Target Recognition Performance for FLIR and Laser Radar Systems
by
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Submitted to the Department of Electrical Engineering and Computer Science
in partial fulfillment of the requirements for the degree of
Master of Engineering in Electrical Engineering and Computer Science
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Abstract

There has been much interest in analyzing the performance of target recognition systems based on imaging sensors. This thesis considers recognition for ground-based targets based on data from a forward-looking infrared (FLIR) intensity image and a laser radar (LADAR) range image. A Bayesian approach to the problem of target recognition involves comparing ratios of likelihood functions in the presence of pose parameters. Computing the resulting nuisance integrals requires numerical approximation techniques. Previous work done on likelihood-based recognition has relied on time-consuming simulations of target CAD models to demonstrate algorithm performance. This thesis obtains asymptotic performance results analytically in the region of high signal-to-noise ratio (SNR), using Laplace’s approximation to integrate out the pose parameters. In the limit as sensor noise becomes small, the conditional probability of recognition error given a target at a true pose is determined by the distance to the closest occurrence of the incorrect target. The asymptotic probability of error decreases exponentially at a rate given by this minimum distance. Rather than using detailed CAD target models, we use target block-models that simplify the target geometry. A comparison of these theoretical results with simulations using these simplified target models shows good agreement in the region of high SNR. A second application of Laplace’s approximation over the pose parameter space yields an analytical form for the unconditional probability of error. The asymptotic unconditional performance is shown to be determined by the global minimum distance to the incorrect target.

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Chapter 1

Introduction

This thesis concerns target recognition systems that rely on imagery from forward-looking infrared (FLIR) and laser radar (LADAR) sensors. We are interested in the classification performance of target recognition systems for ground-based targets with a known location but unknown pose orientation. We use the minimum probability of error criterion and the corresponding Bayesian rule to make our recognition decisions.

The pose of ground-based targets is modeled as a rotation angle $\theta$ about the vertical axis of the target, where $\theta$ is unknown to the system a priori. The Bayesian decision rule consists of comparing ratios of likelihood functions. In the presence of a random pose parameter, however, computing these likelihood functions requires the evaluation of pose integrals. In general, these integrals cannot be calculated analytically and require numerical approximation or other techniques.

Previous work done on likelihood-based recognition relied on time-consuming simulations of target CAD models to demonstrate algorithm performance [1, 2]. Further motivation comes from the closely related problem of likelihood-based orientation estimation. In these studies, performance evaluation via the CAD model approach requires a new set of simulations each time a different set of targets or sensor parameters is studied [3, 4]. It is desirable for both pose estimation and target recognition to develop an approach that will give better analytic understanding of these problems. An asymptotic, analytical performance assessment for FLIR and laser radar orientation estimation has recently been reported [5]. This work showed how pose estimation accuracy depends on target and sensor parameters. We want to extend this type of understanding to the target recognition problem. In partic-
ular, we want to know how target and system parameters affect our ability to make correct classification decisions.

The remainder of this thesis is organized as follows. In Chapter 2, we describe the FLIR and LADAR sensors and give statistical models for their respective sensor data. In Chapter 3, we develop our low-noise asymptotic analysis of recognition performance. These results are applied to FLIR-based target recognition systems in Chapter 4. Our main results are developed using the binary recognition setup. We show how we can treat the detection and multi-hypothesis cases as well. We conclude Chapter 4 by providing some insight into how recognition performance depends on FLIR sensor parameters and target geometry in the high SNR regime. In Chapter 5 we offer a preliminary study of systems based on LADAR range imagery. Chapter 6 provides an overview of our accomplishments. We also describe areas of unfinished work and further extensions that could be made to our target recognition research.
Chapter 2

System Models

We are interested in analyzing the performance of target recognition systems based on forward-looking infrared radar (FLIR) and laser radar (LADAR) imagers. Studies of FLIR and LADAR have led to the development of statistical models for the data collected by these sensors [6]. The FLIR is a passive sensor that detects temperature variations between a background scene and any targets embedded therein. For each pixel, the photodetector array converts received light into a photocurrent which is integrated and recorded as intensity data. The FLIR’s intensity image is corrupted by a combination of Gaussian thermal noise and shot noise. Figure 2-1 shows a computer generated FLIR intensity image of a tank embedded in a uniform background. We use this as our infinite signal-to-noise ratio (SNR) truth image. Figure 2-2 is a FLIR image with simulated Gaussian noise representing finite SNR image data.

A coherent laser radar, on the other hand, is an active sensor. It can produce range imagery by raster scanning a field of view in pulsed-imager mode, wherein the roundtrip time-of-flight for each pulse gives the range to its corresponding pixel. The range image is degraded by laser speckle and local-oscillator shot noise. The single-pixel statistical model for laser radar range data has been theoretically developed and experimentally verified in [7]. In Fig. 2-3 we have a computer generated LADAR range image of a tank on a planar background. Simulated Gaussian noise and anomalous pixels were applied to the LADAR image in Fig. 2-4, which represents our finite carrier-to-noise ratio (CNR) range data.
Figure 2-1: FLIR simulated truth image corresponding to SNR = $\infty$.

Figure 2-2: FLIR image with simulated 5 dB SNR. Grayscale indicates intensity: white is the highest intensity, black is the lowest.
Figure 2-3: LADAR simulated range truth image on planar background for CNR = \infty.

Figure 2-4: LADAR range image with simulated 30 dB CNR and random anomalous pixels. Grayscale indicates range: light is farthest away, dark is closest to the sensor.
2.1 FLIR System Model

We first describe a statistical model for FLIR sensor data. The FLIR is a passive sensor that measures thermal radiation from the target scene. We assume that the targets and background are Lambertian surfaces and emit blackbody radiation at uniform known temperature values. It is convenient to consider the effective temperature data, which is the incremental temperature above the constant background temperature. Measurements of this incremental temperature are corrupted by Gaussian-distributed thermal noise and by shot noise. In the high-density shot-noise limit, this gives the following single-pixel statistical model for FLIR data [6],

\[ x_{ij} = x^*_{ij}(\theta) + n_{ij} \]  

(2.1)

where \( x_{ij} \) is the measured effective temperature on the \( ij \)-th pixel, \( x^*_{ij}(\theta) \) is the true value at the target orientation angle \( \theta \), and \( n_{ij} \) is a zero-mean unity-variance Gaussian random variable. Given our uniformity assumptions, we have that,

\[ x^*_{ij}(\theta) = \sqrt{\text{SNR}} F_{ij}(\theta) \]  

(2.2)

where \( F_{ij}(\theta) \) is the fraction of the \( ij \)-th pixel that is on the target and SNR is the single-pixel signal-to-noise ratio. The function \( F_{ij} \) takes value in the range \([0,1]\) where \( F_{ij}(\theta) = 0 \) if the \( ij \)-th pixel is completely off-target at pose \( \theta \) and \( F_{ij}(\theta) = 1 \) if it is completely on-target. The FLIR’s single-pixel signal-to-noise ratio satisfies \( \text{SNR} = (\frac{\Delta T}{\sigma})^2 \), where \( \Delta T \) is the target’s effective temperature relative to the background and \( \sigma^2 \) is the FLIR’s noise-equivalent differential temperature.

For an \( I \times J \)-pixel image lattice, it is convenient to assemble the pixel data into an \( IJ \)-dim column vector \( x \). For a known target at orientation \( \theta \) the FLIR pixels can be taken to be statistically independent [6]. Define vector \( \mathbf{F}(\theta) = [F_{ij}(\theta)] \) as the collection of fractional pixel values and \( \mathbf{n} \) to be a standard white-Gaussian vector \( \sim N(0, I) \). The FLIR’s statistical model is then compactly expressed as,

\[ \mathbf{x} = \sqrt{\text{SNR}} \mathbf{F}(\theta) + \mathbf{n} \]  

(2.3)
with the following joint probability density for \( x \), given the target and its pose,

\[
p(x|\theta) = \prod_{i,j} \frac{\exp\left(-\frac{(x_{ij} - x_{ij}^*(\theta))^2}{2\delta^2}\right)}{\sqrt{2\pi \sigma^2}}.
\] (2.4)

2.2 Laser Radar System Model

The second image sensor we look at is the laser radar. A coherent laser radar transmits a series of laser pulses, one for each pixel in a raster scan. The laser radar range image is formed by recording the time delay between the peaks of the transmitted and received pulse for each pixel. Laser speckle and local-oscillator shot noise combine to degrade the range image. The former is due to the rough-surfaced nature of the reflecting object measured on the scale of the laser wavelength. The latter is the fundamental noise encountered in optical heterodyne detection, and it results in Gaussian noise. Speckle degrades range imagery through range anomalies, i.e., when a deep speckle fade together with a strong noise peak leads to a range measurement far removed from the true range. These noise mechanisms combine to give the following model for the single-pixel measurement of the range value [6],

\[
r_{ij} = \begin{cases} \ r_{ij}^*(\theta) + n_{ij}, & \text{pixel non-anomalous} \\ \ u_{ij}, & \text{pixel anomalous.} \end{cases}
\] (2.5)

Each pixel is a mixture of a uniform and a Gaussian density. Given the pose \( \theta \), the true range value is \( r_{ij}^*(\theta) \). The random variables \( n_{ij} \) are Gaussian zero-mean with standard deviation equal to the local-range accuracy \( \delta R \). The \( u_{ij} \) are uniform over the radar’s range uncertainty interval, which has length \( \Delta R \gg \delta R \).

We assume that the pixel spacing is large enough to give independent range measurements for each pixel. Collecting the pixel range data into a vector \( r \), the joint probability density for \( r \) given the target and its pose is,

\[
p(r|\theta) = \prod_{i,j} \left[ 1 - \Pr(A) \right] \frac{\exp\left(-\frac{(r_{ij} - r_{ij}^*(\theta))^2}{2\delta^2 R^2}\right)}{\sqrt{2\pi (\delta R)^2}} + \frac{\Pr(A)}{\Delta R} \right].
\] (2.6)

Here, \( \Pr(A) \) is the probability of anomaly. The first term on the right side represents the
local range behavior; it is the product of the probability that the pixel is non-anomalous and the density for \( r^*_i(\theta) + n_{ij} \). The second term represents the global range behavior; it is the product of the probability that the pixel is anomalous and the density for \( u_{ij} \).

The parameters given in the statistical model above can be related to the LADAR’s physical sensor as follows [7]. In terms of carrier-to-noise ratio, defined as

\[
\text{CNR} = \frac{\text{average target-return power}}{\text{average local-oscillator shot-noise power}},
\]  

the range resolution \( R_{\text{res}} \), and the number of range resolution bins \( N \equiv \Delta R / R_{\text{res}} \), we have that the local range accuracy and anomaly probability obey

\[
\delta R \approx \frac{R_{\text{res}}}{(\text{CNR})^{1/2}} \quad \text{and} \quad \Pr(A) \approx \frac{1}{\text{CNR}} \left[ \ln N - \frac{1}{N} + 0.577 \right],
\]

respectively. These results are valid in the interesting regime \( N \gg 1 \) and \( \text{CNR} \gg 1 \).
Chapter 3

Target Recognition Analysis

We are interested in target recognition for ground-based targets based on data from the image sensors described in the last chapter. We assume that the location of the target has been determined so that only its identity and rotation orientation are unknown. Uncertainty in the target’s pose results in a wide variety of ways a single target can appear to the sensor. Fig. 3-1 shows a picture of a target at two possible orientations at which it would appear differently to an image sensor. Uncertainty in the target’s orientation relative to the sensor is modeled by an unknown pose parameter $\theta$. In this chapter, we will study the problem of recognition in a Bayesian framework and analyze the asymptotic probability of error for high sensor SNR.

3.1 Bayesian Hypothesis Testing

Assume that we have a finite library of possible targets $\mathcal{L} = \{\alpha_1, \ldots, \alpha_M\}$. Let hypothesis $H_i$ mean that target $\alpha_i$ is present and $x$ be the system’s observed sensor data vector. We will assume equiprobable hypotheses in our analysis, in other words, each target has prior probability $\Pr(H_i) = 1/M$. We model the unknown pose parameter $\theta$ as a random variable and assign it an uninformative uniform prior $\pi(\theta)$. Recognition decisions will be made

Figure 3-1: T-62 tank at two possible pose orientations.
in a manner that minimizes the probability of incorrect decision. Under this criterion, the optimum Bayesian decision is the MAP (maximum a posteriori probability) rule [8]. This rule says to choose the target with the maximum posterior probability \( \Pr(H_i|x) \). To compute these posterior probabilities, we use Bayes' Rule,

\[
\Pr(H_i|x) = \frac{\Pr(H_i)p(x|H_i)}{p(x)}. \tag{3.1}
\]

The marginal density \( p(x|H_i) \) is called the likelihood function and will be denoted as \( L_i(x) = p(x|H_i) \). Under our assumption of equiprobable hypotheses, the terms \( \Pr(H_i) \) and \( p(x) \) are not functions of the hypothesis \( H_i \). Thus, the MAP rule reduces to choosing the target with the maximum likelihood function. This is known as the ML decision rule. We will use the ML rule, which has the form

**ML Rule:** Decide Target \( \alpha_i \) if \( L_i(x) > L_j(x), \) for all \( j \neq i, \)

(3.2)

as the basis for our recognition decisions. To evaluate the ML rule, it is necessary to compute the likelihood functions by integrating out the pose parameter \( \theta \):

\[
L_i(x) = \int_{\Theta} p(x|H_i, \theta)\pi(\theta)d\theta. \tag{3.3}
\]

### 3.2 Asymptotic Analysis

The last section showed that evaluating the optimal Bayesian decision rule requires computing integrals (3.3) over the pose space \( \Theta \). In general, it is not possible to obtain analytical expressions for these pose integrals. The evaluation of such integrals is the main computational obstacle to using Bayesian inference. To avoid this problem, a common non-Bayesian approach is to find the ML estimates \( \hat{\theta}_i \) and evaluate a generalized likelihood ratio test (GLRT) using \( L_i(x) = p(x|H_i, \hat{\theta}_i) \). See [9] for example, a study on LADAR-based detection. As we will see later, the asymptotic performance of a GLRT provides the optimal low noise exponent in the probability of error.

To implement the Bayesian rule, we must resort to either numerical integration or approximation techniques. Target recognition studies with unknown orientation parameters have used Monte Carlo simulations with target CAD models to assess recognition perfor-
mance. However with this approach a new set of simulations is required each time a new set of sensors or targets is used. We will instead use approximations of the pose integral to help us develop an analytic understanding of performance in terms of sensor parameters and target geometry.

3.2.1 Nuisance Parameters

The recognition problem studied in this research involves the unknown pose orientation $\theta$ of a ground-based target. From the viewpoint of the target classification decision, the value of the target’s pose is irrelevant. For this reason, the pose orientation $\theta$ is called a nuisance parameter. However, to perform target recognition, the pose must be dealt with in our statistical model through the target pose integrals (3.3).

Previous studies on hypothesis testing involving nuisance parameters have concluded that asymptotic performance is determined by the value of the parameter most likely to confuse the decision maker. In a classification study [10], recognition in the asymptotic regime is quantified by a measure of the similarity between the correct and incorrect objects. In a communications setting, a large deviations approach was used in [11] to study the probability of error based on conditionally independent, identically distributed observations given phase and delay parameters $\psi \in \Psi$. This study showed that the unconditional error-probability, averaged over $\Psi$, was determined by the value of $\psi$ with the greatest conditional error-probability decay rate.

For our target recognition problem, it will be shown that the unconditional probability of error is determined by a worst case pose parameter $\theta^* \in \Theta$. For the problem of deciding between two possible targets $\alpha_1$ and $\alpha_2$, we can visualize the situation as in Fig. 3-2. In this picture, $S$ represents the range of all possible data vectors. The subset $S_i \subset S$ is the range of noise-free images target $\alpha_i$ can take over the parameter space. Suppose target $\alpha_1$ is the true target. When $\alpha_1$ is at a given pose $\theta_1$, we say that the confuser target is the incorrect target $\alpha_2$ at the pose $c(\theta_1)$ which minimizes the distance to the true target. The minimized distance $D(\theta_1)$ determines the asymptotic conditional probability of error. It will be shown that the unconditional probability of error rate is determined by the parameter $\theta^*$ which minimizes the distance function $D(\theta_1)$. 

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Figure 3-2: Asymptotic performance is determined by confuser distance.

3.2.2 Laplace Method

In this section we introduce the Laplace method of evaluating integrals. This standard analytic tool is described in texts on integration and asymptotic expansion such as [12, 13]. For statistical applications, [14] discusses the use of the Laplace method in evaluating integrals for Bayesian inference, and [15, 16] use the approximation to study Bayesian target recognition. We will be satisfied in giving a heuristic argument to derive the Laplace approximation. In section 4.3, we provide more precise statements on the convergence of the approximation.

Integrals of the form

\[ F(\sigma^2) = \int_{\theta_{\min}}^{\theta_{\max}} g(\theta) \exp \left( -\frac{h(\theta)}{2\sigma^2} \right) d\theta \]  

(3.4)

appear in the likelihood expressions needed to evaluate the Bayesian decision rule we have assumed for our target recognition system. If \( h(\theta) \) is positive in the interval \([\theta_{\min}, \theta_{\max}]\), then we can use the Laplace method to asymptotically evaluate \( F(\sigma^2) \) in the limit \( \sigma^2 \to 0 \). The idea behind the approximation is that most of the integral is contained in a neighborhood of the \( \theta \) value where \( h(\theta) \) is a minimum, \( \theta_0 \equiv \text{argmin}_\theta h(\theta) \).

Assume that the function \( h(\theta) \) achieves a unique global minimum at \( \theta_0 \) in the interior of the pose space interval \([\theta_{\min}, \theta_{\max}]\) with a first derivative equal to zero. In this case the second-order Taylor expansion for \( h(\theta) \) around \( \theta_0 \) is \( h(\theta) = h(\theta_0) + \frac{1}{2} h''(\theta_0)(\theta - \theta_0)^2 \). Using
this two-term expression in the integrand of \( F(\sigma^2) \) and the fact that

\[
\int_{\theta_{\min}}^{\theta_{\max}} \exp \left[ - \frac{h''(\theta)}{4\sigma^2} (\theta - \theta_0)^2 \right] d\theta \to \sqrt{\frac{4\pi\sigma^2}{h''(\theta_0)}},
\]

as \( \sigma^2 \to 0 \), we have the Laplace-method approximation

\[
F(\sigma^2) \approx g(\theta_0) \exp \left( - \frac{h(\theta_0)}{2\sigma^2} \right) \sqrt{\frac{4\pi\sigma^2}{h''(\theta_0)}}.
\]

Equation (3.6) is the main tool we will use in our low-noise approximation of recognition error-probability. Unless otherwise noted, all references to the Laplace method will refer to this result. Two main assumptions were used to derive this expression: \( \theta_0 = \text{argmin}_\theta h(\theta) \) is unique and \( \theta_0 \) is interior to the interval \([\theta_{\min}, \theta_{\max}]\) with \( h'(\theta_0) = 0 \). For problems that do not satisfy these conditions, the following alternative approximations can be used. Suppose \( h(\theta) \) achieves its global minimum at multiple \( \theta \) values, i.e., \( \theta_1, \ldots, \theta_k \), all of which are interior to \([\theta_{\min}, \theta_{\max}]\). In this case we sum expressions of the form (3.6) and obtain,

\[
F(\sigma^2) \approx \sum_{i=1}^{k} g(\theta_i) \exp \left( - \frac{h(\theta_i)}{2\sigma^2} \right) \sqrt{\frac{4\pi\sigma^2}{h''(\theta_i)}}.
\]

On the other hand, if \( h(\theta) \) has a unique global minimum that occurs on an endpoint of the interval \([\theta_{\min}, \theta_{\max}]\), then in general the first derivative will not be zero. Using a first-order Taylor expansion for \( h(\theta) \) gives the following result [13]. For the minimum taken on the lower boundary, the approximation is

\[
F(\sigma^2) \approx \frac{2\sigma^2 g(\theta_{\min})}{h'(\theta_{\min})} \exp \left( - \frac{h(\theta_{\min})}{2\sigma^2} \right),
\]

and for the upper boundary,

\[
F(\sigma^2) \approx \frac{-2\sigma^2 g(\theta_{\max})}{h'(\theta_{\max})} \exp \left( - \frac{h(\theta_{\max})}{2\sigma^2} \right).
\]

### 3.3 System Performance - Gaussian noise

In this section we study recognition performance for systems based on data corrupted by Gaussian noise. The Laplace approximation described in the last section is used here to
develop asymptotic results for the performance of these systems. First, Bayesian likelihood functions are approximated using the Laplace method to evaluate the error-probability conditioned on the true pose orientation. Then a second application of the Laplace method over the pose space yields an expression for the unconditional probability of error.

### 3.3.1 Conditional Probability of Error

Consider the binary recognition problem in which we have a set of two possible targets \( \mathcal{L} = \{\alpha_1, \alpha_2\} \) with unknown rotation pose \( \theta \in \Theta \). Fix the true target as \( \alpha_1 \) at true pose \( \theta_1 \). We want to compute the conditional probability of error \( \Pr(e|H_1, \theta_1) \) using the Bayesian ML rule. Let \( x_k(\theta) \) be the uncorrupted data vector for target \( \alpha_k \) at pose \( \theta \), for \( k = 1, 2 \).

Define

\[
E_k(\theta, \theta_1) = \|x_k(\theta) - x_1(\theta_1)\|^2 = \sum_{i,j} (x_{k,i,j}(\theta) - x_{1,i,j}(\theta_1))^2, \quad k = 1, 2.
\]  

(3.10)

The function \( E_k(\theta, \theta_1) \) is a measure of how differently targets appear with respect to the image sensor. More specifically \( E_k(\theta, \theta_1) \) is the squared Euclidean distance between the uncorrupted data vectors for target \( \alpha_k \) at pose \( \theta \) and the true target \( \alpha_1 \) at pose \( \theta_1 \).

Define the confuser pose as \( c(\theta_1) = \arg\min_\theta E_2(\theta, \theta_1) \). The confuser pose is the orientation at which the incorrect target \( \alpha_2 \) most closely resembles the true target \( \alpha_1 \) at pose \( \theta_1 \). Define the confuser distance \( D(\theta_1) = \sqrt{E_2(c(\theta_1), \theta_1)} \) as the minimum distance achieved by the confuser pose. The confuser distance plays an important role in determining the recognition error-probability for the region of high SNR.

In the following derivation of the conditional error-probability we will suppress the notational dependence on the true pose. Namely we will use the following simpler notation with the assumption that \( \theta_1 \) is the true pose:

\[
E_k(\theta) = E_k(\theta, \theta_1)
\]

\[
\ddot{E}_k(\theta) = \frac{\partial^2}{\partial \theta^2} E_k(\theta, \theta_1)
\]

\[
c = c(\theta_1)
\]

\[
D = D(\theta_1)
\]

(3.11)

The following result considers the additive white Gaussian noise model. Denote the standard
Gaussian noise vector as \( \mathbf{n} \sim N(0, \sigma^2 \mathbf{I}) \). For binary recognition, a sufficient statistic for computing the ML decision rule is the ratio of likelihood functions \( L_2(x)/L_1(x) \). Since the logarithm function is positive and monotonic, the logarithm of the likelihood ratio is also sufficient. Using the Laplace method of Eqn. (3.6) to approximate the likelihood functions, it is shown in [15] that the log-likelihood ratio has the asymptotic form:

\[
\log \frac{L_2(x)}{L_1(x)} \approx \frac{D}{\sigma} \left( -\frac{D}{2\sigma} + z \right) + \frac{1}{2} \log \frac{\hat{E}_1(\theta_1)}{\hat{E}_2(c)} \tag{3.12}
\]

as \( \sigma^2 \to 0 \), where

\[
z = \frac{(x_2(c) - x_1(\theta_1))^T \mathbf{n}}{\sigma D} \tag{3.13}
\]

Here \( z \) is a zero-mean, unity-variance Gaussian random variable. To compute the ML rule, we decide hypothesis \( H_2 \) if the right-hand side of (3.12) is greater than zero and \( H_1 \) otherwise. This approximation shows that, in the limit of low noise and conditioned on the true pose \( \theta_1 \), the sufficient statistic can be expressed in terms of a Gaussian random variable. It follows that the asymptotic conditional probability of error can be computed in terms of the Gaussian Q-function:

\[
\Pr(e|H_1, \theta_1) \approx Q(\kappa) = \int_{\kappa}^{\infty} \frac{\exp(-t^2/2)}{\sqrt{2\pi}} \, dt \tag{3.14}
\]

where

\[
\kappa = \frac{D}{2\sigma} - \frac{\sigma}{2D} \log \frac{\hat{E}_1(\theta_1)}{\hat{E}_2(c)}. \tag{3.15}
\]

It can be shown that the Gaussian Q-function is upper and lower bounded as follows:

\[
\left(1 - \frac{1}{\kappa^2}\right) \frac{1}{\sqrt{2\pi \kappa}} \exp\left(-\kappa^2/2\right) < Q(\kappa) < \frac{1}{\sqrt{2\pi \kappa}} \exp\left(-\kappa^2/2\right). \tag{3.16}
\]

This shows that for large \( \kappa \), the upper bound provides a good approximation for the Q-function. In the next section we will compute the unconditional probability of error by
using

$$Q(\kappa) \approx \frac{1}{\sqrt{2\pi \kappa}} \exp\left(-\frac{\kappa^2}{2}\right)$$

(3.17)

to approximate the conditional error-probability and averaging this result for fixed $\theta_1$ over the pose parameter space $\Theta$.

### 3.3.2 Unconditional Probability of Error

The preceding section yielded a weak-noise approximation for the conditional error-probability when the sensor data is corrupted by white Gaussian noise. Having an expression for the conditional probability of error allows us to compute the unconditional error-probability $\Pr(e|H_1)$. If we assume a uniform pose prior, we can use Eqns. (3.14) and (3.17) to obtain,

$$\Pr(e|H_1) = \int_{\theta_{\min}}^{\theta_{\max}} \Pr(e|H_1, \theta_1) \pi(\theta_1) d\theta_1$$

$$\approx \frac{1}{|\Theta|} \int_{\theta_{\min}}^{\theta_{\max}} \frac{1}{\sqrt{2\pi \kappa(\theta_1)}} \exp\left(-\frac{\kappa^2(\theta_1)}{2}\right) d\theta_1,$$

(3.18)

where $|\Theta| = (\theta_{\max} - \theta_{\min})$ is the length of the pose space interval.

The parameter $\kappa$ is defined in the previous section without its $\theta_1$ argument, but in the unconditional case we use the original, more general notation. This clutters up the equations, but allows us to see the functional dependence more clearly. Equation (3.15) in its full form is

$$\kappa(\theta_1) = \frac{D(\theta_1)}{2\sigma} - \frac{\sigma}{2D(\theta_1)} \log \frac{\tilde{E}_1(\theta_1, \theta_1)}{\tilde{E}_2(c(\theta_1), \theta_1)}.$$

(3.19)

We now approximate the integrand for small values of $\sigma^2$:

$$\frac{1}{\sqrt{2\pi \kappa(\theta_1)}} \exp\left(-\frac{\kappa^2(\theta_1)}{2}\right) \approx \sqrt{\frac{2\sigma^2}{\pi}} \frac{D(\theta_1)}{\tilde{E}_1(\theta_1, \theta_1)} \left[ \frac{1}{2} \left( \frac{D(\theta_1)}{2\sigma} - \frac{\sigma}{2D(\theta_1)} \log \frac{\tilde{E}_1(\theta_1, \theta_1)}{\tilde{E}_2(c(\theta_1), \theta_1)} \right)^2 \right]^{1/4}$$

$$\approx \sqrt{\frac{2\sigma^2}{\pi}} \left( \frac{\tilde{E}_1(\theta_1, \theta_1)}{\tilde{E}_2(c(\theta_1), \theta_1)} \right)^{1/4} \exp\left(-\frac{D^2(\theta_1)}{8\sigma^2}\right).$$

(3.20)

In the first line we approximated $\kappa$ in the denominator by $\kappa \approx D/2\sigma$. In the second line,
after expanding the square in the exponent, we threw away a decaying term and brought out the cross-term. Substituting this approximation of the integrand into (3.18), we have

\[
\Pr(e|H_1) \approx \frac{1}{|\Theta|} \sqrt{\frac{2\sigma^2}{\pi}} \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} \frac{1}{D(\theta_1)} \left( \frac{E_1(\theta_1, \theta_1)}{E_2(c(\theta_1), \theta_1)} \right)^{1/4} \exp \left( -\frac{D^2(\theta_1)}{8\sigma^2} \right) d\theta_1. \tag{3.21}
\]

The integral in (3.21) has the form of \( F(\sigma^2) \) in Eqn. (3.4). If we assume that \( D^2(\theta_1) \) achieves a unique minimum strictly in the pose space interval \( \Theta = [\theta_{\text{min}}, \theta_{\text{max}}] \), then we can apply the Laplace approximation in Eqn. (3.6) to evaluate the unconditional probability of error. This results in the expression

\[
\Pr(e|H_1) \approx \frac{4\sigma^2 |\Theta|^{-1}}{D(\theta^*)^{3/2} \sqrt{D(\theta^*)}} \left( \frac{E_1(\theta^*, \theta^*)}{E_2(c(\theta^*), \theta^*)} \right)^{1/4} \exp \left( -\frac{D^2(\theta^*)}{8\sigma^2} \right), \tag{3.22}
\]

where \( \theta^* = \arg\min_{\theta_1} D(\theta_1) \) is the value of the true pose of target \( \alpha_1 \) at which the confuser distance is minimized. In other words, \( D(\theta^*) \) represents the closest that targets \( \alpha_1 \) and \( \alpha_2 \) can resemble each other over all possible pose orientations.

### 3.3.3 Target Detection

In this section we consider the problem of using sensor data corrupted by Gaussian noise to decide whether or not a target is present in the sensor’s field of view. This generalizes the hypothesis testing problem described in section 3.1 to include an additional hypothesis, \( H_0 \), representing a scene that contains only background radiating at a uniform temperature. The simplest problem of this class is to decide between hypothesis \( H_0 \) and hypothesis \( H_1 \), the presence of a particular target \( \alpha_1 \).

In section 3.1 the ML rule was used for recognition to minimize the probability of error. To perform target detection, we will utilize the Neyman-Pearson criterion instead. The Neyman-Pearson decision rule maximizes the probability of detection \( P_D = \Pr(\text{decide } H_1|H_1) \) given the constraint on the probability of false alarm \( P_F = \Pr(\text{decide } H_1|H_0) \leq \gamma \). The optimum Neyman-Pearson decision rule takes the form of a likelihood ratio test (LRT) [8]
\[ l(x) \equiv \log \left[ \frac{L_1(x)}{L_0(x)} \right]_{H_1 \leq H_0} < \eta \]  

(3.23)

where

\[ L_1(x) = \int_{\Theta} p(x|H_1, \theta) \pi(\theta) d\theta \]  

(3.24)

is the pose integral for target \( \alpha_1 \) with unknown pose parameter \( \theta \). The threshold \( \eta \) is chosen to satisfy the false alarm constraint \( \gamma \) with equality. As we allow the LRT threshold \( \eta \) to vary, the corresponding \( (P_D, P_F) \) values trace out a continuous curve known as the receiver operating characteristic (ROC). For our low-noise asymptotic analysis, it will be more natural to focus on the miss probability defined as \( P_M = 1 - P_D \). We calculate the miss probability \( P_M \) and false alarm probability \( P_F \) using the low noise approximation developed previously for the Gaussian noise case.

First we condition on target \( \alpha_1 \) being present at a true pose \( \theta_1 \). We use a Laplace approximation for the likelihood function \( L_1(x) \) in (3.24) to derive the probability of miss given true pose \( \theta_1 \). In this section we will be more careful in applying the Laplace method.

We assume that the function \( E_1(\theta) = \|x_1(\theta) - x_1(\theta_1)\|^2 \) uniquely achieves its minimum value zero at \( \theta_1 \). For \( \theta_1 \) in the interior of the pose space, this implies using the Laplace method Eqn. (3.6). For \( \theta_1 \) on the boundary of the pose space, the correct approximation is one of the first-order expressions (3.8) or (3.9). Taking into account the location of the true pose \( \theta_1 \), the conditional miss probability is

\[ \Pr(c|H_1, \theta_1) \approx Q(\kappa), \quad \text{where} \quad \kappa = \frac{D}{2\sigma} + \frac{\sigma}{2D} \log \phi(\theta_1) - \frac{\sigma \eta}{D}, \]  

(3.25)

and

\[ \phi(\theta_1) = \begin{cases} 
\frac{4\pi \sigma^2 |\Theta|^2}{E_1(\theta_1)}, & \text{if } \theta_{\text{min}} < \theta_1 < \theta_{\text{max}}, \\
\frac{4\sigma^2 |\Theta|^2}{E_1(\theta_1)^2}, & \text{if } \theta_1 = \theta_{\text{min}} \text{ or } \theta_{\text{max}}.
\end{cases} \]  

(3.26)

Distance \( D \) is defined as the norm of the true data vector, \( D = \|x_1(\theta_1)\| \).

To compute the miss probability \( P_M \), we integrate (3.25) over the pose space \( \Theta \). We
follow the reasoning given previously in section 3.3.2 for the target recognition problem:
1. Approximate the integrand using the Q-function upper bound (3.17),

\[ P_M = \int_{\theta_{\min}}^{\theta_{\max}} \Pr(e|H_1, \theta_1) \pi(\theta_1)d\theta_1 \]
\[ \approx \frac{1}{|\Theta|} \int_{\theta_{\min}}^{\theta_{\max}} \frac{1}{\sqrt{2\pi}\kappa(\theta_1)} \exp \left( -\frac{r^2(\theta_1)}{2} \right) d\theta_1 \]
\[ \approx \frac{e^{\eta/2}}{|\Theta|} \sqrt{\frac{2\sigma^2}{\pi}} \int_{\theta_{\min}}^{\theta_{\max}} \frac{\phi(\theta_1)^{-1/4}}{D(\theta_1)} \exp \left( -\frac{D^2(\theta_1)}{8\sigma^2} \right) d\theta_1. \] (3.27)

The minimizing value of \( \theta_1 \) is defined as \( \theta^* = \arg\min_{\theta_1} D(\theta_1) \). It is the value of \( \theta_1 \) that minimizes the Euclidean norm of the target’s sensor data vector \( \mathbf{x}_1(\theta_1) \). Where \( \theta^* \) falls in the interval \([\theta_{\min}, \theta_{\max}]\) determines which version of the Laplace method we should use to evaluate (3.27). If \( \theta^* \) is achieved in the interior of the pose space, then using Eqn. (3.6) gives the low-noise miss probability

\[ P_M \approx \frac{e^{\eta/2} \pi^{-1/4} \sqrt{8|\Theta|^{-1}\sigma^3}}{D(\theta^*)^{3/2}} \left( \hat{E}_1(\theta^*, \theta^*) \right)^{1/4} \exp \left( -\frac{D^2(\theta^*)}{8\sigma^2} \right). \] (3.28)

Note that the LRT threshold \( \eta \) affects the miss probability through the factor \( e^{\eta/2} \). In the next chapter, we will look at a case where \( \theta^* \) is achieved on the upper boundary of the pose space \( \Theta = [\theta_{\min}, \theta_{\max}] \). For this situation, the first-order Laplace method (3.9) gives a different coefficient term,

\[ P_M \approx -\frac{e^{\eta/2} \pi^{-3/4} \sqrt{16|\Theta|^{-1}\sigma^6}}{D(\theta_{\max})^2 D(\theta_{\max})} \left| \hat{E}_1(\theta_{\max}, \theta_{\max}) \right|^{1/2} \exp \left( -\frac{D^2(\theta_{\max})}{8\sigma^2} \right). \] (3.29)

The minus sign in this expression is due to the negative first derivative \( \dot{D}(\theta_{\max}) \). One technical concern in deriving (3.29) involves the discontinuity of \( \phi(\theta_1) \) at the upper boundary \( \theta_1 = \theta_{\max} \). However, we know that the conditional miss probability \( \Pr(e|H_1, \theta_1) \) should be continuous as a function of \( \theta_1 \). This suggests that \( \phi(\theta_1) \), defined for the low-noise approximation of \( \Pr(e|H_1, \theta_1) \), might behave reasonably well in the vicinity of \( \theta_1 = \pi/2 \) for low values of \( \sigma^2 \).

Now let \( H_0 \) be the true hypothesis. In this case, the function \( E_1(\theta) \) is defined as

\[ E_1(\theta) = \|\mathbf{x}_1(\theta) - \mathbf{0}\| = \|\mathbf{x}_1(\theta)\|^2. \] The confuser pose \( c \) minimizes \( E_1(\theta) \), the squared
Table 3.1: Notation used in section 3.3.3 on target detection.

Euclidean distance from $x_1(\theta)$ to the zero vector. In other words, just like $\theta^*$ above, $c$ minimizes the Euclidean norm of the sensor vector $x_1(\theta)$. Using the Laplace method of Eqn. (3.6) to approximate $L_1(x)$ gives the following expression for the probability of false alarm $Pr(e|H_0)$:

$$P_F \approx Q(\kappa) \quad \text{where} \quad \kappa = \frac{D}{2\sigma} - \frac{\sigma}{2D} \log \frac{4\pi \sigma^2 |\Theta|^2}{E_1(c)} + \frac{\sigma \eta}{D}. \quad (3.30)$$

Again, we also provide a low-noise approximation for $P_F$ when $c$ is achieved on the upper boundary $\theta_{max}$. A first-order approximation for the likelihood function $L_1(x)$ gives a result that will be useful in the next chapter:

$$P_F \approx Q(\kappa) \quad \text{where} \quad \kappa = \frac{D}{2\sigma} - \frac{\sigma}{D} \log \frac{-2\sigma^2 |\Theta|^{-1}}{E_1(c)} + \frac{\sigma \eta}{D}. \quad (3.31)$$

Asymmetry between hypotheses $H_0$ and $H_1$ makes the notation slightly confusing in the detection scenario. For this reason, Table 3.1 provides a summary of the notation used in this section.
Chapter 4

FLIR-Based Recognition

Under our uniformity assumptions, the statistical model for the forward-looking infrared sensor takes on a relatively simple form compared to the laser radar. Hence we first develop results for recognition based on FLIR data.

4.1 Blocks-world Target Modeling

In target recognition performance analysis it is common to perform numerical simulations on target CAD models. In place of CAD modeling, we will simplify analytical calculations by modeling targets with simple geometric building blocks, cf. [5]. We attempt to capture the essential features of the target while making computation more tractable. For example a tank’s main sections, i.e., its body, turret, and barrel, are each modeled as parallelepiped rectangular blocks with given lengths, widths, and heights, as seen in Fig. 4-1.

![Figure 4-1: Blocks-world tank model. The axis of rotation is taken as the center of the body block. Side view (left, θ = 0), front view (right, θ = π/2).](image)

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4.2 FLIR Analysis

A target recognition system based on a forward-looking infrared sensor recognizes targets using a thermal profile of the field of view. A description of the FLIR's statistical model was given in Chapter 2. The low-noise analysis under the assumption of Gaussian noise can be applied directly to the FLIR.

4.2.1 Conditional Probability of Error - FLIR

For the binary recognition problem of deciding between targets \( \alpha_1 \) and \( \alpha_2 \), let us first consider the conditional error probability. Suppose \( \alpha_1 \) is the true target and that \( \theta_1 \) is its orientation. From Chapter 2, the statistical model for the FLIR’s sensor data was given as \( x = \sqrt{\text{SNR}} F(\theta) + n \), where \( F(\theta) \) is the vector of fractional values and \( n \) is distributed as \( N(0, I) \). Define \( D_F = \frac{D}{\sqrt{\text{SNR}}} = \arg \min_\theta \| F(\theta) - F_1(\theta_1) \| \). The normalized distance \( D_F \) represents the underlying geometric difference between the targets \( \alpha_1 \) and \( \alpha_2 \). The results for Gaussian noise in Eqns. (3.14) and (3.15) can be applied using variance \( \sigma^2 = 1 \) and \( D = D_F \sqrt{\text{SNR}} \) when SNR is sufficiently large. The asymptotic conditional probability of error for the FLIR model is then

\[
\Pr(e|H_1, \theta_1) \approx Q(\kappa)
\]

(4.1)

where

\[
\kappa = \frac{D_F \sqrt{\text{SNR}}}{2} - \frac{1}{2D_F \sqrt{\text{SNR}}} \log \frac{\bar{E}_1(\theta_1)}{\bar{E}_2(c)} .
\]

(4.2)

Approximating the Gaussian Q-function with the upper bound (3.17), we see that in the high SNR regime, the conditional error probability decays exponentially as \( \frac{1}{\sqrt{\text{SNR}}} \exp \left( -\frac{\text{SNR}}{8} D_F^2 \right) \).

To apply these results, recognition was simulated using blocks-world target models described in the previous section. We took the pose space to be the interval \( \Theta = [0, \pi/2] \), which allows the target model to rotate over a quarter-circle. To compute the conditional probability of error \( Q(\kappa) \), we see that we need to solve a minimization problem to get the confuser pose \( c \). A search for the minimum distance is performed over the interval \([0, \pi/2]\) with some small spacing \( \Delta \theta \). To find \( c \) to within a few percent is not a time-consuming computation; however in the next section a more time-consuming minimization problem
Figure 4-2: Blocks-world target geometry setup in numerical simulation of FLIR-based recognition. The blocks Body$_k$, Turret$_k$, Barrel$_k$ comprise a three block tank model of target $\alpha_k$, $k = 1, 2$.

Numerical simulations were run with two blocks-world tank models of differing dimensions, denoted as $\alpha_1$ and $\alpha_2$. These models included the three main sections, the body, turret, and barrel. The dimensions of the simulated target block models are summarized in Fig. 4-2. A pixel size of side length $\Delta = 0.6$ block units was used. Tank block model $\alpha_1$ was chosen as the true target and fixed at pose orientation $\theta_1 = 0.8$ radians. Numerical simulations were run to estimate the conditional probability of error of choosing the incorrect target $\alpha_2$. Trapezoidal integration was used to calculate the pose integral needed for the ML decision rule. To calculate the low-noise result (4.1), we found the confuser pose $c = 0.957$ radians and the minimum distance $D_F = 1.40$.

Figure 4-3 displays a log-log plot of the binary recognition conditional probability of error. Each data point represents 2000 trials of Monte Carlo simulation, and the graphed line shows our low-noise approximation of the conditional error-probability. Our analytical approximation closely fits the numerical simulation results for large values of SNR.

4.2.2 Unconditional Probability of Error - FLIR

Assuming a uniform prior density $\pi(\theta_1)$ for the pose parameter, our asymptotic expression for the conditional probability of error can be integrated over the pose space $\Theta = [0, \pi/2]$. A second Laplace approximation is used to evaluate this integral, which from Eqn. (3.22) gives

$$\Pr(c|H_1) \approx \frac{4(2/\pi)}{D_F(\theta^*)^{3/2}\sqrt{D_F(\theta^*)}} \left( \frac{E_1(\theta^*)}{E_2(c(\theta^*), \theta^*)} \right)^{1/4} \frac{1}{\text{SNR}} \exp \left( -\frac{\text{SNR}}{8} D_F^2(\theta^*) \right).$$ (4.3)
Figure 4-3: FLIR low-noise approximation for conditional probability of error $Pr(e|H_1, \theta_1)$ at true pose $\theta_1 = 0.8$ radians.
The value of $\sqrt{\text{SNR}} D_F(\theta^*)$ is the minimum Euclidean distance between the target vectors $\mathbf{x}_1(\theta)$ and $\mathbf{x}_2(\theta)$ over all possible orientations:

$$\sqrt{\text{SNR}} D_F(\theta^*) = \min_{\theta', \theta'' \in \Theta} \| \mathbf{x}_1(\theta') - \mathbf{x}_2(\theta'') \|.$$  

(4.4)

Exchanging the roles of the true target and confuser, the symmetry of $D_F(\theta^*)$ between the targets $\alpha_1$ and $\alpha_2$ implies that conditioned on $\alpha_2$ being the true target, the probability of error $\Pr(e|H_2)$ has the same exponential decay rate as $\Pr(e|H_1)$. Moreover, this means that the completely unconditional recognition probability of error $\Pr(e)$ also has this decay rate:

$$\frac{1}{\text{SNR}} \exp\left(-\frac{\text{SNR}}{8} D^2_F(\theta^*)\right).$$

To find the coefficient factors in the asymptotic expression for the unconditional probability of error in Eqn. (4.3), not only do we need the minimizing value $\theta^*$, we also have to compute the second derivative term $\hat{D}_F(\theta^*)$. Since this second derivative cannot be computed analytically, we instead compute samples of the confuser pose $c(\theta_1)$ in an interval near the minimizer $\theta^*$ and use a quadratic fit to estimate the second derivative $\hat{D}_F(\theta^*)$.

Let the pose space be the interval $\Theta = [0, \pi/2]$. To compute the confuser pose $c(\theta_1)$ in the unconditional case requires searching over both the true pose and the incorrect target’s pose. In addition, it is necessary to find $c(\theta_1)$ very accurately in the vicinity near $\theta^*$ to perform an accurate quadratic fit for the estimation described above. A straightforward search over all poses at the required accuracy is very time-consuming. To help overcome this problem, an efficient coarse-to-fine algorithm was used. An initial search using some spacing $\Delta \theta$ was done over $[0, \pi/2]$. The minimized squared distance $E(\theta, \theta_1)$ was computed for $\theta_1$ at each multiple of $\Delta \theta$ in $[0, \pi/2]$ searching over multiples of $\Delta \theta$ of the incorrect target. Then successive searches were performed, each time halving the spacing $\Delta \theta/2$, and using the results of the previous search as seed values for the new search.

Again we used the two blocks-world tank models $\alpha_1$ and $\alpha_2$, whose parameters are given in Fig. 4-2, and performed numerical simulations to compute recognition performance. We chose $\alpha_1$ to be the true target. To estimate the probability of error $\Pr(e|H_1)$, we took $N = 60$ equally spaced samples $\{\theta_i\}_{i=1}^N$ from the pose space $[0, \pi/2]$ and for each sample performed $K = 50$ Monte Carlo simulation trials. Our estimator in this simulation was

$$\hat{P}_e = \frac{1}{N} \sum_{i=1}^N \frac{1}{K} \sum_{k=1}^K \tau_k(\theta_i).$$  

(4.5)
where

\[ \tau_k(\theta_i) = \begin{cases} 1, & \text{if } k\text{-th trial at pose sample } i \text{ is an error,} \\ 0, & \text{otherwise.} \end{cases} \]  

The expectation and variance of our estimator \( \hat{P}_e \) are thus given by

\[
E[\hat{P}_e] = \frac{1}{NK} \sum_{i=1}^{N} \sum_{k=1}^{K} \Pr(e|H_1, \theta_1) \\
= \frac{1}{N} \sum_{i=1}^{N} \Pr(e|H_1, \theta_1) \approx \Pr(e|H_1),
\]

and

\[
\text{Var}[\hat{P}_e] = \frac{1}{N^2K^2} \sum_{i=1}^{N} \sum_{k=1}^{K} \Pr(e|H_1, \theta_1)(1 - \Pr(e|H_1, \theta_1)) \\
= \frac{1}{N^2K} \sum_{i=1}^{N} \Pr(e|H_1, \theta_1)(1 - \Pr(e|H_1, \theta_1)) \approx \frac{1}{4NK},
\]

respectively. Thus our estimator \( \hat{P}_e \) is approximately unbiased for large values of \( N \) and has a small variance when the product \( NK \) is large. This shows that we can get a good estimate from a moderately large number of trials \( K \) per sample. Numerical simulations were performed to compute our estimator \( \hat{P}_e \) with the parameters \( N = 60 \) and \( K = 50 \). Trapezoidal integration was used to compute the pose integral involved in the decision rule.

Figure 4-4 is a log-log plot of the unconditional probability of error versus SNR. The data points represent the value of our estimator at different values of SNR. The minimizer pose orientation \( \theta^* = 1.2597 \text{ radians} \) was found using a coarse-to-fine algorithm, and a quadratic polynomial fit was used to estimate the second derivative \( \partial^2 P(\theta^*) = 695.62 \). Using these computed parameters, we graphed the low-noise approximation to \( \Pr(e|H_1) \) from Eqn. (4.3), which is the top curve in Fig. 4-4. It can be seen that there is a discrepancy between this low-noise approximation and the Monte-Carlo computation of the unconditional probability of error. To help explain the difference, we numerically integrated both the con-
Figure 4-4: FLIR low-noise approximation for unconditional probability of error $Pr(e|H_1)$ with target geometry shown in Table 4-2. The top curve is computed from the Laplace approximation (4.3), the middle curve is a numerical integration of the Q-function upper bound (3.18), and the bottom curve is a numerical integration of the Q-function.
ditional probability of error Q-function (4.1), and the upper bound approximation to the Q-function (3.18). These numerical integrations are shown in Fig. 4-4 as dashed curves. The Q-function integration shows that averaging the conditional probability of error Q-function over the pose space gives an accurate approximation of the unconditional error-probability. The upper-bound integration shows that our second Laplace approximation is quite accurate for large SNR. We conclude that the Laplace method slightly overestimates the coefficient term for Pr(e|H₁) because of looseness in our Q-function upper-bound approximation.

It is possible to correct the coefficient term by utilizing tighter Q-function approximations. The bounds given in (3.16) are the first of a sequence of upper and lower bounds derived from taking the Taylor expansion:

\[ Q(\kappa) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\kappa^2}{2}\right) \int_0^\infty \exp\left(-t\kappa - t^2/2\right)dt \]

\[ = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\kappa^2}{2}\right) \left\{ \frac{1}{\kappa} - \frac{1}{\kappa^3} + \frac{1 \cdot 3}{\kappa^5} - \frac{1 \cdot 3 \cdot 5}{\kappa^7} + \cdots \right\}. \] (4.9)

The bounds obtained by using more terms in this expansion, although more accurate for large \( \kappa \), are looser than (3.16) for smaller values of \( \kappa \). Various other Q-function bounds are discussed in [17], including the upper bound

\[ Q(\kappa) \leq \sqrt{\frac{\pi}{32}} \left( \sqrt{\kappa^2 + \frac{8}{\pi}} - \kappa \right) \exp\left(-\frac{\kappa^2}{2}\right). \] (4.10)

As shown in Fig. 4-5, this upper bound is tight at small values of \( \kappa \) and converges to the correct value at \( \kappa = 0 \). We calculated the Laplace approximation for the unconditional error-probability Pr(e|H₁) using this alternative Q-function approximation. In Fig. 4-6, we show the improved results for our example using an adjusted coefficient based on the approximation (4.10). Despite this improvement, the original Laplace method expression we gave in (4.3) has a simpler form and provides accurate exponential decay behavior. For this reason, it is easier to use our original expression to interpret the asymptotic behavior of the unconditional probability of error.

4.2.3 FLIR Recognition Generalizations

Some extensions of the problem studied in the previous section are now considered. Some modifications are involved when we consider a more general pose space and when some of
Figure 4-5: A comparison of our Q-function upper bounds (3.17) and (4.10). We have plotted the ratio of the bounds to the value of $Q(\kappa)$. The new upper bound (4.10) provides a good approximation to the Gaussian Q-function at small values of $\kappa$. 

Figure 4-5: A comparison of our Q-function upper bounds (3.17) and (4.10). We have plotted the ratio of the bounds to the value of $Q(\kappa)$. The new upper bound (4.10) provides a good approximation to the Gaussian Q-function at small values of $\kappa$.
Figure 4-6: FLIR low-noise approximation for unconditional probability of error $Pr(e|H_1)$. The dashed line is computed from the Laplace approximation using an adjusted coefficient based on (4.10).
the assumptions that were used to derive the Laplace approximation in Eqn. (4.3) do not hold. Also, target detection is looked at as a special case of the recognition problem, and recognition is extended to the M-ary hypothesis case with $M > 2$.

**General Pose Orientation**

It is more realistic to allow the target to rotate over more than just the quarter-circle $\Theta = [0, \pi/2]$, as we have been thus far considering. For the simple block models that we have been using, a more general pose space introduces symmetry that forces us to be more careful in applying the Laplace approximation.

When the confuser pose is achieved at more than one point, finding a correct approximation to the likelihood ratio requires looking at the squared-distance functions $E_k(\theta, \theta_1)$. To illustrate the idea, let the pose space be the half-circle $\Theta = [0, \pi]$ and consider the target geometry setup in Fig. 4-2. For target $\alpha_1$, the barrel block is obscured by the turret block at orientations $\theta_1 \in (1.27, \pi - 1.27)$. Since a barrel-less tank looks the same to a FLIR sensor at angles symmetrical about $\pi/2$, the confuser pose will be achieved at two distinct points. Figure 4-7 shows the squared-distance functions $E_1(\theta, \theta_1)$ and $E_2(\theta, \theta_1)$ for a fixed true pose $\theta_1 = 1.3$ radians at which the barrel block is not visible to the FLIR sensor.

The likelihood functions can be approximated with the Laplace method by adding contributions from the two confuser poses, cf. Eqn. (3.7).

\[
L_2(x) = \int_0^{\pi} p(x|H_2, \theta)\pi(\theta) d\theta
\]

\[
\approx \frac{1}{(2\pi)^{1/2}} \left( \exp\left( -\frac{1}{2} \|x_1(\theta_1) + n - x_2(c)\|^2 \right) \sqrt{\frac{2\pi}{E_2(c)}} 
+ \exp\left( -\frac{1}{2} \|x_1(\theta_1) + n - x_2(\pi - c)\|^2 \right) \sqrt{\frac{2\pi}{E_2(\pi - c)}} \right)
\]

\[
= \frac{1}{(2\pi)^{1/2}} \left( \exp\left( -\frac{1}{2} \|x_1(\theta_1) + n - x_2(c)\|^2 \right) 
+ \exp\left( -\frac{1}{2} \|x_1(\theta_1) + n - x_2(\pi - c)\|^2 \right) \right) \sqrt{\frac{2\pi}{E_2(c)}},
\]  

(4.11)
and similarly,

\[ L_1(x) = \int_0^\pi p(x|H_1, \theta) \pi(\theta) \, d\theta \]

\[ = \frac{2/\pi}{(2\pi)^{1/2}} \exp \left( -\frac{1}{2} \|n\|^2 \right) \sqrt{\frac{2\pi}{E_1(\theta_1)}}. \] (4.12)

The equality \( \tilde{E}_k(\theta) = \tilde{E}_k(\pi - \theta) \) was used to factor out the square root terms above. This holds because second derivatives of even functions are themselves even, and \( E_k(\theta) \) is even about \( \theta = \pi/2 \). Dividing the expressions in (4.11) and (4.12) gives the likelihood ratio,

\[ \log \frac{L_2(x)}{L_1(x)} \approx D_F \sqrt{\text{SNR}} \left( -\frac{D_F \sqrt{\text{SNR}}}{2} + z \right) + \frac{1}{2} \log \frac{\tilde{E}_1(\theta_1)}{\tilde{E}_2(c)}, \] (4.13)

which is identical to the expression used for the case \( \Theta = [0, \pi/2] \). The effect of the extra contributions cancels out in the ratio, so we can simply choose the confuser pose \( c \) in the range \([0, \pi/2]\) to make our computations as before.

For unconditional binary recognition, the distance function \( D_F(\theta_1) \) is computed over the interval \([0, \pi]\). A plot is shown in Fig. 4-8. Note that the symmetry of the rectangular blocks in our target model implies that the minimum distance function \( D_F(\theta_1) \) will be symmetrical about \( \theta_1 = \pi/2 \). The confuser pose is two-valued when the true pose is such that the barrel block is obscured by the turret block. Equation (4.13) shows that the expression for the conditional probability of error is the same in this case, so this is not troublesome.

To compute the unconditional probability of error, we add terms of the form given in (3.22). The distance function \( D_F(\theta_1) \) achieves its minimum at two distinct values \( \theta^* \) and \( \pi - \theta^* \), so it is necessary to add both of these contributions. Using the fact that \( D_F(\theta_1), D(\theta_1), \) and \( \tilde{E}_k \) are all symmetric about \( \theta_1 = \pi/2 \), we have

\[ \Pr(e|H_1) \approx -\frac{8(\frac{1}{\pi})}{D_F(\theta_1)^{3/2}} \left( \frac{\tilde{E}_1(\theta^*, \theta^*)}{E_2(c(\theta^*), \theta^*)} \right)^{1/4} \frac{1}{\text{SNR}} \exp \left( -\frac{\text{SNR}}{8} D_F^2(\theta^*) \right). \] (4.14)

Comparing Eqns. (4.3) and (4.14) shows that the unconditional probability of error is the same for both quarter-circle and half-circle turns. This makes sense since the tank block-models \( \alpha_1 \) and \( \alpha_2 \) are both symmetric about \( \theta_1 = \pi/2 \).
Figure 4-7: Squared distance functions $E_1(\theta)$ and $E_2(\theta)$ at true pose $\theta_1 = 1.3$ radians. The lower graph shows that there are two confuser poses, 1.40 and $\pi - 1.40$ radians. The confuser distance $D = 0.29$ which is difficult to see from the scale on this plot.
Minimization on Boundary

Another situation that requires more careful application of the Laplace approximation is when the confuser pose \( c \) is not achieved strictly within the pose space. When the confuser pose is taken on a boundary, the first derivative of \( E_2(\theta) \) will not in general be zero, and the Taylor series used to derive our main Laplace result Eqn. (3.6) is not valid. Suppose the function \( E_2(\theta) \) is minimized at the boundary value \( \theta = 0 \), i.e., the confuser pose \( c = 0 \). The alternative result Eqn. (3.8) given for this situation in Chapter 3 can be applied. Our low-noise approximation for the incorrect target's likelihood function is

\[
L_2(x) \approx \frac{2\Theta^{-1}}{(2\pi)^{1/2}E_2(0)} \exp\left(-\frac{1}{2}||x_1(\theta_1) - x_2(0) + n||^2\right). \tag{4.15}
\]

Using the usual Laplace approximation for the likelihood function \( L_1(x) \), the asymptotic log-likelihood ratio is

\[
l(x) \approx D_F\sqrt{\text{SNR}} \left( -\frac{D_F\sqrt{\text{SNR}}}{2} + z \right) + \frac{1}{2} \log \frac{\tilde{E}_1(\theta_1)}{\pi E_2(0)^2}, \tag{4.16}
\]

where \( z \) is a standard Gaussian random variable. It follows that the conditional probability
Figure 4-9: Blocks-world target geometry setup in numerical simulation of FLIR-based recognition. Example of first-order Laplace approximation.

\[
\Pr(e|H_1, \theta_1) \approx Q(\kappa) \quad \text{where} \quad \kappa = \frac{D_F \sqrt{\text{SNR}}}{2} - \frac{1}{2D_F \sqrt{\text{SNR}}} \log \frac{\hat{E}_1(\theta_1)}{\pi \hat{E}_2(0)^2}. \tag{4.17}
\]

With the target geometry setup shown in Fig. 4-9 and a pixel size \( \Delta = 0.5 \), the confuser pose \( c \) is taken on the boundary of the pose space \( \Theta = [0, \pi/2] \) when \( \alpha_1 \) is the true target at pose \( \theta_1 = 0.92 \) radians. A plot of the distance function \( \hat{E}_2(\theta) \) is given in Fig. 4-10. Simulations using 2000 trials for each value of SNR were performed to find the conditional probability of error, and the first-order Laplace method analytical expression (4.17) was computed. These results are compared in Fig. 4-11, showing convergence at higher values of SNR.

**FLIR Target Detection**

In Chapter 3, the target detection scenario was treated as a special case of the target recognition problem. An additional hypothesis \( H_0 \) was introduced that represented the absence of any targets. The case of detection with Gaussian noise was treated in section 3.3.3, and the results derived there can be applied directly to the FLIR system.

We attempt to approximate the ROC curves for varying LRT threshold \( \eta \). The results derived in section 3.3.3 will be applied here using the FLIR parameters \( \sigma = 1 \) and \( D = D_F \sqrt{\text{SNR}} \). Under hypothesis \( H_0 \), we need the value of the confuser pose \( c \), and under \( H_1 \) we need to compute \( \theta^* \). Both of these parameters have the same interpretation as the orientation that minimizes the Euclidean norm \( ||x_1(\cdot)|| \) of the target \( \alpha_1 \). The target model described in Table 4-12 has dimensions such that the block depths are smaller than their widths. Thus to minimize the amount of target surface area facing the FLIR sensor requires
Figure 4-10: Squared-distance function $E_2(\theta)$ for target geometry in Table 4-9 and true pose $\theta_1 = 0.92$ radians. The minimum is achieved at the lower boundary, which implies that the confuser pose $c = 0$. 

\[ \text{Squared-distance function } E_2(\theta) \] 

\[ \begin{array}{c|c|c|c|c|c} \hline \theta \text{ radians} & 0 & \pi/8 & \pi/4 & 3\pi/8 & \pi/2 \\ \hline \text{squared-distance} & 0 & 5 & 10 & 15 & 30 \\ \hline \end{array} \]
Figure 4-11: FLIR low-noise approximation for conditional probability of error $\Pr(e|H_1, \theta_1)$ at true pose $\theta_1 = 0.92$ radians. The analytical curve is based on a first-order Laplace approximation expressed in Eqn. (4.17).
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Figure 4-12: Two-block target model $\alpha_1$ for FLIR detection. The axis of rotation is taken as the center of the body block. Side view (left, $\theta = 0$), front view (right, $\theta = \pi/2$).

Turning $\theta = \pi/2$ so that the smaller side is toward the sensor. To handle the case when $\theta^*$ and $c$ are achieved on the upper boundary, we use the first-order results from Eqns. (3.29) and (3.31). From (3.29), the low-noise miss probability is

$$P_M \approx -\frac{4\sqrt{2}e^{\eta/2\pi}\pi^{-5/4}}{D_F(\pi/2)^2D_F(\pi/2)}\left|\tilde{E}_1(\pi/2, \pi/2)\right|^{1/2}\frac{1}{\text{SNR}^{3/2}} \exp\left(-\frac{\text{SNR}}{8}D_F^2(\pi/2)\right).$$

(4.18)

and from (3.31), the low-noise false-alarm probability is

$$P_F \approx Q(\kappa) \quad \text{where} \quad \kappa = \frac{D_F\sqrt{\text{SNR}}}{2} - \frac{1}{D_F\sqrt{\text{SNR}}} \log \left(\frac{-4/\pi}{\tilde{E}_1(\pi/2)}\right) + \frac{\eta}{D_F\sqrt{\text{SNR}}}.$$  

(4.19)

Detection was simulated with a two-block target model versus a uniform background. Figure 4-12 shows a picture and the dimensions of target $\alpha_1$. The pixel size used was $\Delta = 0.5$ block units. The pose space was taken to be $[0, \pi/2]$ where the axis of rotation is the center of the lower body block. We performed Monte Carlo simulations to compute the ROC curve at $\text{SNR} = 5$. In Fig. 4-13 the results of these computations are compared with the low-noise analytical ROC curves determined from (4.18) and (4.19). It can be seen that the Laplace method ROC curve lies above the Monte Carlo computed values. This can be explained in part from the second Laplace approximation necessary to compute the unconditional miss probability $P_M$. As was illustrated in Fig. 4-4, the upper bound Q-function approximation can overestimate the coefficient term for moderate values of SNR. For large enough levels of SNR, the Laplace method ROC curve should converge to the correct values.
Figure 4-13: FLIR detection of two-block target model with geometry shown in Fig. 4-12. Plot shows a comparison of Laplace method derived ROC curve with Monte Carlo computation at SNR = 5.
M-ary Target Recognition

The results developed to this point have focused on the binary recognition problem of deciding which of two targets $\alpha_1$ and $\alpha_2$ is present in the sensor’s field of view. In this section we extend these results to the case where we have a library of possible targets $\mathcal{L} = \{\alpha_1, ..., \alpha_M\}$ which could be present in our recognition problem.

Let $\alpha_1$ be the true target at a given pose orientation $\theta_1$. The Bayesian ML rule for minimizing probability of error says to choose hypothesis $H_i$ if the likelihood function $L_i(x) > L_j(x)$ for all $j \neq i$. This rule is equivalent to a sequence of likelihood ratio tests $L_j(x)/L_1(x)$, $j = 2, ..., M$, where we eliminate hypothesis $H_j$ if the $j$-th ratio is less than one and $H_1$ if the ratio is greater than one. From our results on binary recognition, we know that for large SNR, the log-likelihood ratio can be approximated by a Gaussian random variable. The $j$-th likelihood ratio test is

$$
H_j
\begin{align*}
\frac{z_j}{\kappa_j} \\
H_1
\end{align*}
$$

where

$$z_j = \frac{(F_j(c_j) - F_1(\theta_1))T n}{DF_{j}}
$$

and

$$
\kappa_j = \frac{DF_{j}}{2} \frac{\log \frac{E_1(\theta_1)}{E_j(c_j)}}{\text{SNR}}
$$

Here $c_j$ is the confuser pose and $DF_{j}$ is the normalized confuser distance of the $j$-th target. We collect the zero-mean, unity-variance Gaussian random variables $z_j$ into a zero-mean Gaussian vector

$$z = [z_2 z_3 ... z_M]^T
$$
with covariance matrix $K = (K_{ij})$ where $K_{ii} = 1$ and for $i \neq j$

$$K_{ij} = \frac{(F_1(\theta_1) - F_i(c_i))^{T}(F_1(\theta_1) - F_j(c_j))}{DF_i DF_j}$$  \hspace{1cm} (4.24)

The ML rule chooses the correct hypothesis $H_1$ when each likelihood ratio test in (4.20) decides on hypothesis $H_1$, so the event of a correct decision can be expressed as the intersection $A = \bigcap_{j=2}^{M}[z_j < \kappa_j]$. Thus the asymptotic conditional probability of error given $H_1$ and $\theta_1$ is: [15]

$$\Pr(e | H_1, \theta_1) \approx 1 - \frac{1}{(2\pi)^{M-1}|K|^{1/2}} \int_A \exp \left( -\frac{1}{2} z^{T} K^{-1} z \right) dz.$$ \hspace{1cm} (4.25)

To simulate multiple target recognition, we created four different blocks-world target models $L = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and fixed $\alpha_1$ as the true target at pose $\theta_1 = \pi/8$. The shapes and dimensions of the target models used for this example are shown in Fig. 4-14. A pixel of side length $\Delta = 0.5$ was used. The low-noise approximation (4.25) was computed numerically using the trapezoidal method. Monte Carlo simulations were also performed using 1000 trials per SNR sample. The results of the simulations and approximations are shown in Fig. 4-15. A comparison of these results shows good agreement.

### 4.3 Convergence Issues

We have used low-noise approximation techniques in this research to gain understanding of recognition performance. The usefulness of these results rely on the asymptotic convergence of the pose integrals we computed for the Bayesian decision rule. At least for the various examples described for FLIR recognition in this chapter, it seems for reasonable values of SNR that the error-probability can be computed quite accurately from the Laplace approximation method. We examine our approximations more carefully in this section to understand the factors that determine sufficient levels of SNR for good convergence.

First we look at convergence of the likelihood function $L_i(x)$ where $x$ is FLIR sensor
Figure 4-14: Four blocks-world target models \{a_1, a_2, a_3, a_4\} for FLIR M-ary recognition. For targets \(a_1, a_2,\) and \(a_4,\) the axis of rotation is the center of the body block. For target \(a_3,\) the axis of rotation is the center of the load block. Due to the tedious calculation required, the tire blocks of targets \(a_3\) and \(a_4\) were not “rotated”.

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Figure 4-15: FLIR low-noise approximation for M-ary conditional probability of error $\Pr(e|H_1, \theta_1)$ at true pose $\theta_1 = \pi/8$ radians.
data. By the Laplace approximation in Eqn. (3.6),

\[ L_i(x) = \int_{\theta} p(x|H_i, \theta) \pi(\theta) \, d\theta \]

\[ \approx \frac{|\theta|^{-1}}{(2\pi)^{J/2}} \exp \left( -\frac{1}{2} \|x - x_i(\hat{\theta}_i)\|_2^2 \right) \sqrt{\frac{4\pi}{d^2 \|x - x_i(\hat{\theta}_i)\|_2^2}}, \tag{4.26} \]

where for a given sample value of \( x \), we have defined \( \hat{\theta}_i = \arg\min_{\theta} \|x - x_i(\theta)\|_2^2 \). Denote the approximation in (4.26) by \( \hat{L}_i(x) \). We note that \( \hat{\theta}_1 \) is the maximum-likelihood estimate of the parameter \( \theta_1 \) and is known to converge to the true value as the noise level goes to zero.

In Chapter 3 we gave a derivation for the Laplace approximation, but did not give precise statements of the convergence of the method. It can be shown using Taylor series expansions that the Laplace method gives the following approximation:

\[ F(\sigma^2) = \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} g(\theta) \exp \left( -\frac{h(\theta)}{2\sigma^2} \right) d\theta \]

\[ = g(\theta_0) \exp \left( -\frac{h(\theta_0)}{2\sigma^2} \right) \sqrt{\frac{4\pi\sigma^2}{h''(\theta_0)}} (1 + a\sigma^2 + b\sigma^4 + \ldots) \]

\[ = g(\theta_0) \exp \left( -\frac{h(\theta_0)}{2\sigma^2} \right) \sqrt{\frac{4\pi\sigma^2}{h''(\theta_0)}} (1 + O(\sigma^2)), \tag{4.27} \]

where, from [12],

\[ a = -\frac{h^{(4)}(\theta_0)}{4h''(\theta_0)} + \frac{5h^{(3)}(\theta_0)^2}{12h''(\theta_0)^3} + \frac{g''(\theta_0)}{g(\theta_0)h''(\theta_0)} - \frac{g'(\theta_0)h^{(3)}(\theta_0)}{g(\theta_0)h''(\theta_0)^2} \tag{4.28} \]

is the coefficient of the first order term in the series above. Equation (4.27) provides the rate of convergence of the (nonprobabilistic) integral approximation in (4.26) and of that used to evaluate the unconditional probability of error (3.22). This shows for example that the likelihood function approximation (4.26) is \( L_i(x) = \hat{L}_i(x)(1 + O(\sigma^2)) \). In [15] it is shown that for \( C^2 \) functions \( f(\cdot) \), we have the probabilistic convergence result \( f(\hat{\theta}) = f(\theta) + O_p(\sigma^2) \). Thus we can use convergence of the ML estimate to approximate functions, such as the LRT, by evaluating them at the true value of \( \theta \).

The results stated above formally show that our low noise approximations are correct in an asymptotic sense and provide rates of convergence. We can also look at the convergence of the Laplace method specifically for the FLIR sensor model to determine at what levels
of SNR we can expect good results. Conditioned on $H_1$ and true pose $\theta_1$, the ML pose estimate for hypothesis $H_i$ is

$$\hat{\theta}_i = \arg\min_{\theta} \| \mathbf{F}_1(\theta_1) - \mathbf{F}_i(\theta) + \frac{\mathbf{n}}{\sqrt{\text{SNR}}} \|^2.$$  

(4.29)

Here $\mathbf{n}$ is a standard white Gaussian vector $\sim \mathcal{N}(0, \mathbf{I})$. In the limit $\text{SNR} \to \infty$, the noise term above is negligible so that $\hat{\theta}_i = \theta_1$ and $\hat{\theta}_i = c_i$, for $i \neq 1$. This is one of the approximations we assumed in deriving the main results for FLIR recognition in section 4.2. From Eqn. (4.29), we see that the relative magnitude between $\mathbf{F}_i$ and $\sqrt{\text{SNR}}$ will affect the convergence of $\hat{\theta}_i$. Since $\mathbf{F}_i$ is a fraction between zero and one and $\mathbf{n}$ takes values on the order of one, we would expect that SNR values less than one will cause the noise to dominate the fractional $\mathbf{F}_i$ values. Looking at our example in Fig. 4-3, we have good convergence for the low noise approximation of $\Pr(e|H_1, \theta_1)$ starting at $\text{SNR} = 0.5$. Convergence in this case appears to be better than expected. In our example of Fig. 4-11, our Laplace approximation shows good agreement starting from $\text{SNR} = 2$. In this example, the likelihood function $L_2(\mathbf{x})$ was computed with a first-order approximation, while $L_1(\mathbf{x})$ was evaluated with the usual Laplace method. The fact that we used different methods to evaluate each likelihood function could explain why a higher level of SNR was required to give us convergence to the true error-probability. In the first example, it is not surprising that we had better convergence results. Both likelihood functions were evaluated with the same Laplace method and the target models have similar geometry. It is possible that even if the likelihood function approximations were not accurate, the relative value of $\hat{L}_1(\mathbf{x})$ to $\hat{L}_2(\mathbf{x})$ tended to remain correct.

### 4.4 Resolution Dependence

We have shown that for large values of the FLIR’s single-pixel SNR, the probability of decision error decays exponentially with a rate dependent on the target geometry. For the conditional case, the decay rate is determined by the confuser distance $D$, which represents how much the incorrect target can resemble the correct target at a fixed pose. In the unconditional case, the decay rate is given by the minimum distance $D(\theta^*)$ between the targets over all possible pose orientations. Here we show that we can use these results to tie binary recognition performance to the FLIR’s resolution.
We measure the FLIR’s resolution by its pixel area $\Delta^2$, where $\Delta$ is the side length of a single pixel. When the pixel size is small compared with the target dimensions, most of the pixel values $F_{ij}(\theta)$ will have value either 0 or 1. The effect that pixels with value strictly within $0 < F_{ij}(\theta) < 1$ will become negligible when $\Delta$ is a few times smaller than the smallest target dimension. In practical situations, this condition might be valid for a tank’s body and turret, but not for its barrel. However, the reasoning here gives some insight into the FLIR’s resolution dependence.

The confuser pose is dependent on the sensor resolution $\Delta^2$. As $\Delta^2 \to 0$, we can consider the limiting situation of having continuous measurements. For continuous measurements, the incorrect target achieves the confuser pose by minimizing the difference in surface area presented to the FLIR sensor. Let $A$ be this minimized difference in surface area. For discrete pixels, the confuser pose does not equal that of the limiting case because a pixel with fraction $F_{ij}$ on-target contributes $F_{ij}\sqrt{\text{SNR}}$ to the Euclidean distance and not $\sqrt{F_{ij}\text{SNR}}$.

For small values of $\Delta^2$ this discrepancy does not have much of an effect because most of the pixels are either $F_{ij} = 0$ or $F_{ij} = 1$. As $\Delta^2 \to 0$, the confuser pose converges to the limiting case. This is the basis of the following approximation. Define the set of pixels $\mathcal{D}_\Delta = \{(i,j) : F_{2,ij}(c) \neq F_{1,ij}(\theta_1) \text{ at resolution } \Delta^2\}$. These are the pixels that differ between targets $\alpha_1$ and $\alpha_2$ when the incorrect target is at the confuser pose $c$. Let $|\mathcal{D}_\Delta|$ be the number of pixels in $\mathcal{D}_\Delta$. Then based on the above discussion, we have the approximation

$$D_{\hat{P}} = \sum_{ij} (F_{2,ij}(c) - F_{1,ij}(\theta_1))^2$$

$$= \sum_{ij \in \mathcal{D}_\Delta} (F_{2,ij}(c) - F_{1,ij}(\theta_1))^2$$

$$\approx \sum_{ij \in \mathcal{D}_\Delta} 1$$

$$= |\mathcal{D}_\Delta|$$

$$\approx \frac{A}{\Delta^2}. \quad (4.30)$$

We now use this result to see how recognition performance depends on the FLIR’s resolution. An approximation to the conditional probability of error Q-function was derived
in Eqn. (3.20). We rewrite this expression here in terms of the FLIR’s sensor parameters,

\[
\Pr(e|H_1, \theta_1) \approx \frac{\sqrt{2 \pi}}{D_F(\theta_1)\sqrt{\text{SNR}}} \left( \frac{\bar{E}_1(\theta_1, \theta_1)}{\bar{E}_2(c(\theta_1), \theta_1)} \right)^{1/4} \exp \left( -\frac{\text{SNR}}{8} D_F^2(\theta_1) \right).
\]  

(4.31)

We analyze this expression to distinguish the factors that depend on target geometry and those that are sensor parameters. Substituting Eqn. (4.30) into (4.31), the asymptotic error-probability is

\[
\Pr(e|H_1, \theta_1) \approx \frac{G(\theta_1)\Delta}{\sqrt{\text{SNR}}} \exp \left[ -\frac{\mathcal{A}(\theta_1)\text{SNR}}{8\Delta^2} \right],
\]

(4.32)

where

\[
G(\theta_1) = \sqrt{\frac{2}{\pi \mathcal{A}(\theta_1)}} \left( \frac{\bar{E}_1(\theta_1, \theta_1)}{\bar{E}_2(c(\theta_1), \theta_1)} \right)^{1/4}.
\]

(4.33)

Both of the second derivatives \( \bar{E}_1 \) and \( \bar{E}_2 \) are proportional to \( 1/\Delta^2 \). This dependence on resolution cancels in their ratio so that \( G(\theta_1) \) is a term that depends only on the target geometry. The area term \( \mathcal{A}(\theta_1) \) is written to explicitly show its dependence on the true pose.

We see that in the regime of low-noise and for pixel length \( \Delta \) on the order of the smallest target dimension, the FLIR’s conditional probability of error exponentially decays with improved resolution. Similarly, from Eqn. (3.22) we have an expression for the asymptotic unconditional probability of error

\[
\Pr(e|H_1) \approx \frac{\tilde{G}\Delta^2}{\text{SNR}} \exp \left[ -\frac{\mathcal{A}^* \text{SNR}}{8\Delta^2} \right],
\]

(4.34)

where

\[
\tilde{G} = \frac{4|\Theta|^{-1}}{\sqrt{\frac{1}{2} \mathcal{A}^* \tilde{A}(\theta^*) - \frac{1}{4} \tilde{A}(\theta^*)^2}} \left( \frac{\bar{E}_1(\theta^*, \theta^*)}{\bar{E}_2(c(\theta^*), \theta^*)} \right)^{1/4}.
\]

(4.35)

For the unconditional case, \( \mathcal{A}^* \) is the minimum possible difference in surface area presented to the FLIR sensor over all orientations. The constant \( \tilde{G} \) depends only on the target geometry, thus (4.34) provides the relationship between the FLIR’s unconditional recognition performance and its sensor resolution \( \Delta^2 \).
4.5 Pose Estimation Dependence

In a recent paper [5], the performance of FLIR and LADAR sensors for pose estimation was studied in the good performance regime. In particular an analytic Cramér-Rao bound was developed as a lower bound on the mean-squared error for rotation estimation. There is an intuitive feeling that pose estimation must be important to the performance of a target recognition system when the target has an unknown random orientation. In this section we study the relationship between estimation performance and target recognition.

The ability to accurately perform pose estimation is often quantified through the Cramér-Rao bound (CRB). In parameter estimation, the CRB specifies a lower bound on the mean-squared error of any unbiased estimator [8]. For hypothesis $H_i$,

$$I_i(\theta) = -E \left[ \frac{\partial^2}{\partial \theta^2} \log p(x|H_i, \theta) \right]$$

is called the Fisher information in $x$ about $\theta$. It is the inverse of the CRB. To estimate the CRB based on the data $x$, the observed information

$$\hat{J}_i = -\frac{\partial^2}{\partial \theta^2} \log p(x|H_i, \theta) \bigg|_{\hat{\theta}_i}$$

is often used, where $\hat{\theta}_i$ is the ML estimate of $\theta$ under hypothesis $H_i$. For noise variance $\sigma^2 \to 0$, the observed information converges to the Fisher information under the true hypothesis.

Approximating the likelihood functions with the Laplace method, the Bayesian ML rule chooses the target according to the following criterion,

$$ML: \min_{i \in \{1, \ldots, M\}} \left\{ \frac{1}{2\sigma^2} ||x - x_i(\hat{\theta}_i)||^2 + \frac{1}{2} \log \hat{J}_i \right\}.$$  \hspace{1cm} (4.38)

We can interpret the ML criterion as combining two different terms. The first term is a measure of how close the data is modeled by target $\alpha_i$. The second term is the observed information for hypothesis $H_i$. This means that under decision criterion (4.38), hypothesis $H_i$ is penalized when the parameter $\theta$ can be more accurately estimated from the data. To help explain the reason for this penalty, suppose we had a target model $\alpha_k$ that was very precisely parameterized or included more components $\theta_k = (\theta_{k,1}, \ldots, \theta_{k,N})$. The added
flexibility in the parameterization might lower the first data term, but would incur a corresponding greater penalty in the information term. In this thesis, the parameterization has been the same for each target model, thus making the data term dominant in the ML decision. We can also see this relationship in our analysis of the recognition performance. The conditional and unconditional low noise probability of error in binary recognition, expressed in (3.20) and (3.22), have the form

\[
\frac{1}{f(D, D)} \left( \frac{\hat{J}_1}{\hat{J}_2} \right)^{1/4} \exp \left( -\frac{D^2}{8\sigma^2} \right). \tag{4.39}
\]

We see that the dominant factor is the exponential term involving the minimum distance \( D \). When the targets have similar geometric characteristics, the ratio of information terms will be negligible compared to the exponential term. In our example in section 4.2.2, the ratio \( \left( \frac{\hat{J}_1}{\hat{J}_2} \right)^{1/4} = 0.98 \). This term will not necessarily always be so close to one, but it illustrates that for our recognition scenario, this ratio of information terms does not strongly impact system performance. There does exist a relationship between estimation and recognition performance in the sense that both will improve as \( \sigma^2 \to 0 \). But we should point out that in the conditional probability of error expression, even though the empirical information term \( \hat{J}_1 \) is known to converge to the Fisher information \( I_1(\theta_1) \) as \( \sigma^2 \to 0 \), there is not such a clear interpretation for \( \hat{J}_2 \). Moreover, in the unconditional case it is difficult to make this interpretation for either term. For these reasons, it seems that there is not a direct quantitative link between target recognition and estimation performance. What we have established is that the dominant factors in recognition performance are noise variance \( \sigma^2 \) and the confuser distance \( D \), which is a geometric measure of the ability of the incorrect target to confuse the decision maker.

We can compare the ML rule to the non-Bayesian GLRT decision rule. The low noise techniques used in Chapter 3 show that the asymptotic performance of the GLRT is equal to the ML rule in the probability of error exponent. The GLRT criterion has the form

\[
\text{GLRT: } \minimize_{\theta_i \in \{1, \ldots, M\}} \frac{1}{2\sigma^2} ||x - x_i(\hat{\theta}_i)||^2. \tag{4.40}
\]

This is the low-noise ML rule without the penalty term for observed information. The
corresponding error-probability of the GLRT is

\[ \frac{1}{f(D, \hat{D})} \exp \left( -\frac{D^2}{8\sigma^2} \right), \]

which is identical to the optimal Bayesian performance without the ratio of information terms. The GLRT bypasses the issue of parameter estimation performance and simply matches the data to the target model as well as possible.

Looking ahead to using target models more general than those studied in this thesis, it will be important to consider how our choice of parameters will affect the Bayesian decision and performance. In a study [18] on asymptotic model selection, the author discusses the ability of the MAP rule to discriminatively penalize model parameters based on our ability to estimate their values.
Chapter 5

LADAR-Based Recognition

A coherent laser radar used in pulsed-imager mode offers the ability to collect range data from a field of view. There has been much interest in studying laser radar (LADAR) as part of an overall automatic target recognition (ATR) system. Previous studies looking at LADAR based system performance have included multiresolution range profiling [19], 3-D object detection [9], and LADAR as part of an overall model-based object recognition system [20]. The low-noise approach in this chapter attempts to extend these results in a new way and offer some insight into performance dependencies on sensor and target parameters.

The statistical model developed for the laser radar consists of a mixture of Gaussian and uniform densities. Thus, the asymptotic results of Chapter 3 for additive white-Gaussian noise are not directly applicable in this case. Our analysis would ideally incorporate the effect of anomalous pixels, but instead we will look at a specific example where our previous results can be applied to the LADAR.

5.1 LADAR Analysis

The recognition problem studied in this thesis is one that assumes the target location is known. For the laser radar, the scenario we envision is one in which a LADAR system has knowledge of the location of a target, possibly cued by another sensor. In this case, a narrow range gate can be used, i.e., the preset range-uncertainty interval $\Delta R$ will be small. This implies that the probability of anomaly will be close to zero. Making the $\Pr(A) \approx 0$ approximation allows us to apply the low-noise approximation results for additive white-
Gaussian noise derived in Chapter 3 to LADAR data.

Consider the situation in which a target has a known location and is positioned adjacent to another large object. For example, a tank could be sitting nearby a grove of trees, a building, etc. A diagram of one possible scene is given in Fig. 5-1. Pulses from the laser radar that miss the target will be returned by the object directly behind the target. Our low-noise analysis will be applied using blocks-world targets in this scenario.

### 5.1.1 Conditional Probability of Error - LADAR

Neglecting pixel anomalies, the LADAR’s sensor noise is modeled as purely Gaussian. From the discussion in Chapter 2, the statistical model for the LADAR’s range data with $\Pr(A) = 0$ is $r = r(\theta) + n$, where $r(\theta)$ is the vector of true range values at pose $\theta$ and $n$ is white Gaussian noise $\sim N(0, (\delta R)^2 I)$. The behavior of the LADAR’s measured range value $r_{ij}(\theta)$, as a function of $\theta$, introduces discontinuities to the squared distance functions $E_k(\theta)$. As a result, the assumptions leading to Eqn. (3.14) will not hold in general. As an example, consider the target geometry for $\mathcal{L} = \{\alpha_1, \alpha_2\}$ in Fig. 5-5 with sensor pixel size $\Delta = 0.5$ block units. To model the scenario shown in Fig. 5-1, we computed off-target pixels as having a true range equal to $(R + 4)$ block units to represent a wall directly behind the target. $R$ is the distance from the LADAR sensor to the target’s axis of rotation. Figures 5-2 to 5-4 show the function $E_2(\theta)$ computed over the quarter-circle $\Theta = [0, \pi/2]$ for different values of $\theta_1$. This example shows that $E_2(\theta)$ can be either continuous or discontinuous at the confuser pose $c$.

This type of discontinuity does not occur in the FLIR sensor’s asymptotic analysis. For the FLIR, as the pose $\theta$ varies, $F_{ij}(\theta)$ increases from zero continuously as the $ij$-th pixel changes from off-target to on-target. For the LADAR sensor, an off-target pixel will jump
Figure 5-2: Squared distance function $E_2(\theta)$ at true pose $\theta_1 = \pi/4$ radians. The lower plot for $E_2(\theta)$ shows the confuser pose $c = 0.810$ radians achieved at the right side of a discontinuity.
Figure 5-3: Squared distance function $E_2(\theta)$ at true pose $\theta_1 = 3\pi/8$ radians. The lower plot for $E_2(\theta)$ shows the confuser pose $c = 1.1141$ radians achieved at the left side of a discontinuity.
Figure 5-4: Squared distance function $E_2(\theta)$ at true pose $\theta_1 = 1.155$ radians. The lower plot shows that $E_2(\theta)$ is continuous at the confuser pose $c = 1.112$ radians.
from \( r_{ij}(\theta) = (R + 4) \) to a range value taken on the target block model. The various Laplace approximation results given in Chapter 3 do not cover this jump discontinuity case. To handle this new situation, we will make the assumption that if the confuser pose is achieved on the left side of a jump discontinuity, then the pose integral has negligible contribution from the interval \([c, \pi/2]\), so that

\[
L_2(r) = \int_{0}^{\pi/2} p(r|H_2, \theta)\pi(\theta) \, d\theta \\
\approx \int_{0}^{c} p(r|H_2, \theta)\pi(\theta) \, d\theta. \tag{5.1}
\]

Similarly, if \( c \) is achieved on the right side of a discontinuity, we use

\[
L_2(r) \approx \int_{c}^{\pi/2} p(r|H_2, \theta)\pi(\theta) \, d\theta. \tag{5.2}
\]

In essence we will truncate the pose space \( \Theta \) at the confuser pose and treat the problem with the result developed for minimization on the boundary. When \( c \) is on a discontinuity, we use a first-order Laplace approximation for the likelihood function \( L_2(r) \). Combining the first-order result with Eqn. (3.15) gives the low-noise approximation,

\[
\Pr(e|H_1, \theta_1) \approx Q(\kappa), \quad \text{where} \quad \kappa = \frac{D}{2\delta R} - \frac{\delta R^2}{2D} \log \varphi(\theta_1), \tag{5.3}
\]

and

\[
\varphi(\theta_1) = \begin{cases} 
\frac{E_1(\theta_1)}{E_2(c)}, & \text{if } E_2(\theta) \text{ is continuous at } c, \\
\frac{\delta R^2 E_1(\theta_1)}{\pi E_2(c)^2}, & \text{if } E_2(\theta) \text{ is not continuous at } c.
\end{cases} \tag{5.4}
\]
Note that the first derivative $E_2(c)$ is evaluated at $\theta = c^-$ or $\theta = c^+$ when the confuser is achieved on the left side or right side of a discontinuity, respectively. The correct expression for the conditional error-probability $\Pr(e|H_1, \theta_1)$ depends on whether $E_2(\theta)$ is continuous at the confuser pose $c$. The need to extend the use of the first-order Laplace method makes determining the correct coefficient term for the LADAR's low-noise approximation somewhat more cumbersome. It is necessary to look at $E_2(\theta)$ in detail before computing (5.4).

Numerical simulations were performed to compute the conditional probability of error $\Pr(e|H_1, \theta_1)$ with true pose $\theta_1 = \pi/4$ radians. The corresponding function $E_2(\theta)$ is plotted in Fig. 5-2, which shows that we should use the first-order version of (5.4). The results in Fig. 5-6 show good agreement between the low-noise approximation and the 5000-trial Monte-Carlo computations for small noise variance.

5.1.2 Unconditional Probability of Error - LADAR

By averaging the conditional error-probability over the pose space $\Theta = [0, \pi/2]$, we can derive an expression for the unconditional error-probability. Applying the same procedure developed in Chapter 3 for integrating the conditional error-probability, we first compute

$$\Pr(e|H_1) = \frac{2}{\pi} \int_0^{\pi/2} \Pr(e|H_1, \theta_1) d\theta_1 \approx \frac{2}{\pi} \int_0^{\pi/2} \frac{1}{\sqrt{2\pi\kappa(\theta_1)}} \exp\left(-\frac{\kappa^2(\theta_1)}{2}\right) d\theta_1 \approx \sqrt{\frac{8\delta R^2}{\pi^3}} \int_0^{\pi/2} \frac{\varphi(\theta_1)^{-1/4}}{D(\theta_1)} \exp\left(-\frac{D^2(\theta_1)}{8\delta R^2}\right) d\theta_1. \quad (5.5)$$

Under the assumption that $\varphi(\theta_1)$ is sufficiently well behaved in the neighborhood of $\theta^* = \arg\min_{\theta_1} D(\theta_1)$, we use Laplace’s method to evaluate (5.5):

$$\Pr(e|H_1) \approx \frac{8\delta R^2}{\pi} \frac{\varphi(\theta^*)^{-1/4}}{D(\theta^*)^{3/2}} \sqrt{\varphi(\theta^*)} \exp\left(-\frac{D^2(\theta^*)}{8\delta R^2}\right). \quad (5.6)$$

5.2 Sensor Fusion

When a target recognition system has sensor data available from both the FLIR and LADAR, we expect that there should be an improvement in performance corresponding
Figure 5-6: LADAR low-noise approximation for conditional probability of error $\Pr(e|H_1, \theta_1)$ at true pose $\theta_1 = \pi/4$ radians. The solid curve was computed from the first-order Laplace approximation in Eqn. (5.4).
to the additional information from the target scene. The intent here is to quantify this performance improvement.

Suppose our observed sensor data vector consists of both FLIR thermal measurements and LADAR range imagery. Let \( x \) be an \( IJ \)-dim thermal image vector and \( r \) be the LADAR’s \( IJ \)-dim range imagery. Combine the thermal and range data into a single vector

\[
x_{SF} = \begin{bmatrix} x \\ r \end{bmatrix}
\]

(5.7)

representing data from both sensors. Let \( \sigma^2 \) be the FLIR’s noise-equivalent differential temperature and \( \delta R \) be the LADAR’s local-range accuracy. Define

\[
\omega_1 = \frac{\delta R^2}{\sigma^2 + \delta R^2}
\]

(5.8)

\[
\omega_2 = \frac{\sigma^2}{\sigma^2 + \delta R^2}
\]

(5.9)

\[
\sigma_{SF}^2 = \left(\frac{1}{\sigma^2} + \frac{1}{\delta R^2}\right)^{-1}
\]

(5.10)

\[
E_k(\theta, \theta_1) = \omega_1 ||x_k(\theta) - x_1(\theta_1)||^2 + \omega_2 ||r_k(\theta) - r_1(\theta_1)||^2.
\]

(5.11)

The weights \( \omega_1, \omega_2 \) reflect the relative certainty of each component of the sensor data based on their respective sensor noise variances \( \sigma^2, \delta R^2 \). The quantity \( \sigma_{SF}^2 \) represents an average (one half the harmonic mean) of the FLIR and LADAR sensor noise variances. Taking into account the discontinuities in \( r_{ij}(\theta) \), the asymptotic conditional error-probability for binary recognition in the limit as \( \sigma_{SF}^2 \to 0 \) is

\[
Pr(\epsilon|H_1, \theta_1) \approx Q(\kappa), \quad \text{where} \quad \kappa = \frac{D}{2\sigma_{SF}^2} - \frac{\sigma_{SF}^2}{2D} \log \bar{\varphi}(\theta_1),
\]

(5.12)

and

\[
\bar{\varphi}(\theta_1) = \begin{cases} \frac{\bar{E}_1(\theta_1)}{\bar{E}_2(c)}, & \text{if } E_2(\theta) \text{ is continuous at } c, \\
\frac{\sigma_{SF}^2 \bar{E}_1(\theta_1)}{\pi E_2(c)^2}, & \text{if } E_2(\theta) \text{ is not continuous at } c.
\end{cases}
\]

(5.13)

The confuser distance \( D \) is the minimum weighted Euclidean distance between the incorrect target \( \alpha_2 \) and the true target \( \alpha_1 \) at pose \( \theta_1 \). In this application of the Laplace method, the weighted distance function \( D \) will be affected by the manner in which we take \( \sigma_{SF}^2 \to 0 \).
However, we assume that the weights $\omega_1, \omega_2$ should be well behaved in carrying out the asymptotics and that the Laplace method results from Chapter 3 will hold here as well.

Analyzing the performance of the ML rule for sensor fusion is difficult. Instead, we will study the GLRT rule, which has probability of error

$$
\Pr(\epsilon|H_1, \theta_1) \approx Q(\kappa), \text{ where } \kappa = \frac{D}{2\sigma_{SF}}. \tag{5.14}
$$

According to our discussion in section 4.5, the GLRT has the optimal asymptotic error decay rate. We can show that using the combination $\mathbf{x}_{SF}$ of FLIR and LADAR sensor data is better than using either sensor alone. First we give an inequality:

$$
\frac{D^2}{\sigma_{SF}^2} = \left( \frac{1}{\sigma^2} + \frac{1}{\delta R^2} \right) \arg\min_{\theta} E_2(\theta) \\
\geq \left( \frac{1}{\sigma^2} + \frac{1}{\delta R^2} \right) (\omega_1 D_1^2 + \omega_2 D_2^2) \\
= \frac{D_1^2}{\sigma^2} + \frac{D_2^2}{\delta R^2} \\
> \max \left\{ \frac{D_1^2}{\sigma^2}, \frac{D_2^2}{\delta R^2} \right\} \tag{5.15}
$$

where $D_1, D_2$ are the confuser distances for FLIR and LADAR sensor data alone. It then follows that the sensor fusion error-probability satisfies

$$
Q \left( \frac{D}{2\sigma_{SF}} \right) < \min \left\{ Q \left( \frac{D_1}{2\sigma} \right), Q \left( \frac{D_2}{2\delta R} \right) \right\}. \tag{5.16}
$$

In other words, for small values of $\sigma_{SF}^2$, there is an improvement in recognition performance when using combined sensor data.
Chapter 6

Conclusions

For target recognition systems based on forward-looking infrared and laser radar image sensors, we have considered the problem of identifying ground-based targets with unknown rotation orientation. This uncertainty in pose orientation leads to an intractable pose integral when computing the optimum Bayesian decision rule. The approach taken in this thesis has been to use a low-noise approximation to analytically evaluate the pose integral. Our purpose is to gain insight into how sensor parameters and target geometry affect system performance.

6.1 Summary of Results

Our low-noise asymptotic analysis of system performance was developed in Chapter 3. In the limit as sensor noise becomes small, the conditional probability of recognition error given a target at a true pose is determined by the distance to the closest occurrence of the incorrect target. The asymptotic probability of error decreases exponentially at a rate given by this minimum distance, which we termed the confuser distance. This shows that the confuser distance is one of the key parameters in evaluating asymptotic recognition performance. A comparison of these theoretical results with simulations using simplified target models shows good agreement in the region of high SNR. A second application of Laplace’s approximation over the pose parameter space yielded an analytical form for the unconditional probability of error. The asymptotic unconditional performance was shown to be determined by the global minimum distance to the incorrect target.

Target recognition based on the FLIR sensor was studied extensively in Chapter 4.
The use of Laplace’s method to approximate Bayesian likelihood functions was applied to binary recognition and extended to more general situations. For the case of target detection, expressions for computing low-noise ROC curves were derived. The conditional probability of error in general M-ary target recognition was derived as well. Simulations using blocks-world target models were performed for most of these various scenarios. Overall, there was agreement between Monte Carlo computations of error-probability and our asymptotic expressions in the limit of large SNR. In section 4.4, we discussed the effects of the FLIR’s resolution on system performance. An expression was derived for the low-noise asymptotic probability of error in binary recognition as a function of pixel area $\Delta^2$.

In Chapter 5, our low-noise techniques were applied to LADAR-based recognition in a limited setting. Under the assumption that the probability of anomaly was small, we looked at how to apply results from Chapter 3 to LADAR range imagery. Since LADAR range data is not continuous as a function of pose orientation $\theta$, it was necessary to extend use of the first-order Laplace approximation to compute the correct exponential coefficient term for the asymptotic error-probability. Simulations show that our approximations are accurate, but that an ideal analysis would take into account the full LADAR sensor model. Target recognition performance in the case of FLIR/LADAR sensor fusion was also looked at. We showed that the probability of error decay rate using a combination of sensor data is better than from using either sensor alone.

### 6.2 Future Work

The goal of this research is to obtain an analytical understanding of the factors affecting target recognition. We mention some areas for further study. There are several possible extensions to the current work on recognition performance. The simple target geometry setup we have assumed can be made more realistic by allowing more degrees of freedom in the target pose or in the geometric building blocks. The simple blocks-world targets consisting of two or three blocks used in our simulations do not capture as many details as more realistic CAD models. We can also consider a pose parameter that takes into account uncertainty in location and orientation. For example, an airborne sensor would require modeling the depression angle and the corresponding effect on the sensor data. We would like to derive a understanding of target geometry that covers these more general models.
Specifically for the FLIR-based system, there is future work to be done on relaxing our assumption that the background and target are uniform reflectors. Often there can be a hot spot on a target due to the engine, for example, that will violate this assumption. For the LADAR-based system, a more accurate analysis would incorporate the effect of anomalous pixels.

Lastly, in our discussion of M-ary target recognition, we provided an expression for the low-noise conditional probability of error $\Pr(e|H_1, \theta_1)$. Due to the complicated integral in Eqn. (4.25), it was not clear how to extend our procedure of integrating the conditional probability to the M-ary case. There is room for more progress on this problem.
Bibliography


