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GEOMETRIC INTERPRETATION OF HALF-PLANE CAPACITY

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Abstract
Schramm-Loewner Evolution describes the scaling limits of interfaces in certain statistical mechanical systems. These interfaces are geometric objects that are not equipped with a canonical parametrization. The standard parametrization of SLE is via half-plane capacity, which is a conformal measure of the size of a set in the reference upper half-plane. This has useful harmonic and complex analytic properties and makes SLE a time-homogeneous Markov process on conformal maps. In this note, we show that the half-plane capacity of a hull $A$ is comparable up to multiplicative constants to more geometric quantities, namely the area of the union of all balls centered in $A$ tangent to $\mathbb{R}$, and the (Euclidean) area of a 1-neighborhood of $A$ with respect to the hyperbolic metric.

1 Introduction

Suppose $A$ is a bounded, relatively closed subset of the upper half plane $\mathbb{H}$. We call $A$ a compact $\mathbb{H}$-hull if $A$ is bounded and $\mathbb{H} \setminus A$ is simply connected. The half-plane capacity of $A$, $\text{hcap}(A)$, is defined in a number of equivalent ways (see [1], especially Chapter 3). If $g_A$ denotes the unique conformal

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2 RESEARCH SUPPORTED BY NATIONAL SCIENCE FOUNDATION GRANT DMS-0734151.
transformation of $\mathbb{H} \setminus A$ onto $\mathbb{H}$ with $g_A(z) = z + o(1)$ as $z \to \infty$, then $g_A$ has the expansion
$$g_A(z) = z + \frac{\text{hcap}(A)}{z} + O(|z|^{-2}), \quad z \to \infty.$$ Equivalently, if $B_t$ is a standard complex Brownian motion and $\tau_A = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus A\}$,
$$\text{hcap}(A) = \lim_{y \to \infty} y \mathbb{E}^{i y} \left[ \text{Im}(B_{\tau_A}) \right].$$ Let $\text{Im}[A] = \sup\{\text{Im}(z) : z \in A\}$. Then if $y \geq \text{Im}[A]$, we can also write
$$\text{hcap}(A) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathbb{E}^{x+iy} \left[ \text{Im}(B_{\tau_A}) \right] dx.$$ These last two definitions do not require $\mathbb{H} \setminus A$ to be simply connected, and the latter definition does not require $A$ to be bounded but only that $\text{Im}[A] < \infty$.

For $\mathbb{H}$-hulls (that is, for relatively closed $A$ for which $\mathbb{H} \setminus A$ is simply connected), the half-plane capacity is comparable to a more geometric quantity that we define. This is not new (the second author learned it from Oded Schramm in oral communication), but we do not know of a proof in the literature. In this note, we prove the fact giving (nonoptimal) bounds on the constant. We start with the definition of the geometric quantity.

**Definition 1.** For an $\mathbb{H}$-hull $A$, let $\text{hsiz}(A)$ be the 2-dimensional Lebesgue measure of the union of all balls centered at points in $A$ that are tangent to the real line. In other words
$$\text{hsiz}(A) = \text{area} \left( \bigcup_{x+i y \in A} B(x+i y, y) \right),$$ where $B(z, \epsilon)$ denotes the disk of radius $\epsilon$ about $z$.

In this paper, we prove the following.

**Theorem 1.** For every $\mathbb{H}$-hull $A$,
$$\frac{1}{66} \text{hsiz}(A) < \text{hcap}(A) < \frac{7}{2\pi} \text{hsiz}(A).$$

**2 Proof of Theorem 1**

It suffices to prove this for weakly bounded $\mathbb{H}$-hulls, by which we mean $\mathbb{H}$-hulls $A$ with $\text{Im}(A) < \infty$ and such that for each $\epsilon > 0$, the set $\{x + iy : y > \epsilon\}$ is bounded. Indeed, for $\mathbb{H}$-hulls that are not weakly bounded, it is easy to verify that $\text{hsiz}(A) = \text{hcap}(A) = \infty$.

We start with a simple inequality that is implied but not explicitly stated in [1]. Equality is achieved when $A$ is a vertical line segment.

**Lemma 1.** If $A$ is an $\mathbb{H}$-hull, then
$$\text{hcap}(A) \geq \frac{\text{Im}[A]^2}{2}. \tag{1}$$

\[3\text{After submitting this article, we learned that a similar result was recently proved by Carto Wong as part of his Ph.D. research.} \]
Proof: Due to the continuity of \( \text{hcap} \) with respect to the Hausdorff metric on \( \mathbb{H} \)-hulls, it suffices to prove the result for \( \mathbb{H} \)-hulls that are path-connected. For two \( \mathbb{H} \)-hulls \( A_1 \subseteq A_2 \), it can be seen using the Optional stopping theorem that \( \text{hcap}(A_1) \leq \text{hcap}(A_2) \). Therefore without loss of generality, \( A \) can be assumed to be of the form \( \eta(0, T] \) where \( \eta \) is a simple curve with \( \eta(0+) \in \mathbb{R} \), parameterized so that \( \text{hcap} \{ \eta(0, t] \} = 2t \). In particular, \( T = \text{hcap}(A)/2 \). If \( g_t = g_{\eta(0,t]} \), then \( g_t \) satisfies the Loewner equation

\[
\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z,
\]

where \( U : [0, T] \to \mathbb{R} \) is continuous. Suppose \( \text{Im}(z)^2 > 2 \text{hcap}(A) \) and let \( Y_t = \text{Im}[g_t(z)] \). Then (2) gives

\[
-\partial_t Y_t^2 \leq \frac{4Y_t}{|g_t(z) - U_t|^2} \leq 4,
\]

which implies

\[
Y_t^2 \geq Y_0^2 - 4T > 0.
\]

This implies that \( z \not\in A \), and hence \( \text{Im}[A]^2 \leq 2 \text{hcap}(A) \).

The next lemma is a variant of the Vitali covering lemma. If \( c > 0 \) and \( z = x + iy \in \mathbb{H} \), let

\[
\mathcal{I}(z, c) = (x - cy, x + cy),
\]

\[
\mathcal{R}(z, c) = \mathcal{I}(z, c) \times (0, y] = \{ x' + iy' : |x' - x| < cy, 0 < y' \leq y \}.
\]

Lemma 2. Suppose \( A \) is a weakly bounded \( \mathbb{H} \)-hull and \( c > 0 \). Then there exists a finite or countably infinite sequence of points \( \{z_1 = x_1 + iy_1, z_2 = x_2 + iy_2, \ldots \} \subseteq A \) such that:

- \( y_1 \geq y_2 \geq y_3 \geq \cdots \);
- the intervals \( \mathcal{I}(x_1, c), \mathcal{I}(x_2, c), \ldots \) are disjoint;
- \( A \subseteq \bigcup_{j=1}^{\infty} \mathcal{R}(z_j, 2c) \).

Proof: We define the points recursively. Let \( A_0 = A \) and given \( \{z_1, \ldots, z_j\} \), let

\[
A_j = A \setminus \left[ \bigcup_{k=1}^{j} \mathcal{R}(z_j, 2c) \right].
\]

If \( A_j = \emptyset \) we stop, and if \( A_j \neq \emptyset \), we choose \( z_{j+1} = x_{j+1} + iy_{j+1} \in A \) with \( y_{j+1} = \text{Im}[A_j] \). Note that if \( k \leq j \), then \( |x_{j+1} - x_k| \geq 2c y_k \geq c(y_k + y_{j+1}) \) and hence \( \mathcal{I}(z_{j+1}, c) \cap \mathcal{I}(z_k, c) = \emptyset \). Using the weak boundedness of \( A \), we can see that \( y_j \to 0 \) and hence (3) holds.

Lemma 3. For every \( c > 0 \), let

\[
\rho_c := \frac{2\sqrt{2}}{\pi} \arctan\left( e^{-\theta} \right), \quad \theta = \theta_c = \frac{\pi}{4c}.
\]

Then, for any \( c > 0 \), if \( A \) is a weakly bounded \( \mathbb{H} \)-hull and \( x_0 + iy_0 \in A \) with \( y_0 = \text{Im}(A) \), then

\[
\text{hcap}(A) \geq \rho_c^2 y_0^2 + \text{hcap} [A \setminus \mathcal{R}(z, 2c)].
\]
Proof. By scaling and invariance under real translation, we may assume that \( \text{Im}[A] = y_0 = 1 \) and \( x_0 = 0 \). Let \( S = S_c \) be defined to be the set of all points \( z \) of the form \( x + iy \) where \( x + i y \in A \setminus \mathcal{R}(i, 2c) \) and \( 0 < u \leq 1 \).

Clearly, \( S \cap A = A \setminus \mathcal{R}(i, 2c) \).

Using the capacity inequality \[ (3.10) \]

\[
\text{hcap}(A \cup A_2) - \text{hcap}(A_2) \leq \text{hcap}(A_1) - \text{hcap}(A_1 \cap A_2),
\]

we see that

\[
\text{hcap}(S \cup A) - \text{hcap}(A) \leq \text{hcap}(A) - \text{hcap}(S \cap A).
\]

Hence, it suffices to show that

\[
\text{hcap}(S \cup A) - \text{hcap}(S) \geq \rho^2.
\]

Let \( f \) be the conformal map of \( \mathbb{H} \setminus S \) onto \( \mathbb{H} \) such that \( x - f(z) = o(1) \) as \( z \to \infty \). Let \( S' := S \cup A \).

By properties of halfplane capacity \[ (3.8) \] and \[ (3.1) \],

\[
\text{hcap}(S') - \text{hcap}(S) = \text{hcap}[f(S' \setminus S)] \geq \frac{\text{Im}[f(i)]^2}{2}.
\]

Hence, it suffices to prove that

\[
\text{Im}[f(i)] \geq \sqrt{2 \rho} = \frac{4}{\pi} \arctan \left( e^{-\theta} \right).
\]

By construction, \( S \cap \mathcal{R}(z, 2c) = \emptyset \). Let \( V = (-2c, 2c) \times (0, \infty) = \{ x + iy : |x| < 2c, y > 0 \} \) and let \( \tau_V \) be the first time that a Brownian motion leaves the domain. Then \[ (3.5) \],

\[
\text{Im}[f(i)] = 1 - \mathbb{E} \left[ \text{Im}(B_{\tau_V}) \right] \geq \mathbb{P} \left\{ B_{\tau_V} \in [-2c, 2c] \right\} \geq \mathbb{P} \left\{ B_{\tau_V} \in [-2c, 2c] \right\}.
\]

The map \( \Phi(z) = \sin(\theta z) \) maps \( V \) onto \( \mathbb{H} \) sending \( [-2c, 2c] \) to \( [-1, 1] \) and \( \Phi(i) = i \sin \theta \). Using conformal invariance of Brownian motion and the Poisson kernel in \( \mathbb{H} \), we see that

\[
\mathbb{P} \left\{ B_{\tau_V} \in [-2c, 2c] \right\} = \frac{2}{\pi} \arctan \left( \frac{1}{\sin \theta} \right) = \frac{4}{\pi} \arctan \left( e^{-\theta} \right).
\]

The second equality uses the double angle formula for the tangent. \( \square \)

**Lemma 4.** Suppose \( c > 0 \) and \( x_1 + iy_1, x_2 + iy_2, \ldots \) are as in Lemma \[ (2) \]. Then

\[
\text{hsiz}(A) \leq [\pi + 8c] \sum_{j=1}^{\infty} y_j^2.
\]

If \( c \geq 1 \), then

\[
\pi \sum_{j=1}^{\infty} y_j^2 \leq \text{hsiz}(A).
\]

**Proof.** A simple geometry exercise shows that

\[
\text{area} \left[ \bigcup_{x+iy \in \mathcal{R}(z, 2c)} \mathcal{R}(x+iy, y) \right] = [\pi + 8c] y_j^2.
\]
Since
\[ A \subset \bigcup_{j=1}^{\infty} \mathcal{R}(z_j, 2c), \]
the upper bound in (6) follows. Since \( c \geq 1 \), and the intervals \( \mathcal{R}(z_j, c) \) are disjoint, so are the disks \( \mathcal{B}(z_j, y_j) \). Hence,

\[
\text{area} \left[ \bigcup_{x + iy \in A} \mathcal{R}(x + iy, y) \right] \geq \text{area} \left[ \bigcup_{j=1}^{\infty} \mathcal{B}(z_j, y_j) \right] = \pi \sum_{j=1}^{\infty} y_j^2.
\]

**Proof of Theorem** Let \( V_j = A \cap \mathcal{R}(z_j, c) \). Lemma 3 tells us that

\[
\mathcal{hcap} \left[ \bigcup_{k=j}^{\infty} V_j \right] \geq \rho^2 c y_j^2 + \mathcal{hcap} \left[ \bigcup_{k=j+1}^{\infty} V_j \right],
\]

and hence

\[
\mathcal{hcap}(A) \geq \rho^2 c \sum_{j=1}^{\infty} y_j^2. \tag{8}
\]

Combining this with the upper bound in (6) with any \( c > 0 \) gives

\[
\frac{\mathcal{hcap}(A)}{\mathcal{hsiz}(A)} \geq \frac{\rho^2}{\pi + 8c}.
\]

Choosing \( c = \frac{8}{5} \) gives us

\[
\frac{\mathcal{hcap}(A)}{\mathcal{hsiz}(A)} > \frac{1}{66}.
\]

For the upper bound, choose a covering as in Lemma 2. Subadditivity and scaling give

\[
\mathcal{hcap}(A) \leq \sum_{j=1}^{\infty} \mathcal{hcap} \left[ \mathcal{R}(z_j, 2cy_j) \right] = \mathcal{hcap}[\mathcal{R}(i, 2c)] \sum_{j=1}^{\infty} y_j^2. \tag{9}
\]

Combining this with the lower bound in (6) with \( c = 1 \) gives

\[
\frac{\mathcal{hcap}(A)}{\mathcal{hsiz}(A)} \leq \frac{\mathcal{hcap}[\mathcal{R}(i, 2)]}{\pi}.
\]

Note that \( \mathcal{R}(i, 2) \) is the union of two real translates of \( \mathcal{R}(i, 1) \), \( \mathcal{hcap}[\mathcal{R}(i, 2)] \leq 2 \mathcal{hcap}[\mathcal{R}(i, 1)] \) whose intersection is the interval \((0, i]\). Using (4), we see that

\[
\mathcal{hcap}(\mathcal{R}(i, 2)) \leq 2 \mathcal{hcap}(\mathcal{R}(i, 1)) - \mathcal{hcap}((0, i]) = 2 \mathcal{hcap}(\mathcal{R}(i, 1)) - \frac{1}{2}.
\]

But \( \mathcal{R}(i, 1) \) is strictly contained in \( A' := \{ z \in \mathbb{H} : |z| \leq \sqrt{2} \} \), and hence

\[
\mathcal{hcap}[\mathcal{R}(i, 1)] < \mathcal{hcap}(A') = 2.
\]
The last equality can be seen by considering \( h(z) = z + 2z^{-1} \) which maps \( \mathbb{H} \setminus A' \) onto \( \mathbb{H} \). Therefore,

\[
\text{hcap}[\mathcal{R}(i, 2)] < \frac{7}{2},
\]

and hence

\[
\frac{\text{hcap}(A)}{\text{hsiz}(A)} < \frac{7}{2\pi}.
\]

\[\square\]

An equivalent form of this result can be stated\(^4\) in terms of the area of the 1-neighborhood of \( A \) (denoted \( \text{hyp}(A) \)) in the hyperbolic metric. The unit hyperbolic ball centered at a point \( x + iy \) is the Euclidean ball with respect to which \( x + iy/e \) and \( x = iy/e \) are diametrically opposite boundary points. For any \( c \), choosing a covering as in Lemma 2,

\[
\text{hyp}(A) < \left( \frac{e}{2} \right)^2 \pi + 4ec \sum_{j=1}^{\infty} y_j^2.
\]

So by (8),

\[
\frac{\text{hcap}(A)}{\text{hyp}(A)} > \rho_c^2 \left( \frac{e}{2} \right)^2 \pi + 4ec \right)^{-1}.
\]

Setting \( c \) to \( \frac{8}{5} \),

\[
\frac{\text{hcap}(A)}{\text{hyp}(A)} > \frac{1}{100}.
\]

For any \( c > \frac{e-e^{-1}}{2} \),

\[
\text{hyp}(A) \geq \pi \left( \frac{e-e^{-1}}{2} \right)^2 \sum_{j=1}^{\infty} y_j^2.
\]

So by (9),

\[
\frac{\text{hcap}(A)}{\text{hyp}(A)} \leq \frac{\text{hcap}[\mathcal{R}(i, 3)]}{\pi \left( \frac{e-e^{-1}}{2} \right)^2}.
\]

\[
\text{hcap}(\mathcal{R}(i, 3)) \leq \text{hcap}(\mathcal{R}(i, 1)) + \text{hcap}(\mathcal{R}(i, 2)) - \text{hcap}((0, i)) \leq 5.
\]

Therefore,

\[
\frac{1}{100} < \frac{\text{hcap}(A)}{\text{hyp}(A)} < \frac{20}{\pi(e-e^{-1})^2}.
\]

References


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\(^4\)This formulation was suggested to us by Scott Sheffield and the anonymous referee.