Dynamics of a Continuum Characterized by a Non-convex Energy Function

by

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by a Non-convex Energy Function

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Abstract

This thesis investigates the dynamics of nonlinearly elastic bars when the energy function
that characterizes the material is a non-convex function of strain. Such energy functions have
been used, for example, to describe the behavior of solids that can undergo stress-induced
phase changes. Dynamic problems for such a material typically involve two types of prop-
agating strain discontinuities, phase boundaries and shock waves. The classical continuum
theory of dynamics involves the field equations and jump conditions stemming from mo-
mentum balance and kinematic compatibility together with the entropy inequality. In the
setting of this theory, even one-dimensional problems often possess an infinite number of so-
lutions when the energy function is non-convex, and the theory is therefore not well-posed.
The theory must be supplemented with additional information from the physics of phase
transitions.

In part A, we study the propagation of a phase boundary separating a stable phase from a
metastable phase. We consider a general non-convex material and show that the propagation
of such a phase boundary is controlled by its kinetics.
In part B, we consider a phase boundary separating a stable phase from an "unstable phase". In this case we show that the propagation is controlled by inertia rather than kinetics.

In part C, we model and analyze a plate impact experiment for this class of materials. The results are compared with experimental observations on limestone undergoing the calcite I $\rightarrow$ calcite II transition.

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# Contents

Abstract 1

Acknowledgments 3

1 General Introduction 10

1.1 General Description 10

1.2 Review of Continuum Modeling of Reversible Phase Transitions 12

1.3 Present Thesis 15

PART A. Propagation of an Interface into a Metastable Phase 17

2 Introduction 18

3 Basic Equations 22

4 Material; Local Properties of Discontinuities 25

5 The Riemann Problem: Construction of Solutions 33

5.1 Formulation 33

5.2 The Structure of Admissible Solutions to the Riemann Problem 36

6 Explicit Solutions to the Riemann Problem 42
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1</td>
<td>Solutions Involving No Phase Change</td>
<td>42</td>
</tr>
<tr>
<td>6.2</td>
<td>Solutions Involving a Phase Change</td>
<td>46</td>
</tr>
<tr>
<td>6.3</td>
<td>Summary of All Solutions</td>
<td>52</td>
</tr>
<tr>
<td>7</td>
<td>Kinetics and Nucleation</td>
<td>58</td>
</tr>
<tr>
<td>8</td>
<td>Introduction</td>
<td>63</td>
</tr>
<tr>
<td>9</td>
<td>Equilibrium States</td>
<td>67</td>
</tr>
<tr>
<td>10</td>
<td>Quasi-static Motions</td>
<td>73</td>
</tr>
<tr>
<td>11</td>
<td>The Regularized Theory: A Traveling Wave Problem</td>
<td>77</td>
</tr>
<tr>
<td>11.1</td>
<td>Construction of the Traveling Wave Problem</td>
<td>77</td>
</tr>
<tr>
<td>11.2</td>
<td>Solutions to the Traveling Wave Problem</td>
<td>80</td>
</tr>
<tr>
<td>12</td>
<td>The Dynamic Theory</td>
<td>88</td>
</tr>
<tr>
<td>12.1</td>
<td>Background</td>
<td>88</td>
</tr>
<tr>
<td>12.2</td>
<td>The Riemann Problem</td>
<td>90</td>
</tr>
<tr>
<td>12.3</td>
<td>The Structure of Admissible Solutions to the Riemann Problem</td>
<td>91</td>
</tr>
<tr>
<td>12.4</td>
<td>Solutions to a Riemann Problem</td>
<td>95</td>
</tr>
<tr>
<td>13</td>
<td>Conclusions</td>
<td>103</td>
</tr>
<tr>
<td>14</td>
<td>Introduction</td>
<td>105</td>
</tr>
<tr>
<td>5</td>
<td>PART C. An Application: The Impact Problem</td>
<td>104</td>
</tr>
<tr>
<td></td>
<td>14 Introduction</td>
<td>105</td>
</tr>
</tbody>
</table>
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Stress-strain curve</td>
<td>30</td>
</tr>
<tr>
<td>4.2</td>
<td>The regions ( \Gamma_i ) in the ((\gamma^-, \gamma^+)-plane)</td>
<td>31</td>
</tr>
<tr>
<td>4.3</td>
<td>Admissible images of ( \Gamma_i ) in the ((\delta, f)-plane)</td>
<td>32</td>
</tr>
<tr>
<td>5.1</td>
<td>General form of solutions to the Riemann problem</td>
<td>41</td>
</tr>
<tr>
<td>6.1</td>
<td>Form of solutions to Riemann problem without phase change</td>
<td>54</td>
</tr>
<tr>
<td>6.2</td>
<td>Form of solutions to Riemann problem with phase change</td>
<td>55</td>
</tr>
<tr>
<td>6.3</td>
<td>The regions ( D_i ) in the ((\gamma^-, \gamma^+)-plane)</td>
<td>56</td>
</tr>
<tr>
<td>6.4</td>
<td>The ((\gamma^-, v_R - v_L)-plane)</td>
<td>57</td>
</tr>
<tr>
<td>8.1</td>
<td>Free-energy versus strain at temperature above and below the critical temperature</td>
<td>66</td>
</tr>
<tr>
<td>9.1</td>
<td>Stress-strain curve</td>
<td>71</td>
</tr>
<tr>
<td>9.2</td>
<td>The ((\delta, \sigma)-plane)</td>
<td>72</td>
</tr>
<tr>
<td>10.1</td>
<td>The ((\delta, \sigma)-plane.) Admissible directions</td>
<td>76</td>
</tr>
<tr>
<td>11.1</td>
<td>The ((\delta, \gamma)-plane)</td>
<td>87</td>
</tr>
<tr>
<td>12.1</td>
<td>Assumed form of solutions to the Riemann problem</td>
<td>100</td>
</tr>
<tr>
<td>12.2</td>
<td></td>
<td>101</td>
</tr>
</tbody>
</table>
12.3 Form of solutions to Riemann problem ........................................ 102

14.1 Plate impact assembly ................................................................. 108

15.1 Stress-strain curve ................................................................. 112

15.2 Initial conditions, boundary conditions and interface conditions of impact problem .................................................. 113

16.1 The \((\gamma, \gamma')\)-plane ......................................................... 117

16.2 The \((\dot{s}, f)\)-plane ............................................................. 118

17.1 Form of solutions to the signalling problem .................................. 128

17.2 The \((\dot{s}, \sigma_B)\)-plane ......................................................... 129

17.3 Form of solutions to Riemann problem involving two distinct materials ... 130

17.4 Form of solutions to Riemann problem involving a single material ...... 131

18.1 Wave pattern without phase change ............................................ 137

18.2 Velocity at free-end of specimen versus time for the case without phase change .................................................. 138

18.3 Velocity at free-end of specimen versus time when both impactor and specimen are composed of the same material .................................................. 139

18.4 Wave pattern with phase change .............................................. 140

19.1 Wave pattern without phase change ............................................ 148

19.2 Velocity at free-end of specimen involving no phase change versus time .................................................. 149

19.3 Wave pattern with phase change .............................................. 150

19.4 Velocity at free-end of specimen involving a phase change versus time .................................................. 151

19.5 Velocity at free-end of specimen versus time (Grady's results) .......... 152
List of Tables

19.1 Comparison between exact and approximate solutions ............... 153
19.2 Grady’s experimental results ............................................. 153
19.3 Results of present calculation .............................................. 153
Chapter 1

General Introduction

1.1 General Description

Many alloys occur in more than one crystal structure, each crystal structure being termed a phase. One phase exists under certain conditions, while another exists under different conditions. Such materials can transform from one phase to another when they are subjected to an appropriate change in either stress or temperature. Examples of such materials are the shape-memory alloy NiTi, the ferroelectric alloy BaTiO$_3$, ferromagnetic alloy FeNi and the high-temperature superconducting ceramic alloy ErRh$_4$B$_4$.

Consider for instance the class of In-Tl alloys described by Burkart and Reed (1953). These alloys can exist in two solid phases. A cubic phase (austenite) is preferred at a stress below the transformation stress $\sigma_0$, and a tetragonal phase (martensite) is favored above it. When a specimen of austenite is subjected to a monotonically increasing stress, the martensite phase is nucleated at the “martensite start stress” $\sigma_{M_s}(\geq \sigma_0)$. The specimen now consists of a mixture of both austenite and martensite. The coexistent phases are separated from each other by one or more interfaces—phase boundaries; the phase boundaries are said to be coherent in the sense that the deformation is continuous across them even though the deformation gradient is not. As these phase boundaries propagate, the entire specimen will
eventually be converted into martensite. If the stress is now decreased, the whole process is reversed, with the martensite to austenite transformation being initiated at the "austenite start stress" $\sigma_{A_s} (\leq \sigma_0)$. This is a reversible or thermoelastic phase transition. The values of the nucleation stresses $\sigma_M$ and $\sigma_{A_s}$, as well as the transformation stress $\sigma_0$ depend critically on the alloy composition, the heat treatment and the temperature (Otsuka (1986)).

The mechanical properties of a material sometimes can be improved by exploiting a phase transformation. For example, if steel in its austenite phase is cooled to below the martensite nucleation temperature, the martensite can precipitate as finely dispersed particles. Since these closely spaced precipitates of stronger martensite can obstruct dislocation motion, there is an increase in the strength of the steel (e.g. Cottrell (1967)). Another example is the toughening of certain ceramics (Evan et al (1986), Green et al (1988)). In PSZ (partially stabilized zirconia), sub-micron sized zirconia particles in their tetragonal phase are dispersed within a zirconia matrix which is in its cubic phase. When sufficiently stressed, the tetragonal zirconia transforms into its monoclinic phase, and there is an accompanying order of magnitude increase in the toughness of the material (Evan et al (1986)). A third example is provided by the thermoelastic behavior of shape memory alloys. Since these materials have very different stress response on different ranges of temperature, it has been possible to design a material to possess various desired mechanical responses at various chosen temperatures (Schetky (1979)).

Many phase transitions, such as those occurring in steel, are not reversible, the deformation associated with the transformation being coupled with plastic deformation, e.g. Stringfellow (1990). We will not consider such transformations in this thesis.
1.2 Review of Continuum Modeling of Reversible Phase Transitions

Various aspects of the theory of finite thermoelasticity associated with reversible phase transformations in crystalline solids have been studied in a number of recent papers; see for example Ericksen (1980,1986), James (1986) and Pitteri (1984). For a thermoelastic material, the Helmholtz free-energy function $\psi$ depends only on the deformation gradient tensor $F$ and the temperature $\theta$: $\psi = \psi(F, \theta)$. If the stress-free material can exist in more than one phase, then the energy function $\psi$ must have multiple energy-wells, each well being associated with a phase. In particular, for a two-phase material, one minimum corresponds to austenite and the other to martensite. At the transformation temperature $\theta_T$, the two energy minima have the same value. For $\theta < \theta_T$, the martensite minimum is smaller; in this case, we speak of the martensite as being stable and the austenite as being metastable. For $\theta > \theta_T$, the austenite minimum is smaller; thus the austenite is stable and the martensite metastable. As the stress-free material is cooled or heated, it often transforms between the martensite and the austenite. In the presence of stress $S$, one must consider the potential energy function $G = G(F, S, \theta)$ involving multiple energy-wells where $S$ is the first Piola-Kirchhoff stress tensor. In this case, the material can transform from one phase to another if it is stressed or heated by a suitable amount.

Since the deformations on either side of a phase boundary are distinct, and there is a finite discontinuity in the deformation gradient tensor across a phase boundary, the constitutive relations which can be used to model phase transitions must have the capability of sustaining such deformations. The occurrence of such deformation fields is closely related to the changing of type of the (displacement) equations of equilibrium from elliptic to non-elliptic. The interpretation of these ellipticity conditions has been a source of considerable difficulty.
Abeyaratne (1980) showed in two-dimensions that strong ellipticity is essentially equivalent to the convexity of the potential energy. The analogous issue within the three-dimensional theory was investigated by Rosakis (1990).

Continuum mechanical studies of reversible phase transitions have been focused on two basic issues; the first is related to energy minimizing deformations corresponding to the stable configurations of a body, the second is associated with the non-equilibrium evolution of a body towards such stable configurations through intermediate states of metastability.

Ericksen (1975) studied the deformation of a bar composed of a two-phase material. In this one-dimensional setting, he showed that, for certain values of the prescribed elongation, the stable configuration of the bar involves a mixture of coexistent phases. In seeking such an absolute minimizer, he showed that an additional jump condition, the “Maxwell condition”, must hold at the phase boundary. The related question of the minimization of energy in the three-dimensional theory was examined by James (1981), Abeyaratne (1983) and Gurtin (1983); they derived a supplementary jump condition (analogous to the Maxwell condition) which should hold at a singular surface if the equilibrium field is to be stable.

Silling (1988) examined certain implications of the Maxwell condition within the setting of anti-plane shear of a two-phase material. He observed that, in many reasonable boundary-value problems, the Maxwell condition cannot be satisfied exactly, and that it can only be satisfied in the sense of a limit of an infinite sequence of increasingly chaotic deformations. Ball and James (1987) examined this same issue in three dimensions and showed that the absolute minimizers of energy can involve a stable configuration of fine mixtures of phases; in particular, they studied an austenite/twinned martensite interface in detail, and showed that the consequences of their theory are in agreement with the crystallographic theory of martensite.
The usual continuum theory of thermoelasticity, though adequate for describing the energy minimizing deformations of the two-phase material, does not, by itself, characterize quasi-static or dynamic processes of a body involving phase transitions. This can be illustrated by the lack of uniqueness of solution to certain initial-boundary-value problems; Abeyaratne and Knowles (1988, 1991).

Quasi-static or dynamic processes generally involve states that are merely metastable and so fall under the category of “non-equilibrium thermodynamic processes.” Considerations pertaining to the rate of entropy production during such a process naturally leads to the notation of the driving force (or Eshelby force) \( f \) acting on a phase boundary, Abeyaratne and Knowles (1990); see also Eshelby (1956), Knowles (1979), Rice (1975). The theory of non-equilibrium processes can then be used to argue for the need for a constitutive equation - a kinetic law - relating the propagating speed \( V_n \) of the phase boundary to the driving force \( f \) and the temperature \( \theta \): \( V_n = V_n(f, \theta) \). The kinetic law controls the rate of progress of the phase transition; the importance of a kinetic law in the description of phase transitions in solids has long been recognized in the materials science literature, e.g. Christian (1975); in fact, some, though not all, micro-mechanical models of kinetics lead to the kinetic laws of the form \( V_n = V_n(f, \theta) \). The kinetic relation controls the progress of the phase transition once it has commenced. A separate nucleation condition is required to signal the initiation of such a transition. This is analogous to the roles played by a flow rule and a yield condition in continuum plasticity theory. A general discussion of nucleation theory in phase transitions, from a materials science point of view, may be found in Christian (1975). Thus a complete constitutive theory which is capable of modeling processes involving thermoelastic phase transitions consists of three ingredients: a Helmholtz free-energy function, a kinetic relation and a nucleation criterion.
Most multi-dimensional problems associated with phase transitions must be solved numerically. Collins and Luskin (1989) have studied energy minimizing deformation in this way, and Silling (1988) has studied questions related to propagation. Molecular dynamics simulations have also been carried out, e.g. Yu and Clapp (1989).

1.3 Present Thesis

Many martensitic phase transformations take place at very high rates; sometimes, phase boundaries move at speeds which are of the order of the shear wave speeds in the solid (Bunshak and Mehl (1952)). According to Grujicik, Olson and Owen (1985), reported measurements of phase boundary velocities vary widely, from values small enough to permit direct optical observation to values approaching the speed of shear waves in the parent phase of the material. Dynamic effects would therefore be very important in the study of such phase transformations.

From the theoretical point of view, there are a number of important questions pertaining to the dynamics of a material undergoing phase transformations. In the mathematical study of systems of conservation laws in one space dimension (see, for example, Dafermos (1983, 1984), Lax (1973)), it is known that the solution to an initial-value problem, subject to the entropy inequality, is unique, provided that the curvature of the underlying stress-strain relation is always of one sign and that it is monotonic (Oleinik (1957)). However, in the absence of monotonicity and convexity (or concavity), the entropy inequality is not strong enough to secure uniqueness. This is, of course, precisely the class of materials that is of interest in studying phase transformations.

The purpose of the present thesis is to explore the effects of inertia on the continuum theory of reversible phase transitions. In parts A and B of this thesis we consider theoretical
issues related to the propagation of a phase boundary. Having thus addressed the formulation
of dynamic problems, in part C, we turn to a specific dynamic problem, viz. an impact
problem. More specific introductions to each of the issues studied in the three parts A, B,
C is provided at the beginning of those parts.
PART A:  
Propagation of an Interface  
into a Metastable Phase
Chapter 2
Introduction

In the simplest one-dimensional theory describing the longitudinal motions of an elastic bar, one employs a pair of conservation laws associated with momentum balance and kinematic compatibility. When the motion of the bar involves a propagating strain discontinuity, it is subject to a pair of jump conditions associated with these conservation laws. In addition, the second law of thermodynamics requires that the dissipation associated with the moving discontinuity be non-negative, a condition usually referred to as the entropy inequality.

The character of the material of the bar enters the conservation laws and jump conditions through the stress-strain relation $\sigma = \hat{\sigma}(\gamma)$. If the material of the bar is such that stress is a monotonically increasing function of strain that is strictly convex or strictly concave, then phase transformations cannot occur, and all propagating discontinuities are shock waves. For a bar made of such a material, it follows from a result of Oleinik (1957) that the Cauchy problem for the associated field equations and jump conditions has at most one piecewise smooth solution that fulfills the entropy inequality; Liu (1976) discussed Oleinik's theorem and related results.

When the material can undergo a reversible, or thermoelastic, phase transformation, the one-dimensional elastic continuum can be characterized by a non-monotonic relation $\sigma = \check{\sigma}(\gamma)$ between stress and strain, or equivalently by a nonconvex elastic potential $W(\gamma)$. 
Typically one encounters a stress response function \( \sigma(\gamma) \) in which stress first increases with increasing strain, then decreases, and finally increases again; for example, Ericksen (1975). The rising branches of such a stress-strain curve are identified with different phases of the material, while the declining branch is associated with an "unstable phase." For suitable value of stress, the associated potential energy \( G(\gamma, \sigma) = W(\gamma) - \sigma\gamma \) has multiple energy-wells, each energy-well being associated with a distinct phase of the material. During a typical thermomechanical process, the material often moves from one energy-well to another, or equivalently, from one branch of the stress-strain curve to another.

When the stress-strain curve is non-monotonic and undergoes a change in the sign of its curvature, the Cauchy problem need no longer have a unique solution, even with the entropy inequality in force; see the remarks of Dafermos (1984). In order to secure uniqueness, many researchers have replaced the entropy inequality with various "admissibility conditions" which are to be satisfied by the weak solutions. For example, two different notions of maximum entropy production have been proposed by Dafermos (1973), and augmenting the elastic theory with viscosity and capillarity effects has been proposed by Slemorod (1983), Truskinovsky (1982). The implications of these criteria for dynamic phase transitions have been examined by, for example, James (1980), Hattori (1986) and Shearer (1986).

A completely different approach has been proposed and studied by Abeyaratne and Knowles (1981, 1988, 1991(a), 1991(b), 1991(c)). They (1981) observed that the lack of uniqueness arises not only in dynamic motions, but in quasi-static motions as well. They (1988) suggested that in addition to the usual constitutive law between stress and strain, further material description in the form of a nucleation criterion and a kinetic relation pertaining to the phase transitions are needed. The importance of a nucleation criterion and a kinetic relation in the description of phase transitions in solids has long been recognized in
the materials science literature, e.g. Christian (1975).

Abeyaratne and Knowles (1988) showed that the inclusion in the continuum theory of the nucleation criterion and the kinetic relation leads to a determinate quasi-static theory whose predictions are in qualitative accord with experiments on shape memory alloys that involve slowly propagating phase boundaries. A similar result in the dynamical setting for the Riemann problem for a special piecewise linear elastic material was established by Abeyaratne and Knowles (1991(b), 1991(c)). They showed that the maximum entropy rate admissibility criterion and the viscosity-capillarity admissibility criterion may in fact be viewed as being two particular examples of kinetic relations.

The study of Abeyaratne and Knowles (1991(a)) was restricted to a piecewise linear elastic material. The nature of this trilinear material model leads to a considerable simplification in the analysis. First, this material does not sustain wave fans. Second, shock waves always travel at the sound speed (and so perhaps ought better to be called acoustic waves). Third, shock waves are dissipation-free. For these reasons, it is natural to question whether the results found by Abeyaratne and Knowles (1991(a)) were special to the trilinear material and inquire whether its conclusions hold for more general rising-falling-rising stress-strain curves. This is the objective of part A of this thesis. We show for a material whose rising-falling-rising stress-strain curve is smooth and has a single inflection point, that the nucleation criterion and the kinetic relation (applied to all subsonic and sonic phase boundaries) serve to single out a unique solution to the Riemann problem with initial data in a single, metastable phase.

We note in passing that if the kinetic law is applied only to phase boundaries that are subsonic and not to those that are sonic, we do not have uniqueness of solution for all initial data. Truskinovsky (1992) takes the view that the kinetic relation should not be applied to sonic phase boundaries and that the accompanying non-uniqueness is an instability.
After dealing with various preliminary issues in Chapters 3 and 4 we turn, in Chapter 5, to deduce the general solution forms to the Riemann problem that are consistent with the entropy inequality. In Chapter 6 we explicitly construct all solutions to the Riemann problem corresponding to initial data with strains in the low-stain phase; the results are summarized, and the nature of the associated non-uniqueness characterized, in Section 6.3. Finally in Chapter 7 we introduce the nucleation criterion and the kinetic relation, thus leading to uniqueness.
Basic Equations

Consider longitudinal motions of an elastic bar that is regarded as a one-dimensional continuum with unit cross-sectional area. The motion of the bar is assumed to take place isothermally. During such a motion, the particle at $x$ in the reference configuration is carried to $x + u(x, t)$ at time $t$, where $u(x, t)$ is the displacement. The displacement is assumed to be continuous with piecewise continuous first and second derivatives throughout the regions of space-time to be considered. The strain and particle velocity are defined by $\gamma = u_x$ and $v = u_t$ at points $(x, t)$ where the derivatives exist. Necessarily, $\gamma(x, t) > -1$ in order to ensure that the mapping $x \rightarrow x + u(x, t)$ is invertible at each instant $t$. The stress $\sigma(x, t)$ is related to the strain through

$$\sigma = \dot{\sigma}(\gamma),$$

where $\dot{\sigma}$ is the stress response function of the material. At points where $\gamma$ and $v$ are smooth, balance of momentum and kinematic compatibility require that

$$\dot{\sigma}'(\gamma) \gamma_x = \rho v_t, \quad v_x = \gamma_t,$$

where the constant $\rho$ is the mass density in the reference configuration. If there is a moving discontinuity at $x = s(t)$, the following jump conditions must hold:
\[ \dot{\sigma}(\gamma) - \sigma(\gamma) = -\rho \dot{s}(\dot{\nu} - \overline{\nu}), \quad (\dot{\nu} - \overline{\nu}) = -\dot{s}(\gamma - \gamma), \] (3.3)

where for any function \( g(x, t) \) we write \( \dot{g} = g(s(t)\pm, t) \) for the limiting values of \( g \) on either side of the discontinuity.

Consider the motion of the piece \( x_1 \leq x \leq x_2 \) of the bar during a time interval \([t_1, t_2]\). Suppose that \( \gamma \) and \( \nu \) are smooth on \([x_1, x_2] \times [t_1, t_2] \) except at the moving discontinuity \( x = s(t) \). Let \( E(t) \) be the total mechanical energy at time \( t \) associated with this piece of bar:

\[ E(t) = \int_{x_1}^{x_2} [W(\gamma(x, t)) + \frac{1}{2} \rho \nu^2(x, t)] \, dx, \] (3.4)

where \( W(\gamma) \) is the strain energy per unit reference volume of the bar, i.e.

\[ W(\gamma) = \int_{0}^{\gamma} \dot{\sigma}(\epsilon) \, d\epsilon, \quad \gamma > -1. \] (3.5)

The following work-energy relation can be readily established:

\[ \sigma(x_2, t) \nu(x_2, t) - \sigma(x_1, t) \nu(x_1, t) - \dot{E}(t) = f(t) \dot{s}(t), \] (3.6)

where the driving force (or driving traction) \( f(t) \) acting on the strain discontinuity is defined by

\[ f = \int_{\gamma}^{\gamma} \dot{\sigma}(\gamma) \, d\gamma - \frac{1}{2} (\dot{\sigma}(\gamma) + \dot{\sigma}(\overline{\gamma}))(\gamma - \overline{\gamma}). \] (3.7)

Note that \( f \) may be interpreted geometrically as the difference between the area under the stress-strain curve between \( \gamma = \gamma \) and \( \gamma = \overline{\gamma} \) and the area of an associated trapezoid having the same base. The right-hand side of (3.6) represents the instantaneous dissipation rate due to the moving discontinuity and the requirement that it be non-negative implies that

\[ f(t) \dot{s}(t) \geq 0. \] (3.8)
Under isothermal conditions, the inequality (3.8) is a consequence of the second law of thermodynamics.

A motion of the bar is governed by the field equations (3.2) at all points of smoothness, and the jump conditions (3.3) and the entropy inequality (3.8) at all discontinuities.
Chapter 4

Material; Local Properties of Discontinuities

In this part, we consider a material whose stress response function $\sigma(\gamma)$ is twice continuously differentiable with $\sigma$ first increasing with increasing $\gamma$, then decreasing, and finally increasing again as shown in Figure 4.1. More specifically we suppose that there are three numbers $\gamma_M$, $\gamma_m$ and $\gamma_{in}$ with $0 < \gamma_M < \gamma_{in} < \gamma_m$ such that

$$\begin{align*}
\sigma'(\gamma) &\begin{cases} 
> 0, & -1 < \gamma < \gamma_M, \\
= 0, & \gamma = \gamma_M, \\
< 0, & \gamma_M < \gamma < \gamma_m, \\
= 0, & \gamma = \gamma_m, \\
> 0, & \gamma > \gamma_m,
\end{cases} \\
\sigma''(\gamma) &\begin{cases} 
< 0, & -1 < \gamma < \gamma_{in}, \\
= 0, & \gamma = \gamma_{in}, \\
> 0, & \gamma > \gamma_{in}.
\end{cases}
\end{align*}$$

and

Moreover, we suppose that $\sigma(0) = 0$ and that

$$\sigma(\gamma) \to -\infty, \quad \sigma'(\gamma) \to \infty \quad \text{as} \quad \gamma \to -1;$$

(4.3)
\[ \dot{\sigma}(\gamma) = \mu_\infty \gamma + \sigma_T + o(1) \quad \text{as} \gamma \to \infty, \]  

(4.4)

where \( \mu_\infty > 0 \) and \( \sigma_T \) are constants. The stress-strain curve therefore consists of three branches, two of which are rising, while the other is declining; it has a single inflection point at the strain-level \( \gamma = \gamma_{in} \) and is asymptotic, at large tensile strains, to the straight line \( \sigma = \mu_\infty \gamma + \sigma_T \). It is useful for later purposes to note that there are two unique values of strain \( R_\infty \) and \( P_\infty \) such that

\[ \mu_\infty R_\infty + \sigma_T = \dot{\sigma}(R_\infty), \quad \dot{\sigma}'(P_\infty) = \mu_\infty, \quad \text{where} \ -1 < R_\infty < P_\infty < \gamma_M; \]  

(4.5)

\( R_\infty \) is the strain-level at which the asymptote \( \sigma = \mu_\infty \gamma + \sigma_T \) intersects the first branch of the stress-strain curve, while \( P_\infty \) is the value of strain on the first branch at which the slope equals the slope of this asymptote. Certain other material parameters are defined in the figure. In particular, the Maxwell stress \( \sigma_o \) is the stress-level for which the two hatched areas of Figure 4.1 are equal.

We shall say that a particle of the bar labeled by \( x \) in the reference state is in the low-strain phase, the "unstable phase" or the high-strain phase at time \( t \) during a motion if \( \gamma(x, t) \) lies in the respective intervals \((-1, \gamma_M], (\gamma_M, \gamma_m) \) or \([\gamma_m, \infty) \). At a moving discontinuity \( x = s(t) \), the jump conditions (3.3) imply

\[ \rho \dot{s}^2 = \frac{\dot{\sigma}(\dot{\gamma}) - \hat{\dot{\sigma}}(\gamma)}{\dot{\gamma} - \gamma} \quad (\geq 0). \]  

(4.6)

A discontinuity is called either a shock wave or a phase boundary according to whether \( \dot{\gamma} \) and \( \dot{\gamma} \) both lie in the same phase or in distinct phases. The sound speed of the material at a strain \( \gamma \) is defined by

\[ c(\gamma) = \sqrt{\frac{\sigma''(\gamma)}{\rho}}, \]  

(4.7)
where it is necessary that \( \gamma \) in (4.7) not belong to the unstable phase. Let \( c_{\infty} = \sqrt{\mu_{\infty}/\rho} \).

Let \( c = c(\frac{\gamma}{\gamma}) \) stand for the sound speeds on the two sides of a discontinuity. The propagation speed \( \dot{s} \) of the discontinuity is said to be subsonic if \( |\dot{s}| < \dot{c} \) and \( \dot{c} \), intersonic if \( \dot{c} < |\dot{s}| < \dot{c} \) or \( \dot{c} < |\dot{s}| < \dot{c} \), and supersonic if \( |\dot{s}| > \dot{c} \) and \( \dot{c} \).

We shall speak of a low-strain shock wave and a high-strain shock wave according to whether the strains \( \gamma, \dot{\gamma} \) both belong to the low-strain phase or to the high-strain phase. For the material (4.1)-(4.4) considered here, it can be readily seen that all shock waves are intersonic. Moreover, it follows from (3.7) and (4.1)-(4.4) that the entropy inequality (3.8) holds at a shock wave if and only if

\[
\text{Low-strain shock:} \begin{cases} 
\dot{\gamma} > \gamma & \text{if } \dot{s} > 0, \\
\dot{\gamma} < \gamma & \text{if } \dot{s} < 0,
\end{cases} \quad (4.8)
\]

\[
\text{High-strain shock:} \begin{cases} 
\dot{\gamma} < \gamma & \text{if } \dot{s} > 0, \\
\dot{\gamma} > \gamma & \text{if } \dot{s} < 0.
\end{cases} \quad (4.9)
\]

This implies in particular that a shock wave always moves into the phase whose sound speed is smaller than the speed \( |\dot{s}| \) of the shock wave.

Turning next to phase boundaries, we will show in the next chapter that a phase boundary for which either \( \gamma, \dot{\gamma} \) is in the unstable phase cannot arise in the problem to be considered here. Thus suppose that \( \gamma \) belongs to the high-strain phase and \( \dot{\gamma} \) to the low-strain phase. In the \( \gamma, \dot{\gamma} \)-plane, the set of all pairs \( (\gamma, \dot{\gamma}) \) for which \( \gamma \) is in the high-strain phase, \( \dot{\gamma} \) is in the low-strain phase and the right side of (4.6) is non-negative is represented by the union \( \Gamma \) of the hatched regions in Figure 4.2; it is the region bounded by the lines \( \dot{\gamma} = -1, \dot{\gamma} = \gamma_M, \gamma = \gamma_m \) and the curve \( \hat{\sigma}(\dot{\gamma}) = \hat{\sigma}(\gamma) \). The region \( \Gamma \) of the \( \gamma, \dot{\gamma} \)-plane will play a major role in the analysis that follows in the next chapters. The boundary segment \( \mathcal{E} \) is defined by

\[
\mathcal{E} : \quad \hat{\sigma}(\dot{\gamma}) = \hat{\sigma}(\gamma) \Rightarrow \dot{\gamma} = T(\gamma), \quad \gamma_m \leq \gamma \leq \gamma_M, \quad (4.10)
\]

27
where the material parameters $\gamma_\beta$ and $\gamma_m$ are defined in Figure 4.1. By (4.6), $s = 0$ at points on $E$ and so this segment represents instantaneously stationary or equilibrium states of the phase boundary. One can verify that $T''(\gamma) > 0$ for $\gamma_m < \gamma < \gamma_\beta$ and that $T'(\gamma_m) = 0$, $T'(\gamma_\beta) = \infty$; the curve $E$ therefore rises monotonically as $\tilde{\gamma}$ increases. Next, consider the curve $F$ which is defined as the set of points $(\tilde{\gamma}, \tilde{\gamma})$ at which the driving force $f(\tilde{\gamma}, \tilde{\gamma})$ introduced in (3.7) vanishes:

$$F: \hat{f}(\tilde{\gamma}, \tilde{\gamma}) = 0 \iff \hat{\gamma} = Q(\tilde{\gamma}), \quad \tilde{\gamma} \geq \gamma_{03},$$

where the material parameter $\gamma_{03}$ is defined in Figure 4.1. One can verify that $Q'(\gamma) < 0$ and that $Q(\gamma) \to R_\infty$, $Q'(\gamma) \to 0$ as $\gamma \to \infty$ where $P_\infty$ is the value of strain defined previously in (4.5). The curve $F$ therefore declines monotonically as $\tilde{\gamma}$ increases as shown in the figure. In view of (3.8), a phase boundary associated with a point on $F$ propagates without dissipation. Since $\hat{f}(\tilde{\gamma}, \tilde{\gamma}) > 0$ above $F$, the entropy inequality indicates that $s = 0$ there; likewise $f < 0$ and $s \leq 0$ below $F$. Consider next the curves $\tilde{S}$ and $\tilde{S}$ which are defined as the ("sonic") curves on which the speed $\hat{s}$ of the phase boundary is equal, respectively, to the sound speeds $\hat{c}$ and $\tilde{c}$:

$$\tilde{S}: \sqrt{(\hat{\sigma}(\tilde{\gamma}) - \hat{\sigma}(\tilde{\gamma}))/(\hat{\gamma} - \tilde{\gamma})} = \sqrt{\hat{\sigma}'(\tilde{\gamma})} \iff \hat{\tilde{\gamma}} = P(\tilde{\gamma}), \quad \tilde{\gamma} \geq \gamma_\beta,$$

$$\tilde{S}: \sqrt{(\hat{\sigma}(\tilde{\gamma}) - \hat{\sigma}(\tilde{\gamma}))/(\hat{\gamma} - \tilde{\gamma})} = \sqrt{\hat{\sigma}'(\tilde{\gamma})} \iff \tilde{\gamma} = R(\tilde{\gamma}), \quad \tilde{\gamma} \geq \gamma_m.$$  

One can verify that $P'(\gamma), R'(\gamma) < 0$ and that $P(\gamma) \to P_\infty, R(\gamma) \to R_\infty, P'(\gamma), R'(\gamma) \to 0$ as $\gamma \to \infty$ where $P_\infty$ and $R_\infty$ were defined earlier in (4.5). The curves $\tilde{S}$ and $\tilde{S}$ therefore decline monotonically with increasing $\tilde{\gamma}$ as shown in the figure. One can also verify that the three curves $\tilde{S}, \tilde{S}$ and $F$ do not intersect each other; necessarily, the curves $F$ and $\tilde{S}$ approach each other asymptotically as $\tilde{\gamma} \to \infty$. The region $\Gamma$ is thus divided into four subregions $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ and $\Gamma_4$ by these curves. The regions $\Gamma_1$ and $\Gamma_2$ correspond to phase boundaries which
propagate into the high-strain phase at, respectively, intersonic and subsonic speeds; the regions $\Gamma_3$ and $\Gamma_4$ correspond to phase boundaries which propagate into the low-strain phase at, respectively, subsonic and intersonic speeds. Points on the curves $\tilde{S}$ and $\hat{S}$ correspond to sonic phase boundaries. For the material (4.1)-(4.4) under consideration, supersonic phase boundaries cannot occur. We note that the figure has been drawn for the case $\gamma_\alpha > P_\infty$ though we do not assume this in the analysis.

Finally we consider the mapping $(\gamma, \gamma) \to (s, f)$ defined by (3.7), (3.8) and (4.6). One can verify that the Jacobian determinant of this mapping vanishes when $\gamma, \gamma$ corresponds to a sonic phase boundary, i.e. on the curves $\tilde{S}$ and $\hat{S}$. Considering the subsonic and intersonic regions separately, one can map each of the regions $\Gamma_i$ into the $s, f$-plane; Figure 4.3 shows the images $\Gamma'_i$ that result from this mapping. Each of the curves $S', \hat{S}', M'$ and $\hat{M}'$ rises monotonically as $s$ increases. As $s \to c_{\infty}$ the curves $\hat{S}'$ and $M'$ rise without bound; when $s \to -c_{\infty}$ the curve $\tilde{S}'$ declines without bound. In Figure 4.3, $c_{\infty} = \sqrt{\mu_{\infty}/\rho}$, $f_M = \hat{f}(\gamma_\beta, \gamma_M)$ and $f_m = \hat{f}(\gamma_m, \gamma_\alpha)$. Note that though this mapping of $\Gamma$ from the $\gamma, \gamma$-plane to the $s, f$-plane is not one-to-one, when restricted to the subsonic region $\Gamma_2 \cup \Gamma_3$, it is one-to-one.
Figure 4.1. Stress-strain curve.
Figure 4.2. The regions $\Gamma_i$ in the $(\bar{\gamma}, \tilde{\gamma})$-plane.
Figure 4.3. Admissible images of $\Gamma_i$ in $(s, f)$-plane.
Chapter 5

The Riemann Problem: Construction of Solutions

5.1 Formulation

We now formulate the Riemann problem for the field equations and jump conditions (3.2)-(3.3). We seek weak solutions of the differential equations (3.2) on the upper half of the $x,t$-plane that satisfy the following initial conditions:

$$
\gamma(x,0), v(x,0) = \begin{cases} 
\gamma_L, v_L, & -\infty < x < 0, \\
\gamma_R, v_R, & 0 < x < \infty,
\end{cases}
$$

(5.1)

where $\gamma_L, \gamma_R, v_L$ and $v_R$ are given constants with $\gamma_L > -1$ and $\gamma_R > -1$.

Since the initial value problem described above is invariant under the scale change $t \rightarrow kt$, $x \rightarrow kx$, we restrict attention to solutions that have this property as well. Where such solutions exist, they must have the form $\gamma(x,t) = \tilde{\gamma}(\xi), v(x,t) = \tilde{v}(\xi)$ where $\xi = x/t$. It then follows from (3.2) that for such solutions, either $\gamma$ and $v$ are both constant, or they are the wave fans which are given by

$$
\dot{\gamma}'(\tilde{\gamma}(\xi)) = \rho \xi^2, 
$$

(5.2)

$$
\dot{v}'(\xi) = -\xi \dot{\gamma}'(\xi).
$$

(5.3)
The left side of (5.2) must necessarily be non-negative for any \( \dot{\gamma} \) that satisfies (5.2). Thus for the material (4.1)-(4.4) wave fans can occur only if \( \dot{\gamma} \) takes values in either the low-strain phase or the high-strain phase. We shall speak of a *low-strain fan* or a *high-strain fan* according to whether \( \dot{\gamma} \) belongs to the low-strain phase or the high-strain phase respectively.

In view of (4.1)-(4.4), it follows that (5.2) defines a unique function \( \dot{\gamma}(\xi) = \gamma^{(L)}(\xi) \) for \(-\infty < \xi < \infty\), such that \( \gamma^{(L)} \in (-1, \gamma_M] \); \( \gamma^{(L)} \) describes the strain field in a low-strain fan. Similarly (5.2) can be uniquely solved for a function \( \dot{\gamma}(\xi) = \gamma^{(H)}(\xi) \) for \(-c_\infty < \xi < c_\infty\), such that \( \gamma^{(H)} \in [\gamma_m, \infty) \); \( \gamma^{(H)} \) describes the strain field in a high-strain fan. Thus (5.2), (5.3) lead to the two wave fans

\[
\dot{\gamma}(\xi) = \gamma^{(i)}(\xi), \quad \dot{v}(\xi) = v^{(i)}(\xi), \quad i = L, H, \quad \text{where}
\]

\[
-(v^{(i)}(\xi) - v^{(i)}(\xi_0)) = \pm \int_{\gamma^{(i)}(\xi_0)}^{\gamma^{(i)}(\xi)} c(\gamma) d\gamma, \quad i = L, H,
\]

and \( \xi_0 \) describes an arbitrary ray \( x/t = \xi_0 \) within the fan; necessarily \( \xi_0 \) must lie in the interval \((-\infty, \infty)\) for a low-strain fan and in the interval \((-c_\infty, c_\infty)\) for a high-strain fan.

The positive and negative signs are taken in the right side of (5.5) according to whether the wave fan occurs in the first quadrant or second quadrant of the \( x, t \)-plane, respectively.

Consider two rays \( x = \bar{c}t \) and \( x = \hat{c}t \) with \( \bar{c} < \hat{c} \) in the \( (x, t) \)-plane between which the field is a fan. Let \( \bar{\gamma} = \gamma(\bar{c}t+, t), \bar{v} = v(\bar{c}t+, t), \bar{\gamma} = \gamma(\bar{c}t-, t), \bar{v} = v(\bar{c}t-, t) \) denote the limiting values from within the fan of strain and particle velocity at these rays. It follows from (4.7) and (5.2) that

\[
\bar{c} = \pm c(\bar{\gamma}), \quad \hat{c} = \pm c(\hat{\gamma}),
\]

and from (5.3) that

\[
-(\bar{v} - \bar{v}) = \pm \int_{\bar{\gamma}}^{\hat{\gamma}} c(\gamma) d\gamma,
\]
where the positive and negative signs are taken according to whether the fan occurs in the first or second quadrant of the \( x,t \)-plane, respectively. Equation (5.7) is the analog for a fan of the kinematic jump condition for a discontinuity in (3.3). Since the field within the fan is smooth, the entropy inequality is trivially satisfied at points within it. For the material (4.1)-(4.4), it follows from (5.2) that necessarily

\[
\text{Low-strain fan: } \begin{cases} 
\frac{\dot{\gamma}}{\dot{\gamma}} < \frac{\dot{\gamma}}{\dot{\gamma}} & \text{if fan is in first quadrant}, \\
\frac{\dot{\gamma}}{\dot{\gamma}} > \frac{\dot{\gamma}}{\dot{\gamma}} & \text{if fan is in second quadrant},
\end{cases} \tag{5.8} \\
\text{High-strain fan: } \begin{cases} 
\frac{\dot{\gamma}}{\dot{\gamma}} > \frac{\dot{\gamma}}{\dot{\gamma}} & \text{if fan is in first quadrant}, \\
\frac{\dot{\gamma}}{\dot{\gamma}} < \frac{\dot{\gamma}}{\dot{\gamma}} & \text{if fan is in second quadrant}.
\end{cases} \tag{5.9}
\]

Equations (5.8)-(5.9) are the analog for fans of equations (4.8)-(4.9) for shocks. Conversely, given numbers \((\tilde{\gamma}, \tilde{v}), (\frac{\dot{\gamma}}{\dot{\gamma}}, \frac{\dot{\gamma}}{\dot{\gamma}})\) which conform to (5.7)-(5.9), one can construct a unique fan between the rays \( x = \tilde{c}t \) and \( x = \frac{\dot{c}}{\dot{c}}t \) where \( \tilde{c} \) is given by (5.6).

The general scale-invariant solution to the Riemann problem has the form shown in Figure 5.1: between any two rays \( x = \dot{s}_i t \) and \( x = \dot{s}_{i+1} t \) the fields \( \gamma, v \) are either constants or fans; the rays themselves may or may not correspond to discontinuities. If \( x = \dot{s}_i t \) is a discontinuity, the jump conditions (3.3) must be satisfied across it so that

\[
\begin{cases} 
\dot{s}_i (\gamma_i - \gamma_{i-1}) = -(v_i - v_{i-1}), \\
\rho \dot{s}_i^2 = (\dot{\sigma}(\gamma_i) - \dot{\sigma}(\gamma_{i-1}))/ (\gamma_i - \gamma_{i-1}),
\end{cases} \tag{5.10}
\]

where \((\gamma_i, v_i)\) and \((\gamma_{i-1}, v_{i-1})\) are the limiting values of strain and particle velocity on the right and left respectively of this discontinuity. Let \( f_i = \dot{f}(\gamma_{i-1}, \gamma_i) \) with \( \dot{f} \) defined by (3.7) stand for the driving force on this discontinuity; the entropy inequality (3.8) then requires that

\[
f_i \dot{s}_i \geq 0. \tag{5.11}
\]

An admissible solution of the Riemann problem is a pair \( \gamma(x,t), v(x,t) \) of the form just described with (5.10)-(5.11) enforced at all discontinuities.
5.2 The Structure of Admissible Solutions to the Riemann Problem

The initial data in (5.1) is said to be metastable if neither of the initial strains $\gamma_L, \gamma_R$ belong to the unstable phase. Before constructing explicit global solutions to the Riemann problem, it is helpful to establish some general results pertaining to the permissible solution forms that are consistent with the entropy inequality.

Let $(\gamma, v)$ be an admissible solution of the Riemann problem with metastable initial data.

(i) The strain $\gamma(x, t)$ does not belong to the unstable phase at any point $(x, t)$ in the upper-half plane.

This result implies that if the initial data does not involve the unstable phase, then at no later time does the solution involve the unstable phase. We prove this claim by contradiction. Suppose that this proposition is false. Since unstable phase fans and shocks do not exist, there must necessarily be two phase boundaries in the $(x, t)$-plane, say $x = s_k t$ and $x = s_{k+1} t$ with $s_k < s_{k+1}$, such that the state between them is constant with the associated strain $\gamma_k$ in the unstable phase and with neither of the strains $\gamma_{k-1} = \gamma(s_k t-, t)$ and $\gamma_{k+1} = \gamma(s_{k+1} t+, t)$ in the unstable phase. From (3.7) and (4.1)-(4.4) one sees that the driving force on the discontinuity $x = s_k t$ is positive; the entropy inequality (5.11) thus implies that $s_k \geq 0$. Similarly the driving force on $x = s_{k+1} t$ must be negative and so $s_{k+1} \leq 0$. Thus $s_k \geq s_{k+1}$ which is a contradiction. This establishes the proposition.

The next six propositions are concerned with the possibility of having shock waves, phase boundaries and wave fans adjacent to each other. When addressing fans, it is sufficient for our purposes to restrict attention to fans which do not terminate on discontinuities. Thus in the propositions that follow, if $x = s_k t$ and $x = s_{k+1} t$ are two rays in the $(x, t)$-plane between
which the field is a fan, these two rays will be assumed not to be discontinuities. Features of fans which do terminate at discontinuities can be deduced from the following results by suitable limiting arguments.

Let \((\gamma, v)\) be an admissible solution of the Riemann problem with metastable initial data.

(ii) Let \(x = \dot{s}_kt, x = \dot{s}_{k+1}t, x = \dot{s}_{k+2}t\) and \(x = \dot{s}_{k+3}t\), be four rays in the same quadrant of the \((x, t)\)-plane. If the field in the interior wedge \(\dot{s}_{k+1}t < x < \dot{s}_{k+2}t\) is constant, then the field in the other two wedges cannot both be fans.

(iii) Let \(x = \dot{s}_kt\) and \(x = \dot{s}_{k+1}t\) be two rays in the same quadrant of the \((x, t)\)-plane between which the field is constant. Then these two rays cannot both be shock waves.

(iv) Let \(x = \dot{s}_kt\) and \(x = \dot{s}_{k+1}t\) be two rays in the same quadrant of the \((x, t)\)-plane between which the field is constant. Then these two rays cannot both be phase boundaries.

(v) Let \(x = \dot{s}_kt, x = \dot{s}_{k+1}t\) and \(x = \dot{s}_{k+2}t\) be three rays in the same quadrant of the \((x, t)\)-plane. Suppose that the field between any two of these rays is a fan. Then the third ray cannot be a shock.

(vi) Let \(x = \dot{s}_kt, x = \dot{s}_{k+1}t\) and \(x = \dot{s}_{k+2}t\) be three rays in the same quadrant of the \((x, t)\)-plane. Suppose that the field between the two slowest rays is a fan. Then the remaining ray cannot be a phase boundary. (The converse case is possible: if the field between the two fastest rays is a fan the remaining ray may be a phase boundary.)

(vii) Let \(x = \dot{s}_kt\) and \(x = \dot{s}_{k+1}t\) be two rays in the same quadrant of the \((x, t)\)-plane. If the slower ray is a shock, then the faster ray cannot be a phase boundary.
(The converse case is possible: if the faster ray is a shock, the slower ray may be a phase boundary.)

The first three of these propositions state that two wave fans, two shock waves and two phase boundaries cannot be adjacent to each other. The next one states that a fan and a shock cannot be adjacent to each other. On the other hand according to proposition (vi), a fan and a phase boundary may be adjacent to each other provided the phase boundary is subsonic. Similarly a shock and a phase boundary may be adjacent to each other provided the phase boundary travels more slowly than the shock.

It is clearly sufficient to prove these results in any one quadrant of the upper-half of the \((x,t)\)-plane and so we shall consider only the first quadrant. Thus in each of these propositions we have \(0 < \dot{s}_k < \dot{s}_{k+1} < \dot{s}_{k+2} < \dot{s}_{k+3}\).

To prove proposition (ii), suppose that it is false so that the field in \(\dot{s}_k t < x < \dot{s}_{k+1} t\) and \(\dot{s}_{k+2} t < x < \dot{s}_{k+3} t\) are fans while the field is constant in the intermediate wedge. It is not possible that one of these fans is a low-strain fan while the other is a high-strain fan, since such fans belong to different phases and so must necessarily be separated by a phase boundary. For two fans of the same type, (5.2) directly shows that they must in fact be smoothly connected to each other to become a single fan with \(\dot{s}_{k+1} = \dot{s}_{k+2}\). This contradicts the assumption that \(\dot{s}_{k+1} < \dot{s}_{k+2}\). The assertion (ii) is thus proved.

We now turn to the proof of proposition (iii). Suppose that the proposition is false so that \(x = \dot{s}_k t\) and \(x = \dot{s}_{k+1} t\) are both shocks and the field between them is constant. Note first that a low-strain shock cannot be adjacent to a high-strain shock, since they must be separated by a phase boundary. Suppose that both \(x = \dot{s}_k t\) and \(x = \dot{s}_{k+1} t\) are low-strain shocks. The strain and velocity field thus have the form
\[ \gamma(x, t), v(x, t) = \begin{cases} \gamma_{k-1}, v_{k-1}, & x = \dot{s}_k t-, \\ \gamma_k, v_k, & \dot{s}_k t < x < \dot{s}_{k+1} t, \\ \gamma_{k+1}, v_{k+1}, & x = \dot{s}_{k+1} t+, \end{cases} \tag{5.12} \]

where \( \gamma_{k-1}, \gamma_k \) and \( \gamma_{k+1} \) are distinct and all three belong to \((-1, \gamma_M]\). According to (5.11), the driving forces acting on these two shock waves must be non-negative. With the help of (3.7) one finds that this implies that \( \gamma_{k-1} < \gamma_k < \gamma_{k+1} \). Since \( \dot{\sigma}'(\gamma) > 0 \) and \( \dot{\sigma}''(\gamma) < 0 \) on \((-1, \gamma_M)\), this, together with the fact that \(-1 < \gamma_{k-1} < \gamma_k < \gamma_{k+1} \leq \gamma_M \), implies that

\[
\frac{\dot{\sigma}(\gamma_{k+1}) - \dot{\sigma}(\gamma_k)}{\gamma_{k+1} - \gamma_k} < \frac{\dot{\sigma}(\gamma_k) - \dot{\sigma}(\gamma_{k-1})}{\gamma_k - \gamma_{k-1}}. \tag{5.13} \]

Equations (5.10) and (5.13) then yield \( \dot{s}_k > \dot{s}_{k+1} \) which is a contradiction. In a similar way, we can prove that two high-strain shocks cannot be adjacent to each other. This proves proposition (iii).

Propositions (iv) and (vii) may be established by arguments that are very similar to the above.

Next we turn to the proof of proposition (v). Suppose that it is false. Note first that a low-strain shock cannot be adjacent to a high-strain fan (or vice versa) since they must be separated by a phase boundary. Suppose that \( x = \dot{s}_k t \) is a low-strain shock and that the field between the rays \( x = \dot{s}_{k+1} t \) and \( x = \dot{s}_{k+2} t \) is a low-strain fan:

\[
\gamma(x, t), v(x, t) = \begin{cases} \gamma_{k-1}, v_{k-1}, & x = \dot{s}_k t-, \\ \gamma_k, v_k, & \dot{s}_k t < x \leq \dot{s}_{k+1} t, \\ \gamma(L)(x/t), v(L)(x/t), & \dot{s}_{k+1} t \leq x \leq \dot{s}_{k+2} t, \\ \gamma_{k+2}, v_{k+2}, & x = \dot{s}_{k+2} t+. \end{cases} \tag{5.14} \]

where \( \gamma_{k-1}, \gamma_k, \) and \( \gamma_{k+2} \) all lie in the low-strain phase and \( \gamma(L) \) is the strain field in a low-strain fan given by (5.4). For the material (4.1)-(4.4), equation (5.2) and the fact that
\( \dot{s}_{k+1} < \dot{s}_{k+2} \) implies that \( \gamma_k > \gamma_{k+2} \). Next, the entropy inequality (5.11) requires the driving force \( \dot{f}(\gamma_{k-1}, \gamma_k) \) at \( x = \dot{s}_k t \) to be non-negative; by (3.7), this implies that \( \gamma_{k-1} < \gamma_k \). Since \( \dot{\sigma}'(\gamma) > 0 \) and \( \dot{\sigma}''(\gamma) < 0 \) on \([\gamma_{k-1}, \gamma_k]\), this, together with the fact that \(-1 < \gamma_{k-1} < \gamma_k \leq \gamma_M\) implies that

\[
\dot{\sigma}'(\gamma_k) < \frac{\dot{\sigma}(\gamma_k) - \dot{\sigma}(\gamma_{k-1})}{\gamma_k - \gamma_{k-1}}. \tag{5.15}
\]

Equations (4.7), (5.10) now yield \( \dot{s}_k > \dot{s}_{k+1} \) which is a contradiction.

The remaining cases where \( x = \dot{s}_{k+2} t \) is a low-strain shock and the field between the rays \( x = \dot{s}_k t \) and \( x = \dot{s}_{k+1} t \) is a low-strain fan, and when the shock and fan are both high-strain ones, can be treated similarly. This establishes proposition (v).

The proof on proposition (vi) is entirely analogous.

The preceding results imply that the form of admissible solutions to the Riemann problem with metastable initial data is in fact much simpler than that described in Figure 5.1. Note that the results (iii)-(vii) depend critically on the entropy inequality (5.11).
Figure 5.1. General form of solutions to the Riemann problem.
Chapter 6

Explicit Solutions to the Riemann Problem

The results established in the preceding chapter allow one to determine all admissible solutions to the Riemann problem in the case of metastable initial data. From here on we shall consider only the special Riemann problem in which the initial strains $\gamma_L$ and $\gamma_R$ are both in the low-strain phase, and $\gamma_R$ is smaller than $\gamma_L$:

$$\gamma_L \in (-1, \gamma_M], \quad \gamma_R \in (-1, \gamma_M], \quad \gamma_R < \gamma_L.$$  \hspace{1cm} (6.1)

At the initial instant, the entire bar is in the low-strain phase. At a later instant, a particle of the bar may or may not change its phase. It is convenient in the following analysis to consider these two cases separately.

6.1 Solutions Involving No Phase Change

In this case, the solution does not involve any phase boundaries. In view of the first proposition in Section 5.2, neither does it involve the unstable phase at any time $t > 0$. Next, in view of propositions (ii), (iii) and (v), the solution $\gamma, v$ can only involve a single low-strain shock wave or a single low-strain fan in each quadrant of the upper half of the $x, t$-plane. Thus the solution must have one of the four forms shown in Figure 6.1(a)-(d).
(i(a)) Solution with two shock waves. Figure 6.1(a). Consider a solution having the form shown in Figure 6.1(a):

\[
\gamma, v = \begin{cases} 
\gamma_L, v_L, & -\infty < x < \hat{s}_1 t, \\
\gamma, v, & \hat{s}_1 t < x < \hat{s}_2 t, \\
\gamma_R, v_R, & \hat{s}_2 t < x < \infty,
\end{cases}
\]

in which \(\bar{\gamma}, \bar{v}, \hat{s}_1\) and \(\hat{s}_2\) are to be found such that \(\bar{\gamma} \in (-1, \gamma_M]\) and \(\hat{s}_1 < 0 < \hat{s}_2\).

The jump conditions (5.10) and the entropy inequality (5.11) at each of the two shock waves require that

\[
-(v_R - \bar{v}) = \hat{s}_2 (\gamma_R - \bar{\gamma}), \quad \hat{s}_2 = \frac{\hat{\sigma}(\gamma_R) - \hat{\sigma}(\bar{\gamma})}{\rho(\gamma_R - \bar{\gamma})}, \quad \gamma_R > \bar{\gamma},
\]

\[
-(\bar{v} - v_L) = \hat{s}_1 (\bar{\gamma} - \gamma_L), \quad \hat{s}_1 = -\frac{\hat{\sigma}(\bar{\gamma}) - \hat{\sigma}(\gamma_L)}{\rho(\bar{\gamma} - \gamma_L)}, \quad \bar{\gamma} < \gamma_L.
\]

The inequalities in (6.1), (6.3), (6.4), together with the requirement that \(\bar{\gamma}\) be in the low-strain phase, imply that

\[-1 < \bar{\gamma} < \gamma_R.
\]

Combining (6.3)-(6.4) yields

\[v_R - v_L = H(\bar{\gamma}),
\]

where \(H(\gamma)\) is defined on \((-1, \gamma_R]\) by

\[H(\gamma) \equiv -\sqrt{[\hat{\sigma}(\gamma_R) - \hat{\sigma}(\bar{\gamma})]/\rho} - \sqrt{[\hat{\sigma}(\bar{\gamma}) - \hat{\sigma}(\gamma_L)](\gamma - \gamma_L)/\rho}.
\]

It can be verified that \(H(\gamma)\) increases monotonically on \((-1, \gamma_R]\) from the value \(-\infty\) at \(\gamma = -1\) to the value \(H_R \equiv H(\gamma_R)\) at \(\gamma = \gamma_R\). Thus if the initial data is such that \(-\infty < \gamma < \gamma_M\),
$v_R - v_L < H_R$, there is a unique root $\tilde{\gamma}$ of (6.6) in the range $-1 < \tilde{\gamma} < \gamma_R (\leq \gamma_M)$. The remaining unknowns \( \tilde{v}, \tilde{s}_1 \) and \( \tilde{s}_2 \) are then given immediately by (6.3) and (6.4).

Thus, there exists a unique admissible solution of the form (6.2) corresponding to Figure 6.1(a) if and only if the given initial data (5.1), (6.1) is such that $-\infty < v_R - v_L < H_R$.

(i(b)) Solution with a shock wave and a wave fan. Figure 6.1(b). Consider next a solution of the form shown in Figure 6.1(b):

$$
\gamma, v = \begin{cases} 
\gamma_L, v_L, & -\infty < x < \bar{c}t, \\
\tilde{\gamma}, \tilde{v}, & \bar{c}t < x \leq c_R t, \\
\dot{\gamma}(x/t), \dot{v}(x/t), & c_R t < x < \infty,
\end{cases} \tag{6.8}
$$

where \( \tilde{\gamma}, \tilde{v}, \tilde{s}, \tilde{c} \) and \( c_R \) are to be determined such that $\tilde{\gamma} \in (-1, \gamma_M]$ and $\bar{c} < \tilde{c} < c_R$. The functions $\dot{\gamma}(x/t)$ and $\dot{v}(x/t)$ are the strain and velocity fields pertaining to a low-strain fan and are given by (5.4)-(5.5).

At the shock wave $x = \bar{c}t$, the jump conditions (5.10) and the entropy inequality (5.11) must hold:

$$
\tilde{v} = v_L - \bar{s}(\tilde{\gamma} - \gamma_L), \quad \bar{s} = -\sqrt{\frac{\dot{\sigma}(\tilde{\gamma}) - \dot{\sigma}(\gamma_L)}{\rho(\tilde{\gamma} - \gamma_L)}}, \quad \tilde{\gamma} < \gamma_L. \tag{6.9}
$$

Turning next to the fan, and by using (5.6)-(5.8), one finds

$$-(\tilde{v} - v_R) = \int_{\gamma_R}^{\tilde{\gamma}} c(\gamma) d\gamma, \quad c_R = c(\gamma_R), \quad \tilde{c} = c(\tilde{\gamma}), \quad \tilde{\gamma} > \gamma_R. \tag{6.10}
$$

Equations (6.9)-(6.10) may now be combined to yield

$$v_R - v_L = H(\tilde{\gamma}) \tag{6.11}
$$

where
\[
H(\gamma) \equiv -\sqrt{\left(\sigma(\gamma_L) - \sigma(\gamma)\right)(\gamma_L - \gamma) / \rho + \int_{\gamma_R}^{\gamma} c(\varepsilon) d\varepsilon}, \quad \gamma_R \leq \gamma \leq \gamma_L. \tag{6.12}
\]

One can verify that \(H(\gamma)\) increases monotonically on \([\gamma_R, \gamma_L]\) from the value \(H_R \equiv H(\gamma_R)\) at \(\gamma = \gamma_R\) to the value \(H_L \equiv H(\gamma_L)\) at \(\gamma = \gamma_L\). Thus if the initial data (5.1), (6.1) is such that \(H_R < v_R - v_L < H_L\), equation (6.11) yields a unique root \(\bar{\gamma}\) in the interval \(\gamma_R < \bar{\gamma} < \gamma_L\). The remaining unknowns \(\bar{v}, s, \bar{c}\) and \(c_R\) are then given by (6.9), (6.10).

Thus, there is a unique admissible solution of the form (6.8) corresponding to Figure 6.1(b) if and only if the given initial data (5.1), (6.1) is such that \(H_R < v_R - v_L < H_L\).

(i(c)) Solution with two wave fans. Figure 6.1(c). We seek solutions in the form of Figure 6.1(c):

\[
\gamma, v = \begin{cases} 
\gamma_L, v_L, & -\infty < x \leq c_L t, \\
\hat{\gamma}_1(x/t), \hat{v}_1(x/t), & c_L t \leq x \leq \bar{c}' t, \\
\check{\gamma}, \check{v}, & \bar{c}' t \leq x \leq \check{c} t, \\
\hat{\gamma}_2(x/t), \hat{v}_2(x/t), & \check{c} t \leq x \leq c_R t, \\
\gamma_R, v_R, & c_R t \leq x < \infty,
\end{cases} \tag{6.13}
\]

where \(\hat{\gamma}, \check{\gamma}, c_L, \bar{c}', \check{c} \) and \(c_R\) are to be determined such that \(\bar{\gamma} \in (-1, \gamma_M]\) and \(c_L < \bar{c}' < 0 < \check{c} < c_R\). The functions \(\hat{\gamma}_1, \hat{v}_1, \hat{\gamma}_2, \hat{v}_2, \) are the fields that are appropriate to a low-strain fan and are given by (5.4)-(5.5).

The analysis of this case is similar to that of the preceding one. One finds that there is a unique admissible solution of the form (6.13) corresponding to Figure 6.1(c) if and only if the given initial data (5.1), (6.1) is such that \(H_L < v_R - v_L < H_M\) where

\[
H(\gamma) \equiv \int_{\gamma_R}^{\gamma} c(\gamma) d\gamma + \int_{\gamma_L}^{\gamma} c(\gamma) d\gamma, \quad \gamma_L \leq \gamma \leq \gamma_M, \tag{6.14}
\]

and \(H_L \equiv H(\gamma_L), H_M \equiv H(\gamma_M)\).
(i(d)) Solution with a shock wave and a wave fan. Figure 6.1(d). For initial strains in the low-strain phase with $\gamma_R < \gamma_L$ as considered here, one finds that solutions having the form of Figure 6.1(d) do not exist. In the reverse case $\gamma_R > \gamma_L$, one finds that such solutions do exist in place of solutions of the form in Figure 6.1(b) which now do not exist.

It is readily seen from (6.7),(6.12) and (6.14) that $H(\gamma)$ is continuous at $\gamma = \gamma_R$ and $\gamma = \gamma_L$. In the respective limits $v_R - v_L \to H_R^-$, and $v_R - v_L \to H_R^+$, the solution forms shown in Figure 6.1(a) and Figure 6.1(b) coincide. The limiting solution involves a single shock wave traveling left and has no other shock waves. Similarly, the solution forms in Figure 6.1(b) and Figure 6.1(c) coincide in the respective limits $v_R - v_L = H_L^-$, $v_R - v_L = H_L^+$. In this case the solution involves a single rightward moving wave fan.

In summary, when initial data (5.1), (6.1) is such that $-\infty < v_R - v_L < H_M$, it has been shown that there is a unique admissible solution that involves no phase change to the Riemann problem. Further discussion of these solutions is postponed until Section 6.3.

In order to find solutions when the initial data is such that $v_R - v_L > H_M$ we must consider solutions which involve a phase change.

### 6.2 Solutions Involving a Phase Change

If a solution involves a phase change, the results (iv), (vi) and (vii) of Section 5.2 show that each quadrant of the upper-half of the $(x, t)$-plane has precisely one phase boundary together with either one shock wave or one wave fan. Moreover, each phase boundary is necessarily subsonic so that the speed of the shock wave or wave fan is greater than that of the phase boundary. Thus the solution must have one of the four forms shown in Figure 6.2(a)-(d).

(ii(a)) Solutions with two phase boundaries and two fans. Figure 6.2(a). Suppose that the solution has the form shown in Figure 6.2(a):
\[\begin{align*}
\gamma, v = & \begin{cases}
\gamma_L, v_L, & -\infty < x \leq c_L t, \\
\gamma_1(x/t), v_1(x/t), & c_L t \leq x \leq c_A t, \\
\gamma_A, v_A, & c_A t \leq x < \delta_1 t, \\
\gamma_2(x/t), v_2(x/t), & \delta_1 t \leq x < \delta_2 t, \\
\gamma_R, v_R, & \delta_2 t \leq x \leq c_R t, \\
\gamma, v, & c_R t \leq x < \infty,
\end{cases}
\end{align*}\]

where \(\gamma_A, v_A, \gamma_1, v_1, \gamma_2, v_2, c_L, c_A, c_R, \delta_1, \delta_2\) are to be determined such that \(\gamma_A, \gamma_1, \gamma_2 \in (-1, \gamma_M), \gamma_1 \in [\gamma_m, \infty)\) and \(c_L < c_A < \delta_1 \leq 0 \leq \delta_2 < c_R \). The functions \(\gamma_1(x/t), v_1(x/t), \gamma_2(x/t), v_2(x/t)\) are the fields appropriate to a low-strain fan and are given by (5.4)-(5.5).

The jump conditions (5.10) at the two phase boundaries lead to
\[\begin{align*}
\dot{\gamma} & = v - \dot{\delta}_2(\gamma_2 - \gamma), \\
\dot{\delta}_2 & = \frac{\dot{\sigma}(\gamma_2) - \dot{\sigma}(\gamma)}{\rho(\gamma_2 - \gamma)},
\end{align*}\]

\[\begin{align*}
\bar{v} & = v_A - \dot{\delta}_1(\gamma_1 - \gamma_A), \\
\dot{\delta}_1 & = -\frac{\dot{\sigma}(\gamma_1) - \dot{\sigma}(\gamma_A)}{\rho(\gamma_1 - \gamma_A)},
\end{align*}\]

while the requirements (5.6), (5.7) at the two fans give
\[\begin{align*}
\dot{c} & = c(\gamma_1), \\
c_R & = c(\gamma_R), \\
-(\dot{v} - v_R) & = \int_{\gamma_R}^{\gamma} c(\gamma)d\gamma, \\
c_A & = -c(\gamma_A), \\
c_L & = -c(\gamma_L), \\
v_A - v_L & = \int_{\gamma_L}^{\gamma_A} c(\gamma)d\gamma.
\end{align*}\]

The roots of these equations are subject to the restrictions imposed by the entropy inequality (5.11) at each phase boundary, the inequality (5.8) at each fan, and the requirement that the strains belong to either low-strain phase or high-strain phase as appropriate. These restrictions lead to the following inequalities
The 10 equations (6.16)-(6.19) are to be solved for the 12 unknown quantities listed below (6.15). Therefore, one anticipates that when there exists a solution, there would in fact be a \textit{two-parameter family of solutions}. For reasons of algebraic simplicity, we assume that

\[ s_2 = -s_1 \equiv s, \]  

(6.23)

where \( s \geq 0 \). This assumption does not change the essential characteristics of the solutions, but leads to a considerable simplification in the analysis; in particular, one now expects a one-, rather than a two-, parameter family of solutions. In what follows, the strain \( \gamma \) shall be treated as this parameter. The assumption (6.23) also leads to \( \gamma_A = \gamma \).

It follows from (6.20)-(6.23) and the requirement \( 0 \leq s < c \) in (6.15) that

\[ c(\gamma) > s, \quad f(\gamma, \gamma) \geq 0, \quad \gamma > \gamma_L, \quad \sigma(\gamma) > \sigma(\gamma) \]  

(6.24)

necessarily hold. These inequalities define a region \( D_1 \subset \Gamma_3 \) in the \( \gamma, \gamma \)-plane; Figure 6.3 displays this region in the case

\[ \gamma_a > P_\infty > \gamma_L, \quad \gamma_R > R_\infty, \]  

(6.25)

and for reasons of definiteness we shall frame our analysis from hereon for this particular case; the analysis can be trivially modified to handle the case when the ordering of the strains differs from (6.25).

By combining equations (6.16)-(6.19), one can reduce the question of their solvability to the following problem: given \( \gamma \), find a root \( \gamma \) with \( (\gamma, \gamma) \in D_1 \) of the equation

\[ f(\gamma, \gamma) \geq 0, \quad f(\gamma, \gamma) \leq 0, \]  

(6.20)

\[ \gamma > \gamma_R, \quad \gamma_A > \gamma_L, \]  

(6.21)

\[ -1 < \gamma \leq \gamma_M, \quad -1 < \gamma_A \leq \gamma_M, \quad \gamma \geq \gamma_M. \]  

(6.22)
\[ v_R - v_L = G(\gamma, \dot{\gamma}), \]  
(6.26)

where \( G \) is defined for \((\gamma, \dot{\gamma}) \in D_1 \) by
\[
G(\gamma, \dot{\gamma}) \equiv \int_{\gamma_L}^{\gamma} c(\gamma) d\gamma + \int_{\gamma_R}^{\dot{\gamma}} c(\gamma) d\gamma + 2\sqrt{[\dot{\delta}(\gamma) - \delta(\gamma)](\dot{\gamma} - \gamma)/\rho}. \]  
(6.27)

The curves \( AD, AI \) and \( CD \) of Figure 6.3 which comprise part of the boundary of the region \( D_1 \) were defined previously in (4.10)-(4.12). At each fixed \( \gamma > \gamma_{03} \), the function \( G \) in (6.27) increases monotonically with increasing \( \dot{\gamma} \). Therefore, for each \( \gamma \geq \gamma_{03} \), (6.26) can be solved uniquely for \( \dot{\gamma} \) provided \( v_R - v_L \) lies in a suitable range: for \( \gamma_{03} \leq \gamma \leq \gamma_{03} \), this range is \( G(\gamma, Q(\gamma)) \leq v_R - v_L \leq G(\gamma, T(\gamma)) \); for \( \gamma_{03} \leq \gamma < \gamma_I \), it is \( G(\gamma, Q(\gamma)) \leq v_R - v_L < G(\gamma, P(\gamma)) \); and for \( \gamma > \gamma_I \), it is \( G(\gamma, \gamma_L) < v_R - v_L < G(\gamma, P(\gamma)) \). Here \( \gamma_I > \gamma_{m} \) is the strain-level at which the driving force \( \dot{f}(\gamma, \gamma_L) \) vanishes, see Figure 6.3. Once \( \dot{\gamma} \) has been thus determined, all of the other unknowns can be found in terms of \( \gamma \) and the initial data without further restriction. Further discussion of this solution is postponed until Section 6.3.

(ii(b)) Solutions with two phase boundaries, one shock wave and one wave fan. Figure 6.2(b). Consider next solutions having the form of Figure 6.2(b):
\[
\gamma, v = \begin{cases} 
\gamma_L, v_L, & -\infty < x < \dot{s}_1 t, \\
\gamma_A, v_A, & \dot{s}_1 t < x < \dot{s}_2 t, \\
\tilde{\gamma}, \tilde{v}, & \dot{s}_2 t < x < \dot{s}_3 t, \\
\dot{\gamma}, \dot{v}, & \dot{s}_3 t < x \leq \dot{c} t, \\
\dot{\gamma}, \dot{v}, & \dot{c} t \leq x \leq c_R t, \\
\gamma_R, v_R, & c_R t \leq x < \infty,
\end{cases} \]  
(6.28)

where \( \gamma_A, v_A; \tilde{\gamma}, \tilde{v}, \dot{\gamma}, \dot{v}, \dot{s}_1, \dot{s}_2, \dot{s}_3, \dot{c} \) and \( c_R \) are to be determined such that \( \gamma_A, \dot{\gamma} \in (-1, \gamma_M] \), \( \tilde{\gamma} \in [\gamma_m, \infty) \) and \( \dot{s}_1 < \dot{s}_2 \leq 0 < \dot{s}_3 < \dot{c} \leq c_R \). The functions \( \dot{\gamma}, \dot{v} \) correspond to the fields in a low-strain fan and are given by (5.4)-(5.5).
The analysis of this case is similar to the previous one and so we merely state the results. Again, there is a two-parameter family of solutions in general, and for algebraic simplicity we assume that $\dot{s}_3 = -\dot{s}_2 \equiv \dot{s}$, thus reducing it to a one-parameter family of solutions.

The strains $(\bar{\gamma}, \dot{\gamma})$ must lie in the region $D_2 \subset \Gamma_3$ shown in Figure 6.3. Given the strain $\bar{\gamma}$, one is to find a root $\dot{\gamma}$ with $(\bar{\gamma}, \dot{\gamma}) \in D_2$ of the equation

$$v_R - v_L = G(\bar{\gamma}, \dot{\gamma}),$$

where $G$ is defined by

$$G(\bar{\gamma}, \dot{\gamma}) \equiv -\sqrt{[\dot{s}(\gamma_L) - \dot{s}(\dot{\gamma})](\gamma_L - \dot{\gamma})/\rho + \int_{\gamma_R}^{\gamma} c(\gamma)d\gamma + 2\sqrt{[\dot{s}(\bar{\gamma}) - \dot{s}(\dot{\gamma})](\bar{\gamma} - \dot{\gamma})/\rho}, \quad \text{for } (\bar{\gamma}, \dot{\gamma}) \in D_2. \quad (6.29)$$

At each fixed $\bar{\gamma} > \gamma_I$ the function $G$ increases monotonically with $\dot{\gamma}$. Therefore for each $\bar{\gamma}$ in the respective ranges $\gamma_I < \bar{\gamma} < \gamma_K$ and $\bar{\gamma} > \gamma_K$, (6.29) can be uniquely solved for $\dot{\gamma}$ provided $G(\bar{\gamma}, Q(\bar{\gamma})) \leq v_R - v_L < G(\bar{\gamma}, \gamma_L)$ and $G(\bar{\gamma}, \gamma_R) < v_R - v_L < G(\bar{\gamma}, \gamma_L)$ respectively. Here $\gamma_K(>\gamma_m)$ is the value of strain at which the driving force $\dot{f}(\gamma_K, \gamma_R)$ vanishes, see Figure 6.3. Once $\dot{\gamma}$ has been thus determined, all of the other unknowns can be found in terms of $\bar{\gamma}$ and the given initial data without further restriction. Further discussion of this solution is postponed until Section 6.3.

(ii(c)) Solutions with two phase boundaries and two shock waves. Figure 6.2(c). Here we seek solutions in the form of Figure 6.2(c):

$$\gamma, v = \begin{cases} 
\gamma_L, v_L, & -\infty < x < \dot{s}_1 t, \\
\gamma_A, v_A, & \dot{s}_1 t < x < \dot{s}_2 t, \\
\bar{\gamma}, \bar{v}, & \dot{s}_2 t < x < \dot{s}_3 t, \\
\dot{\gamma}, \dot{v}, & \dot{s}_3 t < x < \dot{s}_4 t, \\
\gamma_R, v_R, & \dot{s}_4 t < x < \infty,
\end{cases} \quad (6.31)$$
in which $\gamma_A, v_A, \bar{\gamma}, \bar{v}, \hat{\gamma}, \hat{v}, \hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3, \hat{\delta}_4$ are to be determined such that $\gamma_A, \hat{\gamma} \in (-1, \gamma_M], \bar{\gamma} \in [\gamma_m, \infty)$ and $\hat{\delta}_1 < \hat{\delta}_2 \leq 0 \leq \hat{\delta}_3 < \hat{\delta}_4$.

The analysis of this case is again similar to the previous ones. There is again a two-parameter family of solutions in general, and for algebraic simplicity we assume that $\hat{\delta}_3 = -\hat{\delta}_2 \equiv \hat{\delta}$, thus reducing it to a one-parameter family of solutions.

The strains $\bar{\gamma}, \hat{\gamma}$ must lie in the region $D_3 \subset \Gamma_3$ shown in Figure 6.3. Given the strain $\bar{\gamma}$, one is to find a root $\hat{\gamma}$ with $(\bar{\gamma}, \hat{\gamma}) \in D_3$ of the equation

$$v_R - v_L = G(\bar{\gamma}, \hat{\gamma})$$

(6.32)

where $G$ is defined by

$$G(\bar{\gamma}, \hat{\gamma}) = -\sqrt{[\hat{\sigma}(\bar{\gamma}) - \hat{\sigma}(\hat{\gamma})](\gamma_L - \bar{\gamma})/\rho - \sqrt{[\hat{\sigma}(\gamma_R) - \hat{\sigma}(\hat{\gamma})](\gamma_R - \bar{\gamma})/\rho}$$

$$+ 2\sqrt{[\hat{\sigma}(\bar{\gamma}) - \hat{\sigma}(\hat{\gamma})](\bar{\gamma} - \hat{\gamma})/\rho}, \text{ for } (\bar{\gamma}, \hat{\gamma}) \in D_3.$$  

(6.33)

At each fixed $\bar{\gamma} > \gamma_K$ the function $G$ increases monotonically with increasing $\hat{\gamma}$ and therefore (6.32) can be uniquely solved for $\hat{\gamma}$ provided $G(\bar{\gamma}, Q(\bar{\gamma})) \leq v_R - v_L < G(\bar{\gamma}, \gamma_R)$. Once $\hat{\gamma}$ has been determined, all of the other unknowns can be found without further restriction.

(ii(d)) Solutions with two phase boundaries, one fan and one shock waves. Figure 6.2(d).

One can show that solutions having the form shown in Figure 6.2(d) do not exist for the initial data (5.1), (6.1) being studied here. Such solutions do exist in the case $\gamma_R > \gamma_L$ in which event solutions of the form of Figure 6.2(b) do not exist.

It is clear from (6.27), (6.30) and (6.33) that $G$ is continuous across the lines $IJ$ and $KL$ in the $(\bar{\gamma}, \hat{\gamma})$-plane shown in Figure 6.3. In the respective limits, for $\bar{\gamma} > \gamma_I$, $v_R - v_L \to G(\bar{\gamma}, \gamma_L -)$ and $v_R - v_L \to G(\bar{\gamma}, \gamma_L +)$, the solution forms shown in Figure 6.2(a) and Figure 6.2(b) coincide. This limiting solution involves a wave fan traveling right and two phase boundaries. Similarly, the solution forms in Figure 6.2(b) and Figure 6.2(c) coincide in the
limits \( v_R - v_L \rightarrow G(\bar{\gamma}, \gamma_R-) \) and \( v_R - v_L \rightarrow G(\bar{\gamma}, \gamma_R+) \). In this case, the solution involves a leftward moving shock wave and again two phase boundaries.

### 6.3 Summary of All Solutions

In the preceding two sections we constructed all solutions to a Riemann problem and it is useful to examine these solutions on the \((\bar{\gamma}, v_R - v_L)\)-plane. Note that \( \bar{\gamma} \) is the final strain in the bar at large time while \( v_R - v_L \) is part of the given initial data. For solutions that involve a phase change, we map the regions \( D_1, D_2 \) and \( D_3 \) of the \((\bar{\gamma}, \gamma)\)-plane into the respective regions \( D'_1, D'_2 \) and \( D'_3 \) in the \((\bar{\gamma}, v_R - v_L)\)-plane by using the mapping \( v_R - v_L = G(\bar{\gamma}, \gamma) \), \( \bar{\gamma} = \bar{\gamma} \). The resulting regions are shown hatched in Figure 6.4. The boundary curves \( A'D', D'C', A'B', I'J' \) and \( K'L' \) are the images of the respective curves \( AD, DC, AB, IJ \) and \( KL \), and are defined by \( v_R - v_L = G(\bar{\gamma}, T(\bar{\gamma})) \), \( v_R - v_L = G(\bar{\gamma}, P(\bar{\gamma})) \), \( v_R - v_L = G(\bar{\gamma}, Q(\bar{\gamma})) \), \( v_R - v_L = G(\bar{\gamma}, \gamma_L) \) and \( v_R - v_L = G(\bar{\gamma}, \gamma_R) \), respectively.

The solution corresponding to a point in \( D'_1 \) involves two phase boundaries and two wave fans (Figure 6.2(a)). At a point in \( D'_2 \) the solution involves two phase boundaries, one wave fan and one shock wave (Figure 6.2(b)), while in \( D'_3 \) it involves two phase boundaries and two shock waves (Figure 6.2(c)). It is useful to note that the solution corresponding to a point on the respective curves \( A'D', D'C' \) and \( A'B' \) involves phase boundaries that are stationary, sonic and dissipation-free respectively.

Solutions which do not involve a phase change can also be described on this plane by plotting, for each type of solution, the curve \( v_R - v_L = H(\bar{\gamma}) \). The result is the curve \( P'Q'R'S' \) shown in Figure 6.4.

Each of the curves \( P'Q'R'S', A'B', A'D', D'C', I'J' \) and \( K'L' \) can be shown to be monotonically increasing. The figure reveals two distinct types of non-uniqueness. For all initial
data such that \( v_R - v_L < H_M \) there is a unique admissible solution that involves no phase change. On the other hand, when the initial data is such that \( v_R - v_L > G_0 \equiv G(\gamma_{01}, \gamma_{03}) \), there is a one-parameter family of solutions all of which involve a phase change; the lack of uniqueness here arises because of the undetermined velocities of the phase boundaries. For initial data on the intermediate interval \( H_M > v_R - v_L > G_0 \) both of these types of solutions are available, in one of which the bar ultimately changes phase, while in the other it does not.
Figure 6.1  Form of solutions to Riemann problem without phase change.
Figure 6.2. Form of solutions to Riemann problem with phase change.
Figure 6.3. The regions $D_i$ in $(\bar{\gamma}, \dot{\gamma})$-plane on which the various solutions exist. The solution forms on $D_1$, $D_2$ and $D_3$ are, respectively, as described in Figure 6(a), (b) and (c).
Figure 6.4. The $(\gamma, v_R - v_L)$-plane. The strain $\gamma$ is the final strain in the bar at large times, $v_R - v_L$ is part of the initial data. The solution forms at points in $D_1', D_2'$ and $D_3'$ are, respectively, as described in Figure 6(a), (b) and (c). Points on $P'S'$ describe solutions that do not change phase.
Chapter 7

Kinetics and Nucleation

The propagation of a phase boundary is controlled by a kinetic relation which, in the simplest models, is a constitutively prescribed relation between the driving force $f$ acting on the phase boundary and its propagation speed $\dot{s}$:

$$f(t) = \phi(\dot{s}(t)).$$  \hspace{1cm} (7.1)

The continuum theory does not provide an explicit expression for $\phi$; this is obtained through suitable micro-mechanical modeling. It is sufficient for our purposes to merely assume that $\phi$ is a continuous function on $(-c_\infty, c_\infty)$ that increases with $\dot{s}$ and that the graph of $f = \phi(\dot{s})$ is a curve that lies in the region $\Gamma'_2 \cup \Gamma'_3$ described previously of the $(\dot{s}, f)$-plane; see Figure 4.3. The entropy inequality (5.11) implies that $\phi(\dot{s})\dot{s} \geq 0$ and, assuming $\phi$ to be continuous, that $\phi(0) = 0$.

The kinetic relation controls the propagation of an existing phase boundary. A nucleation criterion is needed in order to signal the onset of a phase transformation when the bar involves only a single phase. We assume that the transformation from the low-strain phase to the high-strain phase is nucleated whenever there exists a solution, corresponding to the given data, with the associated driving force $f$ at least as great as a given critical value $f_{cr}$; the parameter $f_{cr} > 0$ is also determined by the constitutive details of the material. Once the
nucleation criterion indicates that we should consider solutions that do change phase, the kinetic relation selects the particular one of these solutions that is appropriate.

The nucleation criterion \( f(\bar{\gamma}, \dot{\gamma}) = f_{cr} \) describes a curve \( \mathcal{N} \) in the \( \bar{\gamma}, \dot{\gamma} \)-plane that lies in the hatched portion \( \Gamma_3 \) in Figure 6.3. For clarity of the figure, this curve shall not be displayed; it can be readily verified that it has one end at a point \( N \) on \( AD \), and declines monotonically as \( \bar{\gamma} \) increases to \( \bar{\gamma} \to R_\infty \) as \( \dot{\gamma} \to \infty \). The coordinates of the point \( N \) are found by solving the pair of equations

\[
\begin{align*}
f(\bar{\gamma}, \dot{\gamma}) &= f_{cr}, \\
\sigma(\bar{\gamma}) &= \sigma(\dot{\gamma});
\end{align*}
\]

let \( (\bar{\gamma}_{cr}, \dot{\gamma}_{cr}) \) be this point and let \( G_{cr} \equiv G(\bar{\gamma}_{cr}, \dot{\gamma}_{cr}) \) be the corresponding value of \( G \) at this point. The image \( \mathcal{N}' \) of the curve \( \mathcal{N} \) in the \( (v_R - v_L, \bar{\gamma}) \)-plane is a curve in the hatched region shown in Figure 6.4 that rises monotonically commencing at a point \( N' \) on \( A'D' \). Again, for clarity of the figure, this image \( \mathcal{N}' \) shall not be displayed. The ordinate of the point \( N' \) is \( G_{cr} \). Thus according to the nucleation criterion, a solution with a phase change is selected whenever \( v_R - v_L \geq G_{cr} \), a solution without phase change is selected when \( v_R - v_L < G_{cr} \).

For \( v_R - v_L \geq G_{cr} \), we must consider the 1-parameter family of solutions that involve a phase change. In order to select the relevant solution from among this family, we begin by applying the kinetic relation (7.1) to the phase boundaries \( x = \dot{s}t \) in each of the solutions in Figure 6.2(a),(b) and (c); kinetic relation at the other phase boundary \( x = -\dot{s}t \) holds automatically. On using (3.7) and (4.6) in the kinetic relation (7.1) and ensuring that \( \dot{s} \geq 0 \), leads to

\[
\mathcal{K}: \quad \hat{f}(\bar{\gamma}, \dot{\gamma}) = \phi \left( \sqrt{\frac{\sigma(\dot{\gamma}) - \sigma(\bar{\gamma})}{\mu(\bar{\gamma} - \dot{\gamma})}} \right) \quad \Leftrightarrow \quad \dot{\gamma} = K(\bar{\gamma}) \quad \text{for} \quad \bar{\gamma} \geq \gamma_{03}.
\]

This describes a curve \( \mathcal{K} \) in the \( \bar{\gamma}, \dot{\gamma} \)-plane that lies in the region \( \Gamma_3 \) (the union of the hatched portions in Figure 6.3); it commences at the point \( A \) and declines monotonically.
with \( K(\tilde{\gamma}) \rightarrow R_\infty \) as \( \tilde{\gamma} \rightarrow \infty \). When mapped from here to the \((v_R - v_L, \tilde{\gamma})\)-plane, this yields the curve \( \mathcal{K}' \) defined by \( v_R - v_L = G(\tilde{\gamma}, K(\tilde{\gamma})) \), \( \tilde{\gamma} \geq \gamma_{03} \), see Figure 6.4. This curve commences at the point \( A' \) and the monotonicity of the kinetic response function \( \phi \) ensures that it rises monotonically without bound.

Thus since \( \mathcal{K}' \) increases monotonically for any given \( v_R - v_L > G_{cr} \), there is a unique strain \( \tilde{\gamma}(>\gamma_{cr}) \) which satisfies the kinetic relation.

Thus, in summary, we have shown that for the metastable initial data (5.1), (6.1), the Riemann problem has a unique solution that is consistent with the kinetic relation and the nucleation criterion. For initial data such that \( v_R - v_L < G_{cr} \) this solution involves no phase change and has the form given by Figures 6.1; the bar commences in the low-strain phase and remains there for all time. For \( v_R - v_L \geq G_{cr} \) the solution involves a phase change and has the form given by Figures 6.2; in this case the bar, which was initially in the low-strain phase, transforms eventually to the high strain phase. In either case, the eventual strain in the bar is \( \tilde{\gamma} \) and can read off Figure 6.4.

In closing, we note that some solutions to the present problem involved phase boundaries that were subsonic while others involved sonic phase boundaries (the latter correspond to points on \( D'C' \) in Figure 6.4). In the present analysis the kinetic relation was imposed on both of these types of phase boundaries and this leads to uniqueness. If the kinetic law had been applied only to subsonic phase boundaries, then solutions corresponding to points on both curves \( \mathcal{K}' \) and \( D'C' \) would be allowed. In this case the Riemann problem corresponding to given data would have a unique solution for \( v_R - v_L < H_M \) (see Figure 6.4) but for \( v_R - v_L \geq H_M \) the problem would have two solutions, one satisfying the kinetic relation and the other involving a sonic phase boundary. This was first observed, in a particular setting, by Truskinovsky (1992); he takes the view that the kinetic relation should only be
applied to subsonic phase boundaries and suggests that the accompanying non-uniqueness (for $v_R - v_L \geq H_M$ in the present setting) describes an instability.
Part B
Propagation of an Interface into an unstable Phase
Chapter 8

Introduction

Consider a material which is in its austenite phase at a temperature above the critical temperature. The free-energy function describing this material at this temperature has many local minima, one of which corresponds to austenite; see Figure 8.1(a). If the material is suddenly cooled to a temperature below the critical temperature, the free-energy function loses the local minimum corresponding to austenite. Since the material was cooled suddenly, it has not had sufficient time to rearrange its crystal structure and so, the instant after cooling, finds itself at the local maximum which has replaced the local austenite minimum (see Figure 8.1(b)). We shall speak of such material as being in an “unstable phase”. This unstable material transforms to one of the stable phases corresponding to a new local energy minimum. It is this process of transition from an unstable phase to a stable phase that we are concerned with in this portion of this thesis.

If the transition is controlled by diffusion, this process is referred to as “spinodal decomposition”, spinodal being the unstable phase. A vast literature on this subject exists, e.g. see the articles by Cahn (1961, 1962, 1966, 1968), Hilliard (1970), Cahn and Hillard (1958), and Hillert (1956, 1961). The physical field of greatest interest is the concentration \( c(x,t) \) of the diffusing species, and the evolution of \( c \) is controlled by the underlying kinetics.

Here we address the corresponding issue when the transition is displacive (structural
rearrangement) rather than diffusive such as is true in martenitic transformations. The physical field of primary interest is now the strain $\gamma(x,t)$ and the questions of interest in this investigation is whether the evolution of strain is controlled by kinetics, and if so, what is the nature of the kinetics.

We begin by studying the quasi-static evolution of material from the unstable (spinodal) phase to a stable phase. In our analysis we control the process by equilibrium and geometric compatibility requirements alone, and as expected, we find that the associated processes are non-unique. This implies that there is a lack of information in our description of the process, pointing presumably, to a need for kinetics.

In order to construct a description of the kinetics we “regularize” the theory by including in it the effects of strain gradient and viscosity. A sharp interface in the “un-regularized theory” goes over into a travelling wave in the regularized theory. A standard technique for deriving kinetics based on strain-gradient and viscous effects is to study such travelling waves and then take the limit as the travelling wave goes into a sharp discontinuity. The kinetics of metastable $\rightarrow$ stable transitions have been previously studied in this way, e.g. Abeyaratne and Knowles (1991(b)), Truskinovsky (1985). Surprisingly, the analysis carried out in this part shows that no kinetics are inherited when this technique is applied to the unstable $\rightarrow$ stable transitions. This means one of two things: either the mechanism controlling the kinetics is not related to strain-gradient and viscosity or, the process of evolution is not controlled by kinetics at all.

Finally we incorporate the effects of inertia into our analysis and study a Riemann problem describing the evolution of an interface which at $t = 0$ separates unstable material from stable material. We find that this problem is well-posed and has a unique solution without any need to incorporate a description of the kinetics of the unstable $\rightarrow$ stable transition.
We thus conclude from this study that the unstable $\rightarrow$ stable transition is controlled by inertial effects when the transition is displacive in character. This is in contrast to the analogous situation in diffusive transitions where the process is controlled by kinetic effects.

In Chapter 9 we consider a specific equilibrium problem for an elastic bar composed of a multi-phase material and we construct all possible equilibrium states. In Chapter 10 we then consider the quasi-static evolution of the bar and show that this problem is not well-posed. Assuming that this lack of well-posedness is due to a need for kinetics, in Chapter 11 we consider a regularized theory and study travelling waves within this theory. Since this fails to lead to a description of kinetics, in Chapter 12, we turn finally to the dynamic theory which includes inertial effects. The results are summarized in Chapter 13.
Figure 8.1. Free energy versus strain at temperature above and below the critical temperature.
Chapter 9
Equilibrium States

An elastic bar of unit cross-sectional area occupies the interval \([0, L]\) in an unstressed reference configuration. Let \(x\) denote the coordinate of a generic point of the bar in this configuration. In a deformation of the bar, the particle at \(x\) is carried to a new location \(x + u(x)\), where the displacement \(u\) is to be continuous with piecewise continuous first and second derivatives throughout the bar. The left-hand end of the bar is taken to be fixed and the bar elongates by an amount of \(\delta\). Thus

\[ u(L) = \delta, \quad u(0) = 0. \]  

The strain \(\gamma\) at a particle \(x\) is defined by

\[ \gamma(x) = u'(x) > -1 \]  

at points \(x\) where the derivative exists. The inequality in (9.2) assures the invertibility of the mapping \(x \rightarrow x + u(x)\). According to the assumptions on displacement, we allow for the possibility that a finite strain jump in the bar may occur; let \(x = s\) denote the location of this strain discontinuity when it occurs.

Let \(\sigma(x)\) denote the nominal stress field in the bar. Then equilibrium of the bar in the absence of body forces requires
\[ \sigma = F = \text{constant}, \quad 0 \leq x \leq L; \quad (9.3) \]

\( F \) is the force in the bar.

The elastic material is characterized by a stress-strain relation \( \sigma = \dot{\sigma}(\gamma) \). In order to allow for the possible occurrence of stationary strain discontinuities in a bar, we shall consider a material whose stress response function \( \dot{\sigma}(\gamma) \) at first increases, then decreases, and finally increases again as \( \gamma \) increases from the value \( \gamma = -1 \). In part B of this thesis we restrict attention to a special case in which \( \dot{\sigma}(\gamma) \) has a piecewise linear form shown in Figure 9.1. Specifically, suppose that there are positive numbers \( \mu_1, \mu_2, \gamma_M, \gamma_m, \gamma_T \) with \( \gamma_M < \gamma_m \) such that

\[
\dot{\sigma}(\gamma) = \begin{cases} 
\mu_1 \gamma, & -1 < \gamma \leq \gamma_M, \\
-\mu_2 \gamma + \sigma_2, & \gamma_M \leq \gamma \leq \gamma_m, \\
\mu_1(\gamma - \gamma_T), & \gamma \geq \gamma_m,
\end{cases} \quad (9.4)
\]

and with \( \sigma_2 = (\mu_1 + \mu_2)\gamma_M = (\mu_1 + \mu_2)\gamma_m - \mu_1\gamma_T \). Moreover, we set

\[
\sigma_m \equiv \dot{\sigma}(\gamma_m) > 0, \quad \sigma_M \equiv \dot{\sigma}(\gamma_M) > \sigma_m. \quad (9.5)
\]

We shall say that a particle of the bar labeled by \( x \) is in the low-strain phase, the "unstable phase" or the high-strain phase according to whether the strain \( \gamma(x) \) is in the respective intervals \((-1, \gamma_M), (\gamma_M, \gamma_m)\) or \([\gamma_m, \infty)\) associated with the three branches of the stress-strain curve. The special stress level \( \sigma_0 \) shown in Figure 9.1 is such that the two shaded areas are equal; it is termed the Maxwell stress. The numbers \( \gamma_{01}, \gamma_{02}, \gamma_{03} \) represent the strain values corresponding to the Maxwell stress \( \sigma_0 \) in the low-strain phase, unstable phase or high-strain phase, respectively.

As the equilibrium condition (9.3) implies, for the material of Figure 9.1, a stationary strain discontinuity corresponds to an interface between two distinct phases; it is therefore a phase boundary. In this chapter, we shall only consider an interface between the low-strain phase and the unstable phase.
Let $\dot{\gamma}_1, \dot{\gamma}_2, \dot{\gamma}_3$ be the functions that are inverse to $\dot{\gamma}(\gamma)$ in the respective intervals $(-1, \gamma_M], [\gamma_M, \gamma_m], [\gamma_m, \infty)$. Then equations (9.3) and (9.4) show that in particular,

\begin{equation}
\dot{\gamma}_1(F) = \frac{F}{\mu_1}, \quad F \leq \sigma_M,
\end{equation}

and

\begin{equation}
\dot{\gamma}_2(F) = \frac{\sigma_2 - F}{\mu_2}, \quad \sigma_m \leq F \leq \sigma_M.
\end{equation}

First consider the case where the entire bar belongs to the low-strain phase. Substituting (9.6) into (9.2) and then integrating (9.2) yields

\begin{equation}
\dot{u}(x) = U_1(x; F) \equiv \frac{Fx}{\mu_1}, \quad 0 \leq x \leq L,
\end{equation}

where $F \leq \sigma_M$ so that the associated strain field belongs to the low-strain phase. By (9.1) and (9.8), the force $F$ and the elongation $\delta$ are related by

\begin{equation}
\delta = \Delta_1(F) \equiv \frac{FL}{\mu_1}, \quad F \leq \sigma_M,
\end{equation}

which defines the straight segment $OAB$ in the $(\delta, F)$-plane; see Figure 9.2.

Similarly, when the entire bar is completely in the unstable phase, it follows from (9.2) and (9.7) that the displacement field is

\begin{equation}
\dot{u}(x) = U_2(x; F) \equiv \frac{\sigma_2 - F}{\mu_2} x, \quad 0 \leq x \leq L,
\end{equation}

in which $\sigma_m \leq F \leq \sigma_M$ in order for the corresponding strain field to lie in the unstable phase. By (9.1) and (9.10), the force-elongation relation is

\begin{equation}
\delta = \Delta_2(F) \equiv \frac{\sigma_2 - F}{\mu_2} L, \quad \sigma_m \leq F \leq \sigma_M,
\end{equation}

which corresponds to the straight segment $BC$ in the $(\delta, F)$-plane, see Figure 9.2.

We turn next to the case where the bar contains a single phase boundary at $x = s$ such that the portion $[0, s)$ of the bar is in the low-strain phase while the other portion $(s, L]$ is in the unstable phase. By (9.6), (9.7) and (9.2), one finds
where \( \sigma_m \leq F \leq \sigma_M \) in order that the associated strain field belongs to the low-strain phase for \( 0 \leq x < s \) and the unstable phase for \( s < x \leq L \). The force-elongation relation corresponding to (9.12) is

\[
\delta = \Delta_{12}(F, s) \equiv \frac{F}{\mu_1}s + \frac{\sigma_2 - F}{\mu_2}(L - s), \quad \sigma_m \leq F \leq \sigma_M.
\]

For each fixed \( s \) in the interval \( 0 < s < L \), (9.13) describes a straight-line segment in the \((\delta, F)\)-plane as shown in Figure 9.2. The one-parameter family of such line segments, as the parameter \( s \) ranges over the interval \((0, L)\), fills the region \( \mathcal{M} \) with boundary segments \( AB \), \( BC \) and \( CA \) in the \((\delta, F)\)-plane as shown in Figure 9.2.

In summary, we see from Figure 9.2 that, given the force \( F \), the problem of determining the elongation \( \delta \) has a unique solution for \( F < \sigma_m \) in which the entire bar belongs to the low-strain phase. On the other hand, when \( \sigma_m \leq F \leq \sigma_M \), this problem has a one-parameter family of solutions with parameter \( s \). Observe from (9.8), (9.10), (9.12) that for \( \sigma_m \leq F \leq \sigma_M \), as the phase boundary recedes to the right-hand end of the bar \((s \to L)\) the weak solution \( U_{12} \) merges with the smooth solution \( U_2 \), while as the phase boundary recedes to the left-hand end of the bar \((s \to 0)\) the weak solution \( U_{12} \) merges with the smooth solution \( U_1 \). For larger values of the force \( F \), i.e. \( F > \sigma_M \), there are no solutions to the problem of the type being considered here. Solutions corresponding to this case necessarily involve the high-strain phase.

In summary, the problem has a unique smooth solution for values of force in the range \( F < \sigma_m \), while for values of force in the intermediate range \( \sigma_m \leq F \leq \sigma_M \), we encounter a major breakdown of uniqueness.
Figure 9.1. Stress-strain curve.
Figure 9.2. The $(\delta, \sigma)$-plane.
Chapter 10

Quasi-static Motions

We now turn attention to quasi-static motions of the bar. At each instant \( t \), the displacement field \( u(., t) \) corresponds to one of the equilibrium states constructed in the preceding chapter. Let \( F(t) \) be a given continuous, piecewise continuously differentiable force history for \( t_1 \leq t \leq t_2 \). Suppose first that the force is sufficiently small so that \( F(t) < \sigma_m \) for all \( t \) in \([t_1, t_2]\). Then by (9.8), \( u(x, t) \) is necessarily given by the smooth field

\[
    u(x, t) = U_1(x, F(t)), \quad 0 \leq x \leq L, \quad t_1 \leq t \leq t_2,
\]

(10.1)

associated with the low-strain phase. Next, assume that \( \sigma_m < F(t) < \sigma_M \) for all \( t \) in \([t_1, t_2]\). Then by (9.12), \( u(x, t) \) must have the form

\[
    u(x, t) = U_{12}(x; F(t), s(t)), \quad 0 \leq x \leq L, \quad t_1 \leq t \leq t_2.
\]

(10.2)

For quasi-static motions of the form (10.1) or (10.2), the assumed smoothness of \( F(t) \) guarantees that \( u(x, t) \) is continuous and piecewise continuously differentiable on \([t_1, t_2]\) for each \( x \).

Suppose that during such a motion, the strain \( \gamma(x, t) \) is smooth on \([0, L] \times [t_1, t_2]\) except at the phase boundary \( x = s(t) \). Let \( W(\gamma) \) be the strain energy per unit reference volume of the bar:
\[ W(\gamma) \equiv \int_{0}^{\gamma} \dot{\sigma}(\varepsilon) d\varepsilon, \quad \gamma > -1. \]  

(10.3)

Then the total strain energy stored in the bar is

\[ E(t) = \int_{0}^{L} W(\gamma(x, t)) dx. \]  

(10.4)

The following work-energy relation can be readily established

\[ F(t) \dot{\delta}(t) - \dot{E} = f(t) \dot{s}(t), \]  

(10.5)

where \( f(t) \) denotes the driving traction acting on the phase boundary and is found in the present case to be given by

\[ f(t) = \dot{f}(\sigma(t)) = \frac{1}{2} \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} (\sigma_M - \sigma(t))^2, \quad \sigma_m \leq \sigma \leq \sigma_M. \]  

(10.6)

The right-hand side of (10.5) represents the rate of dissipation of energy due to the moving phase boundary and the requirement that it be non-negative implies that

\[ f(t) \dot{s}(t) \geq 0, \quad t_1 \leq t \leq t_2. \]  

(10.7)

The inequality (10.7) is a consequence of the second law of thermodynamics (under isothermal conditions) and is referred to as the entropy inequality.

It follows from (10.6) that \( \dot{f}(\sigma) \) is non-negative. Thus the entropy inequality (10.7) is equivalent to

\[ \dot{s} \geq 0, \quad t_1 \leq t \leq t_2. \]  

(10.8)

This implies that a phase boundary of the type being considered in this part (separating the unstable phase on the right from the low-strain phase on the left) can move only into the unstable phase, thus transforming particles from the unstable phase to the low-strain phase.

A loading path is the curve traced by the point \((\delta(t), F(t))\) during a quasi-static motion. The permissible directions of a loading path allowed by (10.8) are shown in Figure 10.1.
We now conclude that for a material having the form of Figure 9.1, the entropy inequality imposes a restriction on the direction of motion of the phase boundary, but that this imposition is not sufficient to guarantee the uniqueness of the response to a prescribed force.
Figure 10.1. The $(\delta, \sigma)$-plane. Admissible directions.
Chapter 11

The Regularized Theory: A Traveling Wave Problem

11.1 Construction of the Traveling Wave Problem

In order to better understand the nature of a strain discontinuity, we regularize the elastic theory by admitting viscous and strain-gradient effects and studying traveling waves in this regularized theory. In the regularized theory, the elastic constitutive statement $\sigma = \dot{\sigma}(\gamma)$ employed in the preceding chapters is replaced by

$$\sigma = \dot{\sigma}(\gamma) + \rho \nu \gamma_t - \rho \lambda \gamma_{xx},$$

where $\dot{\sigma}(\gamma)$ is again the piecewise linear function of Figure 9.1, and $\nu \geq 0$ and $\lambda \geq 0$ are the viscosity and strain-gradient coefficients, respectively. Let $c_1 = (\mu_1/\rho)^{\frac{1}{2}}$ and $c_2 = (\mu_2/\rho)^{\frac{1}{2}}$; note that $c_1$ is the sound wave speed in the low-strain and high-strain phases, while $c_2$ is not a wave speed. The dimensionless parameter $\omega$ defined by

$$\omega = 2\lambda^{\frac{1}{2}}/\nu$$

will be useful in the following analysis. In this chapter, we shall restrict attention to the study of travelling waves in the quasi-static theory of this bar.

For material (11.1), equilibrium of the bar in the absence of body forces, $\partial \sigma(x,t)/\partial x = 0$, requires
\[ \dot{\sigma}(\gamma) + \rho \nu \dot{\gamma} - \rho \lambda \gamma_{xx} = F, \]  

(11.3)

where constant \( F \) is the force in the bar.

When considering the material governed by (11.1), we shall strengthen the smoothness requirement on displacement; \( u(x, t) \) will be twice continuously differentiable with piecewise continuous third and fourth derivatives in the region of space-time to be considered. Equation (11.3) further requires that \( \gamma_{xx} \), and therefore \( u_{xxx} \), be continuous.

Now we seek travelling wave solutions of differential equation (11.3) in the form

\[ \gamma = \gamma(\xi), \quad \xi = x - \dot{s}t, \]  

(11.4)

subject to the boundary conditions

\[ \gamma(-\infty) = \overline{\gamma}, \quad \gamma(+\infty) = \overline{+\gamma}, \]  

(11.5)

where \( \dot{s} \) is a unknown constant, and \( \overline{\gamma}, \overline{+\gamma} \) are given strains with \(-1 < \overline{\gamma} \leq \gamma_M \) and \( \gamma_M < \overline{+\gamma} < \gamma_m \). Thus the strain at \( \xi = +\infty \) is in the unstable phase, the strain at \( \xi = -\infty \) is in the low-strain phase, and the travelling wave in this regularized theory corresponds to a strain-discontinuity connecting \( \overline{\gamma} \) to \( \overline{+\gamma} \) in the elastic theory. The question of interest is as follow: given \( \overline{\gamma}\in(-1, \gamma_M) \) and \( \overline{+\gamma}\in(\gamma_M, \gamma_m) \) does there exist a travelling wave connecting them, and if so, is its propagation speed \( \dot{s} \) determined. If the analysis produces a relation \( \dot{s} = h(\overline{\gamma}, \overline{+\gamma}) \) we speak of a kinetic relation controlling the propagation of the phase boundary. If not, the regularization does not produce a mechanism controlling the propagation speed \( \dot{s} \).

In conformity with the smoothness assumptions made in this chapter, we suppose that \( \gamma(\xi) \) is continuous and continuously differentiable on \((-\infty, \infty)\), and that it has piecewise continuous second and third derivatives there. In view of (11.3), \( \gamma''(\xi) \) must in fact be continuous for all \( \xi \), provided \( \lambda \neq 0 \).
From (11.3), solutions of the form (11.4) must satisfy
\[ \lambda \gamma'' + \nu \dot{\gamma}' + \frac{F - \dot{\gamma}}{\rho} = 0. \]  
(11.6)

From the boundary conditions (11.5), it follows that the constant F is given by
\[ F = \dot{\gamma}(\infty) = \dot{\gamma}(\infty). \]  
(11.7)

We speak of (11.4)-11.7 as the travel wave problem.

Now we seek a solution of the travel wave problem for which \( \gamma(\xi) \) is in the low-strain phase for \(-\infty < \xi \leq 0\) and in the unstable phase for \(0 \leq \xi < +\infty\). In view of our smoothness requirements on \( \gamma(\xi) \), the following continuity conditions must hold at the interface between the low-strain phase and the unstable phase:
\[ \gamma(0-) = \gamma(0+) = \gamma_M, \quad \gamma'(0-) = \gamma'(0+). \]  
(11.8)

In order to construct such a solution, we shall first consider two subsidiary problems. By (9.4), (11.5), (11.6) and (11.8), these two subsidiary problems can be described as follow.

**Problem 1:**
\[ \lambda \gamma'' + \nu \dot{\gamma}' - c_1^2 (\gamma - \bar{\gamma}) = 0, \quad \xi \leq 0, \]  
(11.9)
\[ \gamma(-\infty) = \bar{\gamma}, \quad \gamma(0-) = \gamma_M; \]  
(11.10)

**Problem 2:**
\[ \lambda \gamma'' + \nu \dot{\gamma}' + c_2^2 (\gamma - \bar{\gamma}) = 0, \quad \xi \geq 0, \]  
(11.11)
\[ \gamma(0+) = \gamma_M, \quad \gamma'(0+) = \gamma'(0-). \]  
(11.12)

The function \( \gamma(\xi) \) on \((-\infty, \infty)\) obtained by solving these two problems will satisfy the differential equation (11.6), all of the interface conditions (11.8) as well as the first of the boundary conditions (11.5). Thus after solving Problems 1 and 2, we shall only need to enforce the boundary condition.
\( \gamma(+\infty) = \dot{\gamma}. \) \hfill (11.13)

and the strain range requirements

\[-1 < \gamma(\xi) \leq \gamma_M, \quad \xi \leq 0, \] \hfill (11.14)

and

\[ \gamma_M \leq \gamma(\xi) \leq \gamma_m, \quad \xi \geq 0, \] \hfill (11.15)

insuring that the strain belongs to the appropriate phase in each interval.

### 11.2 Solutions to the Taveling Wave Problem

Problem 1 is a boundary-value problem. Solving it, one finds the solution is given by

\[ \gamma(\xi) = \ddot{\gamma} + (\gamma_M - \ddot{\gamma})e^{p_1 \xi}, \quad \xi \leq 0, \] \hfill (11.16)

where the positive number \( p_1 \) is defined by

\[ p_1 = -\frac{\nu}{2\lambda}(\dot{s} - P) > 0, \quad P = P(\dot{s}) = (\dot{s}^2 + \omega^2 c_1^2)^{1/2}. \] \hfill (11.17)

and \( \omega \) is defined in (11.2). Clearly, \( \gamma(\xi) \) monotonically increases on \((-\infty, 0]\) from the value \( \ddot{\gamma} \) at \( \xi = -\infty \) to the value \( \gamma_M \) as \( \xi \to 0 \) so that the strain range condition (11.14) holds automatically.

We turn next to Problem 2. For the differential equation (11.11), the associated characteristic equation is \( \lambda q^2 + \nu \dot{s}q + c_2^2 = 0 \) which has two roots

\[ q_1 = -\frac{\nu}{2\lambda}(\dot{s} - Q), \quad q_2 = -\frac{\nu}{2\lambda}(\dot{s} + Q), \] \hfill (11.18)

where \( Q \) is defined by

\[ Q = Q(\dot{s}) \equiv (\dot{s}^2 - \dot{s}_c^2)^{1/2}, \quad \dot{s}_c \equiv \omega c_2 \geq 0. \] \hfill (11.19)
If \( \dot{s} < 0 \), it can be verified that even if there exists a solution to (11.11) with the initial conditions (11.12) satisfied, the requirement (11.15) can never be satisfied. Therefore the present traveling wave problem has no solutions in this case. This result corresponds to the entropy inequality obtained in the preceding chapter: the interface between the low-strain phase and the unstable phase cannot move towards the low-strain phase. For \( \dot{s} \geq 0 \), there are three cases to be considered. We consider them separately in the following analysis.

(i) \( 0 \leq \dot{s} < \dot{s}_c \). In this case, \( Q \) is imaginary and can be rewritten as \( Q = i|Q| \) where \( |Q| = (\dot{s}_c^2 - \dot{s}^2)^{1/2} \). The solution of (11.11) now has the form

\[
\gamma(\xi) = \gamma + Ae^{-\alpha\xi}\sin(\beta\xi + \phi), \quad \xi \geq 0,
\]

where \( A \) and \( \phi \) are constants, and \( \alpha \) and \( \beta \) are given by

\[
\alpha = \alpha(\dot{s}) = \frac{\nu \dot{s}}{2\lambda} \geq 0, \quad \beta = \beta(\dot{s}) = \frac{\nu|Q|}{2\lambda} \geq 0.
\]

The constants \( A \) and \( \phi \) are chosen so that the initial conditions (11.12) for Problem 2 are satisfied. This leads to

\[
A = A(\dot{s}) = \pm(\gamma_M - \dot{s})(1 + Z^2(\dot{s}))^{1/2}, \quad \phi = \phi(\dot{s}) = \tan^{-1} \frac{1}{Z(\dot{s})}, \quad 0 \leq \dot{s} < \dot{s}_c
\]

in which the positive and negative signs are taken according to whether \( \phi \) is positive or negative, respectively. Note that \( \phi(\dot{s}) \) is restricted to the interval \([-\pi, \pi]\) for \( 0 \leq \dot{s} < \dot{s}_c \). In (11.22), \( Z(\dot{s}) \) is defined by

\[
Z = Z(\dot{s}) = \frac{1}{|Q|}[(1 + \frac{c_2^2}{c_1^2})\dot{s} - \frac{c_2^2}{c_1^2}P(\dot{s})], 0 \leq \dot{s} < \dot{s}_c
\]

where \( P(\dot{s}) \) was given previously in (11.17). One can verify that \( Z(\dot{s}) \) monotonically increases on \([0, \dot{s}_c^+\) from the negative value \(-c_2/c_1\) at \( \dot{s} = 0 \) to positive infinity as \( \dot{s} \to \dot{s}_c^- \). This indicates that there exists a unique positive number \( \dot{s}_0 \) on \([0, \dot{s}_c)\) such that \( Z(\dot{s}_0) = 0 \). By (11.23), \( \dot{s}_0 \) is given by
\[
\dot{s}_0 = \frac{c_2 \dot{s}_c}{(c_1^2 + 2c_2^2)^{1/2}}.
\] (11.24)

Then (11.22) shows that \(\phi(\dot{s})\) is negative on \([0, \dot{s}_0]\), and positive on \((\dot{s}_0, \dot{s}_c)\), respectively. Thus function \(\phi(\dot{s})\) experiences a finite jump \(\pi\) across \(\dot{s} = \dot{s}_0\). By (11.22), the solution (11.20) becomes

\[
\gamma(\xi) = \gamma + (\gamma_M - \gamma_m \xi)|1 + Z^2(\dot{s})|^{1/2} e^{-\alpha \xi} \sin[\beta \xi + \phi(\dot{s})], \quad \xi \geq 0,
\] (11.25)

where the positive sign is taken for \(\dot{s}_0 < \dot{s} < \dot{s}_c\), while the negative sign is taken for \(0 \leq \dot{s} < \dot{s}_0\).

Clearly, the remaining requirement (11.13) holds automatically in this case. In order to see whether the other requirement (11.15) is satisfied, it is necessary to first find all of the extrema of the solution (11.25). These extrema are located at

\[
\xi = \xi_n(\dot{s}) \equiv \frac{1}{\beta} \left[n \pi + \psi(\dot{s}) - \phi(\dot{s})\right], \quad n = \text{integers},
\] (11.26)

where \(\psi(\dot{s})\) is defined by

\[
\psi(\dot{s}) = \psi(\dot{s}) = \tan^{-1}\left(\frac{Q}{\dot{s}}\right), \quad 0 \leq \dot{s} < \dot{s}_c.
\] (11.27)

For \(0 \leq \dot{s} < \dot{s}_c\), \(\psi(\dot{s})\) lies in the interval \((0, \frac{\pi}{2})\). Equation (11.27) shows that \(\psi(\dot{s})\) monotonically decreases on \([0, \dot{s}_c]\) from the value \(\frac{\pi}{2}\) at \(\dot{s} = 0\) to zero as \(\dot{s} \to \dot{s}_c^-\). Comparing (11.27) to (11.22), one finds

\[
\psi(\dot{s}) - \phi(\dot{s}) > 0, \quad 0 \leq \dot{s} \leq \dot{s}_0^-,
\] (11.28)

and

\[
\psi(\dot{s}) - \phi(\dot{s}) < 0, \quad \dot{s}_0^+ \leq \dot{s} < \dot{s}_c.
\] (11.29)

Thus the maximum and minimum values of the solution \(\gamma(\xi)\) given by (11.25) can be selected from its extremes. Since the amplitude \((\gamma - \gamma_M)(1 + Z^2)^{1/2} e^{-\alpha \xi}\) of the second term of solution
(11.25) exponentially decays with increasing $\xi (\geq 0)$, one only needs to find the smallest positive numbers $\xi_{\text{max}}$, $\xi_{\text{min}}$ so that the strain $\gamma(\xi)$ becomes maximum at $\xi_{\text{max}}$ and minimum at $\xi_{\text{min}}$. It follows from (11.25), (11.28) that

$$\xi_{\text{max}} = \xi_0 = \frac{1}{\beta} \left[ \psi(\dot{\xi}) - \Phi(\dot{\xi}) \right] > 0, \quad 0 \leq \dot{s} \leq \dot{s}_0 - , \quad (11.30)$$

and

$$\xi_{\text{min}} = \xi_1 = \frac{1}{\beta} \left[ \pi + \psi(\dot{\xi}) - \Phi(\dot{\xi}) \right] > \xi_0, \quad 0 \leq \dot{s} \leq \dot{s}_0 - . \quad (11.31)$$

Similarly, when considering $\dot{s}_0 + \leq \dot{s} < \dot{s}_c$, equations (11.25) and (11.29) yield

$$\xi_{\text{max}} = \xi_1 = \frac{1}{\beta} \left[ \pi + \psi(\dot{\xi}) - \Phi(\dot{\xi}) \right] > 0, \quad \dot{s}_0 + \leq \dot{s} < \dot{s}_c, \quad (11.32)$$

and

$$\xi_{\text{min}} = \xi_2 = \frac{1}{\beta} \left[ 2\pi + \psi(\dot{\xi}) - \Phi(\dot{\xi}) \right] > \xi_1 \quad \dot{s}_0 + \leq \dot{s} < \dot{s}_c. \quad (11.33)$$

The maximum and minimum values of $\gamma(\xi)$ on $[0, \infty)$ now become

$$\gamma_{\text{max}} = \dot{\gamma} + (\gamma_M - \dot{\gamma}) \left[ 1 + Z^2 \right]^{\frac{1}{2}} e^{-\alpha \xi_{\text{max}} sin\psi} \quad (11.34)$$

and

$$\gamma_{\text{min}} = \dot{\gamma} - (\gamma_M - \dot{\gamma}) \left[ 1 + Z^2 \right]^{\frac{1}{2}} e^{-\alpha \xi_{\text{min}} sin\psi} \quad (11.35)$$

in which $\xi_{\text{max}}$, $\xi_{\text{min}}$ are given in (11.30), (11.31) if $0 < \dot{s} \leq \dot{s}_0 -$ or in (11.32), (11.33) if $\dot{s}_0 + \leq \dot{s} < \dot{s}_c$.

The second inequality $\gamma(\xi) \leq \gamma_m$ in (11.15) holds on $[0, \infty)$ if and only if $\gamma_{\text{max}} \leq \gamma_m$.

Thus

$$\dot{\gamma} + (\gamma_M - \dot{\gamma}) \left[ 1 + Z^2(\dot{s}) \right]^{\frac{1}{2}} e^{-\alpha(\dot{\xi}) \xi_{\text{max}} sin\phi(\dot{s})} \leq \gamma_m. \quad (11.36)$$
It follows from (11.22), (11.23), (11.27), (11.30) and (11.32) that 0 < \sin(\psi(\delta)) \leq 1 and 0 < e^{-\alpha(\delta)} \xi < 1 hold in both cases 0 < \delta \leq \delta_0 - \delta_0 + \leq \delta < \delta_c. Thus (11.36) holds if the following statement holds

\[ \hat{\gamma} + (\gamma_M - \hat{\gamma})[1 + Z^2(\delta)]^{1/2} \leq \gamma_m, \]  

(11.37)

which is a restriction on \delta and \hat{\gamma}. Equation (11.37) is equivalent to

\[ \hat{\gamma} \leq g(\delta), \]  

(11.38)

where g is

\[ g(\delta) = \frac{\gamma_m + \gamma_M[1 + Z^2(\delta)]^{1/2}}{1 + [1 + Z^2(\delta)]^{1/2}}, \quad 0 \leq \delta < \delta_c. \]  

(11.39)

By (11.39), (11.23), one can show that g(\delta) is a single-valued function of \delta and \gamma_M \leq g(\delta) \leq \gamma_{02} always holds for 0 \leq \delta < \delta_c.

In order that the leftmost inequality in (11.15) holds, \gamma_{\min} \geq \gamma_M must hold. Then by (11.35) and the associated equations, one can easily show that \gamma_{\min} \geq \gamma_M holds if and only if \hat{\gamma} \geq \gamma_M.

Note that the remaining boundary condition (11.13) holds automatically in (11.25) without any further restrictions on \tilde{\gamma}, \hat{\gamma} and \delta.

Thus in summary, given the strains \( -1 < \tilde{\gamma} \leq \gamma_M, \gamma_M < \hat{\gamma} < \gamma_M \), the present travelling problem has a one-parameter family of solutions with parameter \delta lying in the range 0 \leq \delta < \delta_c, \hat{\gamma} \leq g(\delta). This is described in Figure 11.1.

(ii) \delta = \delta_c. In this case, \( q_1 = q_2 = q_c \) where

\[ q_c = \frac{\nu \delta_c}{2\lambda} \geq 0. \]  

(11.40)

The general solution for Problem 2 is then

\[ \gamma(\xi) = \hat{\gamma} + (B_1 + B_2 \xi) e^{-q_c \xi}, \quad \xi \geq 0. \]  

(11.41)
After applying the initial conditions (11.12), one finds that the solution (11.41) becomes

\[ \gamma(\xi) = \dot{\gamma} - (\gamma_M - \dot{\gamma})[1 - \xi(\frac{c^2}{c^1}p_1 - q_c)]e^{-2\xi}, \quad \xi \geq 0. \]  
(11.42)

One can show that the strain \( \gamma(\xi) \) monotonically increases on \( [0, \infty) \) from the value \( \gamma_M \) at \( \xi = 0 \) to the value \( \dot{\gamma} \) as \( \xi \to \infty \). It thus follows that the remaining requirements (11.13), (11.15) hold automatically in this case.

Thus, the present traveling wave problem with \( \dot{\gamma} \leq \gamma_M \) and \( \gamma_M < \dot{\gamma} < \gamma_m \) has a solution with \( \dot{s} = \dot{s}_c \). This solution consists of (11.16) and (11.42).

(iii) \( \dot{s} > \dot{s}_c \). In this case, \( Q(\dot{s}) > 0 \) so that \( q_1, q_2 \) are real and distinct. Then the general solution of (11.11) has the form

\[ \gamma(\xi) = \dot{\gamma} + C_1 e^{q_1\xi} + C_2 e^{q_2\xi}, \quad \xi \geq 0. \]  
(11.43)

The constants \( C_1 \) and \( C_2 \) are chosen so that the initial conditions (11.12) are satisfied. This leads to

\[ C_1 = -\frac{\lambda}{\nu Q}(\gamma_M - \dot{\gamma}) \frac{q_2}{p_2}(p_2 - q_1), \quad C_2 = \frac{\lambda}{\nu Q}(\gamma_M - \dot{\gamma}) \frac{q_1}{p_2}(p_2 - q_2), \]  
(11.44)

where \( p_2 \) is given by

\[ p_2 = -\frac{\nu}{2\lambda}(\dot{s} + P) < 0, \]  
(11.45)

and \( P \) is defined in (11.17). It can be verified that in this case, the remaining requirements (11.13) and (11.15) hold automatically.

Thus the present traveling wave problem with the given boundary strains \( \dot{\gamma} \leq \gamma_M \) and \( \gamma_M < \dot{\gamma} < \gamma_m \) has a one-parameter family of solutions with the parameter \( \dot{s} > \dot{s}_c \).

In summary, we have showed that there are no traveling waves of the assumed form (11.4) with \( \dot{s} < 0 \), while there is a one-parameter family of traveling waves connecting a strain \( \dot{\gamma} \) in the low-strain phase at \( \xi = -\infty \) to a strain \( \dot{\gamma} \) in the unstable phase at \( \xi = \infty \) with the
parameter \( \hat{s} \). The values of the parameter \( \hat{s}(\geq 0) \) must be such that \((\hat{s}, \hat{\gamma})\) lies the shaded portion of Figure 11.1 but is otherwise unrestricted. Thus in particular the propagation speed \( \hat{s} \) is not determined by the strains \( \hat{\gamma}, \hat{\gamma} \) and so the regularization does not induce a kinetic relation governing the motion of a phase boundary separating the unstable phase from the low-strain phase.
Figure 11.1. The \((s, \dot{\gamma})\)-plane.
Chapter 12
The Dynamic Theory

12.1 Background

We now incorporate the effects of inertia into our analysis and consider longitudinal motions of an elastic bar made of the material of Figure 9.1. The displacement $u(x, t)$ is now assumed to be continuous with piecewise continuous first and second derivatives throughout the regions of space-time to be considered. The strain and particle velocity are defined by

$$\gamma = u_x \text{ and } v = u_t \text{ at points } (x, t) \text{ where the derivatives exist. Again, } \gamma(x, t) > -1 \text{ in order to ensure that the mapping } x \rightarrow x + u(x, t) \text{ is invertible at each instant } t. \text{ At points where } \gamma \text{ and } v \text{ are smooth, balance of momentum and kinematic compatibility require that}

$$\dot{\sigma}'(\gamma)\gamma_x = \rho v_t, \quad v_x = \gamma_t, \quad (12.1)$$

where the constant $\rho$ is the mass density in the reference configuration. If there is a moving discontinuity at $x = s(t)$, the following jump conditions must hold:

$$\dot{\sigma}(\gamma) = -\rho \dot{s}(\dot{v} - \bar{v}), \quad (\dot{v} - \bar{v}) = -\dot{s}(\dot{\gamma} - \bar{\gamma}), \quad (12.2)$$

where for any function $g(x, t)$ we write $\dot{g} = g(s(t)\pm, t)$ for the limiting values of $g$ on either side of the discontinuity.
Consider the motion of the piece \( x_1 \leq x \leq x_2 \) of the bar during a time interval \([t_1, t_2]\) and suppose that \( \gamma \) and \( v \) are smooth on \([x_1, x_2] \times [t_1, t_2]\) except at the moving discontinuity \( x = s(t) \). The total mechanical energy at time \( t \) associated with this piece of bar is

\[
E(t) = \int_{x_1}^{x_2} \left[ W(\gamma(x,t)) + \frac{1}{2} \rho v^2(x,t) \right] dx,
\]

where \( W(\gamma) \) is defined in (10.3). The following relation between work and energy can be derived

\[
\sigma(x_2, t)v(x_2, t) - \sigma(x_1, t)v(x_1, t) - E(t) = f(t)\dot{s}(t),
\]

where the (dynamic) driving force \( f(t) \) acting on the strain discontinuity is defined by

\[
f = \int_{\gamma}^{\gamma'} \dot{\sigma}(\gamma)d\gamma - \frac{1}{2}(\dot{\sigma}(\gamma') + \dot{\sigma}(\gamma))(\gamma' - \gamma).
\]

Note that \( f \) may be interpreted geometrically as the difference between the area under the stress-strain curve between \( \gamma = \gamma' \) and \( \gamma = \gamma'' \) and the area of an associated trapezoid having the same base. The entropy inequality

\[
f(t)\dot{s}(t) \geq 0,
\]

is required to hold at each discontinuity, thus ensuring that the rate of dissipation of mechanical energy is non-negative.

A motion of the bar is governed by the field equations (12.1) at all points of smoothness, and the jump conditions (12.2) and the entropy inequality (12.6) at all discontinuities.

We note from (12.2) that the velocity of a moving discontinuity \( x = s(t) \) satisfies

\[
\rho \dot{s}^2 = \frac{\dot{\sigma}(\gamma') - \dot{\sigma}(\gamma)}{\gamma' - \gamma}.
\]

From this it follows that the slope of the chord joining the points \((\gamma, \dot{\sigma}(\gamma))\) and \((\gamma', \dot{\sigma}(\gamma'))\) on the stress-strain curve must be non-negative. Conversely, one can show that, if a pair
\((\tilde{\gamma}, \tilde{\gamma})\) satisfies this condition, there exist numbers \(\tilde{v}, \tilde{v}\) and \(\tilde{s}\) that, together with the pair \((\tilde{\gamma}, \tilde{\gamma})\), satisfy the jump conditions (12.2). It is readily seen from (12.7) and Figure 9.1 that necessarily \(|\tilde{s}| \leq c_1\). The propagation speed \(\tilde{s}\) of the discontinuity is said to be subsonic if \(|\tilde{s}| < c_1\) and sonic if \(\tilde{s} = c_1\).

Let phase-1, phase-2 and phase-3 denote the low-strain phase, the unstable phase and the high-strain phase, respectively. Then, if \(\tilde{\gamma}\) is in phase \(p\) and \(\tilde{\gamma}\) in phase \(q\), we shall call the associated \(p,q\)-discontinuity a shock wave if \(p = q\), a phase boundary if \(p \neq q\). The condition (12.7) clearly rules out shock wave of 2,2-type. Moreover, for the material shown in Figure 9.1, it follows from (12.7) that shock waves must be sonic.

### 12.2 The Riemann Problem

We now formulate a Riemann problem that pertains to the 1,2-discontinuities being studied in this part. We seek weak solutions of the differential equations (12.1) on the upper half of the \((x, t)\)-plane that satisfy the following initial conditions:

\[
\gamma(x, 0), v(x, 0) = \begin{cases} 
\gamma_L, v_L, & -\infty < x < 0, \\
\gamma_R, v_R, & 0 < x < \infty,
\end{cases}
\]  

(12.8)

where \(\gamma_L, \gamma_R, v_L\) and \(v_R\) are given constants with \(\gamma_L \in (-1, \gamma_M]\) and \(\gamma_R \in (\gamma_M, \gamma_m)\).

Since the initial value problem described above is invariant under the scale change \(t \rightarrow kt\), \(x \rightarrow kx\), we restrict attention to solutions that have this property as well. Where such solutions exist, they must have the form \(\gamma(x, t) = \gamma(x/t), v(x, t) = v(x/t)\). It then follows from (12.1) that for such solutions, either \(\gamma\) and \(v\) are both constant, or that the equality \(\tilde{\sigma}'(\gamma(x/t)) = \rho(x/t)^2\) must hold. But for the piecewise linear material shown in Figure 9.1, \(\tilde{\sigma}'(\gamma)\) is piecewise constant, so that the latter equality cannot hold on any region of the \((x, t)\)-plane. Therefore the fields are necessarily piecewise constant on the \((x, t)\)-plane. If either \(\gamma\) or \(v\)
jumps at $x = s(t)$, then the fact that $\gamma$ and $v$ are constant on either side of $x = s(t)$ necessarily requires that $s(t) = \dot{s}t$, where $\dot{s}$ is constant. We conclude that the general scale-invariant solution to the Riemann problem must have the following form:

$$
\gamma(x, t) = \gamma_j, \quad v(x, t) = v_j, \quad \dot{s}_j t < x < \dot{s}_{j+1} t, \quad j = 0, 1, \ldots, N, \quad (12.9)
$$

where $\gamma_j$, $v_j$, $\dot{s}_j$ and $N$ are constants, with $N$ a positive integer, and $\gamma_0 = \gamma_L$, $\gamma_N = \gamma_R$, $v_0 = v_L$, $v_N = v_R$, $\dot{s}_0 = -\infty$, $\dot{s}_{N+1} = +\infty$; see Figure 12.1. The $\gamma_j$'s are required to satisfy $\gamma_j > -1$ for $j = 0, 1, \ldots, N$ and $\gamma_j \neq \gamma_{j+1}$, $j = 0, \ldots, N - 1$. In the field given by (12.9), there are $N$ discontinuities on lines $x = \dot{s}_j t$; they may be shock waves or phase boundaries.

We seek solutions of the Riemann problem in the class of all functions described above. At each discontinuity $x = \dot{s}_j t$, the jump conditions (12.2) must be satisfied across it so that

$$
\begin{align*}
\dot{s}_j (\gamma_j - \gamma_{j-1}) &= -(v_j - v_{j-1}), \\
\rho \dot{s}_j^2 &= (\dot{s}(\gamma_j) - \dot{s}(\gamma_{j-1}))/((\gamma_j - \gamma_{j-1})),
\end{align*}
\tag{12.10}
$$

where $(\gamma_j, v_j)$ and $(\gamma_{j-1}, v_{j-1})$ are the limiting values of strain and particle velocity on the right and left respectively of this discontinuity. Let $f_j = \dot{f}(\gamma_{j-1}, \gamma_j)$ with $\dot{f}$ defined by (12.5) stand for the driving force on this discontinuity; the entropy inequality (12.6) then requires that

$$
f_j \dot{s}_j \geq 0. \tag{12.11}
$$

An admissible solution of the Riemann problem is a pair $\gamma(x, t), v(x, t)$ of the form just described with (12.10)-(12.11) enforced at all discontinuities.

## 12.3 The Structure of Admissible Solutions to the Riemann Problem

Before constructing explicit global solutions to the Riemann problem formulated above, it is helpful to establish some general results pertaining to the permissible solution forms that
are consistent with the entropy inequality. For this purpose we begin by considering the following modified Riemann problem: let \( x = \dot{s}_0 t \) and \( x = \dot{s}_{N+1} t \) be two rays in the \((x, t)\)-plane with \(-\infty < \dot{s}_0 < 0\) and \(0 < \dot{s}_{N+1} < \infty\). Let \( \gamma(x, t) = \gamma_0, v(x, t) = v_0 \) on \( x = \dot{s}_0 t \) and \( \gamma(x, t) = \gamma_N, v(x, t) = v_N \) on \( x = \dot{s}_{N+1} t \). See Figure 12.2. Suppose finally that neither of the strains \( \gamma_0 \) and \( \gamma_N \) belongs to the unstable phase so that one may speak of the problem at hand as being a modified Riemann problem with metastable data.

Let \((\gamma, v)\) be an admissible solution of the modified Riemann problem with metastable data. Then

(i) The strain \( \gamma(x, t) \) does not belong to the unstable phase at any point \((x, t)\) in the wedge \( \dot{s}_0 t < x < \dot{s}_{N+1} t \) of the upper-half plane.

(ii) Let \( x = \dot{s}_k t \) and \( x = \dot{s}_{k+1} t \) be two discontinuities in the same quadrant of the \((x, t)\)-plane. Then these two discontinuities cannot both be shock waves.

(iii) Let \( x = \dot{s}_k t \) and \( x = \dot{s}_{k+1} t \) be two discontinuities in the same quadrant of the \((x, t)\)-plane. Then these two discontinuities cannot both be phase boundaries.

(iv) Let \( x = \dot{s}_k t \) and \( x = \dot{s}_{k+1} t \) be two discontinuities in the same quadrant of the \((x, t)\)-plane. If the slower discontinuity is a shock wave, then the faster discontinuity cannot be a phase boundary. (The converse case is possible: if the faster discontinuity is a shock wave, the slower discontinuity may be a phase boundary.)

The first result implies that if the given data does not involve the unstable phase, then at no later time does the solution involve the unstable phase. The next two propositions state that two shock waves and two phase boundaries cannot be adjacent to each other. On the other hand according to proposition (iv), a shock and a phase boundary may be adjacent to each other provided the phase boundary is subsonic.
It is clearly sufficient to prove these results in any one quadrant of the upper-half of the \((x, t)\)-plane and so we shall consider only the first quadrant. Thus in each of these propositions we have \(0 < \dot{s}_k < \dot{s}_{k+1} < \dot{s}_{N+1}\).

We prove the claim (i) by contradiction. Suppose that this proposition is false. Since 2,2-shocks do not exist, there must necessarily be two phase boundaries in the \((x, t)\)-plane, say \(x = \dot{s}_k t\) and \(x = \dot{s}_{k+1} t\) with \(\dot{s}_k < \dot{s}_{k+1}\), such that the state between them is constant with the associated strain \(\gamma_k\) is in the unstable phase and with neither of the strains \(\gamma_{k-1} = \gamma(\dot{s}_k t-, t)\) and \(\gamma_{k+1} = \gamma(\dot{s}_{k+1} t+, t)\) in the unstable phase. From (12.5) and (9.4) one sees that the driving force on the discontinuity \(x = \dot{s}_k t\) is positive; the entropy inequality (12.11) thus implies that \(\dot{s}_k \geq 0\). Similarly the driving force on \(x = \dot{s}_{k+1} t\) must be negative and so \(\dot{s}_{k+1} \leq 0\). Thus \(\dot{s}_k \geq \dot{s}_{k+1}\) which is a contradiction. This establishes the proposition.

To prove proposition (ii), note first that a 1,1-shock wave cannot be adjacent to a 3,3-shock wave, since they must be separated by a phase boundary. If two shock waves are both of the same type, they must travel at the same sonic speed \(c_1\) and thus become a single shock wave. Thus this contradicts the assumption that \(\dot{s}_k < \dot{s}_{k+1}\). The assertion (ii) is thus proved.

We now turn to the proof of proposition (iii). Suppose that the proposition is false so that \(x = \dot{s}_k t\) and \(x = \dot{s}_{k+1} t\) are both phase boundaries. The strain and velocity field thus have the form

\[
\gamma(x, t), v(x, t) = \begin{cases} 
\gamma_{k-1}, v_{k-1}, & x = \dot{s}_k t-, \\
\gamma_k, v_k, & \dot{s}_k t < x < \dot{s}_{k+1} t, \\
\gamma_{k+1}, v_{k+1}, & x = \dot{s}_{k+1} t+. 
\end{cases}
\]

(12.12)

First suppose that \(\gamma_{k-1}\) and \(\gamma_{k+1}\) belong to the low-strain phase, and \(\gamma_k\) to the high-strain phase so that \(0 < \dot{s}_k\) and \(0 < \dot{s}_{k+1}\) hold. According to (12.12), the driving forces acting on these two phase boundaries must be non-negative. If \(\gamma_{k-1}, \gamma_k\) and \(\gamma_{k+1}\) can be chosen
so that the driving forces are non-negative, it follows from (12.5) that \( \gamma_{k-1} < \gamma_{k+1} \). Then equation (12.7) yields \( s_k > s_{k+1} \) which is a contradiction. In a similar way, we can prove that proposition (iii) is true if \( \gamma_{k-1}, \gamma_{k+1} \) are in the high-strain phase and \( \gamma_k \) in the low-strain phase. This establishes the proposition (iii).

Proposition (vi) may be established by arguments that are very similar to the above.

We now return to the original Riemann problem formulated in Section 12.2 with \( \gamma_L \) in the low-strain phase, \( \gamma_R \) in the unstable phase. Let \( (\gamma, v) \) be an admissible solution of this Riemann problem. The following results can be readily established.

(v) Let \( x = s_k t \) and \( x = s_{k+1} t \) be two rays in the first quadrant of the \((x,t)\)-plane. If the slower ray is a \( p,2 \)-phase boundary where \( p = 1 \) or \( 3 \), then the faster ray cannot be either a phase boundary or a shock wave.

(vi) Let \( x = s_k t \) and \( x = s_{k+1} t \) be two rays in the first quadrant of the \((x,t)\)-plane. If the faster ray is a \( p,2 \)-phase boundary where \( p = 1 \) or \( 3 \), then the slower ray cannot be a shock wave.

(vii) Let \( x = s_k t \) and \( x = s_{k+1} t \) be two rays in the first quadrant of the \((x,t)\)-plane. If the faster ray is a \( p,2 \)-phase boundary where \( p = 1 \) or \( 3 \), then the slower ray may be a \( q,p \)-phase boundary. If so, these two phase boundaries are the only two discontinuities allowed in the first quadrant.

The result (v) implies that if a \( p,2 \)-phase boundary exists in the first quadrant, then there exist no shocks and phase boundaries traveling faster than it. The last two results state that if a \( p,2 \)-phase boundary exists, then a \( q,p \)-phase boundary is the only other discontinuity allowed in that same quadrant and that the \( q,p \)-phase boundary must travel slower than the \( p,2 \)-phase boundary.
The proofs of propositions (v)-(vii) are entirely analogous to the proofs of propositions (i)-(iii).

If the $p,2$- and $q,p$-phase boundaries described in the above propositions (v)-(vii) are replaced by the $2,p$- and $p,q$-phase boundaries respectively, then results parallel to (v)-(vii) can be readily established for the second quadrant of the upper-half of the $(x,t)$-plane.

Thus we conclude that the solutions to the Riemann problem formulated in Section 12.2 must involve at least one phase boundary separating the unstable phase from either the low-strain phase or the high-strain phase in each of the two quadrants of the upper-half of the $(x,t)$-plane. In each case, the entropy inequality implies that the interface between the unstable phase and other phase must travel into the unstable phase.

### 12.4 Solutions to a Riemann Problem

The results established above allow one to determine all admissible solutions to the Riemann problem. Recall that the initial strains $\gamma_L$ and $\gamma_R$ are such that

\[
\gamma_L \in (-1, \gamma_M], \quad \gamma_R \in (\gamma_M, \gamma_m).
\]  

(12.13)

To describe the forms of admissible solution of the Riemann problem with initial data (12.13), we shall first consider the first quadrant of the upper-half of the $(x,t)$-plane. In view of propositions (v)-(vii), this quadrant must involve at least one phase boundary; this phase boundary may be of 1,2-type or 3,2-type. However this quadrant can have at most two phase boundaries. If a 1,2-phase boundary occurs, then it may involve a 3,1-phase boundary that moves slower than the 1,2-phase boundary. If a 3,2-phase boundary occurs in the first quadrant, then it may involve a 1,3-phase boundary that moves slower than the 3,2-phase boundary.
We turn next to the second quadrant of the upper-half of the \((x, t)\)-plane. At the initial instant, the portion of the bar \(x \leq 0\) is in the low-strain phase. At a later instant, the bar may or may not change its phase. If there is no phase change, it follows from proposition (ii) that this quadrant can involve only a single 1,1-shock wave. If there is phase change, proposition (i) implies that the unstable phase cannot occur, but propositions (iii), (iv) show that this quadrant may involve only a 1,3-phase boundary possibly with a 1,1-shock wave.

Based on the above analysis, the solutions must necessarily have one of the forms shown in Figure 12.3(a)-(c). The propagation speed of the 1,3-phase boundary in the solution form of Figure 12.3(b) may be negative, zero or positive.

In the preceding chapters, we have focused our attention on the propagation of an interface that separates the unstable phase from the low-strain phase. In the dynamic theory, we have formulated a related Riemann problem. The question at hand is whether, after imposing the jump conditions (12.10) and the entropy inequality (12.11) at all discontinuities as well as the kinetic laws \(s = V(f)\) at 1,3- and 3,1-type phase boundaries, the Riemann problem has a unique solution. If it does have a unique solution this indicates that a kinetic relation is not required to describe the propagation of a 1,2- or 2,1-type phase boundary. The quasi-static theory would be misleading in this event, and the ingredient that was missing was inertial effects rather than kinetic effects. On the other hand if the Riemann problem does not have a unique solution, the dynamic theory also points to a need for kinetics.

Consider solutions having the form shown in Figure 12.3(a):

\[
\gamma, v = \begin{cases} 
\gamma_L, v_L, & -\infty < x < \hat{s}_1 t, \\
\gamma, v, & \hat{s}_1 t < x < \hat{s}_2 t, \\
\gamma_R, v_R, & \hat{s}_2 t < x < \infty,
\end{cases}
\]  

(12.14)

in which \(\gamma, v, \hat{s}_1\) and \(\hat{s}_2\) are to be found such that \(\gamma \in (-1, \gamma_M]\) and \(\hat{s}_1 < 0 \leq \hat{s}_2\).

The jump conditions (12.10) at the shock wave and the phase boundary lead to
\[-(v_R - v) = \dot{s}_2 (\gamma_R - \gamma), \quad \dot{s}_2 = \sqrt{\frac{\dot{\sigma}(\gamma_R) - \dot{\sigma}(\gamma)}{\rho(\gamma_R - \gamma)}}, \quad (12.15)\]

\[-(v - v_L) = \dot{s}_1 (\gamma - \gamma_L), \quad \dot{s}_1 = -c_1. \quad (12.16)\]

By (12.15), the condition \(\dot{s}_2 \geq 0\) requires

\[-1 < \gamma \leq \gamma'_R \equiv \gamma_M - \frac{c_1^2}{c_1^2} (\gamma_R - \gamma_M).\]

(12.17)

Since \(\gamma_\alpha \leq \gamma'_R \leq \gamma_M\), the above inequality guarantees that the strain \(\gamma\) lies in the low-strain phase. (See Figure 9.1 for meaning of \(\gamma_\alpha\)).

Next, combining equations (12.15) and (12.16) yields

\[v_0 = G(\gamma; \gamma_L, \gamma_R),\]

(12.18)

where \(G(\gamma; \gamma_L, \gamma_R)\) is defined for \(\gamma\) in \((-1, \gamma'_R)\) by

\[G(\gamma; \gamma_L, \gamma_R) \equiv c_1 (\gamma - \gamma_L) - \sqrt{c_1^2 (\gamma_M - \gamma) + c_2^2 (\gamma_M - \gamma_R)(\gamma_R - \gamma)}.\]

(12.19)

It can be verified that given the initial strains \(\gamma_L\) and \(\gamma_R\) with \(\gamma_L \in (-1, \gamma_M]\) and \(\gamma_R \in (\gamma_M, \gamma_m)\), \(G(\gamma; \gamma_L, \gamma_R)\) increases monotonically with \(\gamma\) on \((-1, \gamma'_R]\) from the negative value \(G(-1; \gamma_L, \gamma_R)\) at \(\gamma = -1\) to the value \(G(\gamma'_R; \gamma_L, \gamma_R)\) at \(\gamma = \gamma'_R\); \(G(\gamma'_R; \gamma_L, \gamma_R)\) is positive if \(-1 < \gamma_L < \gamma'_R\), zero if \(\gamma_L = \gamma'_R\) and negative if \(\gamma'_R < \gamma_L < \gamma_M\). Thus if the initial data is such that \(G(-1; \gamma_L, \gamma_R) < v_R - v_L < G(\gamma'_R; \gamma_L, \gamma_R)\), there is a unique root \(\gamma\) of (12.19) in the range \(-1 < \gamma < \gamma'_R(\leq \gamma_M)\). The remaining unknowns \(v, \dot{s}_1\) and \(\dot{s}_2\) are then given immediately by (12.15) and (12.16).

It can be showed that the entropy inequality (12.11) holds automatically at the phase boundary in this case; it is trivially satisfied at the shock wave for the piecewise linear material of Figure 9.1.

In summary, there exists a unique admissible solution of the form (12.14) corresponding to Figure 12.3(a) if the given initial data (12.8), (12.13) is such that \(G(-1; \gamma_L, \gamma_R) < v_R - v_L < \)
\( G(\gamma'_R; \gamma_L, \gamma_R) \). It is thus unnecessary for a kinetic relation to apply at the phase boundary separating the unstable phase from the low-strain phase.

We turn next to a solution having the form shown in Figure 12.3(b):

\[
\gamma, v = \begin{cases} 
\gamma_L, v_L, & -\infty < x < \dot{s}_1 t, \\
\gamma_A, v_A, & \dot{s}_1 t < x < \dot{s}_2 t, \\
\gamma, v, & \dot{s}_2 t < x < \dot{s}_3 t, \\
\gamma_B, v_B, & \dot{s}_3 t < x < \dot{s}_4 t, \\
\gamma_R, v_R, & \dot{s}_4 t < x < \infty,
\end{cases}
\tag{12.20}
\]

in which \( \gamma_A, v_A, \gamma, v, \gamma_B, v_B, \dot{s}_1, \dot{s}_2, \dot{s}_3, \dot{s}_4 \) are to be determined such that \( \gamma_A \in (-1, \gamma_M], \gamma \in [\gamma_m, \infty), \gamma_B \in (-1, \gamma_M] \) and \( \dot{s}_1 < \dot{s}_2 < 0 < \dot{s}_3 < \dot{s}_4 \). Thus there are ten unknowns associated with such a solution.

Equation (12.10) indicates that there are two jump conditions associated with each discontinuity. One must also apply a kinetic relation at the 1,3-phase boundary and another at the 3,1-phase boundary. Thus eight jump conditions plus two kinetic relations must be enforced thus leading to ten equations which must be solved for the 10 unknowns. The system just described may or may not have solution. In any case, it certainly does not have a one-parameter family of solutions and so again there is no room for a kinetic relation to apply at the 1,2-phase boundary.

Finally consider a solution of the form of Figure 12.3(c). Just as in the preceding case one can apply six jump conditions corresponding to the three discontinuities plus one kinetic relation controlling the 1,3-phase boundary to obtain seven equations which control the seven unknowns. Again, irrespective to whether or not this system of equations has a solution, it does not have a one-parameter family of solutions. Thus again one finds that there is no room for a kinetic relation to apply at the phase boundary separating the unstable phase from
the high-strain phase.
Figure 12.1. Assumed form of solutions to Riemann problem.
Figure 12.2.
Figure 12.3. Form of solutions to Riemann Problem.
Chapter 13

Conclusions

In this part of this thesis we have been concerned with the propagation of an interface that separates a metastable phase from the unstable "phase". The equilibrium and quasi-static theories indicated a lack of uniqueness that suggested the need for additional information. We begin by assuming that this missing information was kinetic in nature and attempted to derive a kinetic law based on a regularization that included strain-gradient and viscous effects. Surprisingly this standard procedure did not lead to a kinetic law. Finally we included, instead, the effect of inertia and found that the dynamic theory, at least in the setting studied, did not require a kinetic law. Thus the physical effects that were missing from the quasi-static theory were inertial, not kinetic.
Part C
An Application: The Impact Problem
Chapter 14

Introduction

In parts A and B of this thesis we addressed certain fundamental questions regarding the proper mathematical modeling of a propagating phase boundary. Part A considered a phase boundary separating a metastable phase from a stable phase; part B considered an interface between a stable phase and an unstable phase.

Having addressed these questions concerning formulation, we now turn, in this part of the thesis, to a specific initial-boundary-value problem.

In order to study the dynamics of reversible (or thermoelastic) phase transitions, Professor R. J. Clifton of Brown university is presently carrying out plate impact experiments on single crystals of a Cu-Al-Ni shape memory alloy. By properly orienting the single crystal, and by adjusting the angle of the impact, he can generate plane waves, which would therefore be amenable to a one-dimensional analysis.

In this part of the thesis we model and analyze the plate impact experiment. Here an impactor of length $l$ propagates at a velocity $v_0$ and strikes a stationary specimen of length $L$ at time $t = 0$. The history of the particle velocity at the free-surface is monitored. The impactor is made of a single-phase material while the specimen is composed of a two-phase material. Depending on whether $v_0$ is smaller than or greater than a certain critical value $v_{cr}$, a phase change is nucleated by the impact. The particle velocity history at the free-surface
is thus qualitatively and quantitatively different in the two cases \( v_0 < v_{cr} \) and \( v_0 > v_{cr} \). We study this problem.

Some years ago, Grady et al (1977, 1978, 1983) carried out experimental tests on compressional waves in calcium carbonate rocks using the plate impact techniques. Calcium carbonate is a natural mineral which may occur in the earth's upper crust. It exists in two crystalline structures at ambient temperature and pressure; aragonite and the more common calcite I. Static volumetric compression studies on calcite I carried out by Bridgman (1939) have identified two additional phases, calcite II and calcite III, with transformation pressures of approximately 1.44 and 1.77 GPa, respectively. Three types of calcite, Solemhofen limestone, Oakhall limestone and Vermont marble, were studied by Grady et al (1977, 1978, 1983).

The experimental methods used by Grady et al to investigate wave propagation properties under plate impact conditions are illustrated in Figure 14.1. A 100 mm diameter light gas gun was used to impel a flat-nosed aluminum projectile at a target. The projectiles were faced with thin plates of a well-known impact material, which in turn are backed with low impedance solid foam. Fused quartz was the impact material used in the experiment. The calcite specimen is backed with a laser window. The particle velocity at the right-hand (free) end of the sample is continuously recorded with diffuse velocity interferometry. The impact tests were performed at different impact velocities. Moreover, the sample thickness varied between 5 and 25 mm to provide for measurement of evolution of the wave.

These experiments involved both shear and dilatational effects. Moreover, at the loading rates involved here, calcite is thought to be strain-rate dependent (Grady (1983)). Finally, failure of the calcite due to crushing and cracking occurs as a result of impact. For these reasons one expects our one-dimensional, elastic analysis to be not a very good model. Even
so, we shall compare our analysis with Grady et al's results. For this purpose we shall consider the calcite I–II transformation occurring in Oakhall limestone. The microstructure of this material is fine grained micrite with a few larger recrystallized calcite grains.
Figure 14.1. Plate impact assembly.
Chapter 15

Formulation of Impact Problem

In this part of the thesis, we consider an impact problem. Here an impactor of length \( l \) strikes a stationary specimen of length \( L \). The impactor is made of a single phase elastic material that is characterized by the stress response function

\[
\dot{\sigma}(\gamma) = \mu' \gamma, \quad \gamma \geq -1, \quad (15.1)
\]

where \( \mu' > 0 \) is the elastic modulus of the impactor. Let \( \rho' \) be its mass density. Then \( c' = (\mu'/\rho')^{\frac{1}{2}} \) is the corresponding sonic speed. The impactor is initially stress-free, and moves at the constant speed \( v_0 \) to the right. Thus we have the initial conditions

\[
\gamma(x, 0), v(x, 0) = \begin{cases} 
0, & -l < x < 0, \\
0, & 0 < x < L.
\end{cases} \quad (15.2)
\]

At the instant \( t = 0 \), the impactor strikes the specimen which is initially at rest. For some time thereafter, \( 0 < t < t_* \), the impactor moves in contact with the specimen. At \( t = t_* \) the impactor separates from the specimen. The interface conditions are therefore

\[
\begin{cases}
\sigma(0+, t) = \sigma(0-, t) < 0, & \text{for } 0 < t < t_*, \\
v(0+, t) = v(0-, t),
\end{cases} \quad (15.3)
\]

and

\[
\begin{cases}
\sigma(0+, t) = \sigma(0-, t) = 0, & \text{for } t \geq t_*, \\
v(0+, t) > v(0-, t),
\end{cases} \quad (15.4)
\]
The forward surface of the specimen and the rear surface of the impactor are always traction-
free:

\[ \sigma(-l, t) = \sigma(L, t) = 0, \quad t \geq 0. \]  \hfill (15.5)

The specimen is composed of a two-phase material whose stress-strain relation is given by

\[ \hat{\sigma}(\gamma) = \begin{cases} 
\mu_1 \gamma, & \gamma \geq \gamma_M, \\
-\mu_2 \gamma + \sigma_2, & \gamma_m \leq \gamma \leq \gamma_M, \\
\mu_1 (\gamma - \gamma_T), & -1 < \gamma \leq \gamma_m.
\end{cases} \]  \hfill (15.6)

The elastic modulus of both phases of this material is \( \mu_1 \), and \( \gamma_T < 0 \) is the transformation strain. The meaning of the constants \( \mu_2, \gamma_M < 0 \) and \( \gamma_m < 0 \) is shown in Figure 15.1. We set \( \sigma_2 = (\mu_1 + \mu_2) \gamma_M = (\mu_1 + \mu_2) \gamma_m - \mu_1 \gamma_T \). Each branch of the stress-strain curve is identified with a material phase and we say that the particle \( x \) is in the low-strain phase at time \( t \) if \( \gamma(x, t) \geq \gamma_M \), in the "unstable phase" if \( \gamma_m < \gamma(x, t) < \gamma_M \), and in the high-strain phase if \( -1 < \gamma(x, t) \leq \gamma_m \). We let \( c_1 = (\mu_1 / \rho)^{\frac{1}{2}} \) be the sonic speed in the low-strain and high-strain phases, and set \( c_2 = (\mu_2 / \rho)^{\frac{1}{2}} \); \( c_2 \) is not a wave speed. A discontinuity is a shock wave or a phase boundary if the strains on both sides of this discontinuity belong to the same phase or to distinct phases, respectively.

The impact problem involves finding fields \( \gamma(x, t), v(x, t) \) subject to the initial conditions (15.2), the interface conditions (15.3) and (15.4), the boundary conditions (15.5) together with appropriate field equations and jump conditions. Figure 15.2 describes this problem. At points where \( \gamma \) and \( v \) are both smooth, the field equations associated with balance of momentum and kinematic compatibility are

\[ \hat{\sigma}'(\gamma) \gamma_x = \rho v_t, \quad v_x = \gamma_t. \]  \hfill (15.7)

If either \( \gamma \) or \( v \) is discontinuous across a curve \( x = s(t) \) in the \((x, t)\)-plane, balance of momentum and the smoothness properties of \( u \) yield the following jump conditions:
\[ \dot{\sigma}(\dot{\gamma}) - \dot{\sigma}(\ddot{\gamma}) = -\rho \dot{s} (\dot{\gamma} - \ddot{\gamma}), \quad (\dot{\gamma} - \ddot{\gamma}) = -\dot{s} (\dot{\gamma} - \ddot{\gamma}). \] (15.8)

In addition to the jump conditions (15.8), at the discontinuity \( x = s(t) \), the following entropy inequality must also hold:

\[ f(t) \dot{s}(t) \geq 0, \] (15.9)

where the driving force \( f \) acting at the discontinuity is given by

\[ f = f(\ddot{\gamma}, \dot{\gamma}) = \int_{\gamma}^{\dot{\gamma}} \dot{\sigma}(\gamma) d\gamma - \frac{1}{2} (\dot{\sigma}(\dot{\gamma}) + \ddot{\sigma}(\ddot{\gamma}))(\dot{\gamma} - \ddot{\gamma}). \] (15.10)

In the field equations (15.7) and jump conditions (15.8), \( \dot{\sigma}(\gamma) \) is given by either (15.1) or (15.6) depending on whether one is referring to the impactor \(-l < x < 0\) or the specimen \(0 < x < L\).
Figure 15.1. Stress-strain curve.
Figure 15.2. Initial conditions, boundary conditions and interface conditions of impact problem.
Chapter 16

Local Properties at a Phase Boundary

Before attempting to solve the impact problem it is convenient to first consider certain local characteristics at a phase boundary (in the specimen) that has \( \tilde{\gamma} \) in the high-strain phase and \( \gamma^+ \) in the low-strain phase. At any moving discontinuity \( x = s(t) \), the jump conditions (15.8) imply

\[
\rho \dot{s}^2 = \frac{\sigma^+(\gamma) - \sigma(\tilde{\gamma})}{\gamma^+ - \tilde{\gamma}} \geq 0.
\]  

(16.1)

The right side of (16.1) is thus necessarily non-negative for any pair of strains \((\tilde{\gamma}, \gamma^+)\) that can occur at a strain jump.

When (16.1) is specialized to the case where \( \tilde{\gamma} \) is in the high-strain phase and \( \gamma^+ \) in the low-strain phase, and to the two-phase material shown in Figure 15.1, equation (16.1) becomes

\[
\dot{s}^2 = c_i^2(1 + \frac{\gamma^+}{\gamma^+ - \tilde{\gamma}}).
\]  

(16.2)

In the \((\tilde{\gamma}, \gamma^+)-plane\), the set of pairs \((\tilde{\gamma}, \gamma^+)\) for which \( \tilde{\gamma} \) is in the high-strain phase, \( \gamma^+ \) is in the low-strain phase and the right side of (16.2) is non-negative is represented by the shaded...
region \( \Gamma \) shown in Figure 16.1. At any point on the boundary segment \( BC \), we have \( \dot{s} = 0 \).

The corresponding phase boundary is thus instantaneously stationary.

The driving force \( f \) acting on a phase boundary of the present type can be found from (15.10) and the explicit form of \( \dot{\sigma}(\gamma) \) for the two-phase material:

\[
f = \frac{1}{2} \mu_1 \gamma_T (\gamma + \dot{\gamma} - \gamma_M - \gamma_m). \tag{16.3}
\]

It follows that the driving traction vanishes on the straight line \( \gamma + \dot{\gamma} = \gamma_M + \gamma_m \). In view of the entropy inequality (15.9) this means that the points in \( \Gamma \) on this straight line correspond to values of \( \dot{\gamma} \) and \( \gamma \) for which the associated phase boundary \( x = s(t) \) propagates without dissipation. At points off this straight line, \( f \neq 0 \), the entropy inequality (15.9) determines the sign of \( \dot{s} \); see Figure 16.1. For points on the straight line \( f = 0 \), the sign of \( \dot{s} \) is not determined by (15.9), and propagation in either direction is possible.

We now use (16.2) and (16.3) to map the region \( \Gamma \) of the \((\gamma, \dot{\gamma})\)-plane into the \((\dot{s}, f)\)-plane. Each point \((\gamma, \dot{\gamma})\) that does not lie on \( BC \) (Figure 16.1) is carried to two points \((\dot{s}, f)\) and \((-\dot{s}, f)\) in the \((\dot{s}, f)\)-plane; if \( f \neq 0 \), only one of these satisfies the entropy inequality (15.9). If \( f = 0 \), the point \((\gamma, \dot{\gamma})\) maps to a pair of points \((\pm \dot{s}, 0)\) in the \((\gamma, \dot{\gamma})\). Each point on \( BC \) maps to a single point \((0, f)\) on the \(f\)-axis. Figure 16.2 shows the image of \( \Gamma \) in the \((\gamma, \dot{\gamma})\)-plane that satisfies the entropy inequality (15.9).

As discussed in Part A of this thesis, the physical basis for the elucidation of phase transitions in solids involve both a nucleation criterion governing the initiation of the transition and a kinetic relation controlling the rate at which it proceeds. In the simplest models, a kinetic relation is a constitutively described relation between the driving force acting on the phase boundary and its propagation speed:

\[
f = \phi(\dot{s}). \tag{16.4}
\]

Because of the entropy inequality (15.9), \( \phi(\dot{s}) \) must satisfy
\( \phi(\dot{s}) \dot{s} \geq 0. \) \hspace{1cm} (16.5)

Note that if \( \phi(\dot{s}) \) is continuous at \( \dot{s} = 0 \) as we shall assume in this part, (16.5) requires that \( \phi(0) = 0 \). It is required that the curve represented by (16.4) lies in the hatched region of the \((\dot{s}, f)\)-plane shown in Figure 16.2. The pre-image of this curve in the \((\gamma, \dot{\gamma})\)-plane under the mapping (15.9), (16.2), (16.3) thus comprise the locus of all strain-pairs \((\gamma, \dot{\gamma})\) at a phase boundary that are consistent with the kinetic relation (16.4).

For purposes of illustrating our results, we will find it convenient to consider the particular kinetic relation

\[
\phi(\dot{s}) = \begin{cases} 
\frac{1}{2} b \mu_1 \gamma_T^2 \frac{\dot{s}^2}{c_t^2 - \dot{s}^2}, & 0 \leq \dot{s} \leq c_1, \\
-\frac{1}{2} b \mu_1 \gamma_T^2 \frac{\dot{s}^2}{c_t^2 - \dot{s}^2}, & -c_1 \leq \dot{s} \leq 0,
\end{cases}
\] \hspace{1cm} (16.6)

where \( b \) is a kinetic material parameter. The curve \( \mathcal{K} \) described by (16.6) is shown in Figure 16.2. In particular, for \( b = 1 \), the two pieces of \( \mathcal{K} \) in the first and third quadrants are parallel to the upper and lower boundaries (respectively) of the hatched region shown in the figure.
Figure 16.1. The $(\gamma, \gamma^\dagger)$-plane.
Figure 16.2. The $(s, f)$-plane.
Chapter 17

Two Preliminary Problems

In order to construct a solution of the impact problem described in Chapter 15 it is convenient to first consider two preliminary problems: one is a signalling problem and the other is a Riemann problem.

17.1 The Signalling Problem

Here we consider a semi-infinite bar $x > 0$ composed of the two-phase material and seek weak solutions of the differential equations (15.7) that satisfy the following initial and boundary conditions:

$$\gamma(x, 0) = \gamma_0, \quad v(x, 0) = v_0, \quad x \geq 0,$$

and

$$\sigma(0, t) = \sigma_B, \quad t > 0,$$

where $\gamma_0$, $v_0$ and $\sigma_B$ are given constants.

With the help of the jump conditions (15.8) and the entropy condition (15.9), we can determine the form of all scale-invariant solutions of the signalling problem. In this section we shall consider two specific cases of initial data: First, we consider the case in which the
initial strain $\gamma_0$ is in the low-strain phase, and then we consider the case in which $\gamma_0$ is in the high-strain phase.

### 17.1.1 The Initial Data in the Low-strain Phase

In this case, a scale-invariant solution of the signalling problem must have one of the forms shown in Figures 17.1(a)-(b). The case of Figure 17.1(a) does not involve a phase change, while the case of Figure 17.1(b) involves a phase change.

(i) Solution without phase change. First consider the case of Figure 17.1(a). The boundary condition (17.2) together with the stress-strain relation (15.6) yields $\gamma(0, t) = \sigma_B/\mu_1$ for $t > 0$ and so

$$\gamma_1 = \frac{\sigma_B}{\mu_1}. \quad (17.3)$$

It will be helpful for latter analysis to set

$$h = v_0 + c_1 \gamma_0 - \frac{c_1 \sigma_B}{\mu_1}. \quad (17.4)$$

The jump conditions (15.8) when applied to the shock wave $x = \dot{s}t$ give

$$\dot{s} = c_1, \quad v_1 = h. \quad (17.5)$$

In order to have the strain $\gamma_1$ in the low-strain phase, it follows from (17.3) that the stress at the boundary $x = 0$ must satisfy

$$\sigma_B \geq \sigma_M. \quad (17.6)$$

(ii) Solution with phase change. We turn next to the case of Figure 17.1(b). The jump conditions (15.8) when applied to the two discontinuities give

$$-(v_1 - v_3) = \dot{s}(\gamma_1 - \gamma_3), \quad -(v_0 - v_1) = c_1(\gamma_0 - \gamma_1), \quad (17.7)$$
and
\[ \gamma_3 - \gamma_1 = \frac{c_1^2}{c_1^2 - \dot{s}^2} \gamma_T. \] (17.8)

The boundary condition (17.2) leads to
\[ \gamma_3 = \frac{\sigma_B}{\mu_1} + \gamma_T. \] (17.9)

Combining (17.9) and (17.8) yields
\[ \gamma_1 = \frac{\sigma_B}{\mu_1} - \frac{\dot{s}^2}{c_1^2 - \dot{s}^2} \gamma_T. \] (17.10)

The particle velocities can now be expressed as
\[ v_1 = h + \frac{c_1 \dot{s}^2}{c_1^2 - \dot{s}^2} \gamma_T, \quad v_3 = h. \] (17.11)

In order that the strains \( \gamma_1 \) and \( \gamma_3 \) lie in the appropriate phases, it follows from (17.9), (17.10) that the stress \( \sigma_B \) at the boundary \( x = 0 \) is required to satisfy
\[ Y_1(\dot{s}) \leq \sigma_B \leq \sigma_m, \] (17.12)

where \( Y_1(\dot{s}) \) is defined by
\[ Y_1(\dot{s}) = \sigma_M + \frac{\dot{s}^2}{c_1^2 - \dot{s}^2} \mu_1 \gamma_T. \] (17.13)

The function \( Y_1(\dot{s}) \) describes a monotonically decreasing curve in the \((\dot{s}, \sigma_B)\)-plane, see Figure 17.2.

By (17.9), (17.10) and (16.3), one finds that the driving force acting at the phase boundary becomes
\[ f = \gamma_T(\sigma_B - \sigma_0 - \frac{1}{2} \mu_1 \gamma_T \frac{s^2}{c_1^2 - s^2}). \] (17.14)

Since \( \dot{s} \geq 0 \), then the entropy inequality (15.9) requires that
\[ \sigma_B \leq Y_2(\dot{s}), \quad (17.15) \]

where \( Y_2(\dot{s}) \) is defined by

\[ Y_2(\dot{s}) = \sigma_0 + \frac{1}{2} \mu_1 \gamma T \frac{\dot{s}^2}{c_1^2 - \dot{s}^2} < \sigma_0. \quad (17.16) \]

The function \( Y_2(\dot{s}) \) describes a monotonically decreasing curve in the \((\dot{s}, \sigma_B)\)-plane shown in Figure 17.2. It can be shown that \( Y_2(\dot{s}) > Y_1(\dot{s}) \) for \( 0 < \dot{s} < c_1 \). Combining (17.12) and (17.15) yields

\[ Y_1(\dot{s}) \leq \sigma_B \leq Y_2(\dot{s}). \quad (17.17) \]

Thus the stress \( \sigma_B \) at boundary \( x = 0 \) and \( \dot{s} \) must be chosen to lie in the hatched region shown in Figure 17.2.

We now apply the kinetic relation (16.6) at the phase boundary and find from (17.14) that

\[ \sigma_B = Y(\dot{s}) = \sigma_0 - \frac{1}{2} \mu_1 \gamma T \frac{\dot{s}^2}{c_1^2 - \dot{s}^2} + \frac{\phi(\dot{s})}{\gamma_T}, \quad (17.18) \]

which describes a curve in the hatched region of Figure 17.2. This equation can be solved to give

\[ \dot{s} = c_1 \sqrt{\frac{R}{1 + R}}, \quad R \equiv \frac{2(\sigma_B - \sigma_0)}{\mu_1 \gamma T (1 + b)}. \quad (17.19) \]

All remaining unknowns \( \gamma_1, \gamma_3, v_1 \) and \( v_3 \) are now determined by (17.9)-(17.11).

Thus there exists a unique solution of the form shown in Figure 17.1(b) whenever the boundary stress \( \sigma_B \) is such that (17.17) holds.

On the other hand, from (17.6) and (17.17), it follows that, when the boundary stress \( \sigma_B \) satisfies

\[ \sigma_M \leq \sigma_B \leq \sigma_0, \quad (17.20) \]
the problem also has a solution of the form shown in Figure 17.1(a) without a phase change. Thus for $\sigma_B > \sigma_0$ the signalling problem with initial data in the low-strain phase never leads to a phase transformation; for $\sigma_B < \sigma_M$, it always does. For the intermediate values of $\sigma_B$ where both types of solutions exist, a nucleation condition for the phase transition is necessary to select the appropriate solution. In this one-dimensional problem, this nucleation criterion can be stated in terms of a critical value of stress which can in turn to be related to a critical value of the driving force. We take the critical stress $\sigma_{cr}$ for nucleation to be $\sigma_{cr} = \sigma_0$ where $\sigma_0$ is the equal-energy (or Maxwell) stress shown in Figure 15.1. The corresponding critical driving force is $f_{cr} = 0$.

According to this nucleation criterion, if the boundary stress $\sigma_B \geq \sigma_0$ the solution has the form of Figure 17.1(a), it does not involve a phase change, and it has a long time equilibrium limit which is in the low-strain phase. This solution is

$$\gamma(x, t) = \begin{cases} \frac{\sigma_B}{\mu_1}, & 0 < x < c_1 t, \\ \gamma_0, & c_1 t < x < \infty, \end{cases}$$

and

$$v(x, t) = \begin{cases} h, & 0 < x < c_1 t, \\ v_0, & c_1 t < x < \infty. \end{cases}$$

If $\sigma_M < \sigma_0$ the solution has the form of Figure 17.1(b), it has a long time equilibrium limit which is in the high-strain phase. The solution is

$$\gamma(x, t) = \begin{cases} \frac{\sigma_B}{\mu_1} + \gamma_T, & 0 < x < c_1 t \sqrt{R/(1 + R)}, \\ \frac{\sigma_B}{\mu_1} - R \gamma_T, & c_1 t \sqrt{R/(1 + R)} < x < c_1 t, \\ \gamma_0, & c_1 t < x < \infty, \end{cases}$$

and
\[ v(x, t) = \begin{cases} 
  h + c_1 \gamma_T R, & 0 < x < c_1 t \sqrt{R/(1 + R)}, \\
  h, & c_1 t \sqrt{R/(1 + R)} < x < c_1 t, \\
  v_0, & c_1 t < x < \infty. 
\] (17.24)

17.1.2 The Initial Data in the High-strain Phase

Consider again the signalling problem described in the previous subsection but now with \( \gamma_0 \) in the high-strain phase. In this case, the problem has two possible scale-invariant solution forms shown in Figures 17.1(c)-(d), one without phase change and the other with phase change.

In an entirely similar way as in the above subsection, one can find all solutions to this signalling problem. We shall only summarize the results below.

(i) Solution without phase change. When \( \sigma_B \leq \sigma_0 \), the solution has the form of Figure 17.1(c) and is given by

\[ \gamma(x, t) = \begin{cases} 
  \sigma_B / \mu_1 + \gamma_T, & 0 < x < c_1 t, \\
  \gamma_0, & c_1 t < x < \infty, 
\] (17.25)

and

\[ v(x, t) = \begin{cases} 
  h - c_1 \gamma_T, & 0 < x < c_1 t, \\
  v_0, & c_1 t < x < \infty, 
\] (17.26)

where \( h \) is defined in (17.4).

(ii) Solution with phase change. When \( \sigma_B > \sigma_0 \), the solution has the form of Figure 17.1(d). The corresponding fields are

\[ \gamma(x, t) = \begin{cases} 
  \sigma_B / \mu_1, & 0 < x < c_1 t \sqrt{R/(1 + R)}, \\
  \sigma_B / \mu_1 + \gamma_T, & c_1 t \sqrt{R/(1 + R)} < x < c_1 t, \\
  \gamma_0, & c_1 t < x < \infty, 
\] (17.27)
and

\[
v(x, t) = \begin{cases} 
    h - c_1 \gamma_T/(1 + \sqrt{R/(1 + R)}), & 0 < x < c_t \sqrt{R/(1 + R)}, \\
    h - c_1 \gamma_T, & c_t \sqrt{R/(1 + R)} < x < c_t, \\
    v_0, & c_t < x < \infty,
\end{cases}
\]

(17.28)

where \( h \) is defined in (17.4), and \( R \) is given in (17.19). The phase boundary speed \( \dot{s} \) is also given in (17.19).

### 17.2 The Riemann Problem

Next we need to consider two preliminary Riemann problems. The first one will be related to the interface between the two bars, while the second problem pertains to the interior of the specimen. In either case we look for weak solutions of the differential equations (15.7) on the upper half of the \((x, t)\)-plane that satisfy the following initial conditions:

\[
\gamma(x, 0), v(x, 0) = \begin{cases} 
    \gamma_L, v_L, & -\infty < x < 0, \\
    \gamma_R, v_R, & 0 < x < \infty,
\end{cases}
\]

(17.29)

where \( \gamma_L, \gamma_R, v_L \) and \( v_R \) are given constants, with \( \gamma_L > -1, \gamma_R > -1 \).

#### 17.2.1 A Riemann Problem Involving Two Distinct Materials

In this case the left semi-infinite bar \( x < 0 \) is characterized by the single phase stress-strain relation (15.1) and the right semi-infinite bar \( x > 0 \) by the two-phase material (15.6). For our purposes, we need only consider the case where the two are bars always in contact at \( x = 0 \) for \( t \geq 0 \) and where the initial strain \( \gamma_R \) is in the low-strain phase. The interface at \( x = 0 \) may be considered as a stationary discontinuity. Therefore the jump conditions must hold at the interface.

It follows from the jump conditions (15.8), the entropy inequality (15.9) that this problem has two scale-invariant solution forms shown in Figures 17.3(a) and (b), where the former
does not involve a phase change while the latter does. We consider these two solution forms separately and summarize the associated results below. For describing these results it is useful to introduce the quantities

\[
h = (v_R - v_L + c_1 \gamma_R + c' \gamma_L)/(\alpha c_1 + c'), \quad \alpha \equiv \mu'/\mu_1. \tag{17.30}
\]

(i) Solution with no phase change. In a manner similar to Subsection 17.1.1 one can use two jump conditions, kinetic relation and nucleation criterion to show that the solution has the form shown in Figure 17.3(a) when \( h \geq \sigma_0/\alpha_1 \) and is given by

\[
\gamma(x, t) = \begin{cases} 
\gamma_L, & -\infty < x < -c_1 t, \\
h, & -c_1 t < x < 0, \\
\alpha h, & 0 < x < c_1 t, \\
\gamma_R, & c_1 t < x < \infty,
\end{cases} \tag{17.31}
\]

and

\[
v(x, t) = \begin{cases} 
v_L, & -\infty < x < -c_1 t, \\
v_L + c'(h - \gamma_L), & -c_1 t < x < 0, \\
v_L + c'(h - \gamma_L), & 0 < x < c_1 t, \\
v_R, & c_1 t < x < \infty.
\end{cases} \tag{17.32}
\]

(ii) Solutions with a phase change. When \( h < \sigma_0/\alpha_1 \), the solution has the form of Figure 17.3(b). In this case, the propagation speed of the phase boundary in the right-hand bar is given by

\[
\dot{s} = -\frac{Q + \sqrt{Q^2 + 4PR}}{2P}, \tag{17.33}
\]

where \( P, Q \) and \( R \) are defined respectively by

\[
P = R - Q + b + 1, \quad Q = \frac{2\alpha c_1}{\alpha c_1 + c'}, \quad R = \frac{2(\mu_1 \alpha h - \sigma_0)}{\mu_1 \gamma_T}; \tag{17.34}
\]
here \( \alpha \) and \( h \) are defined in (17.30). Let
\[
H = h - \frac{c_1 \dot{s}}{(\alpha c_1 + c')(c_1 + \dot{s})} \gamma_T.
\]
Then the solution can be expressed as
\[
\gamma(x, t) = \begin{cases} 
\gamma_L, & -\infty < x < -c_1 t, \\
H, & -c_1 t < x < 0, \\
\alpha H + \gamma_T, & 0 < x < \dot{s} t, \\
\alpha H - \frac{\dot{s}^2}{c_1^2 - \dot{s}^2} \gamma_T, & \dot{s} t < x < c_1 t, \\
\gamma_R, & c_1 t < x < \infty,
\end{cases}
\]
and
\[
\gamma(x, t) = \begin{cases} 
v_L, & -\infty < x < -c_1 t, \\
v_L + c'(H - \gamma_L), & -c_1 t < x < 0, \\
v_L + c'(H - \gamma_L), & 0 < x < \dot{s} t, \\
v_R + c_1(\gamma_R - (\alpha H - \frac{\dot{s}^2}{c_1^2 - \dot{s}^2} \gamma_T)), & \dot{s} t < x < c_1 t, \\
\gamma_R, & c_1 t < x < \infty.
\end{cases}
\]

17.2.2 A Riemann Problem for a Two-phase Material

Here the entire bar \(-\infty < x < \infty\) is made of the two-phase material (15.6). There are a number of different cases to be considered depending on which phase the strains \( \gamma_L \) and \( \gamma_R \) belong to. One can show that the solution of the Riemann problem may have one of the six forms shown in Figures 17.4(a)-(f). The solution forms of Figures 17.4(a) and 8(b) correspond to initial data in the low-strain phase. When the initial data is in the high-strain phase, the solution has one of the forms of Figure 17.4(c) or Figure 17.4(d). The solution forms shown in Figures 17.4(e)-(f) correspond to cases where the initial data involves both the low-strain and the high-strain phases. Each of these problems can be solved explicitly. We will not show these results here. Some of these results are given in (Jiang(1989)).
Figure 17.1. Form of solutions to signalling problem.
Figure 17.2. The \((s, \sigma_B)\)-plane.
$1 = \text{low-strain phase}$  
$3 = \text{high-strain phase}$

Figure 17.3. Form of solutions to Riemann problem involving two distinct materials.
Figure 17.4. Form of solutions to Riemann problem involving a single phase material.
Chapter 18

The Impact Problem: Construction of Solution

According to the results established in the preceding chapter, one now can construct the solution to the impact problem described in Chapter 15. We seek a weak solution of the differential equations (15.7) in the region $-l < x < L, t > 0$ of the $(x, t)$-plane that satisfies the initial and boundary conditions (15.2), (15.5). We speak of the left piece of the bar $-l \leq x < 0$ as the impactor and the right piece $0 \leq x < L$ as the specimen. The impactor is made of the single phase material (15.1) that is unable of undergoing phase transitions. The specimen on the other hand is made of the two-phase material (15.6) that can undergo phase transitions in the compressional region, see Figure 15.1. The specimen is initially unstressed and so is in the low-strain phase.

Instantaneously after the impactor strikes the specimen at $t = 0$, as the results in Section 17.2.1 indicated, there are two possible local wave structures near $x = 0, t = 0$ depending on the impact velocity $v_0$. Let

$$v_{cr} = \frac{|\sigma_0|/(\alpha c_1 + c')}{\alpha \mu_1}. \quad (18.1)$$

Then the impact problem has a local wave structure near $x = 0, t = 0$ of the form shown in Figure 17.3(a) if $0 < v_0 \leq v_{cr}$ or of the form shown in Figure 17.3(b) if $v_0 > v_{cr}$. We consider
these two cases separately.

18.1 No Phase Change in the Specimen

This case occurs if the velocity of the impactor is not too large: \( v_0 \leq v_{cr} \). In view of the results in Section 17.2.1, shortly after \( t = 0 \), the impact problem can be viewed locally as a Riemann problem having the wave structure in the \((x, t)\)-plane shown in Figure 17.3(a). Two shock waves are generated, the left one \( x = -c't \) is in the single phase material, the right one \( x = c_1 t \) involves the low-strain phase of the two-phase material. The rightward moving shock wave in the specimen travels at the sonic speed \( c_1 \) and reaches the right-hand end of the specimen at time \( t_A = L/c_1 \); see Figure 18.1. For \( t > t_A \), the wave structure near the point \( A \) in the upper \((x, t)\)-plane is determined by a signalling problem. Since the right-hand end of the specimen is a free surface, the signalling problem near point \( A \) for \( t > t_A \) has the form shown in Figure 17.1(a). In this case only a low-strain shock wave reflects from the point \( A \).

Meanwhile, the leftward moving shock wave \( x = -c't \) in the impactor travels with the speed \( c' \) and reaches the end \( x = -l \) at time \( t_B = l/c' \). For \( t > t_B \), the signalling problem near point \( B \) must have the wave form of Figure 17.1(a) since the corresponding material \((15.1)\) has a single phase. The reflected shock wave \( BC \) hits the interface \( x = 0 \) at time \( t_C = 2l/c' \). Our analysis shows that, at \( t_C \), the particle velocity on the left-hand side of the shock wave \( BC \) is smaller than that on the right-hand side of the interface at \( x = 0 \) for any values of the impactor velocity in the range \( 0 < v_0 \leq v_{cr} \). Thus the impactor separates from the specimen at time \( t_C \). The left-hand end of the specimen thus becomes a free surface for \( t > t_C \). Thus the instant of separation \( t_* \) in Chapter 15 is \( t_C \):

\[
t_* = 2l/c'.
\]
From now on, one only needs to consider the specimen subjected to the stress-free boundary conditions at its two ends.

The boundary condition (15.5) implies that the wave structure of the signalling problem locally near point $C$ for $t > t_C$ has the form shown in Figure 17.1(a) and a low-strain shock wave $CD$ is generated. Then this shock wave meets the low-strain shock wave $AD$ reflected from the point $A$ at time $t_D$, where $t_D \equiv L/c_1 + l/c'$. The local wave structure near point $D$ for $t > t_D$ is determined by the solution of this Riemann problem described in Subsection 17.2.2. One can show that the solution of this Riemann problem has the form of Figure 17.4(a).

Similarly, the local wave structures at points $E$, $F$, $G$ in the upper $(x, t)$-plane can be determined by utilizing an appropriate signalling problem or Riemann problem, respectively. It can be shown that for times $t > t_G \equiv 2L/c_1 + l/c'$ the wave pattern is a periodic repetition of the pattern between $t_D$ and $t_G$. Thus the solution has periodicity $L/c_1$. The entire wave pattern in the impact problem may be constructed in this way from the many local signalling and Riemann problems.

The history of the particle velocity $v(L, t)$ at the free end $x = L$ versus time $t$ is shown in Figure 18.2. The figure is drawn for $v_0 = 0.11 \text{ km/s}$.

It is interesting to note in passing that, if the impactor and the specimen both have the same modulus $\mu_1 = \mu'$, the history of particle velocity $v(L, t)$ versus time becomes the one shown in Figure 18.3.

### 18.2 With phase change in the specimen

When the impactor velocity $v_0$ is sufficiently large, $v_0 > v_{cr}$, a phase change is nucleated by the impact. In this case, the local wave structure near $x = 0$, $t = 0$ is determined by
a Riemann problem and the local solution has the form shown in Figure 17.3(b); thus two shock waves $OA$, $OB$ and one phase boundary $OD$ in the specimen are generated (see Figure 18.4). The shock wave $OA$ reaches the end $x = L$ at $t_A = L/c$. Then the wave structure near point $A$ is determined by a signalling problem and one finds that a shock wave $AD$ is reflected. This reflecting shock wave intersects the phase boundary $OD$ at point $D$ at time $t_D$. The waves generated by this intersection are determined by solving a Riemann problem. This allows us to calculate the low-strain shock wave, high-strain shock wave and phase boundary emerging from $D$.

On the other hand, the shock wave $OB$ reaches the end at $x = -l$ and then a shock wave $BC$ reflects from point $B$. As in the last section, when this shock wave $BC$ reaches the interface $x = 0$, the impactor separates from the specimen. For $t > t_C$ both ends of the specimen are free surfaces. For $t > t_C$ we need consider only the specimen. In the present case, the wave structure near point $C$ is given by a signalling problem and has the form of Figure 17.1(b) and involves an (unloading, i.e. high-strain phase $\rightarrow$ low-strain phase) phase boundary and a high-strain shock wave. Thus the reverse phase transition is nucleated at time $t_C$ in the specimen.

For $t > t_C$, many propagating discontinuities meet the ends of the specimen or intersect each other inside the specimen. When a shock wave or a phase boundary reaches one of the two ends of the specimen, the corresponding local wave structure is determined by a signalling problem of one of the forms described in the Section 17.1.1; and when two discontinuities meets inside the specimen, the corresponding local wave structure is determined by a Riemann problem of one of the forms described in Section 17.1.2.

The solution generated after each of these wave intersections can be calculated explicitly from the results of Chapter 17. All that remains is to follow the intersection sequentially.
Since there are infinite number of intersections, and since there is no periodic pattern or recursive relation associated with the solution, one cannot write down an analytic expression for all times. Thus we assigned the bookkeeping task of keeping track of all the intersections to a computer. At each wave intersection, one of the preliminary problems of Chapter 17 immediately gives the results.
Figure 18.1. Wave pattern without phase change.
Figure 18.2. Velocity at free-end of specimen versus time for the case without phase change.
Figure 18.3. Velocity at free-end of specimen versus time when both impactor and specimen are composed of the same material.

The equation for the velocity is:

\[ v_a = \frac{2u'c_1}{\mu c_1 + \mu_1 c'} v_0 \]
Figure 18.4. Wave pattern with phase change.
Chapter 19

Results and Discussion

19.1 Non-dimensionalization

In order to describe the results of the impact problem for the material models (15.1) and (15.6), it is useful to non-dimensionalize the variables in the problem. Let $\Gamma$ and $\Sigma$ be the dimensionless strain and stress:

$$
\Gamma = \frac{\gamma}{|\gamma_M|}, \quad \Sigma = \frac{\sigma}{|\sigma_M|}.
$$

We define four dimensionless material parameters:

$$
\alpha = \frac{\mu'}{\mu_1}, \quad \beta = \frac{c'}{c_1}, \quad R_1 = \frac{\sigma_m}{\sigma_M}, \quad K = \frac{\mu_2}{\mu_1},
$$

and the geometric parameter

$$
R_2 = \frac{l}{L}.
$$

This leads to the following related dimensionless material constants:

$$
\Gamma_m = \frac{\gamma_m}{|\gamma_M|} = R_1 - 2, \quad \Gamma_T = \frac{\gamma_T}{|\gamma_M|} = 2(R_1 - 1),
$$

$$
\Sigma_m = \frac{\sigma_m}{|\sigma_M|}, \quad \Sigma_2 = \frac{\sigma_2}{|\sigma_M|}, \quad \Sigma_0 = \frac{\sigma_0}{|\sigma_M|} = \frac{1}{2}(1 + R_1).
$$
By (19.1)-(19.5), the non-dimensional forms of the stress-strain relations (15.1) and (15.6) become respectively

\[
\hat{\Sigma}(\Gamma) = \alpha \Gamma, \quad \Gamma > \frac{-1}{|\gamma_M|},
\]

(19.6)

and

\[
\hat{\Sigma}(\Gamma) = \begin{cases} 
\Gamma, & \Gamma \geq -1, \\
-\Gamma + \Sigma_2, & \Gamma_m < \Gamma < -1, \\
\Gamma - \Gamma_T, & -1/|\gamma_M| < \Gamma \leq \Gamma_m.
\end{cases}
\]

(19.7)

Let dimensionless time and position in the reference configuration be denoted by \(T\) and \(X\):

\[
T = \frac{t}{t_L}, \quad X = \frac{x}{L}, \quad t_L \equiv \frac{L}{c_1}.
\]

(19.8)

Then, as long as the strain does not belong to the unstable phase, the two field equations given in (15.7) have the following non-dimensional forms

\[
\frac{\partial \Gamma}{\partial X} = \frac{\partial V}{\partial T}, \quad \frac{\partial V}{\partial X} = \frac{\partial \Gamma}{\partial T}.
\]

(19.9)

The dimensionless particle velocity \(V\) and the dimensionless propagation speed \(\hat{S}\) of a discontinuity are defined as

\[
V = \frac{v}{c_1|\gamma_M|}, \quad \hat{S} = \frac{\dot{s}}{c_1}.
\]

(19.10)

It then follows from (19.1), (19.10) that the non-dimensional forms of the jump conditions (15.8) become

\[
\hat{\Sigma}(\Gamma^+) - \hat{\Sigma}(\Gamma^-) = -\hat{S} (\hat{\nu} - \hat{\nu}), \quad (\hat{\nu} - \hat{\nu}) = -\hat{S} (\Gamma^+ - \Gamma^-).
\]

(19.11)

Next we define the non-dimensional driving force \(F\) and kinetic response function \(\Phi\):

\[
F = \frac{f}{|\sigma_M||\gamma_M|}, \quad \Phi = \frac{\phi}{|\sigma_M||\gamma_M|}.
\]

(19.12)
Thus the entropy inequality (15.9) has the non-dimensional form

\[ F \dot{S} \geq 0, \]  

and the non-dimensional driving force \( F \) is defined by

\[ F = \dot{F}(\Gamma, \dot{\Gamma}) = \int_{\Gamma}^{\dot{\Gamma}} \Sigma(\Gamma)d\Gamma - \frac{1}{2}(\dot{\Sigma}(\dot{\Gamma}) + \dot{\Sigma}(\Gamma))(\dot{\Gamma} - \dot{\Gamma}). \]  

The particular expression (19.14) for the driving force becomes

\[ F = \frac{1}{2}\Gamma_T(\Gamma + \dot{\Gamma} - \Gamma_m - \Gamma_M). \]  

The non-dimensional form of the kinetic relation (16.4) now becomes

\[ F = \Phi(\dot{S}), \]  

where the function \( \Phi \) corresponding to (16.6) is given by

\[ \Phi(\dot{S}) = \begin{cases} \frac{1}{2}b\Gamma_T^2\dot{S}^2/(1 - \dot{S}^2), & 0 \leq \dot{S} \leq 1, \\ -\frac{1}{2}b\Gamma_T^2\dot{S}^2/(1 - \dot{S}^2), & -1 \leq \dot{S} \leq 0. \end{cases} \]  

Finally, the non-dimensional impact velocity \( V_0 \) and critical impact velocity \( V_{cr} \) are

\[ V_0 = v_0/c_1|\gamma_M|, \quad V_{cr} = -\Sigma_0(1 + \beta)/\alpha. \]  

### 19.2 Results

Given the material parameters \( \alpha, \beta, R_1, b \) and the geometric ratio \( R_2 \), the solution of the impact problem depends only on the value of the dimensionless impact velocity \( V_0 \). For example, let \( \alpha = 0.1364, \beta = 0.409, b = 1.0, R_1 = 0.65 \) and \( R_2 = 0.3 \). In this case, (19.18) leads to the critical impact velocity \( V_{cr} = 3.325 \). Then for \( V_0 \leq 3.325 \) the specimen does not undergo a phase change and the results are given in Section 18.1. On the other hand, if \( V_0 > 3.325 \), the specimen will change phase and the solution is determined by the procedure outlined in Section 18.2.
We now show results corresponding to each of these cases. The wave pattern in the 
\((X, T)\)-plane and the dimensionless particle velocity at the right-hand end of the specimen versa dimensionless time are shown for \(V_0 = 3.0 < V_{cr}\) in the Figures 19.1 and 19.2, and for 
\(V_0 = 5.6 > V_{cr}\) in Figures 19.3 and 19.4. Observe that the velocity profile has the general 
shape of a pulse. The width \(T_w\) and the amplitude \(V_a\) of the particle velocity profile in the 
\((T, V)\)-plane (see Figure 19.4) are of particular interest.

In the case of \(V_0 < V_{cr}\), there are two different types of shock waves shown in Figure 19.1; 
one is generated due to the collision between the two bars, while another is generated due to 
the separation between them. We shall speak of the wave generated due to the collision as 
a loading wave, and the wave generated due to separation as an unloading wave. Thus the 
shock wave \(OABCD\cdots\) is a loading wave, while the shock wave \(PQRS\cdots\) is an unloading 
wave. In this case, our results show that the particle velocity at \(X = 1\) increases each time 
when the loading wave \(OABCD\) meets the free surface \(X = 1\), and decreases each time 
when the unloading wave \(PQRS\) intersects the free surface \(X = 1\). This can be observed in 
Figures 19.1 and 19.2. For \(V_0 < V_{cr}\), the width \(T_w\) and the amplitude \(V_a\) can be found to be 

\[
T_w = T_\ast = \frac{2R_2}{\beta}, \quad V_a = \frac{2\alpha}{\alpha + \beta}V_0. \tag{19.19}
\]

Similarly, in the case of \(V_0 > V_{cr}\) as shown in Figure 19.3, the low-strain shock wave 
\(OABCDE\) and the phase boundary \(OBD\) are loading waves, and the high-strain shock wave 
\(PR\), phase boundary \(PQ\) and the low-strain shock wave \(MN\) are unloading waves. Thus the 
particle velocity at \(X = 1\) increases at respective instants \(T_A, T_C, \) and \(T_E\), and then decreases 
at instant \(T_N\); see Figure 19.4. Our analysis also shows that after the loading shock wave 
\(OABCDE\) bounces back and forth several times between the loading phase boundary \(OBD\) 
and the free surface \(X = 1\), it becomes weaker and weaker in the sense that the strain 
jump across it becomes smaller and smaller. This result can also be found by knowing that
the increase in the particle velocity at \( X = 1 \) becomes smaller and smaller. Based on this observation, one can approximate the amplitude \( V_a \) of the velocity profile. For example, in the case of \( V_0 = 5.6 < V_r \), one only needs to calculate two Riemann problems corresponding to the points \( O \) and \( B \) in the \((X, T)\)-plane, and two signalling problems corresponding to the points \( A \) and \( C \). For the case \( V_0 > V_r \), one can also approximate the width \( T_w \) of the velocity profile as follow: Extending the high-strain shock wave \( PR \) to point \( S \) as shown in Figure 19.3, the time interval between points \( S \) and \( A \) is taken to be the approximate width \( T_w \). Since the shock waves \( PR \) and \( OA \) are parallel to each other, this approximate width \( T_w \) of particle velocity profile is equal to \( T_r = \frac{2R_2}{\beta} \). The approximate results for the case \( V_0 = 5.6 > V_r \) are shown on Figure 19.4. In general, how many Riemann problems and signalling problems need to be considered in order to approximate the amplitude \( V_a \) and width \( T_w \), it depends on the value of initial velocity \( V_0 \). As a good approximation, only two Riemann problems and two signalling problems (as described above for \( V_0 = 5.6 \)) need to be considered for \( (V_0 - V_r)/V_r < 70\% \). For \( (V_0 - V_r)/V_r > 70\% \), more Riemann and signalling problems need to be considered; see Table 19.1. From Table 19.1, observe that for fixed \( R_2 \), the amplitude \( V_a \) increases with the impact velocity \( V_0 \).

19.3 Experimental results

We now turn to some of Grady's experimental results, from six experiments associated with the calcite I-II transition in limestone. The results are taken from (Grady, 1983) and are presented in Table 19.2.

In the first three tests, the projectile impact velocity is approximately \( 0.11 \) \( km/s \). These three tests were performed with sample thickness of about \( 5, 10, 15 \) \( mm \). No phase transitions occur in these specimens and the samples remained in the calcite I-phase. The velocity
profiles corresponding to the right-hand end of the specimen are shown in Figure 19.5(a).

In the last three tests, the impact velocity is about $0.35\ km/s$, and the sample thicknesses are approximately $5, 10, 15\ mm$. The Calcite I–II phase transition occurs in these specimens. This calcite I–II transition is identified as the displacive phase transition initiating at approximately $v_{cr} = 0.13\ km/s$. The velocity profiles are shown in Figure 19.5(b).

### 19.4 Comparison

The results of the Grady's *plate impact experiment* on limestone cannot be rigorously compared with our analysis. This is due to the presence of both shear and tensile effects in Grady's specimen, the rate-dependence of calcite and the fracture of calcite by crushing. According to the experimental results (Singh, 1974) and some analysis based on elasticity theory, the stress-strain curve of calcite can be approximately represented by a piecewise linear model analogous to that shown in Figure 15.1 except that the elastic modulus of calcite II is different from that of calcite I. For all of these reasons, our present model is only an approximation.

In order to find the material parameters involved in our model, we fit the predictions of the model to one set of Grady's experimental observations. For this purpose we consider the experiment in which $v_0 = 0.106\ km/s$, $L = 5.029\ mm$, $l = 3.246\ mm$ for which the height and width of the observed pulse (no phase change) were $v_a = 0.053\ km/s$ and $t_w = 1.25\ \mu s$. The arrival time of this pulse was $0.8\ \mu s$. Then by using (19.19) and the fact that the arrival time must be $L/c_1$, we get

$$c_1 = 6.29\ km/s, \quad \alpha = \frac{\mu'}{\mu_1} = 0.2678, \quad \beta = \frac{c'}{c_1} = 0.8035. \quad (19.20)$$

According to Grady the density of calcite I is $\rho = 2700\ kg/m^3$. From this and (19.20) we find

$$\mu' = 30.19\ GPa, \quad c' = 5.18\ km/s, \quad \mu_1 = 112.74\ GPa, \quad c_1 = 6.29\ km/s \quad (19.21)$$

146
Next, the critical impact velocity measured by Grady was \( v_{cr} = 0.13 \text{ km/s} \). Therefore from (18.1), (19.21) we get

\[
|\sigma_0| = 0.568 \text{ GPa.
}
\]  

(19.22)

The value of the kinetic parameter \( b \) was chosen for best fit of the data; we took \( b=1 \).

The numerical results of our calculation for the impact problem associated with the calcite I-II transition are listed in Table 19.3. The six cases in Table 19.3 correspond to the six sets of experimental results in Table 19.2. In the first three cases no calcite I-II transition occurs, and in the last three cases the calcite I-II transition does occur.

Comparing the results given in Table 19.3 with the results given in Table 19.2, one first notes that the theoretical width \( t_w \) of the velocity profile is close to that from Grady's experiments, the difference between the theoretical and experimental results relative to Grady's results is less than 12 percent. Second, the amplitude \( v_a \) of the velocity profile from the theoretical study is higher than that from Grady's experiments, the difference between them is less than 16 percent.
Figure 19.1. Wave pattern without phase change.
Figure 19.2 Velocity at free-end of specimen involving no phase change versus time.
Figure 19.3. Wave pattern with phase change.
Figure 19.4. Velocity at free-end of specimen involving a phase change versus time.
Figure 19.5. Velocity at free-end of specimen versus time (Grady’s results).
Table 19.1: Comparison between exact and approximate solutions.

<table>
<thead>
<tr>
<th>( R_2 )</th>
<th>( V_0 )</th>
<th>( V_a )</th>
<th>( T_w )</th>
<th>( V_{a\text{app}}(m,n) )</th>
<th>( T_{a\text{app}} )</th>
<th>( E_\alpha(%) )</th>
<th>( E_w(%) )</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>0.3</td>
<td>2.8</td>
<td>1.4</td>
<td>1.4670</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>2.8</td>
<td>1.4</td>
<td>0.9780</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.3</td>
<td>5.6</td>
<td>3.2323</td>
<td>1.5732</td>
<td>3.1164 (2,2)</td>
<td>1.467</td>
<td>3.5</td>
</tr>
<tr>
<td>4</td>
<td>0.2</td>
<td>5.6</td>
<td>3.1164</td>
<td>1.1415</td>
<td>3.1164 (2,2)</td>
<td>0.978</td>
<td>0.0</td>
</tr>
<tr>
<td>5</td>
<td>0.3</td>
<td>7.28</td>
<td>4.0877</td>
<td>1.5542</td>
<td>4.0600 (3,3)</td>
<td>1.467</td>
<td>0.6</td>
</tr>
<tr>
<td>6</td>
<td>0.2</td>
<td>7.28</td>
<td>4.0877</td>
<td>1.1390</td>
<td>4.0600 (3,3)</td>
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<td>7</td>
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<tr>
<td>8</td>
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<td>8.95</td>
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<td>1.1388</td>
<td>4.7995 (3,3)</td>
<td>0.978</td>
<td>2.6</td>
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</tbody>
</table>

- \( E_\alpha = \frac{V_a - V_{a\text{app}}}{V_a} \), \( E_w = \frac{T_a - T_{a\text{app}}}{T_a} \).

- \( m \) is the number of Riemann problems to be considered for approximation.

- \( n \) is the number of signalling problems to be considered for approximation.

Table 19.2: Grady's experimental results.

<table>
<thead>
<tr>
<th>( l(\text{mm}) )</th>
<th>( L(\text{mm}) )</th>
<th>( v_0(\text{km/s}) )</th>
<th>( v_a(\text{km/s}) )</th>
<th>( t_w(\mu s) )</th>
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</thead>
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<tr>
<td>1</td>
<td>3.246</td>
<td>5.029</td>
<td>0.106</td>
<td>0.053</td>
</tr>
<tr>
<td>2</td>
<td>3.239</td>
<td>10.003</td>
<td>0.116</td>
<td>0.055</td>
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<tr>
<td>3</td>
<td>3.205</td>
<td>15.024</td>
<td>0.113</td>
<td>0.055</td>
</tr>
<tr>
<td>4</td>
<td>3.068</td>
<td>4.968</td>
<td>0.216</td>
<td>0.108</td>
</tr>
<tr>
<td>5</td>
<td>3.231</td>
<td>9.985</td>
<td>0.212</td>
<td>0.105</td>
</tr>
<tr>
<td>6</td>
<td>3.239</td>
<td>14.963</td>
<td>0.208</td>
<td>0.105</td>
</tr>
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</table>

Table 19.3: Results of present calculation.

<table>
<thead>
<tr>
<th>( l(\text{mm}) )</th>
<th>( L(\text{mm}) )</th>
<th>( v_0(\text{km/s}) )</th>
<th>( v_a(\text{km/s}) )</th>
<th>( t_w(\mu s) )</th>
</tr>
</thead>
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<td>5.029</td>
<td>0.106</td>
<td>0.0530</td>
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<tr>
<td>2</td>
<td>3.239</td>
<td>10.003</td>
<td>0.116</td>
<td>0.0580</td>
</tr>
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<td>15.024</td>
<td>0.113</td>
<td>0.0565</td>
</tr>
<tr>
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<td>4.968</td>
<td>0.216</td>
<td>0.1248</td>
</tr>
<tr>
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<td>9.985</td>
<td>0.212</td>
<td>0.1189</td>
</tr>
<tr>
<td>6</td>
<td>3.239</td>
<td>14.963</td>
<td>0.208</td>
<td>0.1190</td>
</tr>
</tbody>
</table>
Chapter 20

Concluding Remarks

In contrast to the trilinear material model employed by Abeyaratne and Knowles (1989), we have considered a more general material model in part A of this thesis. This material can sustain not only shock waves and phase boundaries, but wave fans as well. Moreover, shock waves in such a material are dissipative and they travel at a speed that has to be determined. In part A, we considered phase boundaries separating a stable phase from a metastable phase. It was found that an intersonic phase boundary can move only into the phase whose sound speed is smaller than the propagation speed of the phase boundary. The entropy inequality selects the direction in which a propagating shock wave or phase boundary must move.

In order to better understand the behavior of an interface between a stable phase and a metastable phase, in part A, we considered a Riemann problem in which the initial strains belong to a single phase. It was seen that, for our material model and for such a Riemann problem, all phase boundaries are subsonic. We showed that such a Riemann problem requires more than the entropy inequality to render solutions unique; this is in contrast to classical dynamics of an ideal gas in one dimension, where the imposition of the entropy inequality at shock fronts is sufficient to rule out all but one solution. We have further shown that, among the many admissible solutions corresponding to the same initial data
and involving phase transitions, a kinetic relation controlling the rate at which the transition occurs will pick a unique solution. However, under certain conditions, two types of solutions can arise from a given set of initial data, one of which involves a phase change while the other does not. A nucleation criterion for the relevant phase transition is needed to choose between these two types of solutions: when such a criterion mandates the occurrence of a phase change, the kinetic relation controls the evolution of the resulting phase boundaries.

The fact that the mathematics needs and can accommodate these two supplementary mechanisms— a nucleation criterion and a kinetic relation— is consistent with the view taken by material scientists, for whom models of nucleation and of growth of the product phase in the parent phase are an important part of the explanation of the phase transition process.

The results (in part A) associated with permissible wave structures in the upper-half \((x,t)\)-plane for solutions of the Riemann problem can also be applied to the signalling problem. In part B, we studied the behavior of an interface separating a stable phase from an unstable phase. In our analysis we used a piecewise linear stress-strain relation. The nature of this trilinear material model leads to considerable simplification in the analysis. We first considered equilibrium and quasi-static states involving both stable and unstable configurations of an elastic bar. It was shown that there is a one-parameter family of solutions with the location of the phase boundary as the parameter. The lack of uniqueness indicates the need for additional information. Based on our results of part A, it is natural to assume that this missing information is kinetic in nature. Thus we attempted to derive a kinetic relation based on a regularization that included strain-gradient and viscous effects. In this approach, one first regularizes the conventional constitutive law for elastic materials capable of undergoing isothermal phase transitions in such a way that stress \(\sigma\) depends not only on the strain \(\gamma_i\), but also on the strain-rate \(\gamma_t\) and the second spatial gradient \(\gamma_{22}\). A kinetic
relation is inherited by studying travelling waves in this theory and then taking its limit. Surprisingly this procedure did not lead to a kinetic law. Finally we turned to inertial effects and found that the dynamic theory, at least in the setting studied, did not require a kinetic law. Thus the physical effects that were missing from quasi-static theory were inertial, not kinetic. We also noted that by the entropy inequality an interface between the stable and the unstable phases can move only into the unstable phase.

In parts A and B, we studied certain fundamental issues regarding the proper mathematical modeling of a propagating phase boundary. In part C, we turned to a specific initial-boundary-value problem, an impact problem. Our predicted results of the impact problem are qualitatively similar to the results observed by Grady (1983) in plate impact experiments on calcite rocks undergoing the calcite I → calcite II transition. The difference between the results obtained from our one-dimensional dynamic theory and from Grady’s experiments is less than twenty percent.

This thesis has investigated the dynamics of elastic bars characterized by a non-convex energy function or equivalently by a non-monotonic stress-strain relation. Such material models can be utilized, in particular, to model certain features associated with phase transitions in solids. Some suggestions for possible extensions of this investigation are as follows:

First, we recall that all studies in this thesis involved only plane waves (i.e. one-dimensional theory). In many circumstances it would be important to consider the effects of both dilatation and shear. For example, Grady’s results (1983) indicated that both dilatational and shear effects are important in his experiments. If the shear effect is included in the material model, it would be more difficult to find the condition(s) on the constitutive law (including the nucleation criterion and the kinetic relation) such that the corresponding material can sustain deformations involving phase transitions, even for a special case.
in which the dilatational and shear effects are uncoupled. However, better agreement with the experimental observations would be expected. It would be important to reconsider the impact problem studied in part C by including the shear effect as well and compare with the observations again.

Secondly, the kinetic relation and the nucleation criterion stated in the present thesis were purely phenomenological. It would be useful to relate them to processes at the microscopic level and to thereby deduce forms for these relations at the continuum mechanical level. In particular, rate equations based on thermal activation models for the kinetics of phase transformations could be investigated.

Thirdly, we note that the entire discussion of this thesis was carried out in a purely mechanical setting. Since temperature plays an important role in phase transformations, dynamic problems coupled to thermal effects should be considered.

Fourth, strain-rate or strain-gradient effects or both may be important for certain solids. For example, in the experiments carried out by Grady (1983), the calcite was found to be strain-rate dependent. In order to understand the behavior of such solids, it is important to consider material models involving these effects and capable of undergoing phase transitions. A simple model of this type was used in the regularized theory of part B; in this model, the stress depends nonlinearly on the strain but linearly on the strain-rate and strain-gradient. The classical “standard solid model” could be modified so that it can sustain deformations involving phase transitions. For such a modified standard solid, the mathematical treatment of many relevant problems would be interesting and useful.

Fifth, it is well-known that, when a material is suddenly cooled to below its transformation temperature, the resulting state of the material immediately after cooling could be momentarily unstable. The material then decomposes into the stable phases. In the case of
diffusive transformations, this leads to the classical problems associated with spinodal decomposition. The corresponding issues for displacive transformations need to be examined.

Sixth, even in the one-dimensional setting, any initial-boundary-value problem involving a finite (rather than infinite) bar would require numerical solution. The propagating discontinuities will have to be tracked, kinetic relations will have to be applied at phase boundaries but not shock waves, and many challenging issues will arise.

Finally, all of this work should be generalized to the case of multi-dimensions.
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