

ACOUSTICS IN MOVING MEDIA

by

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ABSTRACT

This thesis is concerned with two types of problems involving moving fluids and sound. The first problem is to determine the qualitative conditions necessary for instability due to infinitesimal acoustic disturbances. The particular technique used here is to assume that the disturbances are fixed in space while the fluid flows past them. This viewpoint is applied to the questions of turbulence onset and resonator excitation.

The second problem is to examine some of the quantitative properties of an interface between two media in relative motion. In particular, the sound field due to a source near such an interface is calculated. It is shown that the sound field can be approximated in a closed form by a saddle point integration, and this will be a valid approximation for distances from the source which are large as compared to the wave length of the emitted sound. This situation is generalized to the study of the characteristics of a channel formed by two parallel velocity discontinuity interfaces. Then, it is further generalized to include the possibility of a plate or membrane along the interface separating the two media in relative motion. The stability of the interface is discussed as each of the various cases arises.

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INTRODUCTION

The main problem to be discussed here is the determination of the far field sound due to point and line sources in moving media near various types of velocity discontinuities. The main motivation for this research is the study of jet noise. The sound sources are inside the jet, and it is most practical to make the measurements outside the jet. Thus, it is necessary to know the effect of the jet boundary on the directionality of the source. If this boundary is idealized to a sharp velocity discontinuity, then the far field can be calculated. A comparison of the experimental and theoretical results is found in the conclusion at the end of this thesis.

This problem of velocity discontinuities raises the question of stability, which, properly, should (and will) be taken up first. As an introduction to the velocity discontinuity instability, it is interesting to examine a few well-known examples of instability from a unified physical point of view. The particular examples of turbulence onset and resonator excitation are discussed in Chapter 1, and the remainder of the thesis is devoted to the velocity discontinuities mentioned above.

The turbulence problem has been treated in detail by a number of authors,⁽¹⁾ but K. Schuster⁽²⁾ has recently suggested a different concept of the problem that leads to some interesting physical insight. In

particular, this concept leads to a natural explanation of some recent data⁽³⁾ on resonator excitation by steady flow. A discussion of the validity and possible extension of this concept is presented in the conclusion.

WAVE EQUATION AND INSTABILITIES

I. Derivation of Wave Equation in Moving Media

The wave equation for sound in a nonuniform moving medium has been derived in many places, so it would hardly seem necessary to give another derivation here. However, there are two reasons for reviving old memories at this time. In the first place, the wave equation is the basis for all of the work presented in this thesis. The other reason is that it is interesting to see how some of the terms which cause instability arise.

We start with the Navier - Stokes equation for a compressible fluid, neglecting viscosity (for the time being) and heat conduction, and considering the sound to be isentropic.

$$(\partial \bar{u} / \partial t) + (\bar{u} \cdot \nabla) \bar{u} = -(\nabla p / \rho)$$

(where \bar{u} is the total velocity vector). In addition, we must use the continuity equation

$$(\partial \rho / \partial t) + \nabla \cdot (\rho \bar{u}) = 0.$$

The usual procedure is to eliminate the linear terms in \bar{u} . This gives

$$(\partial^2 \rho / \partial t^2) - \nabla^2 p = \nabla \cdot [\rho (\bar{u} \cdot \nabla) \bar{u} + \bar{u} \nabla \cdot (\rho \bar{u})]. \quad (1.1)$$

The pressure can be expressed in terms of the density (or vice versa) to any desired degree of accuracy, but it is still difficult to disentangle the nonlinear term on the right-hand side of Eq. (1.1). One procedure is to divide the total velocity into the steady flow \bar{V} and the fluctuating, or sound, velocity \bar{v} . Then, the wave equation is linearized in the sound variables.

One example of frequently encountered steady velocity distribution is a two-dimensional flow with the velocity vector along the x-axis and the magnitude of the velocity depending only upon y. This is the case for flow in channels and many similar situations. Since the steady flow is incompressible, ρ and p are constant except for sound fluctuations. Thus, to first order in acoustic variables, $p = c^2 \rho$, and the left-hand side of Eq. (1.1) becomes a scalar wave equation. If $V \ll c$, the right-hand side of Eq. (1.1) can be treated as a perturbation, and the scalar wave solutions can be substituted in it to get

$$\rho_0 V (\partial^2 v_x / \partial x^2) + \rho_0 V (\partial^2 v_y / \partial x \partial y) + \rho_0 (\partial V / \partial y) (\partial v_y / \partial x) , \quad (1.1a)$$

for the perturbing terms. All of the terms in V^2 have been omitted since they are of higher order in this approximation. We return to them later when considering higher-order approximations and homogeneous flow distributions.

The question of stability of steady flows can be considered as the question of growth of small disturbances. With this concept in mind, we can consider the terms of (1.1a) as driving terms for the scalar wave equations. As shown in Eq. (1.1), the scalar wave equation has no damp-

ing. But a phenomenological damping term can always be introduced. Then, the sound wave will grow exponentially (and the flow will be unstable) whenever the driving terms are larger than the damping term. In addition, it is necessary for the driving terms to be in phase with the damping term.

II. Instability due to Inhomogeneous Terms

To clarify this reasoning with an example, we consider ordinary parabolic flow in a two-dimensional channel. In keeping with the above restriction of flow in the x -direction varying in the y -direction, the width of the channel is taken from $y = -(d/2)$ to $y = (d/2)$. Then, the steady flow can be written as

$$V = V_0 (1 - 4y^2/d^2).$$

Considering the fundamental mode of the channel as a wave guide, the sound variables can be written as

$$\begin{aligned}
 p &= A \sin k_y y e^{i(\omega t - k_x x)} & k_x^2 + k_y^2 &= (\omega_0^2/c^2) \\
 v_x &= [A k_x \sin k_y y e^{i(\omega t - k_x x)}] / \rho_0 \omega & \omega &= \omega_0 + \alpha \\
 v_y &= - [A k_y \cos k_y y e^{i(\omega t - k_x x)}] / i \rho_0 \omega & k_y &= \pi/d \\
 p &= (A/c^2) \sin k_y y e^{i(\omega t - k_x x)} & & (1.2)
 \end{aligned}$$

α is a frequency shift (real or imaginary) caused by the damping and

driving (perturbation) terms. Equation (1.1) can be written as [using the terms of (1.1a) for the right-hand side and using the damping term $R(\partial\rho/\partial t)$]

$$(\partial^2\rho/\partial t^2) + R(\partial\rho/\partial t) - c_0^2 \nabla^2 \rho = \rho_0 V(\partial^2 v_x/\partial x^2) + \rho_0 V(\partial^2 v_y/\partial x\partial y) + \rho_0(\partial V/\partial y)(\partial v_y/\partial x) .$$

(If the viscosity had been included in Eq. (1.1), we would have $R = -\gamma \nabla^2$, as is explained later.) With the substitution of all of the appropriate previously defined quantities, this becomes

$$[-(2\omega\alpha/c^2) + (i\omega R/c^2)] \sin k_y y = (k_x \omega V_0/c^2) [1 - (4y^2/d^2)] \sin k_y y - (8k_x k_y/\omega d^2) \cos k_y y . \quad (1.3)$$

[That we are not concerned that the terms of (1.3) do not have the same y dependence is shown later.] It is now apparent why the perturbation terms on the right-hand side of (1.3) must have the same phase as the damping term in order to produce instability. This will cause α to be imaginary, and, if the perturbation terms are larger than the damping term, α will be negative imaginary which gives an exponential increasing in time. Of course, this derivation has been sloppy, and all of the terms in Eq. (1.3) do not have the same dependence on y , so the equation implies that the perturbation excites higher modes. At this point, the most convenient procedure is to make the approximation that both sides have the same y dependence. Once y is eliminated, we can take the usual lumped parameter form for $R = \omega/Q$. Then, we can obtain a crude ap-

proximation from Eq. (1.3):

$$- 2\omega a / c^2 = - (i\omega^2 / c^2 Q) + (V_0 k_x \omega / c^2) . \quad (1.4)$$

The approximation used thus far may seem outrageous, but it is shown later that there is a sufficiently rigorous, but intuitively obscure, method for obtaining a remarkably similar result. At the present point, it must be noted that k_x must be imaginary to produce instability. This means that the mode is below cut off for the guide. If we make the substitutions

$\epsilon = \omega d / \pi c$; $M = V_0 / c$; and $\gamma = \text{kinematic viscosity}$ with $Q = 3/2r\epsilon$ and $r = \pi\gamma / cd$ for damping due to viscosity, then, for the onset of instability, $a = 0$ in Eq. (1.4), which gives

$$M/r = 2\epsilon^2 / 3\sqrt{1 - \epsilon^2} = R_k^* / 2\pi \quad (1.5)$$

(where R_k^* is the critical Reynolds number with a sound wave, of frequency given by ϵ , below cut off). If the channel is considered as a resonator (standing wave across the width), R_k^* can be averaged over that part of the frequency spectrum which has $\epsilon < 1$ (or frequency below cut off) to obtain the critical Reynolds number for the flow

$$R_k = \int_0^1 R_k^* \phi(\epsilon) d\epsilon = (4\pi/3) \sqrt{Q_0} ; \quad (Q_0 = 3/2r) . \quad (1.6)$$

In keeping with the crudeness of this derivation, the frequency distribution has been approximated by

$$\phi(\epsilon) = 0 \quad \text{for} \quad 0 < \epsilon < 1 - 2/Q_0$$

$$\phi(\epsilon) = Q_0 \quad \text{for} \quad 1 - (2/Q_0) < \epsilon < 1 .$$

The above result agrees reasonably well with experiment. The method is similar to that used by K. Schuster⁽²⁾ and the comparison with experimental results is given in his paper. After Eq. (1.4), the present treatment is identical with that of Schuster, but the derivation of Eq. (1.4) is rather novel in its direct approach showing the particular terms in the wave equation which are responsible for the amplification and instability.

III. Solution of Wave Equation for Channels

The derivation of Eq. (1.4) by Schuster is more rigorous than is mine, but it is possible to give a much more rigorous treatment than Schuster did by solving the wave equation for parabolic steady flow. For this treatment, the y dependence of ρ is left unspecified, and the density fluctuation is taken to be of the form

$$\rho = F(y) e^{i(\omega t - k_x x)} \quad . \quad (1.7)$$

Now, the two components of the Navier - Stokes equation can be written as

$$(\partial v_x / \partial t) + V(\partial v_x / \partial x) + v_y(\partial V / \partial y) = - (c^2 / \rho_0) (\partial \rho / \partial x) ,$$

$$(\partial v_y / \partial t) + V(\partial v_y / \partial x) = - (c^2 / \rho_0) (\partial \rho / \partial y) , \quad (1.8a)$$

and the continuity equation can be written as

$$(\partial \rho / \partial t) + V(\partial \rho / \partial x) + \rho_0(\partial v_x / \partial x) + \rho_0(\partial v_y / \partial y) = 0. \quad (1.8b)$$

Since v_x and v_y must have the same form as ρ [given in Eq. (1.7)], they can be eliminated from Eq. (1.8a) and Eq. (1.8b) to give

$$(\partial^2 F / \partial y^2) + [(2k_x (2V / \partial y) (\partial F / \partial y)) / (\omega - k_x V)] + [(\omega - k_x V)^2 (1/c^2) - k_x^2] F = 0 . \quad (1.9a)$$

In order to solve this equation, we must make the approximation $\omega \gg k_x V$. Then, using the explicit $V(y)$, Eq. (1.9a) becomes

$$F'' - (16k_x V_0 / \omega d^2) y F' + [(\omega^2 / c^2) - k_x^2 - (2\omega k_x V_0 / c^2) + (8\omega k_x V_0 y^2 / c^2 d^2)] F = 0 . \quad (1.9b)$$

(where the prime denotes differentiation with respect to y). If the first derivative term is removed by the substitution

$$F = \psi(y) e^{(4k_x V_0 / \omega d^2) y^2} ,$$

then Eq. (1.9b) becomes

$$\psi'' + [(\omega^2 / c^2) - k_x^2 - (2\omega k_x V_0 / c^2) + (8k_x V_0 / \omega d^2) + (8k_x V_0 y^2 / \omega^2 d^2) ((\omega^2 / c^2) + (8k_x V_0 / \omega^2 d^2))] \psi = 0 . \quad (1.10)$$

This can be put in a standard form by the substitution

$$\zeta = \kappa^{1/4} y \quad \kappa = - (32k_x V_0 / d^2) [(\omega^2 / c^2) + (8k_x V_0 / \omega^2 d^2)] ,$$

which gives, in place of Eq. (1.10)

$$(d^2 \psi / d\zeta^2) + [\kappa^{-1/4} ((\omega^2 / c^2) - k_x^2 - (2\omega k_x V_0 / c^2) + (8k_x V_0 / \omega d^2)) - (\zeta^2 / 4)] \psi = 0 . \quad (1.10a)$$

This equation is the same as the Schroedinger equation for the harmonic oscillator, and the solutions ψ are called parabolic cylinder functions or Weber - Hermite functions. The eigen values which satisfy the boundary conditions

$$(\partial\psi/\partial\zeta) = 0 \quad \text{at} \quad \zeta = \pm (\kappa_x^{1/2} d/2)$$

have been approximated by F. C. Auluck⁽⁴⁾ for the lowest few orders in the parameter $\kappa_x V/\omega$. If $\omega = \omega_0 + \alpha + i\omega/Q$, then the eigen value equation is

$$-(2\omega\alpha/c^2) = - (i\omega^2/Qc^2) + (\kappa_x V_0/\omega) [(\omega^2/c^2)(4/3 + 4/\pi^2) - 8/d^2] \quad (1.11)$$

This is seen to be similar to Eq. (1.4), except that the amplifying term will only have the proper sign if

$$\epsilon > 2/(1 + \pi^2/3) \quad .$$

Since only values of ϵ close to 1 were used in the calculation of critical Reynolds number, the instability calculation is still valid, and the intuitive approach leading to Eq. (1.4) is seen to give fairly accurate results.

It should be noted that the treatment of Sections II and III differs radically from the conventional stability analysis, as has been mentioned in the Introduction. Usually, the stability of the propagating modes of the channel is examined. The present discussion considered only the nonpropagating mode stability.

IV. Resonator Excitation

The instability of nonpropagating disturbances leads to the instability of systems like resonators which create stationary disturbances while a fluid flows past them. Such a situation for a closed pipe resonator is shown in Fig. 1. The fluid has a constant velocity in the x direction as shown, and, for resonance, there is a standing wave, with variation along the y -axis, in the resonator. Since the velocity is constant in the moving fluid, the wave equation (including viscosity) can be written quite simply as

$$(\partial^2 \rho / \partial t^2) - \gamma \nabla^2 (\partial \rho / \partial t) - c^2 (\partial^2 \rho / \partial y^2) = c^2 (\partial^2 \rho / \partial x^2) - 2V (\partial^2 \rho / \partial x \partial t) - V^2 (\partial^2 \rho / \partial x^2) - \gamma V \nabla^2 (\partial \rho / \partial x) . \quad (1.12)$$

As shown in Fig. 1, the moving fluid moves a short distance into the mouth of the pipe; beyond that, there is no x variation in the pipe for the fundamental mode, so all terms on the right-hand side of Eq. (1.12) are zero. Furthermore, the left-hand side of Eq. (1.12) can be written in lumped parameter form as a resonator equation, and set equal to zero for the lowest order solution,

$$(\partial^2 \rho / \partial t^2) + (\omega_0 / Q) (\partial \rho / \partial t) + \omega_0^2 \rho = 0 \quad (1.13)$$

(where $\omega_0 = c\pi/2d$ for the pipe fundamental). The value of Q is found from the viscous and radiation damping, but left unspecified for the moment. The solutions satisfying Eq. (1.13) are of the form

$$\rho_1 = (\rho_0 A_1 / c) \sin(\pi y / 2d) (\sin \omega_0 t + \frac{1}{Q} \cos \omega_0 t) . \quad (1.14)$$

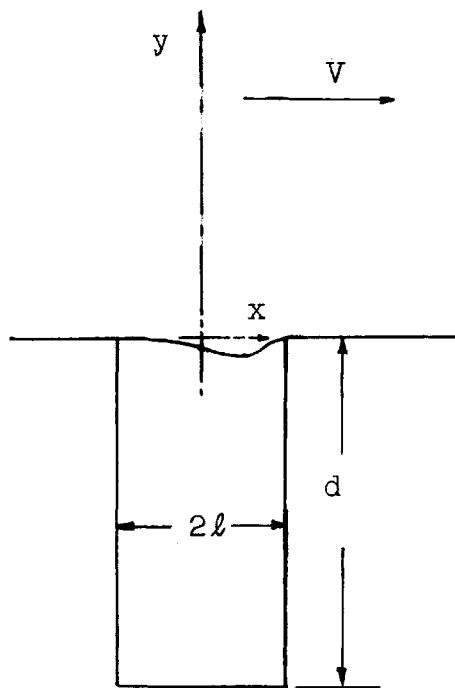


Figure 1. FLOW PAST A CLOSED PIPE RESONATOR

We next consider the terms on the right-hand side of Eq. (1.12) in the moving fluid (in particular, that part of the fluid which extends slightly into the resonator, as shown in Fig. 1). These terms can cause instability if they have the same time phase as the damping term on the left of Eq. (1.12). We find their phase just inside the resonator mouth by using the ρ_1 time dependence from Eq. (1.14). The first term on the right-hand side of (1.12) is not of interest, since it is not multiplied by the velocity, and so does not contribute to the excitation. The next term $V(\partial^2 \rho / \partial x \partial t)$ has the required time phase. The term $V^2(\partial^2 \rho / \partial x^2)$ has the reactive time phase for the large term of ρ_1 . (This merely produces the well-known frequency shift.) The small term of ρ_1 gives the damping phase for this term. The last term is a viscosity term which has the reactive time phase for the large term of ρ_1 , and the damping time dependence for the small term. If the viscous effects are small, this term can be neglected. The terms of interest are now the second and the small part of the third term on the right-hand side of Eq. (1.12). To put these terms in a lumped parameter form such as Eq. (1.13) would require a knowledge of the exact wave field over the interaction volume. Therefore, it is most convenient to leave the parameters of these terms unspecified, except for sign. A look at Fig. 1 reveals that the interaction volume will be greatest on the $x > 0$ side of the pipe (where the moving fluid comes the furthest into the pipe). At this side, $(\partial \rho / \partial x) \simeq -(\rho / \ell)$, since the field is expected to be a maximum at the center and fall off for large x . Thus, Eq. (1.12) becomes (considering only terms with damping time phase)

$$(\omega_0 / Q) \simeq (2Vg / \ell) - (V^2 h / \ell^2 Q \omega_0)$$

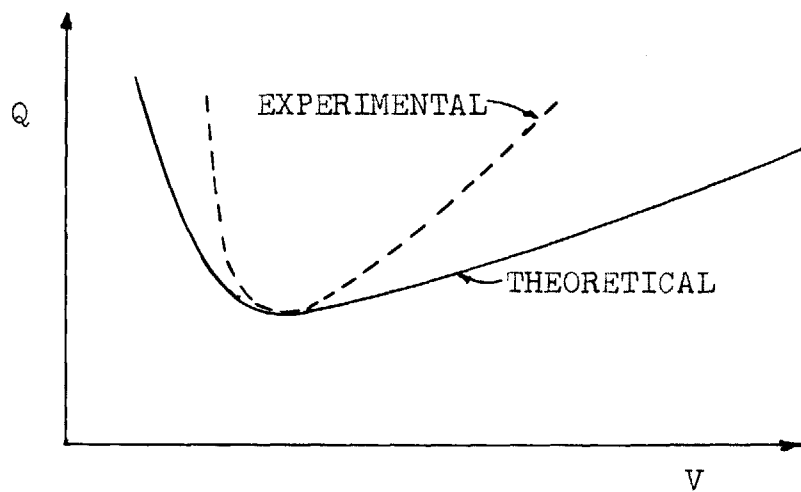


Figure 2. EXPERIMENTAL AND THEORETICAL CURVES OF NEUTRAL STABILITY FOR A CLOSED PIPE RESONATOR

(The minimum is at $\frac{V}{l\omega_0} = 1.$)

(where g and h are unspecified positive parameters).

Just as in the previous example, it is the disturbance which is exponentially damped in space which causes the unstable, exponentially increasing disturbance in time. In Fig. 2, Q is plotted as a function of V , with g and h adjusted to make the minimum of the theoretical and experimental curves coincide. The experimental data are due to L. W. Dean.⁽³⁾

GENERAL DISCUSSION OF A TANGENTIAL VELOCITY DISCONTINUITY

I. Boundary Conditions for the Reflection and Refraction of Sound

The remainder of this thesis is devoted to fluids with constant velocity in each of two regions, but with a different velocity in each region. In addition, the boundary between the two regions will be taken parallel to the flow direction, which is taken to be the same in the two regions. The problem of reflection and refraction of a plane sound wave from such an interface has been considered recently in papers by Miles⁽⁵⁾ and by Ribner,⁽⁶⁾ but it is of interest to repeat the derivation here in a slightly different manner.

The geometry is pictured in Fig. 3. Medium 1 is at rest, and medium 2 is identical with medium 1 in all properties (for the sake of simplicity) but has velocity V in the positive y direction as shown. In medium 1, the solution of the wave equation for the acoustic velocity potential can be written quite simply. For the wave incident at the angle as shown,

$$\phi_0 = e^{i\omega t -iky - \gamma_1 x} = e^{i\omega t - (i\omega/c)(y \cos \theta_1 + x \sin \theta_1)} \quad , \quad (2.1)$$

$$\text{(with } \gamma_1 = \sqrt{k^2 - (\omega^2/c^2)} \text{ , } \quad k = (\omega/c)\cos \theta_1 \text{ .}$$

By the same reasoning, the reflected wave can be written as

$$\phi_1 = A e^{i\omega t -iky + \gamma_1 x} = A e^{i\omega t - (i\omega/c)y \cos \theta_1 + (i\omega/c)x \sin \theta_1} \quad . \quad (2.2)$$

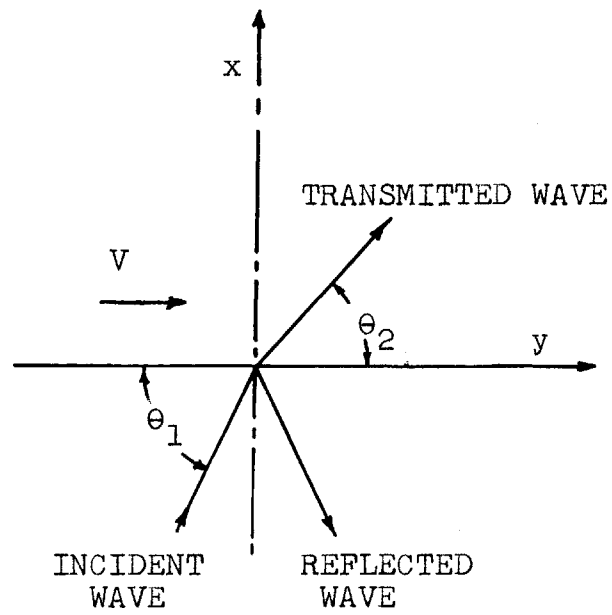


Figure 3. A VELOCITY DISCONTINUITY WITH INCIDENT, REFLECTED, AND TRANSMITTED SOUND WAVES

A must be determined from the boundary conditions at $x = 0$.

Thus far, the coordinate system has been considered at rest relative to medium 1. If the coordinate system is transformed to motion with velocity V , then, the potential in medium 2 can be written just as simply as Eq. (2.1). Then, the transmitted wave with frequency ω_2 can be written as

$$\phi_2 = B e^{i\omega_2 t - (i\omega_2/c)y_2 \cos \theta_2 - (i\omega_2/c)x_2 \sin \theta_2} \quad (2.3)$$

(with θ_2 , x_2 , y_2 being coordinates at rest with respect to medium 2, $x_2 = x$, $y_2 = y - Vt$). Since all values of y and t must give the same phase for ϕ_1 and ϕ_2 at the boundary $x = 0$,

$$\omega = \omega_2 (1 + M \cos \theta_2) \quad (\text{coefficient of } t) \quad ,$$

$$\omega \cos \theta = \omega_2 \cos \theta_2 \quad (\text{coefficient of } y) \quad (2.4)$$

(where $M = V/c$). These equations were first given by Rayleigh⁽⁷⁾ to determine the angle of refraction as a function of incidence angle. Using the relations of Eq. (2.4), it is evident that ϕ_2 may be written as

$$\phi_2 = B e^{i\omega t - iky + i\gamma_2 x} \quad (2.5)$$

$$[\gamma_2 = \sqrt{k^2 - (\omega_2^2/c^2)} \quad , \quad \omega_2 = \omega - kV] \quad .$$

This is the form most convenient for the application of the boundary conditions.

The pressure in a stationary medium is given by

$$p = \rho (\partial\phi / \partial t),$$

but in a moving medium

$$p = \rho (\partial\phi / \partial t) + \rho V (\partial\phi / \partial y),$$

from the linearized Bernoulli equation. Thus, the continuity of pressure boundary condition is

$$\omega(\phi_0 + \phi_1) = \omega_2 \phi_2 \quad . \quad (2.6)$$

If we call the boundary displacement η in the coordinate system at rest, then the velocity of the boundary in a coordinate system fixed to medium 2 is $(\partial\eta / \partial t) + V(\partial\eta / \partial y)$. Now, η must have the same exponent as ϕ_0 , ϕ_1 , ϕ_2 for $x = 0$, and the x component of velocity in medium 2 is given by $\partial\phi_2 / \partial x$, so the definition of η gives

$$i(\omega - kV) \eta = -\gamma_2 \phi_2$$

$$i\omega \eta = \gamma_1 (\phi_1 - \phi_0) \quad .$$

We can eliminate η from these equations to obtain the second boundary equation

$$(\gamma_1 / \omega) (\phi_0 - \phi_1) = (\gamma_2 / \omega_2) \phi_2 \quad . \quad (2.7)$$

Now, Eqs. (2.6) and (2.7) may be solved simultaneously to give

$$A = (\gamma_1 \omega_2^2 - \gamma_2 \omega^2) / (\gamma_1 \omega_2^2 + \gamma_2 \omega^2) \quad \text{and}$$

$$B = (2\omega \omega_2 \gamma_1) / (\gamma_1 \omega_2^2 + \gamma_2 \omega^2) \quad . \quad (2.8)$$

II. Helmholz Instability

If the denominator in Eq. (2.8) becomes zero, we have the well-known Helmholtz instability. This denominator can be factored to $(k^2 - \gamma_1\gamma_2)(\gamma_1 + \gamma_2)$, so that the Helmholtz instability is given by

$$k^2 = \gamma_1\gamma_2 \quad .$$

The roots of this equation are

$$(kc)/\omega = [(M/2) \pm i \sqrt{\sqrt{1+M^2} - 1 - (M^2/4)}] / [\sqrt{1+M^2} - 1] \quad . \quad (2.9)$$

These will be complex for $M < 2\sqrt{2}$, which means that the interface is unstable. This instability has been discussed by a number of writers, and the most recent and most thorough discussion was given by J. Miles.⁽⁸⁾ He showed that the complex frequency given by Eq. (2.9), with the imaginary part both positive and negative, causes an initial disturbance of the interface to grow exponentially in time. This leads one to question the meaning of the reflection and transmission coefficients calculated above. However, we can get physically sensible results by ignoring the poles entirely. This is shown in the next section. The effect upon the Helmholtz instability, of a plate or membrane separating medium 1 and medium 2, is discussed in Section II of Chapter 6.

III. Transmission through Many Layers

It is interesting to use Eq. (2.8) in an experimentally known situation, and to check the results by an analytic calculation. One possi-

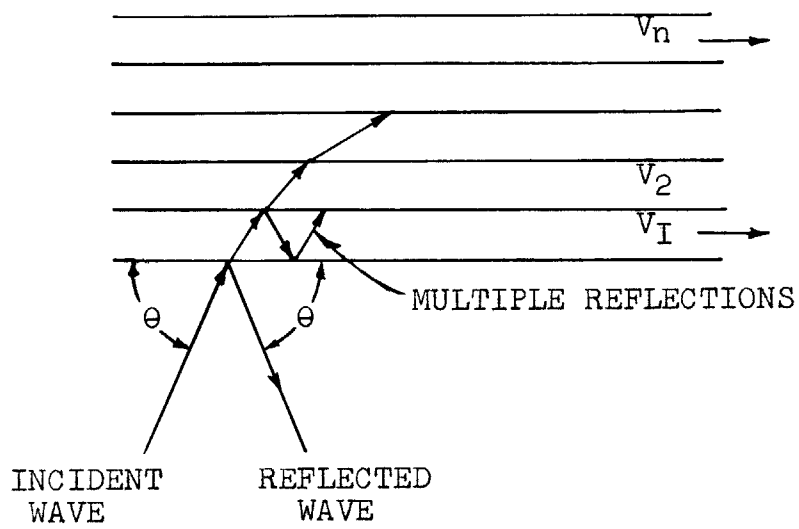


Figure 4. FLOW WITH CONSTANT GRADIENT BROKEN INTO LAYERS OF CONSTANT VELOCITY

bility is shown in Fig. 4. For $x > 0$, the fluid has a velocity in the positive y direction given by $M = bx$ (with b constant). If we have nearly normal incidence ($\cos \theta \ll 1$), and, if $M \ll 1$ for the region under consideration, we can take a plane wave solution of the form

$$p = e^{i\omega t - (i\omega \cos \theta / c)F(x)}, \quad (2.10)$$

and $\cos \theta$ will be approximately constant. The wave equation becomes (neglecting terms of order $M^2 \cos^2 \theta$)

$$(d^2 F/dx^2) + 2b \cos \theta (dF/dx) + (\omega^2/c^2) (\sin^2 \theta - 2bx \cos \theta) F = 0. \quad (2.11)$$

The exact solution of Eq. (2.11) is well-known as

$$F(x) = Nu^{1/2} Z_{1/3} \left(\frac{2}{3} u^{3/2} \right) e^{-bx \cos \theta}$$

[where N is a normalizing factor, $Z_{1/3}$ is an appropriate Bessel or Hankel function of $1/3$ order, and where

$$u = (\sin^2 \theta - 2bx \cos \theta) / [(2bc \cos \theta) / \omega]^{2/3} \quad .$$

Now, if the approximation $\cos \theta \ll 1$ is made again, the asymptotic form of the Hankel function may be used to obtain

$$H_{1/3}^{(2)} \left[\left(\frac{2}{3} \right) u^{3/2} \right] \longrightarrow \left[A(\theta) / u^{3/4} \right] e^{-i\omega x / c} \quad .$$

If the pressure is given at $x = 0$ and a specified value of θ , then the normalizing factors are determined, and Eq. (2.10) becomes

$$p(x, \theta) / p(0, \theta) \simeq \left[1 - (M \cos \theta / 2) \right] e^{i\omega t - (i\omega y \cos \theta / c) - (i\omega x / c)} \quad (2.12)$$

Now, Eq. (2.8) can be applied to this same problem. First, the continuous flow distribution must be broken into layers of constant velocity, as shown in Fig. 4. Then the sound field in layer n is found by taking the product of all of the transmission coefficients for the interfaces between the layers preceding n . With the use of B from Eq. (2.8), the pressure transmission coefficient at an interface can be written, to first approximation in $M \cos \theta$, as

$$p_1/p_0 \simeq 1 - (M_1 \cos \theta/2) . \quad (2.13)$$

Now, we cannot match solutions at the interfaces in Fig. 4 in the manner used for the interface of Fig. 3 to derive Eq. (2.8). This is because there must be waves travelling in both x directions for each layer. In other words, the amplitude in the second layer is not found simply from the product of the transmission coefficients of the first and second layers, but must also be found from the transmission coefficients after each of the multiple reflections shown in Fig. 4. However, since $M \cos \theta \ll 1$ throughout the region under consideration, it is seen that the reflection coefficient at the n^{th} layer is of the order of magnitude $M_n \cos \theta$. Since any fraction of the wave which is reflected once must be reflected a second time before it can contribute to the transmitted fraction, the contribution, to the transmitted wave, due to multiple reflections will be of order of magnitude $(M \cos \theta)^2$. Since the present approximation is only to the order $M \cos \theta$, these multiple reflections can be neglected entirely, and the amplitude can be calculated from the product of transmission coefficients, as was suggested to begin with. To the same approximation as

Eq. (2.13), the transmission coefficient from the $(n - 1)$ to the n layer can be written as

$$(p_n/p_{n-1}) \simeq 1 - [(M_n - M_{n-1}) \cos \theta/2],$$

and the product of the transmission coefficients up to the n^{th} layer can be approximated by

$$(p_n/p_0) \simeq 1 - (M_n \cos \theta/2) . \quad (2.14)$$

This is seen to agree with the amplitude factor in Eq. (2.12), so the two different methods give the same result.

In physical situations, viscosity makes a flow like that shown in Fig. 4 stable, for low enough Reynolds numbers. Thus, the results of Section I have physical meaning, in spite of the Helmholtz instability. In most of the following, we neglect this instability entirely.

LINE SOURCE

I. Formal Integral Solutions

The geometrical situation for a line source located near a velocity discontinuity is shown in Fig. 5. The coordinate system is fixed with respect to the source and the two media have Mach numbers $M_1 = (V_1/c)$, $M_2 = (V_2/c)$ with respect to this fixed system. The plane wave expansion of a line source in a medium with velocity V_1 relative to a coordinate system fixed in that source is

$$\phi_0 = e^{i\omega t} \int_{-\infty}^{+\infty} [e^{-iky - \gamma_1|x+h|} / \gamma_1] dk \quad (3.1)$$

$$[\text{where } \gamma_1 = \sqrt{k^2(1 - M_1^2) - (\omega^2/c^2) + (2kM_1\omega/c)}].$$

The total field will be made up of ϕ_0 plus the reflected wave in medium 1. In medium 2, the field will be simply the transmitted wave. The reflected wave can be written as

$$\phi_1 = e^{i\omega t} \int_{-\infty}^{+\infty} A(k) e^{-iky + \gamma_1 x} dk. \quad (3.2)$$

And the transmitted wave is

$$\phi_2 = e^{i\omega t} \int_{-\infty}^{+\infty} B(k) e^{-iky - \gamma_2 x} dk \quad (3.3)$$

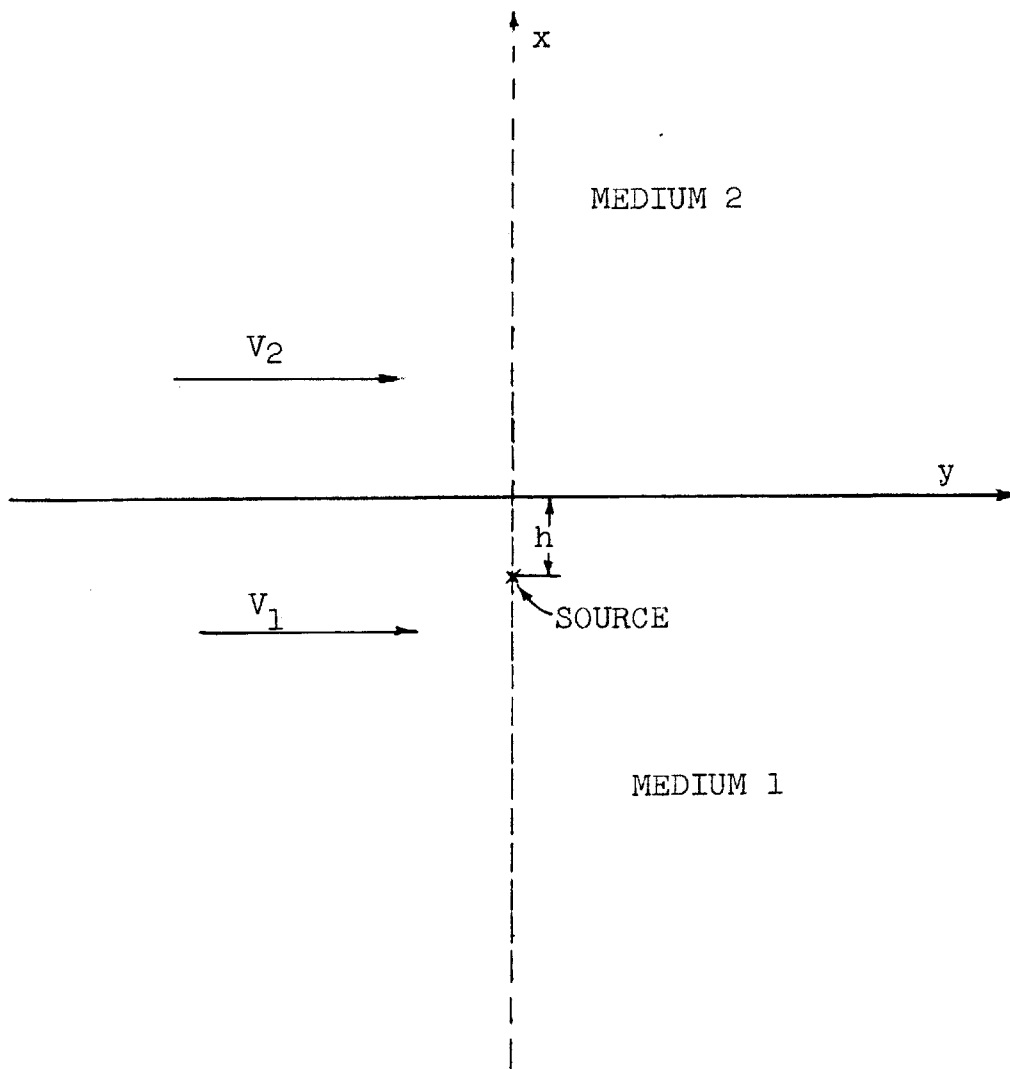


Figure 5. SOUND SOURCE NEAR A VELOCITY DISCONTINUITY

[with $\gamma_2 = \sqrt{k^2(1 - M_2^2) - (\omega^2/c^2) + (2kM_2\omega/c)}$].

If we make the simplification of notation

$$\omega_1 = \omega - kM_1c \qquad \omega_2 = \omega - kM_2c$$

then, the continuity of pressure boundary condition becomes

$$(\omega_1 e^{-\gamma_1 h} / \gamma_1) + \omega_1 A = \omega_2 B,$$

and the continuity of the velocity normal to the interface gives

$$-(e^{-\gamma_1 h} / \omega_1) + (\gamma_1 A / \omega_1) = -(\gamma_2 B / \omega_2).$$

Solving the boundary equations for A and B gives

$$A = (\gamma_1 \omega_2^2 - \gamma_2 \omega_1^2) / [\gamma_1 (\gamma_1 \omega_2^2 + \gamma_2 \omega_1^2)] e^{-\gamma_1 h} \quad ,$$

$$B = [2\omega_1 \omega_2 / (\gamma_1 \omega_2^2 + \gamma_2 \omega_1^2)] e^{-\gamma_1 h} \quad .$$

This result is seen to be similar to Eq. (2.8), and the derivation leading to it is also similar to the derivation of Eq. (2.8).

II. Evaluation of Integral for Transmitted Wave

The next step is to evaluate the integrals of (3.2) and (3.3).

The integral for ϕ_2 can be written as

$$\int_{-\infty}^{+\infty} g(k) e^{-f(k)} dk \qquad f(k) = iky + \gamma_2 x \quad . \qquad (3.4)$$

Since this integral is incapable of an exact analytic evaluation, the natural thing is to try the asymptotic approximation for the far field.

For the saddle point method,

$$f(k) = f(k_0) + (k - k_0)f'(k_0) + [(k - k_0)^2/2]f''(k_0) \quad .$$

And we solve for k_0 given by $f'(k_0) = 0$. If we substitute $x = r \sin \theta$,
 $y = r \cos \theta$,

$$k_0 = [\omega/c(1 - M_2^2)] [-M_2 \pm (\cos \theta / \sqrt{1 - M_2^2 \sin^2 \theta})] \quad (3.5)$$

(where the positive square root is always understood). It is noted that k_0 has a branch point at $\theta = \pi/2$, and the proper branch is chosen as the positive sign to satisfy the subsonic radiation conditions for $y \gg c/\omega$. We can then calculate

$$f''(k_0) = -i(c/\omega)[(1 - M_2^2 \sin^2 \theta)^{3/2} / \sin^2 \theta] r \quad .$$

The path of integration must be chosen so that the phase of $(k - k_0)^2$ is i . The two possible paths are shown in Fig. 6. The appropriate path goes in the positive real k direction, since the integral is over real values of k from $-\infty$ to $+\infty$. We must be careful about crossing branch points of the integrand. (Poles are neglected entirely in this analysis, as was suggested in Section II, Chapter 2.) If we assume an infinitesimal amount of damping, the branch points are as shown in Fig. 7. When the saddle point is to the left of $-(\omega/c)$, we will have to cross the $-(\omega/c)$ branch point in order to distort the real axis path into the saddle path. Thus, the

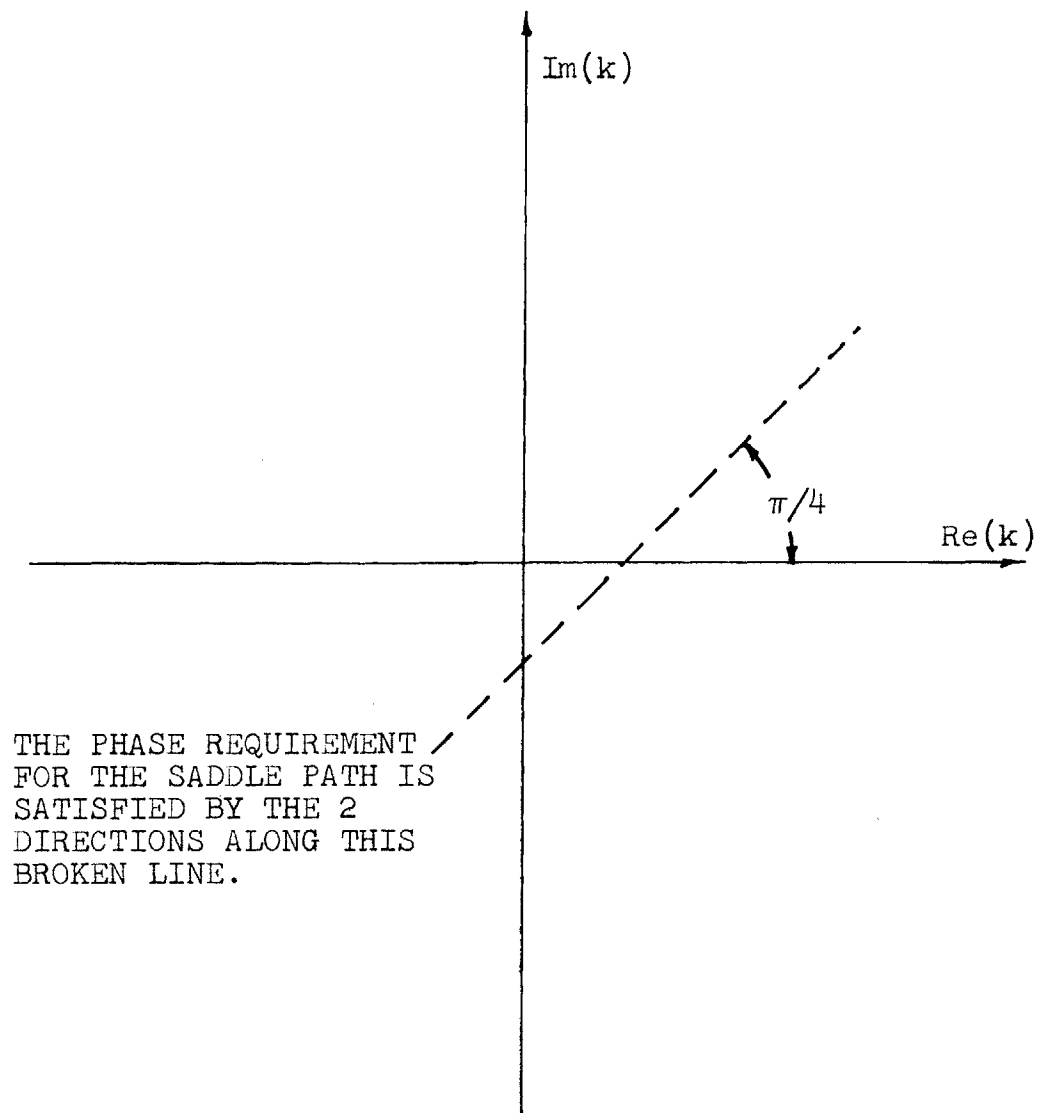


Figure 6. THE PATH OF INTEGRATION IN THE k PLANE

(The path of integration is originally along the real axis, but in the neighborhood of the saddle point it is distorted so that it crosses the real axis at the saddle point, making an angle of 45° with the real axis.)

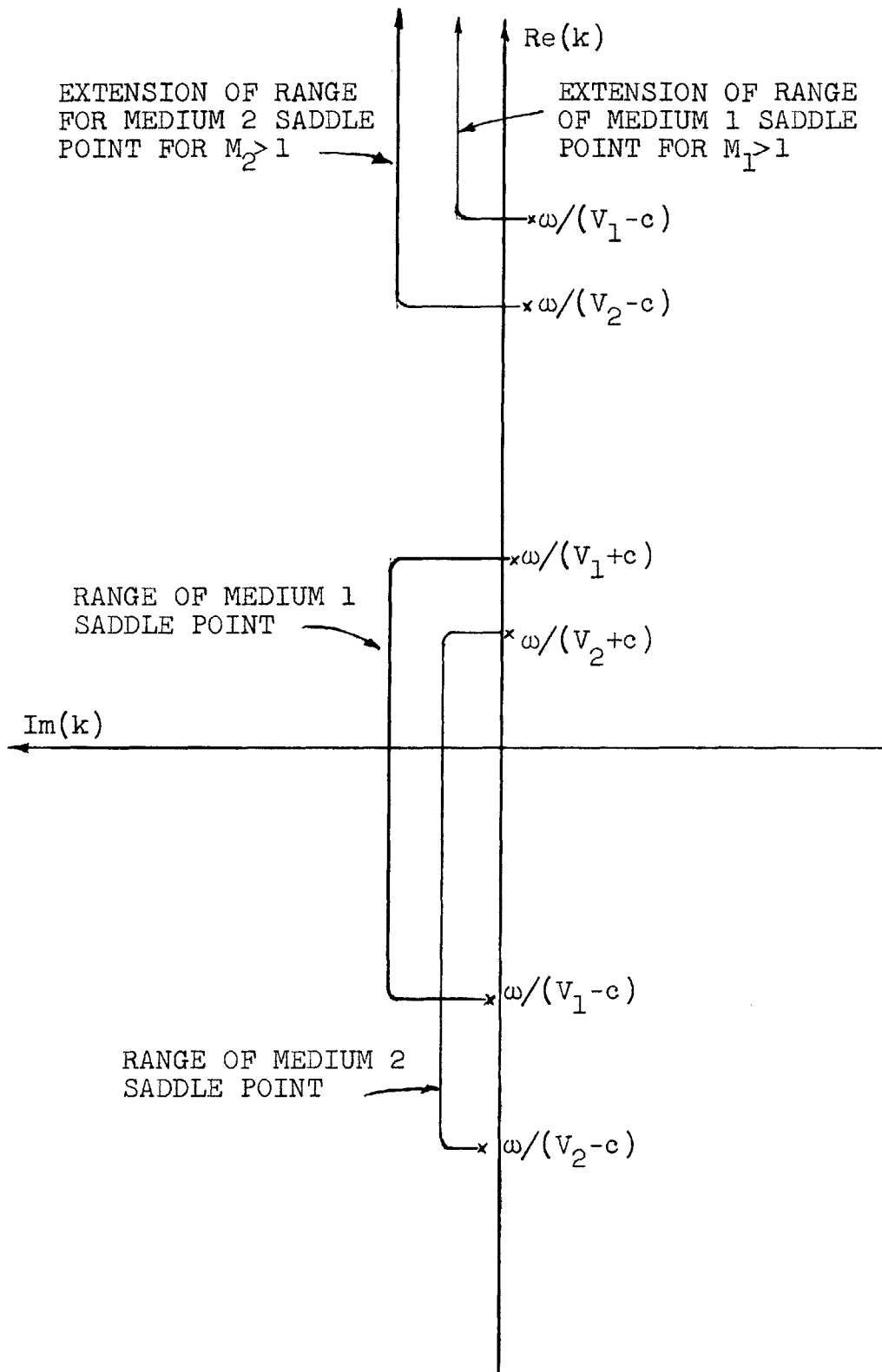


Figure 7. BRANCH POINTS OF THE INTEGRAND FOR ϕ_1 OR ϕ_2
 (FOR Θ INSIDE THE MACH ANGLE)

sound field will have an extra term coming from an integration over a cut surrounding the $-(\omega_1/c)$ branch point. (This is explained in detail in Section V of this chapter.) This will be found to correspond to the "refraction arrival" (also explained in Section V), and will have a straight wave front with a fall off of $r^{-3/2}$ so that it makes no contribution to the far field. The integral of Eq. (3.4) can now be approximated by the integration over the saddle point to give

$$\phi_2 = \sqrt{[2\pi/f''(k_0)]} g(k_0) e^{-f(k_0) + i\omega t} \quad (3.6)$$

$$\text{with } g(k) = [2\omega_1\omega_2 e^{-\gamma_1 h} / (\gamma_1\omega_2^2 + \gamma_2\omega_1^2)]$$

$$c\gamma_2/\omega = i \sin \theta / \sqrt{1 - M_2^2 \sin^2 \theta}$$

$$c\gamma_1/\omega = \left(\left[\frac{(1 - M_1^2)}{(1 - M_2^2)^2} \right] \left[M_2^2 - (2M_2 \cos \theta / \sqrt{1 - M_2^2 \sin^2 \theta}) + \right. \right. \\ \left. \left. (\cos \theta / [1 - M_2^2 \sin^2 \theta]) \right] - 1 - (2M_1 M_2 / [1 - M_2^2]) + [2M_1 \cos \theta / \right. \\ \left. (1 - M_2^2) \sqrt{1 - M_2^2 \sin^2 \theta} \right] \right)^{1/2} .$$

Note: In most of the remaining work, we use the terminology

$$\Delta = (1 - M_2^2 \sin^2 \theta)^{-1/2} .$$

It should be noted that γ_1 is real for k_0 to the left of $-(\omega_1/c)$ in Fig. 7. This means that all waves in this region are reduced by a factor $e^{-\gamma_1 h}$. The physical reason for this is that for the "least time" path $k_0 > \omega_1/c$, so the wave is attenuated in medium 1. The expression for ϕ_2

is rather complicated, but two cases permit straightforward interpretation. First, consider $M_2 = 0$.

$$\phi_2 = \left[2\sqrt{2\pi} e^{i\omega t + (i\pi/4) - h\sqrt{\cos^2\theta - (1 - M_1 \cos\theta)^2} - (i\omega/c)y \cos\theta - (i\omega/c)x \sin\theta} \right] / \left[(1 - M_1 \cos\theta) \left(i \sin\theta + \sqrt{\cos^2\theta - (1 - M_1 \cos\theta)^2} / (1 - M_1 \cos\theta)^2 \right) \sqrt{r\omega/c} \right]. \quad (3.7)$$

This equation is plotted in Figs. 8 and 9 for constant r . The factor multiplying h in the exponential is real for $\cos\theta > \{1/(1 + M_1)\}$ which gives the "zone of silence" at $\cos\theta = \{1/(1 + M_1)\}$ and ϕ_2 has a maximum. However, the saddle point integration is a poor approximation in the neighborhood of this point, since it is here that $v_1(k_0) = 0$ (branch point). This expression is valid for all values of M_1 . Equation (3.7) is the leading term of the far field approximation, so the $r^{-1/2}$ dependence is as expected. The next order terms have an $r^{-3/2}$ dependence. In all of these cases, we have assumed $r \gg h, c/\omega$.

The other straightforward situation is for $M_1 = 0$. In this case,

$$\phi_2 = \left[2\sqrt{2\pi}\Delta^3 (1 - M\Delta \sin\theta) e^{i\omega t - (i\pi/4) - h\sqrt{(\Delta \cos\theta - M)^2/(1 - M^2) - 1 - [(i\omega/(1 - M^2))(\Delta \cos\theta - M) - i\Delta \sin\theta]} \right] / \left((1 - M^2) \left(\Delta \sin\theta + \left[(1 - M\Delta \cos\theta) / (1 - M^2) \right]^2 \sqrt{1 - \left[(M - \Delta \cos\theta) / (1 - M^2) \right]^2} \right) \sqrt{r\omega/c} \right). \quad (3.8)$$

(Where no ambiguity is likely, the subscripts denoting the medium will be

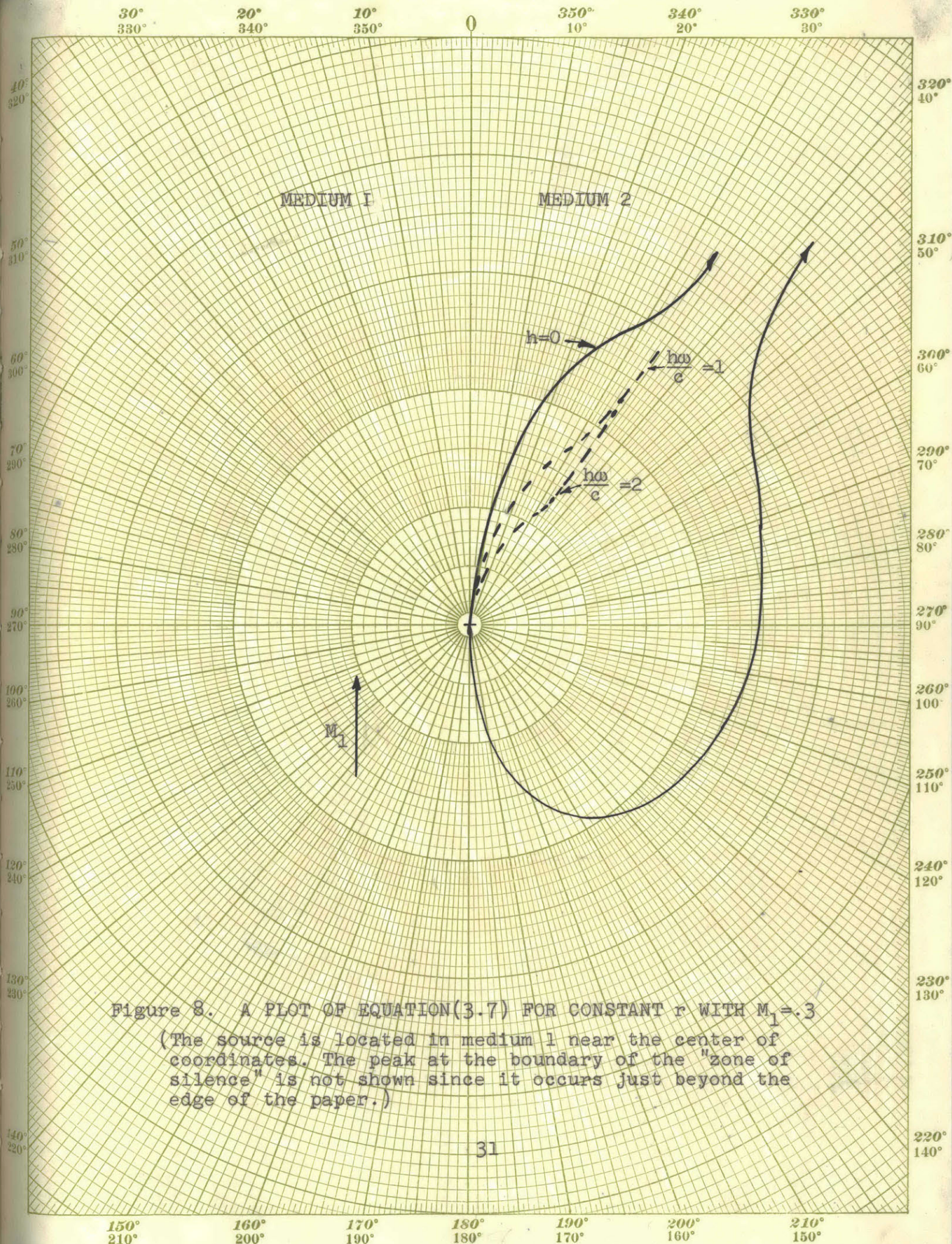


Figure 8. A PLOT OF EQUATION(3.7) FOR CONSTANT r WITH $M_1 = .3$
 (The source is located in medium 1 near the center of coordinates. The peak at the boundary of the "zone of silence" is not shown since it occurs just beyond the edge of the paper.)

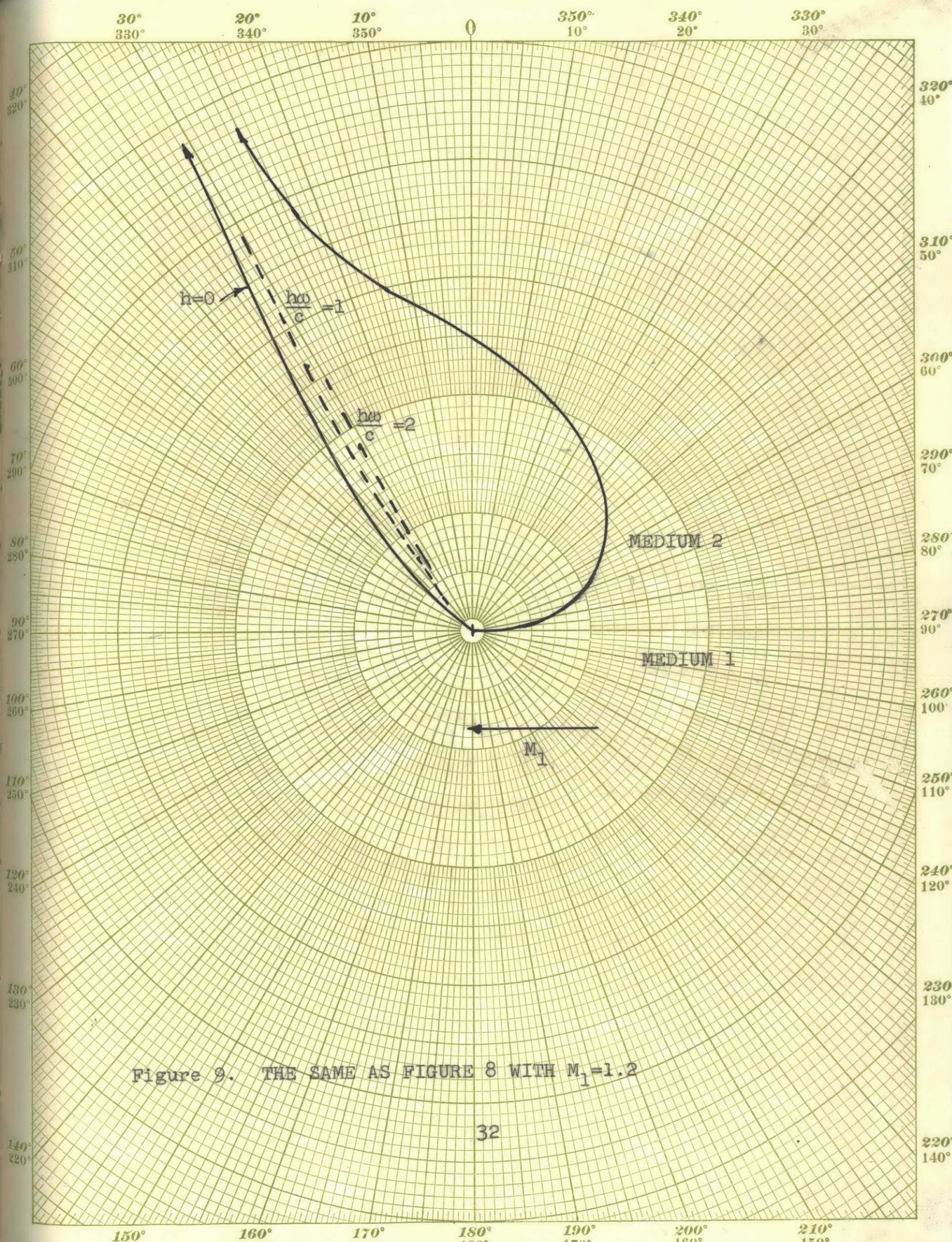


Figure 9. THE SAME AS FIGURE 8 WITH $M_1=1.2$

omitted.) This rather forbidding expression can be simplified by application of a transformation to retarded coordinates. This is done by taking a coordinate system fixed in medium 2 so that medium 1 is moving with velocity $-M_2$ in the new coordinate system, and fixing the x-axis at the point where the source was when the sound was emitted which is received at time t . Denoting the coordinates in this system by θ_R and R , we find, by means of well-known transformations, that

$$r = R\sqrt{1 + M_2^2 + 2M_2\cos\theta_R}$$

$$\sin\theta = (\sin\theta_R)/\sqrt{1 + M_2^2 + 2M_2\cos\theta_R}$$

$$\cos\theta = (\cos\theta_R + M_2)/\sqrt{1 + M_2^2 + 2M_2\cos\theta_R} \quad .$$

With these substitutions, Eq. (3.8) becomes

$$\phi_2 = \left[2\sqrt{2\pi} e^{i\omega[t - (R/c)] - i\omega/c \sqrt{\cos^2\theta_R - (1 + M\cos\theta_R)^2}} / (1 + M\cos\theta_R) - (i\pi/4) \right] / \left[(\omega R/c)^{1/2} (1 + M\cos\theta_R)^{3/2} \left(\sin\theta_R + \sqrt{(1 + M\cos\theta_R)^2 - \cos^2\theta_R} / (1 + M\cos\theta_R)^2 \right) \right] \quad . \quad (3.9)$$

Equation (3.9) is plotted in Figs. 10 - 12. Eqs. (3.8) and (3.9) are only valid for $M < 1$ thus far, but the slight modification shown in Section IV does not affect the amplitude factor. Again, we see that there is a "zone of silence" for Eq. (3.7).

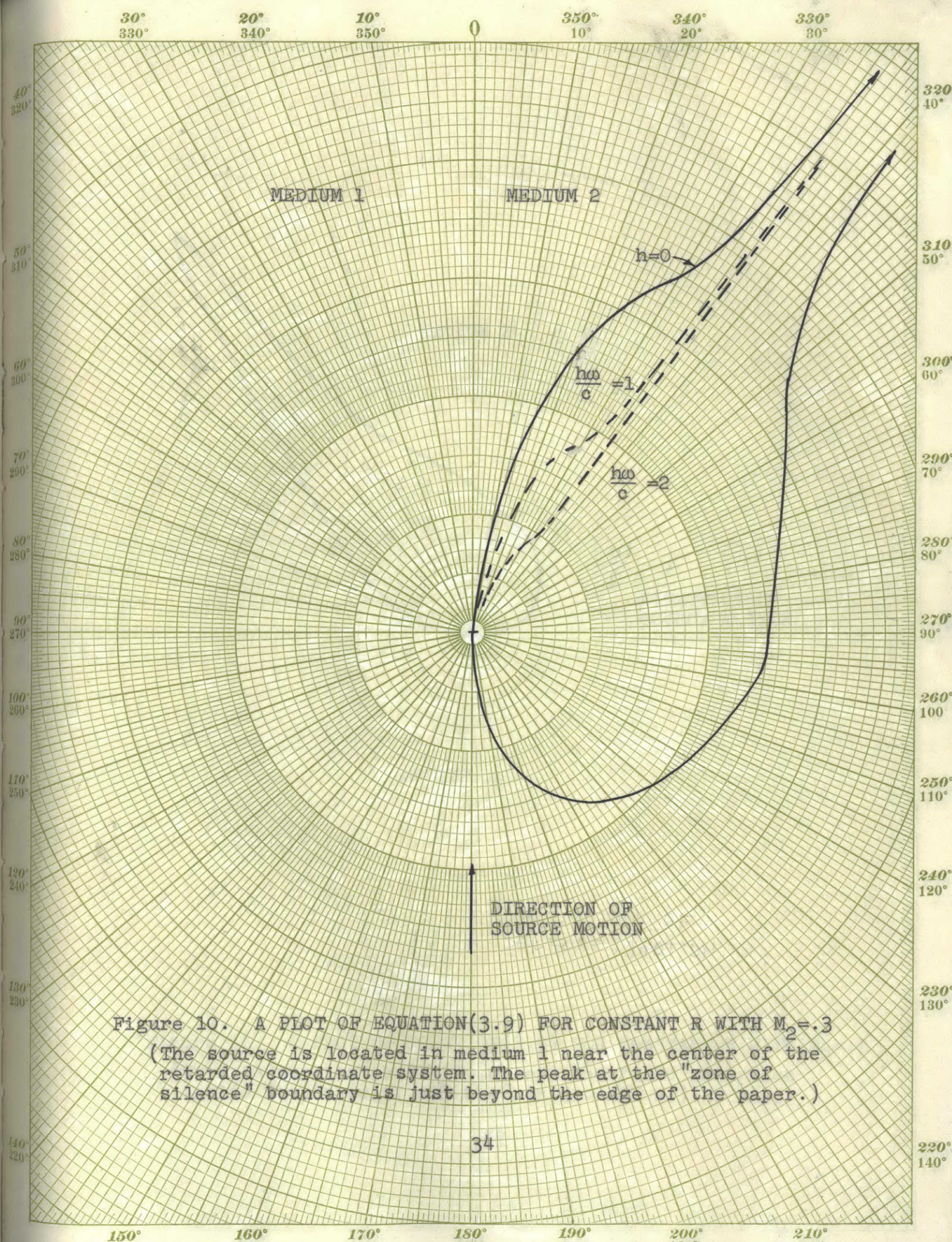


Figure 10. A PLOT OF EQUATION(3.9) FOR CONSTANT R WITH $M_2 = .3$
 (The source is located in medium 1 near the center of the retarded coordinate system. The peak at the "zone of silence" boundary is just beyond the edge of the paper.)

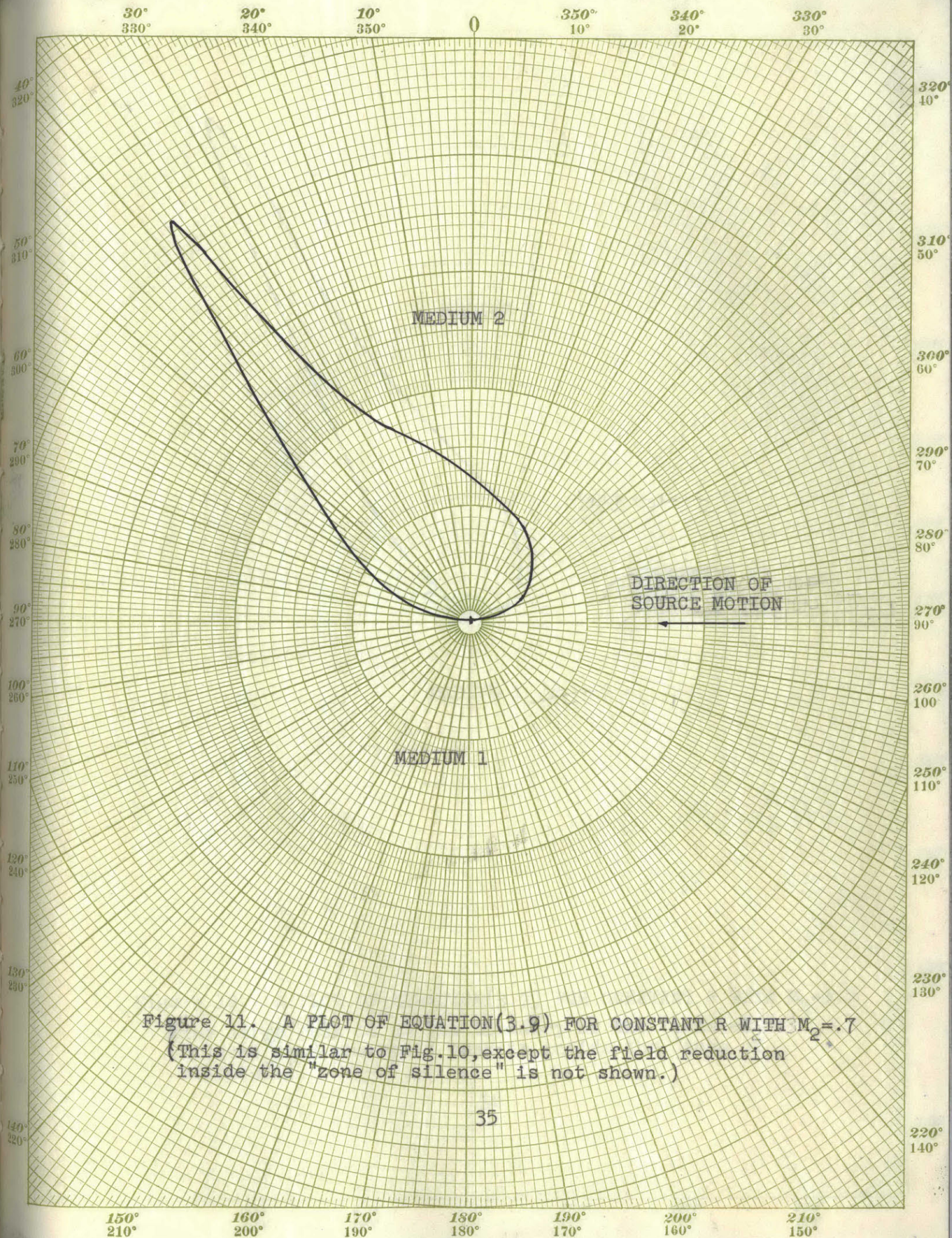


Figure 11. A PLOT OF EQUATION(3.9) FOR CONSTANT R WITH $M_2 = .7$
 (This is similar to Fig.10, except the field reduction inside the "zone of silence" is not shown.)

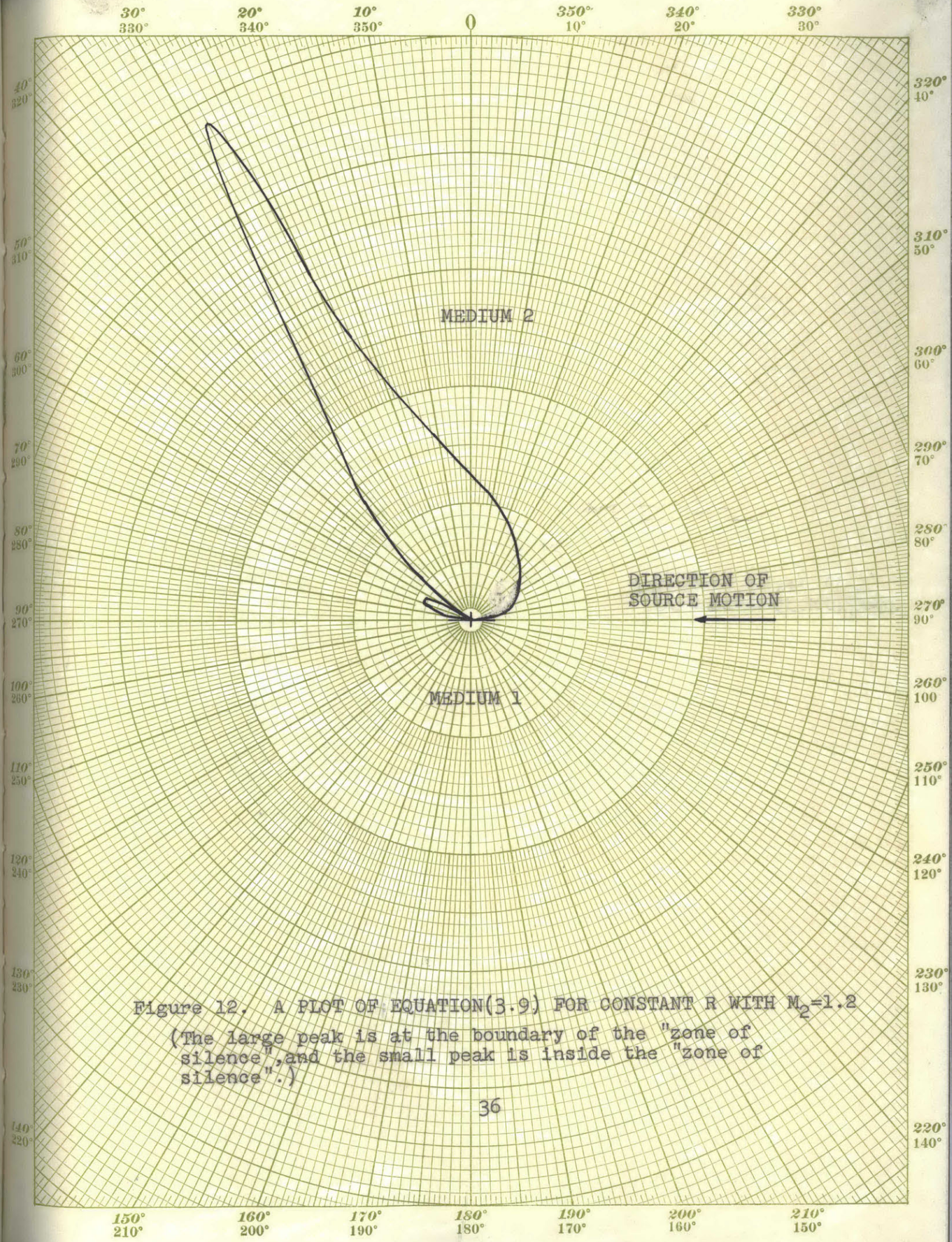


Figure 12. A PLOT OF EQUATION(3.9) FOR CONSTANT R WITH $M_2=1.2$
 (The large peak is at the boundary of the "zone of silence", and the small peak is inside the "zone of silence".)

III. Evaluation of Integral for Reflected Waves

The procedure for calculating the reflected wave ϕ_1 is quite similar. First, the saddle point for the integral of Eq. (3.2) is found to be

$$k_0 = [\omega/c(1 - M_1^2)] \left(-M + [\cos \theta / (\sqrt{1 - M_1^2 \sin^2 \theta})] \right) . \quad (3.10)$$

This is the same as Eq. (3.5) with M_1 replacing M_2 . $f^{\#}(k_0)$ is the same as calculated previously with M_1 again replacing M_2 . The same is true for γ_1 and γ_2 : we simply interchange all 1 and 2 subscripts, with the extra replacement of $\sin \theta$ by $-\sin \theta$ (since $\sin \theta$ is negative for the reflected wave). Again, the general expression for the asymptotic expansion of the integral is quite complex, so the same simplifications will be made as for the transmitted field. For $M_1 = 0$, the solution is quite simple.

$$\begin{aligned} \phi_1 = & - [\sqrt{2\pi}/(r\omega/c)^{1/2}] \left\{ [\sqrt{(1 - M_2 \cos \theta)^2 - \cos^2 \theta} + \sin \theta(1 - M_2 \cos \theta)^2] / \right. \\ & \left. [\sqrt{(1 - M_2 \cos \theta)^2 - \cos^2 \theta} - \sin \theta(1 - M_2 \cos \theta)^2] \right\} e^{i\omega t - (i\omega/c)[y \cos \theta + \\ & (x - \mathbf{h}) \sin \theta] - (i\pi/4)} . \end{aligned} \quad (3.11)$$

The factor multiplying the exponential becomes complex for $\cos \theta > 1/(1 + M_2)$. This is the angle for total reflection, and beyond this angle, the absolute value of the amplitude factor becomes equal to

$\sqrt{2\pi}$, which is the source strength as can be seen from Eq. (3.14). Thus, the only reduction in amplitude for these totally reflected rays is the cylindrical spreading $r^{-1/2}$, as would be expected. In this region, there is also a refraction arrival wave; the mathematical reason for this is again the distortion of a saddle path around a branch point. The case $M_2 = 0$ is more complicated. Then, ϕ_1 can be written as

$$\phi_1 = \left[-\sqrt{2\pi} \Delta^{1/2} \left(\Delta \sin \theta + [(1 - M\Delta \cos \theta)/(1 - M^2)]^2 \sqrt{1 - [(\Delta \cos \theta - M)/(1 - M^2)]^2} \right) e^{i\omega t - (i\omega/c) [y(\Delta \cos \theta - M)/(1 - M^2)] + (x-h)\Delta \sin \theta - (i\pi/4)} \right] / \left[(r\omega/c)^{1/2} \left(-\Delta \sin \theta + [(1 - M\Delta \cos \theta)/(1 - M^2)]^2 \sqrt{1 - [(\Delta \cos \theta - M)/(1 - M^2)]^2} \right) \right]. \quad (3.12)$$

This is plotted in Fig. 13. This result is simpler in the retarded coordinate system at rest with respect to medium 1. Using the same transformations as previously,

$$\phi_1 = \left[-\sqrt{2\pi} \left[\sin \theta_R + \sqrt{(1 + M \cos \theta_R)^2 - \cos^2 \theta_R} / (1 + M \cos \theta_R)^2 \right] \times e^{i\omega [t - (R/c)] + \left((i\omega h \sin \theta_R) / [c(1 + M \cos \theta_R)] \right) - (i\pi/4)} \right] / \left[(1 + M \cos \theta_R)^{1/2} (R\omega/c)^{1/2} \left(\left[\sqrt{(1 + M \cos \theta_R)^2 - \cos^2 \theta_R} / (1 + M \cos \theta_R)^2 \right] - \sin \theta_R \right) \right]. \quad (3.13)$$

We notice that for total reflection the amplitude factor is

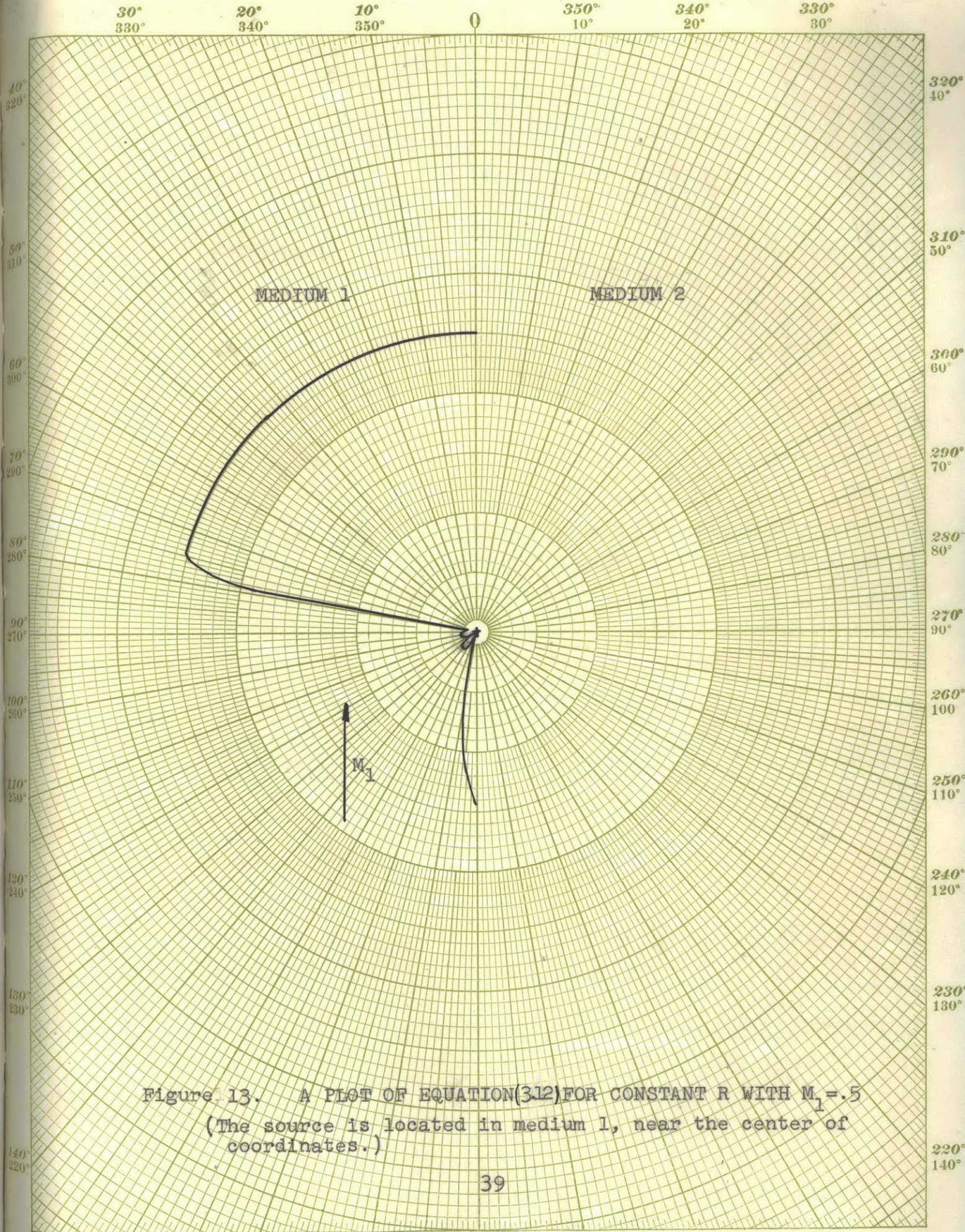


Figure 13. A PLOT OF EQUATION(3.12) FOR CONSTANT R WITH $M_1 = .5$
 (The source is located in medium 1, near the center of
 coordinates.)

$$1/(1 - M^2 \sin^2 \theta)^{1/4} r^{1/2} \quad \text{or} \quad 1/(1 + M \cos \theta_R)^{1/2} R^{1/2} ,$$

which is just the expression for a point source in a moving medium (or moving source) in each of the coordinate systems, as would be expected from simple ray analysis. The angle for total reflection is given by

$$\cos \theta_R < -[1/(1 + M)] \quad \text{or} \quad \cos \theta < \left[\left(-[1/(1 + M)] + M \right) / \sqrt{1 + M^2 - [2M/(1 + M)]} \right] .$$

The total field in medium 1 is the sum of the source term and the reflected wave ϕ_1 . The asymptotic form for ϕ_0 is

$$\phi_0 = \Delta_1^{1/2} \sqrt{2\pi} e^{i\omega t - (i\omega/c) - [y/(1 - M_1^2)] (\Delta_1 \cos \theta - M_1) - (h+x)\Delta_1 \sin \theta - (i\pi/4)} (i\omega/c)^{1/2} \quad (3.13a)$$

Equation (3.1) can also be evaluated exactly to give

$$\phi_0 = \pi e^{i\omega t + iM_1 \omega y / [c(1 - M_1^2)]} H_0^{(2)} \left[\left(\frac{\omega}{c} \sqrt{1 - M_1^2} \right) \sqrt{[y^2/(1 - M_1^2)] + (x+h)^2} \right] \sqrt{1 - M_1^2} \quad (3.14)$$

provided $M_1^2 < 1$.

IV. Modification Necessary for the Supersonic Case

For the supersonic case $M_1^2 > 1$, the exact source function is

$$\phi_0 = \pi e^{i\omega t - (iM_1 \omega y)/[c(M_1^2 - 1)]} J_0 \left(\frac{[\omega/(c\sqrt{M_1^2 - 1})] \sqrt{[y^2/(M_1^2 - 1)] - (x+h)^2}}{\sqrt{M_1^2 - 1}} \right) \quad (3.14a)$$

for $\theta < \arcsin 1/M_1$, and

$$\phi_0 = 0$$

for $\theta > \arcsin 1/M_1$.

This solution can be obtained asymptotically, by an extension of the previous saddle point technique, from Eq. (3.1). It is first necessary to note, from Eqs. (3.5) or (3.10), that supersonic speeds imply an additional branch point in k_0 as a function of θ at the Mach angle. The proper branch is determined from the physical requirement that the field be zero outside the Mach cone. Inside the Mach cone, we can obtain outgoing waves at infinity when the radical of Eq. (3.5) or of Eq. (3.10) has either positive or negative sign. This implies that there are two saddle points on the same Riemann surface. These saddle points tend to $+\infty$ and $-\infty$ as θ approaches the Mach angle ($\sin \theta = 1/M$). Thus, the branch point in k_0 is at $k_0 = \pm \infty$, and an appropriate cut is a large semicircle in the lower half of the k plane, as shown in Fig. 14. This cut excludes the values of k_0 corresponding to θ outside the Mach cone.

Thus, the supersonic evaluation of Eq. (3.1) gives two terms, corresponding to the two saddle points shown in Fig. 14. It is easily seen that the new saddle point has $f^{\mathbf{x}}(k_0)$ the opposite sign from the old saddle point; this means that the saddle path is shifted 90° as shown in Fig. 14.

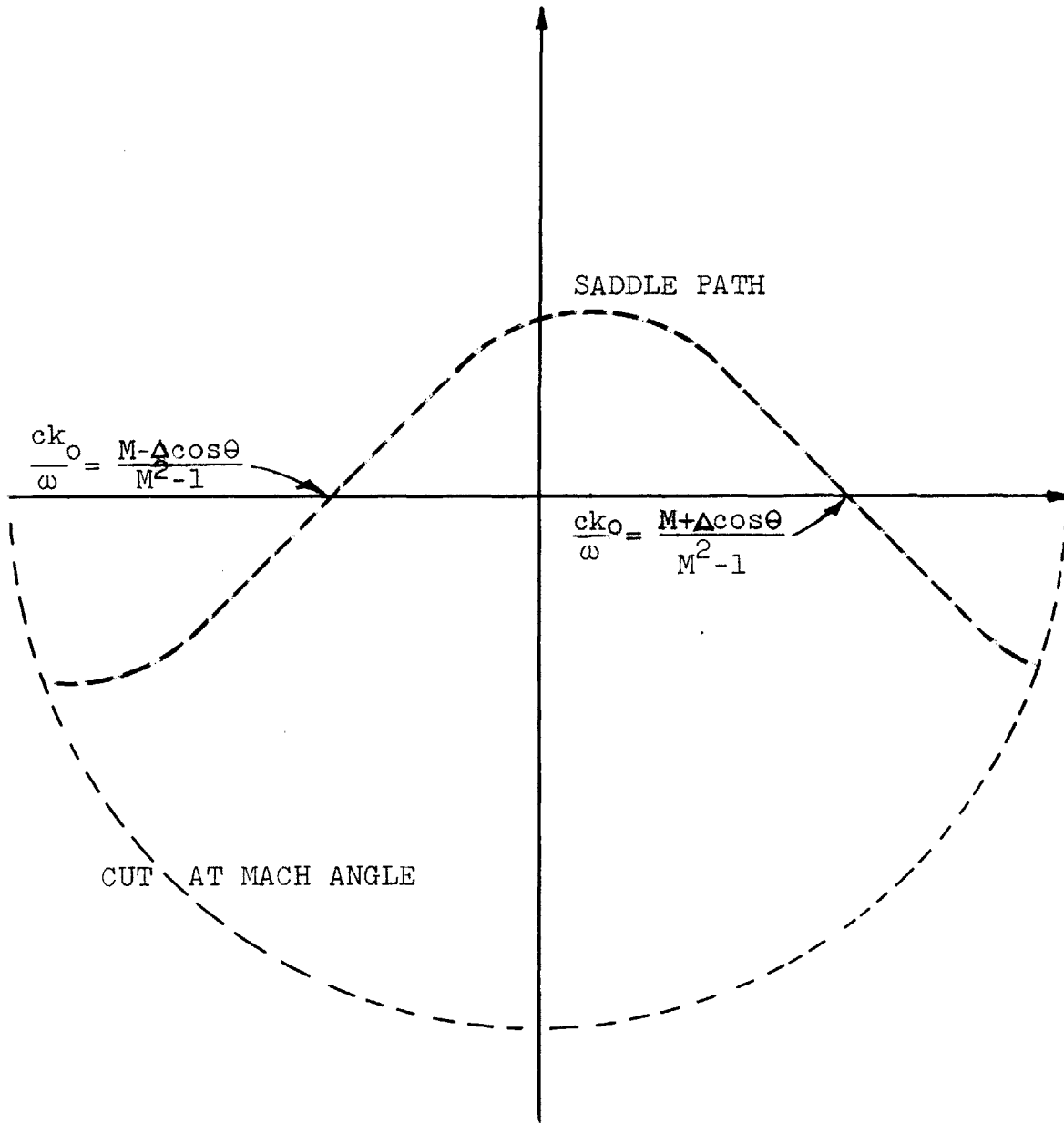


Figure 14. SADDLE PATH FOR $M > 1$ WITH BRANCH CUT AT MACH ANGLE

This means that the supersonic evaluation of Eq. (3.1) is the sum of Eq. (3.13a) and a term identical with Eq. (3.13a) except for the sign of Δ_1 changed wherever it appears and a phase factor of $+i\pi/4$ instead of $-i\pi/4$.

This analysis indicates that there is an extra term added to Eqs. (3.8), (3.9), (3.12), (3.13) for supersonic flow in the medium which the particular equation applies to. Equations (3.9) and (3.13) are written in retarded coordinates, so the second term is identical in form with the term already given. But, since there are two retarded source points for the supersonic case, the second term refers to a coordinate system with the origin at the new second retarded source point. For Eqs. (3.8) and (3.12), the second term is obtained by taking the negative square root for Δ . (In addition, the phase must be shifted as is described above.) Both terms become equal (except for phase) at the Mach angle which is a branch point for the expressions in the instantaneous coordinate system. Beyond this point, the saddle point k_0 obviously becomes complex, and the causality condition requires that there be no sound field. This will be the case for the cut shown in Fig. 14.

V. Some Additional Terms for the Near Field

It was mentioned earlier that it is sometimes necessary to cross a branch point while moving the real k -axis to the saddle path indicated in Figs. 6 or 14. Examples of the proper distortion of the saddle path are given in Figs. 15 and 16. These examples are taken for the evaluation of ϕ_2 , but the situation for ϕ_1 is similar. It is evident from these figures that the effect of the distortion around the branch

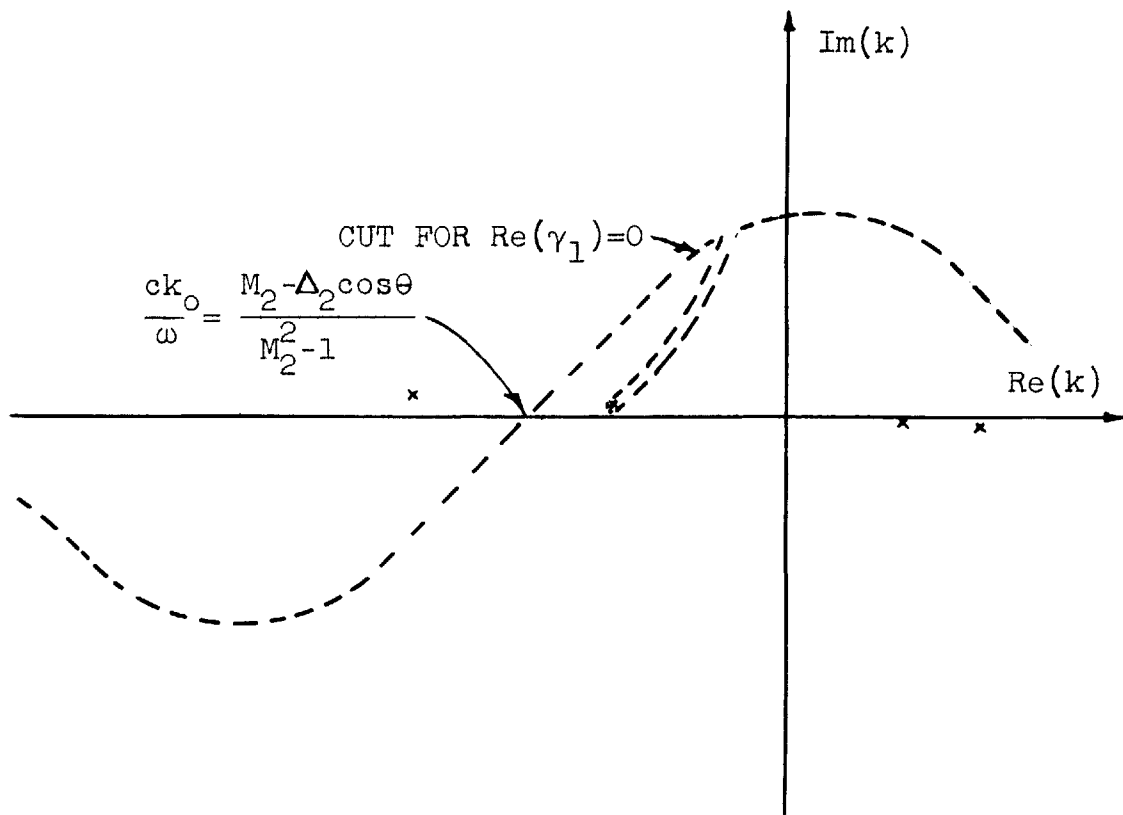


Figure 15. SADDLE PATH FOR ϕ_2 AND CUT FOR ϕ_2'

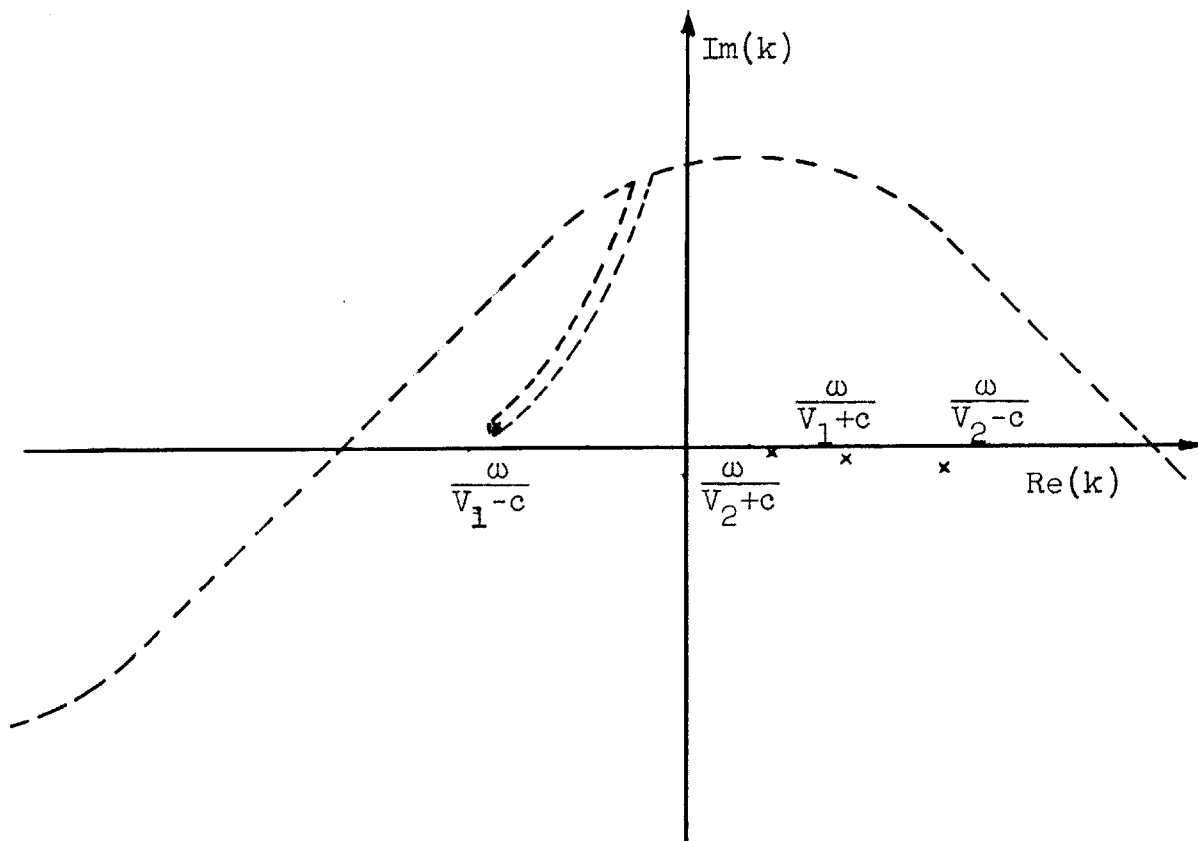


Figure 16. THE CUT AND THE SADDLE PATH FOR MEDIUM 2
WITH $M_1 < 1$ AND $M_2 > 1$

point can be accounted for by the addition of the integral around the branch cut.

As a simple example, we set $M_1 = 0$ and solve the integral for ϕ_2 . The necessity for the extra integral arises when k_0 is to the left of the branch point for $\gamma_1 = 0$. Such a situation is shown in Fig. 15. It is convenient to integrate along both sides of the cut $\text{Re}\gamma_1 = 0$. To do the integration for ϕ_2 , it is convenient to make the change of variable

$$k^2 = \omega^2/c^2 - u^2 \quad (\text{which means } \gamma_1 = \pm iu)$$

and approximate the integrand for small values of u . This assumes that the major contribution to the integral comes from the vicinity of the branch point $u = 0$. To this approximation,

$$k = -(\omega/c) + (u^2 c/2\omega) \quad \text{and}$$

$$\gamma_2 = (i\omega/c) \sqrt{M_2(2 + M_2)} + [icu^2(1 - M_2^2 - M_2)]/[2\omega\sqrt{M_2(2 + M_2)}] .$$

With these substitutions, the integral becomes

$$\begin{aligned} \phi_2^{\dagger} = & 2c e^{i\omega t + (i\omega y/c) - (i\omega x/c) \sqrt{M(2+M)}} \int_0^{\infty} \left[e^{-[(ic/2\omega)y + \{x(1 - M^2 - M)/\right. \\ & \left. \sqrt{M(2+M)}\}u^2]/[\omega_2 \sqrt{(\omega_2/c_2)^2 - u^2}]} \left[e^{-iu h}/[iu + (\gamma_2 \omega^2/\omega_2^2)] - \right. \right. \\ & \left. \left. e^{iu h}/[-iu + \gamma_2(\omega^2/\omega_2^2)] \right] \right] u du \end{aligned}$$

(where $M = M_2$ has been understood). Since it is assumed that the

major contribution to the integral comes from $u \ll \omega/c$, the integrand can be further simplified by neglecting u wherever it occurs in the denominator. Then, the integral may be evaluated in a straightforward manner to yield

$$\phi_2 = 8\sqrt{\pi i} h(1 + M) e^{i\omega \left[t + (y/c) - [x\sqrt{M(2+M)}/c] + h^2/2c \left([x(1 - M^2 - M)/\sqrt{M(2+M)}] + y \right) \right]} / \left[\sqrt{[\omega M(2+M)/c]} \left([x(1 - M^2 - M)/\sqrt{M(2+M)}] + y \right)^{3/2} \right] . \quad (3.15)$$

It is evident that ϕ_2^{r} becomes infinite for

$$\tan \theta = -\sqrt{M_2(2+M_2)}/(1 - M_2^2 - M_2) .$$

This corresponds to the branch point $k = \omega/c$, being the same point as the saddle point k_0 . For larger angles, the saddle point will be to the left of the branch point as depicted in Fig. 15, and ϕ_2^{r} is added to the saddle point integration ϕ_2 to obtain the more complete solution. For smaller angles, the branch point is to the left of the saddle point, and ϕ_2^{r} gives no contribution. This wave ϕ_2^{r} is just in the "zone of silence" as was predicted earlier. At the angle of the boundary of the "zone of silence," given by (3.15a), ϕ_2^{r} becomes a poor approximation just as the saddle point integration becomes a poor first-order approximation. This is to be expected, since the two integrations overlap as this critical angle is approached. ϕ_2^{r} is a wave that travels most of the distance

in medium 1, as can be easily seen from the exponential. Consequently, this wave has a shorter travel time than a wave travelling most of the distance in medium 2, and so ϕ_2^{\dagger} violates causality, for medium 2. Thus, it is expected that the amplitude of ϕ_2^{\dagger} should vanish as $h \rightarrow 0$, and should fall off as $r^{-3/2}$ so that the wave can carry no energy to $r = \infty$.

A somewhat different solution for the higher-order approximation can be found for ϕ_1 . For simplicity, we set $M_2 = 0$, and find that the saddle path and branch points are the same as in the previous case with M_2 replaced by M_1 . Thus, a term arising from the branch cut must be added to Eqs. (3.12) and (3.13). The cut is similar to that shown in Fig. 15, but M_2 is replaced by M_1 and the cut is for $\text{Re} y_2 = 0$. Thus, we use the same type of approximations as were used to derive Eq. (3.15). The integral can be evaluated in a straightforward manner to yield

$$\phi_1^{\dagger} = \left[2\sqrt{2\pi i} (1 + M_1)^2 e^{i\omega t + (i\omega y/c) + [i\omega(x-h)/c] \sqrt{M_1(2+M_1)}} \right] / \left[M_1(2+M_1) (\omega/c)^{3/2} \left([x(1 - M_1^2 - M_1)/\sqrt{M_1(2+M_1)}] - y \right)^{3/2} \right] \quad (3.16)$$

(where $y \leq [x(1 - M_1^2 - M_1)/\sqrt{M_1(2+M_1)}]$, and x is negative).

Equation (3.16) for ϕ_1^{\dagger} becomes infinite for $M_1 = 0$, but this is to be expected since we assumed that u could be neglected compared with $M_1(\omega/c)$. The result also becomes a poor approximation and blows up when the inequality given after Eq. (3.16) becomes an equality. There is no "zone of silence" for medium 1 since the integrand for ϕ_1 does

not contain γ_2 in the exponential. The region in which ϕ_1^* exists also contains the direct and reflected cylindrical waves, so that the distinguishing feature of ϕ_1^* is the shorter travel time. Since ϕ_1^* has an $r^{-3/2}$ fall off, just like ϕ_2^* , it does not actually carry energy to $r = \infty$. This is to be expected, since the wave travels most of the distance in medium 2 which is at rest. This phenomenon is well-known experimentally with the modification of two different media with different sound velocities, rather than the present method of having differing sound velocities caused by relative motion of the two media. With two different media, this wave is known as the "refraction arrival," and a thorough discussion of this has been given by Ewing, Jardetzky, and Press.⁽⁹⁾ For supersonic flows, the calculation proceeds in the same manner as the derivation of Eq. (3.15) and Eq. (3.16), with the saddle path similar to that shown in Fig. 16.

There are additional higher-order terms obtained by the second term of the asymptotic expansion in the saddle point integration. These also have a fall off of $r^{-3/2}$, but they have a cylindrical wave front with the same phase dependence as already given for the leading term of the asymptotic expansion. Their calculation is straightforward, but complex, and there is no need to note them here.

POINT SOURCE

I. Formal Integral Solutions

The geometrical situation is similar to that shown in Fig. 5 (page 24), except that a z-axis extends out from the page. The integral expansion of the source can be written in a manner similar to the two-dimensional case. Thus,

$$\phi_0 = e^{i\omega t} \int_{-\infty}^{+\infty} [e^{-i(\lambda y + \mu z) - \gamma_1 |x+h|} d\lambda d\mu / \gamma_1] \quad (4.1)$$

(where $\omega_1 = \omega - \lambda V_1$, and $\gamma_1 = \sqrt{\lambda^2 + \mu^2 - (\omega_1^2/c^2)}$.

This integral is the Fourier transform of

$$\phi_0 = \left(2\pi e^{i\omega t + [i\omega M_1 y / (1 - M_1^2)] - [i\omega/c \sqrt{1 - M_1^2}] \sqrt{[y^2 / (1 - M_1^2)] + z^2 + (x+h)^2}} \right) / \left(\sqrt{1 - M_1^2} \sqrt{[y^2 / (1 - M_1^2)] + z^2 + (x+h)^2} \right) . \quad (4.1a)$$

This can be found by exact evaluation or by the saddle point approximation to be used on the integrals for the reflected and transmitted wave.

The reflected wave is written as

$$\phi_1 = e^{i\omega t} \int_{-\infty}^{+\infty} A(\lambda, \mu) e^{-i(\lambda y + \mu z) + \gamma_1 x} d\lambda d\mu, \quad (4.2)$$

and the transmitted wave in medium 2 is written as

$$\phi_2 = e^{i\omega t} \int_{-\infty}^{+\infty} B(\lambda, \mu) e^{-i(\lambda y + \mu z) - \gamma_2 x} d\lambda d\mu \quad (4.3)$$

(where $\gamma_2 = \sqrt{\lambda^2 + \mu^2 - (\omega_2^2/c^2)}$, and $\omega_2 = \omega - \lambda V_2$).

Matching boundary conditions is exactly the same as in the two-dimensional case. Thus, the results are given by

$$A = [(\gamma_1 \omega_2^2 - \gamma_2 \omega_1^2) / (\gamma_1 \omega_2^2 + \gamma_2 \omega_1^2)] e^{-\gamma_1 h},$$

and

$$B = [(2\omega_1 \omega_2) / (\gamma_1 \omega_2^2 + \gamma_2 \omega_1^2)] e^{-\gamma_1 h}$$

II. Evaluation of Integrals for Transmitted Wave

We can now proceed with the saddle point approximations in a manner similar to that of the two-dimensional case. It is found to be much simpler to perform the μ integration first. Considering ϕ_2 , we can write the integral as

$$\int_{-\infty}^{+\infty} g(\lambda, \mu) e^{-f(\lambda, \mu)} d\lambda d\mu$$

$$f(\lambda, \mu) = i\lambda y + i\mu z + \gamma_2 x.$$

The saddle point for the μ integration is found from

$$[\partial f(\lambda, \mu)/\partial \mu]_{\mu = \mu_0} = 0 \quad .$$

This gives a value for μ_0 with λ remaining unspecified. If we use spherical coordinates with

$$x = r \cos \theta \quad , \quad y = r \sin \theta \cos \psi \quad , \quad z = r \sin \theta \sin \psi \quad ,$$

and let

$$u = \tan \theta \sin \psi \quad ,$$

then,

$$\mu_0 = \pm (iu/\sqrt{1+u^2}) \sqrt{\lambda^2(1-M_2^2) - (\omega^2/c^2) + (2\lambda M_2 \omega/c)} \quad . \quad (4.4)$$

If the positive square root is taken, then only the negative sign need be used. Thus,

$$\mu_0 = (u/\sqrt{1+u^2}) \sqrt{(\omega^2/c^2) - \lambda^2} \quad .$$

It is useful to calculate the saddle point for the λ integration. This is given by

$$[(\partial f(\lambda, \mu_0)/\partial \lambda)]_{\lambda = \lambda_0} = 0 \quad .$$

Making the substitution

$$v = \tan \theta \cos \psi \quad ,$$

the solution is

$$\lambda_0 = [1/(1 - M_2^2)] \left(- (M_2 \omega / c) \pm [v \omega / c \sqrt{v^2 + (1 - M_2^2)(1 + u^2)}] \right). \quad (4.5)$$

This has a branch point at $v = 0$, but only the branch given by the + sign in Eq. (4.5) is necessary for the subsonic case (as it was in two dimensions). Before the saddle point integrations are performed, the integrals are over real paths for λ and μ . (The saddle point integrations require distortions of these paths.) Thus, there is some insight to be gained by examining Fig. 17 which shows the positions of the branch points in the real λ, μ plane. Points inside the closed curve $\gamma_1 = 0$ give imaginary values of γ_1 , whereas the points outside give real values of γ_1 .

The correct way to perform the μ integration first is to leave the value of λ unspecified, since the λ integration is over all real λ . However, if λ is outside the range of values for λ shown in Fig. 17, then μ_0 , as given in Eq. (4.4), becomes imaginary. In order to simplify the calculation, λ will be assumed in the range of λ_0 . This procedure appears valid when it is noted that this range contains the saddle point and the branch point for $\gamma_1 = 0$ (which sometimes must be used as in the two-dimensional case), and the main contribution to the integration comes from the vicinity of these points. To continue with the μ integration, we calculate

$$\left(\frac{\partial^2 f}{\partial \mu^2} \right)_{\mu = \mu_0} = - [iu \cos \theta (1 + u^2) / \mu_0] r.$$

Since μ_0 and u always have the same sign, this indicates that the distorted path crosses the $\text{Re} \mu$ -axis at 45° at the saddle point, as indicated

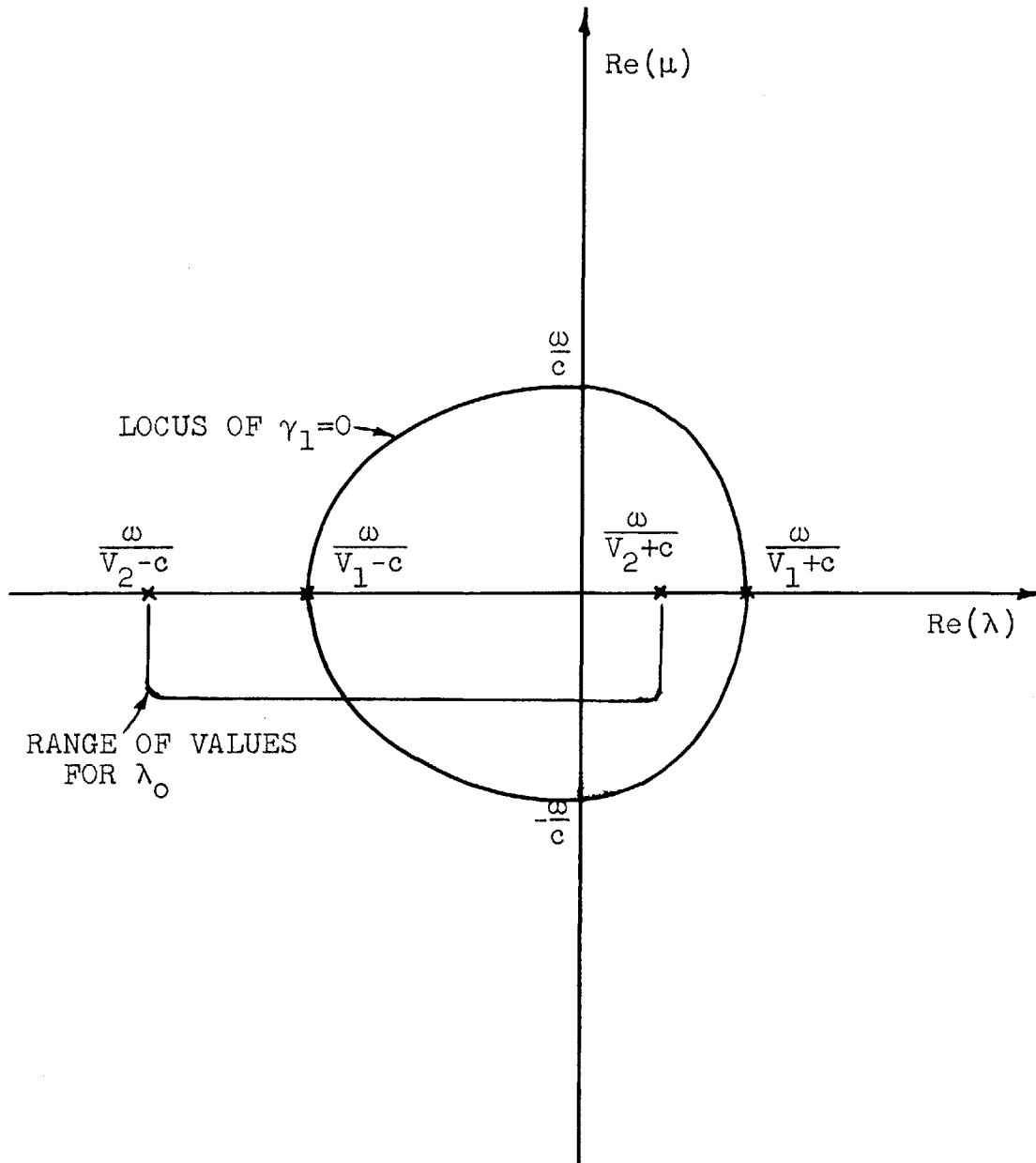


Figure 17. THE RANGE OF VALUES OF λ_0 AND THE LOCUS OF BRANCH POINTS FOR $\gamma_1=0$ (FOR $M_1, M_2 < 1$)

in Fig. 18. Also shown in Fig. 18 are the branch points for $\gamma_1 = 0$ and the cut required for $\mu_0 < -\sqrt{(\omega_1^2/c^2) - \lambda^2}$ or $\mu_0 > \sqrt{(\omega_1^2/c^2) - \lambda^2}$. If we draw a line parallel to the μ -axis of Fig. 17, through a given point on the λ -axis, the two intersections with the closed curve give the branch points shown in Fig. 18. As λ approaches $-\omega/c(1 - M_1)$, these branch points approach the origin, and for $-\omega/c(1 - M_2) < \lambda < -\omega/c(1 - M_1)$, the branch points lie on the imaginary axis of Fig. 18. Thus, the origin of the μ plane is a branch point of the curve $\gamma_1 = 0$ as a function of the parameter λ , and the sheet of the Riemann surface which this curve moves onto is determined by the physical condition that there be no infinitely growing waves. Since μ is imaginary, this means that the waves given by the branch cut must decay exponentially in y . The integration along the branch cut for the case in Fig. 18 will give a higher-order term, and are neglected for the present.

The result of integration over the μ saddle point is

$$\phi_2 = \sqrt{\pi} \int_{-\infty}^{+\infty} [g(\lambda, \mu_0) e^{-f(\lambda, \mu_0)} d\lambda] / \sqrt{\partial^2 f / \partial \mu^2} \quad . \quad (4.5)$$

The saddle point for the λ integration was given in Eq. (4.5). The branch points are given by $\gamma_1(\lambda, \mu_0) = 0$, which has the solution

$$c\lambda/\omega = - [(1 + u^2)M_1 - M_2 u^2 \pm \sqrt{1 + u^2(1 + u)(M_1 - M_2)^2}] / [1 + u^2 M_2^2 - M_1^2(1 + u^2)] \quad . \quad (4.6)$$

If $u = 0$, it is seen that these reduce to the $\gamma_1 = 0$ branch points of the

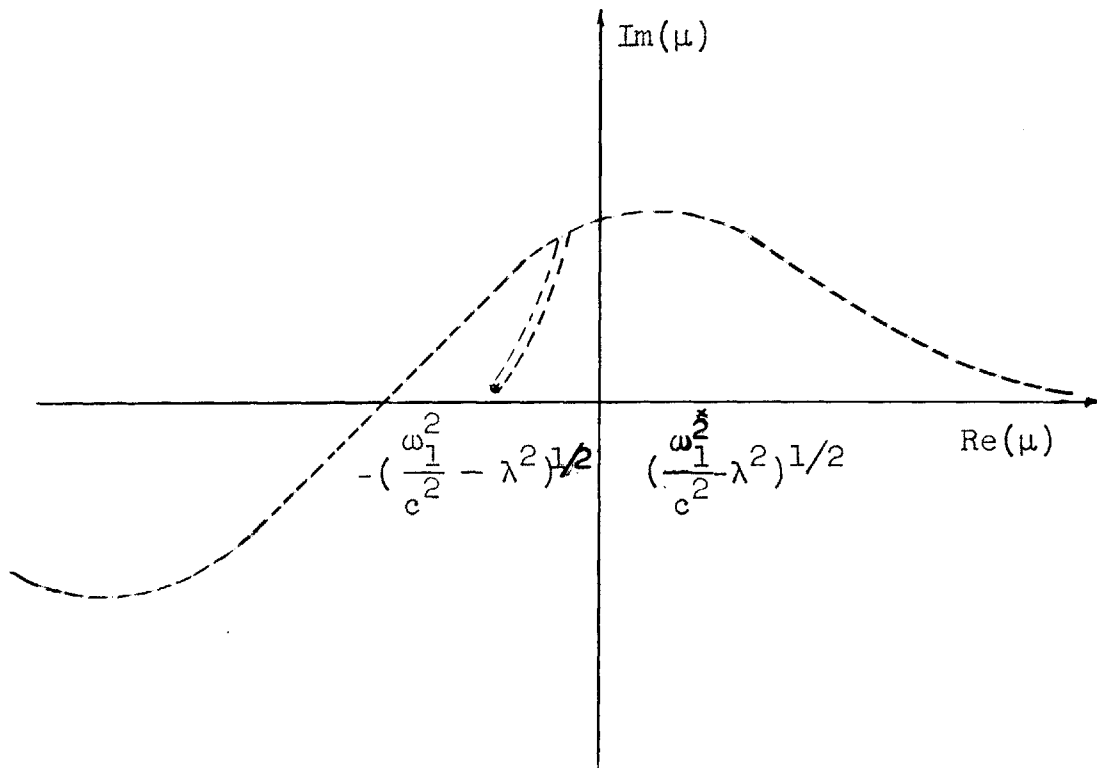


Figure 18. SADDLE PATH, BRANCH POINTS, AND BRANCH CUT FOR THE μ INTEGRATION

two-dimensional case. Since these branch points are always real (with small imaginary parts to be precise), the integration procedure for λ is the same as used previously. For the saddle point integration, we calculate

$$(\partial^2 f / \partial \lambda^2) = -[i \cos \theta u^3 / (1 + u^2) \mu_0^3] (\omega^2 / c^2) r \quad ,$$

$$(\omega_2 / \omega) = [1 / (1 - M_2^2)] \left(1 - [v M_2 / \sqrt{v^2 + (1 - M_2^2)(1 + u^2)}] \right) \quad ,$$

$$(\omega_1 / \omega) = \left(1 - M_2^2 + M_1 M_2 - [M_1 v / \sqrt{v^2 + (1 - M_2^2)(1 + u^2)}] \right) (1 - M_2^2)^{-1} \quad ,$$

$$\gamma_1 = \sqrt{[u^2 / (1 + u^2)] (\omega^2 / c^2) + [\lambda_0^2 / (1 + u^2)] - (\omega_1^2 / c^2)} \quad ,$$

so that

$$\phi_2 = \frac{-2\pi i \omega_1 \omega_2 e^{i\omega t - i\lambda_0 y - i\mu_0 z - \gamma_2 x - \gamma_1 h}}{\left[\cos \theta [v^2 + (1 - M_2^2)(1 + u^2)] [\gamma_1 \omega_2^2 + \gamma_2 \omega_1^2] r \right]} \quad . \quad (4.7)$$

As in the two-dimensional case, this expression can be greatly simplified by changing to retarded coordinates. If we make the substitutions

$$v = \tan \theta_R \cos \psi_R + M_2 \sec \theta_R \quad , \quad y = y_R + M_2 R \quad ,$$

$$u = \tan \theta_R \sin \psi_R \quad , \quad r = R \sqrt{1 + M_2^2 + M_2 \sin \theta_R \cos \psi_R} \quad .$$

Then, Eq. (4.7) can be written as

$$\begin{aligned}
\phi_2 = & \left[- 2\pi \cos \theta [1 + (M_2 - M_1) \sin \theta \cos \psi] e^{i\omega[t - (R/c)] -} \right. \\
& \left. h \sqrt{\sin^2 \theta - [1 + (M_2 - M_1) \sin \theta \cos \psi]^2} / (1 + M_2 \sin \theta \cos \psi) \right] / \\
& \left[(R\omega/c)(1 + M_2 \sin \theta \cos \psi) \left(\sqrt{[1 + (M_2 - M_1) \sin \theta \cos \psi]^2 - \sin^2 \theta} + \right. \right. \\
& \left. \left. \cos \theta [1 + (M_2 - M_1) \sin \theta \cos \psi]^2 \right) \right] \quad (4.7a)
\end{aligned}$$

(where θ and ψ are measured in the retarded coordinate system).

For $\psi_R = 0$ or π , Eq. (4.7a) reduces to the two-dimensional case with an additional factor $\sqrt{[R\omega(1 + M_2 \sin \theta_R)/c]}$ in the denominator. This is to be expected; if the field for a moving point and line source are compared, the same factor is obtained. If we set $\psi_R = \pi/2$, all θ_R dependence cancels out of the amplitude, and the field is unchanged by the velocity, except for the phase factor multiplying h in the exponential.

The calculation for the reflected wave is made by the same method as was used to derive Eq. (4.7a). The extension to supersonic flows is exactly the same as for the two-dimensional case. One simply uses both saddle points given by Eq. (4.5) for the λ integration, which results in an extra term being added to Eq. (4.7) with the sign of v reversed, but otherwise the same as the original term in Eq. (4.7).

We note that, whenever γ_1 is real, the amplitude is reduced proportional to h . This is similar to the "zone of silence" discussed in the two-dimensional case. In terms of Fig. 17, it means $\lambda = \lambda_0$, $\mu = \mu_0$ lies outside the closed curve of $\gamma_1 = 0$. This implies that there is a contribution from the branch cut integral of μ or λ in the form of an

extra wave, with plane wave front, decaying faster than $1/r$. The calculation of these waves is much more complicated than in the two-dimensional case.

LINE SOURCE IN A VELOCITY DISCONTINUITY CHANNEL

I. Formal Integral Solutions

The solution for a line source near a velocity discontinuity can be generalized to the case of two discontinuities. Such a situation is shown in Fig. 19, with $M_1 \neq M_2$. We must satisfy boundary conditions at $x = +a$ and $x = -a$. The field of the source is written as

$$\phi_0 = e^{i\omega t} \int_{-\infty}^{+\infty} (e^{-iky - \gamma_1 |x-h|} / \gamma_1) dk \quad (5.1)$$

[with $\gamma_1 = \sqrt{k^2 - (\omega_1^2/c^2)}$, and $\omega_1 = \omega - M_1 kc$] .

It should be noted that h would be a negative number for the situation which is shown in Fig. 19, as distinct from the positive h shown in Fig. 5 (page 24). The rest of the field must be determined from the boundary conditions.

$$\phi_2 = e^{i\omega t} \int_{-\infty}^{+\infty} A(k) e^{-iky - \gamma_1 x} dk \quad (\text{for } x > a) \quad (5.2)$$

[with $\gamma_2 = \sqrt{k^2 - (\omega_2^2/c^2)}$, and $\omega_2 = \omega - M_2 kc$] .

$$\phi_1 = e^{i\omega t} \int_{-\infty}^{+\infty} [B(k) e^{-\gamma_1 x} + C(k) e^{\gamma_1 x}] e^{-iky} dk \quad (a > x > -a) \quad (5.3)$$

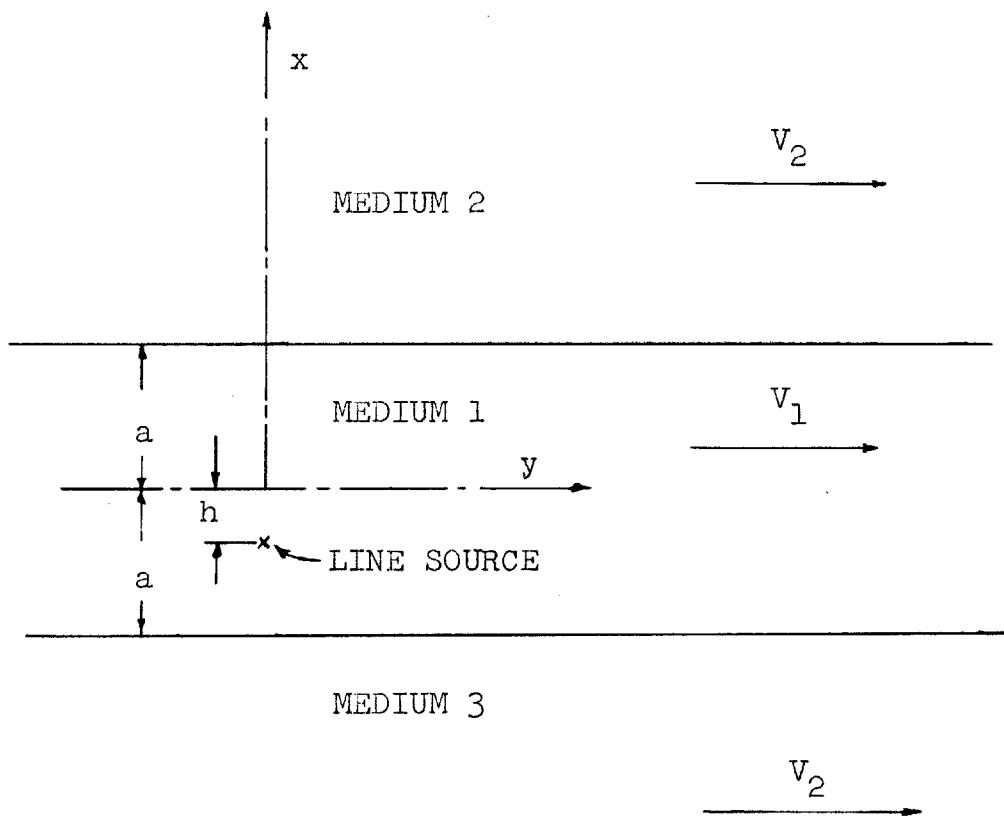


Figure 19. CHANNEL FORMED BY 2 VELOCITY DISCONTINUITIES

$$\phi_3 = e^{i\omega t} \int_{-\infty}^{+\infty} D(k) e^{\gamma_2 x - iky} dk \quad . \quad (5.4)$$

The continuity of pressure at $x = a$ gives

$$\omega_2 A e^{-\gamma_2 a} = \omega_1 \left(+B e^{-\gamma_1 a} + C e^{\gamma_1 a} + [e^{-\gamma_1(a-h)}/\gamma_1] \right) \quad .$$

The continuity of velocity normal to the interface at $x = a$ gives

$$-(\gamma_2 A e^{-\gamma_2 a} / \omega_2) = (\gamma_1 / \omega_1) \left(-B e^{\gamma_1 a} + C e^{-\gamma_1 a} - [e^{-\gamma_1(a+h)}/\gamma_1] \right) \quad .$$

Applying the same boundary conditions at $x = -a$ results in

$$\omega_2 D e^{-\gamma_2 a} = \omega_1 \left(B e^{\gamma_1 a} + C e^{-\gamma_1 a} + [e^{-\gamma_1(a+h)}/\gamma_1] \right)$$

$$(\gamma_2 D / \omega_2) e^{\gamma_2 a} = (\gamma_1 / \omega_1) \left(-B e^{\gamma_1 a} + C e^{-\gamma_1 a} + [e^{-\gamma_1(a+h)}/\gamma_1] \right) \quad .$$

These four equations can be solved simultaneously to yield

$$A = \frac{\omega_1 \omega_2 e^{\gamma_2 a} [\omega_2^2 \gamma_1 \cosh \gamma_1 (a+h) + \gamma_2 \omega_1^2 \sinh \gamma_1 (a+h)]}{[\gamma_1 \omega_2^2 \cosh \gamma_1 a + \gamma_2 \omega_1^2 \sinh \gamma_1 a] [\gamma_2 \omega_1^2 \cosh \gamma_1 a + \gamma_1 \omega_2^2 \sinh \gamma_1 a]} \quad (5.5)$$

$$B = \frac{(\gamma_1 \omega_2^2 - \gamma_2 \omega_1^2) e^{-\gamma_1 a} [\omega_2^2 \gamma_1 \cosh \gamma_1 (a-h) + \gamma_2 \omega_1^2 \sinh \gamma_1 (a-h)]}{2\gamma_1 [\gamma_1 \omega_2^2 \cosh \gamma_1 a + \gamma_2 \omega_1^2 \sinh \gamma_1 a] [\gamma_2 \omega_1^2 \cosh \gamma_1 a + \gamma_1 \omega_2^2 \sinh \gamma_1 a]} \quad (5.6)$$

$$C = \frac{(\gamma_1 \omega_2^2 - \gamma_2 \omega_1^2) e^{-\gamma_1 a} [\omega_2^2 \gamma_1 \cosh \gamma_1 (a+h) + \gamma_2 \omega_1^2 \sinh \gamma_1 (a+h)]}{2\gamma_1 ([\gamma_1 \omega_2^2 \cosh \gamma_1 a + \gamma_2 \omega_1^2 \sinh \gamma_1 a] [\gamma_2 \omega_1^2 \cosh \gamma_1 a + \gamma_1 \omega_2^2 \sinh \gamma_1 a])} \quad (5.7)$$

$$D = \frac{\omega_1 \omega_2 e^{\gamma_2 a} [\omega_2^2 \gamma_1 \cosh \gamma_1 (a-h) + \omega_1^2 \gamma_2 \sinh \gamma_1 (a-h)]}{[\gamma_1 \omega_2^2 \cosh \gamma_1 a + \gamma_2 \omega_1^2 \sinh \gamma_1 a] [\gamma_2 \omega_1^2 \cosh \gamma_1 a + \gamma_1 \omega_2^2 \sinh \gamma_1 a]} \quad (5.8)$$

II. Discussion of Characteristic Equation

The poles of the denominator of these expressions correspond to the propagation modes of the channel. For $h = 0$, the denominator becomes

$$\gamma_2 \omega_1^2 \cosh \gamma_1 a + \gamma_1 \omega_2^2 \sinh \gamma_1 a \quad .$$

Thus, the condition

$$\gamma_2 \omega_1^2 \cosh \gamma_1 a + \gamma_1 \omega_2^2 \sinh \gamma_1 a = 0 \quad (5.9a)$$

gives the symmetric modes, and

$$\gamma_1 \omega_2^2 \cosh \gamma_1 a + \gamma_2 \omega_1^2 \sinh \gamma_1 a = 0 \quad (5.9b)$$

gives the antisymmetric modes. The lowest symmetric mode corresponds to a modification of a plane wave, since it exists for arbitrarily small values of $(\omega a/c)$. The solutions, k , for this lowest mode are shown in Fig. 20. To simplify the result, we have used $M_1 = 0$. These modes only exist for waves travelling in the same direction as M_1 . This is expected since this is the only direction which can give total reflection. It is noted, in Figs. 20 and 21, that when $M > 2$ the modes split and have a high frequency cut off. This can be explained by examining the geometrical conditions for propagating modes, as shown in Fig. 22, if a plane wave is incident along \overline{AB} , with perpendicular wave front \overline{AD} , then a propagating mode requires that the phase change along the path ABCD be a multiple of 2π . The phase shift ϵ at each total reflection is given by the

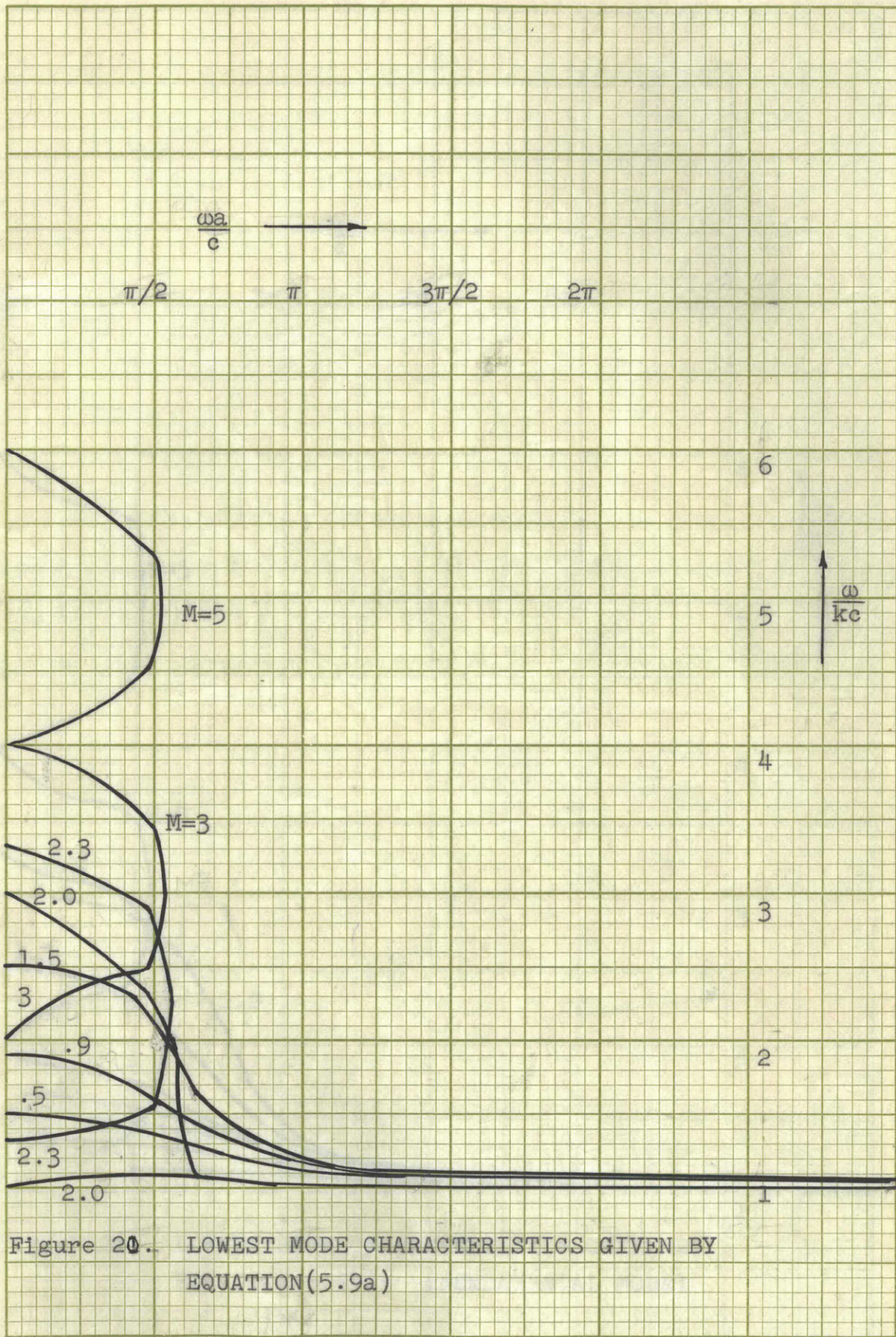


Figure 20. LOWEST MODE CHARACTERISTICS GIVEN BY EQUATION(5.9a)

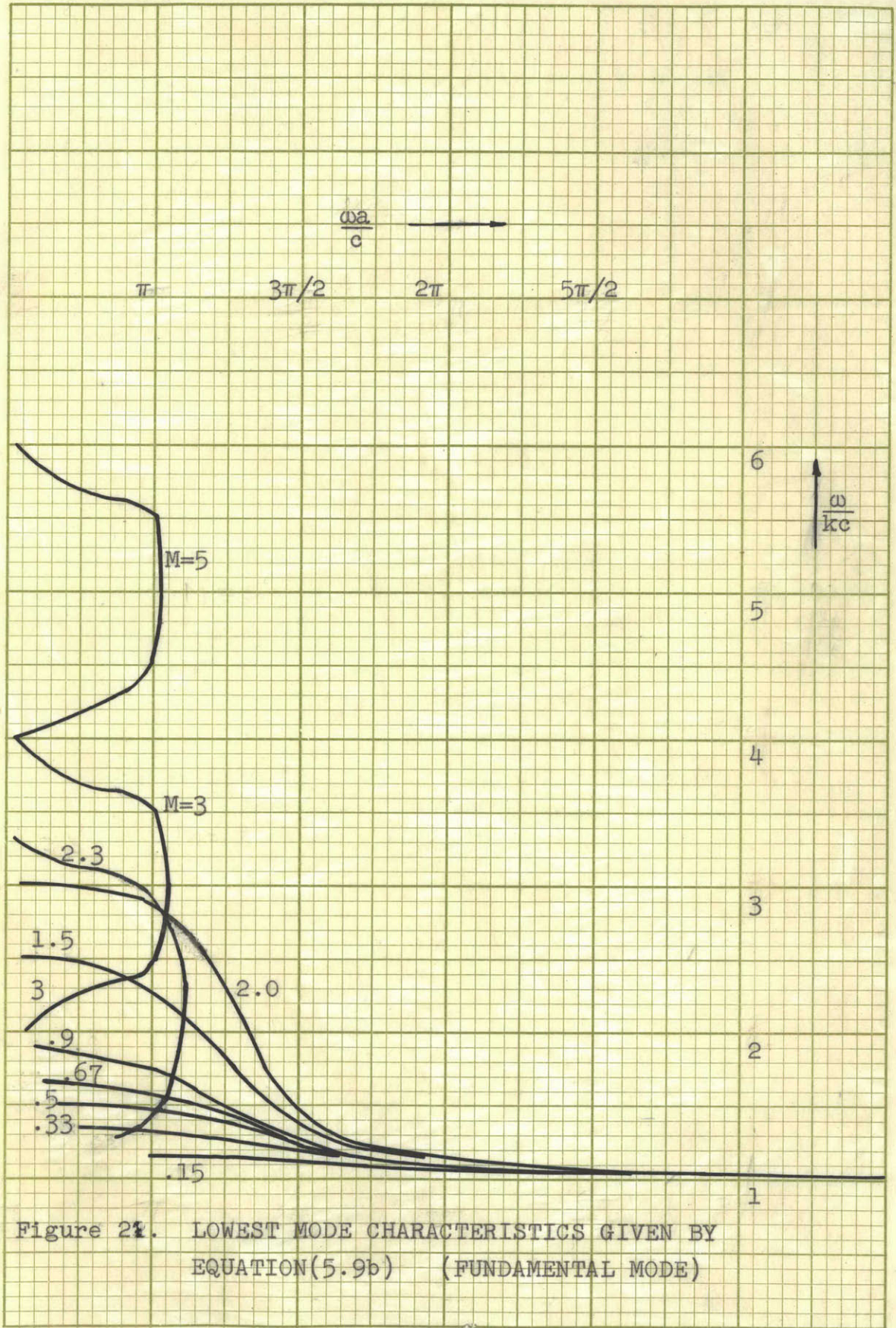


Figure 21. LOWEST MODE CHARACTERISTICS GIVEN BY EQUATION(5.9b) (FUNDAMENTAL MODE)

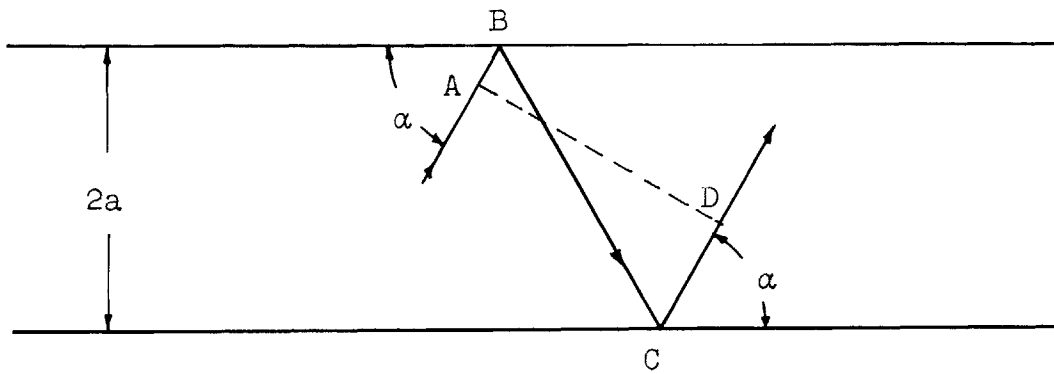


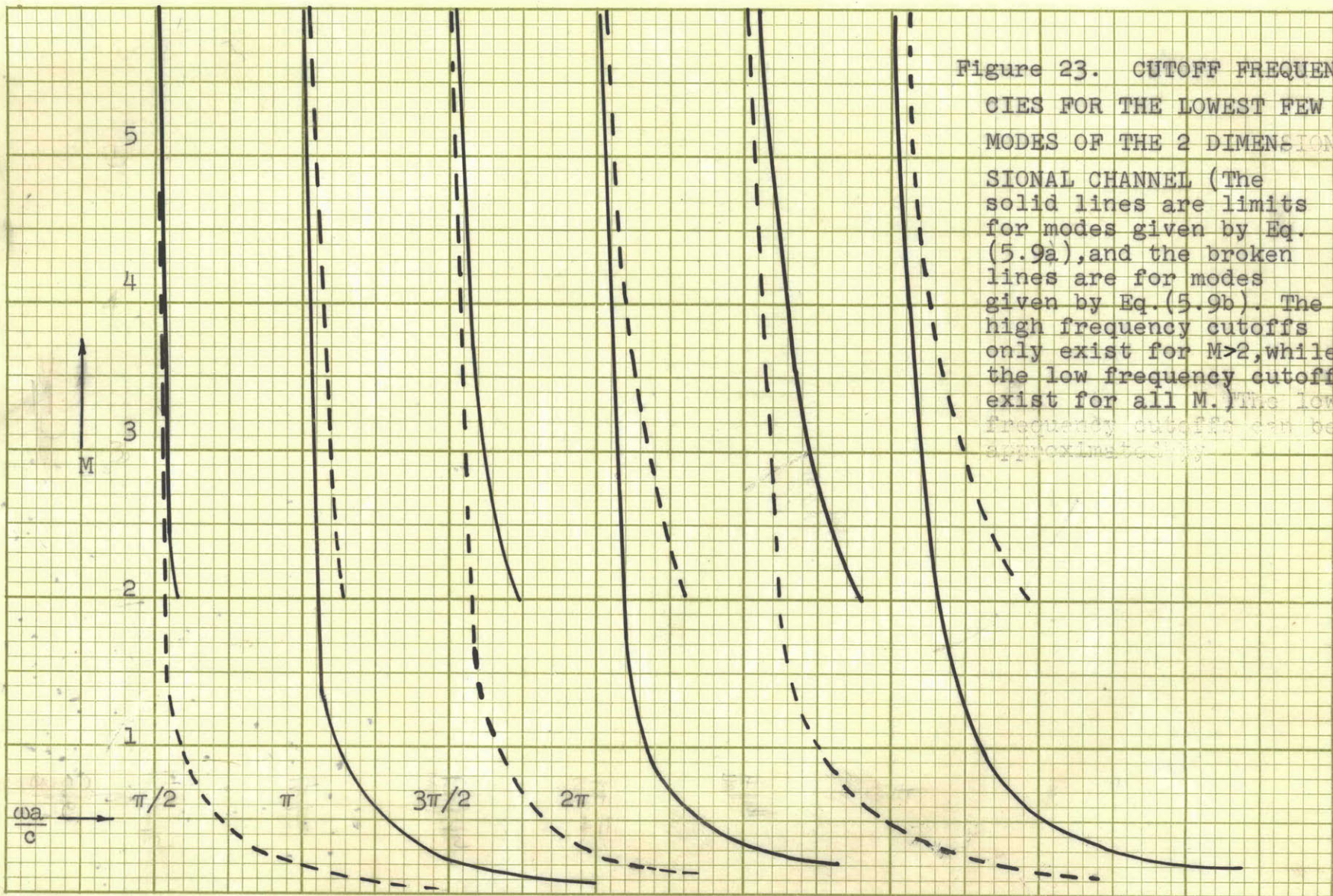
Figure 22. ILLUSTRATION OF THE GEOMETRICAL CONDITIONS NECESSARY FOR A PROPAGATING MODE

amplitude factor of Eq. (3.11), and the rest of the phase change of ABCD is due to the path lengths $\overline{AB} + \overline{BC} + \overline{CD}$. This gives the equation

$$(4a\omega/c) \sin \alpha + 2\epsilon + 2n\pi = 0. \quad (5.9c)$$

For $n = 0$, this gives Eq. (5.9a), and, for $n = 1$, this gives Eq. (5.9b). If $M < 2$, then it is only necessary for α to be less than the critical angle for total reflection given by $\cos \alpha = 1/(1 + M_2)$, and the above discussion applies. However, if $M > 2$, then α must satisfy the additional requirement $\cos \alpha < 1/(M_2 - 1)$ in order to have total reflection. Now, as $a\omega/c$ increases, $\sin \alpha$ must decrease to satisfy Eq. (5.9c) for any given mode, so if $M > 2$, large enough $a\omega/c$ will not allow a solution of Eq. (5.9c), hence the high frequency cut off. Furthermore, the phase shift ϵ now has the same value for two different values of α , and thus the total phase shift from A to D can be the same for two different values of α , and the modes will be split.

The lowest antisymmetric mode [solution of Eq. (5.9b)] is shown in Fig. 21. This corresponds to the fundamental mode of an ordinary guide, with the usual low frequency cut off at $(\omega a/c) = \pi/2$. This mode also exhibits the splitting and high frequency cut off of the previous case, and, for the same reasons as the previous case. The cut offs for typical modes are plotted in Fig. 23. It is seen that there are low frequency cut offs for all of the modes except the lowest symmetric mode already discussed.



There is no general process for finding the complex roots of a transcendental equation such as Eq. (5.9). The equations can be solved approximately for very large or for very small relative velocities, but these solutions are not very illuminating.

III. Evaluation of Integrals

The asymptotic expansions of the integrals of ϕ_2 and ϕ_3 can be done in exactly the same manner as was used for the transmitted waves due to a line source near a single interface. There is no convenient asymptotic approximation for ϕ_1 if we try to include all of the waves due to the poles of the integrand. But, if we neglect the contribution of the poles, the calculation can be done just as for ϕ_2 and ϕ_3 . We simply take the values $\gamma_1, \gamma_2, \omega_1, \omega_2$, and k_0 from the two-dimensional saddle point calculation and substitute into the saddle point formula. Thus, the leading term of the asymptotic expansion can be expressed in closed, but rather complicated, form. If we use the polar coordinates $x = r \sin \theta$, $y = r \cos \theta$, then the sound field can be written as

$$\phi_2 = \left(\sqrt{2\pi i} \sin \theta e^{i\omega t - \gamma_2 x - ik_0 y} / \sqrt{(rc/\omega)} (1 - M_2^2 \sin^2 \theta)^{3/4} \right) A_{k=k_0} \quad (5.10)$$

$$\phi_1 = \left(\sqrt{2\pi i} \sin \theta e^{i\omega t - ik_0 y} / \sqrt{(rc/\omega)} (1 - M_1^2 \sin^2 \theta)^{3/4} \right) \left(e^{-\gamma_2 x} B_{k=k_0} + e^{\gamma_2 x} C_{k=k_0} \right) \quad (5.11)$$

$$\phi_3 = \left(\sqrt{2\pi i} \sin \theta e^{i\omega t + \gamma_2 x - ik_0 y} / \sqrt{(rc/\omega)(1 - M_2^2 \sin^2 \theta)^{3/4}} \right) D_{k=k_0} \quad (5.12)$$

It must be remembered that the values of $\gamma_1, \gamma_2, \omega_1, \omega_2, k_0$ used in Eq. (5.11) are not the same as those used in Eq. (5.10) and in Eq. (5.12). Although these expressions appear formidable, it is possible to obtain some simple physical insight from them. Since $\cosh z = \sinh z = (1/2)e^z$ for $z \gg 1$, we see that when γ_1 is real and when $\gamma_1 a, \gamma_1 h \gg 1$ [in Eq. (5.10)], then the amplitude factor of ϕ_2 is reduced by the factor $e^{\gamma_1(h-a)}$. This corresponds to the "zone of silence" in the one interface case. This is to be expected since, if the source is near one wall of the channel, the opposite wall has little effect if it is many wave lengths away. Under such conditions, we are essentially back to the one-interface problem.

The extension to supersonic flows proceeds exactly as for the one-interface problem. We can also calculate the higher-order plane wave front terms which turn out to be similar to those done previously.

IV. Cylindrical Channel

The above treatment can be extended to a point source located in a cylindrically symmetric channel or jet. Such a geometrical situation is shown in Fig. 24. The point source is located at $r = r_0, z = 0$, and $\psi = 0$, with $r_0 < a$ (the radius of the jet). The field of a point source can be written in cylindrical coordinates as

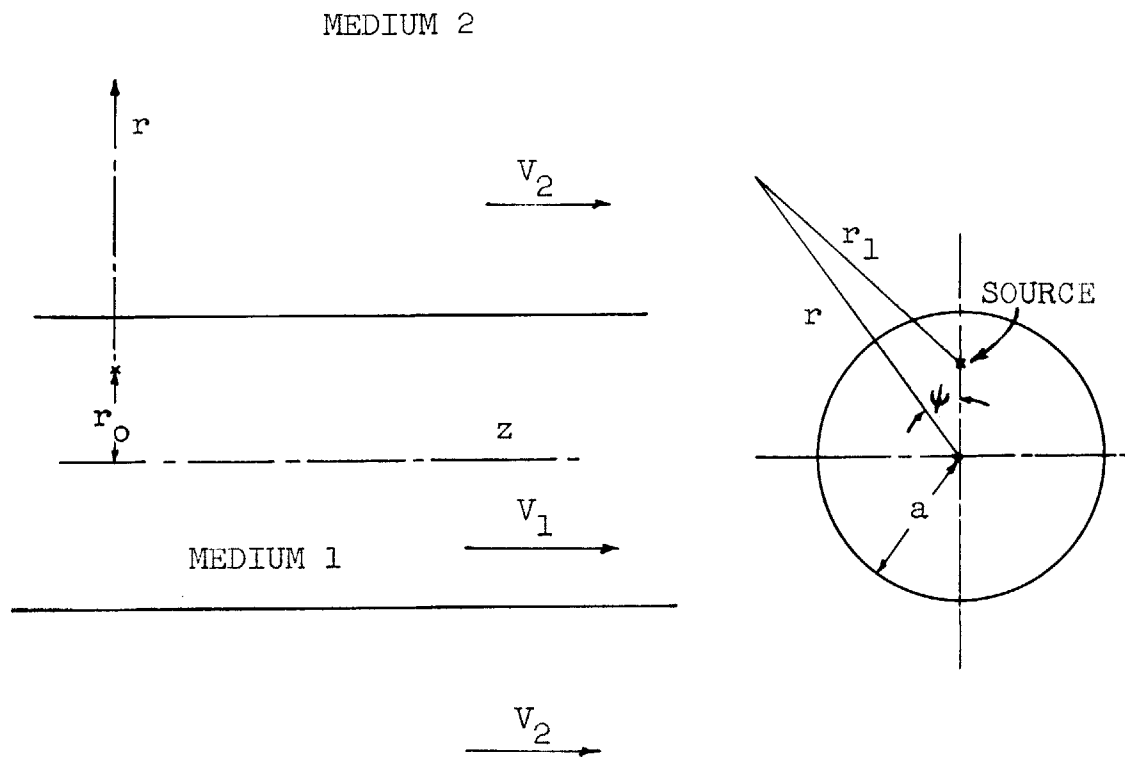


Figure 24. CYLINDRICALLY SYMETRIC VELOCITY DISCONTINUITY CHANNEL OR JET

$$[(2ie^{i\omega t - (i\omega/c)\sqrt{r_1^2 + z^2}})/\sqrt{r_1^2 + z^2}] = \int_{-\infty}^{+\infty} e^{+ikz} H_0^{(2)}(r_1 \lambda_1) dk \quad .$$

$$\lambda_1 = \sqrt{(\omega_1^2/c^2) - k^2}$$

where r_1 is the distance from the source to the point of observation.

By the well-known wave expansion,

$$H_0^{(2)}(\lambda_1 r) = \sum_{-\infty}^{+\infty} J_m(\lambda_1 r_0) H_m^{(2)}(\lambda_1 r) e^{im\psi} \quad (\text{for } r > r_0) \quad .$$

The source term may be written, in terms of the appropriate coordinates for the jet, as

$$\phi_0 = \sum_{m=-\infty}^{m=+\infty} e^{im\psi} \int_{-\infty}^{+\infty} e^{-ikz} J_m(\lambda_1 r_0) H_m^{(2)}(\lambda_1 r) dr \quad (\text{for } r > r_0) \quad . \quad (5.13)$$

This is seen to reduce to cylindrical symmetry when $r_0 = 0$, since only $J_0 \neq 0$. For $r_0 \neq 0$, the reflected wave in medium 1 is written

$$\phi_1 = \sum_{m=-\infty}^{\infty} e^{im\psi} \int_{-\infty}^{+\infty} A_m(k) e^{-ikz} H_m^{(1)}(\lambda_1 r) dk \quad (\text{for } r < a) \quad . \quad (5.14)$$

The transmitted wave in medium 2 is written as

$$\phi_2 = \sum_{m=-\infty}^{+\infty} e^{im\psi} \int_{-\infty}^{+\infty} B_m(k) e^{-ikz} H_m^{(2)}(\lambda_2 r) dk \quad (\text{for } r > a,$$

$$\text{and with } \lambda_2 = \sqrt{(\omega_2^2/c^2) - k^2} \quad). \quad (5.15)$$

The boundary conditions become a set of pairs of equations

$$\omega_1 [J_m(\lambda_1 r_0) H_m^{(2)}(\lambda_1 a) + A_m H_m^{(1)}(\lambda_1 a)] = \omega_2 B_m H_m^{(2)}(\lambda_2 a)$$

$$(\lambda_1/\omega_1) [J_m(\lambda_1 r_0) H_m^{(2)\prime}(\lambda_1 a) + A_m H_m^{(1)\prime}(\lambda_1 a)] = (\lambda_2/\omega_2) B_m H_m^{(2)\prime}(\lambda_2 a) .$$

These give the solution

$$A = \frac{J_m(\lambda_1 r_0) [\lambda_2 \omega_1^2 H_m^{(2)}(\lambda_1 a) H_m^{(2)}(\lambda_2 a) - \lambda_1 \omega_2^2 H_m^{(2)}(\lambda_1 a) H_m^{(2)}(\lambda_2 a)]}{\lambda_1 \omega_2^2 H_m^{(1)\prime}(\lambda_1 a) H_m^{(2)}(\lambda_2 a) - \lambda_2 \omega_1^2 H_m^{(1)}(\lambda_1 a) H_m^{(2)\prime}(\lambda_2 a)} \quad (5.16)$$

$$B_m = \frac{\lambda_1 \omega_1 \omega_2 J_m(\lambda_1 r_0) [H_m^{(2)}(\lambda_1 a) H_m^{(1)\prime}(\lambda_1 a) - H_m^{(2)\prime}(\lambda_1 a) H_m^{(1)}(\lambda_1 a)]}{\lambda_1 \omega_2^2 H_m^{(2)}(\lambda_2 a) H_m^{(1)\prime}(\lambda_1 a) - H_m^{(2)\prime}(\lambda_2 a) H_m^{(1)}(\lambda_1 a) \lambda_2 \omega_1^2} . \quad (5.17)$$

The vanishing of the denominator gives the propagation modes of the channel, just as in Eqs. (5.9a) and (5.9b) for the plane two-dimensional case. This characteristic equation is much more complicated than Eq. (5.9a) and Eq. (5.9b) because the derivatives of Hankel functions are not simply Hankel functions of the same order, while sin and cos go into each other upon differentiation.

If we use the asymptotic form of $H_m^{(2)}$ for large $\lambda_2 r$, in Eq. (5.15), the integrand assumes the familiar form which is directly capable of a saddle point integration. If we let $i\lambda = \gamma$, we can use the saddle point calculation of the quantities $\gamma_1, \gamma_2, \omega_1, \omega_2, k_0, f^{\#}(k_0)$, just as was done to obtain Eqs. (5.10), (5.11), and (5.12). Thus, we can write

$$\phi_{2m} = \left(2i \sin \theta / [R \sqrt{\lambda_2 c / \omega} (1 - M_2^2 \sin^2 \theta)^{3/4}] \right) e^{i\omega t - ik_0 z - i\lambda_2 r} B_{m(k=k_0)} \quad (5.18)$$

$$\phi_2 = \sum_{m=-\infty}^{+\infty} e^{im\psi} \phi_{2m} \quad .$$

(To retain the unity of presentation, we have made the substitution $r = R \sin \theta$, $z = R \cos \theta$. The sound field has the dimensions of $1/R$ since that is how the source was defined.)

FLOW DISCONTINUITY WITH PLATE OR MEMBRANE SEPARATION

I. Formal Integral Solutions

Most of the preceding results concerning a velocity discontinuity interface can be generalized to include a plate or membrane at the interface. Such a situation is the same as that shown in Fig. 5 (page 24) with the plate or membrane at the interface between medium 1 and medium 2. Only the two-dimensional case (line source) is discussed here since the extension to three dimensions follows the same pattern as the simple interface discussed in Chapter 4. The integral expressions for the sound field can be written in the same manner as before:

$$\phi_0 = e^{i\omega t} \int_{-\infty}^{+\infty} (e^{-iky - \gamma_1|x+h|} / \gamma_1) dk \quad x < 0 \quad , \quad (6.1)$$

$$\phi_1 = e^{i\omega t} \int_{-\infty}^{+\infty} A(k)e^{-iky + \gamma_1 x} dk \quad x < 0 \quad , \quad (6.2)$$

$$\phi_2 = e^{i\omega t} \int_{-\infty}^{+\infty} B(k)e^{-iky - \gamma_2 x} dk \quad x < 0 \quad . \quad (6.3)$$

In addition, the displacement of the plate can be written as

$$W = e^{i\omega t} \int_{-\infty}^{+\infty} D(k)e^{-iky} dk \quad x = 0 \quad . \quad (6.4)$$

A plate may have thickness τ , but we assume that τ is much smaller than any other length in the problem. Consequently, if σ is used for the mass per unit area, the quantity τ need not appear in the equations ($\sigma = \rho_p \tau$ for the plates). The equation of motion for the plate can be written as

$$\sigma(\partial^2 w / \partial t^2) = -[E\tau^3 / 12(1 - S^2)]\nabla^4 w + p_1 - p_2 \quad (6.5)$$

(where E = Young's modulus and S = Poisson's ratio for the plate

$$p_1 = i\omega_1(\phi_0 + \phi_1)\rho \quad p_2 = i\omega_2\phi_2\rho \quad). \quad \text{If}$$

$$[E\tau^2 / 12\rho_p(1 - S^2)] = (V_f^4 / \omega^2) \quad ,$$

then Eq. (6.5) can be written as

$$(\partial^2 w / \partial t^2) = - (V_f^4 / \omega^2)\nabla^4 w + (i\rho/\sigma)[\omega_1(\phi_0 + \phi_1) - \omega_2\phi_2] \quad ,$$

and this equation can be Fourier transformed to

$$D\omega^2[-1 + (k^4/k_f^4)] = (i\rho/\sigma)(\omega_1[A + (e^{-\gamma_1 h}/\gamma_1)] - \omega_2 B) \quad (6.5a)$$

(where $k_f = \phi/V_f$). This equation replaces the continuity of pressure boundary condition in the simple interface case. The displacement on each side of the plate must be equal to the displacement of the plate w . Since w does not vary through the negligible thickness of the plate, the remaining boundary conditions become

$$D = (\gamma_2 B / i\omega_2) \quad , \quad (6.6)$$

and

$$D = (e^{-\gamma_1 h}/i\omega_1) - (A\gamma_1/i\omega_1) \quad . \quad (6.7)$$

The case of the membrane is quite similar. Instead of Eq. (6.5), we have

$$\sigma(\partial^2 w/\partial t^2) = T\nabla^2 w + p_1 - p_2 \quad (T = \text{tension of membrane}). \quad (6.8)$$

If we write $c_m^2 = T/\sigma$, then Eq. (6.8) becomes

$$D\omega^2 [(k^2/k_m^2) - 1] = (i\rho/\sigma)(\omega_1[A + (e^{-\gamma_1 h}/\gamma_1)] - \omega_2 B) \quad . \quad (6.8a)$$

The remaining two boundary conditions for the membrane are Eqs. (6.6) and (6.7), the same as for a plate. The case of no plate or membrane, but with surface tension, is simply obtained from Eq. (6.8a) by letting $\sigma \rightarrow 0$. This leads to

$$Dk^2 T = (i\rho/\sigma)(\omega_1[A + (e^{-\gamma_1 h}/\gamma_1)] - \omega_2 B) \quad . \quad (6.8b)$$

If (6.5a), (6.6), and (6.7) are solved simultaneously, one obtains

$$A = \frac{\gamma_1 \omega_2^2 - \gamma_2 \omega_1^2 + (\sigma\gamma_1\gamma_2\omega^2/\rho)[1 - (k^4/k_f^4)]}{\gamma_1 \omega_2^2 + \gamma_2 \omega_1^2 + (\sigma\gamma_1\gamma_2\omega^2/\rho)[1 - (k^4/k_f^4)]} (e^{-\gamma_1 h}/\gamma_1) \quad , \quad (6.9)$$

$$B = 2\omega_1\omega_2 e^{-\gamma_1 h} / \left(\gamma_1 \omega_2^2 + \gamma_2 \omega_1^2 + (\sigma\gamma_1\gamma_2\omega^2/\rho)[1 - (k^4/k_f^4)] \right) \quad . \quad (6.10)$$

For the membrane [Eq. (6.8a) instead of (6.5a)], the solutions are identical except that k^4/k_f^4 is replaced by k^2/k_m^2 .

II. Discussion of Characteristic Equation

Before evaluating the integrals for ϕ_1 and ϕ_2 asymptotically, we must examine the poles of A and B. These are given by

$$\gamma_1 \omega_2^2 + \gamma_2 \omega_1^2 + (\sigma \gamma_1 \gamma_2 \omega^2 / \rho) [1 - (k^4 / k_f^4)] = 0. \quad (6.11)$$

If the plate is very light and flexible, the first two terms of (6.11) will dominate, and the roots will be close to those of the Helmholtz instability. From intuitive reasoning, we expect that a plate or membrane would tend to inhibit this instability. To check this, we make the approximation

$$\omega / kc = \beta = \beta_0 + \epsilon,$$

and assume $\epsilon \ll \beta_0$, so that a perturbation calculation can be made with powers of $\sigma \omega / \rho c$. Using

$$\beta_0 = (M/2) \pm i[\sqrt{1 + M^2} - 1 - (M^2/4)]^{1/2}; \quad M = M_2, \quad M_1 = 0,$$

the first-order perturbation for the membrane gives

$$\epsilon = \frac{(\sigma \omega / \rho c) [\mp (iM/2) - \sqrt{\sqrt{1 + M^2} - 1 - (M^2/4)}] [c_m^2 / (c^2 \sqrt{1 + M^2} - 1)]}{2\sqrt{2} \sqrt{1 + \sqrt{1 + M^2} - (M^2/2)} \sqrt{\sqrt{1 + M^2} - 1 - (M^2/4)} \sqrt{1 + M^2}} \quad (6.12)$$

Thus, the imaginary part of ϵ is opposite in sign from $\text{Im} \beta_0$ and the instability is inhibited provided

$$c_m^2 / [c^2 (\sqrt{1 + M^2} - 1)] > 1.$$

This means that the membrane must be more springlike than mass-like. For the surface tension case [Eq. (6.8b)], this inequality is always satisfied, since the right-hand side is zero, so that the surface tension will always inhibit the instability.

If the same perturbation is made for a plate, one obtains

$$\epsilon = \left(\mp (i\sigma\omega/\rho c) [(V_f^4/c^4)\beta_0^{-3} - \beta_0] \right) / \left(2\sqrt{2}\sqrt{1+\sqrt{1+M^2}} - (M^2/2) \sqrt{\sqrt{1+M^2} - 1} - (M^2/4)\sqrt{1+M^2} \right) \quad (6.13)$$

Since

$$\beta_0^{-3} = \frac{\frac{M^3}{2} + \frac{3}{2}M(1 - \sqrt{1+M^2}) \mp i\sqrt{\sqrt{1+M^2} - 1} - \frac{M^2}{4}(M^2 + 1 - \sqrt{1+M^2})}{\frac{M^3}{2} + \frac{3}{2}M(1 - \sqrt{1+M^2}) + (\sqrt{1+M^2} - 1 - \frac{M^2}{4})(M^2 + 1 - \sqrt{1+M^2})^2}$$

it is evident that $\text{Re}(\beta_0^{-3})$ is negative. Thus, Eq. (6.13) implies that the imaginary part of ϵ is the same sign as the imaginary part of β_0 . This, in turn, seems to imply the paradoxical result that the plate enhances the Helmholtz instability. This, however, is not so surprising when we realize that the force which restores a plate to equilibrium is of a fundamentally different nature from the membrane. The plate has a stiffness which bends a displacement back into equilibrium while the membrane has a tension which pulls the displacement back to equilibrium. Thus, a cusp-shaped disturbance (as could be obtained by poking the membrane with a needle) is pulled back to equilibrium by the membrane, but such a disturbance can only be obtained by breaking a plate, and can-

not be inhibited unless the plate strength is so large that the above approximation does not apply. Now, the Helmholtz instability is not cusp-shaped, but the increasing amplitude of a given wave length makes it tend toward that shape. Thus, the plate cannot slow down the Helmholtz instability, but it can suppress it entirely, if the plate strength is large enough.

In the opposite limit from that considered above, we can consider the surrounding fluid as a perturbation on the plate or membrane. In this case, the perturbation parameter is just the inverse of the one used previously, namely, $\rho c/\sigma\omega$. We naturally start the perturbation from the zeroes of the last term of Eq. (6.11). These are the plate zeroes

$k = k_f, -k_f, ik_f, -ik_f$ (for the membrane only the first two roots apply), and the familiar branch points for $\gamma_1 = 0, \gamma_2 = 0$

$k = (\omega/(c + V_1), \omega/(V_1 - c), \omega/(V_2 + c), \omega/(V_2 - c))$.

These branch points become poles when $V_1 = V_2$ so that they coalesce in pairs to give two poles from four branch points. These poles correspond to waves travelling along the plate with the same wave length and phase velocity as sound waves in the fluid travelling parallel to the plate. To calculate the perturbation of these poles, let

$$k = \omega/(V \pm c) + \epsilon = k_0 + \epsilon \quad \gamma_1 = \gamma_2 \approx \sqrt{\pm 2\omega\epsilon/c} \quad (6.14)$$

then

$$\epsilon = \pm 2c^5 k_0^4 k_f^8 \rho^2 / \sigma^2 \omega^5 (k_f^4 - k_0^4)^2 .$$

If we use the same perturbation for the membrane, then

$$\epsilon = \pm 2c^5 k_0^4 k_m^4 \rho^2 / \sigma^2 \omega^5 (k_m^2 - k_0^2)^2 . \quad (6.15)$$

In the limit $V = 0$, Eq. (6.14) becomes the solution given by George Lamb in his thesis.⁽¹⁰⁾ It should also be noted, in passing, that when $V_1 \neq V_2$ this solution is replaced by the straight wave-front waves with faster than cylindrical fall off, which we discussed previously. Physically speaking, the wave of Eqs. (6.14) or (6.15) can't exist for $V_1 \neq V_2$, since it would have to radiate all of its energy into the slower moving medium.

If we follow the above perturbation procedure starting with $k = \pm k_m$, then, we get

$$k = \pm k_m \left[1 + \rho [1 \mp (V_1 k_m / \omega)]^2 / \left(2\sigma \sqrt{k_m^2 - (\omega - V_1 k_m / c)^2} \right) + \rho [1 \mp (V_2 k_m / \omega)]^2 / \left(2\sigma \sqrt{k_m^2 - (\omega - V_2 k_m / c)^2} \right) \right] \quad (6.16)$$

for the membrane, and if we use $k = \pm k_f$, we get

$$k = \pm k_f \left[1 + \rho [1 \mp (V_1 k_f / \omega)]^2 / \left(4\sigma \sqrt{k_f^2 - (\omega - V_1 k_f / c)^2} \right) + \rho [1 \mp (V_2 k_f / \omega)]^2 / \left(4\sigma \sqrt{k_f^2 - (\omega - V_2 k_f / c)^2} \right) \right] \quad (6.17)$$

for the plate. In the case of the plate, we can also start from

$k = \pm ik_f$ to get the same thing as Eq. (6.17), but with k_f replaced by ik_f . Equation (6.17) goes over into the expression given by George Lamb for the case $V_1 = V_2 = 0$. From Eqs. (6.16) and (6.17), we can observe the conditions necessary for k to be real (propagating wave). This is just the requirement that γ_1 and γ_2 be real. This means that (for the membrane)

$$k_m > \omega/c(1+M_1) \quad \text{or} \quad k_m < \omega/c(M_1-1) \quad (\gamma_1 \text{ real}) \quad (6.18)$$

and

$$k_m > \omega/c(1+M_2) \quad \text{or} \quad k_m < \omega/c(M_2-1) \quad (\gamma_2 \text{ real}) .$$

(The conditions for the plate are exactly the same; merely change k_m to k_f .) These results are in agreement with the well-known fact that the phase velocity in a plate must be less than the sound velocity in the surrounding medium in order for the flexural wave to propagate unattenuated. When flow is present, the critical phase velocity is increased in the direction of flow and decreased in the opposite direction, as shown by the conditions (6.18). This change in critical phase velocity agrees, quantitatively, with the change in sound velocity relative to the stationary plate, as would be expected. The position of the poles given by Eq. (6.17) is shown in Fig. 25, along with the position of the branch points. The perturbation term of Eq. (6.17) has been denoted by a , and the perturbation term for the poles near the imaginary axis has been denoted by a^{\dagger} . The figure is drawn for the case that the in-

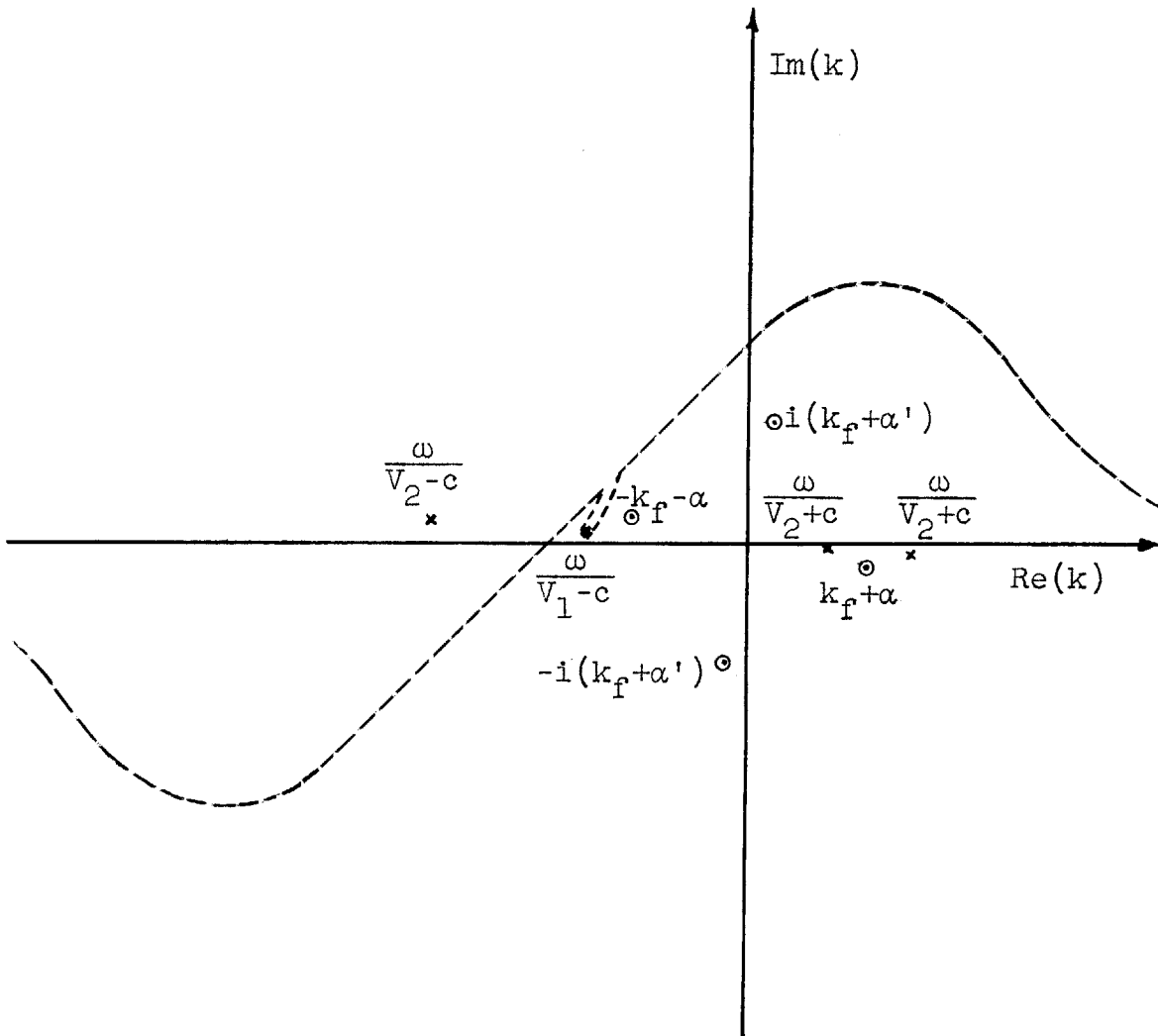


Figure 25. POLES, BRANCH POINTS, AND THE SADDLE PATH FOR THE INTEGRATION OF EQUATION(6. 2) OR (6. 3)

equalities of Eq. (6.18) are not satisfied and α is imaginary so that the poles are displaced from the real axis as shown. We also note, in passing, that the pairs of branch points on each side of the imaginary axis coalesce into poles when $V_1 = V_2$, as was mentioned earlier.

III. Evaluation of Transmitted Field

The integration to obtain the sound field now proceeds in a method similar to that used for the simple interface problem. The saddle point is found for the exponential of Eqs. (6.2) or (6.3). This k_0 is the same function of M_1 , M_2 , and the space coordinates as was obtained previously for the line source without the plate. Then, the path of integration is distorted from the real axis so that it crosses the saddle point in the proper direction. A typical situation is shown in Fig. 25, which is the same as previous saddle paths for the simple interface case. The only additional feature is that, to distort the path from the real axis to that shown in the figure, we must cross the two poles shown circled in the upper half plane. This will happen whenever $k_0 < \text{Re}(k_f + \alpha)$, and a similar situation will occur in the lower half plane when k_0 is positive and $k_0 > \text{Re}(k_f + \alpha)$. Recalling that $[\omega/(V_2 - c)] < k_0 < [\omega/(V_2 + c)]$ for the transmitted field, it is evident that the poles must always be between these branch points to satisfy this condition. Thus, α is complex with imaginary part displacing the poles from the real axis as shown. For $\text{Re}(-k_f - \alpha) > \omega/(V_1 - c)$, α will be totally imaginary, but even if this pole is between the two branch points of the left

half plane, the real part of a is quite small as compared with k_f .

Thus, the condition for the poles to contribute to the transmitted field is

$$k_0 \leq -k_f \quad \text{or} \quad k_0 \geq k_f .$$

To obtain this relation in terms of spatial coordinates, we change to $x = r \sin \theta$, $y = r \cos \theta$. Then, the previously calculated value of k_0 is used to obtain

$$\begin{aligned} \left(\frac{\omega}{[c(1 - M_2^2)]} \right) (-M_2 + \Delta_2 \cos \theta) &\leq -k_f && \text{or} \\ k_f &\leq \left(\frac{\omega}{[c(1 - M_2^2)]} \right) (-M_2 + \Delta_2 \cos \theta) && (6.19) \end{aligned}$$

(where Δ was defined on page 29). The angles which do not satisfy this relation are called the "shadow zone" in Fig. 26. Of course, if the poles are not between the branch points, there will be no angles which satisfy the necessary conditions so that the "shadow zone" of Fig. 26 will expand to cover all of medium 2. This is because the flexural waves in the plate or membrane will not radiate since their phase velocity will always be less than the velocity of sound. The tilting of the shadow zone away from symmetry is in accord with a generalization of Eq. (6.18).

Inside the "shadow zone" the field is simply that due to the saddle point integration at k_0 and the branch cut around $\omega/(V_1 - c)$. Outside the shadow zone we must also add the residue from the poles in the upper half plane. If the saddle point integration is performed exactly

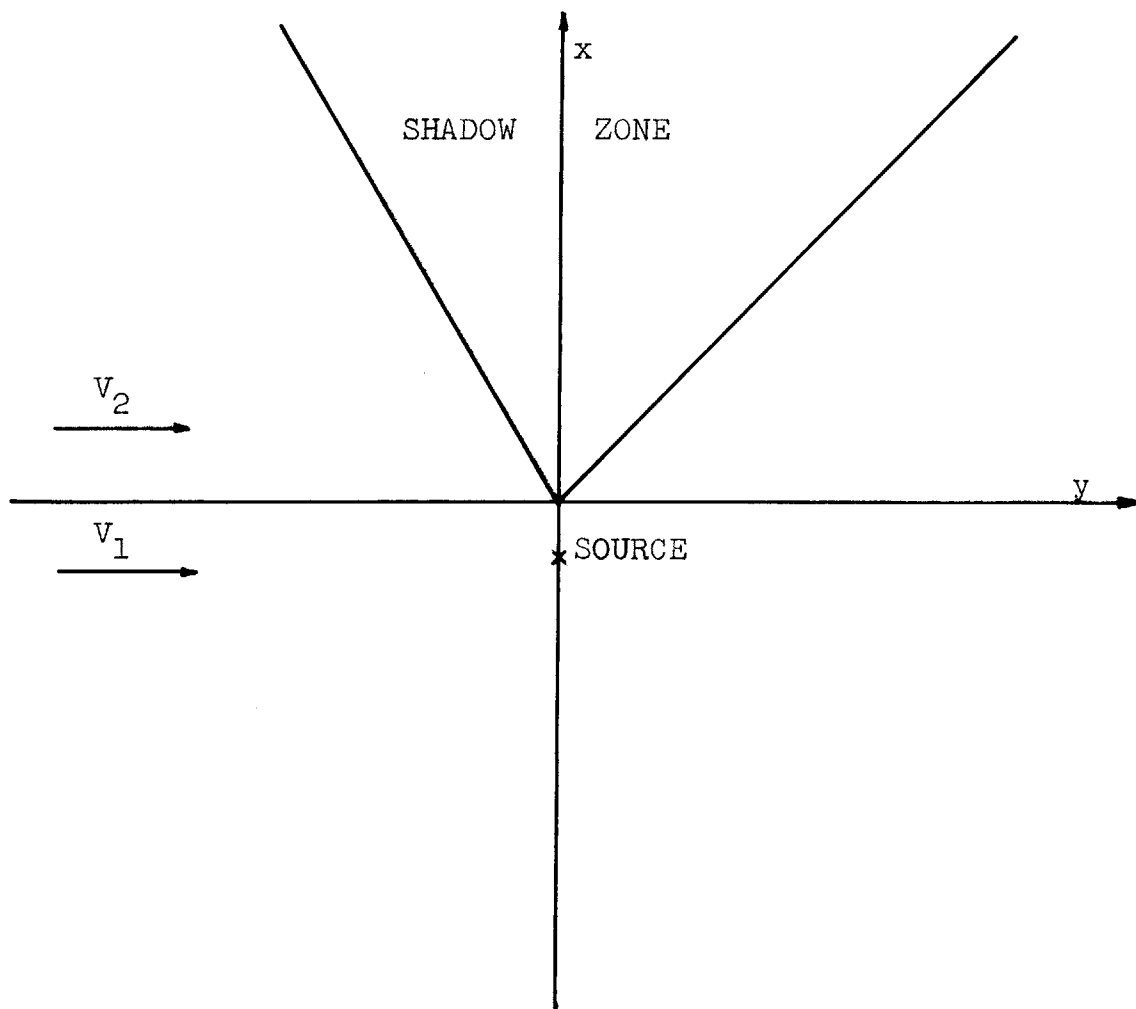


Figure 26. SHADOW ZONE WHERE SURFACE WAVES DO NOT RADIATE (The boundaries of this shadow zone are given by the equalities of equation(6.19).)

as before, we obtain the contribution

$$\phi_2^{(s)} = \frac{2\sqrt{2\pi i} \omega_1 \Delta_2^{3/2} \omega_2 e^{i\omega t - ik_0 y - \gamma_2 x - \gamma_1 h}}{\left[\sqrt{rc/\omega} \left(\gamma_1 \omega_2^2 + \gamma_2 \omega_1^2 + (\sigma \gamma_1 \gamma_2 / \rho) [1 - (k_0^4 / k_f^4)] \right) \right]}. \quad (6.20)$$

$\gamma_1, \gamma_2, \omega_1, \omega_2$ are evaluated at

$$k_0 = \left(\omega / [c(1 - M_2^2)] \right) (-M_2 + \Delta_2 \cos \theta)$$

which makes Eq. (6.20) quite complicated when written out in detail.

We can see that the denominator will have a minimum when $\gamma_1 = 0$ just as in the simple interface case; this will also be the boundary of the "zone of silence," since the exponential in Eq. (6.20) is the same as for the simple interface case. In addition, the denominator of Eq. (6.20) will have another minimum at $k_0 = k_f$ (the boundary of the "shadow zone" described previously). These minima will give peaks in the angular distribution of $\phi_2^{(s)}$ and the relative strength of these peaks will be determined by the size of the parameter $\sigma\omega/\rho c$ which determines the dominance of the plate or the surrounding medium. For the case of a membrane, we replace k_0^4/k_f^4 by k_0^2/k_m^2 in Eq. (6.20), as is explained following Eq. (6.10).

In the lowest order, the presence of the plate or membrane does not effect the integral along the branch cut. This can be seen by recalling the calculation of ϕ_2^f in the simple interface case. The branch cut was along $\text{Re}\gamma_1 = 0$, so $\gamma_1 = \pm iu$, and all terms multi-

plied by u in the denominator of the integrand were neglected. But, it is seen in Eq. (6.10) that, if terms multiplied by γ_1 in the denominator are neglected, all of the effect of the plate of membrane is also neglected. Thus, the field inside the "shadow zone" is just the sum of $\phi_2^{(s)} + \phi_2^r$, where ϕ_2^r was calculated for the line source near a velocity discontinuity without plate or membrane.

The contribution of the poles is simply $2\pi i$ times the sum of the residues. From the pole close to the real axis, we get

$$\phi_2^{(p)} = 2\pi i \omega_1 \omega_2 e^{i\omega t - iky \gamma_2 x - \gamma_1 h} / (\partial F / \partial k)_{k = -k_f} \quad , \quad (6.21)$$

in which

$$\begin{aligned} (\partial F / \partial k)_{k = -k_f} &= \left([-k_f(1 - M_1^2) + (\omega M_1 / c)] / \gamma_1^0 \right) \omega_2^2 + \gamma_1^0 \omega_2 u_2 + \\ &\left([(\omega M_2 / c) - k_f(1 - M_2^2)] / \gamma_2^0 \right) \omega_1^2 + u_1 \gamma_2^0 \omega_1 + (4\sigma u_1 \gamma_2^0 \omega^2 / \rho k_f) \\ k &= -k_f \left(1 + (\rho / 4\sigma) [(\omega_1^2 / \gamma_1^0) + (\omega_2^2 / \gamma_2^0)] \right) \\ \gamma_1 &= \gamma_1^0 + (\rho k_f / 4\sigma \gamma_1^0) [(\omega_1^2 / \gamma_1^0) + (\omega_2^2 / \gamma_2^0)] [k_f(1 - M_1^2) - (M_1 \omega / c)] \\ \gamma_2 &= \gamma_2^0 + (\rho k_f / 4\sigma \gamma_2^0) [(\omega_1^2 / \gamma_1^0) + (\omega_2^2 / \gamma_2^0)] [k_f(1 - M_2^2) - (M_2 \omega / c)] \quad , \end{aligned}$$

and $\omega_1, \omega_2, \gamma_1^0, \gamma_2^0$ are to be evaluated at $k = -k_f$. In case we wish to consider the region outside the "shadow zone" in the right half plane of Fig. 26, the saddle point must move over to the right half plane of Fig. 25, and the two poles below the real axis are used, so we simply change

the sign of k_f wherever it appears in Eq. (6.21). Equation (6.21) is also valid for a membrane if we use k_m instead of k_f , and replace $\rho/4\sigma$ by $\rho/2\sigma$, as is evident upon comparison of Eqs. (6.16) and (6.17).

Since γ_2^0 is imaginary (or else medium 2 would be all "shadow zone"), γ_2 has a real part and k has an imaginary part, and these cause the exponential to have a real part,

$$(iyk_f\rho/4\sigma)[(\omega_1^2/\gamma_1^0) + (\omega_2^2/\gamma_2^0)] - (xk_f\rho/4\sigma)[(\omega_1^2/\gamma_1^0) + (\omega_2^2/\gamma_2^0)] \times \\ \left(k_f[1 - M_2^2] - (M_2\omega/c) \right)$$

in case γ_1^0 is real, ω_1^2/γ_1^0 is left out of the parentheses in each term. This real part must always be less than zero to avoid the absurdity of an exponentially growing wave. It will equal zero for

$$iy = (x/\gamma_2^0)[k_f(1 - M_2^2) - (M_2\omega/c)]$$

which is the same as $-k_f = k_0$. Thus, (6.21) will always have a decaying exponential outside the "shadow zone." Inside the "shadow zone," the wave does not exist, so there is no problem with the growing exponential. This seems to imply that the wave is propagated unattenuated, and with no spreading, along the boundary of the shadow zone. However, the pole and saddle point are too close together for this calculation to give an accurate answer near the "shadow zone" boundary. The basic result of performing the more exact calculation in this neighborhood is that the amplitude of the pole contribution is reduced as the pole approaches the

saddle path. For the limiting value with the pole precisely on the saddle point, the Cauchy principle value is used, which means the limiting reduction is one-half. This is shown by H. Ott⁽¹¹⁾ in his calculation of a saddle point integral in the neighborhood of a pole.

The contribution from the pole near the imaginary axis has the same form as $\phi_2^{(p)}$ of Eq. (6.21), but k_f is replaced by $-ik_f$, so the wave is strongly damped in all directions.

SUMMARY AND CONCLUSION

It seems rather unsatisfactory to end the thesis at this point, because it has raised more questions than it has answered. The theory of instabilities developed in Chapter 1 must be extended to more-general resonator frequency distributions than that used for Eq. (1.6). Then, one can experiment with the effect of various types of noise spectra upon the onset of instability. Equation (1.6) indicates that the critical Reynolds number is proportional to the square root of the channel thickness. I am not aware of any experimental results which check this dependence directly, and the conventional stability theory gives a result which is independent of thickness, so Eq. (1.6) may prove to be invalid as it stands. However, this would only imply that the nonpropagating disturbances are not primarily responsible for turbulence onset; the nonpropagating disturbance viewpoint is still a natural one for resonator excitation, as is explained in Section IV of Chapter 1.

The results of Chapters 3, 4, and 5 cannot be accurately compared with the jet noise experimental data because the jet does not have a sharp boundary. The present data on the angular dependence of the far field sound pressure is consistent with my results for a source and observer at rest while the medium, which the source is in, is mov-

ing, and it is also consistent with the result for the source at rest with respect to the medium it is in which is moving with respect to the observer, provided the retarded position of the source is taken at the mouth of the jet. This is to be expected when one notes the similarity of Figs. 8 and 10 (pages 31 and 34). (Actually, one should use the much more complicated results of Chapter 5 for comparison with the jet data, but the previously mentioned simplification, using a sharp boundary, makes the results of Chapter 4 inaccurate for a real jet to the extent that the simple interface results of Chapter 3 are probably not much worse.) It would be of great interest to check these results experimentally with the more ideal condition of a sharp boundary.

The theoretical and experimental problems suggested by the results of Chapter 6 are also of great interest. The effect of the Helmholtz instability should be observable by the breaking of weak membranes or plates in flows. The theoretical work should also be extended to semi-infinite or finite plates or membranes, since it is the finite cases which are of practical importance. The problem of a semi-infinite plate is currently under study.

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BIOGRAPHY

Peter Gottlieb was born 29 November 1935 in Cleveland, Ohio. He graduated from Fitzgerald Grammar School, Detroit, Michigan, in 1948; from Le Conte Junior High School, Hollywood, California, in 1950; and from Hollywood High School, Hollywood, California, in 1953. He attended the University of Chicago for one year, and then transferred to California Institute of Technology where he received a Bachelor of Science degree majoring in Physics in June 1956.

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