Counting Statistics of a System to Produce Entangled Photon Pairs

by

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Submitted to the Department of Electrical Engineering and Computer Science
in Partial Fulfillment of the Requirements for the Degree of
Master of Science in Electrical Engineering and Computer Science

Abstract

Doubly-resonant optical parametric amplifiers have been proposed to generate the entangled photon pairs needed for quantum magic bullets and quantum teleportation. This thesis extends the analysis of these entanglement sources to include singly-resonant configurations. Singly-resonant systems have the advantage of being simpler to operate than their doubly-resonant counterparts, a benefit that can outweigh the deterrent of their need for higher pump power.

The special filtering property of light called the magic bullet effect is investigated for the case of doubly-resonant operation. Single-pole and Kth-order Butterworth filters are applied to entangled photon pairs to attempt to demonstrate the existence of this effect. A setup that includes a prism, lens and detector of finite size is analyzed to realize a filter with the requisite photon counting statistics and a detector that could be used in an actual system.

Finally, polarization-entangled photon pairs are used in conjunction with a trapped-atom quantum memory to create a quantum optical communication system capable of high-fidelity long-distance teleportation and quantum information storage over relatively long periods of time. The advantages and disadvantages of the singly- and doubly-resonant configurations are considered as the performances of the two are compared.
ACKNOWLEDGEMENTS

I would like to take this opportunity to recognize the many people whose support over the past two years made my journey through MIT the incredible experience that it was.

First, I would like to thank my thesis supervisor, Professor Jeffrey Shapiro, without whose patience and encouragement I would not know what an entangled photon pair is to this day. His hard work and devotion to the subject have been an inspiration to me as I have struggled to make my own contribution to the field.

Dr. Franco Wong and his group – Elliott Mason, Chris Kuklewicz, Eser Keskiner and Marius Albota – have kept life as a theorist interesting by bringing to life that which I had spent hours trying to understand. Through them I have gained insight into the challenges of experimental work and the full complexity of our project.

Finally, I would like to thank my family for always believing in me. I would not be where I am today if my parents had not taught me the value of education and fostered my interest in the world of science from an early age. I am greatly indebted to my parents, Thomas and Carolyn; my sister and brother, Elizabeth and Alexander; and, last but not least, Benjamin, for all their love and support.
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1. INTRODUCTION

In classical physics, it is possible (in principle) to have a completely noise free electromagnetic wave. In quantum physics, the Heisenberg uncertainty principle precludes that possibility [1]. At optical wavelengths, noises of quantum mechanical origin set the sensitivity limits of many photodetection systems. However, for certain classes of light-beam states – called classical states – these noises can be described by a semiclassical shot-noise theory in which the light beam is treated classically. There are non-classical states, on the other hand, whose detection-sensitivity analysis requires the full quantum theory [2].

Two beams of non-classical light whose joint photon count distribution is narrower than a Poissonian counting distribution can be produced and used to advantage in the realization of quantum magic bullets [3] and quantum teleportation [4]. Two beams – the signal (S) and the idler (I) – exhibit the magic bullet effect if, when the signal photon passes through a scattering barrier, the idler photon will pass through a corresponding barrier with probability one. As we will see later, this property is important in accomplishing quantum teleportation, a process by which a quantum state of light is transported from one place to another without the state itself being sent.

Early efforts in the generation of this type of non-classical light relied on the use of an optical parametric downconverter source to generate the two beams [5]. To produce a much brighter, narrowband source of such light, attention is now shifting to the use of an optical parametric amplifier (OPA) configuration, in which downconversion is performed within a resonant cavity [6]. In either the downconverter or OPA cases, the
individual photon count distribution of each of the two output beams is in a classical state whose counting statistics are super-Poissonian. However, the signal and idler photons are entangled; this entanglement leads to a reduced noise level in appropriate joint measurements on the two beams. This thesis will address two types of entangled photon sources—time-entangled photon sources and polarization-entangled photon sources. At the most fundamental level, both of these types of systems rely on the entanglement of two photons (the signal and the idler from a parametric interaction) to produce the desired photon counting statistics.

In order to generate a pair of time-entangled photons, a single optical parametric interaction is sufficient. Signal and idler photon pairs generated in this way have definite polarizations set by the phase-matching conditions. Thus there is no polarization entanglement in this case. Instead, there is a high degree of correlation between the time at which the signal photon is generated and the time at which the idler photon is generated that is only permitted in quantum mechanics. This correlation is shown in Figure 1.1, which is drawn for a parametric downconverter, in which signal and idler photons are created in near-ideal simultaneity.

![Figure 1.1: Time Entanglement - Although (a) signal photons are generated at random time intervals by the downconverter, (b) idler photons are, in fact, generated at the same random time intervals.](image-url)
Photons exhibit perfect time entanglement in pairs if the conditional probability of counting a particular number, \( m \), photons over some time interval at the idler output of a system given that \( n \) photons were counted over this time interval at the signal output of the system is 1 if \( n = m \) and 0 if \( n \neq m \). This can be seen quantitatively by looking at two modes, the signal and the idler output modes at frequencies \( \omega_s \) and \( \omega_i \) from an ideal OPA, with photon-number states \( |n>_s \) and \( |n>_{i} \). The joint signal-idler state is then [7]:

\[
|\Psi\rangle_{SI} = \sum_{n=0}^{\infty} \sqrt{\frac{\overline{N}^n}{(\overline{N} + 1)^{n+1}}} |n\rangle_s |n\rangle_i
\]

The probability of counting \( n \) signal photons in the interval \( 0 \leq t \leq T \) at the output of the OPA can be found from the number operator \( \hat{N}_s = \int_0^T \hat{A}_s^\dagger(t)\hat{A}_s(t)dt \) (where \( \hat{A}_s(t) \) is the baseband signal-field creation operator and \( \hat{A}_s(t) \) is the baseband signal-field annihilation operator) and is denoted \( \text{Pr}[\hat{N}_s = n] \). The probability of counting a particular number of idler photons in this same interval can be found accordingly. When only the two modes whose joint signal-idler state is the one given above are considered, \( \hat{N}_s \) and \( \hat{N}_i \) reduce to \( \hat{N}_s = \hat{a}_s^\dagger \hat{a}_s \) and \( \hat{N}_i = \hat{a}_i^\dagger \hat{a}_i \), where \( \hat{a}_s \) and \( \hat{a}_i \) are the annihilation operators for the signal and idler modes under consideration. It then follows that the system has a Bose-Einstein counting distribution for each individual mode:

\[
\text{Pr}[\hat{N}_s = n] = \text{Pr}[\hat{N}_i = n] = \frac{\overline{N}^n}{(\overline{N} + 1)^{n+1}}
\]

where the average number of photon counts per mode is \( \overline{N} \). The variance of this counting distribution is \( \overline{N}^2 + \overline{N} \), which is always larger than the shot-noise limited.
value, $\bar{N}$. This counting distribution also has the unique conditional counting probability mentioned above:

$$\Pr[\hat{N}_t = m | \hat{N}_s = n] = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

Thus, by knowing the state of one subsystem (the signal), it is possible to obtain complete information about the other subsystem (the idler.) Of course, this perfect entanglement is degraded if the measurements are made with less than the unity quantum efficiency that we have assumed thus far.

The role of the counting-interval duration, $T$, is important for an OPA. Signal-idler pairs are created in near simultaneity within a parametric downconverter, but the addition of the resonant cavity that turns this downconverter into an OPA smears out this simultaneity because signal and idler photons may stay inside for several cavity lifetimes before escaping. Thus, in OPA measurements we must count over an interval that is significantly longer than the OPA cavity lifetime to ensure that each detected idler photon always accompanies a detected signal photon [6].

It has been shown that polarization entanglement can be obtained by polarization combining the outputs from two matched, doubly-resonant OPAs [6]. In this scheme, each OPA produces time-entangled signal and idler beams that are orthogonally polarized. The two signal and two idler beams – from the two OPAs – are then combined, respectively, in orthogonal polarizations. While the individual signal and idler photons of these combined beams are randomly polarized, the polarizations of the photons are entangled such that if you determine the signal (idler) polarization, say by measurement, then you can deduce the idler (signal) polarization. The OPA setup allows the generation of polarization-entangled photon pairs at a rate sufficiently high as to permit useful quantum communication rates in conjunction with a trapped-atom quantum
memory [8]. Because quasi-phase-matched nonlinear optics allows OPAs to be built at many wavelengths, OPA entanglement sources are compatible with fiber optic technology transmission. Still, this result leaves many questions unanswered about the generation of entangled photons using OPAs. The original entanglement source proposed in [6] used two OPAs, both doubly-resonant. Recently it has been suggested that a single, degenerate OPA could be used to produce polarization entanglement [9]. Because singly-resonant operation may be another simpler alternative in the dual-OPA source arrangement, this thesis will address the possibility of using a singly-resonant OPA (SRO) to generate entangled photon pairs.

There is much interest in the special filtering properties of entangled light, a topic now being called quantum magic bullets [3]. Quantum entanglement and quantum magic bullets have the potential to transform the field of quantum information processing, particularly in the area of quantum teleportation. Bennett et al. [10] proposed a scheme wherein teleportation of quantum information from a sender, Alice, to a receiver, Bob, is accomplished by the following protocol. First, the two parties share a singlet state, 

$$|\Psi\rangle_{AB} = \left( |0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B \right)/\sqrt{2},$$

via a quantum channel. Then, Alice accepts an input particle and performs the Bell-state measurements ($\hat{B}$) on the joint state of the input particle and her portion of the singlet state.

$$\hat{B} = \sum_{n=0}^{3} n |B_n\rangle \langle B_n|$$ where

$$B_0 = \left( |1\rangle_{IN} |0\rangle_A - |0\rangle_{IN} |1\rangle_A \right)/\sqrt{2}$$

$$B_1 = \left( |1\rangle_{IN} |0\rangle_A + |0\rangle_{IN} |1\rangle_A \right)/\sqrt{2}$$

$$B_2 = \left( |1\rangle_{IN} |1\rangle_A - |0\rangle_{IN} |0\rangle_A \right)/\sqrt{2}$$

$$B_3 = \left( |1\rangle_{IN} |1\rangle_A + |0\rangle_{IN} |0\rangle_A \right)/\sqrt{2}$$

Alice communicates the results of the measurements to Bob via a classical channel. With
a small amount of manipulation, the original state can then be recreated at Bob’s location, and teleportation is achieved. The magic bullet effect is important because in order for this protocol to work, Alice and Bob must share a singlet state. Two quantum particles exhibit the quantum magic bullet effect if when one particle passes through a scattering potential, the other particle will pass through a corresponding barrier with probability one, no matter how low the probability of the first event. This is clearly a desirable property when sharing a singlet state. It was demonstrated in [3] that if entangled photons are measured in a high-Q cavity with no excess loss, the magic bullet effect occurs. However, the occurrence of the effect is limited to the regime where the measurement-cavity linewidth is much smaller than the OPA cavity linewidth, and where the cavity-loading time is long enough for statistical steady-state to be reached. This result also leaves many questions unanswered. In particular, it is unclear whether this effect could be achieved using a singly-resonant OPA and how filtering the OPA output could affect this result. This thesis will also address these issues.

Until recently, no system had been proposed that was capable of performing high-fidelity teleportation over long distances and of storing quantum information for a significant period of time. Researchers at MIT have proposed such a system that uses the narrowband dual OPA described above in conjunction with a trapped-atom memory scheme [8]. Entangled photons are generated by the OPA setup and transported over long distances to be received and stored in a quantum memory. In the proposed memory, photons are loaded into a cavity occupied by an ultra-cold trapped rubidium atom and stored in its long-lived hyperfine levels. The cavity-loading analysis of the quantum memory has been carried using the doubly-resonant OPA (DRO) to generate the
entangled photons, but this thesis will extend the analysis to the case of a general dual-resonator and will investigate a protocol in which the SRO is used.

The thesis is organized in the following way. Chapter 2 describes OPA operation in more detail and presents the derivation of SRO quantum correlation functions from the general dual-resonator equations of motion. Chapter 3 presents a more thorough account of magic bullets and investigates the filter penetration properties of the OPA output to determine whether magic bullets can be realized with this setup. Chapter 4 compares the performances of the DRO and the SRO in the context of quantum teleportation. Finally, Chapter 5 concludes the thesis and looks at potential future research in the area of quantum optical communications.
2. OPTICAL PARAMETRIC AMPLIFIER SOURCE

The optical parametric amplifier is fundamental to the generation of entangled photon pairs. Photon counting statistics are determined by both the internal equations of motion of the OPA and the input/output relations, all of which depend on the parameters of the OPA. This chapter begins with an overview of the operation of entangled photon sources and describes why the OPA was chosen over the parametric downconverter for the analyses in this thesis. General dual-resonator equations of motion are then introduced and the derivation of their solutions is presented. Finally, the quantum correlation functions are obtained for this general resonator.

2.1 Entangled Photon Sources

Until recently, entangled photons were produced primarily by parametric downconversion, in which power from an input laser beam at frequency \( \omega_P \) is transferred to signal and idler beams at frequencies \( \omega_S \) and \( \omega_I \), respectively [11]. A schematic overview of this process is shown in Figure 2.1.

![Figure 2.1: Downconversion by \( \chi^{(2)} \) interactions](image)

Energy and momentum must be conserved in this process. Conservation of energy dictates that \( \omega_P = \omega_S + \omega_I \), where the energy of a photon is \( E = \hbar \omega \) and \( \hbar \) is
Planck’s constant divided by $2\pi$. The conservation of momentum requires that the wave propagation vectors be conserved, $k_p = k_s + k_i$. This phase matching sets the signal and idler polarizations. When entangled photons are generated in this manner, there is no need for a device to separate them because they generally leave the nonlinear crystal traveling in different directions, as shown in Figure 2.1.

Photon-pair generation rates for sources relying on spontaneous downconversion were exceedingly low until Kwiat et al. [5] reported a source in which two $\chi^{(2)}$ nonlinear crystals were employed to accomplish the desired frequency conversion. The polarization-entangled photon pair generation rate of this source was $1.5 \times 10^6 \text{ s}^{-1}$ over a 5 nm bandwidth at a 702 nm center wavelength for 150 mW of pump power. Yet even this source, which was ten times brighter per unit of pump power than previous sources, is not sufficiently narrowband to be useful for trapped atom storage applications. In particular, for a trapped atom quantum memory with a 30 MHz bandwidth, the Kwiat et al. source only produces 15 pairs/sec.

![Figure 2.2: Downconversion in an OPA](image)

The OPA, with a pair production rate of $1.5 \times 10^6 \text{ s}^{-1}$ over a 30 MHz bandwidth at a 795 nm center wavelength using only 0.7 mW of pump power, was proposed as a source of entangled photons in [5] to combat this problem. The premise of this scheme was that optical parametric amplification in a doubly resonant cavity was used instead of using
spontaneous downconversion.

Optical parametric amplification is shown schematically in Figure 2.2. Pump photons enter the OPA through M1. They then travel through the nonlinear crystal, where some experience downconversion and become entangled signal and idler photons. To improve performance, entangled photons are resonated in the cavity consisting of M1, a totally reflecting mirror, and M2, an almost totally reflecting mirror. Entangled photons are reflected back and forth several times before they escape through M2. There is a small amount of timing jitter associated with this process due to the fact that signal and idler photons may not escape the cavity on the same pass.

Figure 2.3: Entangled Photon Pair Sources – The single OPA configuration (a) produces time-entangled photon pairs, while the dual OPA configuration (b), in which PBS indicates a polarizing beam splitter, produces polarization-entangled photon pairs. Bullets and arrows indicate horizontal and vertical polarization, respectively.

Energy and momentum are conserved in this process and are required in order for the OPA to operate efficiently. When a single OPA is operated, these conditions lead to entanglement in the time domain. Signal and idler photons are generated in the OPA; they are then separated into signal and idler components by a beamsplitter, which is necessary because the photon paths are collinear at the OPA output. Optical parametric amplification can be accomplished with either type-I or type-II phase matching. A single type-II phase-matched OPA is shown in Figure 2.3(a). Here the signal and idler photons
have definite, orthogonal polarizations and can therefore be separated with a polarizing beam splitter. To make polarization-entangled light with the type-II arrangement, we bring the outputs from a second OPA into the polarizing beam splitter, as shown in Figure 2.3(b), arranged so that we get the signal beams from both OPAs out from one port of the beam splitter and the idler beams from both OPAs out from the other port of the beam splitter. The photon generation rate of the dual-OPA arrangement is sufficiently high, over a narrow emission spectrum, for the source to be practical for a trapped-atom quantum memory application.

The photon counting statistics of the OPA are essential to quantifying the generation of entangled photon pairs. For a type-II phase-matched doubly-resonant OPA, the normally ordered and phase-sensitive correlation functions in the ideal lossless case were calculated in [6] based on the internal equations of motion of the OPA plus the signal and idler output equations. It was found that although the signal-minus-idler photon-count difference is classically shot-noise limited for photon-counting intervals much shorter than the cavity lifetime, the normalized variance of this difference is very non-classical in that it approaches zero for photon counting intervals much longer than the cavity lifetime. It is this non-classical property that yields the desired statistics that, when generated using two OPAs, will be used for quantum teleportation. This chapter extends the analysis of the single OPA to the case in which the signal and idler have different cavity linewidths, and shows that the same non-classical property applies to this new case.

2.2 General Dual-Resonator Output Solutions for the Lossless Case

As a starting point in the calculation of the general dual-resonator output
solutions, let us define some field operators. The photon-unit, positive-frequency scalar field operators $\hat{E}_S(t)$ and $\hat{E}_I(t)$ for the signal and idler output beams are defined in the following way:

$$\hat{E}_S(t) = \hat{A}_S(t) \exp(-i\omega_S t)$$
$$\hat{E}_I(t) = \hat{A}_I(t) \exp(-i\omega_I t)$$

These definitions neglect the spatial mode and focus only on the non-vacuum polarizations from the OPA. The field quantity $\hat{A}_k(t)$ has units of photons/sec. The field $\hat{A}_k(t)$ is centered at zero frequency. The commutator brackets of the input field operators are as follows:

$$[\hat{A}_k^\dagger(t), \hat{A}_k^\dagger(u)] = \delta(t-u)$$
$$[\hat{A}_j^\dagger(t), \hat{A}_k(u)] = [\hat{A}_j^\dagger(t), \hat{A}_k^\dagger(u)] = 0$$

In these commutator brackets, $k$ and $j$ can indicate either signal or idler and $k \neq j$.

The next step in the calculation of the output fields of the general dual-resonator case is to introduce the internal equations of motion and the signal and idler output equations. Counting statistics of the general dual-resonator can then be obtained from the solution of these equations. This was done in [6] for the case in which the signal and idler cavity linewidths were equal. The equations and their solutions become, as we shall see, more complicated when different cavity linewidths are introduced. Therefore, the effects of internal cavity losses will not be considered until after a complete derivation has been done for the lossless case.

To facilitate the introduction of the relevant equations, Figure 2.4 illustrates the relevant operators for a single OPA in a block diagram of the setup.
The applicable internal equations of motion for the lossless case are [12]:

$$\begin{align*}
\left( \frac{d}{dt} + \Gamma_s \right) \hat{a}_s(t) &= G \sqrt{\Gamma_s \Gamma_i} \hat{a}_i^+(t) + \sqrt{2\Gamma_s} \hat{A}_s^{IN}(t) \\
\left( \frac{d}{dt} + \Gamma_i \right) \hat{a}_i(t) &= G \sqrt{\Gamma_s \Gamma_i} \hat{a}_s^+(t) + \sqrt{2\Gamma_i} \hat{A}_i^{IN}(t)
\end{align*}$$

In the Figure and the equations, $\hat{a}_s(t)$ and $\hat{a}_i(t)$ are the intracavity annihilation operators of the signal and idler; $G^2$ is the normalized OPA gain, i.e. the ratio of pump power to the oscillation threshold, $G^2=P_p/P_T$; and $\Gamma_s$ and $\Gamma_i$ are the linewidths of the signal and idler cavities, respectively. $\hat{A}_s^{IN}(t)\exp(-i\omega_s t)$ and $\hat{A}_i^{IN}(t)\exp(-i\omega_i t)$ are the vacuum-state, photon-units, positive-frequency input-field operators. The signal and idler outputs from the OPA are [12]:

$$\begin{align*}
\hat{A}_s(t) &= \sqrt{2\Gamma_s} \hat{a}_s(t) - \hat{A}_s^{IN}(t) \\
\hat{A}_i(t) &= \sqrt{2\Gamma_i} \hat{a}_i(t) - \hat{A}_i^{IN}(t)
\end{align*}$$

The OPA produces signal and idler that are in an entangled, zero-mean, Gaussian pure state [6]. This system of equations does not characterize this state in a straightforward manner; the remainder of this section will be devoted to simplifying these equations to a point where they can be used to obtain the two output correlation functions of the general dual-resonator case. These functions will characterize the state completely. This is
accomplished by reducing the four equations to two equations relating the output-field operators to the input-field operators, not involving the internal-mode annihilation operators. It is possible to avoid solving differential equations by working in the Fourier domain. The convention used is shown here for the intracavity annihilation operator of the signal:

\[
\hat{a}_s(\omega) = \int \hat{a}_s(t)e^{-i\omega t}dt
\]

\[
\hat{a}_s(t) = \int \hat{a}_s(\omega)e^{i\omega t}\frac{d\omega}{2\pi}
\]

All other operators follow the same convention. The four equations above then become:

\[
(i\omega + \Gamma_s)\hat{a}_s(\omega) = G\sqrt{\Gamma_s \Gamma_s}\hat{a}_s^\dagger(-\omega) + \sqrt{2}\Gamma_s \hat{A}_s^\text{IN}(\omega)
\]

\[
(i\omega + \Gamma_f)\hat{a}_f(\omega) = G\sqrt{\Gamma_s \Gamma_s}\hat{a}_s^\dagger(-\omega) + \sqrt{2}\Gamma_f \hat{A}_f^\text{IN}(\omega)
\]

\[
\hat{A}_s(\omega) = \sqrt{2}\Gamma_s \hat{a}_s(\omega) - \hat{A}_s^\text{IN}(\omega)
\]

\[
\hat{A}_f(\omega) = \sqrt{2}\Gamma_f \hat{a}_f(\omega) - \hat{A}_f^\text{IN}(\omega)
\]

These equations are straightforward to solve in the frequency domain; the resulting equations of output as a function of input are:

\[
\hat{A}_s(\omega) = \frac{2G\Gamma_s\Gamma_f\hat{A}_f^\text{IN}(\omega) + [G^2\Gamma_s\Gamma_f - (i\omega - \Gamma_s)(i\omega + \Gamma_f)]\hat{A}_s^\text{IN}(\omega)}{(i\omega + \Gamma_s)(i\omega + \Gamma_f) - G^2\Gamma_s\Gamma_f}
\]

\[
\hat{A}_f(\omega) = \frac{2G\Gamma_s\Gamma_f\hat{A}_s^\text{IN}(\omega) + [G^2\Gamma_s\Gamma_f - (i\omega - \Gamma_s)(i\omega + \Gamma_f)]\hat{A}_s^\text{IN}(\omega)}{(i\omega + \Gamma_s)(i\omega + \Gamma_f) - G^2\Gamma_s\Gamma_f}
\]

These are the dual-resonator output equations in their most general form. In accordance with the Heisenberg Uncertainty Principle, the commutator brackets of the input field operators are preserved by the output field operators.
2.3 Quantum Correlation Functions for the Lossless Case

The normalized variance of the signal-minus-idler photon-count difference, which was calculated and used as the performance standard for the DRO in [5], is the ratio of the variance of the photon count difference divided by the sum of the signal mean photon count plus the idler mean photon count. (Each of these measures will be defined quantitatively later in this section.) It was found that in the case of the doubly-resonant OPA, in which the signal and idler have identical linewidths, the normalized variance approaches zero for photon counting intervals much longer than the cavity lifetime, $1/\Gamma$, i.e.,

$$\frac{\langle \Delta N^2 \rangle}{\langle \hat{N}_s \rangle + \langle \hat{N}_i \rangle} \rightarrow \begin{cases} 1 & \Gamma T \ll 1 \\ 0 & \Gamma T \gg 1 \end{cases}$$

If this non-classical property holds for singly-resonant OPAs as well, this could greatly simplify the generation of entangled photon pairs because it is much easier to stabilize the cavity of a singly-resonant system than that of a doubly-resonant system. Of course this simplification comes at the expense of a higher threshold power for the SRO as compared to the DRO. Hence, for the same $G^2$ value a stronger pump beam will be needed for the SRO case. In order to determine if the SRO has the desired photon-count-difference variance behavior, it is necessary to find the quantum correlation functions of the general dual-resonator OPA. These are found from the output equations using the input field operator commutator brackets introduced in Section 2.2. The correlation functions of the input field operators in the frequency domain can be found using the Fourier transform relations defined in Section 2.2. They are all zero except:

$$\langle \hat{A}_k^{\text{IN}}(\omega_1)\hat{A}_k^{\text{IN}^\dagger}(\omega_2) \rangle = 2\pi\delta(\omega_1 - \omega_2)$$
Let $\Lambda = \sqrt{(\Gamma_s - \Gamma_i)^2 + 4G^2\Gamma_s\Gamma_i}$. The normally ordered and phase-sensitive correlation functions, respectively, then become:

$$\langle \hat{A}_s^+(t + \tau)\hat{A}_s(t) \rangle = \langle \hat{A}_s^+(t + \tau)\hat{A}_s(t) \rangle = K^{(n)}(\tau)$$

$$= \frac{2G^2\Gamma_s\Gamma_i}{(1 - G^2)(\Gamma_s + \Gamma_i)} \left[ \cosh\left(\frac{\Lambda|\tau|}{2}\right) + \frac{\Gamma_s + \Gamma_i}{\Lambda} \sinh\left(\frac{\Lambda|\tau|}{2}\right) \right]$$

$$= \frac{2G^2\Gamma_s\Gamma_i}{(1 - G^2)(\Gamma_s + \Gamma_i)} \left[ \cosh\left(\frac{\Lambda|\tau|}{2}\right) + \frac{(\Gamma_s + 2G^2\Gamma_i - \Gamma_s)}{\Lambda} \sinh\left(\frac{\Lambda|\tau|}{2}\right) \exp\left(-\frac{\Gamma_s + \Gamma_i}\Lambda \right) \right]$$

These correlation functions completely characterize the state of the entangled signal and idler. From the normally ordered correlation function equation, the individual statistics of the signal and idler photon counts can be calculated by the following equations, assuming unity quantum efficiency photon counting over the time interval $0 \leq t \leq T$:

$$\hat{N}_s = \int_0^T \hat{A}_s^+(t)\hat{A}_s(t)dt$$

$$\hat{N}_i = \int_0^T \hat{A}_i^+(t)\hat{A}_i(t)dt$$

The average signal and idler photon counts and are found to be, in the general case:

$$\langle \hat{N}_s \rangle = \langle \hat{N}_i \rangle = \frac{2G^2\Gamma_s\Gamma_i T}{(1 - G^2)(\Gamma_s + \Gamma_i)}$$

This implies that the photon-count-difference, $\Delta \hat{N} = \hat{N}_s - \hat{N}_i$, is zero mean, a prerequisite for the desired application.
The other component that is required to assess the potential of the SRO as an alternative to the doubly-resonant OPA is the difference count variance. Calculation of the variance provides a conclusive theoretical result with regard to the utility of an OPA for generating entangled photons. This calculation is done using the normally ordered and phase-sensitive correlation functions in equation (29) of [13]. In the case of the general dual-resonator, the equations involved in this computation are quite lengthy; the details are therefore relegated to Appendix A. The final result is an equation for normalized variance, $\langle \Delta \hat{N}^2 \rangle$, that depends only on the dimensionless parameters $\Gamma_s / \Gamma_1$, $\Gamma_s T$ and $G$. This makes sense from a physical point of view. $\Gamma_s / \Gamma_1$ is the ratio of the signal cavity linewidth to the idler cavity linewidth. $\Gamma_s T$ is the photon counting period in units of signal cavity bandwidth. $G^2$ is the normalized gain of the OPA, as discussed in Section 2.2.

For a singly-resonant OPA in which the signal resonates (that is, $\Gamma_s$ is much smaller than $\Gamma_1$) the normalized variance can be shown to have the following asymptotic behavior:

$$\frac{\langle \Delta \hat{N}^2 \rangle}{\langle \hat{N}_s \rangle + \langle \hat{N}_i \rangle} \rightarrow \begin{cases} 1 & \Gamma_s T \ll 1 \\ 0 & \Gamma_s T \to \infty \end{cases}$$

Clearly, the difference count is not shot-noise limited when the photon counting interval is much longer than the signal cavity lifetime.

In order to generate a more complete picture of the dependence of variance on the input parameters, I first plotted the normalized variance as a function of the normalized counting interval, $\Gamma_s T$, for several ratios of signal-to-idler linewidth at a fixed gain value. The plots use the notation $R= \Gamma_s / \Gamma_1$. The $R=1$ (DRO) curve is shown for comparison. As
Figure 2.5(a) shows, the DRO curve has the lowest normalized variance. As $\Gamma_s/\Gamma_1$ decreases, the normalized variance increases overall at fixed $\Gamma_s T$. The normalized variance is highest in the SRO operation regime ($\Gamma_s << \Gamma_1$). However, it is clear from this plot that if photons are counted over a long period of time relative to the signal cavity linewidth, the normalized variance approaches zero for any type of resonator.

Next, I plotted normalized variance as a function of counting interval for several values of gain at a fixed signal-to-idler linewidth ratio. This is shown in Figure 2.5(b). The plot shows that the normalized variance approaches zero in the limit as the photon counting interval increases without bound for all relevant gain values. Yet it appears that the best system performance occurs when gain is extremely low. This low-gain preference is not a problem because quantum communication using an OPA source requires low-gain operation to keep the probability that there is more than one signal-idler photon pair in a counting interval at an acceptably low value.
Figure 2.5(a): Normalized Variance vs. Time-Bandwidth Product of the Signal Cavity for Several Signal-to-Idler Cavity Linewidth Ratios. The curves for $R = 0.01$ and $R = 0.001$ are indistinguishable.
Normalized Variance vs. Time-Bandwidth Product of the Signal Cavity for $R = 0.0001$

$G^2 = 0.01, 0.05, 0.1, 0.5$

Figure 2.5(b): Normalized Variance vs. Time-Bandwidth Product of the Signal Cavity for Several Values of Gain. The curves for $G^2 = 0.01$ and $G^2 = 0.05$ are indistinguishable.
2.4 General Dual-Resonator Output Solutions

The analysis above did not account for the effects of internal cavity losses, which can have a significant impact on system performance. The goal of this section is to quantify that impact so that the behavior of a more realistic system can be understood.

Once again, the starting point of the analysis is the internal equations of motion and the signal and idler output equations. These equations take on a slightly more complicated form when internal cavity loss is included in the analysis. Figure 2.6 shows the relevant operators that appear in these equations.

The applicable internal equations of motion are [12]:

\[
\left( \frac{d}{dt} + \Gamma_s \right) \hat{a}_s(t) = G \sqrt{\Gamma_s \Gamma_l} \hat{a}_i^+(t) + \sqrt{2 \gamma_s} \hat{A}_s^\text{IN}(t) + \sqrt{2 (\Gamma_s - \gamma_s)} \hat{A}_s^\text{loss}(t)
\]

\[
\left( \frac{d}{dt} + \Gamma_I \right) \hat{a}_I(t) = G \sqrt{\Gamma_s \Gamma_l} \hat{a}_s^+(t) + \sqrt{2 \gamma_I} \hat{A}_I^\text{IN}(t) + \sqrt{2 (\Gamma_I - \gamma_I)} \hat{A}_I^\text{loss}(t)
\]

In Figure 2.6 and in the equations, \( \hat{A}_s^\text{loss}(t) \) and \( \hat{A}_I^\text{loss}(t) \) are the vacuum-state field operators that represent the effect of intracavity loss. The signal and idler outputs from the OPA are [12]:

\[
\hat{A}_s^L(t) = \sqrt{2 \gamma_s} \hat{a}_s(t) - \hat{A}_s^\text{IN}(t)
\]

\[
\hat{A}_I^L(t) = \sqrt{2 \gamma_I} \hat{a}_I(t) - \hat{A}_I^\text{IN}(t)
\]
where the coupling constants $\gamma_s$ and $\gamma_i$ obey $0 < \gamma_s < \Gamma_s$ and $0 < \gamma_i < \Gamma_i$.

The solutions to these four equations can be found following the steps outlined in Section 2.2. The quantum correlation functions can be found by the same methods used in Section 2.3. The only additional information required to perform the analysis is the commutator brackets of the loss operators. These are:

\[
\left[ \hat{A}_k^{\text{loss}}(t), \hat{A}_k^{\text{loss}\dagger}(u) \right] = \delta(t-u)
\]
\[
\left[ \hat{A}_j^{\text{loss}}(t), \hat{A}_k^{\text{loss}}(u) \right] = \left[ \hat{A}_j^{\text{loss}}(t), \hat{A}_k^{\text{loss}\dagger}(u) \right] = 0
\]

In these commutator brackets, $k$ and $j$ can indicate either signal or idler and $k \neq j$. The resulting normally-ordered and phase-sensitive correlation functions have a very simple relation to the correlation functions found when internal cavity loss was not considered:

\[
\left\langle \hat{A}_s^{L\dagger}(t + \tau) \hat{A}_s^L(t) \right\rangle = \frac{\gamma_s}{\Gamma_s} \left\langle \hat{A}_s^\dagger(t + \tau) \hat{A}_s(t) \right\rangle
\]
\[
\left\langle \hat{A}_i^{L\dagger}(t + \tau) \hat{A}_i^L(t) \right\rangle = \frac{\gamma_i}{\Gamma_i} \left\langle \hat{A}_i^\dagger(t + \tau) \hat{A}_i(t) \right\rangle
\]
\[
\left\langle \hat{A}_s^L(t + \tau) \hat{A}_i^{L\dagger}(t) \right\rangle = \frac{\gamma_s \gamma_i}{\Gamma_s \Gamma_i} \left\langle \hat{A}_s(t + \tau) \hat{A}_i(t) \right\rangle
\]

In these equations, $\hat{A}_s(t)$ and $\hat{A}_i(t)$ are the signal and idler output field operators for the lossless case, and $\hat{A}_s^L(t)$ and $\hat{A}_i^L(t)$ are the corresponding field operators for the lossy case. Figure 2.7 shows the impact that $\gamma_s/\Gamma_s$ and $\gamma_i/\Gamma_i$ on the normalized variance, plotted as a function of $\Gamma_s \tau$. As $\gamma_s/\Gamma_s$ and $\gamma_i/\Gamma_i$ decrease, the minimum attainable normalized variance increases dramatically. When $\gamma_s/\Gamma_s = \gamma_i/\Gamma_i = 0.5$, the lowest normalized variance that can be achieved over long counting times is 0.5. Figure 2.8(a) shows the impact of the ratio $\Gamma_s/\Gamma_i$ on the normalized variance. As in the lossless case, the
normalized variance is always lowest for the DRO and increases as the ratio decreases. But here again, if photons are counted over a long period of time relative to the signal cavity linewidth, the normalized variance approaches the same constant value for any type of resonator. Figure 2.8(b) shows normalized variance for several gain values. As in Figure 2.5(b), the best system performance occurs when gain is extremely low.

It should not be surprising that intracavity loss has a dramatic adverse effect on the normalized difference count variance. If $\gamma_S/\Gamma_S = \gamma_I/\Gamma_I = 0.5$, then, on average, half of the signal and half of the idler photons are lost prior to their exiting the OPA cavity. Because these losses are statistically independent, this means that, on average, half the signal (idler) photons will not have their idler (signal) companion photons present at the photodetection stage.

Thus far in the analysis, parametric amplification looks promising as a means of generating entangled photon pairs if excess losses can be kept small, using either a singly- or doubly-resonant OPA. Chapter 3 takes the investigation one step further, to determine whether an OPA setup can generate time-entangled pairs that demonstrate the magic bullet effect.
Normalized Variance vs. Time-Bandwidth Product of the Signal Cavity
With the Effects of Excess Loss Included for $G^2 = 0.5$ and $R = 0.1$
for the Signal and Idler Coupling-to-Cavity Ratios 0.25, 0.5, 0.75, and 1

Figure 2.7: Normalized Variance vs. Time-Bandwidth Product of the Signal Cavity for Several Signal and Idler Coupling-to-Cavity Linewidth Ratios. The Coupling Rate to Cavity Linewidth Ratio of 1 indicates that no excess loss is present.
Normalized Variance vs. Time-Bandwidth Product of the Signal Cavity
With the Effects of Excess Loss Included for $G^2 = 0.5$
$R = 1, 0.1, 0.01, 0.001$

Figure 2.8(a): Normalized Variance vs. Time-Bandwidth Product of the Signal Cavity for Several Signal-to-Idler Cavity Linewidth Ratios. The curves for $R = 0.01$ and $R = 0.001$ become indistinguishable as the time-bandwidth product increases. The Coupling Rate to Cavity Linewidth Ratio is 0.25.
Figure 2.8(b): Normalized Variance vs. Time-Bandwidth Product of the Signal Cavity for Several Values of Gain. The curves for $G^2 = 0.01$, $G^2 = 0.05$ and $G^2 = 0.1$ become indistinguishable as the time-bandwidth product increases. The Coupling Rate to Cavity Linewidth Ratio is 0.25.
3. QUANTUM MAGIC BULLETS

Recently, it has been shown that entangled photon pairs have an interesting filtering property [3] called the magic bullet effect; when the signal photon passes through a scattering barrier, the idler will automatically pass through a corresponding barrier. This can be shown, for example, by analyzing the performance (as determined by the normalized variance) of the OPA output after being filtered by one of two filters: a single-pole filter or a Kth-order Butterworth filter. After introducing some background on quantum magic bullets, the quantum magic bullet effect will be demonstrated by examining the performance of the filtered OPA output. The last section of the chapter will present one way to implement the type of filter that would enable the realization of the quantum magic bullet effect.

3.1 Magic Bullets and the Einsten-Podolsky-Rosen Analogy

Entanglement in the time domain is required to achieve the magic bullet effect. The single OPA setup shown in Figure 2.3(a) can be used to produce time-entangled photon pairs. It was not until recently that an OPA setup was used to produce entangled photon pairs, but the concept of entanglement was introduced in the 1930s.

Einstein, Podolsky, and Rosen (EPR), who first described the phenomenon of entanglement in 1935 in [14], are responsible for the earliest work related to the quantum magic bullet effect. Their original paper centered on the entanglement of continuous variables, position and momentum. There is a strong analogy between quantum magic bullets and EPR pairs which becomes clear when examined quantitatively.
The EPR joint state of two particles formed by the decay of a zero momentum particle is:

\[ |\Psi\rangle_{EPR} = \int_{-\infty}^{\infty} |x\rangle_1 |x\rangle_2 dx = \int_{-\infty}^{\infty} |p\rangle_1 |-p\rangle_2 dp \]

where \( x \) denotes position and \( p \) denotes momentum. Thus, these two particles are perfectly correlated in position and perfectly anti-correlated in momentum. Therefore, the position of particle two (one) can be determined to a great degree of accuracy by measuring the position of particle one (two). In addition, the momentum of particle two (one) can be determined to a great degree of accuracy by measuring the momentum of particle one (two). This dual correlation is paradoxical in that it seems as though both position and momentum can be found precisely. However, because only position or momentum can be determined to a great degree of accuracy, the Heisenberg uncertainty principle is not violated.

The analogy between the joint signal-idler state introduced in Chapter 1 and the EPR state was shown in [3] by obtaining the field-quadrature representation for \( |\Psi\rangle_{SL} \). A photon annihilation operator \( \hat{a} \) has quadrature components \( \hat{a}_1 \equiv \text{Re}(\hat{a}) \) and \( \hat{a}_2 \equiv \text{Im}(\hat{a}) \) that behave like normalized versions of position and momentum. The eigenkets of the quadrature components are related by Fourier transformation:

\[ |\alpha_2\rangle = \left( \frac{1}{\sqrt{\pi}} \right) \int_{-\infty}^{\infty} e^{2i\alpha_1 \alpha_2} |\alpha_1\rangle d\alpha_1 \]

and vice-versa. The joint signal-idler state then becomes:

\[ |\Psi\rangle_{SL} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi(\alpha_{s_1}, \alpha_{t_1}) |\alpha_{s_1}\rangle |\alpha_{t_1}\rangle d\alpha_{s_1} d\alpha_{t_1} \]

with \( \Psi(\alpha_{s_1}, \alpha_{t_1}) \equiv \exp\left[-(1+2\bar{N})\alpha_{s_1}^2 + 4\sqrt{\bar{N}(\bar{N}+1)}\alpha_{s_1}\alpha_{t_1} - (1+2\bar{N})\alpha_{t_1}^2 \right]/\sqrt{\pi/2} \)
These equations show a nonclassical correlation. When the $\hat{a}_{s_i}$ and $\hat{a}_{i_i}$ measurements are made on the state, the $\hat{a}_{s_i}$ and $\hat{a}_{i_i}$ observations have identical unconditional statistics: both are zero-mean Gaussian distributions with variance $(1+2\bar{N})/4$. However, the nonclassical behavior comes about when the joint statistics of the two measurements are considered. Given the outcome of the $\hat{a}_{s_i}$ measurement is $\alpha_{s_i}$, the conditional statistics of the $\hat{a}_{i_i}$ measurement are still Gaussian, but with conditional mean $\alpha_{s_i}\sqrt{4\bar{N}(\bar{N}+1)/(1+2\bar{N})}$ and conditional variance $1/4(1+2\bar{N})$. This variance is always below the shot-noise limited value of $\frac{1}{4}$, and goes to zero as $\bar{N} \to \infty$.

Because the entangled photon equations above are not identical to the EPR state equation, it is not immediately obvious that the two are analogous. However, [3] showed that as $\bar{N} \to \infty$, the state $|\Psi\rangle_{si}$ approaches a normalized version of the EPR state. Thus the analogy is established between the two-mode signal/idler state and the two-particle EPR state.

### 3.2 Filter Penetration of a Single-Pole Filter

In order to theoretically demonstrate the existence of the ideal magic bullet effect, it is necessary to show that when the signal photon penetrates a filter, the idler photon penetrates a corresponding filter with probability one, no matter how small the probability that the signal photon penetrated the first filter. For a realistic system, it is sufficient to show that this is almost always true. A quantitative definition of almost depends on the requirements of the system but is generally taken to be when the photon-count-difference variance is below $0.05$.

The first step toward understanding the impact of filtering on the photon counting
statistics of the system is to calculate the modified normally ordered and phase-sensitive correlation functions, which account for the fact that the source outputs have been filtered. The correlation functions derived in Chapter 2 correspond to the case in which an all-pass filter is used, for which the transfer function has a value of one for all frequencies. These correlation functions are the inverse Fourier transforms of the associated spectra of the OPA output.

\[
\left\langle \hat{A}_{s}^{\dagger}(t+r)\hat{A}_{s}(t)\right\rangle = \left\langle \hat{A}_{j}^{\dagger}(t+r)\hat{A}_{j}(t)\right\rangle = K^{(n)}(\tau) = \int \frac{d\omega}{2\pi} S^{(n)}(\omega)e^{-i\omega \tau}
\]

\[
\left\langle \hat{A}_{s}(t+r)\hat{A}_{j}(t)\right\rangle = K^{(p)}(\tau) = \int \frac{d\omega}{2\pi} S^{(p)}(\omega)e^{-i\omega \tau}
\]

\(K^{(n)}(\tau)\) and \(K^{(p)}(\tau)\) were defined in Chapter 2 for the general dual-resonator source.

By doing some straightforward analysis in the Fourier domain, it can be seen that the modified correlation functions of the filtered OPA output field operators, \(\hat{A}_{s}^{F}(t)\) and \(\hat{A}_{j}^{F}(t)\), for general filters \(h_{s}(t)\) and \(h_{j}(t)\), whose Fourier transforms are \(H_{s}(\omega)\) and \(H_{j}(\omega)\), are:

\[
\left\langle \hat{A}_{s}^{F}(t+r)\hat{A}_{s}^{F}(t)\right\rangle = \int \frac{d\omega}{2\pi} S^{(n)}(\omega)|H_{s}(\omega)|^{2}e^{-i\omega \tau}
\]

\[
\left\langle \hat{A}_{j}^{F}(t+r)\hat{A}_{j}^{F}(t)\right\rangle = \int \frac{d\omega}{2\pi} S^{(n)}(\omega)|H_{j}(\omega)|^{2}e^{-i\omega \tau}
\]

\[
\left\langle \hat{A}_{s}^{F}(t+r)\hat{A}_{j}^{F}(t)\right\rangle = \int \frac{d\omega}{2\pi} S^{(p)}(\omega)H_{s}(-\omega)H_{j}(\omega)e^{-i\omega \tau}
\]

As a starting point for understanding how filtering affects the photon counting statistics of the system, standard single-pole filters were chosen for analysis. Because \(\hat{A}_{s}(t)\) and \(\hat{A}_{j}(t)\) are baseband field operators, and because the signal light at \(\omega_{s} - \Delta \omega\) is entangled
with the idler light at $\omega_i + \Delta \omega$, the filters used were

$$H_s(\omega) = \frac{\Gamma_c}{j(\omega - \Delta \omega) + \Gamma_c} \quad \text{and} \quad H_i(\omega) = \frac{\Gamma_c}{j(\omega + \Delta \omega) + \Gamma_c}$$

where $\Gamma_c$ is the linewidth of the filter and $\Delta \omega$ is the filter detuning from the signal and idler cavity resonances $\omega_s$ and $\omega_i$, respectively. A schematic of the filter setup to produce magic bullets for the detuned DRO case is shown in Figure 3.1(a).

\[\text{Figure 3.1: Magic Bullet Setup} - \text{A block diagram of the filter setup (a) that detects signal and idler photons in the wings of the associated spectra (b) to ensure that there is almost always only one signal-idler pair in a given counting interval.}\]

As indicated in the Figure, a filter is added to both the signal and idler OPA outputs. The OPA setup described in Chapter 2 remains the same. Figure 3.1(b) shows quantitatively how the magic bullet setup operates. The signal and idler filters act on the wings of the OPA output fluorescence spectrum, $S^{(n)}(\omega)$. This tests the magic bullet effect in that the probability that an emitted signal (idler) photon will successfully penetrate the filter $H_s(\omega)$ ($H_i(\omega)$) is low. Thus if the normalized photon-count difference variance is well below the shot-noise level then there is a magic bullet effect. In other words, we would like to show that every time a signal photon penetrates the signal filter, the idler photon from the same entangled photon pair penetrates the corresponding idler filter, despite the fact that the probability of the signal photon penetrating the filter is low. Because we are
not considering an ideal system, we will recognize that the magic bullet effect has taken place this occurs at least 95% of the time.

We now turn to the calculation of the normalized photon-count-difference variance. The same calculations that were done in Chapter 2 are done with the filtered output correlation functions. Again, the mean photon count difference is zero. The resulting equation for normalized variance as a function of the dimensionless parameters $\Gamma_C/\Gamma$, $G^2$, and $\Gamma T$ is quite involved. In addition, both the cavity time-bandwidth product and the filter time-bandwidth product are present in this equation, making it difficult to fully understand the impact of each linewidth on the normalized variance.

In order to take a more unified approach toward understanding the behavior of the system, the plot in Figure 3.2 includes a new variable, the parallel cavity bandwidth:

$$\Gamma_p = \frac{\Gamma_C \Gamma}{\Gamma_C + \Gamma}$$

In this plot, the product of the photon counting interval and the parallel cavity bandwidth, $\Gamma_p T = P$, is kept constant and the filter is not detuned. The plot in Figure 3.2 shows normalized variance as a function of $\Gamma_C/\Gamma$ when $\Gamma_p T = P$ equals 10, 20 and 100.

With this new time-bandwidth product the impact of the filter becomes clear. When $\Gamma \ll \Gamma_C$, the variance decreases to a value of $1/(2\Gamma T)$ as predicted by Shapiro and Wong [6]. For the parameters in this plot, the value of $1/(2\Gamma T)$ is about 0.05. Unfortunately, the condition $\Gamma_C \gg \Gamma$ means that the filter is an all-pass filter (i.e. the filter has no effect on the system.) We have already shown in Chapter 2 that when unfiltered photon pairs are counted over an interval that is long compared to the cavity bandwidth, the conditional probability of counting an idler photon given a signal photon is counted is
one. Thus the $\Gamma_C/\Gamma >> 1$ part of Figure 3.2 is not a demonstration of the magic bullet effect.

Normalized Variance vs. $\log(\Gamma_C/\Gamma)$

![Figure 3.2: Normalized Variance vs. $\log(\Gamma_C/\Gamma)$ For Different Values of $\Gamma p T$](image)

It is desirable to operate the system in the regime where $\Gamma >> \Gamma_C$ because a narrowband filter ensures that only a small fraction of the signal and idler photons are detected. In the limit of $\Gamma >> \Gamma_C$, the normalized variance asymptotically approaches a constant value of 0.5, which shows noise reduction, but not enough to demonstrate the magic bullet effect.

Despite the fact that a normalized variance of 0.5 is not low enough to be of great interest, it would be edifying to see how the introduction of detuning impacts the normalized variance in the region of interest, where $\Gamma_C/\Gamma << 1$. The plot of normalized variance as a function of $\Gamma_C/\Gamma$ is shown in Figure 3.3 for different values of normalized detuning ($\text{detun} = \Delta \omega / \Gamma$), where $\Gamma p T = 10$. Clearly, the detuning has an adverse impact on the counting statistics of the system.
Since a value of 0.5 for normalized variance is unacceptable, and both changing the ratio of $\Gamma_3$ to $\Gamma_1$ and introducing detuning were seen to negatively impact system performance, the obvious conclusion is that the magic bullet effect cannot be demonstrated with this simple single-pole filter. Presumably, the failure of the single-pole filter to exhibit the magic bullet effect is due to the slow fall-off of its frequency response. In particular, because of this slow fall-off, a signal photon that penetrates $H_S(\omega)$ might not have come from through the high-transmission part of the filter. As a result, the companion idler photon will have a substantially reduced conditional probability of penetrating $H_I(\omega)$. The next step is to turn to a more steep-skirted filter to see if that improves the results.

### 3.3 Penetration of a Kth-Order Butterworth Filter

The Butterworth filter holds more promise in the demonstration of the magic bullet effect because the order of the filter can be increased to achieve a more steep-skirted filter, as we think is required. I analyzed the general case of using a Kth-order Butterworth filter to try to achieve the magic bullet effect.
The analysis of the Kth-order Butterworth filter is similar to the one performed in Section 3.2, except that the baseband transmission functions are given by:

\[
|H_s(\omega)|^2 = \frac{1}{1 + \left(\frac{\omega - \Delta \omega}{\Gamma_c}\right)^{2k}} \quad |H_i(\omega)|^2 = \frac{1}{1 + \left(\frac{\omega + \Delta \omega}{\Gamma_c}\right)^{2k}}
\]

The mean signal and idler photon counts are straightforward to calculate. In the time domain, the variance of the photon-count-difference of the transmitted photons is [13]:

\[
\langle \Delta \hat{N}_s^2 \rangle = \langle \hat{N}_s \rangle + \langle \hat{N}_i \rangle \\
+ T \int_{-\tau}^{T} d\tau \left(1 - \frac{|\tau|}{T}\right) \left\{ \left| \hat{A}_s^{F'}(t + \tau) \hat{A}_s^{F}(t) \right|^2 + \left| \hat{A}_s^{F'}(t + \tau) \hat{A}_s^{F}(t) \right|^2 - 2 \left| \hat{A}_s^{F'}(t + \tau) \hat{A}_s^{F}(t) \right|^2 \right\}
\]

Let \( S \) denote Fourier transformation. In the frequency domain, this variance equation then becomes:

\[
\langle \Delta \hat{N}_s^2 \rangle = T \int \frac{d\omega}{2\pi} S^{(n)}(\omega)|H_s(\omega)|^2 + T \int \frac{d\omega}{2\pi} S^{(n)}(\omega)|H_i(\omega)|^2 \\
+ T \int \frac{d\omega}{2\pi} T \left( \frac{\sin(\omega T / 2)}{\omega T / 2} \right)^2 \left\{ \left| \hat{A}_s^{F'}(t + \tau) \hat{A}_s^{F}(t) \right|^2 + \left| \hat{A}_s^{F'}(t + \tau) \hat{A}_s^{F}(t) \right|^2 - 2 \left| \hat{A}_s^{F'}(t + \tau) \hat{A}_s^{F}(t) \right|^2 \right\}
\]

Since we have already determined that desirable photon counting statistics only occur when photons are counted over long intervals, let us further simplify the calculation by looking at the variance in the limit as \( T \to \infty \). In this limit:

\[
T \left( \frac{\sin(\omega T / 2)}{\omega T / 2} \right)^2 \to 2\pi \delta(\omega)
\]

and we get
\[
\langle \Delta \hat{N}^2 \rangle = T \int \frac{d\omega}{2\pi} S^{(n)}(\omega) |H_s(\omega)|^2 + T \int \frac{d\omega}{2\pi} S^{(m)}(\omega) |H_1(\omega)|^2 \\
+ T \int d\tau \left\{ \left| \hat{A}_s^p(t+\tau) \hat{A}_s^p(t) \right|^2 + \left| \hat{A}_1^p(t+\tau) \hat{A}_1^p(t) \right|^2 - 2 \left| \hat{A}_s^p(t+\tau) \hat{A}_1^p(t) \right|^2 \right\} \\
= T \int \frac{d\omega}{2\pi} S^{(n)}(\omega) |H_s(\omega)|^2 + T \int \frac{d\omega}{2\pi} S^{(m)}(\omega) |H_1(\omega)|^2 \\
+ T \int \frac{d\omega}{2\pi} \left\{ S^{(n)}(\omega) |H_s(\omega)|^4 + |S^{(m)}(\omega) |H_1(\omega)|^4 - 2 |S^{(p)}(\omega) |^2 |H_s(-\omega)|^2 |H_1(\omega)|^2 \right\} 
\]

This makes the variance very straightforward to calculate using a standard software package to perform the integration.

![Normalized Variance vs. K](image)

**Figure 3.4:** Normalized Variance vs. K for the DRO, at several values of \( \Gamma_c \)

Using a Kth-order Butterworth filter instead of a single-pole filter has a significant impact on the resulting photon counting statistics. Figure 3.4 demonstrates the dependence of the normalized variance on K for a DRO, with all other variables held constant, for three values of \( c = \Gamma_c \). The parameter \( \Gamma_c \) impacts the behavior of the system, but the normalized variance exhibits the same type of 1/(2K) dependence for all cases. Clearly a steep-skirted filter is needed to see the magic bullet effect; one possible way to accomplish this is to use a high-order Butterworth filter. The plot also shows the
expected result that as $\Gamma_C$ gets large, the filter essentially is an all-pass filter, and the variance is exceedingly low. This plot is for the doubly-resonant OPA where $\Gamma_S/\Gamma_1 = 1$. 

![Normalized Variance vs. K for C = 1](image)

Figure 3.5: Normalized Variance vs. K for doubly- and singly-resonant OPAs

Figure 3.5 shows a comparison of the normalized variance when the doubly-resonant and singly-resonant oscillators are employed for a single value of $\Gamma_C = 1$. The ratio $\Gamma_S/\Gamma_1$ is taken to be 0.001 for the singly-resonant oscillator. As we might have expected based on the results of Chapter 2, the normalized variance of the singly-resonant oscillator is not quite as low as that of the doubly-resonant oscillator, but as K gets larger than about 8 this figure reaches acceptably low levels below about 0.05. While slight modifications may need to be made to a system in order for the SRO to exhibit the quantum magic bullet effect, this effect is clearly possible in systems that rely on either the DRO or the SRO to generate entangled photon pairs.

### 3.4 Realization of a Steep-Skirted Filter

Based on the analyses presented in this chapter, it is clear that it is necessary to use a steep-skirted filter in conjunction with the entangled photon source in order to achieve the magic bullet effect. This section will investigate one way to create the
desired steep-skirted filter to be used with the OPA source. The setup, shown in Figure 3.6 for the signal field, includes a prism, lens and detector of finite size.

![Diagram of Steep-Skirted Filter Realization](image)

Figure 3.6: Setup of the Steep-Skirted Filter Realization

The prism spatially separates incoming light based on frequency. It is then possible to capture a narrow band of frequencies by using a lens to focus them onto a detector. The detector size and the prism properties can be adjusted to obtain a filter that is sufficiently narrow.

![Diagram of Steep-Skirted Filter Setup, Including Relevant Dimensions](image)

Figure 3.7: Block Diagram of the Steep-Skirted Filter Setup, Including Relevant Dimensions

We take a systematic approach to analyzing the setup. A block diagram of the system for isolating and detecting the signal frequency is shown in Figure 3.7, which includes relevant distances and sizes. For simplicity, we only consider a two-dimensional spatial dependence, i.e., a single transverse coordinate $x$ and a single
longitudinal coordinate \( z \). The spatial frequency separation effects of the prism are modeled by the parameter \( \beta \), the amount of tilt that is applied to frequencies that differ from the frequency of interest, \( \omega_s \). That is to say, suppose that the frequency \( \omega \) OPA output \( \hat{E^s}_s(x, z = 0, \omega) \) enters the prism of width \( D \) at \( z = 0 \), as shown in Figure 3.6. We use a Fourier optics technique described in Appendix B to analyze the behavior of the field as it travels through each of the elements to the point just before the detector, at \( z = l+f \). At this point, the baseband output field can be rewritten as the convolution of the baseband input field and an effective filter, \( h_s(t) \).

\[
\hat{A^F}_s(x', t) = \int du \hat{A}^{OPA}_s(u)h_s(x', t - u) = \int \frac{d\omega}{2\pi} \hat{A}^{OPA}_s(\omega)H_s(x', \omega)e^{-j\omega t}
\]

where

\[
H_s(x', \omega) = \frac{\exp\left[ j \left( \frac{(\omega + \omega_s)}{c} \right) f + j(\omega + \omega_s) \frac{x'^2}{2cf} \right]}{\sqrt{j \frac{2\pi c}{(\omega + \omega_s)} f}} \sqrt{D} \left( \sin \left( \frac{(\omega + \omega_s)D}{2cf} (x' - \beta \omega f) \right) \right)
\]

The \( x' \) in this expression is the position in the \( z = l+f \) plane and will be integrated over the location of the detector. Using this result, we can modify the equations derived earlier in this Chapter to determine the output correlation functions in the presence of this type of filter. When spatial variations are present, the output correlation functions become

\[
\langle \hat{A}^{F^+}_s(x', t)\hat{A}^F_s(x, u) \rangle = \int \frac{d\omega}{2\pi} S^{(s)}(\omega)H_s(x, \omega)H^*_s(x', \omega)e^{j\omega(t-u)}
\]

\[
\langle \hat{A}^{F^+}_i(x', t)\hat{A}^F_i(x, u) \rangle = \int \frac{d\omega}{2\pi} S^{(i)}(\omega)H_i(x, \omega)H^*_i(x', \omega)e^{j\omega(t-u)}
\]

\[
\langle \hat{A}^F_s(x', t)\hat{A}^F_s(x, u) \rangle = \int \frac{d\omega}{2\pi} S^{(p)}(\omega)H_s(x', -\omega)H_s(x, \omega)e^{j\omega(t-u)}
\]
In order to facilitate the use of these expressions in calculating the photon-count-difference variance, let us make a few simplifying assumptions. First, assume that the bandwidths of the filters are much narrower than the signal and idler fluorescence bandwidths, so the fluorescence spectra at the filter’s center frequency can be taken outside the integrals. The fluorescence bandwidths are much narrower than $\omega_s$ and $\omega_i$, the fluorescence center frequencies. We assume that OPA operation is reasonably close to frequency degeneracy, so $\lambda_s = \lambda_i = 2\lambda_p$. Finally, we choose the center locations of the pinholes symmetrically to achieve center frequency detuning $+\Delta\omega = x/\beta f$ on the signal filter and $-\Delta\omega = -x/\beta f$ on the idler filter. This allows us to create a single effective filter and to rewrite the correlation functions in a greatly simplified form that will assist us in calculating the photon-count-difference variance

$$\langle \Delta \hat{N}^2 \rangle = \langle \hat{N}_s^2 \rangle + \langle \hat{N}_i^2 \rangle - 2 \langle \hat{N}_s \hat{N}_i \rangle$$

$\langle \hat{N}_s^2 \rangle$ can be written as

$$\langle \hat{N}_s^2 \rangle = \int_0^T dt \int_0^T du \int_S dx \int_S dx' \langle \hat{A}_s^F (x,t) \hat{A}_s^F (x,t) \hat{A}_s^F (x',u) \hat{A}_s^F (x',u) \rangle$$

where $S$ indicates the spatial region where the pinhole is for the signal-beam filter. When Gaussian moment factoring is applied, this expression becomes

$$\langle \hat{N}_s^2 \rangle = \langle \hat{N}_s \rangle$$

$$+ \int_0^T dt \int_0^T du \int_S dx \int_S dx' \left[ \langle \hat{A}_s^F (x,t) \hat{A}_s^F (x,t) \hat{A}_s^F (x',u) \hat{A}_s^F (x',u) \rangle + \langle \hat{A}_s^F (x,t) \hat{A}_s^F (x',u) \rangle^2 \right]$$

$$= \langle \hat{N}_s \rangle + \left[ \int_0^T dx \langle \hat{A}_s^F (x,t) \hat{A}_s^F (x,t) \rangle \right]^2 + \int_0^T dt \int_0^T du \int_S dx \int_S dx' \langle \hat{A}_s^F (x,t) \hat{A}_s^F (x',u) \rangle^2$$

When the counting interval is assumed to be long (as we have shown is desirable for
obtaining the magic bullet effect) the assumptions we have made allow this expression to be reduced to

\[
\left\langle \hat{N}_S^2 \right\rangle = \left\langle \hat{N}_S \right\rangle + \left[ \int_0^T dt \int_S dx \langle \hat{F}_S^r (x, t) \hat{F}_S^r (x, t) \rangle \right]^2
\]

\[
+ \left[ S^{(n)} (\Delta \omega) \right]^2 \int_{-d/2}^{d/2} dx \int_{-d/2}^{d/2} dx' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |H_{\text{eff}} (x, \omega)|^2 |H_{\text{eff}} (x', \omega)|^2
\]

where \( H_{\text{eff}} (x, \omega) = \exp \left[ \frac{j \omega_p f + j \omega_p x'^2}{\sqrt{2c}} \right] \sqrt{\frac{\pi D}{2\lambda_p f}} \frac{(x' - \beta \omega f)}{\sqrt{2\lambda_p f}} \).

Similar expressions can be derived for \( \left\langle \hat{N}_i^2 \right\rangle \) and \( \left\langle \hat{N}_S \hat{N}_i \right\rangle \). Adding these expressions together and canceling out like terms yields the dramatically streamlined expression

\[
\left\langle \Delta \hat{N}^2 \right\rangle = \left\langle \hat{N}_S \right\rangle + \left\langle \hat{N}_i \right\rangle
\]

\[
+ 2T \left[ S^{(n)} (\Delta \omega) \right]^2 \int_{-d/2}^{d/2} dx \int_{-d/2}^{d/2} dx' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |H_{\text{eff}} (x, \omega)|^2 |H_{\text{eff}} (x', \omega)|^2
\]

\[
- 2T \left[ S^{(p)} (\Delta \omega) \right]^2 \int_{-d/2}^{d/2} dx \int_{-d/2}^{d/2} dx' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |H_{\text{eff}} (x, \omega)|^2 |H_{\text{eff}} (x', \omega)|^2
\]

\[
= \left\langle \hat{N}_S \right\rangle + \left\langle \hat{N}_i \right\rangle + 2T \left[ S^{(n)} (\Delta \omega) \right]^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |H_{\text{eff}} (\omega)|^4 - 2T \left[ S^{(p)} (\Delta \omega) \right]^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |H_{\text{eff}} (\omega)|^4
\]

where \( |H_{\text{eff}} (\omega)|^2 = \int_{-d/2}^{d/2} dx |H_{\text{eff}} (x, \omega)|^2 \).

We know that \( \left\langle \hat{N}_S \right\rangle = \left\langle \hat{N}_i \right\rangle = S^{(n)} (\Delta \omega) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |H_{\text{eff}} (\omega)|^2 \), so we now have all of the necessary expressions to determine the normalized variance. All of these expressions are functions of the effective filter. Thus, if we can show that the effective filter is steep-
skirted enough to reduce the normalized difference count variance to below 0.05, then this setup will generate quantum magic bullets.

Before looking at the normalized variance, let us look more closely at the properties of the effective filter. In order to understand the behavior of the effective filter, let us introduce some normalized variables. First, \( \tilde{x} = x'D/2\lambda_p f \) is the transverse coordinate in the pinhole measured in units of diffraction lengths. Similarly, \( d_o = dD/2\lambda_p f \) is the width of the pinhole filter in units of diffraction lengths. Finally, \( \beta_0 = \beta f / (2\lambda_p f / D) \) is the tilt effect of the prism normalized to units of diffraction length. Figure 3.8 shows a plot of \( |H_{EFF}(\omega)|^2 \) (where \( \omega = \omega \))

\[
|H_{EFF}(w)|^2 = \int_{-d_o/2}^{d_o/2} d\tilde{x} \left[ \sin[\pi(\tilde{x} - \beta_0 w)] / \pi(\tilde{x} - \beta_0 w) \right]^2
\]

for different values of \( \beta_0 \) and \( d_o \). As the Figure shows, increasing \( \beta_0 \) decreases the bandwidth of the filter. This makes sense because increasing \( \beta_0 \) increases the tilt effect of the prism that spatially separates the different frequencies emitted by the OPA. Therefore, a narrower bandwidth is captured by the narrow pinhole detector. Increasing \( d_o \) makes the filter more steep-skirted because the intensity pattern of the light consisting of the frequency of interest is a sinc function centered at the pinhole detector in the \( z = l + f \) plane. Therefore, increasing \( d_o \) makes it more likely that the filter will capture the light that is in the sidelobes of the intensity pattern. Based on this plot, it seems that we can make this type of filter as steep-skirted as we need in order to generate the requisite photon counting statistics.

Let us take this analysis one step further and look at the normalized variance of the photon difference count.
\[
\frac{\langle \Delta \hat{N}^2 \rangle}{\langle \hat{N}_S \rangle + \langle \hat{N}_I \rangle} = 1 + \frac{2T \left( \left| S^{(r)}(\Delta \omega) \right|^2 - \left| S^{(p)}(\Delta \omega) \right|^2 \right) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |H_{\text{EFF}}(\omega)|^4}{2TS^{(r)}(\Delta \omega) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |H_{\text{EFF}}(\omega)|^2}
\]

In the absence of excess losses in the OPA, the results of Chapter 2 can be used to show that \( \left( \left| S^{(r)}(\Delta \omega) \right|^2 - \left| S^{(p)}(\Delta \omega) \right|^2 \right) = -S^{(s)}(\Delta \omega) \), hence

\[
\frac{\langle \Delta \hat{N}^2 \rangle}{\langle \hat{N}_S \rangle + \langle \hat{N}_I \rangle} = 1 - \frac{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |H_{\text{EFF}}(\omega)|^4}{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |H_{\text{EFF}}(\omega)|^2}
\]

We would like to evaluate this expression to determine if the normalized variance is low enough to demonstrate the magic bullet effect. We look first to the denominator of the fraction.

\[
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |H_{\text{EFF}}(\omega)|^2 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\frac{\omega}{\beta_0}}^{\frac{\omega}{\beta_0}} \left[ \frac{\sin[\pi(x - \beta_0 \omega)]}{\pi(x - \beta_0 \omega)} \right]^2 dx\]

As it stands, this expression requires a double integration. However, if we perform the integration over the frequency domain first, we get an expression that is not a function of \( x \). Integrating over \( x \) then amounts to multiplying the result of the frequency integration by \( d_0 \). Thus, we can write an equivalent, simpler expression, in which we have introduced a new variable, \( z = \beta_0 \omega \).

\[
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |H_{\text{EFF}}(\omega)|^2 = d_0 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[ \frac{\sin[\pi\beta_0 \omega]}{\pi\beta_0 \omega} \right]^2 = d_0 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[ \frac{\sin[\pi\xi]}{\pi\xi} \right]^2
\]

The Fourier transform of a sinc function is a square, which makes the integral straightforward to evaluate. We find that
Next, we turn to the numerator of the fraction. Changing the order of integration and introducing \( z = \beta_0 \omega \) gives the form

\[
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |H_{\text{EFF}}(\omega)|^4 = \frac{d_0}{2\pi \beta_0}
\]

We can again use Fourier transformation to advantage to obtain an integral that is easier to evaluate.

\[
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |H_{\text{EFF}}(\omega)|^4 = \frac{1}{2\pi \beta_0} \int_{-d_0/2}^{d_0/2} \int_{-d_0/2}^{d_0/2} d\bar{x} \int_{-d_0/2}^{d_0/2} dz \left[ \frac{\sin[\pi(\bar{x} - z)]}{\pi(\bar{x} - z)} \right]^2 \left[ \frac{\sin[\pi(\bar{x}' - z)]}{\pi(\bar{x}' - z)} \right]^2
\]

Let \( W = 2\pi v \). Then the integral becomes

\[
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |H_{\text{EFF}}(\omega)|^4 = \frac{1}{2\pi \beta_0} \int_{-\infty}^{\infty} dv \left( 1 - |v|^2 \right)^2 \left[ \frac{\sin(\pi v d_0)}{\pi v} \right]^2 d\bar{x}
\]

Similar to the asymptotic behavior found earlier in this Chapter, in the limit as \( d_0 \to \infty \)

\[
d_0 \left[ \frac{\sin(Wd_0 / 2)}{Wd_0 / 2} \right]^2 \to 2\pi \delta(W)
\]

Thus, we obtain the asymptotic result

\[
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |H_{\text{EFF}}(\omega)|^4 = \frac{d_0}{2\pi \beta_0}
\]

as \( d_0 \to \infty \). Hence in this limit we find that

\[
\frac{\langle \Delta \hat{N}^2 \rangle}{\langle \hat{N}_s \rangle + \langle \hat{N}_l \rangle} \to 0
\]

and we have shown that the magic bullet effect can be demonstrated with the prism-lens-
pinhole filter. In particular, not only it is possible to generate quantum magic bullets with a theoretical filter, but also with a filter that can be built in the laboratory.
Figure 3.8: Magnitude Squared of the Effective Filter as a Function of Difference from the Center Frequency. \( f(w) \) corresponds to \( \beta_0 = 10 \) and \( d_0 = 10 \). \( g(w) \) corresponds to \( \beta_0 = 2 \) and \( d_0 = 20 \). \( h(w) \) corresponds to \( \beta_0 = 10 \) and \( d_0 = 20 \). \( k(w) \) corresponds to \( \beta_0 = 2 \) and \( d_0 = 10 \).
4. QUANTUM OPTICAL COMMUNICATION

One of the most important fields that the concepts of quantum entanglement and quantum magic bullets will impact is the field of quantum information processing. In particular, the use of entangled photons will facilitate the transmission and receipt of quantum information across noisy and lossy quantum communication channels – quantum teleportation – as well as the storage of this information in a quantum memory.

The next section describes a quantum communication system in more detail; the focus of the remainder of the chapter is on the cavity loading analysis of the singly-resonant oscillator and a comparison of the performance of the singly- and doubly-resonant oscillators. This is the final step of the comparative analysis of singly- and doubly-resonant oscillators in this thesis.

4.1 Teleportation and Quantum Memory

Bennett et al. [10] are responsible for the idea that teleportation, which in this context means transporting a quantum state of light from one place to another without sending the state itself, can be accomplished using singlet states. This was done in theory in their paper using a classical channel to share two bits of classical information and a quantum channel to share a quantum state. First, the sender and receiver, Alice and Bob, share an EPR singlet state over the quantum channel: 

\[ |\psi\rangle_{AB} = \left( |0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B \right)/\sqrt{2}. \]

Alice then accepts an input particle whose state can be expressed most easily as 

\[ |\psi\rangle_{IN} = \alpha |0\rangle_{IN} + \beta |1\rangle_{IN}, \] where \( |\alpha|^2 + |\beta|^2 = 1. \) At this point, before Alice’s measurement, the input particle, and Alice and Bob’s EPR state are in the joint state:
\[ |\psi\rangle_{AB} = \frac{\alpha}{\sqrt{2}} (|0\rangle_A |0\rangle_B - |0\rangle_A |1\rangle_B) + \frac{\beta}{\sqrt{2}} (|1\rangle_A |0\rangle_B - |1\rangle_A |1\rangle_B) \]

\[ = \frac{1}{\sqrt{8}} (\alpha |0\rangle_B + \beta |1\rangle_B) (|1\rangle_A |0\rangle_B - |0\rangle_A |1\rangle_B) - \frac{1}{\sqrt{8}} (\alpha |0\rangle_B - \beta |1\rangle_B) (|1\rangle_A |0\rangle_B + |0\rangle_A |1\rangle_B) \]

Alice then performs the Bell-state measurements on the joint state. The Bell-state measurement result gives Alice the information that she needs to give to Bob in order to enable him to reconstruct the original state. If the result of the measurement is zero, then Bob does not need to do anything to his EPR state to get the original state. If the Bell-state measurement result is one, then Bob must flip the phase of his \( |1\rangle_b \) state. If the result of the measurement is two, then Bob must exchange his \( |0\rangle_b \) and \( |1\rangle_b \) states to obtain the original state. If the result of Alice’s measurement is three, then Bob must both flip the phase of his \( |1\rangle_b \) state and exchange his \( |0\rangle_b \) and \( |1\rangle_b \) states to reconstruct the original state. Yet neither the measurements nor the phase flip and exchange operations require any information about the values of \( \alpha \) and \( \beta \). All that is needed to complete the teleportation is the result of Alice’s Bell-state measurement.
Although teleportation has been demonstrated experimentally [15,16], the demonstrations have been very limited in the context of long-distance quantum-state communication. Work is currently underway to develop a quantum communication system to perform singlet-based long-distance teleportation that supports long-duration quantum storage [8]. The system relies on an ultrabright narrowband source of entangled photon pairs ($P$), like the one discussed in Chapter 3, to create singlet-state entangled photons, as shown in the block diagram in Figure 4.1. The signal and idler photons are each transmitted through a different $L$-km long standard telecommunication fibers to a quantum memory, $M$, where separated EPR particles can be stored a relatively long time.
In the quantum memory, photons are stored in long-lived hyperfine levels of an ultra-cold trapped rubidium atom. Figure 4.2 shows a simplified schematic of the quantum memory. The memory starts out free of photons, with the atom in the ground state A, and is clocked. Light from the OPA is loaded into the memory cavity for a short interval, on the order of a few cavity lifetimes. B-to-D pumping takes place for about 100 nsec to transfer the coherence from the upper levels B to the storage levels D. Then the A-to-C cycling transition is run repeatedly. Photon absorption by the atom in the cavity can be determined by whether or not fluorescence occurs from this cycling transition. In the protocol, all coherence of the input photon is transferred to the memory atom.

The proposed quantum teleportation and quantum memory can be modified to be compatible with fiber-optic transmission, and will therefore yield a complete communication system. Bell state measurements can be made in the atoms and qubit logic can be applied to complete the teleportation process, so the system will capable of both long-distance teleportation and long-duration quantum storage.

4.2 Cavity Loading Analysis of the Quantum Memory

Before examining the cavity loading in more detail, it is important to establish a basis by which the quality of the system will be measured. The figures of merit of the quantum communication system are throughput and fidelity. Throughput is the rate at which entangled photon pairs are loaded into the memory cavity, while fidelity quantifies how true the stored photon is to the original photon. These figures of merit are presented more thoroughly in the next subsection. The second subsection consists of the cavity
loading analysis of the SRO. Finally, the figures of merit of quantum communication systems based on the SRO and DRO are compared.

4.2.1 Definition of the Figures of Merit

The quantum communication protocol involves the use of two OPAs. One OPA produces an x-polarized signal photon and a y-polarized idler photon with the density operator $\hat{\rho}_{S_{x}I_{y}}$. The other OPA produces a y-polarized signal photon and an x-polarized idler photon with the density operator $\hat{\rho}_{S_{y}I_{x}}$. The joint state of the output of the double OPA is $\hat{\rho} = \hat{\rho}_{S_{x}I_{y}} \hat{\rho}_{S_{y}I_{x}}$.

On any trial of the protocol, one of three things will happen. Success occurs if both memory atoms absorb a photon and the photons come from an entangled pair. This occurs with probability

$$P_{\text{SUCCESS}} = \langle \Psi | \hat{\rho} | \Psi \rangle_{SI}$$

where $| \Psi \rangle_{SI}$ is the singlet signal-idler state,

$$| \Psi \rangle_{SI} = \left( \frac{|1 \rangle_{S_{x}} |0 \rangle_{I_{y}} |0 \rangle_{S_{y}} |1 \rangle_{I_{x}} - |0 \rangle_{S_{y}} |1 \rangle_{S_{x}} |1 \rangle_{I_{x}} |0 \rangle_{I_{y}}}{\sqrt{2}} \right).$$

Erasure is defined as when one or both atoms from the memories fail to absorb photons, i.e. either the signal or the idler photon (or both) is lost en route to the memory. Thus we have that

$$P_{\text{ERASURE}} = P_{0_{S_{x}}} P_{0_{S_{y}}} + P_{0_{I_{x}}} P_{0_{I_{y}}} - P_{0_{S_{x}}} P_{0_{S_{y}}} P_{0_{I_{x}}} P_{0_{I_{y}}}$$

where $P_{0_{S_{x}}} = s_{x} \langle 0 | \hat{\rho}_{S_{y}} | 0 \rangle_{S_{x}}$ and $\hat{\rho}_{S_{x}} = tr_{y} (\hat{\rho}_{S_{x}I_{y}})$;

$P_{0_{I_{x}}} = s_{x} \langle 0 | \hat{\rho}_{I_{y}} | 0 \rangle_{I_{x}}$ and $\hat{\rho}_{I_{x}} = tr_{y} (\hat{\rho}_{S_{x}I_{y}})$;

$P_{0_{S_{y}}} = s_{y} \langle 0 | \hat{\rho}_{S_{y}} | 0 \rangle_{S_{y}}$ and $\hat{\rho}_{S_{y}} = tr_{x} (\hat{\rho}_{S_{x}I_{y}})$;

$P_{0_{I_{y}}} = s_{y} \langle 0 | \hat{\rho}_{I_{y}} | 0 \rangle_{I_{y}}$ and $\hat{\rho}_{I_{y}} = tr_{x} (\hat{\rho}_{S_{x}I_{y}})$;

$P_{0_{S_{x}}} P_{0_{S_{y}}} = s_{x} s_{y} \langle 0 | \hat{\rho}_{S_{x}I_{y}} | 0 \rangle_{S_{x}} | 0 \rangle_{S_{y}}$; and $P_{0_{S_{x}}} P_{0_{S_{y}}} = s_{y} s_{x} \langle 0 | \hat{\rho}_{S_{y}I_{x}} | 0 \rangle_{S_{y}} | 0 \rangle_{S_{y}}$.
gives the erasure probability. Appendix C shows this calculation in more detail. If both atoms absorb photons, but the photons come from different entangled pairs, the result is an error. Because these are the only possible outcomes, the sum of the probabilities of these events is one, and the probability of error is:

\[ P_{\text{ERROR}} = 1 - (P_{\text{SUCCESS}} + P_{\text{ERASURE}}) \]

The figures of merit of the system are defined in terms of these outcomes. Throughput is the product of the loading protocol clock rate, \( R \), and the probability of success, \( P_{\text{SUCCESS}} \): \( N_{\text{SUCCESS}} = R \times P_{\text{SUCCESS}} \). If the input state to the teleportation process is \( |\psi\rangle_{\text{IN}} \), and the teleportation process produces a pure state \( |\Psi\rangle_B \), both of unit length, fidelity is the magnitude squared of the projection of the received state onto the input state:

\[ \text{Fidelity} = \left| \langle \psi | \Psi \rangle \right|^2 \]

If the teleportation produces a mixed state, i.e. \( |\Psi\rangle_k \) is produced with probability \( p_k \) for \( 1 \leq k \leq K \) then

\[ \text{Fidelity} = \sum_k p_k \left| \langle \psi | \Psi \rangle \right|^2 = \sum_k p_k |\langle \psi | \hat{\rho}_k |\psi\rangle \rangle_{\text{IN}} | \text{where } \hat{\rho}_k \equiv \sum_k p_k |\Psi\rangle_k \langle \Psi \rangle \]

The conditional fidelity given that the cavity was loaded successfully is one, while the conditional fidelity given that an error occurred when the cavity was loaded is one half. To see why the latter is true, suppose \( |\psi\rangle_{\text{IN}} = |0\rangle_{\text{IN}} \). Then, the conditional fidelity is:

\[ \text{Fidelity} = \left| \langle \psi | \Psi \rangle \right|^2 = |\langle \psi | \hat{x} \rangle |^2 = \frac{1}{2} \]

where the last equality prevails for a randomly polarized state. In terms of the probabilities of success, erasure and error, fidelity is calculated to be:
The following discussion quantifies these definitions and applies them to analyze the performance of the SRO.

4.2.2 Cavity Loading Analysis

In order to analyze cavity loading in terms of the figures of merit outlined above, it is necessary to derive the photon counting statistics of the memory and apply them to calculate the probabilities of erasure, success and error. These, in turn, will lead to the throughput and fidelity of the system.

The entangled photon pair source assumed in the analysis is a general double OPA that produces entanglement between the x-polarized signal beam and the y-polarized idler beam, and vice-versa. When excess loss is included in the memory cavity, the equation that governs the cavity loading is:

\[
\frac{d\hat{a}_{k,\text{pol}}(t)}{dt} = -\Gamma_C \hat{a}_{k,\text{pol}}(t) + \sqrt{2\gamma_C} \hat{A}_{\text{pol}}(t) + \sqrt{2(\Gamma_C - \gamma_C)} \hat{A}_{\text{vac}}(t)
\]

In this equation, \( k \) indicates either the signal or the idler and \( \text{pol} \) indicates either x or y polarization. \( \Gamma_C \) is the total loss rate (which is the cavity coupling rate plus the excess loss rate associated with the memory cavity.) \( \gamma_C \) is the cavity coupling rate, \( \hat{a}_k(t) \) is the internal-mode annihilation operator, \( \hat{A}_k(t) \) is the input signal-mode field operator and \( \hat{A}_{\text{vac}}(t) \) is a vacuum-state field operator that is required quantum mechanically to preserve the commutator brackets. Without the introduction of this vacuum-state noise, the uncertainty principle would be violated. Solving this differential equation leads to the result:
\[ \hat{a}_{\text{vac0}}(t) = \hat{a}_{\text{vac0}}(0) \exp(-\Gamma_c t) + \int_0^t d\tau \left( \sqrt{2\gamma_c} \hat{A}_{\text{vac0}}(\tau) + \sqrt{2(\Gamma_c - \gamma_c)} \hat{A}_{\text{vac}}(\tau) \right) \exp[-\Gamma_c (t-\tau)] \]

The annihilation operators are in vacuum state at \( t=0 \), and the only non-zero correlation functions are those found in Chapter 2. Thus, because the state is zero-mean and Gaussian, the correlation functions are all that we need to find the cavity loading statistics. When the cavity loading time \( T_c \) is long compared to the cavity lifetime \( (T_c >> 1/\Gamma_c) \) the loading statistics simplify significantly. When a loss factor, \( \eta \), is included to account for fixed and propagation losses in the fiber (source cavity loss is accounted for by \( K^{(n)}(\tau) \) and \( K^{(p)}(\tau) \), these equations become:

\[ \langle \hat{a}_{s_x}^+ (T_c) \hat{a}_{s_x} (T_c) \rangle = \langle \hat{a}_{s_y}^+ (T_c) \hat{a}_{s_y} (T_c) \rangle = \langle \hat{a}_{t_x}^+ (T_c) \hat{a}_{t_x} (T_c) \rangle = \langle \hat{a}_{t_y}^+ (T_c) \hat{a}_{t_y} (T_c) \rangle = \int_{-\infty}^{\infty} d\tau \eta K^{(n)}(\tau) \exp[-\Gamma_c |\tau|] = \bar{n} \]

\[ \langle \hat{a}_{s_x} (T_c) \hat{a}_{t_x} (T_c) \rangle = -\langle \hat{a}_{t_x} (T_c) \hat{a}_{s_x} (T_c) \rangle = \int_{-\infty}^{\infty} d\tau \eta K^{(p)}(\tau) \exp[-\Gamma_c |\tau|] = \bar{n} \]

The sign flip in the last equation is due to the fact that these correlation functions are out of phase; this is what enables a singlet state to be generated.

The next step is to use these correlation functions to calculate the probabilities of the three outcomes defined above as a function of the parameters of the system. This is done in Appendix C. Once the figures of merit of the system have been found as a function of various input and loss parameters, optimal system operation can be determined based on the requirements of the system.

In addition to cavity losses mentioned above, a realistic analysis must include both fixed and propagation losses in the fiber that transmits the photon from the sender to the receiver. The losses were assumed to be 5 dB of excess loss in each path and 0.2 dB/km loss in the fiber. So, \( \eta = 10^{(-0.5-0.02L)} \). This information allows us to calculate \( \bar{n} \)
and \( \tilde{n} \) as a function of the dimensionless system parameters \( A = \frac{\Gamma_s}{\Gamma_t}, C = \frac{\Gamma_c}{\Gamma_s}, G^2, \) and \( \eta \) (which in turn depends on the total path length.) For notational simplicity, let \( Z = \sqrt{(A-1)^2 + 4G^2 A/A} \). Then:

\[
\tilde{n} = \frac{4\eta G^2}{(1 - G^2)(A + 1)Z} \left[ \frac{1 + \frac{1}{A} - \frac{1}{A+1}}{\frac{2A + 1}{Z} + 1} + \frac{1 + \frac{1}{A} - \frac{1}{A+1}}{\frac{2A + 1}{Z} - 1} \right]
\]

\[
\tilde{n} = \frac{4\eta G}{(1 - G^2)(A + 1)Z} \left[ \frac{1 - \frac{G^2(1 + A^{-1})}{Z}}{\frac{2A + 1}{Z} + 1} + \frac{1 + \frac{G^2(1 + A^{-1})}{Z}}{\frac{2A + 1}{Z} - 1} \right]
\]

Before trying to optimize the operation of the system, it is helpful to develop a general understanding about how different input parameters affect the figures of merit at hand. With this in mind, Figure 4.3 shows how system path length affects fidelity and throughput, with all other parameters fixed. In general, systems with longer total path lengths have lower throughput (i.e. the total number of photons that are loaded into the cavity are lower) because longer paths have higher loss, which means that fewer photons reach the memory cavity. However, fidelity increases slightly with increasing path length because, while there are fewer total photons, there is also a smaller probability that there will be more than one signal-idler photon pair in a given counting interval.

Figures 4.4 and 4.5 are mesh plots that show the impact of both \( G^2 \) and \( \Gamma_c/\Gamma \) on throughput and fidelity, respectively. By increasing gain, it is possible to increase throughput by a small amount because increasing the gain increases the total number of photon pairs that are created. However, there is a tradeoff involved (as there is in the case of total path length) in that increasing gain decreases fidelity because increasing the
number of photon pairs created increases the probability that there will be more than one signal-idler pair in a given counting interval. Fidelity can be improved by increasing the ratio of the memory cavity linewidth to the cavity linewidth, but at the cost of losing throughput.

The trends identified by these three plots also hold for the general dual-resonator, and are important to keep in mind as we try to determine the best point of operation of the communication system. That is, for a fixed signal-to-idler cavity linewidth ratio, we want to find out what values of $\Gamma_C/\Gamma_S$ and $G^2$ will give the highest throughput for a given fidelity requirement. Because we are interested in comparing the performance of the singly-resonant OPA and the doubly-resonant OPA as an entangled photon source for a quantum optical communication system, we will look at two signal-to-idler cavity linewidth ratios which represent these two cases. In order to compare the SRO and the DRO, we will look at optimizing system operation while maintaining a minimum fidelity of 0.95 in each case.

Figure 4.6 shows the throughput that can be attained for each total path length using the SRO, while maintaining the desired fidelity. The plots in Figure 4.7 show how (a) $G^2$ and (b) $\Gamma_C/\Gamma_S$ must change (as a function of path length) to maintain optimal system operation. The gain required to achieve optimal system operation increases as total path length increases. $\Gamma_C/\Gamma_S$ is approximately constant at a value of 0.0001 as a function of total path length, although if examined closely enough extremely small variations can be seen, as shown in Figure 4.7(b).

Figure 4.8 shows the highest throughput that can be obtained while maintaining fidelity of 0.95 for the DRO. Figure 4.9 shows the input parameters used to achieve these
throughputs. One interesting difference between the SRO and the DRO is that in the latter case, the gain increases as a function of total path length for optimal operation. Also, throughput is slight lower for the DRO than it was for the SRO when total path length is less than about 30 km. However, beyond this point, the throughputs for the two systems are indistinguishable. With both, it is possible to achieve a throughput of ~600 pairs/sec out to an end-to-end path length of 50 km.

Based on this analysis, it would indeed be possible to use the SRO as a source of entangled photon pairs in a quantum optical communication system.

### 4.3 Comparison of the Singly- and Doubly-Resonant Oscillators

While these results are quite valuable, the comparison isn’t entirely complete without looking at the advantages and the disadvantages of using an SRO versus a DRO. There are two primary differences between operating an SRO and operating a DRO. The first is the pump power requirement. The second is ease of operation.

A typical type-II phase-matched DRO has a threshold power of ~65 mW [17]. In order to run a communication system with a total path length of 50 km using a DRO as the entangled photon pair source, a gain of 0.016 was found to be optimal. This means that the pump power required is ~1.04 mW. On the other hand, the threshold of a similar SRO is ~6500 mW [18]. In order to run a communication system with a total path length of 50 km using an SRO as the entangled photon pair source, a gain of 0.016 was found to be optimal. This means that the pump power required is ~104 mW. This is a significant difference. It is much more desirable to operate a system with a significantly lower power requirement.
This higher power requirement is offset by the fact that an SRO is significantly easier to operate than a DRO. This is because in an SRO, only one frequency must be resonant inside the cavity. It is much simpler to stabilize a cavity with only one resonance than it is to stabilize a cavity with two resonances.
Figure 4.3: Impact of Total Path Length on (a) Fidelity, and (b) Throughput.
Figure 4.4: Impact of Input Parameters on Throughput at $2L = 0$
Figure 4.5: Impact of Input Parameters on Fidelity at $2L = 0$

Fidelity vs. $G^2$ and $\Gamma_C/\Gamma$ for the DRO
Throughput vs. 2L at SRO ($\Gamma_{S}/\Gamma_{I} = 0.001$) Optimal Operation ($F = 0.95$)

Figure 4.6: Throughput of the SRO at Optimal Operation as a Function of Total Path Length
Figure 4.7: Input Parameters for SRO Optimal Operation – (a) Gain, and (b) Memory-to-Signal Cavity Linewidth Ratio
Figure 4.8: Throughput of the DRO at Optimal Operation as a Function of Total Path Length

Cycling Rate = 500 kHz
Figure 4.9: Input Parameters for DRO Optimal Operation – (a) Gain, and (b) Memory-to-Signal Cavity Linewidth Ratio
5. CONCLUSION

5.1 Summary

Entangled photon pairs are needed to achieve the magic bullet effect and to produce a quantum optical communication system capable of high-fidelity long-distance teleportation. We have extended the analysis of the optical parametric amplifier as a source of entangled photon pairs to include the singly-resonant configuration. We have found that while the performance of the SRO is slightly inferior to that of its DRO counterpart, the SRO is still a viable source of entangled photon pairs for applications such as quantum magic bullets and quantum teleportation. In addition, the SRO has the advantage being simpler to operate than the DRO, which makes it a preferable source despite its higher pump power requirement.

We have shown that the special filtering property of light called the magic bullet effect can be observed for the case of doubly-resonant operation when the appropriate filter is used. Although a single-pole filter is not sufficiently steep-skirted to illustrate this property, a Kth-order Butterworth filters can be applied to entangled photon pairs to demonstrate the magic bullet effect. A setup that includes a prism, lens and detector of finite size was developed to realize a filter with the requisite photon counting statistics and a detector that could be used in an actual system.

Finally, polarization-entangled photon pairs were used in conjunction with a trapped-atom quantum memory to create a quantum optical communication system capable of high-fidelity long-distance teleportation and quantum information storage over relatively long periods of time. It was shown that either the singly- or doubly-resonant
source could be used for this application. For a given fidelity requirement, we found that the SRO was able to generate slightly more throughput. However, we found that depending on the system requirements it might still be advantageous to use the doubly-resonant configuration.

### 5.2 Future Work

There is still a great deal of work to be done to improve our understanding of OPA systems. We have learned a great deal about the photon-count-difference statistics of both singly- and doubly-resonant OPAs. Still, the theoretical results remain to be compared to what can be achieved by an actual system. It is the experimental results that will ultimately determine the capabilities of both OPAs as entangled photon sources.

We have shown in theory that in order to achieve the desired photon counting statistics from a parametric amplifier source, it is necessary to count over an interval that is long compared to the bandwidth of the OPA. The theory of OPA systems can be expanded by investigating the time statistics of the signal and idler photon coincidence counts from both singly- and doubly-resonant OPAs. This analysis would enable us to better determine an appropriate counting interval in order to maximize the likelihood of a coincidence while maintaining a sufficiently low probability of encountering more than one signal-idler pair in a given counting interval.

Finally, an interesting application that warrants further exploration is the use of parametric amplifiers to generate higher-order entangled states. We would like to know how these photon sources could be used to enable three or more parties to share an entangled state. This multi-party entanglement could be useful if one party wanted to send information via a quantum state to two or more other parties but only wanted the
recipients to be able to recreate the state and decode the transmitted information with the cooperation of one another.
APPENDIX A – DERIVATION OF COUNT VARIANCE

The photon count difference is defined in the following way:

$$\Delta N = \int_0^T \left[ \hat{A}_s^\dagger(t)\hat{A}_s(t) - \hat{A}_r^\dagger(t)\hat{A}_r(t) \right] dt$$

As mentioned in Chapter 2, the mean photon count difference is zero: $\langle \Delta \hat{N} \rangle = 0$.

However, another important measure of the OPA system is the variance of the photon-count-difference, given by:

$$\text{var}(\Delta \hat{N}) = \left\langle (\Delta \hat{N})^2 \right\rangle = \int_0^T \int_0^T \left[ \hat{A}_s^\dagger(t)\hat{A}_s(t) - \hat{A}_r^\dagger(t)\hat{A}_r(t) \right] dt \int_0^T \left[ \hat{A}_s^\dagger(u)\hat{A}_s(u) - \hat{A}_r^\dagger(u)\hat{A}_r(u) \right] du$$

When the two integrals are brought together and Gaussian moment factoring is used, this photon-count-difference variance is equivalent to equation (29) of [11]:

$$\langle (\Delta \hat{N})^2 \rangle = \int_0^T dt \int_0^T du \left[ \langle \hat{A}_s^\dagger(t)\hat{A}_s(t) \rangle + \langle \hat{A}_r^\dagger(t)\hat{A}_r(t) \rangle \delta(t - u) + \langle \hat{A}_s^\dagger(t)\hat{A}_s(u) \rangle \langle \hat{A}_r^\dagger(u)\hat{A}_r(t) \rangle \right]$$

$$+ \langle \hat{A}_s^\dagger(t)\hat{A}_s(u) \rangle \langle \hat{A}_r^\dagger(u)\hat{A}_r(t) \rangle - \langle \hat{A}_s^\dagger(t)\hat{A}_s(u) \rangle \langle \hat{A}_r^\dagger(t)\hat{A}_r(u) \rangle$$

Taking advantage of a change of variables $s = \frac{t + u}{2}$ and $\tau = t - u$ allows the integral to be simplified significantly to become a single integral:

$$\langle (\Delta \hat{N})^2 \rangle = \int_0^T dt \int_0^T du \left[ \langle \hat{A}_s^\dagger(t)\hat{A}_s(t) \rangle + \langle \hat{A}_r^\dagger(t)\hat{A}_r(t) \rangle \delta(t - u) + \langle \hat{A}_s^\dagger(t)\hat{A}_s(u) \rangle \langle \hat{A}_r^\dagger(u)\hat{A}_r(t) \rangle \right]$$
\[
\langle \Delta \hat{N}^2 \rangle = \int_0^T dt \left[ \langle \hat{A}_s^+ (t) \hat{A}_s (t) \rangle + \langle \hat{A}_t^+ (t) \hat{A}_t (t) \rangle \right] \\
+ \int_{-T}^T d\tau \left[ 1 - \frac{|\tau|}{T} \right] \left\{ \langle \hat{A}_s^+ (t + \tau) \hat{A}_s (t) \rangle \langle \hat{A}_s^+ (t + \tau) \hat{A}_s (t) \rangle + \langle \hat{A}_t^+ (t + \tau) \hat{A}_t (t) \rangle \langle \hat{A}_t^+ (t + \tau) \hat{A}_t (t) \rangle \right\} \\
- \left\langle \hat{A}_s^+ (t + \tau) \hat{A}_t^+ (t) \right\rangle \left\langle \hat{A}_s (t + \tau) \hat{A}_t (t) \right\rangle - \left\langle \hat{A}_t^+ (t + \tau) \hat{A}_s^+ (t) \right\rangle \left\langle \hat{A}_t (t + \tau) \hat{A}_s (t) \right\rangle \right\}
\]

It is easy to recognize that performing the first integral produces the sum of the mean signal and idler photon counts. Thus, the normalized variance, \( \langle \Delta \hat{N}^2 \rangle / \langle \hat{N}_s \rangle + \langle \hat{N}_t \rangle \), is less than one (i.e. the light has a sub-Poissonian count distribution) if the second integral is negative.

Once the variance is in this form, only a straightforward calculation is required. Using the equations derived in Chapter 2 for mean signal and idler photon counts, and the normally ordered and phase-sensitive correlation functions, the variance can be computed as a function of gain, signal and idler linewidths, and the length of the photon counting interval.

\[
\langle \Delta \hat{N}^2 \rangle = \frac{4G^2 \Gamma_s \Gamma_t T}{(1 - G^2)(\Gamma_s + \Gamma_t)} + \frac{4G^4 \Gamma_s^2 \Gamma_t^2}{(1 - G^2)^2(\Gamma_s + \Gamma_t)^2 \Lambda^2} \left\{ \frac{(\Lambda - (\Gamma_s + \Gamma_t))^2}{(\Gamma_s + \Gamma_t) + \Lambda} e^{-(\Gamma_s + \Gamma_t) + \Lambda)T} + \frac{(\Lambda + (\Gamma_s + \Gamma_t))^2}{(\Gamma_s + \Gamma_t) - \Lambda} e^{-(\Gamma_s + \Gamma_t) - \Lambda)T} \right\} \\
- \left\{ \frac{(\Lambda - (\Gamma_s + \Gamma_t))^2}{(\Gamma_s + \Gamma_t) + \Lambda} \right\} + \frac{(\Lambda + (\Gamma_s + \Gamma_t))^2}{(\Gamma_s + \Gamma_t) - \Lambda} \right\} + 2 \left\{ \frac{(\Lambda^2 - (\Gamma_s + \Gamma_t))^2}{(\Gamma_s + \Gamma_t)} \right\} \\
- \left\{ \frac{(\Lambda - (\Gamma_s + \Gamma_t))^2}{(\Gamma_s + \Gamma_t) + \Lambda} \right\} + \frac{(\Lambda + (\Gamma_s + \Gamma_t))^2}{(\Gamma_s + \Gamma_t) - \Lambda} \right\} + 2 \left\{ \frac{(\Lambda^2 - (\Gamma_s + \Gamma_t))^2}{(\Gamma_s + \Gamma_t)} \right\} 
\]
\[
- \frac{G^2 \Gamma_s^2 \Gamma_i^2}{(1-G^2)^2(\Gamma_s+\Gamma_i)^2} \left\{ \frac{A}{((\Gamma_s+\Gamma_i)+\Lambda)^2} e^{-(\Gamma_i+\Gamma_s+\Lambda)^2} + \frac{B}{((\Gamma_s+\Gamma_i)-\Lambda)^2} e^{-(\Gamma_i+\Gamma_s-\Lambda)^2} + 2 \frac{C}{(\Gamma_s+\Gamma_i)^2} e^{-(\Gamma_s+\Gamma_i)^2} \right\} \\
+ \frac{B}{((\Gamma_s+\Gamma_i)-\Lambda)^2} e^{-(\Gamma_i+\Gamma_s-\Lambda)^2} + 2 \frac{C}{(\Gamma_s+\Gamma_i)^2} e^{-(\Gamma_s+\Gamma_i)^2}
\]
SUPPLEMENTARY MATERIAL

**APPENDIX B - DERIVATION OF FILTER OUTPUT**

Suppose that the frequency ω OPA output \( \hat{E}_S(x, z = 0, \omega) \) enters the prism of width \( D \) at \( z = 0 \), as shown in Figure 3.6. We use a Fourier optics technique to analyze the behavior of the field as it travels through each of the elements to the point just before the detector. At \( z = l \), the field is

\[
\hat{E}_S(x, z = l, \omega) = \hat{E}_S(x, z = 0, \omega) \exp \left[ j \frac{\omega}{c} \beta(\omega - \omega_s) \right]
\]

At the output of the prism, a lens is applied to focus the desired frequencies. At the output of the thin lens, still at \( z = l \), the field is

\[
\hat{E}^{LENS}_S(x, z = l, \omega) = \hat{E}_S(x, z = l, \omega) \exp \left[ - j \frac{\omega x^2}{2 f} \right]
\]

Prior to entering the detector, at \( z = l + f \), the field is

\[
\hat{E}_S(x, z = l + f, \omega) = \frac{\int_{-D/2}^{D/2} dx'}{\sqrt{j \frac{2\pi c}{\omega} f}} \exp \left[ - j \frac{\omega x^2}{2 f} \right] \exp \left[ j \frac{\omega f}{c} \right] \exp \left[ j \frac{\omega (x - x')^2}{2 f} \right]
\]

With these Fourier optics tools, we would like to understand the behavior of our filter system in the context of the photon counting statistics of the filter output. Before proceeding further, we would like to simplify the problem somewhat, if possible. The full field at \( z = 0 \) is of the form

\[
\hat{E}_S(x, z = 0, \omega) = \frac{1}{\sqrt{D}} \hat{E}_S^{OPA}(\omega) + \hat{E}_{VAC}(x, \omega) \text{ for } |x| \leq \frac{D}{2}
\]

In this equation, \( \frac{1}{\sqrt{D}} \hat{E}_S^{OPA}(\omega) \) is the normalized field at the OPA output, and \( \hat{E}_{VAC}(x, \omega) \)
is the vacuum term that is required by commutator preservation. However, only the excited mode will contribute to the photon counting statistics because the correlation between different vacuum modes and between the excited mode and the vacuum mode is zero. Therefore, this vacuum term can be neglected in the analysis. We would now like to look at the field $\hat{E}_F^x (x, \omega) = \hat{E}_S (x, z = l + f, \omega)$, considering only the excited mode input field. In the time domain, this field is

$$E_S^x (x', t) = \int du \hat{E}_S^{OPA} (u) \frac{d\omega}{2\pi} \frac{\exp[-j\omega(t-u)]}{\sqrt{2\pi c \omega f}} \exp \left[ j \frac{\omega}{c} f + j \frac{\omega x'^2}{2cf} \right] Q$$

where $Q = \int_{-D/2}^{D/2} dx \frac{1}{\sqrt{D}} \exp \left[ j \frac{\omega}{c} \beta (\omega - \omega_S) x - j \frac{\omega}{c} \frac{xx'}{f} \right]$

This equations look complicated, but $Q$ is a straightforward integral to solve, and the output field can be rewritten as the convolution of the input field and a filter, $h_S (x', t)$.

$$\hat{E}_F^x (x', t) = \int du \hat{E}_S^{OPA} (u) h_S (x', t-u)$$

where

$$h_S (x', t-u) = \int \frac{d\omega}{2\pi} e^{-j\omega(t-u)} \exp \left[ j \frac{\omega}{c} f + j \frac{\omega x'^2}{2cf} \right] \frac{\sin \left( \frac{\omega D}{2cf} (x' - \beta (\omega - \omega_S) f) \right)}{\sqrt{\frac{\omega D}{2cf} (x' - \beta (\omega - \omega_S) f)}}$$

Converting to baseband field operators gives the results used in Chapter 3.
APPENDIX C – CAVITY LOADING STATISTICS

In computing the cavity loading statistics of the quantum memory, the first step is to compute the probabilities that different combinations of photon counts occur. The probabilities of success, erasure and error can then be determined based on these probabilities. The only possible outcomes are success, erasure and error; therefore, the sum of the probabilities of these events is one.

The probability of erasure is the probability that no photons were loaded into the signal cavity \( P_{o_s} \) plus the probability that no photons were loaded into the idler cavity \( P_{o_i} \) minus the probability that no photons were loaded into either cavity \( P_{o,0} \).

\[
P_{\text{ERASURE}} = P_{o_s} + P_{o_i} - P_{o,0}
\]

Using the fact that \( S_x \) and \( I_y \) are independent of \( S_y \) and \( I_x \) we find that

\[
Pr(Y, s' = I 0) = Pr(a, s = 0) = Pr(a, t = 0) = P_0 = \frac{1}{1 + n}
\]

\[
Pr(\hat{a}_s^\dagger \hat{a}_x = 0 \& \hat{a}_i^\dagger \hat{a}_t = 0) = Pr(\hat{a}_s^\dagger \hat{a}_y = 0 \& \hat{a}_i^\dagger \hat{a}_i = 0) = P_{oo} = \frac{1}{(1 + n)^2 - n^2}
\]

Hence, \( Pr[\text{erasure}] = 2P_0^2 - P_{oo}^2 \).

The success probability is more complicated because it involves determining whether or not the two absorbed photons originated from the same entangled pair.
When calculated, the probability of success turns out to be $P_{11}P_{00} + \tilde{n}^2/[(1 + \bar{n})^2 - \bar{n}^2]^4$ where $P_{11}$ is the probability that both the signal and the idler photons from an entangled pair from one OPA were absorbed into the memory cavity. This probability is a function of $P_{00}$ and $P_{10}$, the probability that either the signal or the idler photon from an entangled pair from one OPA was absorbed, but not both:

$$P_{10} = \Pr(\hat{a}^+_{S_y} \hat{a}_{S_x} = 1 \& \hat{a}^+_{I_y} \hat{a}_{I_x} = 0) = \Pr(\hat{a}^+_{S_y} \hat{a}_{S_x} = 0 \& \hat{a}^+_{I_y} \hat{a}_{I_x} = 1)$$

$$= \Pr(\hat{a}^+_{S_y} \hat{a}_{S_y} = 1 \& \hat{a}^+_{I_x} \hat{a}_{I_x} = 0) = \Pr(\hat{a}^+_{S_x} \hat{a}_{S_x} = 0 \& \hat{a}^+_{I_x} \hat{a}_{I_x} = 1) = \frac{\bar{n}(1 + \bar{n}) - \tilde{n}^2}{[(1 + \bar{n})^2 - \bar{n}^2]^2}$$

$$P_{11} = \Pr(\hat{a}^+_{S_x} \hat{a}_{S_x} = 1 \& \hat{a}^+_{I_y} \hat{a}_{I_y} = 1) = \Pr(\hat{a}^+_{S_y} \hat{a}_{S_y} = 1 \& \hat{a}^+_{I_y} \hat{a}_{I_y} = 1) = 2P_{10} - P_{00} + \frac{\bar{n}(1 + \bar{n}) + \tilde{n}^2}{[(1 + \bar{n})^2 - \bar{n}^2]^2}$$

Finally, the probability of error is calculated in the simplest possible way:

$$\Pr[\text{error}] = 1 - P_{\text{SUCCESS}} - P_{\text{ERASURE}}$$

Once all of these probabilities have been calculated, it is straightforward to calculate the figures of merit:

$$\text{Throughput} = N_{\text{SUCCESS}} = \text{Rate} \times P_{\text{SUCCESS}}$$

$$\text{Fidelity} = \frac{P_{\text{SUCCESS}}}{P_{\text{ERROR}} + P_{\text{SUCCESS}}} + \frac{P_{\text{ERROR}}}{2 \left( P_{\text{ERROR}} + P_{\text{SUCCESS}} \right)}$$

These figures depend on the gain and the linewidths of the signal, idler and memory cavities, and can be used to determine optimal system operation.
REFERENCES


   p. 120


   and Polzik, E.S. 1998 *Science*. 282 706


   2354

   published in Proc. 5th International Conf. On Quantum Comm., Measurement &
   Computing, P. Tombesi and O. Hirota, eds.

   photons,” MEng Thesis Proposal, MIT, August 2000


[18] Conversation with E. Mason, March 16, 2001