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ANALYTIC METHODS FOR CALCULATING PERFORMANCE MEASURES OF PRODUCTION LINES WITH BUFFER STORAGES

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Abstract

A Markov Chain model of an unreliable transfer line with interstage buffer storages is introduced. The system states are defined as the operational conditions of the stages and the levels of material in the storages. The steady-state probabilities of these states are sought in order to establish relationships between system parameters and performance measures such as production rate (efficiency), forced-down times, and expected in-process inventory.

A matrix solution that exploits the sparsity and block tri-diagonal structure of the transition matrix is discussed. The steady-state probabilities of the system states are also found analytically, by guessing a sum-of-products form solution for a class of states and deriving the remaining expressions by using the transition equations.

1. Introduction

As complex manufacturing and assembly systems gain more and more importance and as automation develops and enters more areas of production, the optimal design and control of such systems acquires great significance. It is important to understand the relationships between design parameters and the production rate and other performance measures of such systems. The problem is particularly complex when the manufacturing on assembly systems under study involves unreliable components, i.e., parts that fail at random times for random periods.

The work presented here concerns transfer lines. These consist of a series of work stations which serve, process, or operate upon material which flows through these stations. This material may consist of jobs in a computer system, workpieces in an industrial transfer line, vehicles in inspection stations or toll booths, etc. Transfer lines are the simplest non-trivial manufacturing systems. At the same time, they are widespread in industry and have become one of the most highly utilized ways of manufacturing or processing large quantities of standardized items at low cost (Goff [1970]).

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The work stations in the transfer line are assumed to be unreliable, in the sense that they fail at random times and remain inoperable for a random length of time. While the effects of such failures on the production rate of the system may be compensated by providing alternate paths and spare work stations, this practice may often be prohibitively expensive. It is possible to reduce somewhat the inefficiency introduced by the unreliability of the stations by providing the line with interstage buffer storages. These buffers act as temporary storage elements for upstream stations when a downstream station breaks down. Similarly, they provide downstream stations with a temporary supply of jobs or workpieces when upstream stations fail. As a result, the production rate of the system is improved to a certain extent; however, the cost of providing buffers, as well as costs associated with keeping inventory in these buffers, can be significant. It is thus necessary to optimally allocate storage space in order to maximize profit.

This optimization problem may only be solved if the relationship between design parameters (such as the efficiencies, and the average up and down times of individual stations, the capacities of interstage storages) and performance measures (such as line productions rate, in-process inventory) can be adequately quantified. The purpose of this paper is to present exact methods for calculating performance measures given the design parameters of a transfer line. The approach uses an extension of networks of queues theory for finite buffers and service stations subject to failures.

2. Modelling of the Transfer Line

The transfer line is sketched in figure 1. Parts (or jobs, etc.) enter the first station from outside the system. Each part is processed by station 1, after which it is moved into storage 1. The part proceeds in the downstream direction, from station i to storage i to station i+1 and so on. Finally, it is processed by station k and leaves the system.

When station i breaks down, the level in storage i-1 goes up as parts continue to be produced by the upstream portion of the line. At the same time, the level in storage i goes down as the downstream portion of the line continues to drain its contents.

If the failure lasts long enough, storage i-1 may fill up, blocking station i-1 and causing it to be forced down. Similarly, storage i may empty, starving station i+1 and causing it to be forced down. For each station, the variable α_i is defined such that

$$\alpha_i = \begin{cases} 0 \text{ if station i is under repair} \\ 1 \text{ if station i is operational} \end{cases} \tag{2.1}$$

Furthermore, the level of storage is given by n_i, so that

$$0 \leq n_i \leq N_i \tag{2.2}$$

where N_i is the capacity of storage i. Then, the state of a k-stage transfer line is given by the set of numbers

$$s = (n_1, \ldots, n_{k-1}, \alpha_1, \ldots, \alpha_K) \tag{2.3}$$

It follows from equation (2.1) and (2.2) that the number of distinct system states for a k-stage line is

$$m = 2^k (N_1+1) \ldots (N_{k-1}+1) \tag{2.4}$$

The following assumptions are made:

i) The first station is never starved, and the last station is never blocked.

ii) All stations operate at equal and deterministic rates. Transportation takes negligible time compared to the service time.

iii) Stations have geometrically distributed times between failures and times to repair. Thus, at every time cycle, there is a constant probability of failure p_i given that the station is processing a part. Similarly, there is a constant probability of repair r_i given that the station is down. Furthermore, a station can only fail if it is processing a part (i.e., a starved or blocked station cannot fail).

iv) Parts are not destroyed or rejected by stations, or added from outside to intermediate points in the line.

v) The probabilistic model of the system is analyzed in steady state .

Under the above assumptions, a Markov chain model may be formulated. The steady-state probabilities of the system states (as defined by equation (2.3)) are used to compute the probability of producing a finished part during any time cycle, the probability of a station being forced down, and the average in-process inventory.

3. A Matrix Solution

The state probabilities discussed above may be obtained by simply solving simultaneously all the state transition equations. For all states s(t+1) at time t+1, it is possible to write transition equations of the form

$$p[s(t+1),t+1] = \sum_{\text{all } s(t)} p[s(t+1)|s(t)] \ \ p[s(t),t] \tag{3.1}$$

If the steady-state probability vector is denoted by $\underline{p}$, so that

$$\underline{p} \triangleq \begin{bmatrix} p[s_1] \\ \vdots \\ p[s_m] \end{bmatrix} \tag{3.2}$$

and the one-step transition matrix (from time t to t+1) is denoted by T, so that

$$T \triangleq \begin{bmatrix} T_{11} & \cdots & T_{1m} \\ \vdots & & \vdots \\ T_{m1} & \cdots & T_{mm} \end{bmatrix} \tag{3.3}$$

where

$$T_{ij} \triangleq p[s_i(t+1)|s_j(t)]$$

it follows that

$$\underline{p} = T\underline{p}$$
$$(T-I)\underline{p} = \underline{0} \tag{3.4}$$

Furthermore, all probabilities must sum up to unity. Thus, if $\underline{\nu}$ is defined as an m-dimensional vector of 1's,

$$\underline{\nu}^T \underline{p} = 1 \tag{3.5}$$

Equations (3.4) and (3.5) may be shown to fully determine the solution of the problem (Schick and Gershwin [1978]). However, the number of equations may be very large: for a three-stage line with two buffers each of capacity 10, equation (2.4) is used to give m = 968. For a four-stage line with the same buffer capacities, the number of equations to be solved is m = 21, 296. This complexity may be reduced by making use of the special structure of the transition matrix T. Because the changes in storage levels are determined once the α_i make their transitions, most $n_i(t) \rightarrow n_i(t+1)$ transition probabilities are zero, so that T is very sparse. Furthermore, the following observations imply that T has a certain structure which is useful in solving equations (3.4) and (3.5):

(i) During a single transition, a storage level is incremented (up or down) by at most 1.

(ii) Adjacent storages cannot change in the same direction, i.e., the levels of adjacent storages cannot both increase or both decease, within a single transition.

It follows that if the $\underline{p}$ vector is arranged such that the states are listed lexicograhically, the T matrix is block tri-diagonal. Furthermore, if there is more than one storage, the main diagonal blocks in T are themselves block tri-diagonal and this nested structure persists k-1 times. Similarly, if there is more than one storage, the off-diagonal blocks in T are block bi-diagonal, and the nested structure persists k-1 times. It is shown below (see also Navon [1977], Varah [1972], Schick and Gershwin [1978]) that the solution of a system of equations with a block tri-diagonal matrix may be obtained with considerably less work than needed for a general system of equations. In this manner, the system of equations given by (3.4) and (3.5) is solved with significant computational savings.

Since T is block tri-diagonal, so is (T-I). However, equation (3.4) implies that (T-I) is singular (since $\underline{p} \neq \underline{0}$ because of (3.5)). It can be shown that a minor modification may be made in (T-I) involving only one of its rows and not disturbing the block tri-diagonal structure. If this modification renders the matrix invertible, then the solution vector $\underline{\pi}$ of the matrix equation

$$M\underline{\pi} = \underline{b} \tag{3.6}$$

(where M is the modified (T-I) and $\underline{b}$ is a vector of 0's except for a 1 at the modified row) is a scalar multiple of the steady-state probability vector $\underline{p}$. Calculating $\underline{p}$ is thus equivalent to normalizing $\underline{\pi}$. (Schick and Gershwin [1978]). Furthermore, since M^{-1} is post multipled by $\underline{b}$ (which is equivalent to reading off one column of the matrix) the entire inverse matrix need not be computed; this further simplified the computation of $\underline{p}$.

It is shown below that obtaining $M^{-1}\underline{b}$ involves knowledge of the inverses of the main diagonal blocks. Since these blocks are themeselves block tri-diagonal, a procedure for obtaining the inverse of a block tri-diagonal matrix is outlined below. The block tri-diagonal matrix Q is partitioned as follows:

$$Q \triangleq \begin{bmatrix} A_0 & C_1 & 0 & \cdots & 0 \\ B_0 & A_1 & C_2 & & \\ 0 & & \ddots & & 0 \\ \vdots & & B_{N-2} & A_{N-1} & C_N \\ 0 & & 0 & B_{N-1} & A_N \end{bmatrix} \tag{3.7}$$

The rectangular matrices Y and E are partitioned as

$$Y \triangleq \begin{bmatrix} Y_0 \\ \vdots \\ Y_N \end{bmatrix} \qquad E \triangleq \begin{bmatrix} E_0 \\ \vdots \\ E_N \end{bmatrix} \tag{3.8}$$

and are defined so as to satisfy

$$QY = E \tag{3.9}$$

Equation (3.9) may be rewritten as

$$A_0Y_0 + C_1Y_1 = E_0 \tag{3.10}$$

$$\vdots$$

$$B_{N-2}Y_{N-2} + A_{N-1}Y_{N-1} + C_NY_N = E_{N-1} \tag{3.11}$$

$$B_{N-1}Y_{N-1} + A_NY_N = E_N \tag{3.12}$$

Equation (3.12) is solved for Y_N in terms of Y_{N-1}, and the result is substituted into (3.11). The system is thus solved backwards until (3.10), and the following recursions is obtained:

$$\left.\begin{aligned} Y_0 &= D_N \\ Y_i &= D_{N-i} - X_{N-i}^{-1}B_{i-1}Y_{i-1} \end{aligned}\right\} i = 1, \ldots, N \tag{3.13}$$

where

$$\left.\begin{aligned} X_0 &= A_N \\ X_i &= A_{N-i} - C_{N-i+1}\, X_{i-1}^{-1}B_{N-i} \end{aligned}\right\} i = 1, \ldots, N \tag{3.14}$$

$$\left.\begin{aligned} D_0 &= A_N^{-1}E_N = X_0^{-1}E_N \\ D_i &= X_i^{-1}[E_{N-i} - C_{N-i+1}D_{i-1}] \end{aligned}\right\} i = 1, \ldots, N \tag{3.15}$$

To obtain Q^{-1}, it is sufficient to sequentially set $E_i = I$, $E_j = 0$, $j \neq i$, for $i=1,\ldots,N$. The Y matrices thus found are the block columns of Q^{-1}. The computational burden has been reduced to that of obtaining X_i^{-1}, where the dimensions of X_i are only as large as the blocks in Q. It may be shown that because of the sparsity of C_i (only about a quarter of the rows are non-zero), the inverses X_i^{-1} can be obtained with less computation than would be necessary for a general matrix of the same dimensions under the conditions that A_{N-i}^{-1} is known; this is done by using the matrix inverse lemma (Householder [1975]).

Matrix X has a nested block tri-diagonal structure, in which the block tri-diagonal structure persists for k-1 levels. The diagonal blocks at all levels but the lowest are themselves block tri-diagonal. Thus, their inverses may be obtained by a recursive application of the procedure summarized by equations (3.13)-(3.15). At the lowest level, the diagonal blocks are only $2^k \times 2^k$ for the present problem, and may be inverted easily.

Although the procedure outlined above involves less computation than a straight-forward solution of the equation system (3.4)-(3.5), the computational complexity of this algorithm is significant for large storages or large numbers of stages. In addition, the X_i^{-1} are generated upwards (i.e., from i=0 to i=N) but are used downwards (i.e., from i=N to i=0) in equations (3.13). This necessitates the storage of all X_i^{-1} and may cause important computer memory problems.

4. An Analytical Solution

The matrix method described in section 3 has the advantage of being flexible and applicable to any length of transfer line; yet, computation and memory problems arising in the implementation of this algorithm may sometimes be prohibitive. The analytical solution presented here is considerably more complex to derive, but easier to implement. The main disadvantage here is that, at least at the present stage it appears that a large amount of analytical derivation is necessary for obtaining the general solution for each specific value of k. Though the approach is general, the problem has been investigated only for $k \leq 3$.

For all states s(t+1) at time t+1, a transition equation is written of the form

$$p[s(t+1),t+1] = \sum_{\text{all } s(t)} p[s(t+1)|s(t)] \cdot p[s(t),t] \quad (4.1)$$

The first factor in the above summation is the product of the transition probabilities for each α_i and n_i. For reasons that become apparent later in this development, states are subdivided into two general classes, as follows:

internal states: $2 \le n_i \le N_i - 2$ for all i

boundary states: $n_i \le 1$ or $n_i \ge N_i - 1$ for at least one i

In transitions between internal states, the initial and final storage levels are related by

$$n_i(t+1) = n_i(t) + \alpha_i(t+1) - \alpha_{i+1}(t+1) \quad (4.2)$$

with probability one. The set $S(s(t+1))$ is defined as the set of all states s(t) such that given the final storage level $n_i(t+1)$ and station operational conditions $\alpha_i(t+1)$ and $\alpha_{i+1}(t+1)$, the initial storage level $n_i(t)$ satisfies (4.2). Then, for given failure and repair probabilities p_i and r_i respectively, equation (4.1) becomes

$$p[s(t+1)] = \sum_{s(t) \in S(s(t+1))} \prod_{i=1}^{k} \left[(1-r_i)^{1-\alpha_i(t+1)} r_i^{\alpha_i(t+1)}\right]^{1-\alpha_i(t)} \cdot \left[(1-p_i)^{\alpha_i(t+1)} p_i^{1-\alpha_i(t+1)}\right]^{\alpha_i(t)} p[s(t)] \quad (4.3)$$

where steady-state has been assumed.

The form of the steady-state probabilities of internal states is guessed to be a sum-of-products (Many queuing theory problems yield product form solutions. See, for example, Jackson [1963], Gordon and Newell [1967], Baskett, Chandy, Muntz and Palacios [1975] and others.)

$$p[s] = p[n_1,\ldots,n_{k-1},\alpha_1,\ldots,\alpha_k] = \sum_{j=1}^{\ell} C_j X_{1j}^{n_1} \cdots X_{k-1,j}^{n_{k-1}} Y_{1j}^{\alpha_1} \cdots Y_{kj}^{\alpha_k} \quad (4.4)$$

where C_j, X_{ij} and Y_{ij} are parameters to be determined. It is assumed that each term in the summation in (4.4) by itself satisfies (4.3). Thus, one term from the summation in (4.4) is substituted into (4.3) and after some manipulation, the equation becomes:

$$\prod_{i=1}^{k} X_{ij}^{\alpha_i(t+1)-\alpha_{i+1}(t+1)} Y_{ij}^{\alpha_i(t+1)} = \prod_{i=1}^{k} [(1-r_i)^{1-\alpha_i(t+1)} r_i^{\alpha_i(t+1)} + (1-p_i)^{\alpha_i(t+1)} p_i^{1-\alpha_i(t+1)} Y_{ij}] \quad (4.5)$$

where for convenience, $X_{kj} \triangleq 1$. Since equation (4.5) is derived without specifying the value of $\alpha_i(t+1)$, it must hold for all values. In particular, if $\alpha_i(t+1) = 0$ for all i, (4.5) reduces to:

$$1 = \prod_{i=1}^{k} [(1-r_i) + p_i Y_{ij}] \quad (4.6)$$

For $\alpha_i(t+1) = 1$, $\alpha_q(t+1) = 0$, $q \ne i$, (4.5) becomes, using (4.6):

$$\frac{X_{ij} Y_{ij}}{X_{i-1,j}} = \frac{r_i + (1-p_i) Y_{ij}}{(1-r_i) + p_i Y_{ij}} \quad ; \; i = 1,\ldots,k \quad (4.7)$$

where for convenience, $X_{0j} \triangleq 1$.

Equations (4.6) and (4.7) comprise a set of k+1 equations in 2k-1 unknowns. When k=2, this system may be solved in a straight-forward manner, and X_{ij} and Y_{ij} are obtained as functions of p_i and r_i. The constant terms C_i are then obtained by using boundary conditions, i.e., transition equations involving at least one boundary state. It is found that there is a single term in the summation in (4.4), i.e., $\ell = 1$ (see Buzacott [1967], Artamonov [1977]). When $k \ge 3$, the solution is not uniquely determined by equations (4.6) and (4.7). It is assumed that the boundary state probability expressions have a sum-of-terms form, analogous to the sum of products for internal states given in (4.4).

Thus, for any state s,

$$p[s] = \sum_{j=1}^{\ell} C_j \xi[s, X_{1j},\ldots,X_{k-1,j}, Y_{1j},\ldots,Y_{kj}] \quad (4.8)$$

where $\xi[\cdot]$ is given by $X_{1j}^{n_1} \cdots X_{k-1,j}^{n_{k-1}} Y_{1j}^{\alpha_1} \cdots Y_{kj}^{\alpha_k}$ when $s = (n_1,\ldots,n_{K-1}, \alpha_1,\ldots,\alpha_K)$ is internal. Expressions $\xi[\cdot]$ can be found that satisfy most (but not all) boundary equation. Most of the analytical derivation towards the solution for a k-stage line occurs here; so far, only two and three-stage (Gershwin and Schick [1978]) lines have been analyzed. The $\xi[\cdot]$ expressions do not all satisfy all the transition equations. Thus a specific linear combination, as given by equation (4.8), is sought to satisfy them all. In other words, given ℓ sets of numbers

$$U_j \triangleq \{X_{1j},\ldots,X_{k-1,j}, Y_{1j},\ldots,Y_{kj}\}, j=1,\ldots,\ell \quad (4.9)$$

there is a set of C_j, $j=1,\ldots,\ell$, such that the linear combinations given by equation (4.4) and (4.9) satisfy the set of transition equations (3.4).

The steady-state probability vector is expressed as

$$\underline{p} = \sum_{j=1}^{\ell} C_j \underline{\xi}[U_j] \quad (4.10)$$

where

$$\underline{\xi}[U_j] \triangleq \begin{bmatrix} \xi[s_1, U_j] \\ \vdots \\ \xi[s_m, U_j] \end{bmatrix} \quad (4.11)$$

Then, equation (3.4) may be rewritten as

$$(T-I) \sum_{j=1}^{\ell} c_j \underline{\xi}[U_j] = \underline{0} \quad (4.12)$$

Defining

$$\underline{c} \triangleq \begin{bmatrix} c_1 \\ \vdots \\ c_\ell \end{bmatrix} \quad (4.13)$$

$$\Xi \triangleq \left[\underline{\xi}[u_1] \vdots \ldots \vdots \underline{\xi}[U_\ell]\right] \quad (4.14)$$

equation (4.12) becomes

$$(T-I)\ \Xi\ \underline{c} = \underline{0} \quad (4.15)$$

Since the $\xi[\cdot]$ expressions satisfy most transition equations, most rows of (4.15) are satisfied trivially. That is, most rows of $(T-I)\Xi$ are identically zero. If λ equations are not satisfied identically, then the number of terms is the summation in (4.8) is

$$\ell = \lambda - 1 \quad (4.16)$$

Then, $\underline{c}$ is obtained (to within a scalar multiple) by solving (4.15). Since the sets U_j, $j=1,\ldots,\ell$ are determined a priori by finding ℓ distinct solutions of the system (4.6)-(4.7), the $\underline{p}$ vector is completely determined (after normalization).

The number of equations to be solved, ℓ, is linear in storage capacities for $k=3$, while the total number of transition equations, m, is quadratic in storage capacities. In general, $\ell \ll m$ and ℓ is of lower degree in storage capacities than m. As a result, the computational complexity of the problem is decreased significantly. Nevertheless, solving (4.15) causes numerical problems. A least square solution (Golub and Kahan [1965]) is obtained, and because the smallest singular values are very small, extended precision arithmetic is required to obtain the solutions. Ways of avoiding this problem are under investigation.

5. Design Parameters and System Performance Measures

The efficiency of the system is the steady-state probability that a part emerges from the last station in the line during any time cycle. This is equal to the sum of the probabilities of those events in which the last station is operational and the next-to-last storage was non-empty at the preceeding time cycle. It may be shown that this quantity is equal to the sum of the probabilities of all the states in which the last station is operational and the next-to-last storage is non-empty at the same time cycle (Schick and Gershwin [1978]). The computation of steady-state efficiency is simplified by this identity.

In a transfer line with no buffer storages, the entire line is forced down if any one station fails. This corresponds to the lowest possible case, and the efficiency is given by (Buzacott [1968]):

$$E(0) = \frac{1}{1+\sum_{i=1}^{k} p_i/r_i} \quad (5.1)$$

In the case where all storages have infinite capacities, it may be shown that the line efficiency is equal to that of the least efficient station (Schick and Gershwin [1978]). Thus, (Buzacott [1967]),

$$E(\infty) = \frac{1}{1 + \max_i\{p_i/r_i\}} \quad (5.2)$$

corresponds to the highest possible efficiency.

As storage capacities increase, the line efficiency increases from $E(0)$ to $E(\infty)$ (figure 2). While these limiting values depend on the ratios of p_i and r_i (see equations), the rate at which efficiency approaches the asymptotic at $E(\infty)$ depends on the magnitude of these probabilities. The effectiveness of a storage capacity configuration is used as a measure of how close to the limit the line efficiency is, given a set of storage capacities. This is useful as it gives a measure of how much may be gained by incrementing the storage capacities. Effectiveness is defined as (Freeman [1964], Buzacott [1969]):

$$\eta(N_1,\ldots, N_{k-1}) = \frac{E(N_1,\ldots, N_{k-1}) - E(0)}{E(\infty) - E(0)} \quad (5.3)$$

The difference between $E(0)$ and $E(\infty)$ may be used as a measure of how much efficiency may be improved given large enough buffers. From equations (5.1) and (5.2), it may be verified that the difference is largest when the individual stations are each not very efficient and no station is significantly less efficient than all others.

The forced-down times of stations are related to the probability that during any time cycle, the station is either starved or blocked. As storage capacities increase, the forced down times of the least efficient station approach zero while those of the others approach positive asymptotes (figure 3). At the limit with infinite storage capacities, the least efficient station is fully utilized (i.e., it is never forced down). This may be proved analytically, at least for the two-stage case (Schick and Gershwin [1978]). The efficiency in isolation of a single station is the probability that it is operational within any time cycle, given that it is never starved or blocked. When the efficiency of a station is much lower than that of the other stations, the system efficiency increases linearly with the efficiency in isolation of the limiting station; as the station stops being limiting, the system efficiency approaches an asymptote (figure 4). Quantitative results illustrating this behavior may be obtained by using the state probabilities. These are of value in determining the optimizing efficiencies of the stations. In some cases, it is seen that even small storage capacities improve the line efficiency as much as improving the efficiency of individual stations; since the latter involves additional cost which may be high, it is important to consider buffer storages as an option in some cases. This is especially true of balanced lines.

Another important factor in the cost functional for optimizing a transfer line is in-process inventory: there is usually a cost associated with storing material, and this is especially true if the material is expensive or if the processing in the line gives the material a high added value. This cost, however, is not generally linear with storage capacity. For some cases, the inventory may be shown to approach a limiting value as storage capacity increases (Figure 5). The inventory level in a particular storage depends on the efficiencies of the upstream and downstream portions of the line. If these portions include storages, the inventory is found to be related also to the capacities of these other storages. The expected inventory is obtained by using the state probabilities, since expected inventory is simply the average level of material in the storage.

6. Conclusions

The work presented here is directed towards obtaining exact methods for calculating system performance measures given system parameters. The methodology may be used in the optimal design of transfer lines, as well as in the more complex problem of the real time control of the line, say, the control of the speeds of stations.

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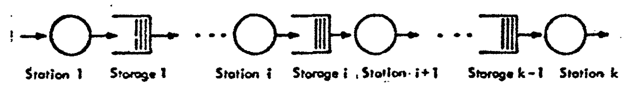

Figure 1: A k-stage transfer line.

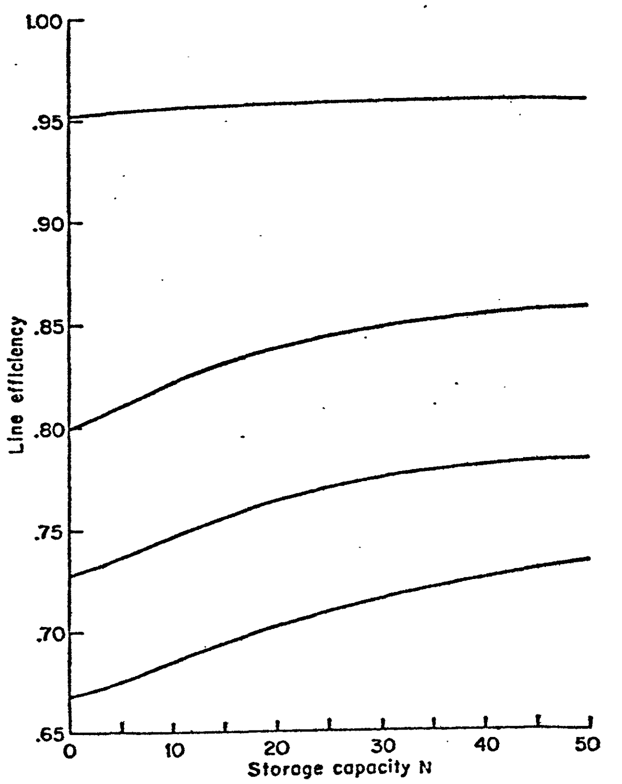

Figure 2: Efficiency v.s. storage capacity for a two-stage line (Gershwin [1973]).

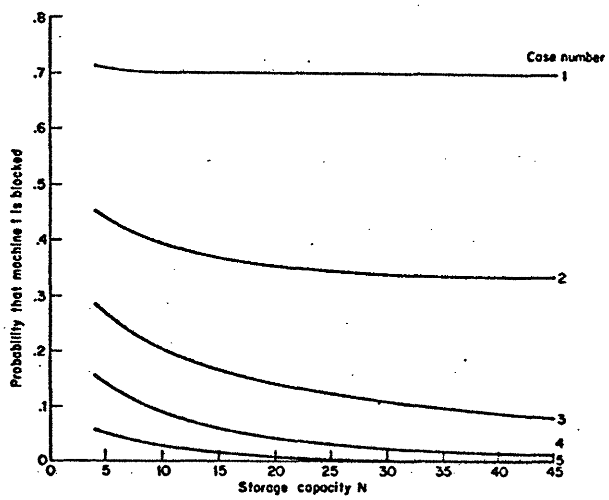

Figure 3: Forced down times for station 1 v.s. storage capacity for a two-stage line. (In cases 1 and 2, station 1 is more efficient. In cases 4 and 5, station 2 is more efficient.)

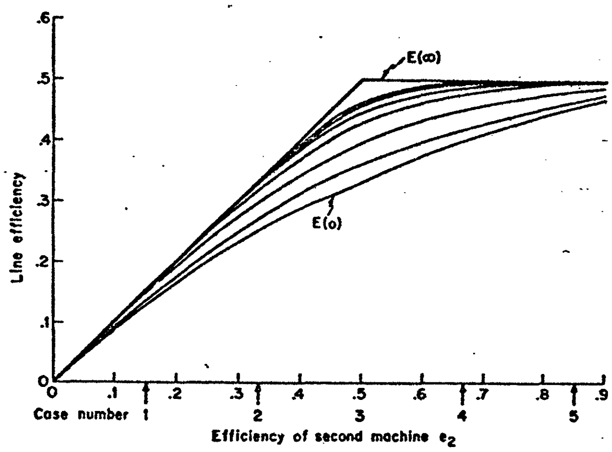

Figure 4: Line efficiency v.s. efficiency in isolation of station 2 for a two-stage line. (curves for N = 4,10,20,30,40 and 50).

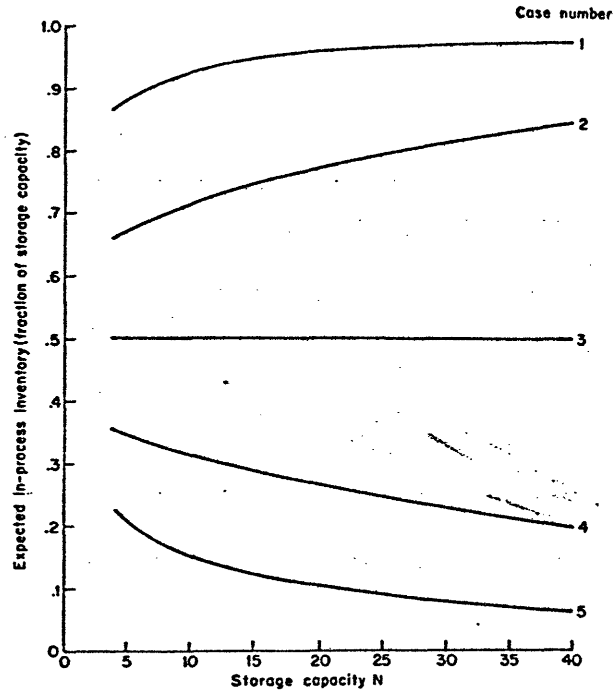

Figure 5: Expected in-process inventory as fraction of storage capacity, v.s. storage capacity, for a two-stage line. (In cases 1 and 2, station 1 is more efficient. In cases 4 and 5, station 2 is more efficient.)