigma_n^2} \right) \right]^{-1} \exp \left[ \sum_{i=1}^{N} \frac{\sigma_{n}^2(k)}{\sigma_n^2} \frac{u_{a_i}(k) \hat{\Gamma}_r u_{a_i}(k)}{\sigma_n^2(k) + \sigma_n^2} \right]}{\int_{-2\pi/\lambda}^{+2\pi/\lambda} \left[ \prod_{i=1}^{N} \left( 1 + \frac{\sigma_{n}^2(k)}{\sigma_n^2} \right) \right]^{-1} \exp \left[ \sum_{i=1}^{N} \frac{\sigma_{n}^2(k)}{\sigma_n^2} \frac{u_{a_i}(k) \hat{\Gamma}_r u_{a_i}(k)}{\sigma_n^2(k) + \sigma_n^2} \right] dk}. \]  

(3.17)

for various values of signal strength, \( \rho \), and signal wavenumber spread, \( \Delta k_a \). In this simulation, a 30 element line array with inter-element spacing of \( \lambda/2 \) was assumed. A random source was synthesized at \( k = 0 \), with a background noise variance of \( \sigma_n^2 = 0 \) dB. As the wavenumber spreads, \( \Delta k_a \) increases, as does the variance of \( l(k) \). Consequently, \( l(k) \) flattens out. This corresponds to the spatial distribution of the signal occupying a larger and larger angular sector, until it fills the physical wavenumber space. A similar result occurred as the signal strength \( \rho \) decreases; \( l(k) \) flattens out. Assuming the noise variance is \( \sigma_n^2 = 0 \) dB, the detection function is nearly flat at a strength level of \( \rho = -20 \) dB, but obtains a sharp peak at the incident wavenumber as \( \rho \) rises.
Figure 3-15: Normalized detection statistic, \( I(k) \), with \( \rho \) varied.
As one increases signal strength (\( \rho \)), the peak of \( I(k) \) becomes sharper, making estimation of \( k_0 \) easier. One assumes the receiver covariance matrix estimate, \( \hat{\Gamma}_r \), is perfectly known.
Figure 3-16: Detection Function $l(k)$, with $\Delta k_s$ varied. As one increases signal spread ($\Delta k_s$), the peak of $l(k)$ flattens out, making estimation of $k_0$ more difficult.
3.6.1 Estimator Performance Metrics

The quality of the AOA estimation can be described in terms of its accuracy and precision. The accuracy of the estimate can be measured in terms of its root mean square error – how closely the estimated incident wavenumber \( \hat{k}_0 \) is to the true value \( k \). The estimate of the incident wavenumber corresponds to the peak of the detection function \( f(k) \). This is shown by example in Figures 3-15 and 3-16, where the peak of the detection statistic corresponds to the true value of incident wavenumber at \( k = 0 \).

The precision of an estimate is measured in terms of its variance. The smaller the estimator variance, the higher its quality. Provided the mean error of the estimate is zero, and the input signal to noise ratio is high enough, then at its asymptotic limit, the ML estimate can satisfy the Cramér-Rao Lower Bounds for nonrandom parameter estimation,

\[
\sigma^2_{k_0}(k_0) = \left\{ \nabla \Gamma^{-1}(k_0) \frac{\partial \Gamma_r(k_0)}{\partial k_0} \nabla \Gamma^{-1}(k_0) \frac{\partial \Gamma_r(k_0)}{\partial k_0} \right\}^{-1} \tag{3.18}
\]

provided the receiver covariance matrix \( \Gamma_r(k_0) \) is known. The estimator variance, \( \sigma^2_{k_0} \) is a function of the spatial angle of the signal source, \( \hat{k}_0 \).

Figure 3-17: Contour plot (in dB) of Cramér-Rao lower bound on variance of incident angle estimate. The minimum variance of \( \hat{k} \) decreases as signal strength \( (\rho) \) increases, and decreases when signal spread \( (\Delta k_s) \) increases.
Figure 3-17 plots the Cramér-Rao lower bound on the variance of the estimate of $k_0$. The bound is shown (in dB) with respect to the signal spread parameter, $\Delta k_s$, and the signal strength parameter, $\rho$. Ambient background noise is held constant at $\sigma_n^2 = 0$ dB. One can observe the minimum variance rises with $\Delta k_s$ when $\rho$ is held constant, indicating the quality of the estimate decreases as signal spread increases. As one increases $\rho$, while holding $\Delta k_s$ constant, the minimum variance decreases. Not surprisingly, the quality of the estimate improves as one increases the input Signal to Noise Ratio (SNR), reflected by the signal strength $\rho$. Figures 3-18 and 3-19 illustrate the same information, both with respect to the incident wavenumber, $k_0$.

![Cramer-Rao Lower Bound on Variance of $k_0$ with no interferer](image)

**Figure 3-18**: Contour plot (in dB) of Cramér-Rao lower bound on variance of incident angle estimate, plotted with respect to signal strength ($\rho$)

This illustrates how the minimum variance decreases as signal strength ($\rho$) increases.
Figure 3-19: Contour plot (in dB) of Cramér-Rao lower bound on variance of incident angle estimate, plotted with respect to signal strength.

This illustrates how the minimum variance increases with signal spread, $\Delta k_w$. 
3.6.2 Incident Wavenumber Estimation in the Presence of an Interferer

![Graph showing the normalized detection function $l(k)$ with source $k_0 = 0$, signal $A_k = 0$ dB, interference strength $p_i = 10$ dB, and interferer $A_{k_i} = 0$ dB.]

Figure 3-20: Normalized Detection Function, $l(k)$, with interferer present and desired signal strength ($\rho$) varied.

Greater interference strength ($\rho_i = 10$ dB) causes it to dominate the detection function.

The behavior of the estimator can also be demonstrated when an interferer is present. Figures 3-20 and 3-21 plot the normalized detection function $l(k)$, showing the effect of an interferer. One can see that an unwanted signal can corrupt the estimate of the true signal's spatial location; the peak of $l(k)$ is shifted to the wavenumber of the interferer, at 0.5. The effects of the interferer are most detrimental when its strength $\rho$ is large. If the spatial location, spread and strength of the interferer are known in advance, a whitening filter can be applied to the received signal. This will have the effect of nulling out the interference signal.
Figure 3-21: Normalized Detection Function, \( l(k) \), with interferer present and desired signal spread \( (\Delta k) \) varied.

A narrow signal spread \( (\Delta k = -20\, \text{dB}) \) allows both the desired signal and interferer to be shown by \( l(k) \).

As \( \Delta k \) increases, the stronger interferer dominates the detection function.
Figure 3-22: Normalized Detection Function, $l(k)$, with interferer present and interference signal spread ($\Delta k_s$) varied.

In this scenario, interference signal spread has little effect on the detection function, $l(k)$.
Figures 3-23, 3-24 and 3-25 show the normalized detection function $l(k)$ after application of the whitening filter. Most effects of the interference signal have been removed from the detection function, allowing an accurate estimation of the spatial location of the incoming signal. A notable exception is when the original signal strength is low, $\rho = -20$ dB: although the whitening filter removes the influence of the interference signal, there is insufficient information remaining for an accurate estimate of $k_0$, and $l(k)$ is flat. Thus, in most cases, whitening can preserve the correct answer of the original estimator. Unfortunately, the presence of the interferer will still have a negative effect on the variance of $\hat{k}_0$.

Figures 3-26 and 3-27 illustrate the CRLB when a discrete interferer is placed at a normalized wavenumber of $k_{0i} = 0.5$. This interferer has the same covariance structure as is given in Equation 3.8, with $\rho = 10$ dB, and $\Delta k_s = 0$ dB. Figure 3-26 assumes a desired signal with spread held constant at $\Delta k_s = 0$ dB, while Figure 3-27 assumes a desired signal with strength held constant, at $\rho = 0$ dB. Compare Figures 3-18 and 3-19 with Figures 3-26 and 3-27. The first set shows the CRLB when no interferer is present; one can see the bounds are constant with respect to bearing. The second set shows a $k$ dependence on the CRLB when the interferer is inserted. Not surprisingly, the variance of the estimate increases near the location of the
interferer.

3.7 Conclusion

The purpose of this chapter was to illustrate signal detection and localization in a straightforward scenario: angle of arrival estimation using a linear array. Examples were given for estimation in the absence and presence of colored Gaussian noise, showing the effects of signal strength and spatial spread on the estimator variance.

Several major concepts were covered in this chapter. First, random signal detection was demonstrated using a linearly spaced, 30 element array. The number of degrees of freedom needed to represent the signal was shown to increase with signal randomness or “spread.” Performance metrics, in terms of ROC curves, were shown. The output SNR of the detector was found to rise with signal amplitude, and decrease with signal spread. From there, the random signal detector was extended to form an Angle of Arrival estimator through the use of a Maximum Likelihood estimator. Its Cramér-Rao lower bounds were used to assess the performance of the estimator, in cases where the output SNR was sufficiently high. Here, the minimum
of a signal. This will be accomplished using Matched Field Processing: correlation of the received signal was shown to be effective, allowing proper "nulling" most traces of the weaker, desired signal. The use of a whitening filter to null out the interfering signal was shown to be detrimental to correct variance decreased with increasing signal strength, but increased with increasing signal spread. Additionally, an interfering signal was shown to be detrimental to correct AOA estimation: the stronger signal "washed out" most traces of the weaker, desired signal. The use of a whitening filter to null out the interfering signal was shown to be effective, allowing proper AOA estimation.

The next chapter will extend spatial parameter estimation from incident angle to both the range and depth of a signal. This will be accomplished using Matched Field Processing: correlation of the received signal with the output of a propagation model. The same detection and nonrandom parameter estimation concepts used here will be needed for successful localization.

Figure 3-25: Normalized Detection Function, $l(k)$, after the received signal has been passed through a whitening filter, with interference signal spread $\Delta k_s$ varied. Compare this with Figure 3-22; the whitening filter has successfully nulled out the interfering signal at $k = 0.5$. For all interfering $\Delta k_s$, the peak of $l(k)$ corresponds to the correct location of the desired signal.
Figure 3-26: Contour plot (in dB) of Cramér-Rao lower bound on variance of incident angle estimate with respect to $\rho$, with +10 dB interferer present at normalized $k = 0.5$
Figure 3-27: Contour plot (in dB) of Cramér-Rao lower bound on variance of incident angle estimate, with respect to $\Delta k_s$, with +10 dB interferer present at normalized $k = 0.5$
Chapter 4

Source Localization and Nulling in a Random Ocean Environment

4.1 Introduction

In the previous chapter, random signal detection and nonrandom parameter estimation methods were applied to incident angle estimation using a linearly spaced hydrophone array. The random signal had three parameters: its strength, $\rho$; the amount of angular spread, $\Delta k_\theta$; and the incident wavenumber, $k_0$. Both $\rho$ and $\Delta k_\theta$ were assumed to be known perfectly; only $k_0$ was estimated. A Maximum Likelihood (ML) method was employed to achieve an estimate of $k_\theta$ with low root mean square (RMS) error. This could asymptotically satisfy the Cramér-Rao bounds on the variance of the estimate, provided the signal to noise ratio was high enough. The performance of the estimator in the presence of a random interferer was considered, and the effects of the interferer were mitigated through the use of a whitening filter.

This chapter applies this estimation framework to source localization using Matched Field Processing (MFP). Rather than a simple three parameter signal model, MFP uses a propagation model with many input parameters to calculate the signal covariance matrix, $\Gamma_\phi$. The additional complexity of the propagation model is compensated by MFP's ability to estimate the spatial location, i.e., range and depth, of a source.

This chapter is divided into three parts. First, a short tutorial of MFP will be given. Secondly, estimator performance metrics will be calculated from simulations performed in a shallow water environment. Source localization in both the presence and absence of random interfering signals will be shown. Third, the estimator will be applied to source localization and interference nulling using data collected during the 1998 Santa Barbara Channel Experiment.
4.2 Matched Field Processing

Matched Field Processing is a method to obtain the spatial location of a signal source. Although MFP can be applied to a variety of propagation scenarios, ranging from radar to optical fibers, the focus here will be on underwater acoustic signal propagation. MFP correlates or “matches” the received pressure “field” from a hydrophone array with a modeled signal. The acoustic source’s location, as well as parameters of the propagation environment, are used as inputs to the acoustic model. One varies the localization parameters, producing a new predicted acoustic field for each candidate source location, then correlates this modeled field with the received signal. With conventional MFP, the estimated location of the acoustic source corresponds to where the output of the correlator is maximized.

One can view MFP as an application of the parameter estimation theory reviewed in the last chapter. Recall the signal model,

\[ r = \zeta s(r, z) + n \]  

where \( r \) is the received signal vector, and \( s \) is the pressure field vector generated by an acoustic source at range \( r \) and depth \( z \), normalized to unit magnitude. \( \zeta \) is a scalar deterministic parameter which corresponds to the amplitude of the received signal. The noise vector is uncorrelated, zero mean Gaussian with variance \( \sigma_n^2 \). A Maximum Likelihood estimate of nonrandom parameters \( r \) and \( z \) can be accomplished by evaluating the likelihood ratio test

\[ l(r, z) = \max \left[ \ln \Lambda(R) \right] = \max \left[ \frac{\zeta}{\sigma_n^2} \left( s^t(r, z)R - \frac{\zeta}{2} \right) \right] \]  

across the range of \( r \) and \( z \), and choosing the value of \( r \) and \( z \) which maximize \( l \). If noise is not an issue, one can multiply through by \( \sigma_n^2/\zeta \), then set \( \sigma_n^2 \) to zero. Squaring the result and removing the constant bias term yields an ML surface equation

\[ l'(r, z) = |s^t(r, z)R|^2. \]  

This quadratic estimator is the basis for Conventional MFP. It was first applied to source localization in underwater acoustics by Bucker[1].

Figure 4-1 illustrates a simple shallow water scenario, taken from the 1998 Santa Barbara Channel Experiment (SBCX). Here, a submerged acoustic source emits a signal which is recorded on a 30 element vertical line array. The elements are linearly spaced 5 meters apart, with the bottommost element at a depth of 195 meters. The propagation environment is composed of water and three sediment layers. The 200 meter layer of water has a speed of sound which is varies with respect to depth. This average, segmented “downward refracting” sound profile was taken from the 1998 Santa Barbara Channel Experiment. The uppermost sediment layer is 89 meters thick, with a linearly increasing compressional propagation speed, starting at 1560
meters/sec and ending at 1821 meters/sec, a density of 1.85 g/cm³, and attenuation of 0.18 dB/λ. The middle, 300 meter thick layer also has a linearly increasing compressional sound speed, starting at 1862 meters/sec, and ending at 2374 meter/sec, with a density of 1.88 g/cm³, and an attenuation of 0.03 dB/λ. The bottom halfspace (extending down to infinity) has a constant compressional speed of 2374 meters/sec, a density of 2.03 g/cm³, and an attenuation of 0.04 dB/λ. Shear propagation is neglected. One assumes propagation environment does not vary with respect to range from the source or the receiver.

Figure 4-2 reviews the order of operations for MFP. A signal is sent from the source, passes through a propagation channel, and is recorded by an array of receivers. Using a mathematical model of the signal
source and propagation channel, the signal can be modeled as it appears at the receiver. The normalized, modeled signal and the actual received signal are compared, and the spatial parameters of the model varied until an optimal match is obtained. The result is plotted on an ambiguity surface, which arranges the output of the correlation operation according to the spatial source parameters. If the model parameters are correct, the source location will correspond to the peak of the ambiguity surface.

Figure 4-3: Conventional MFP Ambiguity Surface.

Figure 4-3 shows an example of an MFP ambiguity surface. Range is plotted on the horizontal axis, and depth on the vertical axis. The intensity of the plot contours correspond to the output of the correlation operation, Equation 4.3. Here, a synthetic source has been placed at a range of 15 kilometers and depth of 50 meters. The dashed lines on the ambiguity surface point to this location. The peak of the ambiguity surface, outlined with a small box, corresponds to the actual location of the source.

4.3 Stochastic Matched Field Processing

The MFP scenario illustrated in Figure 4-3 assumes the received signal and the propagation model are perfectly matched. Unfortunately, the realities of ocean acoustics dictate otherwise. Mismatch occurs when the parameters, environmental or otherwise, which are used in the signal model are not the same as those in the received signal. Environmental parameter mismatch specifically, of the sound velocity profile, is the single largest cause of localization error using MFP in shallow water.

In this thesis, the Sound Velocity Profile (SVP) is modeled as a random process. In reality, due to measurement limitations, the profile at a single point in time is rarely known with precision. Thus, it is prudent to think of the actual SVP as a single realization of a random process. Although the actual SVP may change with respect to range between source and receiver, the changes are assumed to be small and are neglected...
here. Provided the source signal is stationary, the random SVP will be the primary source of variability in the resulting random pressure field.

The same random signal detection algorithms which were employed in incident wavenumber estimation form the basis of MFP. Consider the similarities, diagrammed in Figure 4-4: The objective of both Angle of Arrival (AOA) Estimation and MFP are to estimate deterministic signal parameters. These parameters, as well as others which are assumed to be known, are inputs to a signal model, in the case of AOA estimation, or a propagation model, for MFP. The output of both models is a signal covariance matrix, $\Gamma_s$, which can be used in the random signal detection models discussed in Appendix C.

![Figure 4-4: Signal Model Comparison between AOA estimation and MFP](image)

Recall the signal model in Equation 4.1. With Stochastic MFP, some inputs to the signal model are random, resulting in a random signal vector, $s$, normalized to unity magnitude. This random vector is multiplied by the known received signal amplitude $\zeta$, then summed with the uncorrelated Gaussian random noise vector $n$ to form the received signal vector $r$. One assumes $s$, $n$, and $r$ are all zero mean Gaussian random variables, which can be characterized by their respective covariance matrices, $\Gamma_r$, $\Gamma_s$, and $\Gamma_n$.

The signal covariance matrix $\Gamma_s$ is determined from the random signal $s$. Parameters of the signal include: the speed of sound and bottom properties, both with respect to range and depth; receiver array element positions; and source location. In this document, only the speed of sound with respect to depth and the source location are treated as random quantities; the bottom properties are assumed to be deterministic. This assumption does not imply that all bottom properties can be treated as deterministic quantities in all cases.
Rather, this assumption was made only in an to simplify the problem at hand.

One wishes to use a ML estimator to find the source location. One can find an applicable ML estimator by employing random signal detection theory. First, one determines whether a signal is present or absent at a particular location,

\[ r = n \quad \text{if no signal is present (Hypothesis } H_0) \]
\[ r = \xi s(a) + n \quad \text{if a signal is present (Hypothesis } H_1) \]

Vectors \( r, s \) and \( n \) are random, complex, zero-mean, and Gaussian. The received signal probability density function (PDF) can be written as

\[ p_r(R) = \frac{1}{\pi^N |\Gamma_r|} \exp \left[ -R^\dagger \Gamma_r^{-1} R \right]. \] (4.6)

One can use a likelihood ratio test (LRT) to determine the presence or absence of the signal \( s \). Given the signal and noise are uncorrelated, and the noise is uncorrelated with itself, then

\[ \Gamma_r = \sigma_n^2 I \quad \text{Hypothesis } H_0 \]
\[ \Gamma_r = \xi^2 \Gamma_s(a) + \sigma_n^2 I \quad \text{Hypothesis } H_1. \]

The LRT would then take the form

\[ l(a) = \frac{\ln p_{r|H_1}(R|H_1)}{\ln p_{r|H_0}(R|H_0)} = \sum_{i=1}^{N} \left[ \frac{\xi^2 \sigma_s^2(a)}{\sigma_n^2} \right] \frac{u_i^\dagger(a) \hat{\Gamma}_r u_i(a)}{\xi^2 \sigma_s^2(a) + \sigma_n^2} - \ln \left[ 1 + \frac{\xi^2 \sigma_s^2(a)}{\sigma_n^2} \right], \] (4.7)

with \( a \) representing the spatial location of the source, \( u_i(a) \) the \( i^{th} \) singular vector, \( \sigma_s^2 \) the \( i^{th} \) singular value corresponding to the signal covariance matrix \( \Gamma_s \), \( \xi \) corresponding to the received signal amplitude, \( \sigma_n^2 \) the noise variance, and \( \hat{\Gamma}_r \) the estimated receiver covariance matrix, with \( \hat{\Gamma}_r = \hat{C} [R R^\dagger] \). The ML estimate of the source location \( \hat{a} \) corresponds to the value of \( a \) which maximizes \( l(a) \),

\[ \hat{a} = \max_a \sum_{i=1}^{N} \left[ \frac{\xi^2 \sigma_s^2(a)}{\sigma_n^2} \right] \frac{u_i^\dagger(a) \hat{\Gamma}_r u_i(a)}{\xi^2 \sigma_s^2(a) + \sigma_n^2} - \ln \left[ 1 + \frac{\xi^2 \sigma_s^2(a)}{\sigma_n^2} \right]. \] (4.8)

For vertical line arrays, one typically performs spatial estimation in range \((r)\) and depth \((z)\). Equation 4.8 is the Rank-N Stochastic MFP equation. Further details on the derivation of the equation can be found in Section C.6 of Appendix C. As with AOA estimation in the previous chapter, successful utilization requires knowledge of \( \Gamma_s \) for each candidate source location, \( a \). The AOA estimation example related the nonrandom parameters to \( \Gamma_s \) using a simple analytical expression. Unfortunately, with MFP the relationship between acoustic source location and \( \Gamma_s \) is much more complex. The next two sections outline the steps needed to estimate \( \Gamma_s \) for MFP, assuming the sound velocity profile \( c(z) \) is random.
4.3.1 Estimation of $\Gamma_\eta$

Rather than focus on the statistics of the sound speed profile, $c(z)$, it is more appropriate to consider the squared index of refraction. Consider the Helmholtz Equation in cylindrical coordinates,

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + k_0^2 n^2 p = 0,$$

where $p$ is the acoustic pressure, $r$ is the range from the source (in meters), $z$ is the depth in the water column (also in meters), $k_0$ is the reference acoustic wavenumber, $2\pi f/c_0$, and $n^2$ is the squared index of refraction, $[c_0/c(r, z)]^2$. This equation governs propagation of the pressure field through the water column. One can express the squared index of refraction as the dimensionless quantity $\eta(r, z)$,

$$\eta(r, z) = \left(\frac{c_0}{c(r, z)}\right)^2 - 1$$

(4.10)

with $c_0$ as a reference sound speed. Assuming $\eta$ is a random quantity, one can estimate its vertical mean and vertical covariance from the environment,

$$\bar{\eta}(z) = \frac{1}{N} \sum_{n=1}^{N} \eta_n(z)$$

(4.11)

$$\hat{\Gamma}_\eta(z_1, z_2) = \frac{1}{N} \sum_{n=1}^{N} [\eta_n(z_1) - \bar{\eta}(z_1)][\eta_n(z_2) - \bar{\eta}(z_2)]^*.$$  

(4.12)

where $\eta_n(z)$ are individual measurements of $\eta(z)$, $\bar{\eta}(z)$ is the estimated mean index of refraction, and $\hat{\Gamma}_\eta$ is the estimated covariance of the index of refraction. Here, one assumes $\eta$ is a Gaussian random variable, which is completely characterized by its mean and covariance.

For the shallow water environment shown in Figure 4-1, a sample random sound velocity profile was derived from individual sound profile measurements obtained during the 1998 Santa Barbara Channel Experiment (SBCX). During the experiment, 59 individual sound speed measurements were taken, distributed over space and time. Figure 4-5 shows the location and dates of each measurement. The color of the dot denotes the day the measurement was taken. Red: April 9. Orange: April 10. Blue: April 11. Dark Green: April 13. Brown: April 14. The bright green symbol to the right center indicates the position of the FFP receiver array.

The set of SVPs is plotted in Figure 4-6. Following a method outlined by LeBlanc and Middleton[47], Equations 4.10, 4.11 and 4.12 were employed to estimate $\bar{\eta}$ and $\hat{\Gamma}_\eta$.

Figures 4-7 and 4-8 illustrate the sample statistics from the SBCX dataset. Figure 4-7 shows the mean index of refraction, illustrating $\bar{\eta}$ increasing with depth. This indicates acoustic energy will interact with the bottom sediment, which further leads to signal scattering and attenuation. One can assume the signal will lose coherence as the distance from the source increases.

Figure 4-8 shows the covariance of $\eta$ with respect to depth, i.e. the vertical covariance. Note the profile is not vertically homogeneous. Instead, the highest degree of variance is shown at 40-60 meters in depth.
Figure 4-5: Sound Velocity Profile measurement locations during the 1998 Santa Barbara Channel Experiment.

While many SVPs were recorded to 200 meters depth and beyond, some were taken in shallower water. Consequently, the quality of the sample mean and variance degrades with depth. The discontinuity at 145 meters in \( \hat{\rho} \) is an example of this.
Figure 4-6: Sound Velocity Profile measurements recorded during the 1998 Santa Barbara Channel Experiment
Figure 4-7: Sample Mean of $\eta(z)$ from the 1998 Santa Barbara Channel Experiment.
Figure 4-8: Estimated Vertical covariance of $\eta(z)$ from the Santa Barbara Channel Experiment.
4.3.2 Estimation of $\Gamma_p$

One can estimate the vertical covariance $\Gamma_p$ of the pressure field, $p(z)$, given the mean and variance of the squared index of refraction, $\eta(z)$. Tatarskiĭ, Uscinski, and their colleagues have written extensively on the calculation of $\Gamma_p$ from $\Gamma_\eta$. Unfortunately, to date their work has focused on cases where $\Gamma_\eta$ is vertically stationary; Appendix B reviews their work. While this simplification allows $\Gamma_p$ to be written in a closed form expression, Figure 4-8 shows the assumption of vertical stationarity is not valid for the shallow water environment under study. Reynolds, Flatté, and Dashen [12] derived an analytical expression for $\Gamma_p$, but only in cases where there was no interaction with the water surface or bottom of the sound channel. This was not applicable to the low frequency, shallow water environment under consideration.

An analytical method for calculation of $\Gamma_p$ in the given environment would be ideal, but its derivation would likely encompass an entire dissertation and is left for future work. Rather than focus on such a currently nonexistent analytical technique, this section employs Monté-Carlo estimation techniques to obtain the vertical covariance of the pressure field.

Pressure Field Estimation

Propagation through an underwater acoustic waveguide can be modeled in several ways. All methods involve solving the Helmholtz Equation (Equation 4.9) for the acoustic pressure field, $p$. In a horizontally stratified range-independent environment, it is convenient to solve for the pressure field using a separation of variables technique. This would represent the pressure field $p$ as two functions multiplied together: one dependent only on range, and the other only on depth. The solution to the depth function can be posed as a Sturm-Liouville eigenvalue problem, with eigenvalues (or wavenumbers) $k_m$ and eigenvectors $\phi_m(z)$. The solution to the range function can be expressed in terms of a Hankel function of the first kind, $H_0^{(1)}(k_m r)$. Placing the range and depth solutions together, one obtains the so-called Normal Mode representation of the acoustic pressure field,

$$p(r, z_s, z_r) = \sum_{m=0}^{M} \frac{\phi_m(z_r)\phi_m(z_s)}{\sqrt{k_m r}} e^{-jk_m r},$$

where $z_s$ and $z_r$ are the depths of the source and receiver (in meters), $r$ is the range between the source and receiver, $\phi_m(z)$ is the $m^{th}$ eigenvector and $k_m$ is the $m^{th}$ eigenvalue of the Sturm-Liouville eigenvalue problem. The pressure field exhibits a high amount of oscillation with range, due to the presence of the $e^{-jk_m r}$ term. One can reduce the oscillatory effect of $k_m r$ by considering the spatially demodulated pressure field, $\psi$,

$$\psi(r, z_s, z_r) = p(r, z_s, z_r) e^{jk_0 r} = \sum_{m=0}^{M} \frac{\phi_m(z_r)\phi_m(z_s)}{\sqrt{k_m r}} e^{-j(k_m - k_0) r},$$

with $k_0$ as the reference acoustic wavenumber.
Covariance Matrix Estimation

The procedure for Monté-Carlo estimation of $\Gamma_p$ from $\Gamma_{\eta}$ is illustrated in Figure 4-9. One employs a Karhunen-Loeve expansion, with $\Gamma_{\eta}$ as input to synthesize multiple realizations of the random sound speed profile, $c(z)$. These profiles are used as inputs to an acoustic propagation model, which calculates a different pressure field, $p(r, z)$ for each input $c(z)$. The pressure field is scaled so its 2-norm is 1,

$$\bar{p}(r, z) = \frac{p(r, z)}{\int_0^D |p(r, z)|^2 \, dz}$$  \hspace{1cm} (4.15)

Such scaling assumes the amplitude of the signal amplitude is known at the receiver; the acoustic propagation simulation code provides the complex phase distribution with respect to depth. The pressure field covariance matrix $\Gamma_p$ is estimated from each $\bar{p}(z)$, using

$$\hat{\Gamma}_p(r, z_1, z_2) = \frac{1}{N} \sum_{n=0}^{N} \bar{p}_n(r, z_1)\bar{p}_n^*(r, z_2).$$  \hspace{1cm} (4.16)

One assumes

$$\lim_{N \to \infty} \hat{\Gamma}_p = \Gamma_p$$  \hspace{1cm} (4.17)

Unfortunately, performing a Monté-Carlo simulation an infinite number of times is not practical. Of interest here is determining the value of $N$ which results in an accurate estimated covariance matrix.

To quantify the effect of $N$ on the accuracy of $\hat{\Gamma}_p$, 1000 random sound velocity profiles were generated, using the statistics from the SBCX shown in Figures 4-7 and 4-8. These, in turn, were fed into the KRAKEN[48] normal mode simulation program to generate a complex pressure field from 1 to 20 kilometers range and 0 to 200 meters depth, assuming the environment shown in Figure 4-1. The pressure field covariance matrix $\Gamma_p$ was estimated using Equation 4.16. Due to the large number of simulations, the 1000
realization result was assumed to be the "correct" estimation of $\Gamma_p$. Next, the singular values of $\Gamma_p$ were calculated, using a Singular Value Decomposition,

$$\Gamma_p = U_p \Sigma_p U_p^T$$

(4.18)

which separates the covariance matrix into its singular values, which are contained in the entries of the diagonal matrix $\Sigma_p$ and singular vectors, which are the columns of the unitary matrix $U_p$.

Estimation of the singular values of $\Gamma_p$ yields significant insight into the accuracy of $\hat{\Gamma}_p$. As $\hat{\Gamma}_p$ converges to $\Gamma_p$ with increasing $N < 1000$, so do the singular values of both covariance matrices. Consequently, one can focus on the difference between $\hat{\Sigma}_p$ and $\Sigma_p$.

Figure 4-10 plots the singular values of $\Gamma_p$ for an acoustic source with a received signal level of 85 dB, located at 20 kilometers range and 100 meters depth, for 79, 166, and 338 Hertz. These were calculated from 1000 Monté-Carlo pressure field calculations, and are assumed to be the "correct" values of $\Sigma_p$. As one increases frequency, the number of singular values required to characterize the pressure field resulting from a random environment increases. In the three plots, one can observe the presence of a "knee," where the singular values drop off precipitously. This occurs around the $7^{th}$, $12^{th}$, and $22^{nd}$ singular value at 79, 166, and 338 Hertz, respectively.

Figures 4-11 through 4-13 illustrate the normalized absolute error of the estimated singular values, $\sigma_p^2$. In this manner, one can view the overall level of error for each singular value and realization. From the illustrations, the quality of the estimation appears to change little as one steps through frequency. Rather, the error depends more on the number of realizations ($N$). Quantitatively, one can see at 79 Hertz, up to 400 realizations are needed to drive the estimation error for all eigenvalues to 10% of its actual value. At 166 and 338 Hertz, only 300 realizations are needed to achieve this level of error. If one is interested only in the first 5 eigenvalues, then approximately 100 realizations of the pressure field would be necessary to keep the estimation error below 10%.

The need for computational accuracy must be balanced with that of computational tractability. While these simulations illustrate the number of realizations needed for accurate singular value estimation, they do not show the connection between number of realizations and accurate source localization. Later in this chapter it will be shown that only the singular values which are above the background noise level will contribute the majority of information necessary for source localization. With this in mind, all estimates of the 30x30 covariance matrix $\Gamma_p$ used at least 100 realizations of the sound speed profile.
Many of the singular values are relatively high. One must include these higher singular values in the location estimate to obtain accurate results.
Figure 4-11: Normalized Absolute Error (in dB), $|\hat{\sigma}_{p_1}(N) - \sigma_{p_1}^2|/\sigma_{p_1}^2$, for 79 Hertz.

Figure 4-12: Normalized Absolute Error (in dB), $|\hat{\sigma}_{p_1}(N) - \sigma_{p_1}^2|/\sigma_{p_1}^2$, for 166 Hertz.
Figure 4-13: Normalized Absolute Error (in dB), $|\hat{\sigma}_p^2(N) - \sigma_p^2|/\sigma_p^2$, for 338 Hertz.
4.4 Degrees of Freedom in a Stochastic Environment

The previous section went through the details of calculating $\Gamma_p$, the pressure field covariance matrix, given a source at a particular spatial location $(r, z)$ and a signal propagation environment with a random, vertically dependent index of refraction. Through this chapter, $\Gamma_p$ is used as the signal covariance matrix, $\Gamma_s$. This section examines the effect of the stochastic propagation environment on the dimensionality of the source signal.

Given a deterministic propagation environment, a received signal covariance matrix can be represented by the expected value of the outer product of the source signal, plus additive white Gaussian noise,

$$\Gamma_r = \xi^2 ss^\dagger + \sigma_n^2 I.$$  \hspace{1cm} (4.19)

This follows the assumption that the information expressed in the received signal can be characterized by a single Degree of Freedom (DOF), or Rank-1 covariance matrix $ss^\dagger$. Here, a stochastic propagation environment is assumed. The primary effect of this type of environment is to spread the energy of the source signal out among several singular values, increasing the DOF. Thus, the received signal covariance matrix can be expressed as

$$\Gamma_r = \xi^2 \Gamma_s + \sigma_n^2 I$$ \hspace{1cm} (4.20)

where $\Gamma_s$ is a Rank-$N$ signal covariance matrix. Brillinger[44] derived a method for estimating the DOF, given $\Gamma_s$.

$$DOF(r, z) = \frac{\left[\sum_{i=1}^{N} \sigma_{n_i}^2 (r, z)\right]^2}{\sum_{i=1}^{N} \left[\sigma_{n_i}^2 (r, z)\right]^2}$$ \hspace{1cm} (4.21)

where $\sigma_{n_i}^2$ is the $i^{th}$ singular value of $\Gamma_s$.

Figure 4-14 illustrates the DOF plotted for 79, 166, and 338 Hertz for an acoustic source. The stochastic propagation environment used in the previous section was employed here. The value of the DOF is shown from 1 to 20 kilometers, and depths from 0 to 200 meters in a range independent waveguide. One can observe the DOF increases with range, especially at lower depths. DOF also increases with frequency. From the illustration one can quantitatively assume that in the simulation environment at 79 Hertz for all depths and ranges below 6 kilometers, one can use a single DOF to model the received signal. At ranges beyond 6 kilometers, one would need to employ a stochastic propagation model to calculate the additional DOF. For higher frequencies, such as 166 and 338 Hertz, one would need a stochastic propagation model except for shallow sources at close range.
Figure 4.14: Degrees of Freedom plotted versus range and depth for an acoustic source, at 79, 166, and 338 Hertz, respectively.
4.5 Source Localization

4.5.1 Output SNR

The deflection coefficient ($d^2$) is a measurement of the Signal to Noise Ratio (SNR) at the output of a signal detector. For a given source location, it rates the quality of the detection result. One calculates the squared difference between the average detector output when a signal is present and when a signal is absent, then divides by the variance of the detector output when no signal is present. Both high values and consistency across the parameter space are desirable qualities of $d^2$. Recall the zero-mean complex random signal detector,

$$l(R, a) = \sum_{i=1}^{N} \left[ \alpha_i(a) \left| u^i(a)R \right| - \beta_i(a) \right]$$

with:

$$\alpha_i(a) = \left[ \frac{\zeta^2 \sigma^2_{\zeta}(a)}{\sigma^2_{n}} \right] \frac{1}{\zeta^2 \sigma^2_{\zeta}(a) + \sigma^2_{n}}$$

and

$$\beta_i(a) = \ln \left[ 1 + \frac{\zeta^2 \sigma^2_{\zeta}(a)}{\sigma^2_{n}} \right]$$

where $u_i(r, z)$ and $\sigma^2_{\zeta}$ are the $i^{th}$ singular vectors and values of the depth and range dependent signal covariance matrix $\Gamma(a)$, $\sigma^2_{n}$ is a measurement of the variance of the ambient noise, $\zeta$ is the received signal amplitude, and $R$ is the received signal vector. In this scenario of narrowband signal analysis, $\sigma^2_{n}$ can be determined by measuring the spectral level at a nearby frequency from which the source is not radiating energy.

Assuming the case when a signal is present at spatial location ($a$) is labeled $H_1$, and the case where only ambient noise is present labeled as $H_0$, the output SNR can be calculated as:

$$d^2(r, z) = \frac{\left\{ E[l(r, z) | H_1] - E[l(r, z) | H_0] \right\}^2}{\sigma^2_{n} |H_0|}.$$  \hspace{1cm} (4.23)

Substituting Equation 4.22 into Equation 4.23, one can solve for the output SNR in the case of a zero mean random signal,

$$d^2(r, z) = \frac{\left\{ \sum_{i=1}^{N} \alpha_i(r, z) \zeta \sigma^2_{\zeta}(r, z) \right\}^2}{\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i(r, z) \alpha_j(r, z) \left[ \eta_{ij} - (\sigma^2_{n})^2 \right]}.$$  \hspace{1cm} (4.24)

with $\eta_{ij} = E\left[ u^i_n n^i_n u^j_n n^j_n u^j_n \right]$, the fourth order statistic of the Gaussian random noise vector $n$, which in this case is independent of both range and depth.

Figure 4-15 plots the deflection coefficient in the form of an ambiguity surface for 79, 166, and 338 Hertz. In all three cases, a received signal strength of 85 dB is assumed, with an ambient noise variance of $\sigma^2_{n} = 80$ dB. Ideally, one would prefer $d^2$ to be constant with range and depth, so one does not introduce unnecessary error into a ML estimate. Considering the environment, the dynamic range of $d^2$ is relatively small: approximately 6 dB. Note $d^2$ drops as a function of range from the receiver. This is due to the increase in variability as the source signal passes through the propagation medium. One can see qualitative similarities between Figure 4-15 and the Degrees of Freedom plot, Figure 4-14. Figure 4-16 illustrates the positive effect
of reducing background noise. When $\sigma_n^2$ is reduced by 20 dB to 60 dB, the output SNR of the Rank-$N$ stochastic processor increases significantly.
Output SNR $10\log_{10}(d^2)$, at 79Hz for 85dB received signal in 80dB noise

Output SNR $10\log_{10}(d^2)$, at 166Hz for 85dB received signal in 80dB noise

Output SNR $10\log_{10}(d^2)$, at 338Hz for 85dB received signal in 80dB noise

Figure 4-15: Deflection Coefficient ($d^2$) showing the output SNR of the Rank-N Stochastic Matched Field Processor, at 79, 166, and 338 Hertz, respectively, with $\sigma_n^2 = 80$ dB.
Output SNR $10\log_{10}(d^2)$, at 79Hz for 85dB received signal in 60dB noise

Output SNR $10\log_{10}(d^2)$, at 166Hz for 85dB received signal in 60dB noise

Output SNR $10\log_{10}(d^2)$, at 338Hz for 85dB received signal in 60dB noise

Figure 4-16: Deflection Coefficient ($d^2$) showing the output SNR of the Rank-N Stochastic Matched Field Processor, at 79, 166, and 338 Hertz, respectively, with $\sigma_n^2 = 60$ dB.
4.5.2 Cramér-Rao Lower Bounds Applied to Source Location Estimate

The Cramér-Rao Lower Bound (CRLB) measures the minimum variance of any unbiased parameter estimator. The Maximum Likelihood Estimator reviewed in Appendix C and shown in Equation 4.8 should attain the Cramér-Rao Lower Bound, provided (1) the received signal is Gaussian and there is no significant mismatch between the actual and modeled signal input parameters, and (2) the input Signal to Noise Ratio is sufficiently high. It is important to understand that the CRLB are not optimal bounds when these conditions are not met. Section 4.5.4 outlined cases where the mean estimation error was nonzero. One could not apply the CRLB in this case, as the bounds assume the location estimate is correct. The CRLB are evaluated here to show that the uncertainty of source localization increases with range.

One first calculates the Fisher Information Matrix,

$[J(a)]_{mn} = \text{Tr} \left[ \Gamma_r^{-1}(a) \frac{\partial \Gamma_r(a)}{a_m} \Gamma_r^{-1}(a) \frac{\partial \Gamma_r(a)}{a_n} \right].$ \hspace{1cm} (4.25)

where \( a = [r \ z]^T \), and then takes the inverse of this matrix to solve for the CRLB,

$$\sigma_a^2 \geq [J^{-1}] :$$ \hspace{1cm} (4.26)

Applied to MFP source localization, the parameter vector \( a \) is made of two elements: the range, \( r \) and the depth, \( z \). The two diagonal entries of the inverse Fisher Information Matrix correspond to the minimum variance of the range and depth estimates, respectively. The top two contour plots of Figure 4-17 illustrate the CRLB for all possible source locations, from 1 to 20 kilometers in range and 0 to 200 meters in depth. The source was modeled with a received signal strength of 85 dB at 166 Hertz, and an uncorrelated Gaussian noise variance of 80 dB. The top surface shows the minimum variance of the range estimate, plotted on a logarithmic scale. Note how the minimum variance of the range estimate increases substantially with range, up to a variance of 50 dB m² at 20 kilometers, but is relatively independent of depth. The minimum variance of the depth estimate (center plot) is substantially smaller, with a maximum variance of 30 dB m² near the surface at 20 km. Here, too, the minimum variance increases with range.

As one reduces the uncorrelated Gaussian random noise \( \sigma_a^2 \), the Cramér-Rao Lower Bounds decrease. The gain of signal information caused by the lowering of ambient noise results in a decline of the minimum variance of the parameter estimator. Figure 4-18 illustrates the result when one drops the background noise in the current scenario from 80 to 60 dB.

Similarly, the CRLB decrease as the frequency increases. The number of propagating modes in the acoustic waveguide rises with frequency. One can utilize the information provided by the additional modes to reduce the variance of the estimated parameters. The quantitative effect of increasing the operating frequency is shown in Figure 4-19. One should be cautioned, however, into realizing the limits of the normal mode propagation model. As one increases frequency, the accuracy of the model compared to experimental data...
Off-diagonal entries of the Fisher Information Matrix are a measure of the minimum covariance between two parameters. One can determine the degree of coupling between two parameters through the calculation of the correlation coefficient,

\[ \rho = \frac{[\Gamma_a]_{01}}{\sqrt{[\Gamma_r][\Gamma_z]}} \quad \text{with} \quad J^{-1} = \begin{bmatrix} \Gamma_r & [\Gamma_a]_{01} \\ [\Gamma_a]_{01}^* & \Gamma_z \end{bmatrix}. \quad (4.27) \]

where \([\Gamma_a]_{01}\) is the off-diagonal entry of the 2 \times 2 inverse Fisher Information Matrix, and \(\Gamma_r\) and \(\Gamma_z\) are the CRLB for the range and depth, respectively. The value of \(\rho\) should range between \(-1\) and \(1\). A calculated value of \(\rho = \pm 1\) indicates the two parameters, range and depth, are perfectly correlated, while a value of \(\rho = 0\) indicates the parameters are uncorrelated. The third surfaces plotted in Figures 4-17 through 4-21 depict the correlation coefficient for ranges to 20 kilometers and depths to 200 meters, assuming a received signal level of 85 dB. With the exception of a shallow region at ranges below 2 kilometers, the magnitude of \(\rho\) does not rise above 0.2, showing the range and depth localization parameters are, for all practical purposes, uncorrelated.
Figure 4-17: Cramér-Rao Lower Bounds and Correlation Coefficient for Estimator Variance, given a 166 Hertz source with 85 dB received signal strength and noise variance of 80 dB.
Figure 4-18: Cramér-Rao Lower Bounds and Correlation Coefficient for Estimator Variance, given a 166 Hertz source with 85 dB received signal strength and noise variance of 60 dB.
Figure 4-19: Cramér-Rao Lower Bounds and Correlation Coefficient for Estimator Variance, given a 338 Hz source with 85 dB received signal strength and noise variance of 80 dB.
Figure 4-20: Cramér-Rao Lower Bounds and Correlation Coefficient for Estimator Variance, given a 79 Hertz source with 85 dB received signal strength and noise variance of 80 dB.
Figure 4-21: Cramér-Rao Lower Bounds and Correlation Coefficient for Estimator Variance, given a 79 Hertz source with 85 dB received signal strength and noise variance of 60 dB.
4.5.3 Effect of Parameter Mismatch on Source Localization

Successful source localization using MFP relies heavily on accurate modeling of the pressure field, which in turn is dependent on the quality of the input parameters. Environmental parameters include the sound velocity profile, bottom sediment properties, and bathymetry between the source and the receiver. Non-environmental parameters include array position, as well as the bearing to the signal source.

The speed of sound in water varies as a function of depth and range. This variation is caused by changes in pressure, temperature, and salinity. Typically, the temperature is highest at the surface, and drops as one descends into the ocean. Pressure increases with depth. Atmospheric disturbances can cause the upper layers of water to mix, further changing the sound speed. Seasonal variations are also present. In a shallow water environment, the differences in sound speed can be remarkable.

For example, consider measured sound velocity profiles from the 1998 Santa Barbara Channel experiment. During this exercise, 59 separate measurements were taken over six days, spread over a large area (see Figure 4-5). The individual sound speed measurements are plotted in Figure 4-6, showing the high degree of spatial and temporal variability. This poses a unique problem in modeling the sound propagation between a source and receiver. Referring back to Figure 4-4, other input parameters, such as the bathymetry and bottom sediment properties, are less likely to change over the time interval between survey and tracking exercise.

The most prominent negative effect of sound velocity mismatch is the introduction of error into the localization estimate. In other words, the peak of the MFP ambiguity surface will not correspond to the actual location of the acoustic source. Figure 4-22 illustrates an environmental mismatch example which is typical in MFP. After taking a series of sound speed measurements, an average sound speed profile was calculated and broken into a piecewise line approximation. In this simulation, a monochromatic source radiating at 160 dB and 166 Hertz was assumed. The ambiguity surface in Figure 4-3 illustrates the localization result if both the true and modeled sound speeds are matched perfectly (in this case, both used the dashed-line sound speed profile in Figure 4-22). The ambiguity surface in Figure 4-22 shows the effect of sound speed mismatch; the modeled profile (dashed line) differs from the true profile (solid line). This mismatch has two effects: first, the peak is shifted to an incorrect location, and second, the maximum value of the ambiguity peak is substantially reduced.
Employing an incorrect sound velocity profile in the pressure field model causes source localization errors.
4.5.4 Estimated Mismatch-Induced Root Mean Squared Error

Figure 4-22 gives anecdotal evidence of the negative effects of sound velocity profile mismatch. However, the ambiguity surface shows the error effects of only a single realization of a random SVP. This section estimates the RMS localization error resulting from a random SVP applied to both Conventional and Stochastic MFP, showing how Rank-N Stochastic MFP is far more robust to random SVP than Conventional MFP.

Here also will be shown the benefits of using more than one degree of freedom for source localization. Equation 4.8 has \( N \) weighted correlations summed together, with each singular vector of \( \Gamma_s \) orthogonal to one another. Many MFP localization algorithms seek to find a single “optimal” replica vector by using constraint methods [32, 36]. If one were limited to choosing a single correlation vector in random signal estimation, one would choose the singular vector which corresponds to the highest singular value of \( \Gamma_s \).

\[
\hat{\mathbf{a}} = \max_{\mathbf{a}} \left[ \frac{\xi^2 \sigma_n^2(\mathbf{a})}{\sigma_n^2} \right] \mathbf{u}_n^\dagger(\mathbf{a}) \hat{\mathbf{\Gamma}}_r \mathbf{u}_n(\mathbf{a}) - \ln \left[ 1 + \frac{\xi^2 \sigma_n^2(\mathbf{a})}{\sigma_n^2} \right]. \tag{4.28}
\]

Equation 4.28 neglects to incorporate signal components which may be present in the lower order singular values. Failing to include these components while attempting to localize in a stochastic environment can cause estimation error.

A simulation was conducted to estimate the RMS localization error caused by SVP mismatch. Given the second order statistics of the index of refraction from the 1998 Santa Barbara Channel Experiment, one hundred realizations of the sound speed profile were generated. The SVP shown in Figure 4-22 is an example of one of these realizations. For each realization, an acoustic source with a received signal level of 85 dB was simulated in the range-independent environment shown in Figure 4-1, at 50 meters depth. The source was stepped through range, from 1000 to 20000 meters at 100 meter intervals. At each range, an ambiguity surface was calculated, and the distance, in meters, between the peak of the ambiguity surface and the simulated source location was calculated. This distance is a measurement of the RMS estimation error. This was repeated for each of the 100 sound speed profile realizations, and the average error for each source range plotted. The entire simulation was run at source frequencies of 79, 166, and 338 Hertz.

Figure 4-23 illustrates the RMS error resulting from a conventional beamformer. The poor performance of the estimator is shown by the high degree of error present at all ranges. This is a direct result of sound velocity profile mismatch; in all simulations, a segmented mean SVP (plotted as the dashed SVP in Figure 4-22) is used as the modeled SVP, contrasting with the random realizations from the vertical covariance matrix illustrated in Figure 4-8.

Figure 4-25 illustrates the reduced RMS error resulting from Stochastic MFP, assuming a received signal level of 85 dB and an ambient noise level of 60 (top), 70 (middle), and 80 (bottom) dB. These plots compare the performance of Rank-\( N \) stochastic MFP with that of Rank-1 stochastic MFP. Rank-1 MFP uses only the primary singular vector of \( \Gamma_s \), while Rank-\( N \) MFP uses all \( N \) terms available. RMS error increases with range. Using Rank-1 MFP in this environment at 166 Hertz and 60 dB ambient noise, one can accurately
estimate the location of a towed source at 50 meters depth using either Rank-1 or Rank-N MFP to a range of 4 kilometers. Beyond that range, localization using Rank-1 MFP exhibits a steadily increasing RMS error.

Background noise has a significant effect on the performance of Rank-N MFP. Lowering \( \sigma_n^2 \) allows for additional singular vectors of \( \Gamma_s \) to be incorporated into the location estimate. Figure 4-24 shows only 1 singular value above 80 dB. Nine singular values are above 60 dB. The additional information provided by the singular values improves the quality of the localization estimate. The top illustration in Figure 4-25 shows the effect on RMS estimator error when background noise is dropped to 60 dB. While the Rank-1 estimate still has a high amount of error, all error from the Rank-N estimate has disappeared. The converse is true when one raises background noise. The bottom plot of the same figure illustrates localization error when the background noise is raised to 80 dB. Here, the received signal is only 5 dB above noise, severely reducing the number of singular values incorporated into Equation 4.8. As a result, the performance of Rank-N source localization degrades to that of Rank-1 MFP. Similar results are shown in Figures 4-26 and 4-27 for 79 and 338 Hertz, respectively. These results are consistent with the CRLB plots of the previous section, showing improved performance as frequency increases.

It is important to realize the RMS error plots are valid only for the given random environment, at the frequencies shown. They are meant to show the relative performance of three different source location estimators. One would expect the results to be qualitatively similar in other shallow water environments.

Increased RMS error with respect to range and background noise can be illustrated by examining the singular values of \( \Gamma_s \). Figure 4-28 plots the 30 singular values, \( \sigma_{\omega_s}^2 \), with respect to range, simulated at 166 Hertz. A received level of 85 dB is assumed; this is reflected in the flat level of the first singular value. The levels of the secondary singular values increase with range, reflecting the upward change in pressure field variability. This is not surprising, considering the Degree of Freedom plot in Figure 4-14 reflects an increase...
in pressure field variability with range.

Rank-\(N\) MFP compensates for the increased variability by incorporating additional singular vectors into its location estimate. Figure 4-29 shows the percent contribution of each singular vector with respect to range, assuming a received source level at 85 dB and frequency of 166 Hertz. As one increases the background noise level, fewer singular values are incorporated into the estimate, resulting in a corresponding increase in localization error.

Figures 4-30 and 4-31 illustrate sample ambiguity surfaces for both Rank-\(N\) and Rank-1 stochastic MFP. In both sets of simulations, a single sound speed realization was generated which obeyed the statistics of \(\Gamma_\eta\) from Figures 4-7 and 4-8. This was used to generate a received signal vector at 166 Hertz, for a source located at 15 kilometers range and 50 meters depth. The received source level was held constant at 85 dB, but the simulations were run with the background noise level set to 60, 70, and 80 dB. The localization results are consistent with the RMS error shown in Figure 4-25; Rank-\(N\) MFP correctly localizes the source when \(\sigma^2 = 60\) dB, but for Rank-1 MFP the peak of the ambiguity surface does not match the actual location of the source.
Figure 4-25: RMS Error Using Stochastic MFP at 166 Hertz
Figure 4-26: RMS Error Using Stochastic MFP at 79 Hertz
RMS Error at 338 Hz, Received source strength 85 dB, $\sigma_n^2 = 60$ dB

RMS Error at 338 Hz, Received source strength 85 dB, $\sigma_n^2 = 70$ dB

RMS Error at 338 Hz, Received source strength 85 dB, $\sigma_n^2 = 80$ dB

Figure 4-27: RMS Error Using Stochastic MFP at 338 Hertz
Figure 4-28: Singular values of $\Gamma_s$ plotted versus range, for a source with received signal level of 85 dB at 50 meters depth.
Figure 4-29: Contribution of Singular values of $\Gamma_n$ plotted versus range, for a 50 meter deep source with received signal level of 85 dB.
Figure 4-30: Sample Rank-N ambiguity surface plots
Correct localization occurs when $\sigma_n^2 = 60 \text{ dB}$ (top). Higher levels of background noise (center, bottom) result in poor source localization.
Figure 4-3: Sample Rank-1 ambiguity surface plots.
Insufficient information exists in the first singular vector to yield correct localization results using Rank-1 MFP.
4.6 Source Localization in the Presence of an Interferer

Previous simulations have shown the effect of varying levels of uncorrelated white Gaussian noise on estimator performance. This section continues to illustrate the effect of spatially coherent sources of noise on localization. In the ocean environment, these noise sources are manifested as shipping noise. If the signal strength of the interfering ship is significantly higher than the strength of the desired target, then detection and tracking can become very difficult.

One method to mitigate the effect of correlated noise is to employ a whitening filter. The purpose of a whitening filter is to convert the correlated noise from an interferer to uncorrelated white noise, while retaining as much information about the desired signal as possible. Appendix C reviews the fundamentals of whitening filter derivation; the major points are summarized here.

One assumes a received signal with a covariance matrix,

\[ \Gamma_r = \Gamma_s + \Gamma_b + \Gamma_n \]  \hspace{1cm} (4.29)

where the received covariance matrix is \( \Gamma_r \), the covariance matrix of the source signal of interest is \( \Gamma_s \), the covariance matrix of the interference signal is \( \Gamma_b \), and the uncorrelated Gaussian noise covariance matrix is represented by \( \Gamma_n \). The whitening filter takes the form

\[ H_w = U_b \Sigma_b^{-1/2} \]  \hspace{1cm} (4.30)

where \( \Sigma_b^{-1/2} \) is the inverse matrix square root of the diagonal matrix of singular values of \( \Gamma_{bn} = \Gamma_b + \Gamma_n \), and \( U_b \) is the unitary matrix of singular vectors of \( \Gamma_b \). One employs the whitening filter on the received data,

\[ r_w = H_w^\dagger r \]  \hspace{1cm} (4.31)

where \( r \) is the received signal vector, and \( r_w \) the "whitened" signal vector. One also applies the whitening filter to the model signal covariance matrix,

\[ \Gamma_{sw} = H_w^\dagger \Gamma_s H_w. \]  \hspace{1cm} (4.32)

One then applies \( r_w \) and \( \Gamma_{sw} \) to Equation 4.8, as \( R \) and \( \Gamma_s \), respectively.

With Matched Field Processing, an interfering ship produces a received signal \( b \), from which an interference covariance matrix \( \Gamma_b \) can be estimated. Although the interference is assumed to be a localized point source, its appearance on an ambiguity surface is characterized by the presence of spurious sidelobes – areas where the modeled signal has some degree of correlation with the interference signal. Application of received signal whitening produces a spatial null on the ambiguity surface. In addition to nulling out the ambiguity surface peak present at the interferer, the whitening filter also eliminates its sidelobes.
The effect of whitening on an interferer can be illustrated through simulation, shown in Figure 4-32. As with previous scenarios, an acoustic source of interest with received signal level of 85 dB is placed at a depth of 50 meters, and a range of 15 kilometers from a 30 element vertical line array. Here, an interfering acoustic source with received signal level of 90 dB is placed at 11 kilometers range and 10 meters depth.

As illustrated in Figure 4-33, different sound velocity profiles are used for the desired and interference sources. This is justifiable since the two sources are spatially separated. The cylindrical symmetry of the problem enables one to consider the two sources to be placed on opposite sides of a vertical line array (effectively 26 km apart), when they would appear on a range-depth ambiguity surface separated by only 4 km. Spatial variability of sound velocity profiles is supported by data taken during the Santa Barbara Channel Experiment. Here, both sound velocity profiles are realizations of the same refractive index vertical covariance matrix, $\Gamma_\eta$.

A better understanding of whitening and its effect on source localization can be obtained through the examination of the singular values of the signal covariance matrices. Given apriori information of the location of the interfering signal, one can estimate the covariance matrix $\Gamma_b$ of the interfering signal. Its singular values are shown in the top plot of Figure 4.6. As the source is only 11 kilometers away and at a depth of 10 meters, much of the energy is contained in its first singular value. One assumes any received signal will contain additional white Gaussian noise. The estimated covariance matrix $\Gamma_{bn}$ from the interference plus noise combination is shown in the center plot of Figure 4.6, which assumes an ambient noise level of $\sigma_n^2 = 35$.
Sound Velocity Profiles used in Interference Simulation

Figure 4-33: Sound Velocity Profiles used to simulate desired and interference signals.

dB. One observes the singular values of $\Gamma_b$ which fall below $\sigma_n^2$ are obscured by ambient background noise. $\Gamma_{bn}$ is then used by Equation 4.30 to calculate $H_w$, the whitening filter. When applied to $\Gamma_{bn}$, as shown in the bottom plot of Figure 4.6, the whitening filter succeeds in reducing all singular values of $\Gamma_{bnw}$ to 0 dB. Thus, one can conclude that $H_w$ converts the correlated noise in $\Gamma_{bn}$ to uncorrelated white Gaussian noise.

One then considers $\hat{\Gamma}_r$, estimated from the received signal $r = s + b + n$, which includes the 85 dB desired signal $s$, the 90 dB correlated interferer $b$, and the 35 dB ambient noise $n$. The singular values of $\hat{\Gamma}_r$ are shown in the top plot of Figure 4.6. Note the first singular value is at 90 dB, showing the dominant effect of the interfering signal. Singular values level out at 35 dB, illustrating the loss of information due to background noise. Equation 4.31 is then used to apply the whitening filter to $\hat{\Gamma}_r$, allowing one to obtain $\hat{\Gamma}_{rw}$.

The result is shown in the center plot of Figure 4.6. As a result of the whitening process, the ambient noise level has been reduced to 0 dB. Only the first singular value is above this noise threshold, but the next thirteen singular values which are below the noise threshold contribute toward successful localization of the desired source.

An increase in the ambient background noise $\sigma_n^2$ has a negative effect on the amount of information available for source localization. This holds true in the presence of an interferer. The bottom two plots of Figure 4.6 illustrate the estimated receiver covariance matrix $\hat{\Gamma}_r$ after it has been passed through the whitening filter, $H_w$. The center plot has background noise set to $\sigma_n^2 = 35$ dB, while the bottom plot has background noise set to $\sigma_n^2 = 60$ dB. With $\sigma_n^2 = 35$ dB, 14 singular values contain information which can be used for localization. However, with $\sigma_n^2 = 60$ dB, only 9 singular values contain information. This loss of information significantly degrades localization performance, as shown in the following set of ambiguity surface contour plots.
Figure 4-34: Singular values of $\Gamma_b$.
Top: Singular values of interferer ($\Gamma_b$) without ambient noise. Center: Singular values of interferer ($\Gamma_{bn}$) with ambient noise included. Bottom: Singular values of $\Gamma_{bnw}$, after application of the whitening filter.
Figure 4-35: Singular Values of $\hat{\Gamma}_r$.
Top: $\hat{\Gamma}_r$, estimated from desired and interfering source. Center: $\hat{\Gamma}_{rw}$, after whitening, assuming $\sigma_n^2 = 35$ dB. Bottom: $\hat{\Gamma}_{rw}$, after whitening, assuming $\sigma_n^2 = 60$ dB.
Figure 4-36 shows three ambiguity surfaces generated by the Rank-\(N\) stochastic matched field processor (Equation 4.8). All three assume a desired received signal level of 85 dB, and an unrealistically low ambient noise level of \(\sigma_n^2 = 35\) dB. The top contour plot illustrates the scenario when only the desired source is present. Thick dotted lines point to the correct source location of 15 kilometers range and 50 meters depth, and a thick rectangle surrounds the peak of the contour plot. One can see the source is localized correctly (both the dotted lines and box intersect). The center surface shows the introduction of an acoustic interference source with a received signal level of 90 dB into the environment. The spatial location of the interference signal is 11 kilometers range and 10 meters depth, representing a surface ship. The peak of the contour plot coincides with the location of the interferer; the desired source is completely obliterated. After application of the whitening filter to both the simulated received signal and the simulated source covariance matrices, the interference signal is nulled out, and the desired signal becomes visible once again.

This demonstration of whitening should not be taken as a panacea for nulling of spatially concentrated interference signals. This procedure will fail if the difference in strength between the desired source and the interference signal is too great. It will also fail if the background noise is too high. Figure 4-37 illustrates the effect of nulling when the ambient noise level is increased to a more realistic level of 60 dB. The first ambiguity surface shows correct source localization in the absence of any interfering signal. The second shows attempted localization of the interferer when its signal is added to the received signal. However, the localization effort fails, since the presence of the interference and the high background noise level has left insufficient signal for localization. In the bottom ambiguity surface, there is not enough signal available to localize the desired source. Thus, one needs a sufficiently low level of background noise for successful stochastic nulling.
Figure 4-36: Ambiguity surfaces from Rank-$N$ Stochastic MFP simulations, $\sigma_n^2 = 35$ dB.
Top: Desired signal, alone. Center: Desired signal and interference signal. Bottom: Desired signal, with interference signal nulled. The background noise level is low enough for correct localization.
Figure 4-37: Ambiguity Surfaces from Rank-N Stochastic MFP simulations, $\sigma_n^2 = 60$ dB.
Top: Desired signal, alone. Center: Desired signal and interference signal. Bottom: Desired signal, with interference signal nulled. The background noise level is too high for correct localization.
4.6.1 Output SNR

Figure 4-38: Deflection Coefficient ($d^2$) showing the output SNR of the Stochastic Matched Field Processor, at 79 Hertz, with $\sigma_n^2 = 60$ and 80 dB, respectively.

The presence of an interferer can have a negative impact on the deflection coefficient, $d^2$. For example, acoustic energy radiated from a surface ship can obliterate the signal of interest, resulting in increased detection difficulty. The effect on the output SNR of the Stochastic MFP processor is illustrated in Figure 4-38. Here, an interference source with received signal level of 90 dB was simulated at 11 kilometers range and 10 meters depth. The interference source has been nulled using the whitening technique reviewed earlier. The upper contour plot shows the output SNR when the ambient background noise is $\sigma_n^2 = 60$ dB, while the lower plot illustrates output SNR when $\sigma_n^2 = 80$ dB.

Comparing this to the top ambiguity surfaces in Figures 4-15 and 4-16, one can see the presence of the interferer significantly reduces the detectability of sources. In the $\sigma_n^2 = 60$ dB case, the output SNR has been
reduced from a peak of 49 dB to 40 dB, and from 9 to 5 dB in the $\sigma_n^2 = 80$ dB case. The area immediately around the interferer is a region of low output SNR resulting from the applied spatial null. These contour plots illustrate that source localization in the face of spatial nulling is possible, and lowered background noise will result in higher output SNR.

The null produced by the whitening process is spatially well defined; if the desired source is too close to the interfering source, the desired source could be nulled with the interfering source. Additionally, if the ambient noise level is too great, the weights of the secondary eigenvectors of $U_s, \alpha_i$ will be small; the stochastic processor will not benefit from the added information contained in the secondary eigenvectors, resulting in localization error. One should heed carefully the output SNR ($d^2$) surface for a given scenario to determine whether stochastic nulling using a whitening filter will be successful.
4.6.2 CRLB in the presence of an interferer

Frequently one wishes to localize a source in the presence of colored Gaussian noise. For example, one may wish to locate a relatively quiet submerged target in the middle of an active shipping channel. Other ships act as point-source interferers, adding correlated noise to the received signal.

Surface ships can be extremely loud, with source levels over 200 dB/Hertz at some frequencies. One can model the effect of an interferer by adding its signal covariance as a noise component to the received signal covariance,

\[ \Gamma_r = \Gamma_s + \Gamma_b + \Gamma_n \]  

(4.33)

where the received covariance matrix is \( \Gamma_r \), the covariance matrix of the source signal of interest is \( \Gamma_s \), the covariance matrix of the interference signal is \( \Gamma_b \), and the uncorrelated Gaussian noise covariance matrix is represented by \( \Gamma_n \). Figure 4-39 illustrates the effect of interference caused by a medium-sized container ship (90 dB received signal level at 79 Hertz), when the ship is located 11 kilometers from the receiver array, at a depth of 10 meters below the surface of the ocean. The received signal level of the target of interest is assumed to be 85 dB. Background noise is set at \( \sigma_n^2 = 80 \) dB. Compare the plots shown in Figure 4-39 with those in Figure 4-20. One can see the presence of the interferer increases the minimum variance of the source location estimate.

Figure 4-40 illustrates the CRLB when the ambient noise level is dropped from 80 to 60 dB. Note the decrease in minimum variance for all ranges and depths. When compared with Figure 4-21, one can again see the negative impact of an interference signal on the CRLB.

Through the examples shown, the overall trend of the CRLB is consistent. The minimum variance of the range estimate is relatively independent with depth, and increases with range. The minimum variance of the depth estimate exhibits similar qualities. The correlation coefficient, which measures the coupling between the range and depth source location estimate, is not substantially affected by the presence of the interferer; the range and depth estimates are for all practical purposes uncorrelated.
Figure 4-39: Cramér-Rao Lower Bounds for Estimator Variance, in the presence of an interferer, with $\sigma_n^2 = 80$ dB.
Cramer–Rao Lower Bounds on Variance of Range Estimate (in $m^2$ dB), 79Hz, $\sigma_n^2=60$ dB

Correlation Coefficient between Range and Depth Estimates, 79Hz, $\sigma_n^2=60$ dB

Figure 4-40: Cramér-Rao Lower Bounds for Estimator Variance, in the presence of an interferer, with $\sigma_n^2 = 60$ dB.
4.6.3 Signal to Interference plus Noise Ratio (SINR)

The Signal to Interference plus Noise Ratio (SINR)[49] is a metric of localization performance in the presence of both an interferer and Gaussian random noise. Traditionally, one plots SINR in the context of a deterministic signal and interferer. One separates the received signal into components resulting from source, interference, and noise,

\[ r = s + b + n. \] (4.34)

Assuming the model of \( s \) is correct, the output of a correlation processor can also be separated into source, interference, and noise components. The SINR is the ratio of the expected value of the correlation of the signal and the correlation of the interference and noise,

\[ \text{SINR} = \frac{E[|r^T s|^2]}{E[|r^T b|^2] + E[|r^T n|^2]}. \] (4.35)

One can apply Equation 4.35 to the Stochastic Matched Field Processing problem by assuming a random signal model. Substituting Equation 4.34 into Equation 4.22, and applying the result to Equation 4.35, one obtains

\[ \text{SINR} = \frac{\sum_{i=1}^{N} (\alpha_i u_i^T \Gamma_s u_i - \beta_i)}{\sum_{i=1}^{N} \alpha_i u_i^T \Gamma_b u_i + \sum_{i=1}^{N} \alpha_i u_i^T \Gamma_n u_i}, \] (4.36)

which can be further simplified to

\[ \text{SINR} = \frac{\sum_{i=1}^{N} (\alpha_i \sigma_{\alpha i}^2 - \beta_i)}{\sum_{i=1}^{N} \alpha_i (u_i^T \Gamma_b u_i + \sigma_{\beta i}^2)}. \] (4.37)

Figure 4-41 plots SINR for an acoustic source with a received signal level of 85 dB, in the presence of an interferer with a received signal level of 90 dB. The interferer is located at 11 kilometers range and 10 meters depth. The top ambiguity surface assumes 60 dB of ambient noise, while the bottom uses 80 dB noise. Contours are plotted on a linear scale. One can see the negative effect of background noise on SINR. At 60 dB, SINR is greater than 0 dB throughout the entire ambiguity surface, indicating source localization is possible in the midst of interference noise. At 80 dB, one encounters SINR less than 0 dB for a significant fraction of the ambiguity surface, reflecting lower estimator performance in the presence of increased noise.

SINR plots allow the observer to obtain a qualitative idea of source localization performance in the presence of an interferer. The plots can be thought of as another form of the detector output SNR. Unfortunately, SINR does not take into account signal gain which can be obtained through nulling an interferer by application of a signal whitening filter; the deflection coefficient, covered in the previous section, does account for the presence of a signal whitening filter.
SINR at 79Hz, 85 dB received signal, 90 dB interference signal, $\sigma_n^2=60$ dB

SINR at 79Hz, 85 dB received signal, 90 dB interference signal, $\sigma_n^2=80$ dB

Figure 4-41: Signal to Interference plus Noise Ratio (SINR) plot of the Stochastic Matched Field Processor, at 79 Hertz
4.7 Localization and Nulling Using Experimental Data

In April of 1998, the United States Defense Advanced Research Projects Agency (DARPA) sponsored a shallow water MFP feasibility experiment, named the Santa Barbara Channel Experiment (SBCX). This experiment had a number of objectives, one of them being passive tracking of towed acoustic sources.

Figure 4-42: Engineering drawing of FFP Array (Courtesy SAIC/Maripro)

Five 30 element vertical line arrays (VLAs) were deployed in a pentagon arrangement off Port Hueneme, California, in the Santa Barbara Channel. The nominal water depth at the center of the 5 arrays was 200 meters. At the base of each array was an instrument pod which digitized the acoustic signals from each array, and sent the signals along an optical fiber cabled back to a shore receiver station, located at the Naval Air Warfare Center, Point Mugu. There, a digital recorder wrote all acoustic data to tape for future analysis.

During SBCX, the research ship USS Acoustic Explorer (A/X), was chartered by the Space and Naval Warfare Systems Command (SPAWAR) to act as a controlled acoustic source. A series of narrowband audio signals were composed to simulate sound emanating from rotating machinery. Twelve simultaneous, continuous wave pilot tones were played at 64, 79, 94, 112, 130, 148, 166, 201, 235, 283, 338, and 388 Hertz.
These signals had amplitudes between 150 and 156 dB. Another group of twelve signals was played, offset 3 Hertz from the twelve pilot tones, and reduced by 20 dB amplitude, at 67, 82, 97, 115, 133, 151, 169, 204, 238, 286, 341, and 391 Hertz. These frequencies were chosen by Dr. Newell Booth of SPAWAR to test acoustic propagation through different frequency ranges, and to not interfere with power frequencies (50 or 60 Hertz) and their harmonics. During the experiment, the ship’s location was given by a Global Positioning System (GPS) receiver and recorded on an electronic log. The actual location of the J-15-III towed source did not correspond exactly to the GPS measurement, as the GPS antenna was located on the bridge of the Acoustic Explorer, with the J-15-III hanging from the A-Frame hoist, 102 feet away[50]. Figure 4-43 shows the power spectral density of the acoustic source, as calibrated and recorded by Howard Lynch at SPAWAR’s Transducer Evaluation Center (TRANSDEC).

![Figure 4-43: Power Spectral Density plot of source signals](image)

SBCX presented an opportunity to test the algorithms reviewed in this document with experimental acoustic data. This contrasts with simulations in that experimental data presents additional, unknown variables which are not necessarily known or controllable. Significant differences between simulated and experiement-
tal data include bathymetry, array shape, and method used to calculate the pressure field. These items are described in detail below.

- **Bathymetry:** Simulations shown in this document have presented a propagation environment with a perfectly flat bottom. In reality, the ocean bottom is rarely flat. In shallow water acoustics with downward refracting sound velocity profiles, bottom interaction is an extremely important factor of acoustic propagation. Neglecting changes in the bottom depth contributes to poor source localization results[3].

- **Acoustic Propagation Code:** For this experiment, the range dependent bottom contour dictated which acoustic propagation modeling code was used. An adiabatic normal mode approach, as implemented in the KRAKEN[48] propagation code, is suitable for range-independent environments. A parabolic equation (PE) method, as implemented by the RAM[51] acoustic propagation code, is better suited for range dependent environments. The highly variable bathymetry found in the Santa Barbara Channel (see the bathymetric contour plots in Figure 4-46) suggested using PE methods for accurate acoustic modeling[52]. Although both KRAKEN and RAM yield the same complex acoustic pressure for range independent environments, RAM requires considerably more computational effort. In all previously illustrated simulations in this document, KRAKEN was used to simulate acoustic propagation; however, for propagation simulation using experimental data from the Santa Barbara Channel, RAM was chosen.

- **Array Shape:** Simulations presented in this document assumed the VLA was perfectly straight, with each hydrophone element separated by 5 meters. The array used for SBCX differed, in that it could bend and twist in the presence of underwater currents. Each of the 150 hydrophones was coupled to a two-axis accelerometer, allowing fine measurement of array tilt. Each VLA was equipped with four magnetic compasses, distributed at equal distances along the array. The combination of accelerometer and compass measurements allowed one to establish the orientation and shape of each of the five VLAs. These measurements were processed to generate array shapes for every six minute segment of data. Thus, in sharp contrast to previous MFP experiments, unknown array tilt was much less of an issue during SBCX, and was treated as a known deterministic quantity in this document.

Figure 4-44 shows the orientation and shapes of the arrays derived from accelerometer and compass data. The top plot shows the arrangement of the five VLAs in the shape of a pentagon. The bottom plot shows two projections of the shape of VLA 1; the left contour shows the array projected onto a plane running along the north/south axis, while the right contour shows the array projected along an east/west axis. One can appreciate not only the fine level of detail available from the accelerometer and compass data, but also the degree of current-induced array tilt present in the Channel.

- **Reciprocity:** One of the steps for successful source localization using MFP involves simulating an acoustic source at a candidate location \((r, z)\) and calculating the complex pressure for each receiver element on the acoustic array.
When one is using normal modes for propagation simulation, obtaining the received pressure field is relatively straightforward. One employs Green's function,

\[ p(r, z_s, z_r) = \frac{e^{i\phi/4}}{\sqrt{2\pi}} \sum_{n=1}^{N} \phi_n(z_r) \phi_n(z_s) e^{-j k_n r}, \]  

(4.38)
to solve for the pressure field. The normal-mode simulation code yields the wavenumbers \( k_n \) and mode shapes \( \phi_n(z) \) for the environment. One simply supplies the candidate source depth \( z_s \) and range \( r \) from the receiver array, and the depth of the receiver hydrophone, \( z_r \). Once the mode wavenumbers and mode shapes are computed, the pressure field can be found from Equation 4.38 at any point in the waveguide.

PE differs in the method it uses to calculate the acoustic pressure field. One must specify the depth of the source. The code calculates, or the user supplies, an initial starting field, and then the code “marches forward” in range, calculating the pressure field for the equally spaced vertical points along the entire water column, from the surface to the bottom, at each marching step. This marching operation continues until the range marched matches the receiver range. The pressure field at the depth of the acoustic receiver is then interpolated from the grid of calculated values. To reduce interpolation error the oscillating carrier \( e^{j k \cdot r} \) is removed from the pressure field, and re-inserted after interpolation.

Traditional “direct” MFP involves setting up a grid of candidate source positions, and modeling the propagation from each one of the source locations to a receiver location. Using normal modes, one can evaluate Equation 4.38 quickly to find the pressure field. With PE, one must re-run the propagation simulation for each grid point. This computationally intensive procedure is illustrated in the top of Figure 4-45.

Using “reciprocal” MFP, one can exploit the reciprocal nature of acoustic propagation to save computational effort. Acoustic reciprocity implies the complex pressure field resulting from in-plane propagation between source and receiver is unchanged if the spatial locations of the source and the receiver are exchanged. Thus, if one treats each receiver element of the acoustic array as a source, and the candidate grid points as receivers, then the marching operation of the PE propagation code furnishes the complex pressure field between each candidate grid point and each array element. This is illustrated in the bottom of Figure 4-45. Once the pressure field is calculated for all grid points using all receiver elements as sources, then the source and receiver positions can be exchanged. No interpolation is necessary, as long as all candidate grid positions overlap. The author compensated for array tilt by modifying RAM to change marching steps “on the fly”: one would propagate with whatever step was necessary to arrive at exactly 500 meters range from the center of the array after an integer number of marching steps, then change the marching step to an even 10 meters. With this technique, one could calculate the pressure field at the receiver array for all candidate source points, after running the propagation code \( N \) times, where \( N \) is the number of receiver elements, in this case, 30 for a single VLA.
FFP Array Shapes, T36F7

Figure 4-44: VLA shapes
trial source range/depth combination

Figure 4-45: MFP Candidate Source Selection
4.7.1 Localization

The Santa Barbara Channel Experiment lasted six days, with nearly 1 terabyte of acoustic data recorded. This provided a rich dataset from which to demonstrate passive source localization. Equipment problems with the digital tape recorder employed during the experiment limited the amount of available data which could be used for analysis. Thus, it was determined early on in the processing to focus on a narrowly defined segment of data, which included the USS Acoustic Explorer following a predetermined course through the Santa Barbara Channel. To demonstrate Rank-N Stochastic MFP, a single five minute stationary event along the segment under study was chosen.

Figure 4-46: Bathymetric Contour Map of Stationary Localization Event

Figure 4-46 illustrates the event. Here, the USS Acoustic Explorer was stationary at an average distance of 15.32 kilometers and bearing of 3.112 radians from the FFP array. The “comb tone” sequence shown in Figure 4-43 was played for 4 minutes and 55 seconds, starting at 1:44:59 UTC on April 12, 1998. Data at the receiver was windowed in the time domain with a Hanning window, then Fourier transformed using a 4096 point FFT. This FFT length was dictated by the need to choose a window length which was longer than the time it would take for an acoustic signal, from any arrival angle, vertical or horizontal, to transit the entire
array. To compensate for the loss of data by the time domain window, each snapshot of data was overlapped with the previous one by 50 percent. Approximately 3 minutes of the received signal was used to estimate a receiver covariance matrix, \( \Gamma_r \).

Figure 4-47: Received Power Spectral Density for Stationary Localization Event

Figure 4-47 illustrates the received signal spectrum for the time interval in question. This measurement was averaged over all 30 hydrophones of the first VLA, across all snapshots of data. Vertical dotted lines indicate the frequencies where the twelve pilot tones appeared. One can compare this to Figure 4-43 and see the effects of transmission loss, interference, and uncorrelated noise. Transmission loss, caused by spherical spreading of the acoustic signal and energy absorption by the water, resulted in decreased amplitude levels at the receiver. One can confirm approximately 75 dB transmission loss when comparing the two plots. Below 100 Hertz, the presence of interference from a nearby container ship obscured the pilot tones. Uncorrelated (ambient) noise manifested itself as a noise “floor,” present at approximately 80dB, falling off gradually with increasing frequency. The Doppler effect was not an issue here, as both desired source and receiver array were nearly stationary.
Figures 4-48, 4-49 and 4-50 illustrate the output of Equation 4.3, for conventional MFP at 166, 148 and 130 Hertz, respectively. The sound speed profile used for pressure field estimation was a segmented average of all 59 CTDs taken during SBCX. It is illustrated as the dashed profile line in the upper plot of Figure 4-22. To maintain consistency with other ambiguity surfaces shown in this chapter, a linear scale was used. So that the primary and secondary ambiguity surface peaks could be emphasized, processor output below $3.5 \times 10^8$ was not shown in Figure 4-48, output below $1.3 \times 10^8$ was not shown in Figure 4-49, and output below $2.0 \times 10^8$ was not shown in Figure 4-50.

In Figure 4-48, one can see the peak of the ambiguity surface is at 19 kilometers range and 40 meters depth, whereas the actual source location is 15.3 kilometers range and 60 meters depth. Several sidelobes exist in the ambiguity surface; one sidelobe corresponds with the actual source location. These sidelobes are caused by environmental mismatch between the actual and modeled signals.

Figure 4-49 has a peak at 4.9 kilometers range and 170 meters depth, and Figure 4-50 has a peak at 9.3 kilometers and 70 meters depth. In both plots, no sidelobes correspond with the true source location.

Figure 4-51 shows ambiguity surfaces from both Rank-\(N\) and Rank-1 stochastic MFP. The upper plot shows the output of Equation 4.8 plotted on a linear scale. The dynamic range has been limited to emphasize the primary and secondary peaks in the ambiguity surface. From Figure 4-47, an ambient noise level of $\sigma_n^2 = 80$ dB was measured. The RMS error plotted in Figure 4-26 for both Rank-\(N\) and Rank-1 MFP is nonzero at 15 kilometers. Given this, one is not surprised that Rank-\(N\) MFP fails to correctly localize the source. However, one is encouraged that one of the secondary sidelobes does coincide with the correct location of the acoustic source. One must also remember that this ambiguity surface represents information from one temporal event at one frequency. When combined with ambiguity surfaces evaluated at other frequencies or times, such an ambiguity surface contributes information toward unique and correct localization of the target.
Similar results are shown for 148 and 130 Hertz, in Figures 4-52 and 4-53. Here, the true location and the peak of the Rank-$N$ ambiguity surface match closely.

This dataset was selected to demonstrate the superiority of Rank-$N$ MFP over Rank-1 MFP (Equation 4.28). The middle plot in Figure 4-51 shows the output of Rank-1 MFP. The peak of the surface is the same as that for Rank-$N$ MFP, but the sidelobe which corresponds to the actual location of the source is diminished. The bottom surface of Figure 4-51 shows the contribution of the second singular value of $\Gamma_s$. One can see a significant peak in the ambiguity surface near the source location. The information provided by this second singular vector serves to raise the overall Rank-$N$ MFP ambiguity surface near the actual source location, improving the overall quality of the result.

Figure 4-52 illustrates Stochastic MFP results at 148 Hertz. The top plot shows Rank-$N$ MFP, with the peak of the ambiguity surface nearly corresponding to the true location of the source. As with the 166 Hertz case, correct localization would not occur if only the first term of Equation 4.8 were used, as shown in the middle plot. Rather, the combination of additional information, such as the second term of Equation 4.8, shown in the bottom plot, gives correct localization.

Figure 4-53 shows correct localization at 130 Hertz. Here, the Rank-1 Stochastic MFP result (center plot) corresponds to the same spatial location as Rank-$N$ Stochastic MFP. In this case, sufficient information exists in the first singular vector for correct localization using Rank-1 Stochastic MFP.
Figure 4-50: Conventional ambiguity surface from SBCX Experimental Data (130 Hertz)
Figure 4-51: Rank-1 and Rank-N stochastic ambiguity surfaces from SBCX Experimental Data (166 Hertz)
Figure 4-52: Rank-1 and Rank-$N$ stochastic ambiguity surfaces from SBCX Experimental Data (148 Hertz)
Figure 4-53: Rank-1 and Rank-N stochastic ambiguity surfaces from SBCX Experimental Data (130 Hertz)
4.7.2 Interference Nulling

In the last section, source localization using Rank-N Stochastic MFP was demonstrated with experimental data. Although positive results were shown for frequencies above 100 Hertz, frequencies under 100 Hertz were obscured by interference. Consultation with radar logs provided by Naval Air Warfare Center, Point Mugu, and the crew of USS FLIP showed the presence of an Evergreen container cargo ship in the test area.

Figure 4-54 illustrates this scenario. The container ship was moving at a speed of approximately 10 meters/second away from the array. During the time interval under study, the interferer traveled nearly 2 kilometers. Based on the received signal, the container ship had a source strength of 170 dB at 79 Hertz, 14 dB above the source strength of the J-15 transducer on the USS Acoustic Explorer. The disparity between the two sources was enhanced by the fact that the Evergreen container ship was closer to the array than the USS Acoustic Explorer.

Figure 4.7.2 shows results of Rank-N Stochastic MFP at 79 Hertz. The ambiguity surface illustrates successful localization of the container ship using Rank-N Stochastic MFP, projected in the direction of the ship. The path of the container ship did not take it along a radial from the array; instead, the midpoint of the
ship's path during the three minutes under study was chosen. This point was 11.3 kilometers from the center of the array, at a bearing of -1.1103 radians. The bathymetry, i.e. sediment/water boundary, along the radial from source to receiver is shown in gray. Given an ambient noise level of 80 dB and a received signal level of 90 dB, the container ship was localized to within 300 meters range and 10 meters of its anticipated depth. As the target was in motion during this time, the resulting range error was deemed to be acceptable. Based on information from Evergreen Corporation, the maximum draft of their larger ocean-going container ships was between 12.6 and 12.7 meters. Smearing present in the ambiguity surface is a result of both source motion and bathymetric differences along the propagation path.

Figure 4-56: Singular values of $\Gamma_s$ at interference source location
The next step was to attempt to null out the interfering ship, in an effort to localize the USS *Acoustic Explorer*. The signal covariance matrix associated with the spatial location of the container ship was selected as the interference matrix, $\Gamma_b$. This was used to form a whitening filter, $H_w$, which was applied to the received signal covariance matrix $\Gamma_r$, and simulated signal covariance matrix $\Gamma_s$. The resulting whitened matrices were used as input for Rank-$N$ Stochastic MFP.

The bottom ambiguity surface in Figure 4-57 illustrates the result. Although successful localization was shown in scenarios with lower ambient noise, nulling and localization with experimental data was unsuccessful here. Figure 4-47 suggests an ambient background noise of 80 dB, once the signal from the container ship is taken into account. Successful nulling was shown in simulation to occur only when the ambient noise level was below 40 dB. The loss of signal information from the increased background noise resulted in a lack of information for the processors to localize the J-15 towed source. Another difference between simulation...
and experiment was the speed of the interferer; during simulation, the interference was treated as a stationary source, but during the experiment the source was moving.

4.8 Summary

This chapter applied random signal detection and estimation methods to Matched Field Processing, in an effort to improve the quality of source localization in random shallow water environments. Comparisons between Conventional MFP and Stochastic MFP were made. For Stochastic MFP, improved performance was shown when one sought to include a weighted combination of all singular values and vectors of the modeled signal covariance estimate into the detection estimate. This contrasts with many previous implementations of MFP, which sought to create a single optimal weight vector to correlate with received data. Performance metrics for source localization both with and without a discrete interferer were also shown. Localization results were verified using experimental data from the 1998 Santa Barbara Channel Experiment.
Chapter 5

Conclusion

This document introduces a new method of acoustic underwater source localization which is designed to function when the propagation environment is uncertain. It applies random signal detection and estimation theory summarized in Van Trees[7, 8] to Matched Field Processing (MFP). This localization method, termed Rank-$N$ MFP, correlates a received signal with a weighted sum of multiple orthogonal replica vectors. The replica vectors are based on a received signal covariance matrix modeled using a stochastic propagation model. The resulting location estimate has a lower RMS error than comparable methods which employ a single replica vector.

Application of Rank-$N$ MFP was limited here to a range-independent vertical sound speed profile, as this has traditionally been a high source of localization error for MFP. This method can be applied to mitigate uncertainty in bottom properties and array receiver element location.

Rank-$N$ MFP was performed on experimental array data obtained during the 1998 Santa Barbara Channel Experiment. Successful localization of a 160 dB narrowband towed source was obtained using a single 30 element vertical line array, at multiple frequencies between 100 and 200 Hertz, when the source was at a range of 15.3 kilometers from the array. Previous localization efforts using the same dataset[53, 54, 55] yielded correct localization results only out to a range of 7.8 kilometers. These past efforts sought to find a single optimal weight vector, either through Conventional, MV-EPC, or MV-WNC Matched Field Processing, rather than the multiple weight vector approach emphasized in this document.

A second contribution of this document is the introduction of a new method of robust interference nulling. Using stochastic propagation, a signal covariance matrix is used to represent a discrete interferer. This matrix is employed to construct a signal whitening filter, which acts as a robust spatial null. Negative effects from multiple discrete interferers, whether stationary or in motion, can be neutralized using a single whitening filter, all while operating in an environment with an uncertain sound velocity profile.

Neither contribution should be considered a panacea for problems encountered in localization and nulling of acoustic sources. Successful application of both methods require low ambient background noise. This can
be achieved by limiting oneself to relatively quiet environments, or increasing the number of receiver elements. One could obtain additional gain through the application of Rank-$N$ MFP to a constrained processor, extending the work of Krolik[32] to account for additional orthogonal replica vectors. This is left for future work.

In this thesis, most of signal covariance matrix estimation was performed using Monté-Carlo simulations. At the time of this writing, a method for direct calculation of pressure field statistics in a shallow water environment without vertical stationarity did not exist. Significant computational savings would be realized if a closed form analytical expression existed for calculating the vertical covariance of the complex pressure field in a shallow water waveguide. This too, is left for future work.

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Appendix A

Derived Distributions of \( p(r, z) \) using the Finite Difference Implementation of the Parabolic Wave Equation

A.1 Introduction

The pressure field generated from an acoustic source is dependent on several parameters: the geometry of the water column, source, and receiver, as well as the environmental parameters of the propagation medium. Included in these are the bathymetry, sediment layer density, and the speed of sound in the water column.

Given a random sound speed profile, \( c(z) \), and its probabilistic distribution, \( p_c(c) \), one could calculate the probabilistic distribution of the pressure field \( p_p(p) \) at the receiver. This paper takes one through the steps of this calculation, giving an analytical result. A finite difference parabolic equation (FD-PE) approach is taken to calculating the acoustic pressure field; the probabilistic distribution calculated here would be valid only in situations where the FD-PE model is valid.

A.2 Background

We start with the derivation provided in Chapter 6 of Computational Ocean Acoustics, by Jensen et al. This, in turn, recounts the derivation made by Tappert in 1977. We start with the Helmholtz equation,

\[
\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + k_0^2 n^2 p = 0, \tag{A.1}
\]

where \( p(r, z) \) is the acoustic pressure evaluated at range \( r \) and depth \( z \), \( k_0 = \omega/c_0 \) is a reference wavenumber, and \( n(r, z) = c_0/c(r, z) \) is the index of refraction.
One candidate solution for this equation would be the Hankel function $H_0^{(1)}(k_0r)$ multiplied by an envelope function, $\psi(r, z)$,

$$p(r, z) = \psi(r, z)H_0^{(1)}(k_0r) \quad (A.2)$$

The Hankel function satisfies the Bessel differential equation,

$$\frac{\partial^2 H_0^{(1)}(k_0r)}{\partial r^2} + \frac{1}{r} \frac{\partial H_0^{(1)}(k_0r)}{\partial r} + k_0^2 H_0^{(1)}(k_0r) = 0, \quad (A.3)$$

and its asymptotic form can be substituted for large $k_0r$,

$$H_0^{(1)}(k_0r) \approx \sqrt{\frac{2}{\pi k_0r}} e^{i(k_0r-\frac{\pi}{4})}. \quad (A.4)$$

Substitution of the trial solution (Equation A.2) into the Helmholtz equation (Equation A.1) gives

$$\frac{\partial^2 \psi}{\partial r^2} + \left( \frac{2}{H_0^{(1)}(k_0r)} \frac{\partial H_0^{(1)}(k_0r)}{\partial r} + \frac{1}{r} \right) \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2(n^2 - 1)\psi = 0. \quad (A.5)$$

Next, substitute the asymptotic Hankel function solution (Equation A.4) to yield

$$\frac{\partial^2 \psi}{\partial r^2} + 2ik_0 \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2(n^2 - 1)\psi = 0. \quad (A.6)$$

The final step in Tappert’s derivation is to use a small angle paraxial approximation, which assumes

$$\frac{\partial^2 \psi}{\partial r^2} \ll 2ik_0 \frac{\partial \psi}{\partial r}, \quad (A.7)$$

and allows one to drop the $\frac{\partial^2 \psi}{\partial r^2}$ term to arrive at the standard parabolic wave equation,

$$2ik_0 \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2(n^2 - 1)\psi = 0. \quad (A.8)$$

This solution lends itself to straightforward analytic expressions for simple environments. Unfortunately, the small angle approximation limits the effectiveness of the parabolic wave equation to pressure fields which propagate in a sector 10-15 degrees from the horizontal axis. For long range, deep water acoustics, most low frequency acoustic the energy propagates in the SOFAR channel, which is within this sector. Unfortunately, shorter range, shallow water acoustic propagation results in significant energy distributed outside of the small angle sector.

For complicated environments which do not lend themselves to simple analytical expressions, a finite element or finite difference approach may be warranted. One divides the water column up into a $(r, z)$ grid, iteratively solving for the pressure field in range and depth. By making the grid spacing $(\Delta r, \Delta z)$ small, one can calculate accurate solutions to the parabolic wave equation (see Figure A-1).
With the finite difference approach, one need not make the small angle assumption used in Equation A.8. Starting again with the far field Helmholtz Equation,

\[
\frac{\partial^2 \psi}{\partial r^2} + 2i k_0 \frac{\partial \psi}{\partial r} + k_0^2 (n^2 - 1) \psi + \frac{\partial^2 \psi}{\partial z^2} = 0,
\]

and applying the boundary conditions at an arbitrary grid boundary point \( z_B \) that (1) the pressure, and (2) the vertical derivative of the fields will be equal at this boundary point,

\[
\psi_1 (r, z_B) = \psi_2 (r, z_B) \quad \text{(A.10)}
\]

\[
\frac{1}{\rho_1} \left. \frac{\partial \psi_1}{\partial z} \right|_{z = z_B} = \frac{1}{\rho_1} \left. \frac{\partial \psi_2}{\partial z} \right|_{z = z_B} \quad \text{(A.11)}
\]

For the first medium (1), one can perform a Taylor series expansion of \( \psi_{l+1}^m \) around \( \psi_l^m \),

\[
\psi_{l+1}^m = \psi_l^m - \Delta z \frac{\partial \psi_l^m}{\partial z} + \frac{(\Delta z)^2}{2} \frac{\partial^2 \psi_l^m}{\partial z^2} + \cdots
\]

(A.12)

Solving for the second derivative of \( \psi \) with respect to \( z \),

\[
\frac{\partial^2 \psi_1}{\partial z^2} = -\frac{2}{(\Delta z)^2} (\psi_1 - \psi_{l+1}^m) + \frac{2}{\Delta z} \frac{\partial \psi_1}{\Delta z} \quad \text{(A.13)}
\]

and substituting back into Equation A.9 results in

\[
\frac{\partial \psi_1}{\partial z} = -\frac{\Delta z}{2} \left[ \frac{\partial^2 \psi_1}{\partial r^2} + 2i k_0 \frac{\partial \psi_1}{\partial r} + k_0^2 (n_1^2 - 1) \psi_1 - \frac{2}{(\Delta z)^2} (\psi_1 - \psi_{l+1}^m) \right].
\]

(A.14)

A similar equation can be constructed for medium 2. Equating the two, setting \( \psi_1 = \psi_2 = \psi \) and satisfying the second boundary condition, Equation A.11, one obtains,
\[
\frac{\partial^2 \psi}{\partial r^2} + 2i k_0 \frac{\partial \psi}{\partial r} + k_0^2 \left( \frac{\rho_2}{\rho_1 + \rho_2} \left( n_1^2 + \frac{\rho_1}{\rho_2} n_2^2 \right) \right) \psi - k_0^2 \psi + \frac{2}{(\Delta z)^2} \frac{\rho_2}{\rho_1 + \rho_2} \left( \psi_{i-1}^m - \frac{\rho_1 + \rho_2}{\rho_2} \psi_i^m + \frac{\rho_1}{\rho_2} \psi_{i+1}^m \right) = 0
\]
(A.15)

As the equations become more complex, it is helpful to make use of additional variable substitutions. Consider:

\[
\Gamma_{zz} \psi = \frac{2}{(\Delta z)^2} \frac{\rho_2}{\rho_1 + \rho_2} \left( \psi_{i-1}^m - \frac{\rho_1 + \rho_2}{\rho_2} \psi_i^m + \frac{\rho_1}{\rho_2} \psi_{i+1}^m \right)
\]
(A.16)

\[
\eta = \frac{\rho_2}{\rho_1 + \rho_2} \left( n_1^2 + \frac{\rho_1}{\rho_2} n_2^2 \right) - 1
\]
(A.17)

\[
G = k_0^2 \eta + \Gamma_{zz}
\]
(A.18)

This allows one to compactly express Equation (A.15) as

\[
\frac{\partial^2 \psi}{\partial r^2} + 2i k_0 \frac{\partial \psi}{\partial r} + G \psi = 0.
\]
(A.19)

Setting \( G = k_0^2 (Q^2 - 1) \) and using the generalized operators described in Section 6.2.2 of Computational Ocean Acoustics, one derives

\[
\partial \psi \over \partial r = i k_0 (Q - 1) \psi
\]
(A.20)

which is the generalized parabolic wave equation valid on horizontal surfaces.

Continuing along the derivation supplied in Section 6.6.2 of Computational Ocean Acoustics, the differential equation shown above can be solved using the Crank-Nicholson finite difference scheme, as outlined by Lee and McDaniel in 1988. Assuming

\[
\frac{\psi^{m+1} - \psi^m}{\Delta r} = i k_0 \left( \sqrt{1 + q} - 1 \right) \psi^{m+1} + \psi^m / 2,
\]
(A.21)

one can rearrange terms to obtain an iterative equation

\[
\left[ 1 - \frac{i k_0 \Delta r}{2} \left( \sqrt{1 + q} - 1 \right) \right] \psi^{m+1} = \left[ 1 - \frac{i k_0 \Delta r}{2} \left( \sqrt{1 + q} - 1 \right) \right] \psi^m,
\]
(A.22)

and apply a rational function approximation of the square root operator \( \sqrt{1 + q} \),

\[
\sqrt{1 + q} \approx \frac{a_0 + a_1 q}{b_0 + b_1 q},
\]
(A.23)
which yields

$$
\left[ 1 - \frac{ik_0 \Delta r}{2} \left( \frac{a_0 + a_1 \left( \eta + \frac{\Gamma_{\alpha \beta}}{k_0^2} \right)}{b_0 + b_1 \left( \eta + \frac{\Gamma_{\alpha \beta}}{k_0^2} \right)} - 1 \right) \right] \psi^{m+1} = \left[ 1 - \frac{ik_0 \Delta r}{2} \left( \frac{a_0 + a_1 \left( \eta + \frac{\Gamma_{\alpha \beta}}{k_0^2} \right)}{b_0 + b_1 \left( \eta + \frac{\Gamma_{\alpha \beta}}{k_0^2} \right)} - 1 \right) \right] \psi^m.
$$

(A.24)

Next, assume that $b_0 + b_1 \left( \eta + \frac{\Gamma_{\alpha \beta}}{k_0^2} \right)$ is constant across $\Delta r$, to obtain

$$
\left[ b_0 + b_1 \eta - \frac{ik_0 \Delta r}{2} \left[ (a_0 - b_0) + (a_1 - b_1) \eta \right] \right] \psi^{m+1} + \frac{1}{k_0^2} \left[ b_1 - \frac{ik_0 \Delta r}{2} (a_1 - b_1) \right] \Gamma_{zz} \psi^{m+1} = \left[ b_0 + b_1 \eta - \frac{ik_0 \Delta r}{2} \left[ (a_0 - b_0) + (a_1 - b_1) \eta \right] \right] \psi^m + \frac{1}{k_0^2} \left[ b_1 - \frac{ik_0 \Delta r}{2} (a_1 - b_1) \right] \Gamma_{zz} \psi^m.
$$

(A.25)

One uses the following shorthand notation to further simplify the expression:

$$
\begin{align*}
\frac{w_1}{w_2} & = b_0 + \frac{ik_0 \Delta r}{2} (a_0 - b_0) \quad (A.26) \\
\frac{w_1^*}{w_2^*} & = b_0 - \frac{ik_0 \Delta r}{2} (a_0 - b_0) \quad (A.27) \\
w_2 & = b_1 + \frac{ik_0 \Delta r}{2} (a_1 - b_1) \quad (A.28) \\
w_2^* & = b_1 - \frac{ik_0 \Delta r}{2} (a_1 - b_1), \quad (A.29)
\end{align*}
$$

finally yielding

$$
\begin{align*}
\left( \frac{w_1}{w_2^*} + \eta \right) \psi_{l+1}^{m+1} + \frac{1}{k_0^2} \left( \frac{2}{(\Delta z)^2} \frac{\rho_2}{\rho_1 + \rho_2} \right) & \times \left( \psi_{l-1}^{m+1} - \frac{\rho_1 + \rho_2}{\rho_2} \psi_{l+1}^{m+1} + \frac{\rho_1}{\rho_2} \psi_{l-1}^{m+1} \right) = \\
\left( \frac{w_1 + w_2 \eta}{w_2^*} \right) \psi_l^m + \frac{1}{k_0^2} \left( \frac{w_2}{w_2^*} \right) \left[ \frac{2}{(\Delta z)^2} \frac{\rho_2}{\rho_1 + \rho_2} \right] & \times \left( \psi_{l-1}^m - \frac{\rho_1 + \rho_2}{\rho_2} \psi_l^m + \frac{\rho_1}{\rho_2} \psi_{l+1}^m \right). \quad (A.30)
\end{align*}
$$

Rearranging terms, one can write this in vector form as

$$
\begin{bmatrix}
\psi_{l+1}^{m+1} \\
\psi_{l+1}^{m+1} \\
\psi_{l+1}^{m+1}
\end{bmatrix}
= \frac{w_2}{w_2^*} \begin{bmatrix}
1, \hat{u}, v
\end{bmatrix}
\begin{bmatrix}
\psi_{l+1}^m \\
\psi_{l+1}^m \\
\psi_{l+1}^m
\end{bmatrix},
$$

(A.31)

with

$$
u = \frac{\rho_1 + \rho_2}{\rho_2} \left[ k_0^2 (\Delta z)^2 \left( \frac{w_1}{w_2^*} \right) - 1 \right] + \frac{k_0^2 (\Delta z)^2}{2} \left[ (n_1^2 - 1) + \frac{\rho_1}{\rho_2} (n_2^2 - 1) \right].
$$

(A.32)
\[
\hat{u} = \frac{\rho_1 + \rho_2}{\rho_2} \left[ \frac{k_0^2 (\Delta z)^2}{2} \left( \frac{w_1}{w_2} \right) - 1 \right] + \frac{k_0^2 (\Delta z)^2}{2} \left[ (n_1^2 - 1) + \frac{\rho_1}{\rho_2} (n_2^2 - 1) \right] \quad (A.33)
\]

\[
v = \frac{\rho_1}{\rho_2} \quad (A.34)
\]

Equations A.31 through A.34 represent the starting point for the derived distributions. Marched through range and depth, they express the pressure field envelope \(\psi\) as a function of the refractive index, \(n\). This represents the last of the background material from Computational Ocean Acoustics. We now proceed with derived distributions.

### A.3 Derived Distributions

Before proceeding further, it is useful to review the concept of derived distributions. Given a random variable \(x\), with probability density function (PDF) \(p_x(x)\), and a function \(y = g(x)\), one wishes to find the probability density function \(p_y(y)\) of \(y\).

From Section 5.2 of Papoulis, one must first express \(x\) as a function of \(y\). If the function is not one-to-one, one must account for all possible roots of \(x\),

\[
y = g(x_1) = \cdots = g(x_n) \quad (A.35)
\]

and then the PDF of \(y\) can be calculated from

\[
p_y(y) = \frac{p_x(x_1)}{\left| \frac{\partial y}{\partial x} \right|_{x=x_1}} + \cdots + \frac{p_x(x_n)}{\left| \frac{\partial y}{\partial x} \right|_{x=x_n}} \quad (A.36)
\]

A few examples will illustrate the concept.

- \(y = ax + b\): Here, \(a\) and \(b\) are scalar deterministic variables. Assuming \(y = g(x)\), \(x\) is uniquely determined by \(y\), and vice-versa.

\[
x_1 = \frac{y - b}{a} \quad \frac{\partial y}{\partial x} = a \quad (A.37)
\]

Application of Equation A.36 yields:

\[
p_y(y) = \frac{1}{|a|} p_x \left( \frac{y - b}{a} \right) \quad (A.38)
\]

- \(y = \frac{x^2}{2}\): \(a, b\) are scalar deterministic variables. \(x\) can take two values given a particular \(y\),

\[
x_1 = -\frac{a}{\sqrt{y}} \quad x_2 = +\frac{a}{\sqrt{y}} \quad (A.39)
\]

\[
\left. \frac{\partial y}{\partial x} \right|_{x=x_1} = \frac{2}{a} y^{\frac{3}{2}} \quad \left. \frac{\partial y}{\partial x} \right|_{x=x_2} = -\frac{2}{a} y^{\frac{3}{2}} \quad (A.40)
\]
Substitution of Equation A.36 results in:

\[
p_y(y) = \frac{a}{2y^{3/2}} p_z \left( -\frac{a}{\sqrt{y}} \right) + \frac{a}{2y^{3/2}} p_z \left( \frac{a}{\sqrt{y}} \right)
\]  

(A.41)

For a case where there are two random variables \((x, y)\) and two functions of these two random variables, one can derive a new joint distribution based on their joint PDF, \(p_{xy}(x, y)\). From Papoulis, given

\[
z = g(x, y) \quad w = h(x, y),
\]  

(A.42)

one solves for \(x\) and \(y\) in terms of \(z\) and \(w\). If the relationship between the two pairs of variables is not one-to-one, all possible roots \((x_n, y_n)\) should be considered. The joint distribution \(p_{zw}(z, w)\) can be expressed as

\[
p_{zw}(z, w) = \frac{p_{xy}(x_1, y_1)}{|J(x_1, y_1)|} + \cdots + \frac{p_{xy}(x_n, y_n)}{|J(x_n, y_n)|},
\]  

(A.43)

where \(J(x, y)\) is the Jacobian operator,

\[
J(x, y) = \begin{vmatrix}
\frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y}
\end{vmatrix} = \begin{vmatrix}
\frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial z} & \frac{\partial y}{\partial w}
\end{vmatrix}^{-1}.
\]  

(A.44)

An example will help illustrate the procedure.

- Given \(p_{xy}(x, y)\), and the two functions

\[
z = ax + by + c \quad w = x
\]  

(A.45)

one wishes to solve for \(p_{zw}(z, w)\). The equations are one-to-one, with resulting inverses,

\[
x = w \quad y = \frac{1}{b}(z - aw - c).
\]  

(A.46)

The Jacobian can be easily evaluated as \(J(x, y) = -b\); these result in a derived distribution of

\[
p_{zw}(z, w) = \frac{1}{b} p_{xy} \left( w, \frac{1}{b} (z - aw - c) \right)
\]  

(A.47)

- Given \(p_{xy}(x, y)\), and the two functions

\[
z = ay + bx^{-1}y + cx^{-1} \quad w = x
\]  

(A.48)
one notes the equations are one-to-one, resulting in a single set of inverse equations,

\[ x = w \quad y = \frac{z - cw^{-1}}{a + bw^{-1}}. \]  
(A.49)

One can then solve for the Jacobian,

\[
\begin{aligned}
\frac{\partial z}{\partial x} &= -bx^{-2}y - cx^{-2} \\
\frac{\partial z}{\partial y} &= a + bx^{-1} \\
\frac{\partial w}{\partial x} &= 1 \\
\frac{\partial w}{\partial y} &= 0
\end{aligned}
\]

with

\[
J(x, y) = \begin{vmatrix}
-bx^{-2}y - cx^{-2} & a + bx^{-1} \\
1 & 0
\end{vmatrix}
\]

(A.50)

\[
J(x, y) = |a + bx^{-1}|. 
\]
(A.51)

Thus the derived distribution becomes

\[
p_{zw}(z, w) = \frac{p_{xy}(w, \frac{z - cw^{-1}}{a + bw^{-1}})}{|a + bw^{-1}|}. 
\]
(A.52)

- Given \( p_{xy}(x, y) \), and the two independent functions,

\[
z = a^2x^{-2} \quad w = a^2y^{-2},
\]
(A.53)

solve for \( p_{zw}(z, w) \). These equations are not one-to-one, with four different roots as

\[
x = \pm \frac{a}{\sqrt{z}} \quad y = \pm \frac{a}{\sqrt{z}},
\]
(A.54)

and the Jacobian evaluated to be \( J(x, y) = 4a^4x^{-3}y^{-3} \), resulting in

\[
p_{zw}(z, w) = \frac{1}{4a^4} \left| \left( \frac{a}{\sqrt{z}} \right)^3 \left( \frac{a}{\sqrt{z}} \right)^3 \right| 
\times \left[ p_{xy} \left( -\frac{a}{\sqrt{z}}, -\frac{a}{\sqrt{w}} \right) + p_{xy} \left( -\frac{a}{\sqrt{z}}, \frac{a}{\sqrt{w}} \right) + p_{xy} \left( -\frac{a}{\sqrt{z}}, -\frac{a}{\sqrt{w}} \right) \right]. 
\]
(A.55)

With this background information it is possible to derive the PDF of the output pressure field, \( p_{p}(p) \), given the PDF of the input sound velocity profile \( p_{c}(c) \), in the trivial isovelocity case.
A.4 Trivial Isovelocity Case

In this simplified case, the environment is taken to be range-independent: \( \rho_1, \rho_2 \) are constants, as is the sound speed, \( c = c_1 = c_2 \) and \( n = n_1 = n_2 \). The top and bottom pressure envelopes, \( \psi_{I-1}^m \) and \( \psi_{I+1}^m \) are known and constant for all \( r \) (and \( m \)). One assumes the initial field envelope, \( \psi_I^O \) and its distribution, \( p_{\psi_0}(\psi_I^O) \) is known, and is independent of the sound speed distribution, \( p_c(c) \).

Starting with the input sound velocity PDF \( p_c(c) \) one wishes to find the PDF for the output pressure \( p \) at a point in the middle of the water column.

The first step is to express the PDF of the refractive index, \( p_n(n) \), as a function of \( p_c(c) \). The refractive index is related to the sound speed by:

\[
n^2 = \frac{c_0^2}{c^2}, \tag{A.56}
\]

with \( c_0 \) a reference sound speed. Using Equation A.41, with \( n^2 \) in place of \( y \), \( c_0 \) as \( a \), and \( c \) as \( x \),

\[
p_{n^2}(n^2) = \frac{c_0}{2(n^2)^{3/2}} \left[ p_c \left( -\frac{c_0}{\sqrt{n^2}} \right) + p_c \left( +\frac{c_0}{\sqrt{n^2}} \right) \right]. \tag{A.57}
\]

Recalling Equations A.31 through A.34, we will assume \( n_1 = n_2 = n \), and rewrite Equation A.32 as

\[
u = \frac{\rho_1 + \rho_2}{\rho_2} \left[ \frac{k_0^2(\Delta z)^2}{2} \left( \frac{w_1^*}{w_2^*} \right) - 1 \right] + \frac{k_0^2(\Delta z)^2}{2} \left[ \frac{(n_1^2 - 1) + \frac{\rho_1}{\rho_2} (n_2^2 - 1)}{1 + \frac{\rho_1}{\rho_2} (n^2 - 1)} \right] \]

\[
= \frac{\rho_1 + \rho_2}{\rho_2} \left[ \frac{k_0^2(\Delta z)^2}{2} \left( \frac{w_1^*}{w_2^*} \right) - 1 \right] + \frac{k_0^2(\Delta z)^2}{2} \left[ \frac{(n_1^2 - 1) + \frac{\rho_1}{\rho_2} (n_2^2 - 1)}{1 + \frac{\rho_1}{\rho_2} (n^2 - 1)} \right]
\]
\[
\frac{\rho_1 + \rho_2}{\rho_2} \left[ \frac{k_0^2(\Delta z)^2}{2} \left( \frac{w_t}{w_2^*} \right) - 1 \right] - \frac{k_0^2(\Delta z)^2}{2} \left( 1 + \frac{\rho_1}{\rho_2} \right) - \frac{k_0^2(\Delta z)^2}{2} n^2 = 0
\] (A.58)

Substituting
\[
\alpha = \frac{k_0^2(\Delta z)^2}{2}
\] (A.59)
\[
\beta = \frac{\rho_1 + \rho_2}{\rho_2} \left[ \frac{k_0^2(\Delta z)^2}{2} \left( \frac{w_t}{w_2^*} \right) - 1 \right] - \frac{k_0^2(\Delta z)^2}{2} \left( 1 + \frac{\rho_1}{\rho_2} \right)
\] (A.60)
yields
\[
u = \alpha n^2 + \beta
\] (A.61)

Application of Equation A.38 gives
\[
p_u(u) = \frac{1}{\alpha} p_{n^2} \left( \frac{u - \beta}{\alpha} \right)
\] (A.62)
\[
= \frac{2}{k_0^2(\Delta z)^2 \left( 1 + \frac{\rho_1}{\rho_2} \right)} \times p_{n^2} \left( u - \frac{\rho_1 + \rho_2}{\rho_2} \left[ \frac{k_0^2(\Delta z)^2}{2} \left( \frac{w_t}{w_2^*} \right) - 1 \right] + \frac{k_0^2(\Delta z)^2}{2} \left( 1 + \frac{\rho_1}{\rho_2} \right) \right)
\] (A.62)

A similar PDF can be constructed for \( \hat{u} \) by substituting \( w_1 \) for \( w_t^* \) and \( w_2 \) for \( w_2^* \). However, this results in two correlated variables in Equation A.31, increasing the complexity unnecessarily. Instead, it is better to rewrite \( \hat{u} \) in terms of \( u \). Assuming:
\[
\beta = \frac{\rho_1 + \rho_2}{\rho_2} \left[ \frac{k_0^2(\Delta z)^2}{2} \left( \frac{w_t}{w_2^*} \right) - 1 \right] - \frac{k_0^2(\Delta z)^2}{2} \left( 1 + \frac{\rho_1}{\rho_2} \right)
\] (A.63)
\[
\hat{\beta} = \frac{\rho_1 + \rho_2}{\rho_2} \left[ \frac{k_0^2(\Delta z)^2}{2} \left( \frac{w_t}{w_2} \right) - 1 \right] - \frac{k_0^2(\Delta z)^2}{2} \left( 1 + \frac{\rho_1}{\rho_2} \right)
\] (A.64)

with
\[
u = \alpha n^2 + \beta
\] (A.65)
\[
\hat{u} = \alpha n^2 + \hat{\beta}
\] (A.66)
\[
\Delta \beta = \hat{\beta} - \beta
\] (A.67)

so
\[
\hat{u} = u + \Delta \beta
\] (A.68)

Rewriting Equation A.31 to take advantage of \( \Delta \beta \),
\[
\begin{bmatrix}
[1, u, v] \\
\psi_{i-1}^m+1 \\
\psi_i^m+1 \\
\psi_{i+1}^m
\end{bmatrix} = \frac{w_2}{w_2^2} \begin{bmatrix}
[1, u + \Delta \beta, v] \\
\psi_{i-1}^m \\
\psi_i^m \\
\psi_{i+1}^m
\end{bmatrix} ,
\] (A.69)

and isolating the unknowns \(\psi_i^{m+1}\) and \(u\),

\[
\psi_i^{m+1} = \frac{w_2}{w_2^2} \psi_i^m + \frac{1}{u} \left\{ \frac{w_2}{w_2^2} \left[ \psi_{i-1}^m + \Delta \beta \psi_i^m + v \psi_{i+1}^m \right] - \psi_{i-1}^{m+1} - v \psi_{i+1}^{m+1} \right\}
\]

\[
= \frac{w_2}{w_2^2} \psi_i^m + \frac{1}{u} \Delta \beta \psi_i^m + \frac{1}{u} \left\{ \frac{w_2}{w_2^2} \left[ \psi_{i-1}^m + v \psi_{i+1}^m \right] - \left[ \psi_{i-1}^{m+1} + v \psi_{i+1}^{m+1} \right] \right\}
\] (A.70)

Substituting the variables

\[
A = \frac{w_2}{w_2^2}
\]

\[
B = \Delta \beta
\]

\[
C = \frac{w_2}{w_2^2} \left[ \psi_{i-1}^m + v \psi_{i+1}^m \right] - \left[ \psi_{i-1}^{m+1} + v \psi_{i+1}^{m+1} \right]
\] (A.73)

into Equation A.70 gives

\[
\psi_i^{m+1} = A \psi_i^m + \frac{1}{u} B \psi_i^m + \frac{1}{u} C.
\] (A.74)

Application of Equation A.52 yields the derived distribution

\[
p_{u, \psi_i^{m+1}}(u, \psi_i^{m+1}) = p_{u, \psi_i^m}(u, \psi_i^m) \left( \frac{u, \psi_i^{m+1} - Cu^{-1}}{A + Bu^{-1}} \right).
\] (A.75)

Once one reaches the desired range \(r\) (iteration \(m\)), one can calculate the marginal distribution of \(\psi_i^{m+1}\),

\[
p_{\psi_i^{m+1}}(\psi_i^{m+1}) = \int_{-\infty}^{+\infty} p_{u, \psi_i^{m+1}}(u, \psi_i^{m+1}) du.
\] (A.76)

The final step expresses the complex pressure field, \(p\) as a function of the envelope, \(\psi\). Using the far-field Hankel approximation,

\[
p(r, z) = \frac{\psi(r, z)}{\sqrt{r}} e^{ik_0r - \frac{\pi}{4} i}
\] (A.77)

with Equation A.38 gives

\[
p_p(p) = \sqrt{r} \times p \left( \sqrt{r} e^{-i(k_0r - \frac{\pi}{4})} p \right).
\] (A.78)
A.5 Two speed case

The logical extension to the isovelocity case is one where the sound speeds \( c_1 \) and \( c_2 \) differ. To model the effect on the pressure field envelope \( \psi_i^n \), one would need to know the joint PDF of the two sound speed layers, \( p_{c_1c_2}(c_1, c_2) \).

The first step would be to calculate the joint PDF of \( p_{n_1n_2}(n_1^2, n_2^2) \) in terms of \( p_{c_1c_2}(c_1, c_2) \). Recall the relationship between \( n^2 \) and \( c \) is \( n^2 = \left(\frac{c_0}{c}\right)^2 \), where \( c_0 \) is a known reference sound speed. Using Equation A.55 with \( a = c_0, x = c_1, y = c_2, z = n_1^2, \) and \( w = n_2^2 \), one calculates:

\[
p_{n_1n_2}(n_1^2, n_2^2) = \frac{c_0^2}{4} \left[ \left( \frac{1}{n_1^2 n_2^2} \right)^{\frac{1}{2}} \rho_{c_1c_2} \left( -\frac{c_0}{\sqrt{n_1^2}} - \frac{c_0}{\sqrt{n_2^2}} \right) + \rho_{c_1c_2} \left( -\frac{c_0}{\sqrt{n_1^2}} + \frac{c_0}{\sqrt{n_2^2}} \right) \right. \\
\left. \quad + \rho_{c_1c_2} \left( -\frac{c_0}{\sqrt{n_1^2}} - \frac{c_0}{\sqrt{n_2^2}} \right) \right] .
\] (A.79)

Integrating this with the isovelocity derivation, the next step is to find the PDF \( p_u(u) \) in terms of the joint PDF \( p_{n_1n_2}(n_1, n_2) \). Starting with Equation A.32, and isolating \( u, n_1 \) and \( n_2 \),

\[
u = \frac{\rho_1 + \rho_2}{\rho_2} \left[ \frac{k_0^2(\Delta z)^2}{2} \left( \frac{w_1}{w_2} - 1 \right) \right] + \frac{k_0^2(\Delta z)^2}{2} \left[ (n_1^2 - 1) + \frac{\rho_1}{\rho_2} (n_2^2 - 1) \right]
\] (A.80)

Setting

\[
\alpha_1 = \frac{k_0^2(\Delta z)^2}{2} \quad (A.81)
\]

\[
\alpha_2 = \frac{k_0^2(\Delta z)^2}{2} \frac{\rho_1}{\rho_2} \quad (A.82)
\]

\[
\beta = \frac{\rho_1 + \rho_2}{\rho_2} \left[ \frac{k_0^2(\Delta z)^2}{2} \left( \frac{w_1^2}{w_2^2} - 1 \right) - 1 \right] \quad (A.83)
\]

one can rewrite Equation A.80 as

\[
u = \alpha_1 n_1^2 + \alpha_2 n_2^2 + \beta.
\] (A.84)

Application of Equation A.47 with \( a = \alpha_1, b = \alpha_2, c = \beta, x = n_1^2, y = n_2^2, \) and \( z = u \), one finds
\[ p_{u_1}(u, n_1^2) = \frac{1}{\alpha_2} p_{n_1 n_2}(n_1^2, n_2^2) \left( n_1^2, \frac{1}{\alpha_2} [u - \alpha_1 n_1^2 - \beta] \right). \]  

(A.85)

The marginal distribution \( p_u(u) \) can be calculated by integrating with respect to \( n_1^2 \),

\[ p_u(u) = \frac{1}{\alpha_2} \int_{-\infty}^{\infty} p_{u_1}(u, q) \left( q, \frac{1}{\alpha_2} [u - \alpha_1 q - \beta] \right) dq. \]  

(A.86)

The derivation would follow the isovelocity case, continuing with Equation A.67, except \( \beta \) and \( \hat{\beta} \) would be defined as

\[ \beta = \frac{\rho_1 + \rho_2}{\rho_2} \left[ \frac{k_0^2 (\Delta z)^2}{2} \left( \frac{w_1}{w_2} - 1 \right) - 1 \right] \]  

(A.87)

\[ \hat{\beta} = \frac{\rho_1 + \rho_2}{\rho_2} \left[ \frac{k_0^2 (\Delta z)^2}{2} \left( \frac{w_1}{w_2} - 1 \right) - 1 \right]. \]  

(A.88)

### A.6 N mesh points

Increasing the number of vertical mesh points allows one to improve the resolution and accuracy of both the simulated pressure field and the derived statistics.

Figure A-3 illustrates the \( N \) vertical mesh point case with five points. The first row of points, \( \psi_1^n \), is assumed deterministic and known. This represents the air-water boundary. Additionally, the environment is assumed range-independent; the statistics of \( c_l \) do not vary with range. Density, \( \rho_l \) is assumed known and deterministic. The environment can be any number of mesh points; it is limited to five here for brevity.

With the parabolic equation method, one steps through range, solving for the pressure field envelope at each mesh point. Solving for the PDF is similar; one recursively derives the joint PDF for each range and depth point, marching in range. Every attempt was made to keep the resulting expressions as simple as possible. Unfortunately, the recursive nature of the solution does not lend itself to a closed form expression. Instead, using these equations to numerically evaluate the PDF is recommended.

We start assuming the joint PDF of the sound speed profile, \( p_c(c) \) is known. One uses the relationship

\[ \nu_1 = n_1^2 = \frac{c^2}{c_l^2} \]  

(A.89)

to calculate the joint PDF of \( p_\nu(\nu) \). Using the formula for derived distribution,

\[ p_\nu(\nu) = \frac{p_c(c)}{|J(c)|}, \]  

(A.90)

one must calculate the roots of Equation A.89 and the Jacobian \( c \). The roots can be calculated for each element of \( \nu \) as:
$c_t = \pm \frac{c_0}{\sqrt{\nu_l}} \quad \text{(A.91)}$

The ± term is problematic, as it shows the relationship between the two equations is not one-to-one. Rather, one must include both terms when expressing the distribution. Taking the Jacobian first,

$$J(x) = \begin{vmatrix} \frac{\partial \nu_1}{\partial c_1} & \cdots & \frac{\partial \nu_1}{\partial c_L} \\ \vdots & \ddots & \vdots \\ \frac{\partial \nu_L}{\partial c_1} & \cdots & \frac{\partial \nu_L}{\partial c_L} \end{vmatrix} = \begin{vmatrix} -2c_0c_1^{-3} & 0 \\ & \ddots \\ 0 & -2c_0c_L^{-3} \end{vmatrix}$$

$$= (-2c_0)^L \prod_{l=1}^{L} c_i^{-3} \quad \text{(A.92)}$$

Substitution of Equation A.91, and taking the magnitude of the result gives
\[ |J(c)| = 2^L c_0^{-2L} \prod_{l=1}^{L} \nu_l^{3/2}. \]  

(Equation A.93)

Evaluation of Equation A.90 requires summation over \(2^L\) terms,

\[ p_v(\nu) = \sum_{m=1}^{2^L} p_c(\mathbf{T}_m) \frac{2^L c_0^{-2L} \prod_{l=1}^{L} \nu_l^{3/2}}{} \]  

(Equation A.94)

where:

\[ \mathbf{T}_m = c_0 \times \begin{bmatrix} \pm \nu_1^{-1/2} \\ \vdots \\ \pm \nu_L^{-1/2} \end{bmatrix}, \]

and the sign of each element in \(\mathbf{T}_m\) is determined by the binary value of \(m\), with each binary digit assigned to its corresponding element of \(\mathbf{T}_m\).

The second step is to solve for the joint PDF of \(p_u(u)\) given \(p_v(\nu)\). Recall

\[ u_i = \frac{\rho_{i-1} + \rho_i}{\rho_i} \left[ \frac{k_0^2 (\Delta z)^2}{2} \left( \frac{u_i^*}{w_0^*} \right) - 1 \right] + \frac{k_0^2 (\Delta z)^2}{2} \left[ (\nu_{i-1} - 1) + \frac{\rho_{i-1}}{\rho_i} (\nu_i - 1) \right] \]  

(Equation A.95)

can be expressed as

\[ u_i = \alpha \nu_{i-1} + \beta \nu_i + \gamma_i \]  

(Equation A.96)

with the appropriate substitutions. Using the derived distribution approach outlined above, the Jacobian evaluates as

\[ J(\nu) = \prod_{l=1}^{L} \beta_l. \]  

(Equation A.97)

Inversion of Equation A.95 to solve for the roots of \(\nu_i\) gives

\[ \nu_i = \frac{1}{\beta_i} \left[ u_i - \alpha \nu_{i-1} - \gamma_i \right]. \]  

(Equation A.98)

This equation requires one solves for \(\nu_i\) recursively. One can assume \(\nu_0\) is a known deterministic value. With this information, one can express the PDF of \(p_u(u)\) as

\[ p_u(u) = \frac{p_v(\nu)}{\prod_{l=1}^{L} \beta_l}, \]  

(Equation A.99)

with the elements of \(\nu\) determined by Equation A.98.

With the PDF of \(u\), one can calculate the iterative PDF for the pressure field envelope, \(\Psi\). One starts assuming the joint PDF for the previous range step is known. Letting \(\Psi^m = [\Psi_2^m \Psi_3^m \cdots \Psi_L^m]^T\) and \(u = \)
one can assume the joint PDF of these two random vectors is known: $p_{\Psi, \psi^0}(\Psi^m, \psi)$. At the source, the PDF of $\psi$ and $\psi^0$ are assumed to be independent; this results in their initial joint PDF to be

$$p_{\Psi, \psi^0}(\Psi^0, \psi) = p_{\psi^0}(\Psi^0)p_{\psi}(\psi).$$  \hfill (A.100)

Calculation of the pressure field envelope vector, $\Psi^{m+1}$ is performed by solving the system of tridiagonal matrices,

$$
\begin{bmatrix}
1 & u_1 & v_1 \\
1 & u_2 & v_2 \\
 & \vdots & \ddots \\
1 & u_{L-2} & v_{L-2} \\
1 & u_{L-1} & v_{L-1} \\
1 & u_L & 0
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_{L-2} \\
\psi_{L-1} \\
\psi_L
\end{bmatrix}
= 
\begin{bmatrix}
\hat{u}_1 & v_1 \\
\hat{u}_2 & v_2 \\
& \vdots & \ddots \\
\hat{u}_{L-2} & v_{L-2} \\
\hat{u}_{L-1} & v_{L-1} \\
\hat{u}_L & 0
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_{L-2} \\
\psi_{L-1} \\
\psi_L
\end{bmatrix},
\hfill (A.101)
$$

where:

$$\xi = \frac{w_2}{w_2^*}$$  \hfill (A.102)
$$\hat{u}_l = u_l - \Delta u_l$$  \hfill (A.103)
$$\Delta u_l = \frac{\rho_{l-1} + \rho_l}{\rho_l} \frac{k_0^2 (\Delta z)^2}{2} \left[ \frac{w_1^*}{w_2^*} - \frac{w_1}{w_2} \right]$$  \hfill (A.104)

and

$$v_l = \frac{\rho_{l-1}}{\rho_l}.$$  \hfill (A.105)

Evaluation of the tridiagonal system is straightforward. As the top layer $\psi_1^{m+1}$ is assumed known, subsequent layers, starting with $\psi_2^{m+1}$ can be calculated.
$$\psi_{2}^{m+1} = -\frac{u_{1}}{v_{1}}\psi_{1}^{m+1} + \xi \frac{u_{1}}{v_{1}}\psi_{1}^{m} + \xi \psi_{2}^{m}$$  \hspace{1cm} (A.106)$$

$$\psi_{3}^{m+1} = -\frac{1}{v_{2}}\psi_{2}^{m+1} - \frac{u_{2}}{v_{2}}\psi_{2}^{m+1} + \xi \psi_{2}^{m} + \xi \frac{u_{2}}{v_{2}}\psi_{2}^{m} + \xi \psi_{3}^{m}$$  \hspace{1cm} (A.107)$$

$$\psi_{4}^{m+1} = -\frac{1}{v_{3}}\psi_{3}^{m+1} - \frac{u_{3}}{v_{3}}\psi_{3}^{m+1} + \xi \psi_{3}^{m} + \xi \frac{u_{3}}{v_{3}}\psi_{3}^{m} + \xi \psi_{4}^{m}$$  \hspace{1cm} (A.108)$$

$$\vdots$$

$$\psi_{4}^{m+1} = -\frac{1}{v_{4}}\psi_{4}^{m+1} - \frac{u_{4}}{v_{4}}\psi_{4}^{m+1} + \xi \psi_{4}^{m} + \xi \frac{u_{4}}{v_{4}}\psi_{4}^{m} + \xi \psi_{4}^{m}.$$  \hspace{1cm} (A.109)$$

With this information a derived distribution can be calculated. The set of equations can be expressed in using partitioned vectors,

$$y = g(x)$$  \hspace{1cm} (A.110)$$

with $\Psi^{m} = [\psi_{2}^{m} \psi_{3}^{m} \cdots \psi_{L}^{m}]^{T}$.

$$y = \begin{bmatrix} \Psi^{m+1} \\ u \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} \Psi^{m} \\ u \end{bmatrix}$$  \hspace{1cm} (A.111)$$

and $g(x)$ utilizes Equations A.106 through A.109. The derived distribution takes the form:

$$p_{\psi}(y) = \frac{p_{x}(x)}{|J(x)|}$$  \hspace{1cm} (A.112)$$

where $J(x)$ is the Jacobian operator, evaluated as

$$J(x) = \begin{vmatrix} \frac{\partial g_{1}}{\partial x_{1}} & \cdots & \frac{\partial g_{1}}{\partial x_{n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{n}}{\partial x_{1}} & \cdots & \frac{\partial g_{n}}{\partial x_{n}} \end{vmatrix}$$  \hspace{1cm} (A.113)$$

To evaluate the derived distribution, one must solve both Equation A.113 and $x = g^{-1}(y)$ for Equation A.112. In the example 5 mesh point case, the Jacobian can be expressed as
which evaluates as

\[
J \left( \begin{bmatrix} \Psi^m \\ u \end{bmatrix} \right) =
\begin{bmatrix}
\frac{\partial \psi_1^{m+1}}{\partial \psi_1^m} & \frac{\partial \psi_1^{m+1}}{\partial \psi_2^m} & \frac{\partial \psi_1^{m+1}}{\partial \psi_3^m} & \frac{\partial \psi_1^{m+1}}{\partial \psi_4^m} & \frac{\partial \psi_1^{m+1}}{\partial \psi_5^m} \\
\frac{\partial \psi_2^{m+1}}{\partial \psi_1^m} & \frac{\partial \psi_2^{m+1}}{\partial \psi_2^m} & \frac{\partial \psi_2^{m+1}}{\partial \psi_3^m} & \frac{\partial \psi_2^{m+1}}{\partial \psi_4^m} & \frac{\partial \psi_2^{m+1}}{\partial \psi_5^m} \\
\frac{\partial \psi_3^{m+1}}{\partial \psi_1^m} & \frac{\partial \psi_3^{m+1}}{\partial \psi_2^m} & \frac{\partial \psi_3^{m+1}}{\partial \psi_3^m} & \frac{\partial \psi_3^{m+1}}{\partial \psi_4^m} & \frac{\partial \psi_3^{m+1}}{\partial \psi_5^m} \\
\frac{\partial \psi_4^{m+1}}{\partial \psi_1^m} & \frac{\partial \psi_4^{m+1}}{\partial \psi_2^m} & \frac{\partial \psi_4^{m+1}}{\partial \psi_3^m} & \frac{\partial \psi_4^{m+1}}{\partial \psi_4^m} & \frac{\partial \psi_4^{m+1}}{\partial \psi_5^m} \\
\frac{\partial \psi_5^{m+1}}{\partial \psi_1^m} & \frac{\partial \psi_5^{m+1}}{\partial \psi_2^m} & \frac{\partial \psi_5^{m+1}}{\partial \psi_3^m} & \frac{\partial \psi_5^{m+1}}{\partial \psi_4^m} & \frac{\partial \psi_5^{m+1}}{\partial \psi_5^m}
\end{bmatrix}
\]

where:

\[
\zeta_i = \frac{1}{v_i} \psi_i^{m+1} + \frac{v_i}{v_i} \psi_i^m.
\]
One can divide the matrix up to solve for its determinant, using the identity

\[ |X| = \det [X_{11} - X_{12}X_{22}^{-1}X_{21}] \det [X_{22}] \] \hspace{1cm} (A.117)

\[ X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \] \hspace{1cm} (A.118)

\[ X_{11} = \begin{bmatrix} \xi & 0 & 0 & 0 \\ \xi \frac{\dot{u}_2}{v_2} & \xi & 0 & 0 \\ \xi \frac{\dot{\zeta}_{3}}{v_3} & \xi \frac{\dot{u}_3}{v_3} & \xi & 0 \\ 0 & \xi \frac{\dot{u}_4}{v_4} & \xi \frac{\dot{u}_4}{v_4} & \xi \end{bmatrix} \]

\[ X_{12} = \begin{bmatrix} \zeta_1 & 0 & 0 & 0 \\ 0 & \zeta_2 & 0 & 0 \\ 0 & 0 & \zeta_3 & 0 \\ 0 & 0 & 0 & \zeta_4 \end{bmatrix} \] \hspace{1cm} (A.119)

\[ X_{21} = 0 \]

\[ X_{22} = 1 \]

The determinant of triangular matrix \( X_{11} \) is the product of its diagonal elements. In this example, the determinant evaluates to \( \xi^4 \). For the more general case with \( L \) mesh points, one finds

\[ J \left( \begin{bmatrix} \Psi^m \\ u \end{bmatrix} \right) = \xi^{L-1} \] \hspace{1cm} (A.120)

Solving Equation A.113 and \( x = g^{-1}(y) \) requires one to express \( \psi^m \) in terms of \( \psi^{m+1} \), using the tridiagonal system of Equation A.101:

\[ \psi^m_2 = \frac{u_1}{\xi v_1} \psi^{m+1}_1 + \frac{1}{\xi} \psi^{m+1}_2 - \frac{\dot{u}_1}{v_1} \psi^m_1 \] \hspace{1cm} (A.121)

\[ \psi^m_3 = \frac{1}{\xi v_2} \psi^{m+1}_2 + \frac{\dot{u}_2}{v_2} \psi^{m+1}_3 + \frac{1}{\xi} \psi^{m+1}_4 - \frac{1}{v_2} \psi^m_1 - \frac{\dot{u}_2}{v_2} \psi^m_2 \] \hspace{1cm} (A.122)

\[ \psi^m_4 = \frac{1}{\xi v_3} \psi^{m+1}_3 + \frac{\dot{u}_3}{v_3} \psi^{m+1}_4 + \frac{1}{\xi} \psi^{m+1}_5 - \frac{1}{v_3} \psi^m_2 - \frac{\dot{u}_3}{v_3} \psi^m_3 \] \hspace{1cm} (A.123)

\[ \vdots \]

\[ \psi^m_l = \frac{1}{\xi v_{l-1}} \psi^{m+1}_{l-1} + \frac{\dot{u}_{l-1}}{v_{l-1}} \psi^{m+1}_{l-1} + \frac{1}{\xi} \psi^{m+1}_l - \frac{1}{v_{l-1}} \psi^m_{l-2} - \frac{\dot{u}_{l-1}}{v_{l-1}} \psi^m_{l-1} \] \hspace{1cm} (A.124)

With this information, a recursive expression can be established for the derived joint PDF.

Given the joint PDF \( p_{\Psi,u}(\Psi^m, u) \), one can solve for the joint PDF \( p_{\Psi,u}(\Psi^{m+1}, u) \):

\[ \text{153} \]
\[ p_{\psi_u}(\Psi^{m+1}, u) = \frac{1}{|\xi^{L-1}|} p_{\psi_u}(\Phi, u), \]  

where \( \Phi \) is a column vector with entries \( 2 \leq l \leq L \) which satisfy

\[
\phi_l = \begin{cases} 
\frac{u_1}{\xi v_1} \psi_1^{m+1} + \frac{1}{\xi} \psi_2^{m+1} - \frac{\dot{u}_1}{v_1} \psi_1^m & \text{for } l = 2 \\
\frac{1}{\xi v_{l-1}} \psi_{l-1}^{m+1} + \frac{u_{l-1}}{\xi v_{l-1}} \psi_{l-1}^{m+1} + \frac{1}{\xi} \psi_{l-1}^{m+1} - \frac{1}{v_{l-1}} \phi_{l-2} - \frac{\dot{u}_{l-1}}{v_{l-1}} \phi_{l-1} & 2 \leq l \leq L.
\end{cases}
\]

The marginal PDF \( p_{\psi}(\Psi) \) can be solved by integrating across \( u \):

\[
p_{\psi}(\Psi) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\psi_u}(\Psi, u) \ du.
\]  

### A.7 Computational Complexity

The relations given in this document allows one to calculate the joint PDF of the pressure field envelope, \( p_{\psi}(\Psi) \) at any range and depth, given a range independent environment and a depth varying, random sound speed profile \( c \) with PDF, \( p_c(c) \). The simplicity of the FD/FE approach and the resulting equations hide the computational complexity required in evaluation.

Consider the dimensionality of \( p_{\psi_u}(\Psi, u) \). If one uses \( L \) mesh points, the PDF would have \( 2L - 1 \) dimensions. One objective of solving for \( p_{\psi}(\Psi) \) is to calculate the its covariance matrix, \( \Lambda_{\psi} \). The covariance matrix has \( L(L - 1)/2 \) unique entries, each of which must be derived from a marginal joint PDF. One would need to integrate \( p_{\psi_u}(\Psi, u) \) \( 2L - 3 \) times to yield the marginal joint PDF, and then integrate twice more to evaluate the covariance matrix entry. To fill \( \Lambda_{\psi} \) would require \( L(L - 1)(2L - 1)/2 \) integrations. Assuming each integration across the PDF space requires evaluating \( p_{\psi_u}(\Psi, u) \) at \( L \) different points, the total number of times these expressions would be evaluated would approach \( L^4 \). For an environment requiring 500 mesh points and a computational platform which can evaluate the PDF 100,000 times each second, this would amount to over 170 hours of CPU time.

Steps can be taken to reduce the overall complexity. For example, one could assume the PDF of \( \nu \) were jointly Gaussian. The relationship between \( \nu \) and \( u \) is linear, which would also be Gaussian. Unfortunately, Equations A.106 through A.109, which relate \( \Psi \) to \( u \) are nonlinear, since \( \psi \) and \( u \) are multiplied together in several terms. Thus the output \( \psi \) cannot explicitly be called Gaussian. As range and (and \( m \)) increases, the Central Limit Theorem would likely take a role, making the final envelope statistics \( p_\psi(\Psi) \) Gaussian in nature. Given current computational capabilities, it would be beneficial to investigate how the second moments of \( \Psi \) propagate through range, rather than its entire PDF.
Appendix B

Derivation of Mutual Coherence
Function of the Pressure Field Envelope
using the Parabolic Equation Method

B.1 Introduction

This Appendix outlines the steps needed to derive the Mutual Coherence Function (MCF) of the pressure field envelope, given a random sound speed profile, whose squared index of refraction exhibits Stationary Gaussian statistics in the vertical direction. Waveguide propagation is assumed, using a small angle Parabolic Equation method first proposed by Tappert\[56\]. With mathematical support from Novikov\[57\], Tatarskii\[58\] derived both the mean and the second moment of the pressure field envelope. This was summarized by Barabanenkov\[59\] and later incorporated into Ishimaru’s monograph\[60\]. Macaskill\[16\] extended Tatarskii’s work to incorporate arbitrary deterministic sound velocity profiles. This derivation is included here for two reasons. First, the original Soviet journal articles by Novikov and Tatarskii are not widely available in the United States. Secondly, although Ishimaru summarized both authors’ work, several typographical errors are present in his text. Those errors are corrected here, and the work of Macaskill incorporated.

B.2 The Parabolic Equation

Following the derivation given by Tappert\[56\] and summarized by Jensen\[9\], one starts with the Helmholtz equation in cylindrical coordinates,

\[
\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + k_0^2 n^2 p = 0, \quad (B.1)
\]
with \( p \) as the acoustic pressure, \( r \) is the range from the source (in meters), \( z \) is the depth in the water column (also in meters), \( k_0 \) is the reference acoustic wavenumber, \( 2\pi f/c_0 \), and \( n \) is the index of refraction, \( c_0/c(r, z) \). Acoustic pressure \( p \) can be separated into two parts; the pressure envelope, \( \psi \), and the oscillatory part, expressed as a Hankel function,

\[
p(r, z) = \psi(r, z)H_0^{(1)}(k_0r).
\] (B.2)

The Hankel function satisfies the Bessel differential equation,

\[
\frac{\partial^2 H_0^{(1)}(k_0r)}{\partial r^2} + \frac{1}{r}\frac{\partial H_0^{(1)}(k_0r)}{\partial r} + k_0^2 H_0^{(1)}(k_0r) = 0
\] (B.3)

whose solution can be expressed in asymptotic form for large values of \( k_0r \),

\[
H_0^{(1)}(k_0r) \simeq \sqrt{\frac{2}{\pi k_0 r}} e^{i(k_0r-\pi/4)}.
\] (B.4)

Substitution of Equation B.4 into Equation B.1, and assuming \( k_0r \gg 1 \) yields the simplified elliptic wave equation,

\[
\frac{\partial^2 \psi}{\partial r^2} + j2k_0 \frac{\partial \psi}{\partial r} + j2k_0 \frac{\partial^2 \psi}{\partial z^2} + k_0^2(n^2 - 1)\psi = 0.
\] (B.5)

Finally, one applies a paraxial approximation,

\[
\frac{\partial^2 \psi}{\partial r^2} \ll j2k_0 \frac{\partial \psi}{\partial r}
\] (B.6)

which eliminates the first term of Equation B.5. It restricts consideration to problems where the acoustic energy travels at angles \( 10 - 15^\circ \) from the horizontal axis, which is the case for the acoustic problems under study here. The result is the standard parabolic equation for underwater acoustics, as expressed by Tappert[56],

\[
j2k_0 \frac{\partial \psi}{\partial r} + j2k_0 \frac{\partial^2 \psi}{\partial z^2} + k_0^2(n^2 - 1)\psi = 0.
\] (B.7)

### B.3 Environmental Statistics

One assumes the index of refraction is a function of range and depth, \( n(r, z) \). The parabolic wave equation takes the index of refraction squared as an argument; one should use \( \eta = n^2 - 1 \) here. One can assume \( \eta \) is made up of two parts: a mean deterministic component, \( \eta_0 \), and a zero mean random component, \( \eta_1 \),

\[
\eta(r, z) = \eta_0(r, z) + \eta_1(r, z).
\] (B.8)
The following conditions are assumed on the random component, $\eta$:

\[
\langle \eta (r, z) \rangle = 0 \tag{B.9}
\]

\[
\langle \eta (r, z) \eta (r', z') \rangle = \delta (r - r') \Gamma (z - z') \tag{B.10}
\]

\[
\Gamma (z - z') = \langle \eta (z) \eta (z') \rangle \tag{B.11}
\]

The zero mean condition (Equation B.9) is assumed since $\eta_0$ incorporates any deterministic bias. The delta correlation (Equation B.10) in the propagation direction, $r$, assumes the transverse (vertical) statistics dominate the stochastic nature of the waveguide, and that correlation in depth occurs only across the same range. $\eta$ is assumed to be stationary in the wide sense (Equation B.11); $\Gamma$ depends only in the difference in depth between the two arguments, not on the absolute position in depth. While this does not accurately reflect realistic ocean conditions, it is suitable for the analytical results presented here.

### B.4 Novikov's Functional Derivative Formula

Novikov[57] developed a method to solve for the correlation between Gaussian random functions and dependent functionals. One specific method dealt with functionals which were delta correlated in time and homogeneous in space. Consider the one dimensional case, with Gaussian random function $f(s)$, and its correlation $\langle f(s) f(s') \rangle = R[s, s']$.

\[
\langle f(s) R[f] \rangle = \int ds' F[s, s'] \left( \frac{\partial R[f]}{\partial f(s')} ds' \right) \tag{B.12}
\]

Novikov’s objective was to prove this formula. The first step was to represent the function $R[s]$ as a functional Taylor series,

\[
R[s] = R[0] + \sum_{n=1}^{\infty} \frac{1}{n!} \int ds_1 \ldots \int ds_n R^n_{i_1, \ldots, i_n} (s_1, \ldots, s_n) f_{i_1}(s_1) \cdots f_{i_n}(s_n) \tag{B.13}
\]

where:

\[
R^n_{i_1, \ldots, i_n} (s_1, \ldots, s_n) = \left. \frac{\partial^n R[f]}{\partial f_{i_1}(s_1) ds_1 \cdots f_{i_n}(s_n) ds_n} \right|_{f=0} \tag{B.14}
\]

Multiplying by $f(s)$ and taking the average results in

\[
\langle f(s) R[f] \rangle = \sum_{n=1}^{\infty} \frac{1}{n!} \int ds_1 \ldots \int ds_n R^n_{i_1, \ldots, i_n} (s_1, \ldots, s_n) \langle f(s) f_{i_1}(s_1) \cdots f_{i_n}(s_n) \rangle \tag{B.15}
\]

The product of an even number of jointly Gaussian random variables can be expressed as a sum of the the mean values of all possible pairwise combinations[61]. The product of an odd number of jointly Gaussian random variables is zero. For example, given a jointly Gaussian set of zero mean random variables
with covariances \( A_{ik} = (x_i x_k) \), then for any set of integers \( i_1, i_2, \ldots, i_L \),
\[
\langle x_{i_1} x_{i_2} \cdots x_{i_L} \rangle = \begin{cases} 
0 & \text{L odd} \\
\sum \lambda_{k_1} \lambda_{k_2} \cdots \lambda_{k_{L-1}} k_L & \text{L even}
\end{cases} \tag{B.16}
\]

with the summation over all distinct pairings of \( i_1, i_2, \ldots, i_L \). This can be applied to the right hand side of Equation B.15 and rewritten as
\[
\langle f_i(s) f_i(s_1) \cdots f_{i_n}(s_n) \rangle = \sum_{\alpha=1}^{n} \langle f_i(s) f_{i_\alpha}(s_\alpha) \rangle \langle f_{i_\alpha-1}(s_{\alpha-1}) f_{i_{\alpha+1}}(s_{\alpha+1}) \cdots f_{i_{n}}(s_{n}) \rangle. \tag{B.17}
\]

Assuming \( F_{i_1}(s, s_1) = F_{i_2}(s, s_2) = \cdots = F_{i_{n}}(s, s_n) \), which can be inferred from the jointly Gaussian nature of \( f_i, f_{i_1}, \ldots, f_{i_n} \), then Equation B.17, can be substituted into Equation B.15 to yield
\[
\langle f_i(s) R[f] \rangle = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int ds_1 \int \cdots ds_n R^{(n)}(s_1, s_2, \ldots, s_n) \langle f_i(s) \cdots f_{i_n}(s_n) \rangle. \tag{B.18}
\]

In a parallel approach, the functional derivative of the Taylor series, Equation B.13 can be taken,
\[
\frac{\partial R[f]}{\partial f_k(s')} ds' = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int ds_2 \int \cdots ds_n R^{(n)}(s', s_2, \ldots, s_n) f_{i_2} \cdots f_{i_n}(s_n). \tag{B.19}
\]

Upon taking the average value of Equation B.19 and substituting it into Equation B.12, one obtains
\[
\langle f_k(s) R[f] \rangle = \int ds' F_{i_k}(s, s') \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int ds_2 \int \cdots ds_n R^{(n)}(s', s_2, \ldots, s_n) \langle f_{i_2} \cdots f_{i_n}(s_n) \rangle. \tag{B.20}
\]

Note Equations B.18 and B.20 are equivalent; this shows Equation B.12 to hold true.

### B.5 Mean Pressure Field Envelope

In solving for the second moment (correlation) statistics, of the pressure field envelope \( \psi \), one can start by solving for the first order (mean) statistics. Starting with the small angle, two dimensional Parabolic Wave Equation[56], and assuming \( \eta(r, z) \) is zero mean:
\[
j 2k_0 \frac{\partial \psi(r, z)}{\partial r} + \frac{\partial^2 \psi(r, z)}{\partial z^2} + k_0^2 \eta_1(r, z) \psi(r, z) = 0, \tag{B.21}
\]
one can take the average of the expression, substituting \( \bar{\psi}(r, z) = \langle \psi(r, z) \rangle \) and, in this two-dimensional case, the transverse Laplacian \( \nabla^2 \bar{\psi}(r, z) = \partial^2 \bar{\psi}(r, z)/\partial z^2 \).
\[ j2k_0 \frac{\partial \tilde{\psi}(r, z)}{\partial r} + \nabla^2 \tilde{\psi}(r, z) + k_0^2 \left( \psi_1(r, z) \psi(r, z) \right) = 0 \] (B.22)

One must now solve for \( g_1(r, z) \). Recall Novikov's functional derivative formula[57]:

\[ \langle \psi_1(r, z) Z[\psi] \rangle = \int dr' \int dz' \psi_1(r, z) \psi_1(r', z') \left( \frac{\partial Z[\psi]}{\partial \psi_1(r', z')} \right) \] (B.23)

One can substitute Equations B.10, B.11 and \( Z[\psi] = \psi \) to obtain

\[ g_1(r, z) = \int dz' \Gamma_\eta(z - z') \left( \frac{\partial \psi(r, z)}{\partial \psi_1(r, z')} \right) . \] (B.24)

One must solve for the average functional derivative, \( \left\langle \frac{\partial \psi(r, z)}{\partial \psi_1(r, z')} \right\rangle \). One can start again with the small angle Parabolic Equation B.21, and integrate with respect to \( r \),

\[ j2k_0 \psi(r, z) - j2k_0 \psi(0, z) + \nabla^2 \int_0^r \psi(\xi, z) d\xi + k_0^2 \int_0^r \eta_1(\xi, z) \psi(\xi, z) d\xi = 0 \] (B.25)

One can introduce the integrated delta function,

\[ \Theta(\xi) = \int_{-\infty}^{\xi} \delta(\xi') d\xi' = \begin{cases} 0 & \text{for } \xi < 0 \\ 1/2 & \text{for } \xi = 0 \\ 1 & \text{for } \xi > 0 \end{cases} \] (B.26)

and insert it into the last term of Equation B.25,

\[ j2k_0 \psi(r, z) - j2k_0 \psi(0, z) + \nabla^2 \int_0^r \psi(\xi, z) d\xi + k_0^2 \int_0^r \eta_1(\xi, z) \psi(\xi, z) d\xi \]

\[ + k_0^2 \Theta(r - x) \delta(z - z') \eta_1(\xi, z') \psi(\xi, z) \]

\[ = 0 . \] (B.27)

One then takes the functional derivative, \( \partial / \partial \eta_1(r', z') \), assuming the functional identity,

\[ \frac{\partial}{\partial \eta_1(r', z')} \eta_1(\xi, z') = \delta(\xi - r') \delta(z' - z'), \] (B.28)

to obtain

\[ j2k_0 \frac{\partial \psi(r, z)}{\partial \eta_1(r', z')} + \nabla^2 \int_0^r \frac{\partial \psi(\xi, z)}{\partial \eta_1(r', z')} d\xi \]

\[ + k_0^2 \Theta(r - r') \delta(z - z') \psi(r', z') \]

\[ + k_0^2 \int_0^\infty d\xi \int dz'' \Theta(r - \xi) \delta(z - z'') \eta_1(\xi, z'') \frac{\partial \psi(\xi, z'')}{\partial \eta_1(r', z')} = 0 \] (B.29)
One condition of the small-angle approximation assumes there is no backscatter (energy reflecting back toward the source). With this in mind, one can assume that the values of $\psi(\xi, z)$ depend only on the index of refraction, $\eta(r', z')$ when $r' < \xi$. Consequently, the functional derivative of the pressure field envelope,

$$\frac{\partial \psi(\xi, z)}{\partial \eta_1(r', z)} = 0 \quad \text{when } r' > \xi. \quad (B.30)$$

This allows one to set the lower limit of integration in Equation B.29 to $r'$. Integrating through (taking care to change the upper limit of integration in the last term, in accordance with Equation B.26), one obtains

$$j2k_0 \frac{\partial \psi(r, z)}{\partial \eta_1(r', z')} + \nabla^2 \psi + k_0^2 \Theta(r - r') \delta(z - z') \psi(r', z') + k_0^2 \int_{r'}^r d\xi \eta_1(\xi, z') \frac{\partial \psi(\xi, z)}{\partial \eta_1(r', z')} = 0. \quad (B.31)$$

As $r'$ approaches $r$, the integral terms vanish, resulting in

$$j2k_0 \frac{\partial \psi(r, z)}{\partial \eta_1(r', z')} + \frac{k_0^2}{2} \delta(z - z') \psi(r, z') = 0, \quad (B.32)$$

which can be rewritten and conjugated as

$$\frac{\partial \psi(r, z)}{\partial \eta_1(r', z')} = \frac{jk_0}{4} \delta(z - z') \psi(r, z), \quad (B.33)$$

$$\frac{\partial \psi^*(r, z)}{\partial \eta_1(r', z')} = -\frac{jk_0}{4} \delta(z - z') \psi^*(r, z). \quad (B.34)$$

This can be substituted into Equation B.24 to find

$$g_1(r, z) = \int dz' \Gamma(\eta)(z - z') \left\langle \frac{jk_0}{4} \delta(z - z') \psi(r, z) \right\rangle = \Gamma(\eta) \frac{jk_0}{4} \tilde{\psi}(r, z), \quad (B.35)$$

and finally arrive at the differential equation for the propagation of the mean acoustic field envelope,

$$j2k_0 \frac{\partial \tilde{\psi}(r, z)}{\partial r} + \nabla^2 \tilde{\psi}(r, z) + \frac{jk_0^3}{4} \Gamma(\eta) \tilde{\psi}(r, z) = 0. \quad (B.36)$$

### B.6 Mutual Coherence Function

A similar approach can be used to solve for the transverse mutual coherence function (otherwise known as the vertical covariance matrix),

$$\Gamma_{\psi}(r, z_1, z_2) = \langle \psi(r, z_1) \psi^*(r, z_2) \rangle. \quad (B.37)$$
One starts with the small angle, two-dimensional Parabolic Equation, Equation B.21, and substitutes \( z_1 \) for \( z \),

\[
j 2 k_0 \frac{\partial \psi(r, z_1)}{\partial r} + \frac{\partial^2 \psi(r, z_1)}{\partial z_1^2} + k_0^2 \eta_1(r, z_1) \psi(r, z_1) = 0, \tag{B.38}
\]

and then multiplies it by \( \psi^\ast(r, z_2) \),

\[
j 2 k_0 \frac{\partial \psi(r, z_1)}{\partial r} \psi^\ast(r, z_2) + \frac{\partial^2 \psi(r, z_1)}{\partial z_1^2} \psi^\ast(r, z_2) + k_0^2 \eta_1(r, z_1) \psi(r, z_1) \psi^\ast(r, z_2) = 0, \tag{B.39}
\]

Substituting \( \psi(r, z_2) \) for \( \psi(r, z_1) \) in Equation B.38, taking the conjugate of the equation, and multiplying the result by \( \psi(r, z_1) \), one finds

\[
-j 2 k_0 \frac{\partial \psi^\ast(r, z_2)}{\partial r} \psi(r, z_1) + \frac{\partial^2 \psi^\ast(r, z_2)}{\partial z_2^2} \psi(r, z_1) + k_0^2 \eta_1(r, z_2) \psi^\ast(r, z_1) \psi^\ast(r, z_2) = 0. \tag{B.40}
\]

Subtracting Equation B.40 from Equation B.39, and taking the average,

\[
j 2 k_0 \frac{\partial \Gamma \psi}{\partial r} + (\nabla^2_{z_1} - \nabla^2_{z_2}) \Gamma \psi + k_0^2 g_2(r, z_1, z_2) = 0 \tag{B.41}
\]

where:

\[
\frac{\partial \Gamma \psi}{\partial r} = \left( \frac{\partial \psi(r, z_1)}{\partial r} \psi^\ast(r, z_2) + \psi(r, z_1) \frac{\partial \psi^\ast(r, z_2)}{\partial r} \right) \tag{B.42}
\]

\[
\nabla^2_{z_1, z_2} \Gamma \psi = \frac{\partial^2}{\partial z_1^2} \left( \psi(r, z_1) \psi^\ast(r, z_2) \right) \tag{B.43}
\]

\[
g_2(r, z_1, z_2) = \langle \eta_1(r, z_1) - \eta_1(r, z_2) \rangle \psi(r, z_1) \psi^\ast(r, z_2) \tag{B.44}
\]

One must now evaluate \( g_2 \). Substituting \( Z(r, z_1, z_2) = \psi(r, z_1) \psi^\ast(r, z_2) \), into Equation B.23, one obtains the equation pair,

\[
\langle \eta_1(r, z_1) Z(r, z_1, z_2) \rangle = \int dr' \int dz_1' \langle \eta_1(r, z_1) \eta_1(r', z_1') \rangle \left( \frac{\partial Z(r, z_1, z_2)}{\partial \eta_1(r', z_1')} \right) \tag{B.45}
\]

\[
\langle \eta_1(r, z_1) Z(r, z_1, z_2) \rangle = \int dr' \int dz_2' \langle \eta_1(r, z_2) \eta_1(r', z_2') \rangle \left( \frac{\partial Z(r, z_1, z_2)}{\partial \eta_1(r', z_2')} \right). \tag{B.46}
\]

Using the directional delta assumption of Equation B.10, these can be rewritten as

\[
\langle \eta_1(r, z_1) Z(r, z_1, z_2) \rangle = \int dz_1' \Gamma_\eta(z_1 - z_1') \left( \frac{\partial Z(r, z_1, z_2)}{\partial \eta_1(r, z_1')} \right) \tag{B.47}
\]

\[
\langle \eta_1(r, z_2) Z(r, z_1, z_2) \rangle = \int dz_2' \Gamma_\eta(z_2 - z_2') \left( \frac{\partial Z(r, z_1, z_2)}{\partial \eta_1(r, z_2')} \right). \tag{B.48}
\]
with
\[
\frac{\partial Z(r, z_1, z_2)}{\partial \eta_1(r, z'_1)} = \frac{\partial \psi(r, z_1)}{\partial \eta_1(r, z'_1)} \psi^*(r, z_2) + \psi(r, z_1) \frac{\partial \psi^*(r, z_2)}{\partial \eta_1(r, z'_1)}.
\]

(B.49)

Application of Equations B.33 and B.34 results in

\[
\frac{\partial Z(r, z_1, z_2)}{\partial \eta_1(r, z'_1)} = \frac{j k_0}{4} \delta(z_1 - z'_1) \psi(r, z_1) \psi^*(r, z_2) - \frac{j k_0}{4} \delta(z_2 - z'_1) \psi(r, z_1) \psi^*(r, z_2).
\]

(B.50)

Combining terms and taking the average, one obtains

\[
\left< \frac{\partial Z(r, z_1, z_2)}{\partial \eta_1(r, z'_1)} \right> = \frac{j k_0}{4} \Gamma_\psi(r, z_1, z_2) [\delta(z_1 - z'_1) - \delta(z_2 - z'_1)].
\]

(B.51)

Substitution of this expression into Equation B.47 and evaluating the integral yields

\[
\langle \eta_1(r, z_1) Z(r, z_1, z_2) \rangle = \frac{j k_0}{4} [\Gamma_\eta(0) - \Gamma_\eta(z_1, z_2)] \Gamma_\psi(r, z_1, z_2).
\]

(B.52)

In a similar manner, the functional with respect to \(\eta(r, z'_1)\) can be evaluated and substituted into Equation B.48 to obtain

\[
\langle \eta_1(r, z_2) Z(r, z_1, z_2) \rangle = \frac{j k_0}{4} [\Gamma_\eta(z_1, z_2) - \Gamma_\eta(0)] \Gamma_\psi(r, z_1, z_2).
\]

(B.53)

Subtracting Equation B.53 from Equation B.52 yields an expression for \(g_2(r, z_1, z_2)\),

\[
g_2(r, z_1, z_2) = \langle \eta_1(r, z_1) - \eta_1(r, z_2) \rangle \psi(r, z_1) \psi^*(r, z_2) = \frac{j k_0}{2} [\Gamma_\eta(0) - \Gamma_\eta(z_1, z_2)] \Gamma_\psi(r, z_1, z_2)
\]

(B.54)

which, when substituted into Equation B.41 results in the parabolic equation for the Mutual Coherence Function of the pressure field envelope, \(\psi\),

\[
\left\{ j^2 k_0 \frac{\partial}{\partial r} + (\nabla^2_{z_1} - \nabla^2_{z_2}) + \frac{j k_0^3}{2} [\Gamma_\eta(0) - \Gamma_\eta(z_1, z_2)] \right\} \Gamma_\psi(r, z_1, z_2) = 0.
\]

(B.55)

### B.7 Mutual Coherence Function For Arbitrary Refractive Indices

Equations B.21, B.36, and B.55 all assume \(\eta(r, z)\) is a zero mean Gaussian random variable,

\[
\eta(r, z) = \eta_0(r, z) + \eta_1(r, z)
\]

(B.56)

where \(\eta_0(r, z)\) is zero for all values. Such a simplification does not allow one to consider realistic, depth dependent sound velocity profiles or their associated indices of refraction. Modification of the equations of propagation to allow for nonzero \(\eta_0\) is relatively straightforward; the result was summarized by Macaskill.
and Uscinski[16].

The small angle parabolic wave equation (B.21) can be modified by replacing \( \eta_i(r, z) \) with \( \eta(r, z) \) and expanding the result,

\[
j2k_0 \frac{\partial \psi(r, z)}{\partial r} + \frac{\partial^2 \psi(r, z)}{\partial z^2} + k_0^2 \eta_0(r, z) \psi(r, z) + k_0^2 \eta_1(r, z) \psi(r, z) = 0. \tag{B.57}
\]

A similar approach can be taken with mean pressure field envelope equation (B.36),

\[
\begin{align*}
j2k_0 \frac{\partial \tilde{\psi}(r, z)}{\partial r} + \nabla_t^2 \tilde{\psi}(r, z) + k_0^2 \langle \eta_0(r, z) + \eta_1(r, z) \rangle \psi(r, z) & = 0 \tag{B.58} \\
j2k_0 \frac{\partial \tilde{\psi}(r, z)}{\partial r} + \nabla_t^2 \tilde{\psi}(r, z) + k_0^2 \eta_0(r, z) \psi(r, z) + k_0^2 \langle \eta_1(r, z) \rangle \psi(r, z) & = 0 \\
j2k_0 \frac{\partial \tilde{\psi}(r, z)}{\partial r} + \nabla_t^2 \tilde{\psi}(r, z) + k_0^2 \eta_0(r, z) \psi(r, z) + \frac{jk_0^3}{4} \Gamma_\eta(0) \tilde{\psi}(r, z) & = 0.
\end{align*}
\]

For the Mutual Coherence Function, one can substitute \( \eta_0(r, z) \) into the expression for \( g_2(r, z_1, z_2) \), (Equation B.44), resulting in

\[
g'_2(r, z_1, z_2) = \langle \{ \eta_0(r, z_1) + \eta_1(r, z_1) \} - \{ \eta_0(r, z_2) + \eta_1(r, z_2) \} \rangle \psi(r, z_1) \psi^*(r, z_2)
\]

\[
= \langle \{ \eta_0(r, z_1) - \eta_0(r, z_2) \} + [\eta_1(r, z_1) - \eta_1(r, z_2)] \rangle \psi(r, z_1) \psi^*(r, z_2) 
\tag{B.59}
\]

\[
= \{ \eta_0(r, z_1) - \eta_0(r, z_2) \} \Gamma_\psi(r, z_1, z_2) + [\eta_1(r, z_1) - \eta_1(r, z_2)] \psi(r, z_1) \psi^*(r, z_2)
\]

Finally, placing Equation B.54 into the last term of Equation B.59 yields the modified MCF for an arbitrary deterministic refractive index,

\[
\left\{ j2k_0 \frac{\partial}{\partial r} + (\nabla_{t_1}^2 - \nabla_{t_2}^2) + k_0^2 [\eta_0(r, z_1) - \eta_0(r, z_2)] + \frac{jk_0^3}{2} [\Gamma_\eta(0) - \Gamma_\eta(z_1 - z_2)] \right\} \Gamma_\psi(r, z_1, z_2) = 0. \tag{B.60}
\]

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Appendix C

Detection of Deterministic and Random Signals and Estimation of their Non-Random Parameters

C.1 Introduction

Source localization using Matched Field Processing (MFP) can be characterized as a signal detection problem: one wishes to determine the presence or absence of a noise source at a particular spatial location. Detection Theory allows one to process multidimensional received sensor data, reducing it to a one dimensional detection statistic. The presence or absence of a signal is determined by the magnitude of this detection statistic. An extension to Detection Theory is Estimation Theory, where one attempts to estimate the spatial location (or other nonrandom parameter) of a signal.

This appendix reviews Detection and Estimation Theory, starting with detection of a known signal in white Gaussian noise, and ending with estimation of nonrandom parameters of a random process in colored Gaussian noise. The performance of these detectors and estimators is quantified as well. This provides the theoretical framework needed to perform source localization in a randomized ocean environment. All of this information can be found in the relevant literature[7, 8]. If the reader is familiar with detection and estimation theory, this appendix can be skipped. The formulas are provided primarily as a reference for chapters within the text.
C.2 Scalar Binary Hypothesis Detection

Suppose one wanted to determine if a constant scalar signal \( a \) were present or absent in noise, given a single observation, \( r \). One would have two possible scenarios:

\[
\begin{align*}
\text{if the signal } a \text{ were present,} & \quad (\text{Hypothesis } H_1) \\
\text{if no signal were present,} & \quad (\text{Hypothesis } H_0)
\end{align*}
\]

\( r = n + a \) \quad \text{(C.1)}

\[ r = n \]

where \( n \) is a zero mean, scalar, Gaussian random variable with variance \( \sigma_n^2 \). The probability density function (PDF) of \( n \) is

\[
p_n(N) = \frac{1}{\sqrt{2\pi \sigma_n^2}} \exp \left( \frac{N^2}{2\sigma_n^2} \right), \quad \text{(C.2)}
\]

while \( a \) is a deterministic scalar variable. Thus, the PDF of the received signals would be

\[
p_r(R|H_1) = \frac{1}{\sqrt{2\pi \sigma_n^2}} \exp \left[ \frac{(R - a)^2}{2\sigma_n^2} \right] \quad \text{(C.3)}
\]

for the case with a signal present in noise, \( (H_1) \), and

\[
p_r(R|H_0) = \frac{1}{\sqrt{2\pi \sigma_n^2}} \exp \left[ \frac{R^2}{2\sigma_n^2} \right] \quad \text{(C.4)}
\]

for the case of noise only, \( (H_0) \).

With these two PDF’s, a likelihood ratio test (LRT) can be constructed. The test allows one to set a threshold, \( \gamma \), for determining which outcome is more likely: \( H_0 \) or \( H_1 \), and compare a function of \( r \) with that threshold.

\[
\Lambda(R) = \frac{p_r(R|H_1)}{p_r(R|H_0)} \geq_{H_0} H_1 \gamma \\
= \frac{1}{\sqrt{2\pi \sigma_n^2}} \exp \left[ \frac{(R - a)^2}{2\sigma_n^2} \right] \geq_{H_0} H_1 \gamma
\]

\[ \text{(C.5)} \]

Dividing terms, one obtains

\[
\Lambda(R) = \exp \left[ \frac{(R - a)^2}{2\sigma_n^2} + \frac{R^2}{2\sigma_n^2} \right] \geq_{H_0} H_1 \gamma \\
= \exp \left[ \frac{2Ra - a^2}{2\sigma_n^2} \right] \geq_{H_0} H_1 \gamma
\]

\[ \text{(C.6)} \]

and taking the natural logarithm.
In $A(R-a) - R - \ln y.$ (C.7)

to obtain the LRT in this binary detection case. One can re-arrange the expression to isolate the observed variable, $R,$

$$R \gtrless_{H_0} \frac{\sigma_n^2}{a} \ln \gamma + \frac{a}{2}.$$ (C.8)

In the noiseless case, $\sigma_n = 0,$ the LRT collapses down to a simple test to see if the received signal is greater than one-half the expected signal, $a.$

This illustrated the basic case of detecting the presence of a known constant signal using a single observation. It was intended to demonstrate the formulation of the Likelihood Ratio Test (LRT) as a method for binary hypothesis testing in additive white Gaussian noise. The next step is to extend this example to the case where one must make a binary test using multiple observations.

### C.3 Vector Binary Hypothesis Detection

An extension of the scalar binary hypothesis detection case is one where multiple independent observations can be made to determine the presence or absence of a signal. Each observation is $r_i,$ where $i$ ranges from 1 to $N,$ the total number of observations. The form of $r_i$ is similar to that of the scalar case,

$$r_i = n_i + a \quad \text{if the signal } a \text{ were present, (Hypothesis } H_1)$$

$$r_i = n_i \quad \text{if no signal were present, (Hypothesis } H_0)$$ (C.9)

where $n_i$ is an independent, identically distributed, scalar, zero mean Gaussian random variable with variance $\sigma_n^2.$ The $N$ observations can be grouped into a vector, $r,$ as can the $N$ noise samples, $n.$ Thus, the received signal can be rewritten in vector form,

$$r = n + a \mathbf{1} \quad \text{if the signal } a \text{ were present, (Hypothesis } H_1)$$

$$r = n \quad \text{if no signal were present, (Hypothesis } H_0),$$ (C.10)

with $\mathbf{1}$ as a $N$ by 1 vector of ones. The probability density function for both hypothesis cases is a $N$ dimensional Gaussian PDF, which can be written as

$$p_r(r|H_1) = \frac{1}{\sqrt{(2\pi)^N |\Gamma_n|}} \exp \left[ -\frac{1}{2} (R - a \mathbf{1})^\dagger \Gamma_n^{-1} (R - a \mathbf{1}) \right]$$ (C.11)

$$p_r(r|H_0) = \frac{1}{\sqrt{(2\pi)^N |\Gamma_n|}} \exp \left[ -\frac{1}{2} R^\dagger \Gamma_n^{-1} R \right]$$ (C.12)

with $\Gamma_n = \sigma_n^2 \mathbf{1},$ the covariance matrix of the noise vector, $n,$ and $|\Gamma_n|$ is the determinant (product of the
eigenvalues) of $\Gamma_n$. Following the same procedure as in the scalar case, a likelihood ratio test can be formed by dividing the PDF's of the two hypotheses,

$$
\Lambda(R) = \frac{p_r(R|H_1)}{p_r(R|H_0)} = \exp \left[ -\frac{1}{2} (R - a1)^T \Gamma_n^{-1} (R - a1) + \frac{1}{2} R^T \Gamma_n^{-1} R \right] \geq_{H_0}^{H_1} \ln \gamma
$$

$$
= \exp \left[ -\frac{1}{2} (a1^T \Gamma_n^{-1} 1a - a1^T \Gamma_n^{-1} R - R^T \Gamma_n^{-1} 1a) \right] \geq_{H_0}^{H_1} \ln \gamma. \quad (C.13)
$$

Taking the logarithm and converting the vector operations to summations,

$$
\ln \Lambda(R) = -\frac{1}{2} \sum_{i=1}^{N} \left( \frac{a^2}{\sigma_n^2} - \frac{ar_i}{\sigma_n^2} - \frac{ar_i}{\sigma_n^2} \right)
$$

$$
= -\frac{a}{\sigma_n^2} \sum_{i=1}^{N} r_i - \frac{Na^2}{2\sigma_n^2} \geq_{H_0}^{H_1} \ln \gamma. \quad (C.14)
$$

Isolating the observed statistic $r_i$,

$$
\frac{1}{N} \sum_{i=1}^{N} r_i \geq_{H_0}^{H_1} \frac{\sigma_n^2}{aN} \ln \gamma + \frac{a}{2} \quad (C.15)
$$

shows the LRT can be expressed as sample mean of the observed statistics. As one increases $N$, the number of observations, the effect of $\sigma_n^2$ decreases. As $N$ approaches infinity (or, if $\sigma_n^2 = 0$), the result is identical to the single observation case: if the observation is greater than one-half the expected signal, $a$, then the signal is present; if not, the signal is absent.

### C.4 Detection of Known Signals in Additive White Gaussian Noise

The previous two sections allow one to formulate a binary hypothesis detection method, based on $N$ independent observations of a constant signal in noise. In many cases, the signal to be detected may vary with time or location. Instead of a scalar constant $a$, the signal may be represented by the waveform $a \times s(t)$ or a scaled vector $as$. Assuming $a$ and $s$ are deterministic and real, with $\|s\|_2 = 1$, one can derive a binary hypothesis test for detection.

Consider the received signal model,

$$
r = n + as \quad \text{if the signal } a \times s \text{ were present}, \quad (\text{Hypothesis } H_1)
$$

$$
r = n \quad \text{if no signal were present}, \quad (\text{Hypothesis } H_0), \quad (C.16)
$$

where $r$ is the $N$ element observation vector, and $s$ is the $N$ element deterministic signal to be detected. As in the previous case, each element of $n$ represents a zero mean real, uncorrelated Gaussian random process.
with variance $\sigma_n^2$.

While it would be possible to follow the Vector Binary Hypothesis detection method outlined in the last section, a simpler method is available which reduces the problem down to a Scalar Binary Hypothesis Test. The observed signal, $r$, can be expressed as the weighted sum of $N$ orthonormal vectors,

$$ r = \sum_{i=1}^{N} \lambda_i u_i $$

where: $u_i^\dagger u_j = \delta(i - j)$ \hspace{1cm} (C.17)

One can premultiply Equation C.17 by $u_j^\dagger$ and obtain an expression for $\lambda_j$,

$$ \lambda_j = u_j^\dagger r, \hspace{1cm} (C.18) $$

The basis vectors $u_1 \cdots u_N$ form a complete orthonormal set, and can be used to represent any observed vector $r$, provided the correct weight coefficients $\lambda_1, \lambda_2, \ldots, \lambda_N$ are chosen. In a similar manner, the noise vector $n$ can be expanded using the same basis into a series of coefficients, $\xi_1, \xi_2, \ldots, \xi_N$. The only restriction on the basis vectors is their orthogonality; one can even choose $s$ to be the first basis vector, $u_1$, provided it is orthonormal to all other basis vectors, and they are orthonormal to each other.

Using $s$ as the first basis vector, one can use Equation C.18 to reduce the observed signal to a series of $N$ coefficients, depending on whether the signal is present ($H_1$) or absent ($H_0$).

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>$H_1$: signal present</th>
<th>$H_0$: no signal present</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>$u_1^\dagger (as + n)$</td>
<td>$u_1^\dagger n$</td>
</tr>
<tr>
<td></td>
<td>$= s^\dagger (as + n)$</td>
<td>$= a + \xi_1$</td>
</tr>
<tr>
<td></td>
<td>$= \zeta_1$</td>
<td>$\zeta_1$</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>$u_2^\dagger (as + n)$</td>
<td>$u_2^\dagger n$</td>
</tr>
<tr>
<td></td>
<td>$= \zeta_2$</td>
<td>$\zeta_2$</td>
</tr>
<tr>
<td></td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\lambda_N$</td>
<td>$u_N^\dagger (as + n)$</td>
<td>$u_N^\dagger n$</td>
</tr>
<tr>
<td></td>
<td>$= \zeta_N$</td>
<td>$\zeta_N$</td>
</tr>
</tbody>
</table>

From (C.19) one sees that the only coefficient which provides information for determining which hypothesis is true is $\lambda_1$. Thus, the decision statistic can be reduced to the scalar case,

$$ \lambda_1 = \xi_1 + a \hspace{1cm} \text{if the signal \textit{as} were present,} \hspace{1cm} (\text{Hypothesis } H_1) $$

$$ \lambda_1 = \xi_1 \hspace{1cm} \text{if no signal were present,} \hspace{1cm} (\text{Hypothesis } H_0) $$

and following Equations C.7 and C.8, one can use the LRT expressed in terms of $\lambda_1$,

$$ \ln \Lambda(\lambda_1) = \frac{a}{\sigma_n^2} \left( \lambda_1 - \frac{a}{2} \right) \overset{H_1}{\gtrless} H_0: \ln \gamma. \hspace{1cm} (C.21) $$
\[ \lambda_1 \gtrless_{H_0} a \frac{\sigma_n^2}{a} \ln \gamma + \frac{a}{2}. \]  

(C.22)

Substituting Equation C.18, one arrives at a decision statistic in terms of the received data,

\[ \ln \Lambda(s^\dagger r) = \frac{a}{\sigma_n^2} \left(s^\dagger r - \frac{a}{2}\right) \gtrless_{H_0} \ln \gamma. \]  

(C.23)

\[ s^\dagger r \gtrless_{H_0} a \frac{\sigma_n^2}{a} \ln \gamma + \frac{a}{2}. \]  

(C.24)

In the zero noise case (\(\sigma_n^2 = 0\)), the decision between Hypothesis \(H_1\) and \(H_0\) reduces to whether the output of the received signal correlator, \(s^\dagger r\) is greater than \(a/2\), half the expected gain of the received signal.

### C.5 Detection of Known Signals with Unknown Amplitude in Additive White Gaussian Noise

In many cases the signal which must be detected is known, but not its amplitude. This can occur when the signal is attenuated by an unknown amount. For example, suppose one did not know the distance from the signal source, but did know that the amplitude of the source signal decreases with range. Such a detector would need to distinguish the presence of the signal, regardless of its amplitude.

Consider the following signal model:

\[ r = as + n \quad \text{Hypothesis } H_1 \text{ (signal present)} \]
\[ r = n \quad \text{Hypothesis } H_0 \text{ (signal absent)} \]  

(C.25)

which describes the detection scenario. \(r\) is a \(N \times 1\) column vector which has the received signal, from \(N\) receiver elements in an array. \(s\) is a \(N \times 1\) vector with unit magnitude which contains the known deterministic signal to be detected. \(a\) is a zero mean, scalar Gaussian random variable, with variance \(\sigma_a^2\), representing the unknown amplitude of signal \(s\). The \(N \times 1\) noise vector \(n\) is uncorrelated, random Gaussian noise with variance \(\sigma_n^2\).

Construction of a useful Likelihood Ratio Test (LRT) requires one to have knowledge of the statistics of the received signal. As \(r\) is Gaussian under both hypotheses, one need only determine its mean and variance in order to fully characterize the signal.

\[ E[r|H_0] = E[n] = 0 \]
\[ E[r|H_1] = E[as + n] = 0 \]  

(C.26)
\[
\Gamma_{r|H_0} = E[rr^\dagger|H_0] = E[nn^\dagger] = \Gamma_n \\
\Gamma_{r|H_1} = E[rr^\dagger|H_1] = E[(as + n)(as + n)^\dagger] = \sigma_s^2 ss^\dagger + \Gamma_n
\] (C.27)

The only statistic which differs between \(H_0\) and \(H_1\) is the \(\sigma_s^2 ss^\dagger\) term in \(\Gamma_{r|H_1}\). Using a Karhunen-Loeve expansion, one can decompose the received signal covariance matrix into a weighted orthonormal basis,

\[
\Gamma_{r|H_1} = U \Sigma_{r|H_1} U^\dagger \\
= U_s (\Sigma_s + \Sigma_n) U_s^\dagger
\] (C.28)

where \(\Sigma_{r|H_1}\) is a diagonal matrix containing the singular values of \(\Gamma_{r|H_1}\), and \(U\) is a unitary matrix with orthonormal columns. As \(\Gamma_n\) is already diagonal, one can choose the basis vectors of \(U\) so that they diagonalize \(\Gamma_s\). Since \(\Gamma_s\) is rank 1, one can diagonalize the covariance matrix by setting the first singular vector equal to \(s\), and letting the remaining singular vectors be orthonormal to \(s\) and each other. The resulting basis matrix is \(U_s\), and the signal singular value matrix \(\Sigma_s\) has one entry in the upper-right corner, \(\sigma_s^2\). The same orthogonalization can be applied to the received covariance matrix in the signal-absent hypothesis, \(H_0\),

\[
\Gamma_{r|H_0} = U_s \Sigma_n U_s^\dagger.
\] (C.29)

The only difference between Equations C.28 and C.29 is the presence of the rank 1 diagonal signal matrix \(\Sigma_s\). It would be useful to process the signal in such a way that the received covariance matrix were initially diagonal. This can be done by correlating the received signal \(r\) with the anticipated signal \(s\),

\[
r' = s^\dagger r
\] (C.30)

Likewise, the received signal statistics can be expressed in terms of \(r'\),

\[
E[r'|H_0] = 0 \quad E[r'|H_1] = 0 \\
\sigma_{r'|H_0}^2 = \sigma_n^2 \quad \sigma_{r'|H_1}^2 = \sigma_a^2 + \sigma_n^2.
\] (C.31)

Using the new, uncorrelated statistics, one can obtain the PDF of the processed signal, \(r'\) under both Hypotheses.

\[
P_{r'|H_1}(R'|H_1) = \frac{1}{\sqrt{2\pi \sigma_{r'|H_1}^2}} \exp \left[ -\frac{1}{2} \frac{|R' - E[R'|H_1]|^2}{\sigma_{r'|H_1}^2} \right] \\
= \frac{1}{\sqrt{2\pi (\sigma_a^2 + \sigma_n^2)}} \exp \left[ -\frac{1}{2} \frac{|R'|^2}{\sigma_a^2 + \sigma_n^2} \right]
\]
\[ p_{r|H_0}(R'|H_0) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp \left[ -\frac{1}{2} \frac{|R'|^2}{\sigma_n^2} \right] \]  \hspace{1cm} (C.32)

Dividing the two PDFs, one arrives at the Likelihood Ratio Test (LRT),

\[ \Lambda(R') = \frac{p_{r|H_1}(R'|H_1)}{p_{r|H_0}(R'|H_0)} = \sqrt{\frac{\sigma_n^2}{\sigma_n^2 + \sigma_a^2}} \exp \left[ \frac{1}{2} \left( \frac{\sigma_n^2}{\sigma_n^2} \right) \frac{|R'|^2}{\sigma_n^2 + \sigma_a^2} \right] \geq H_0 \gamma \]  \hspace{1cm} (C.33)

Taking the natural logarithm of the LRT yields the log-likelihood detection statistic,

\[ \ln \Lambda(R') = \frac{1}{2} \left( \frac{\sigma_n^2}{\sigma_n^2 + \sigma_a^2} - \ln \left( 1 + \frac{\sigma_a^2}{\sigma_n^2} \right) \right) \geq H_1 \ln \gamma \]  \hspace{1cm} (C.34)

which provides a criterion for signal detection in Gaussian noise. This expression contrasts with that for a signal with known amplitude in noise (Equation C.23), in the operations performed on \( r \). The detector for the known amplitude case is simply a correlation operation; it is a linear function of \( r \). Consequently, the detection statistic in Equation C.23 is Gaussian. The current scenario of unknown signal amplitude is different. Instead of a linear operation on \( r \), Equation C.34 is a quadratic function of the received data. Thus, the detection statistic is not Gaussian. This will have implications in the future, when evaluating detection performance.

### C.6 Random Process Detection in Additive White Gaussian Noise

Some problems require detection of a signal in noise when only the statistics of the signal are known, and not the signal itself. One example of this problem surfaces when a known signal is perturbed in a random way by the propagation environment.

Under these constraints, one can model both Hypotheses of the received signal as

\[ r = n + s \quad \text{if the signal were present, \ (Hypothesis } H_1) \]

\[ r = n \quad \text{if no signal were present, \ (Hypothesis } H_0) \]  \hspace{1cm} (C.35)

with \( r \) as an \( N \) element vector of observations, \( n \) is a zero mean white Gaussian random process with covariance matrix \( \sigma_n^2 I \), and \( s \) is a Gaussian random process.

The simplest statistics for a random process are its mean, \( \bar{s} \) and covariance, \( \Gamma_s \). One assumes these statistics are known in advance. Our objective is to derive similar statistics for the received signal, and compare them to the signal statistics.

The mean and covariance of the received signal can be derived in a straightforward manner. Consider:

<table>
<thead>
<tr>
<th>Received Signal Statistic</th>
<th>( H_1 ): signal present</th>
<th>( H_0 ): signal absent</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{r} )</td>
<td>( E[n] + E[s] )</td>
<td>( E[n] )</td>
</tr>
</tbody>
</table>
|                           | \( \bar{s} \)            | 0                        |  \hspace{1cm} (C.36)  

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The covariance can be calculated by taking the expected value of the outer product of the unbiased vector \( r \).

Assuming for Hypothesis \( H_1 \) the signal and noise are uncorrelated, the covariance can be calculated as

\[
\Gamma_{r|H_1} = E \left[ (r - \bar{r})(r - \bar{r})^\dagger \right]
= E \left\{ [n + (s - \bar{s})] [n + (s - \bar{s})]^\dagger \right\}
= \Gamma_n + \Gamma_s
\]

(C.37)

For the case of no signal present (Hypothesis \( H_0 \)), the covariance is simply

\[
\Gamma_{r|H_0} = E \left[ (r - \bar{r})(r - \bar{r})^\dagger \right]
= E [nn^\dagger]
= \Gamma_n.
\]

(C.38)

Given the mean and covariance for both Hypotheses, it is possible to give their probability density functions,

\[
p_r(R|H_1) = \frac{1}{\sqrt{2\pi N} |\Gamma_r|} \exp \left\{ -\frac{1}{2} (R - E[R])^\dagger (\Gamma_r^{-1} (R - E[R])) \right\}
\]

\[
= \frac{1}{\sqrt{2\pi N} |\Gamma_n + \Gamma_s|} \exp \left\{ -\frac{1}{2} (R - \bar{s})^\dagger (\Gamma_n + \Gamma_s)^{-1} (R - \bar{s}) \right\}
\]

\[
p_r(R|H_0) = \frac{1}{\sqrt{2\pi N} |\Gamma_n|} \exp \left\{ -\frac{1}{2} R^\dagger \Gamma_n^{-1} R \right\}.
\]

(C.39)

(C.40)

Dividing the two PDF's yields a likelihood ratio test (LRT),

\[
\Lambda(R) = \frac{p_r(R|H_1)}{p_r(R|H_0)} = \sqrt{\frac{|\Gamma_n|}{|\Gamma_n + \Gamma_s|}} \exp \left\{ -\frac{1}{2} \left[ (R - \bar{s})^\dagger (\Gamma_n + \Gamma_s)^{-1} (R - \bar{s}) - R^\dagger \Gamma_n^{-1} R \right] \right\} \geq_{H_0} \gamma
\]

(C.41)

which can be evaluated with the appropriate substitutions. Unfortunately, the large number of matrix multiplications can make solving the LRT cumbersome. The expression could be simplified if the observed vector \( R \) were composed of a series of uncorrelated random processes.

A Karhunen-Loeve (KL) expansion allows one to express a Gaussian covariance matrix as the sum of deterministic orthonormal vectors, weighted by uncorrelated Gaussian random variables. One can perform a KL expansion on \( \Gamma_s \) and express the covariance as the weighted sum of deterministic orthonormal vectors,

\[
\Gamma_s = U_s \Sigma_s U_s^\dagger = \sum_{i=1}^N \sigma_{s_i}^2 u_i u_i^\dagger,
\]

where \( \Sigma_s \) is a diagonal matrix with elements \( \sigma_{s_i}^2 \): zero mean, uncorrelated Gaussian random variables. If one
were to perform a basis rotation of the received vector \( \mathbf{R} \) by premultiplying it with the unitary matrix \( U_L \), then the receiver covariance matrix would become diagonal under both Hypotheses,

\[
\mathbf{R}' = U_L^* \mathbf{R} \quad \quad E[\mathbf{R}'] = U_L^* E[\mathbf{R}] \quad \quad \Gamma_{\mathbf{r}'} = U_L^* \Gamma_r U_L
\]  

(C.43)

<table>
<thead>
<tr>
<th>Received Signal Statistic</th>
<th>( H_1 ): signal present</th>
<th>( H_0 ): signal absent</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{r}' )</td>
<td>( \tilde{s}' = U_L^* \tilde{s} )</td>
<td>0</td>
</tr>
<tr>
<td>( \Gamma_{\mathbf{r}'} )</td>
<td>( \Sigma_u + \Gamma_n )</td>
<td>( \Gamma_n )</td>
</tr>
</tbody>
</table>

Substituting this information into the expression for the LRT, Equation C.41, yields a new LRT with simplified, diagonal matrix inversions,

\[
\Lambda(\mathbf{R}') = \sqrt{\frac{|\Gamma_n|}{|\Gamma_n + \Sigma_u|}} \exp \left\{ -\frac{1}{2} \left[ (\mathbf{R}' - \tilde{s}')^\dagger (\Gamma_n + \Sigma_u)^{-1} (\mathbf{R}' - \tilde{s}') - \mathbf{R}'(\Gamma_n^{-1} \mathbf{R}') \right] \right\} \geq \chi^2_{H_0, \gamma}.
\]  

(C.45)

Taking the natural logarithm of both sides and simplifying,

\[
\ln \Lambda(\mathbf{R}') = -\frac{1}{2} \ln \left( \prod_{i=1}^{N} \frac{\sigma_n^2 + \sigma_{\tilde{s}_i}^2}{\sigma_n^2} \right) - \frac{1}{2} \sum_{i=1}^{N} \frac{(r_i' - \tilde{s}_{i'})^* (r_i' - \tilde{s}_{i'})}{\sigma_n^2 + \sigma_{\tilde{s}_i}^2} + \frac{1}{2} \sum_{i=1}^{N} \frac{r_i'^* r_i'}{\sigma_n^2} \geq \chi^2_{H_1, \gamma}.
\]

(C.46)

Examining Equation C.46, one can see some of the practical aspects of this detector. The first term represents the squared output of the correlation operation, weighted by the SNR of each orthogonal component. \( r_i' \) represents the output of the received signal with the \( i^{th} \) orthogonal component of the covariance matrix, \( \Gamma_r \). The second term is a normalized correlation of the \( i^{th} \) component of the received signal with the mean statistics of the anticipated signal, \( \tilde{s} \). The third term is a bias term based on the natural logarithm of the input Signal to Noise Ratio (SNR).

Finally, one can consider a trivial case. If the anticipated signal were zero mean, Equation C.46 would reduce to the \( N \)-dimensional version of Equation C.34,

\[
\ln \Lambda(\mathbf{R}') = \sum_{i=1}^{N} \left( \frac{\sigma_{\tilde{s}_i}^2}{\sigma_n^2} \right) \frac{r_i'^* r_i'}{\sigma_n^2 + \sigma_{\tilde{s}_i}^2} - \sum_{i=1}^{N} \ln \left( 1 + \frac{\sigma_{\tilde{s}_i}^2}{\sigma_n^2} \right) \geq \chi^2_{H_1, \gamma}.
\]  

(C.47)
C.7 Rank-1 Random Process Detection in the Presence of Colored Noise

One interesting detection scenario is when the source signal is zero mean random, but its covariance matrix has a rank of 1, while the noise is also Gaussian, but colored. The detection algorithm outlined in the previous section must be modified to handle this special case. Equation C.36 illustrates the signal detection model. Since the source and noise are both assumed to be zero mean, their covariance are the only differing statistics,

\[ \Gamma_{r|H_0} = \Gamma_n \quad \Gamma_{r|H_1} = \Gamma_s + \Gamma_n \quad \Gamma_s = \sigma_s^2 u_{s1} u_{s1}^T, \]  

where \( u_{s1} \) is the first column of the unitary matrix of singular vectors, \( U_s \). One can perform a likelihood ratio test (LRT) by dividing the probability density function (PDF) of the two hypotheses,

\[ \Lambda(R) = \frac{Pr[H_1|R,H_0]}{Pr[H_0|R,H_0]} = \sqrt{\frac{(2\pi)^N |\Gamma_n|}{(2\pi)^N |\Gamma_s + \Gamma_n|}} \exp \left[ -\frac{1}{2} \left( R^T [\Gamma_s + \Gamma_n]^{-1} R - R^T [\Gamma_n]^{-1} R \right) \right] \]

\[ = \sqrt{\frac{|\Gamma_n|}{|\Gamma_s + \Gamma_n|}} \exp \left[ -\frac{1}{2} \left( R^T \left( \sigma_s^2 u_{s1} u_{s1}^T + \Gamma_n \right)^{-1} - \Gamma_n^{-1} \right) R \right] \]  

One can evaluate the two term matrix inverse by using Woodbury's Identity,

\[ (A + BCD)^{-1} = A^{-1} - A^{-1} B \left( D A^{-1} B + C^{-1} \right)^{-1} D A^{-1} \]

where \( A, B, C, \) and \( D \) are all matrices, which yields

\[ \Lambda(R) = \frac{\sqrt{|U_s^T \Gamma_n U_s|}}{|\Gamma_s + U_s^T \Gamma_n U_s|} \exp \left[ -\frac{1}{2} \left( R^T \left( \Gamma_n^{-1} - \Gamma_n^{-1} u_{s1} \left( u_{s1}^T \Gamma_n^{-1} u_{s1} + \frac{1}{\sigma_s^2} \right)^{-1} u_{s1}^T \Gamma_n^{-1} - \Gamma_n^{-1} \right) \right) R \right] \]

\[ = \frac{\left[ U_s^T \Gamma_n U_s \right]_{11}}{\sigma_s^2 + \left[ U_s^T \Gamma_n U_s \right]_{11}} \exp \left[ \frac{1}{2} \left( R^T \Gamma_n^{-1} u_{s1} \left( u_{s1}^T \Gamma_n^{-1} u_{s1} + \frac{1}{\sigma_s^2} \right)^{-1} u_{s1}^T \Gamma_n^{-1} R \right) \right] \]

\[ = \left[ 1 + \sigma_s^2 u_{s1} \Gamma_n^{-1} u_{s1} \right]^{-1/2} \exp \left[ \frac{1}{2} \left( \frac{\sigma_s^2 |u_{s1} \Gamma_n^{-1} R|^2}{\sigma_s^2 u_{s1} \Gamma_n^{-1} u_{s1} + 1} \right) \right] \]  

where \( \left[ U_s^T \Gamma_n U_s \right]_{11} \) is the first (upper-left) element of the matrix \( U_s^T \Gamma_n U_s \). Taking the natural logarithm and applying a threshold, one obtains the log-likelihood ratio test,

\[ \ln \Lambda(R) = \frac{1}{2} \left( \frac{\sigma_s^2 |u_{s1} \Gamma_n^{-1} R|^2}{\sigma_s^2 u_{s1} \Gamma_n^{-1} u_{s1} + 1} \right) - \frac{1}{2} \ln \left( 1 + \sigma_s^2 u_{s1} \Gamma_n^{-1} u_{s1} \right) \approx H_0 : \ln \gamma. \]

The rank-1 nature of the source signal permits the detection statistic to be one dimensional; the numerator and denominator of the first term in Equation C.52 are both scalar. This formula is useful for detecting
random rank-1 signals in the presence of colored noise. A more general approach for detection of rank-N signals in colored Gaussian noise will be covered in the next section.

C.8 Rank-N Random Process Detection in the Presence of Colored Noise

Previous detection methods have assumed the noise component $n$ is uncorrelated, white, Gaussian random noise. This is not always the case; unwanted noise can be highly directional or colored. One would like to be able to pose the signal detection problem as a LRT, similar to that given in Equation C.47 for a zero mean signal. One can apply a “whitening” filter to the signal, to convert the noise from colored to white.

Consider noise which is made up of a white and colored part, both of which are Gaussian but uncorrelated with each other:

$$r_n = b + n$$  \hspace{1cm} (C.53)

The colored noise component $b$ can result from any unwanted signal, including single or multiple spatial jammers. Assuming both signals are zero mean, and both noise covariance matrices are known, the total noise covariance matrix is

$$\Gamma_{r_n} = \Gamma_b + \sigma_n^2 I.$$  \hspace{1cm} (C.54)

One can decompose $\Gamma_{r_n}$ into its singular values and singular vectors,

$$\Gamma_{r_n} = \Gamma_{bn} = U_b [\Sigma_b + \sigma_n^2 I] U_b^\dagger = U_b \Sigma_{bn} U_b^\dagger$$  \hspace{1cm} (C.55)

with $U_b$ a unitary matrix of singular values of $\Gamma_b$, and $\Sigma_b$ a diagonal matrix of the singular values of $\Gamma_b$.

Inverting this,

$$\Gamma_{bn}^{-1} = U_b \Sigma_{bn}^{-1} U_b^\dagger$$  \hspace{1cm} (C.56)

one can obtain the inverse matrix square root,

$$\Gamma_{bn}^{-1} = U_b \Sigma_{bn}^{-1/2} \left[ \Sigma_{bn}^{-1/2} \right]^\dagger U_b^\dagger.$$  \hspace{1cm} (C.57)

As $\Gamma_{bn}$ is a positive definite Hermitian matrix, all its singular values are real, so $\Sigma_{bn}^{-1/2} = \left[ \Sigma_{bn}^{-1/2} \right]^\dagger$. The matrix square root is the “whitening” filter,

$$H_w = U_b \Sigma_{bn}^{-1/2}$$  \hspace{1cm} (C.58)
so-called because it converts the Gaussian noise from colored to white. Consider pre-multiplying the received noise vector in Equation C.53 by $H_w^t$,

$$r_w = H_w^t r_n$$  \hspace{1cm} (C.59)

and then solving for the noise covariance matrix,

$$\Gamma_{r_w} = E \left[ H_w^t r_n (H_w^t r_n)^t \right]$$

$$= H_w^t E \left[ r_n r_n^t \right] H_w$$

$$= H_w^t U_b \Sigma_{bn} U_b^t H_w$$

$$= \left[ \Sigma_{bn}^{-1/2} \right]^t U_b^t U_b \Sigma_{bn} U_b^t U_b \Sigma_{bn}^{-1/2}$$

$$= \left[ \Sigma_{bn}^{-1/2} \right]^t \Sigma_{bn} \Sigma_{bn}^{-1/2}$$

$$= I. \hspace{1cm} (C.60)$$

Application of the whitening filter $H_w$ to the received noise vector $r_n$ converts the correlated colored noise to a spatially uncorrelated white noise with unit variance.

When prescribed to a received vector with a desired Gaussian signal component, $H_w$ does not alter the Gaussian nature of the signal. The linear operations do change the signal covariance matrix. Consider a zero mean random signal $s$ with known covariance matrix $\Gamma_s$. Such a signal received in the presence of noise,

$$r = s + b + n$$  \hspace{1cm} (C.61)

would (assuming the signal is uncorrelated with the noise) have a received covariance matrix of

$$\Gamma_r = \Gamma_s + \Gamma_{bn}$$  \hspace{1cm} (C.62)

Application of the whitening filter would modify the received covariance matrix,

$$\Gamma_{r_w} = H_w^t \Gamma_s H_w + H_w^t \Gamma_{bn} H_w$$

$$= \Gamma_{s_w} + I$$  \hspace{1cm} (C.63)

$$= U_{s_w} \Sigma_{s_w} U_{s_w}^t + I$$  \hspace{1cm} (C.64)

with $U_{s_w}$ and $\Sigma_{s_w}$ the matrices of singular vectors and values of $\Gamma_{s_w}$, respectively.

Equations C.60 and C.64 allows one to construct a binary Hypothesis test in the zero mean non-Gaussian noise case. Passing the received signal through both the whitening filter and the basis rotation yields the modified received signal vector and statistics,

$$r'' = U_{s_w}^t H_w^t r$$  \hspace{1cm} (C.65)
Received Signal Statistic  \( H_1 \): signal present  \( H_0 \): signal absent

\[
\begin{align*}
\mathbf{r}'' &= \mathbf{U}_w^\dagger \mathbf{H}_w^\dagger \left[ \mathbf{s} + \mathbf{b} + \mathbf{n} \right] \\
\Gamma_{r''} &= \Sigma_{s_w} + 1
\end{align*}
\]  \hspace{1cm} (C.66)

This is very similar to the signal conditions imposed in the case of random process detection in additive white Gaussian Noise (Equation C.44). Application of Equation C.47 (substituting \( \sigma_n^2 = 1 \)) gives the simplified LRT,

\[
\ln \Lambda (\mathbf{R}'') = \sum_{i=1}^{N} \left( \frac{\sigma_{s_w}^2}{1 + \sigma_{s_w}^2} \right) |r_i''|^2 - \sum_{i=1}^{N} \ln \left( 1 + \sigma_{s_w}^2 \right) \geq H_1 \ln \gamma^2
\]  \hspace{1cm} (C.67)

where \( r_i'' \) is the \( i \)th element of \( \mathbf{r}'' \), and \( \sigma_{s_w}^2 \) is the \( i \)th diagonal element of the singular value matrix \( \Sigma_{s_w} \). This LRT can be used to determine the presence or absence of a Gaussian random signal in colored Gaussian noise.

One can give an example of filter whitening by solving for the rank-1 signal detection statistic, as covered in the previous section. There, the signal covariance matrix was \( \sigma_s^2 \mathbf{u}_s \mathbf{u}_s^\dagger \), and the noise covariance \( \Gamma_n \). Using Equation C.58, the whitening filter for this signal would be

\[
\mathbf{H}_w = \mathbf{U}_n \Sigma_n^{-1/2}
\]  \hspace{1cm} (C.68)

where \( \mathbf{U}_n \) is the unitary matrix of singular vectors of \( \Gamma_n \), and \( \Sigma_n^{-1/2} \) is the square-root matrix of the singular values of \( \Gamma_n \). Application of Equation C.59 gives a whitened received covariance matrix of

\[
\Gamma_{rw} = \mathbf{H}_w^\dagger \Gamma_n \mathbf{H}_w
\]  \hspace{1cm} (C.69)

One can rewrite the first term as an outer product of two orthonormal vectors by substituting,

\[
\sigma_{s_w}^2 = \sigma_s^2 \left\| \Sigma_n^{-1/2} \mathbf{U}_n^\dagger \mathbf{u}_s \right\|^2_2
\]  \hspace{1cm} (C.70)

\[
\mathbf{u}_{s_w} = \frac{\Sigma_n^{-1/2} \mathbf{U}_n^\dagger \mathbf{u}_s}{\sqrt{\left\| \Sigma_n^{-1/2} \mathbf{U}_n^\dagger \mathbf{u}_s \right\|^2_2}}
\]

As the signal matrix is rank-1, one can employ Equation C.34 as a detector, with \( \sigma_s^2 = \sigma_{s_w}^2 \), \( \sigma_n^2 = 1 \), and \( \mathbf{R}' = \mathbf{u}_{s_w}^\dagger \mathbf{H}_w^\dagger \mathbf{R} \),

\[
\ln \Lambda (\mathbf{R}) = \frac{1}{2} \left[ \frac{\sigma_{s_w}^2}{\sigma_{s_w}^2 + 1} \left| \mathbf{u}_{s_w}^\dagger \mathbf{H}_w^\dagger \mathbf{R} \right|^2 - \ln \left( 1 + \sigma_{s_w}^2 \right) \right] \geq H_1 \ln \gamma.
\]  \hspace{1cm} (C.72)
Knowing \( \| \Sigma_n^{-1/2} U_n^1 u_{s1} \|_2 = u_{s1}^\top \Gamma_n^{-1} u_{s1} \), and substituting through Equation C.70, one obtains Equation C.52,

\[
\ln \Lambda(R) = \frac{1}{2} \left( \frac{\sigma_n^2}{\sigma_n^2 u_{s1}^\top \Gamma_n^{-1} u_{s1} + 1} \right) - \frac{1}{2} \ln \left( 1 + \frac{\sigma_n^2 u_{s1}^\top \Gamma_n^{-1} u_{s1}}{\sigma_n^2} \right) \geq \mathcal{H}_1 \ln \gamma.
\]

By equating the whitened rank-1 detection result with its non-whitened counterpart, one demonstrates signal whitening does not negatively impact the performance of the detector.

### C.9 Deterministic Signal Detection Performance

The likelihood ratio tests developed in the previous sections are designed to reduce a multidimensional problem down to one dimension. Given received signals from \( N \) array elements, one feeds them into an equation which generates \( l \), the detection statistic. Whether \( l \) is greater or less than a particular threshold value determines whether the signal is present or absent.

Given the presence of noise in the system, it is possible that the LRT may not give the correct answer at all times. For example, if noise obscures a signal, the calculated value of \( l \) may not exceed the threshold, and the signal would be missed. Similarly, random noise could be mistaken for a signal, and the detector would register a false alarm. These are the two major types of error which are present in an LRT. To evaluate the performance of a particular LRT, it is important to calculate the probability of misses, \( P_M \), as well as the probability of false alarms, \( P_F \).

The calculation can be demonstrated in the simple Scalar Binary Hypothesis Test. Multiplying Equation C.8 by \( 1/\sigma_n \),

\[
l = \frac{1}{\sigma_n} R \geq_{\mathcal{H}_1} \frac{\sigma_n}{a} \ln \gamma + \frac{a}{2\sigma_n} = \gamma^* \tag{C.73}
\]

one obtains the detection statistic, \( l \). This is a Gaussian random variable, with mean 0 or \( d = a/\sigma \), depending on whether the signal is absent (Hypothesis \( H_0 \)) or present (Hypothesis \( H_1 \)). In either case, \( l \) has a variance of 1. The detection threshold, \( \gamma^* \) forms the right half of the equation. Figure C.9 plots the PDFs for both hypotheses with respect to \( l \). \( P_M \) is the area of the \( H_1 \) Gaussian PDF to the left of \( \gamma^* \), and \( P_F \) is the area of the \( H_0 \) PDF to the right of \( \gamma^* \). One can calculate \( P_M \) and \( P_F \) by integrating the Gaussian,

\[

P_M = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\gamma^*} \exp \left[ -\frac{1}{2} (l - d)^2 \right] \, dl \quad P_F = \frac{1}{\sqrt{2\pi}} \int_{\gamma^*}^{\infty} \exp \left[ -\frac{l^2}{2} \right] \, dl \tag{C.74}
\]

These integrals are tabulated in relevant mathematical literature in terms of error functions, or \( \text{erf}(x) \).

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-y^2) \, dy, \quad \text{erfc}(x) = 1 - \text{erf}(x). \tag{C.75}
\]

Here, one uses a normalized version of the error function and its complement.
Figure C-1: Plot of the Probability Density Functions for Detection Statistic \( l \), under Hypotheses \( H_0 \) and \( H_1 \). \( P_M \) is the area under the curve which denotes the probability of a missed detection, and \( P_F \) is the area under the curve which represents the probability of a false alarm. \( d \) is the distance between the means of the two hypotheses.

\[
\text{erf}_* (x) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{x}{\sqrt{2}} \right) \right] = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{y^2}{2} \right) dy. \tag{C.76}
\]

Finally, one can express \( P_M \) and \( P_F \) in terms of the normalized error functions,

\[
P_M = \text{erf}_* (\gamma_* - d) \quad P_F = \text{erfc}_* (\gamma_*) \tag{C.77}
\]

and express them in terms of \( \gamma \),

\[
P_M = \text{erf}_* \left( \frac{\ln \gamma - d}{d} \right) \quad P_F = \text{erfc}_* \left( \frac{\ln \gamma + d}{d} \right). \tag{C.78}
\]

One would like to select \( \gamma \) to minimize the total amount of error, keeping both \( P_M \) and \( P_F \) low. Figure C.9 shows the two values to be closely related; with \( a \) constant, lowering one raises the other. The effects of \( \gamma \) on \( P_M \) and \( P_F \) can be illustrated using a Receiver Operating Characteristic (ROC) curve (Figure C-2). This plots the probability of detection, \( P_D = 1 - P_M \) versus \( P_F \) for various values of \( d \). In this example, \( d = a/\sigma \).

ROC curves have several properties which aid interpretation. For values of \( d > 0 \), the curves are concave downward, and are above the \( P_D = P_F \) line. Also, the value of \( \gamma \) in Equation C.78 increases as the ROC approaches the origin. Figure C-2 illustrates this. As one increases \( d \), the distance between \( E[l|H_1] \) and \( E[l|H_0] \), the quality of the detector improves. In this way, \( d \) can be considered a measure of the detector output signal to noise ratio (SNR). Given \( d \), one can choose a value of \( P_D \) or \( P_F \), and obtain the desired threshold \( \gamma \) from the ROC curve.

Recall the observation statistic for the Vector Binary Hypothesis test,
Following the same method as in the simple binary test case, one arrives at the detection statistic, $l$, after multiplying Equation C.79 by $\sqrt{N}/\sigma_n$,

$$I = \frac{1}{\sigma_n} \sqrt{N} \sum_{i=1}^{N} r_i \geq H_0 \frac{\sigma_n}{a\sqrt{N}} \ln \gamma + \frac{a}{2} = \gamma_\ast. \tag{C.80}$$

$l$ is a scalar Gaussian random variable, with unit variance and mean,

$$E[l|H_0] = 0 \quad E[l|H_1] = \frac{a\sqrt{N}}{\sigma_n} \tag{C.81}$$

A plot of $p_{l|H_0}(L|H_0)$ and $p_{l|H_1}(L|H_1)$ would appear exactly the same as Figure C.9. Using Equation C.74, the probabilities of miss and false alarm, ($P_M$ and $P_F$) would be

$$P_M = \text{erf}_\ast \left( \frac{\ln \gamma}{d} + \frac{d}{2} \right) \quad P_F = \text{erfc}_\ast \left( \frac{\ln \gamma}{d} - \frac{d}{2} \right), \tag{C.82}$$
which are identical to Equation C.78, except $d$ is now $\frac{a\sqrt{N}}{\sigma_n}$.

From Equation C.24, the Deterministic Signal Detector has the observation statistic,

$$s^\dagger r \overset{H_0}{\overset{H_1}{\sim}} \frac{\sigma_n^2}{a} \ln \gamma + \frac{a}{2}.$$  \hspace{1cm} (C.83)

where $r$ is the received signal vector, $s$ is the unit-magnitude known deterministic signal, $\sigma_n^2$ is the variance of the noise, and $a$ is the known amplitude of $s$. Multiplication by $1/\sigma_n$ gives the detection statistic,

$$l = \frac{1}{\sigma_n} \sum_{i=1}^{N} s_i^* r_i \overset{H_0}{\overset{H_1}{\sim}} \frac{\sigma_n}{a} \ln \gamma + \frac{a}{2\sigma_n} = \gamma_a.$$  \hspace{1cm} (C.84)

This equation is remarkably similar to Equation C.73, the detection statistic for the simple binary hypothesis test. As the LRT for the deterministic signal in noise reduces to that of the simple binary LRT, this is not surprising. Evaluation of the detection statistic proceeds as before, with $P_M$ and $P_F$ described by Equation C.77, with $d = a/\sigma_n$, and the ROC curves shown in Figure C-2.

**C.10 Random Signal Detection Performance**

Detection performance has focused on evaluation of the detection statistic, $l$, and the assumption that it is a Gaussian random variable. For the deterministic signal, $l$ is a linear combination of Gaussian random variables. Consider the detection statistic for a random signal from Equation C.46,

$$l(R') = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{\sigma_i^2}{\sigma_n^2} \right) \frac{r_i'^* r_i'}{\sigma_n^2 + \sigma_i^2} + \frac{1}{2} \sum_{i=1}^{N} \left( \frac{r_i'^* \bar{s}_i' + \bar{s}_i'^* r_i' - \bar{s}_i'^* \bar{s}_i'}{\sigma_n^2 + \sigma_i^2} \right) - \frac{1}{2} \sum_{i=1}^{N} \ln \left( 1 + \frac{\sigma_i^2}{\sigma_n^2} \right) \overset{H_0}{\overset{H_1}{\sim}} \ln \gamma.$$  \hspace{1cm} (C.85)

The first term of $l(R')$ in Equation C.85 is a summation of the magnitude squared of $r_i'$. As a result of this squaring term, the PDFs of $p_{L|H_0}(L)_{H_0}$ and $p_{L|H_1}(L)_{H_1}$ are no longer Gaussian. The simple equations for calculating $P_M$ and $P_F$ can not be applied. Instead, one can estimate the PDFs based on the knowledge that $l$ is a sum of random variables. While exact expressions cannot be easily found for $P_M$ and $P_F$, one can estimate their bounds.

Consider the natural logarithm of the moment generating function of $l$,

$$\mu(\alpha) = \ln \int_{-\infty}^{+\infty} e^{\alpha l} p_{L|H_0}(L|H_0) dL$$  \hspace{1cm} (C.86)

with $\alpha$ a real variable. Using the identity,

$$l(R) = \ln \Lambda(R) = \ln \left[ \frac{p_{L|H_1}(L|H_1)}{p_{L|H_0}(L|H_0)} \right] \overset{H_0}{\overset{H_1}{\sim}} \ln \gamma.$$  \hspace{1cm} (C.87)

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and realizing $l$ is a function of $r$, one can formulate the simplified expressions,

$$
\mu(\alpha) = \ln \int_{-\infty}^{+\infty} e^{\alpha l(R)} p_l|H_0(R|H_0) dR \\
= \ln \int_{-\infty}^{+\infty} [p_l|H_1(R|H_1)]^\alpha [p_l|H_0(R|H_0)]^{1-\alpha} dR \\
= \ln \int_{-\infty}^{+\infty} e^{(\alpha-1)l(R)} p_l|H_1(R|H_1) dR.
$$

(C.88)

Taking the derivative of Equation C.86 and using Leibniz’ rule for differentiation of integrals involving a parameter[62], one can solve for the expected value and variance of $l$ under both hypotheses,

$$
E[l|H_0] = \frac{d\mu(\alpha)}{d\alpha} \bigg|_{\alpha=0} \\
E[l|H_1] = \frac{d\mu(\alpha)}{d\alpha} \bigg|_{\alpha=1} \\
\sigma^2_{\alpha|H_0} = \frac{d^2\mu(\alpha)}{d\alpha^2} \bigg|_{\alpha=0} \\
\sigma^2_{\alpha|H_1} = \frac{d^2\mu(\alpha)}{d\alpha^2} \bigg|_{\alpha=1}
$$

(C.89)

(C.90)

The moment generating functions of $p_l|H_0(L|H_0)$ and $p_l|H_1(L|H_1)$ are themselves random variables; their PDFs can be expressed in terms of a new random variable, $x$,

$$
p_x(X) = \frac{e^{\alpha X} p_l|H_0(X|H_0)}{\int_{-\infty}^{+\infty} e^{\alpha X} p_l|H_0(L|H_0) dL} = \frac{e^{\alpha X} p_l|H_0(X|H_0)}{\mu(\alpha)} \\
= \frac{e^{(\alpha-1)X} p_l|H_1(X|H_1)}{\int_{-\infty}^{+\infty} e^{(\alpha-1)X} p_l|H_1(L|H_1) dL} = \frac{e^{(\alpha-1)X} p_l|H_1(X|H_1)}{\mu(\alpha)}
$$

(C.91)

One can rewrite the probability of false alarm and the probability of a missed detection in terms of $\mu(\alpha)$,

$$
P_F = \int_{-\infty}^{+\infty} p_l|H_0(L|H_0) dL = \int_{-\infty}^{+\infty} \exp[\mu(\alpha) - \alpha X] p_x(X) dX.
$$

(C.92)

$$
P_M = \int_{-\infty}^{+\infty} p_l|H_1(L|H_1) dL = \int_{-\infty}^{+\infty} \exp[\mu(\alpha) + (1-\alpha)X] p_x(X) dX.
$$

(C.93)

For values of $\alpha > 0$, $e^{-\alpha X}$ is less than $e^{-\alpha \gamma^*}$, so

$$
P_F \leq \exp[\mu(\alpha) - \alpha \gamma^*] \int_{\gamma^*}^{+\infty} p_x(X) dX \leq \exp[\mu(\alpha) - \alpha \gamma^*]
$$

(C.94)

Similarly, for values of $\alpha < 1$, $e^{(1-\alpha)X}$ is less than $e^{(1-\alpha)\gamma^*}$, so

$$
P_M \leq \exp[\mu(\alpha) + (1-\alpha) \gamma^*] \int_{-\infty}^{\gamma^*} p_x(X) dX \leq \exp[\mu(\alpha) + (1-\alpha) \gamma^*]
$$

(C.95)

A tighter bound on $P_F$ and $P_M$ can be obtained by minimizing the expressions above. Taking their derivatives with respect to $s$ and setting them to zero, one obtains $\frac{d\mu(\alpha)}{d\alpha} = \gamma$, resulting in the Chernoff bounds for $P_F$. 

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and $P_M$, 

$$P_F \leq \exp \left[ \mu(\alpha) - \alpha \frac{d\mu(\alpha)}{d\alpha} \right], \quad \alpha \geq 0 \tag{C.96}$$

$$P_M \leq \exp \left[ \mu(\alpha) + (1 - \alpha) \frac{d\mu(\alpha)}{d\alpha} \right], \quad \alpha \leq 1 \tag{C.97}$$

One can map from $\alpha$ to $l$ (and thus, $\gamma$) through the nonlinear function (based on Equation C.89),

$$l = \frac{d\mu(\alpha)}{d\alpha}. \tag{C.98}$$

This allows one to express the Chernoff bounds in terms of $l$, the detection statistic.

The Chernoff bounds apply to arbitrary $\mu(\alpha)$. For large values of $N$, one can apply the Central Limit Theorem (CLT). The CLT shows the PDF of the sum of $N$ statistically independent random variables approaches Gaussian, for large $N$, regardless of the distribution of the component random variables. Rewriting Equation C.92, the expression of $P_F$ in terms of $\mu(\alpha)$,

$$P_F = \exp \left[ \mu(\alpha) - \alpha \frac{d\mu(\alpha)}{d\alpha} \right] \int_{d\mu(\alpha)/d\alpha}^{+\infty} \exp \left[ \alpha \left( \frac{d\mu(\alpha)}{d\alpha} - X \right) \right] p_x(X)dX \tag{C.99}$$

one recognizes the expression to the left of the integral as the bound from Equation C.96. To prepare for the CLT, one can rewrite the integral in terms of a normalized random variable,

$$y = \frac{x - E(x)}{\sigma_x} = \frac{x - d\mu(\alpha)}{\sqrt{d^2\mu(\alpha)/d\alpha^2}} \tag{C.100}$$

$$P_F = \exp \left[ \mu(\alpha) - \alpha \frac{d\mu(\alpha)}{d\alpha} \right] \int_{0}^{+\infty} \exp \left[ -\alpha \sqrt{\frac{d^2\mu(\alpha)}{d\alpha^2}} Y \right] p_y(Y)dY \tag{C.101}$$

For large $N$, $p_y(Y)$ will be zero mean Gaussian with unit variance; the resulting integral can be evaluated in terms of the complementary error function,

$$\int_{0}^{+\infty} \exp \left[ -\alpha \sqrt{\frac{d^2\mu(\alpha)}{d\alpha^2}} Y \right] \frac{1}{\sqrt{2\pi}} e^{-Y^2/2} dY = \exp \left[ \frac{\alpha^2 d^2\mu(\alpha)}{2 \frac{d\mu(\alpha)}{d\alpha}} \right] \text{erfc} \left( \alpha \sqrt{\frac{d^2\mu(\alpha)}{d\alpha^2}} \right). \tag{C.102}$$

Substituting this back into Equation C.101, one obtains an approximate expression for $P_F$ based on the CLT,

$$P_F \simeq \left\{ \exp \left[ \mu(\alpha) - \alpha \frac{d\mu(\alpha)}{d\alpha} + \frac{\alpha^2 d^2\mu(\alpha)}{2 \frac{d\mu(\alpha)}{d\alpha}} \right] \right\} \text{erfc} \left( \alpha \sqrt{\frac{d^2\mu(\alpha)}{d\alpha^2}} \right), \quad \alpha \geq 0. \tag{C.103}$$

In a similar manner, an expression for $P_M$, the probability of a missed detection, can be derived,

$$P_M \simeq \left\{ \exp \left[ \mu(\alpha) + (1 - \alpha) \frac{d\mu(\alpha)}{d\alpha} + \frac{(\alpha - 1)^2 d^2\mu(\alpha)}{2 \frac{d\mu(\alpha)}{d\alpha}} \right] \right\} \text{erfc} \left( (1 - \alpha) \sqrt{\frac{d^2\mu(\alpha)}{d\alpha^2}} \right), \quad \alpha \leq 1. \tag{C.104}$$
Equations C.103 and C.104 give approximate expressions for \( P_F \) and \( P_M \) in terms of \( \mu(\alpha) \) in the case of a random signal. To evaluate these expressions, it would be beneficial to obtain a closed form expression for \( \mu(\alpha) \) and its derivatives. Substituting

\[
p_{r'|H_1}(R'|H_1) = \frac{1}{\sqrt{(2\pi)^N |\Sigma_r|}} \exp \left[ -\frac{1}{2} (R' - \bar{s}')^\top (\sigma_n^2 I + \Sigma_r)^{-1} (R' - \bar{s}') \right] \quad \text{(C.105)}
\]

and

\[
p_{r'|H_0}(R'|H_0) = \frac{1}{\sqrt{(2\pi)^N |\Sigma_n|}} \exp \left[ -\frac{1}{2} R'^\top (\sigma_n^2 I)^{-1} R' \right]. \quad \text{(C.106)}
\]

into Equation C.88 yields,

\[
\mu(\alpha) = \ln \int_{-\infty}^{+\infty} \left\{ \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi (\sigma_n^2 + \sigma_{n_i}^2)}} \exp \left[ -\frac{1}{2} \sum_{i=1}^{N} \frac{|r'_i - \bar{s}'_i|^2}{\sigma_n^2 + \sigma_{n_i}^2} \right] \right\}^\alpha \times \left\{ \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi \sigma_n^2}} \exp \left[ -\frac{1}{2} \sum_{i=1}^{N} \frac{|r'_i|^2}{\sigma_n^2} \right] \right\}^{1-\alpha} dR'
\]

\[= \frac{1}{2} \sum_{i=1}^{N} \left\{ (1 - \alpha) \ln \left[ 1 + \frac{\sigma_{n_i}^2}{\sigma_n^2} \right] - \ln \left[ 1 + \frac{\sigma_{n_i}^2}{\sigma_n^2} (1 - \alpha) \right] \right\} - \frac{\alpha}{2} \sum_{i=1}^{N} \frac{|\bar{s}'_i|^2 (1 - \alpha)}{\sigma_n^2 + \sigma_{n_i}^2 (1 - \alpha)}, \quad 0 \leq \alpha \leq 1. \quad \text{(C.107)}
\]

Taking derivatives with respect to \( \alpha \),

\[
\frac{d\mu(\alpha)}{d\alpha} = \frac{1}{2} \sum_{i=1}^{N} \left\{ \frac{\sigma_{n_i}^2}{\sigma_n^2 + \sigma_{n_i}^2 (1 - \alpha)} - \ln \left[ 1 + \frac{\sigma_{n_i}^2}{\sigma_n^2} \right] \right\} + \frac{\alpha}{2} \sum_{i=1}^{N} \frac{|\bar{s}'_i|^2 \sigma_n^2}{\sigma_n^2 + \sigma_{n_i}^2 (1 - \alpha)} - \frac{1}{2} \sum_{i=1}^{N} \frac{|\bar{s}'_i|^2 (1 - \alpha)}{\sigma_n^2 + \sigma_{n_i}^2 (1 - \alpha)} \quad \text{(C.108)}
\]

\[
\frac{d^2\mu(\alpha)}{d\alpha^2} = \frac{1}{2} \sum_{i=1}^{N} \frac{\sigma_{n_i}^2 + 2 |\bar{s}'_i|^2 \sigma_n^2}{\left[ \sigma_n^2 + \sigma_{n_i}^2 (1 - \alpha) \right]^2} + \frac{\alpha}{2} \sum_{i=1}^{N} \frac{|\bar{s}'_i|^2 \sigma_n^2 \sigma_{n_i}^2}{\left[ \sigma_n^2 + \sigma_{n_i}^2 (1 - \alpha) \right]^3} \quad \text{(C.109)}
\]

one has the formulas needed to estimate \( P_F \) and \( P_M \) for a random signal.

Figure C.2 illustrates the performance of the simple binary hypothesis test depends on the value of \( d^2 \). In the deterministic signal case, where \( l \) is Gaussian, this quantity would be the detector output signal to noise ratio (SNR). However, in the random signal case, with \( l \) no longer Gaussian, this cannot be guaranteed. When \( l \) is Gaussian, the detector output SNR can be expressed as,

\[
d^2 = \frac{(E[l|H_1] - E[l|H_0])^2}{\sigma_{l|H_0}^2}. \quad \text{(C.110)}
\]

Here, when \( l \) is non-Gaussian, the SNR can be approximated, using Equations C.89, C.90, C.108, and C.109,
The output SNR describes only part of the detection performance. One must also consider the ROC curves, showing $P_F$ and $P_D$, to accurately characterize the effect of a detector.

C.11 Low Energy Coherent Detection

The random signal detectors shown thus far rely on a weighted correlation of a received signal with the singular values of the modeled signal covariance matrix, $\Gamma_s$. The weights which are used to combine the correlations are based on the input signal to noise ratio, represented through the singular values of $\Gamma_s$, $\sigma_n^2$.

These detectors made no assumptions about the relative magnitude of $\sigma_n^2$, when compared with the noise level, represented by $\sigma^2_n$.

This section explores the condition when the singular values of $\Gamma_s$ are significantly less than those from the noise covariance. This negative input signal to noise ratio condition is called low energy coherence (LEC). Certain approximations are allowed which greatly simplify the detection statistic, $l$, as well as the performance metrics, $P_F$ and $P_M$. In this section, only zero mean random signals will be considered.

Recall the zero mean detection statistic from the last section:

$$l(R') = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{\sigma_i^2}{\sigma_n^2} \right) \left[ \frac{1}{\sigma_n^2 + \sigma_i^2} \frac{\left| \alpha_i \right|^2}{\sigma_n^2} \right] \ln \frac{\sigma_n^2}{\sigma_i^2} + \ln \mu \left( \frac{\sigma_n^2}{\sigma_i^2} \right) = \gamma_s.$$  (C.112)

$l(R')$ can be rewritten and expanded in a power series,

$$l(R') = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{\sigma_i^2}{\sigma_n^2} \right) \left( \frac{1}{\sigma_n^2 + \sigma_i^2} \right) \left| \alpha_i \right|^2 = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{\sigma_i^2}{\sigma_n^2} \right) \left[ 1 - \left( \frac{\sigma_i^2}{\sigma_n^2} \right)^2 + \left( \frac{\sigma_i^2}{\sigma_n^2} \right)^4 - \cdots \right] \left| \alpha_i \right|^2,$$  (C.113)

retaining only the first term if $\max(\sigma_i^2) \ll \sigma_n^2$. Similarly, the logarithm sum of $\gamma_s$ can be expanded,

$$\frac{1}{2} \sum_{i=1}^{N} \ln \left( 1 + \frac{\sigma_i^2}{\sigma_n^2} \right) = \frac{1}{2} \left[ \sum_{i=1}^{N} \frac{\sigma_i^2}{\sigma_n^2} - \frac{1}{2} \sum_{i=1}^{N} \left( \frac{\sigma_i^2}{\sigma_n^2} \right)^2 + \cdots \right]$$  (C.114)

Retaining the first two terms of the logarithmic expansion, one can form a new detection statistic, the optimum LEC receiver,

$$l_{sec}(R') = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{\sigma_i^2}{\sigma_n^2} \right) \left| \alpha_i \right|^2 \ln \frac{\sigma_n^2}{\sigma_i^2} + \frac{1}{2} \sum_{i=1}^{N} \left( \frac{\sigma_i^2}{\sigma_n^2} \right)^2 - \frac{1}{4} \sum_{i=1}^{N} \left( \frac{\sigma_i^2}{\sigma_n^2} \right)^4 = \gamma_{sec}.$$  (C.115)

In a similar manner, the performance basis function, $\mu(\alpha)$, for the random detector can be derived in the
LEC case. Recall the zero mean random signal from Equation C.107,

\[ \mu(\alpha) = \frac{1}{2} \sum_{i=1}^{N} \left\{ (1 - \alpha) \ln \left[ 1 + \frac{\sigma_{s_i}^2}{\sigma_n^2} \right] - \ln \left[ 1 + \frac{\sigma_{s_i}^2}{\sigma_n^2} (1 - \alpha) \right] \right\}, \quad 0 \leq \alpha \leq 1. \]  

(C.116)

Expanding the natural logarithms and retaining only the first two terms of each, one obtains

\[ \mu_{\text{lec}}(\alpha) = -\frac{\alpha(1 - \alpha)}{2} \sum_{i=1}^{N} \left( \frac{\sigma_{s_i}^2}{\sigma_n^2} \right)^2 = -\frac{\alpha(1 - \alpha)}{2} d^2. \]  

(C.117)

In this coherent detection case, the summation and its argument can be considered the detector output signal to noise ratio, \( d^2 \). Assuming this is true, then using Equations C.103 and C.104, one can estimate \( P_F \) and \( P_M \),

\[ P_{F_{\text{lec}}} \simeq \text{erfc}(\alpha d) \quad P_{M_{\text{lec}}} \simeq \text{erfc}([1 - \alpha]d), \quad 0 \leq \alpha \leq 1. \]  

(C.118)

\section*{C.12 Derived Distribution of Random Signal Detection Statistic}

The performance metrics derived in the previous sections assume the received signal is a real quantity. If the signal is complex, the Chernoff bounds do not apply (as given). One can derive a more accurate expression for the PDF of the detection statistic, \( p_L(L) \), then use it to find the performance metrics \( P_F \) and \( P_M \).

\subsection*{C.12.1 Random Signal Detection Statistic}

Calculating a detection statistic starts with two received signal hypothesis. The first \( (H_0) \) assumes no signal is present, and that the received signal is composed entirely of uncorrelated Gaussian random noise. The second \( (H_1) \) assumes a random signal is present, combined with uncorrelated Gaussian random noise.

\[ r = n \quad \text{Hypothesis } H_0 \]

\[ r = s + n \quad \text{Hypothesis } H_1 \]  

(C.119)

The random signal vector \( s \) is assumed to be zero mean and complex Gaussian, with a covariance matrix \( \Gamma_s \). The complex Gaussian random noise vector \( n \) is also zero mean, with a diagonal covariance matrix \( \Gamma_n = \sigma_n^2 I \). Regardless of whether the desired random signal is present or absent, the received signal vector is a linear combination of Gaussian random vectors, and is therefore Gaussian. One can completely characterize the statistics of the received signal by its mean and covariance matrix. The desired random signal \( s \) and noise are both zero mean, resulting in a zero mean received signal. As both the signal and noise are uncorrelated, the received signal covariance matrix \( \Gamma_r \) is \( \Gamma_n + \sigma_n^2 I \) if the signal is present, in Hypothesis \( H_1 \), and simply \( \sigma_n^2 \) if no signal is present, in Hypothesis \( H_0 \).
The detection statistic relies on the different received covariance matrices for each Hypothesis to determine when a signal is present or absent. Using a LRT and simplifying, one can derive

\[ l = \sum_{i=1}^{N} \left( \frac{\sigma_{ii}^2}{\sigma_{n}^2} \right) \left| u_{si}^T R \right|^2 - \sum_{i=1}^{N} \ln \left( 1 + \frac{\sigma_{ii}^2}{\sigma_{n}^2} \right) \] (C.120)

where \( N \) is the number of elements in the received vector \( R \), and \( \sigma_{ii}^2 \) and \( u_{si} \) are the singular \( i^{th} \) singular value and vector of the desired random signal covariance matrix \( \Gamma_s \).

### C.12.2 Linear Operations on Gaussian Random Vectors

The overall objective is to find \( p_L(L) \), the PDF of \( l \). The input random quantity is the received signal vector, \( R \), characterized by its estimated covariance matrix, \( \hat{\Gamma}_r \).

If one applies linear operators to a Gaussian random variable, the resulting random variable is still Gaussian. For example, if one is presented with random vectors \( x \) and \( y \), modified by the deterministic matrices \( A \) and \( B \), and shifted by the deterministic vector \( b \), then

\[ z = Ax + By + b. \] (C.121)

The mean of \( z \) would then be

\[ z = Ax + By + b \] (C.122)

and the covariance matrix

\[ \Gamma_z = \Gamma_x A^\dagger + \Gamma_y B^\dagger. \] (C.123)

Using a Karhunen-Loeve expansion, one can express the received signal vector \( R \) as a linear combination of independent, identically distributed (i.i.d.) random variables. One starts by decomposing the received signal covariance matrix into its component singular values and vectors:

\[ \Gamma_r = U_r \Sigma_r U_r^\dagger, \] (C.124)

with \( U_r \) a unitary matrix of singular vectors, and \( \Sigma_r \) a diagonal matrix of real singular values. Each singular vector-value pair represents an uncorrelated random vector, which, when summed together, yields a realization of the random vector \( R \):

\[ r = \sum_{i=1}^{N} \lambda_{i} \sqrt{\sigma_{ri}^2} u_{ri} \] (C.125)

where \( \lambda_{i} \) is a zero mean Gaussian random variable with a variance of 1, and \( \sigma_{ri}^2 \) and \( u_{ri} \) are the \( i^{th} \) singular
value and vector of \( \Gamma_r \). This can be expressed in vector form,

\[
r = U_r D_r \Lambda_1
\]

where \( D_r \) is a \( N \) by \( N \) diagonal matrix with \( \sqrt{\sigma^2_{r_i}} \) as its elements, and \( \Lambda_1 \) is a \( N \) by 1 column vector of uncorrelated zero mean, unit variance Gaussian random variables. One can check the resulting statistic through the application of Equation C.123 to Equation C.126.

Equation C.120 can be simplified to separate its deterministic and random parts. Consider

\[
l = \sum_{i=1}^{N} \alpha_i u_i^t R u_i - \sum_{i=1}^{N} \beta_i
\]

with: 

\[
\alpha_i = \left( \frac{\sigma^2_{r_i}}{\sigma^2_n} \right) \frac{1}{\sigma^2_{r_i} + \sigma^2_n} \quad \text{and} \quad \beta_i = \ln \left( 1 + \frac{\sigma^2_{r_i}}{\sigma^2_n} \right)
\]

which can be further simplified, if one makes the substitution

\[
v_i = \sqrt{\alpha_i} u_i
\]

to yield

\[
l = \sum_{i=1}^{N} v_i^t R v_i - \sum_{i=1}^{N} \beta_i.
\]

Since \( \Gamma_\alpha \) is known, one assumes \( \alpha_i \) and \( \beta_i \) are both deterministic quantities. In matrix form, this becomes

\[
l = V^t R R^t V - b,
\]

with \( V = U_\alpha D_\alpha \), where \( D_\alpha \) is a diagonal matrix with entries of \( \sqrt{\alpha_i} \), and \( b \) the summation of all \( \beta_i \). Shifting \( b \) to the left hand side and expanding Equation C.130 gives

\[
l - b = (D_\alpha U_\alpha^t U_r D_r \Lambda_1)^t D_\alpha U_\alpha^t U_r D_r \Lambda_1
\]

\[
= \Lambda_1^t D_\alpha^t U_r^t U_\alpha D_\alpha U_\alpha^t U_r D_r \Lambda_1
\]

\[
= \Lambda_1^t K \Lambda_1,
\]

where \( K \) is a full rank, deterministic non-diagonal \( N \) by \( N \) matrix and \( \Lambda_1 \) is a \( N \) by 1 column vector of independent, identically distributed zero mean unit variance Gaussian random variables.

Equation C.131 shows a nonlinear combination of the Gaussian random variables in \( \Lambda_1 \). Thus \( l \) is not a Gaussian random variable. If \( K \) were the identity matrix, \( l \) then \( l \) would have a chi-squared \((\chi^2)\) distribution, with \( N \) degrees of freedom,

\[
p_l(L) = \frac{1}{2^{N/2} \Gamma(N/2)} L^{N/2-1} e^{-L/2}
\]
where $\Gamma(n)$ is the Gamma function. This distribution holds for $L > 0$.

Unfortunately, as $K$ is not a diagonal matrix, $l$ does not possess a simple $\chi^2$ distribution. Rather, $l$ is the dot product of two vectors of correlated Gaussian random variables,

$$l - b = a^T a \quad \text{with} \quad a = VR \quad \text{and} \quad \Gamma_a = V \Gamma_r V.$$  \hfill (C.133)

This will require additional derivation to find the appropriate PDF.

### C.12.3 Derived Distributions

Before finding the PDF of $l$, one must first consider a few elementary derived distributions. Given a scalar random variable $x$, with PDF $p_x(x)$, one can find the PDF of scalar random variable $y$, when $y = x^2$. From Papoulis[63], one employs the formula,

$$p_y(y) = \frac{p_x(x_1)}{|g'(X_1)|} + \cdots + \frac{p_x(x_M)}{|g'(X_M)|}, \quad \text{where} \quad g'(X_M) = \left. \frac{\partial g(X)}{\partial X} \right|_{X=X_M}$$  \hfill (C.134)

to obtain a distribution for $y$. Here, $X_1$ represents the first of $M$ roots to $y = g(x)$. In the case where $g(x) = x^2$, there are two roots. Thus, the PDF would evaluate to be

$$p_y(y) = \frac{1}{2\sqrt{y}} \left[ p_x(-\sqrt{y}) + p_x(\sqrt{y}) \right] \quad \text{for} \quad y > 0.$$  \hfill (C.135)

In the case of $x$ and $y$ being random vectors, one must adjust the distribution to compensate for additional random variables. For example, if $x$ and $y$ are each 2 elements, then $x = [x_1 \ x_2]^T$, $y = [y_1 \ y_2]^T$, $p_x(X) = p_{x_1,x_2}(X_1, X_2)$, and $p_y(Y) = p_{y_1,y_2}(Y_1, Y_2)$. Using the relationships $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$, the distribution of $y$ can be calculated as

$$p_y(Y) = p_{y_1,y_2}(Y_1, Y_2) = \frac{p_{x_1,x_2}(X_{11}, X_{21})}{|J(X_{11}, X_{21})|} + \cdots + \frac{p_{x_1,x_2}(X_{1M}, X_{2M})}{|J(X_{1M}, X_{2M})|}.$$  \hfill (C.136)

where $J(X_{1m}, X_{2m})$ is the Jacobian operator, evaluated at the $m^{th}$ root of $y = g(x)$,

$$J(X_{1m}, X_{2m}) = \begin{vmatrix} \frac{\partial g_1(x_1, x_2)}{\partial x_1} & \frac{\partial g_1(x_1, x_2)}{\partial x_2} \\ \frac{\partial g_2(x_1, x_2)}{\partial x_1} & \frac{\partial g_2(x_1, x_2)}{\partial x_2} \end{vmatrix}_{x=[X_{1m} \ X_{2m}]^T}$$  \hfill (C.137)

Following the example above, where $y = g(x) = [x_1^2 \ x_2^2]$, one obtains a Jacobian of $J(x_1, x_2) = 2^2 x_1 x_2$, and a distribution

$$p_y(Y) = \frac{p_{x_1,x_2}(-\sqrt{y_1}, -\sqrt{y_2})}{|2^2 \sqrt{y_1 y_2}|} + \frac{p_{x_1,x_2}(-\sqrt{y_1}, +\sqrt{y_2})}{|2^2 \sqrt{y_1 y_2}|}.$$
with \( p_n = (m - 1) \gg (n - 1) \), the quantity \((m - 1)\) left shifted by \((n - 1)\) bit positions. One can continue and derive the distribution for a vector with \( N \) elements. Assuming \( y = [x_1^2, x_2^2, \ldots, x_N^2]^T \), then

\[
\begin{align*}
py(Y) &= \frac{1}{2^N \sqrt{\prod_{n=1}^{N} y_n}} \sum_{m=1}^{2^N} \sum_{y_m} p_X(X_m), \\
& \text{with: } X_m = [(\pm 1)^{n_1} \sqrt{y_1}, \ldots, (\pm 1)^{n_N} \sqrt{y_N}]^T \text{ and } p_n = (m - 1) \gg (n - 1), \quad y \geq 0.
\end{align*}
\]

One would also like to find the distribution of two random variables summed together. Given a random variable \( y = x_1 + x_2 \), and the joint PDF \( p_{x_1,x_2}(X_1,X_2) \), the PDF of \( y \) can be calculated as,

\[
p_y(Y) = \int_{-\infty}^{\infty} p_{x_1,x_2}(Y - X_2, X_2) dX_2.
\]

If \( x_1 \) and \( x_2 \) are independent random variables, one can replace the joint distribution in Equation C.140 with a convolution, allowing one to solve for \( p_y(Y) \) through the inverse Fourier transform of the product of the moment generating functions of \( p(x_1)(X_1) \) and \( p(x_2)(X_2) \). Conversely, one is restricted to the integral in Equation C.140 if the random variables are correlated. This can be extended to include the sum of \( N \) elements, with \( y = \sum_{n=1}^{N} x_n \) and

\[
p_y(Y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_X(X_N, X_{N-1} - X_N, X_{N-2} - X_{N-1}, \ldots, Y - X_2) dX_2 \cdots dX_N.
\]

These two derived distributions can be used to solve for the PDF of \( l, \quad p_l(L) \). One can apply Equation C.139 to Equation C.133, yielding
\[ p_q(Q) = \frac{1}{2^N \sqrt{\prod_{n=1}^{N} q_n}} \sum_{m=1}^{2^N} \frac{1}{\pi^N |V' \Gamma_r V|} \exp \left[ -A_m^T (V' \Gamma_r V)^{-1} A_m \right] \]  
(C.142)

with

\[
q = [a_1^* a_1, a_2^* a_2, \ldots, a_N^* a_N]^T \\
A_m = [(-1)^{p_1} \sqrt{q_1}, \ldots, (-1)^{p_N} \sqrt{q_N}]^T \\
p_n = (m - 1) \gg (n - 1) \\
\text{and } q_n \geq 0
\]

The distribution for \( I \) can then be found through application of Equation C.141,

\[ p_I(L) = \int_0^\infty \cdots \int_0^\infty p_q(Q_N, Q_{N-1} - Q_N, Q_{N-2} - Q_{N-1}, \ldots, L + b - Q_2) dQ_2 \cdots dQ_N, \]  
(C.143)

with \( L \geq 0, V \) and \( b \) defined in Equation C.130, and \( \Gamma_r \) the signal covariance matrix of the received signal vector \( r \).

### C.13 Parameter Estimation

Up to this point one has been concerned only with signal detection: determining whether a signal is present or absent in a given situation. A logical extension to detection theory is estimation theory, where the parameters of a signal are estimated. These parameters can be random or deterministic, and can correspond to the signal’s spatial or temporal location, or center frequency. In this section, attention will be confined to deterministic signal parameters. These can be represented by the vector parameter, \( a \). One assumes the received signal, \( r \), as well as the source signal, \( s \), are both functions of \( a \).

The Maximum Likelihood (ML) estimate of a parameter is derived by forming a detection statistic, \( l(a) \), and finding the maximum of that statistic with respect to \( a \). Following the random signal detection method,

\[ r = s(a) + n \]  
(C.144)

where \( s(a) \) is the source signal, \( n \) is uncorrelated Gaussian noise, and \( r \) is the received signal. Assuming the source signal is Gaussian and its statistics known, the received signal’s probability density function, conditioned on parameter vector \( a \), would be

\[ p_r(a|R|a) = \frac{1}{\sqrt{(2\pi)^N |\Gamma_r|}} \exp \left[ -\frac{1}{2} (R - E[R|a])^\dagger \Gamma_r^{-1} (R - E[R|a]) \right] \]  
(C.145)

\[ = \frac{1}{\sqrt{(2\pi)^N |\Sigma_s(a) + \Sigma_n|}} \exp \left[ -\frac{1}{2} (R'(a) - \bar{s}'(a))^\dagger (\Sigma_s(a) + \Sigma_n)^{-1} (R'(a) - \bar{s}'(a)) \right]. \]  

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In this equation, the signal covariance matrix is decomposed, $\Gamma_s(a) = U_s(a)\Sigma_s(a)U_s(a)^\dagger$ into a unitary matrix of singular vectors, $U_s(a)$, and a diagonal matrix of singular values, $\Sigma_s(a)$. As $\Gamma_s$ is a function of the parameter vector $a$, so are its singular values and vectors. To diagonalize the received covariance matrix, $\Gamma_r$, one premultiplies the received signal by the transpose of the signal singular vector matrix, $R'(a) = U_s(a)^\dagger R$, as well as its mean $s'(a) = U_s(a)^\dagger s(a)$. Before using Equation C.145 as a likelihood statistic, one can simplify by dividing out the uncorrelated noise,

$$p_r(R) = \frac{1}{\sqrt{(2\pi)^N |\Gamma_n|}} \exp \left[-\frac{1}{2} R^\dagger \Gamma_n^{-1} R \right]$$

and taking the natural logarithm to obtain,

$$l(a) = \frac{1}{2} \sum_{i=1}^N \left( \frac{\sigma_n^2(s_i(a))}{\sigma_n^2} \right) - \frac{1}{2} \sum_{i=1}^N \frac{r_i^* r_i}{\sigma_n^2(s_i(a)) + \sigma_n^2} - \frac{1}{2} \sum_{i=1}^N \ln \left( 1 + \frac{\sigma_n^2(s_i(a))}{\sigma_n^2} \right).$$

If the signal is zero mean, $s = 0$, then the expression reduces to

$$l(a) = \frac{1}{2} \sum_{i=1}^N \left( \frac{\sigma_n^2(s_i(a))}{\sigma_n^2} \right) - \frac{1}{2} \sum_{i=1}^N \ln \left( 1 + \frac{\sigma_n^2(s_i(a))}{\sigma_n^2} \right).$$

The estimation statistic given in Equation C.148 assumes a single observation vector, $R$. If multiple observation vectors are available, then the expected value of $l(a)$ can be taken with respect to $R$. The result is an estimation statistic which is based on the estimated received signal covariance, $\hat{\Gamma}_r$. Realizing $r_i^* r_i^* = r_i^* r_i^*$, and substituting $r_i^* = u_i^\dagger(a)R_i$, one obtains,

$$l(a) = \frac{1}{2} \sum_{i=1}^N \left( \frac{\sigma_n^2(s_i(a))}{\sigma_n^2} \right) - \frac{1}{2} \sum_{i=1}^N \ln \left( 1 + \frac{\sigma_n^2(s_i(a))}{\sigma_n^2} \right).$$

One can estimate the receiver covariance matrix from individual observations by taking the average of the observations' outer product,

$$\hat{\Gamma}_r = E [RR^\dagger] \simeq \frac{1}{L} \sum_{i=1}^L R_i R_i^\dagger$$

where $L$ is the number of observations. The rank of $\hat{\Gamma}_r$ increases with the number of observations, but the contribution of uncorrelated Gaussian random noise decreases. If the process whose parameters are to be estimated is stationary, then increasing the number of observations reduces the effective $\sigma_n^2$. With this, the estimation statistic reduces to

$$l(a) = \frac{1}{2} \sum_{i=1}^N \left( \frac{\sigma_n^2(s_i(a))}{\sigma_n^2} \right) - \frac{1}{2} \sum_{i=1}^N \ln \left( 1 + \frac{\sigma_n^2(s_i(a))}{\sigma_n^2} \right).$$

One can see similarities between Equations C.147-C.148 and Equations C.46-C.47. The maximum likelihood estimate of $a$ occurs where $l(a)$ is maximum. One can find the maximum by taking the derivative of $a$
and setting it to zero,

\[ \hat{a} = \max_a l(a) \quad \text{or} \quad \frac{dl(a)}{da} \bigg|_{a=\hat{a}} = \frac{d\ln L(R|a)}{da} \bigg|_{a=\hat{a}} = 0. \] (C.152)

### C.14 Estimator Performance Metrics

The performance of an estimator can be categorized in two ways: its accuracy and precision. If an estimator is accurate, then the estimated parameter \( \hat{a} \) will always equal the true parameter value, \( a \). To solve for \( \hat{a} \), one must solve for the peak of \( l \) with respect to \( a \). As \( l \) is a scaled, logarithmic version of the Probability Density Function (PDF) of \( a \), finding the peak of \( p_a(A) \) would serve the same purpose. Recall Equation C.145, applied to a zero mean random signal, after having been normalized by Equation C.146,

\[
\frac{p_{R|a}(R|A)}{p_R(R)} = \left[ \prod_{i=1}^{N} \left( 1 + \frac{\sigma_{s_i}^2(A)}{\sigma_n^2} \right) \right]^{-1/2} \exp \left[ \frac{1}{2} \sum_{i=1}^{N} \left( \frac{\sigma_{s_i}^2(A)}{\sigma_n^2} \right) \frac{u_{s_i}(A) \hat{r} u_{s_i}(A)}{\left( \sigma_{s_i}^2(A) + \sigma_n^2 \right)} \right].
\] (C.153)

This normalization by \( p_R(R) \) allows one to consider the estimation PDF in the context of a detection problem. To obtain a true probability density function for \( a \), one must normalize Equation C.153 so its area is unity,

\[
p_a(A) = \frac{\frac{p_{R|a}(R|A)}{p_R(R)}}{\int_{-\infty}^{\infty} \frac{p_{R|a}(R|A)}{p_R(R)} dR} = \frac{p_{R|a}(R|A)}{\int_{-\infty}^{\infty} p_{R|a}(R|A) dR}
\] (C.154)

The Cramér-Rao Lower Bound (CRLB) establishes an absolute lower bound on the variance of any unbiased estimator. The bound is greater than or equal to the inverse of the Fisher Information Matrix, \( J \),

\[
\Gamma_a \geq J^{-1}(a).
\] (C.155)

The elements of the Fisher Information Matrix are given by

\[
[J(a)]_{mn} = -E \left[ \frac{\partial^2}{\partial a_m \partial a_n} \ln p_{R|a}(R|a) \right] = -E \left[ \frac{\partial^2}{\partial a_m \partial a_n} l(a) \right].
\] (C.156)

One can show the relationship between the Maximum Likelihood estimate and the Cramér-Rao bound. Limiting oneself to the scalar nonrandom parameter case \( a = a \) for the moment, one can start with an expression for the bias error of this estimator,

\[
E[\hat{a} - A] = \int_{-\infty}^{\infty} [\hat{a} - A] p_{R|a}(R|A) dR = 0,
\] (C.157)

where \( \hat{a} \) is the output of the estimator, and \( A \) is the true parameter. This is a consequence of the unbiased nature of this estimate. Taking the derivative of the right hand side with respect to \( A \), one obtains
\[- \int_{-\infty}^{\infty} p_{\hat{a}|A}(R|A) dR + \int_{-\infty}^{\infty} \frac{d p_{\hat{a}|A}(R|A)}{dA} [\hat{a} - A] dR = 0, \quad (C.158)\]

which can be further simplified as

\[
\int_{-\infty}^{\infty} \frac{d \ln p_{\hat{a}|A}(R|A)}{dA} p_{\hat{a}|A}(R|A) [\hat{a} - A] dR = 1. \quad (C.159)\]

Recall the Schwarz inequality,

\[
\left[ \int_{-\infty}^{\infty} g(x) h(x) dx \right]^2 \leq \int_{-\infty}^{\infty} g^2(y) dy \int_{-\infty}^{\infty} h^2(z) dz \quad (C.160)\]

for scalar functions \( g(x) \), and \( h(x) \). Expanding Equation \( C.159 \),

\[
\int_{-\infty}^{\infty} \left[ \frac{d \ln p_{\hat{a}|A}(R|A)}{dA} \sqrt{p_{\hat{a}|A}(R|A)} \right] \left\{ \sqrt{p_{\hat{a}|A}(R|A)} (\hat{a} - A) \right\} dR = 1, \quad (C.161)\]

and applying the Schwarz inequality, one obtains

\[
\left\{ \int_{-\infty}^{\infty} \left[ \frac{d \ln p_{\hat{a}|A}(R|A)}{dA} \right]^2 \right\} \left\{ \int_{-\infty}^{\infty} \left[ \hat{a} - A \right]^2 p_{\hat{a}|A}(R|A) dR \right\} \geq 1. \quad (C.162)\]

One can simplify this expression by recalling

\[
\int_{-\infty}^{\infty} \frac{d p_{\hat{a}|A}(R|A)}{dA} dR = \int_{-\infty}^{\infty} \frac{d \ln p_{\hat{a}|A}(R|A)}{dA} \frac{p_{\hat{a}|A}(R|A)}{dA} dR = 0, \quad (C.163)\]

and taking its derivative with respect to \( A \), to find

\[
E \left[ \frac{d^2 \ln p_{\hat{a}|A}(R|A)}{dA^2} \right] = -E \left[ \frac{d \ln p_{\hat{a}|A}(R|A)}{dA} \right]^2 \quad (C.164)\]

which can be substituted into Equation \( C.162 \) to find the Cramér-Rao lower bound for the variance of the estimate of a nonrandom parameter, \( \hat{a} \).

\[
E \left\{ [\hat{a} - A]^2 \right\} \geq \left\{ E \left[ \frac{d \ln p_{\hat{a}|A}(R|A)}{dA} \right]^2 \right\}^{-1} \quad (C.165)\]

The bound is satisfied with equality only if

\[
\frac{d \ln p_{\hat{a}|A}(R|A)}{dA} = [\hat{a} - A] k(A) \quad (C.166)\]

for all \( R \) and all \( A \). Recall from Equation \( C.152 \) that the left hand side of Equation \( C.166 \) must be zero.

As this estimate is unbiased, \( [\hat{a} - A] \) is also zero. Thus, \( k(A) \) is left to be any expression, and the estimate
asymptotically satisfies the ML estimate, and also the Cramér-Rao Lower Bound.

Extension to a vector parameter is relatively straightforward. The results are similar; the ML estimate satisfies the vector CRLB, which can be stated in terms of the Fisher Information Matrix, \( J \).

\[ \Gamma_{\mathbf{a}} \geq J^{-1}(\mathbf{a}). \]  

(C.167)

The elements of the Fisher Information Matrix are given by

\[ [J(\mathbf{a})]_{mn} = -E \left[ \frac{\partial^2}{\partial a_m \partial a_n} \ln p_{\mathbf{R}|\mathbf{a}}(\mathbf{R}|\mathbf{a}) \right]. \]  

(C.168)

When one assumes \( \mathbf{R} \) is a Gaussian signal, the expression for the Fisher Information Matrix can be written as[6]

\[ [J(\mathbf{a})]_{mn} = \frac{1}{2} \text{Tr} \left[ \Gamma_r^{-1}(\mathbf{a}) \frac{\partial \Gamma_r(\mathbf{a})}{a_m} \Gamma_r^{-1}(\mathbf{a}) \frac{\partial \Gamma_r(\mathbf{a})}{a_n} \right]. \]  

(C.169)

where \( \text{Tr}[\mathbf{A}] \) refers to the trace of matrix \( \mathbf{A} \). In future sections, the outputs of Equations C.151 and C.169 will be applied to scenarios involving spatial estimation of a random signal.

One word of caution, however. Although it has been proven the ML estimator satisfies the CRLB, the estimator itself is a function of the estimated receiver covariance matrix, \( \hat{\Gamma}_r \). Here, it has been assumed that \( \Gamma_r \) is known perfectly, with \( \Gamma_r = \hat{\Gamma}_r \), which may not be the case. Thus, errors in estimating \( \Gamma_r \) from data may result in an estimator which is biased, and/or whose variance does not meet the CRLB.

The dependence of each parameter element on another parameter element’s estimate can be shown through the parameter covariance matrix, \( \Gamma_{\mathbf{a}} \). The diagonal elements of the matrix show the variance of each parameter in \( \mathbf{a} \), while the off-diagonal elements show the correlation between two parameters. One can quantitatively measure this through the correlation coefficient,

\[ \rho_{mn}(\mathbf{a}) = \frac{[\Gamma_{\mathbf{a}}]_{mn}}{\sqrt{[\Gamma_{\mathbf{a}}]_{mm}[\Gamma_{\mathbf{a}}]_{nn}}}. \]  

(C.170)

If \( \rho_{mn} \) is zero, the parameters \( a_n \) and \( a_m \) are uncorrelated. If \( \rho_{mn} = \pm 1 \), then the parameters are perfectly correlated.

The deflection coefficient is another estimator performance metric. It is exactly the same as the output SNR in Equation C.111, except each \( \sigma^2_{\mathbf{a}_i} \) is a function of the parameter vector \( \mathbf{a} \). One can evaluate the deflection coefficient across the domain of \( \mathbf{a} \), to determine the strength of the estimator across the parameter space. A desirable estimator would have a value of \( d^2(\mathbf{a}) \) constant for all \( \mathbf{a} \).

C.15 Estimation From Complex Data

To this point, one has assumed the received data \( \mathbf{R} \) and estimated parameters \( \mathbf{a} \) were real-valued. If \( \mathbf{R} \) were complex, small modifications would need to be made to the estimator formulas. Initially, one must
substitute the complex Gaussian PDF for the real Gaussian PDF,

\[ p_r(R) = \frac{1}{\pi^N |\Gamma_r|} \exp \left[ - (R - \bar{s})^\dagger \Gamma_r^{-1} (R - \bar{s}) \right]. \quad (C.171) \]

This has the effect of removing the factor of 1/2 from the detection and estimation statistics. For example, Equation C.151 becomes

\[ l(a) = \sum_{i=1}^{N} \left( \frac{\sigma_{a_i}^2(a)}{\sigma_n^2} \right) \left( \frac{u_{a_i}^\dagger(a) \Gamma_r u_{a_i}(a)}{\sigma_{a_i}^2(a) + \sigma_n^2} \right) - \sum_{i=1}^{N} \ln \left( 1 + \frac{\sigma_{a_i}^2(a)}{\sigma_n^2} \right), \quad (C.172) \]

and the variance of the nonrandom parameter estimate from complex data is given by the Cramér-Rao bound[29],

\[ [J(a)]_{mn} = \text{Tr} \left[ \Gamma_r^{-1} \frac{\partial \Gamma_r(a)}{\partial a_m} \Gamma_r^{-1} \frac{\partial \Gamma_r(a)}{\partial a_n} \right]. \quad (C.173) \]

**C.16 Conclusion**

This appendix was intended to be a short tutorial on detection and estimation theory. Formulas for deterministic and random signals were provided, in the presence of white and colored Gaussian noise. Also reviewed was Maximum Likelihood estimation applied to nonrandom parameter estimation. Performance metrics for both detection and estimation were discussed.

This information is applied throughout this thesis. In the accompanying text, random signal detection and nonrandom parameter estimation are applied to angle of arrival estimation using a linearly spaced, one dimensional array. Additionally, this theory is applied to source localization in range and depth using Matched Field Processing.
Bibliography


