Structural Systems and Conceptual Design of Cantilevers

by

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Massachusetts Institute of Technology

June 2014

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ABSTRACT

Cantilevers are a popular way to express form and create unique feature spaces. From a design perspective, cantilevers are amazing feats for the built environment, and structurally, present many opportunities. However, conceptual cantilever design can be a difficult task for Architects and Structural Engineers because there are many structural systems or strategies designers could choose to carry loads to supports.

This thesis begins with examples of built cantilevers which are distilled into five categories of structural systems. These structural systems serve as the beginning of the design process. In addition to choosing a structural system, there are many parameters of a cantilever that can be altered that all impact the overall structural performance to varying degrees.

This thesis proposes to study these parameters to better understand how they relate to one another through analytical derivations of global deflection and member forces. Secondly, with these analytical relationships, this thesis attempts to quantitatively measure the effectiveness of each structural system through an optimization sequence that takes into account both material use and deflection criteria. This method of optimization can then be applied to particular examples and be used as a systematic approach to conceptual cantilever design. A design example is optimized for material weight while satisfying a given deflection criteria, as a way to illustrate the differences between each structural system.

Thesis Supervisor: Pierre Ghisbain

Title: Lecturer in Civil and Environmental Engineering

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I would like to thank my parents for their unwavering support. Without their encouragement and dedication I would not have the chance to study at MIT.

I would like to thank all the professors that I have had the pleasure of meeting while at MIT. They have taught me so much about engineering. Special thanks to Professors Pierre Ghisbain and Jerome Connor for their guidance both inside and outside the classroom.

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Table of Contents

Table of Figures

Introduction

Cantilever design is primarily driven by aesthetics and form and is wildly popular with architects. From a structural perspective, they have pushed the envelope for great engineers to create record breaking structures. The case studies presented in the following chapters are only a sample of the extraordinary work that Architects/Structural Engineers have built in recent years. It is for this reason that this thesis focuses on cantilever design as a way to encourage the undertaking of further cantilever projects.

Cantilevers in the conceptual stages of a design can be a difficult task for Architects and Structural Engineers because there are many strategies one can use to carry loads to the ground or other supports. The problem or task is often an initial choice of structural system or strategy, as represented in Figure 1.

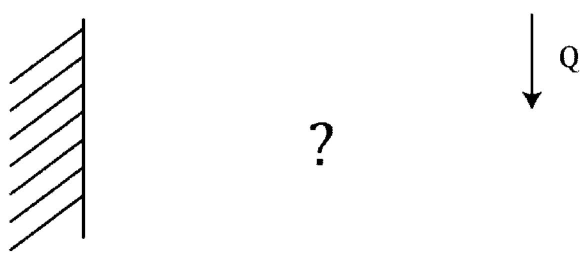
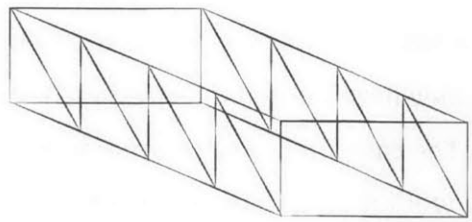


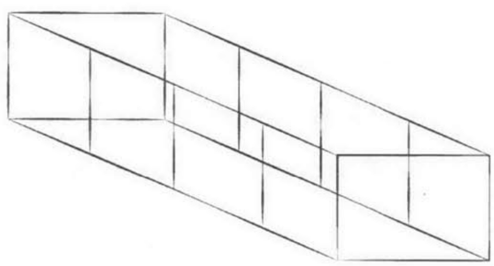
Figure 1: The choice of a structural system

The case studies presented in each of the chapters demonstrate the variety of ways designers have approached the problem posed in Figure 1. They have been distilled down to 5 common structural systems:

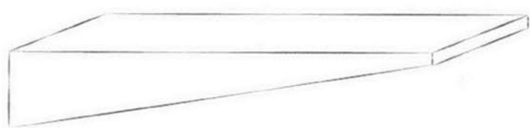
1. Steel Braced Truss (axial loads only)



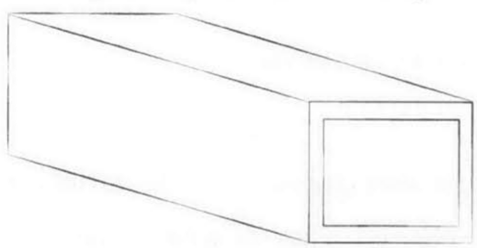
2. Steel Vierendeel Truss (acts in bending)



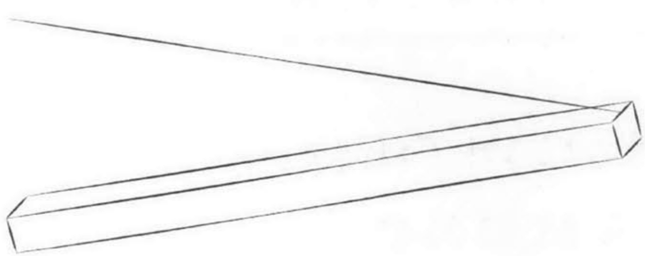
3. Concrete Beam (could also be steel but is limited to short spans)



4. Bending tube (Concrete or Steel)



5. Suspended platform



The case studies presented are either for residential or commercial use and the cantilever is an integral part of useable space (rather than a cantilevered canopy such as those seen in the roofs of sports stadiums). Once a structural system is chosen, the next steps in conceptual design are less understood. Rules of thumb for how to go about cantilever design do exist but are generally vague and are not particularly helpful since they do not apply to a specific structural system. For example, there is the "one-to-three" rule:

> *"Generally speaking, if a cantilever exceeds 1/3 of the total backspan, economy is lost and may lead to design difficulties. So if your beam has a 30' backspan, try to keep an adjacent cantilever to less than 10' long." – Craig Bursch*[1]

This ratio may apply to some structures but it is evident from the case studies shown in the following chapters that many are far more ambitious with cantilevered span than suggested in the one-to-three rule.

In the conceptual design phase, cantilevers have many parameters that can be altered that all greatly impact its structural performance. Such parameters include height, span, width of the global structure, the number of bays, the total deflection under live loading, size of members, thickness of concrete etc. Some of these parameters can be regarded as fixed by constrains either by the owners/designers or the code, while others are variable and open for experimentation. Each parameter's influence on deflection and weight will be discussed for the five structural systems.

In addition to understanding the effect of these parameters on the overall performance, it would also be useful for the designer to be able to approximate member sizes without running a complex analysis model. The analytical relationships provided in each of the categories include determining member forces. Since structural members are often an integral part of a cantilever's aesthetic, being able to size members from their forces is important to the decision making process.

[1] Bursch, Craig. "6 Rules of thumb for structural steel design" *Matrix: the official newsletter of the American Institute of Architects Minnesota.* Matrix., May 2006. Web. 20 Apr 2014.

Comparisons between Structural Systems

The decision of which structural system to pick depends on many factors:

Structural performance	Can members take only tension loads, axial loads, bending, or a combination of these?
Desired member sizes	Members could have small cross sectional area (e.g. cables), medium sized steel sections, or very large concrete beams.
Material choice	Steel, Reinforced Concrete, Timber or a hybrid of these.
Dimension constraints set by the site or the designer	Certain structural systems have depth requirements that may not comply with the intentions of the design. E.g. concrete beams have a minimum depth and can only be implemented if there is enough space or if the inhabitable cantilevered space can be elevated above ground.
Scale of the structure to the human body	Considerations for the designer could be if the structural system is inhabitable or hidden below the floor? Does a person interact with the structure if it is visible?
Programmatic constraints set by the architect or client	Programmatic desires such as minimum ceiling heights or unobstructed views/access could rule out certain structural systems and their associated constraints. For example, minimum ceiling heights for a Vierendeel truss will determine the height the truss before a span can be found.

Primary concerns

It should be noted that cantilever design is a primarily deflection governed task. In the design examples, strength checks are done but indicate that the designs are controlled by deflection due to long spans rather than material yielding or crushing. For this reason, the comparison between structural systems is done on the condition that deflection criteria are met, with the strength check being of secondary concern. With this method, it allows the designer to compare the material use and/or deflection sensitivity between the structural systems knowing that they all satisfy the deflection criteria.

Techniques often used to reduce overall deflection involve cambering the structure under dead loads since they can be accurately determined, reducing the self-weight of structure, stiffening the structure and finally, optimizing geometry (only putting material where it is needed) for structural efficiency. The last technique will be the focus of this thesis and the design example in chapter 6 offers a comparison between the effectiveness of optimized geometry for different structural systems.

The scope of this thesis does not cover vibration analysis, earthquake and wind loading, creep of materials over time, buckling of members, local effects, connection, and foundation design. These issues are important and need to be designed for but are not the basis for an initial conceptual design of a cantilever.

Loading

A live load of 100 psf was applied to each of the structural systems. For trusses (categories 1 and 2), this live load was converted into an equivalent point load of $Q = 10$ Kips applied at the nodes. The span can be lengthened by increasing the number of bays, thereby increasing the number of nodes, but the same load $Q = 10$ Kips is applied. This ensures that as the span increases, there are greater overall loads applied to the structure. For categories 3, 4, 5 this live load was converted into a linear distributed load of $w = 1$ Kip/ft, which will be used throughout the design examples. A width of 10 feet is chosen for the examples but can be changed by adjusting the applied load Q or w using tributary areas.

For categories 1 and 2, the length of each bay is assumed to stay the same, while the number of bays (n) can vary and thus produce longer spans. A diagram of the tributary area used to get nodal loads Q is shown below.

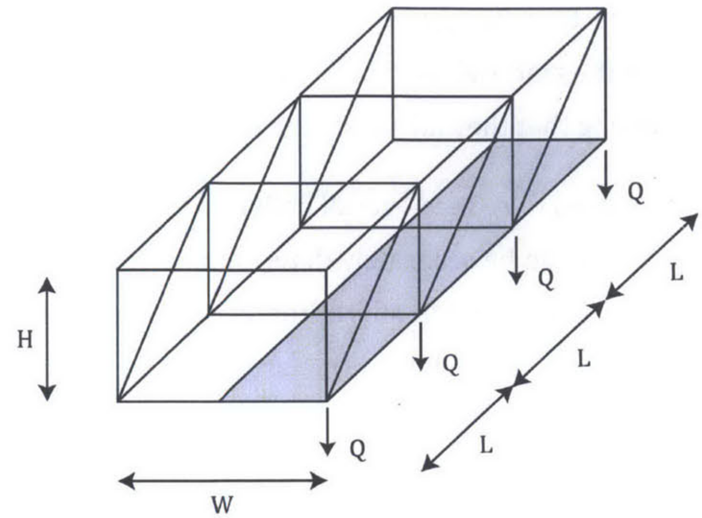


Figure 2: Equivalent nodal loads Q

Live Load is only considered in deflection calculations since dead load can be cambered out. Methods to camber involve bending steel in the shop or specifying certain members to be shorter such that the overall structure is cambered up. In concrete, the formwork can be built to slope upwards, or the use of Post/Pre Tensioning rods located above the center of gravity of the slab can be used to pull the slab upwards.

Category 1: Steel Braced Truss

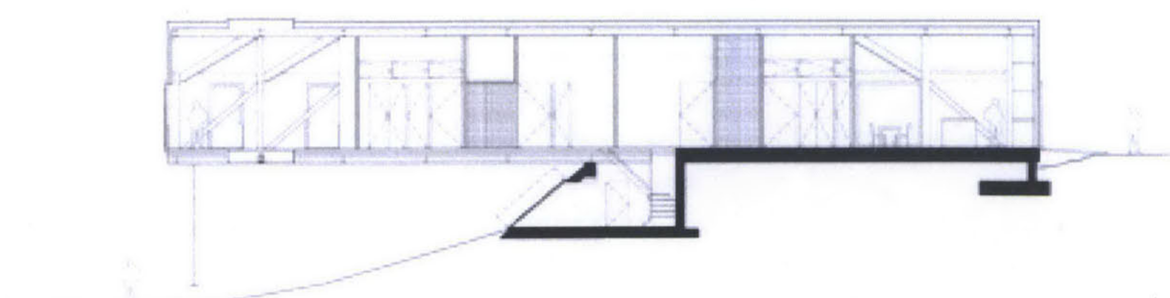
1.1 Case Studies

The following case studies presented are all recently completed projects and are only a sample of built cantilevers that have a steel truss as a structural system. An interior view is provided to show the different ways designers have incorporated the structure into the aesthetic of the space.

Balancing Barn	
Architect	MVRDV and Mole architects
Structural Engineer	Jane Wernick Associates
Location	Suffolk, UK
Year	2007-2010
Cantilever Length	57 ft

http://www.mvrdv.nl/en/

http://www.mvrdv.nl/en/



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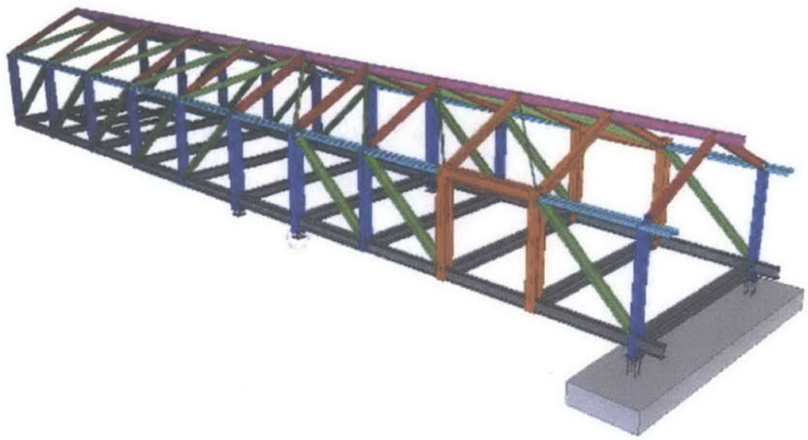

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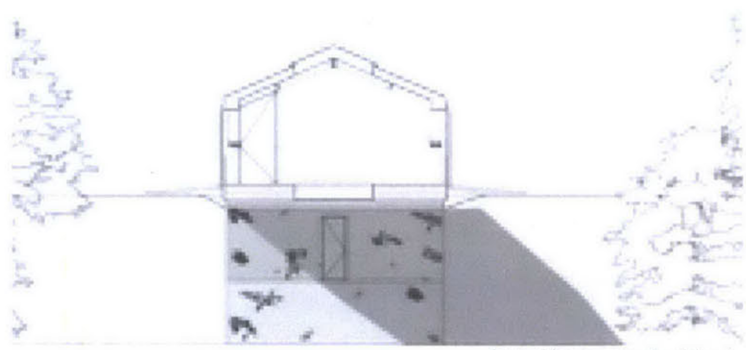
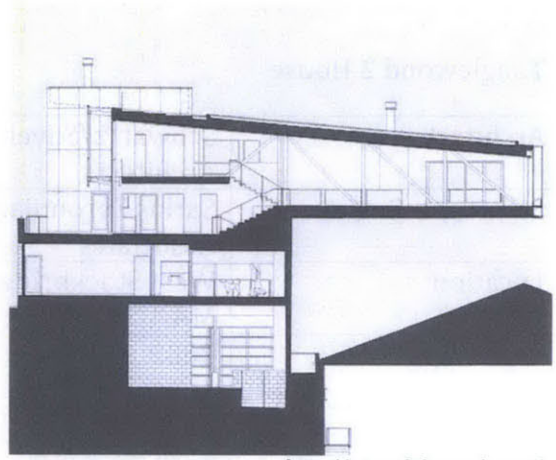


http://www.mvrdv.nl/en/

Emerald Art Glass house

Architect	Fisher Architecture
Structural Engineer	Eric Fisher
Location	Pittsburgh, PA
Year	2011
Cantilever Length	53 ft



http://www.fisherarch.com/

http://www.fisherarch.com/

http://www.fisherarch.com/

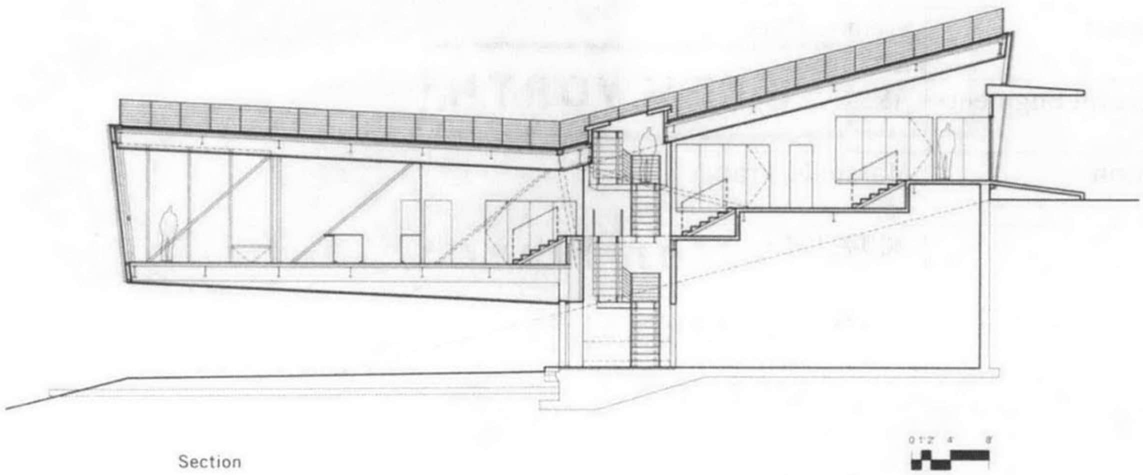
Tanglewood 2 House

Architect	Schwartz/Silver Architects
Structural Engineer	Sarkis Zerounian & Associates
Location	West Stockbridge, MA, USA
Year	2009
Cantilever Length	45 ft

http://schwartzsilver.com/

http://schwartzsilver.com/

http://schwartzsilver.com/



Section

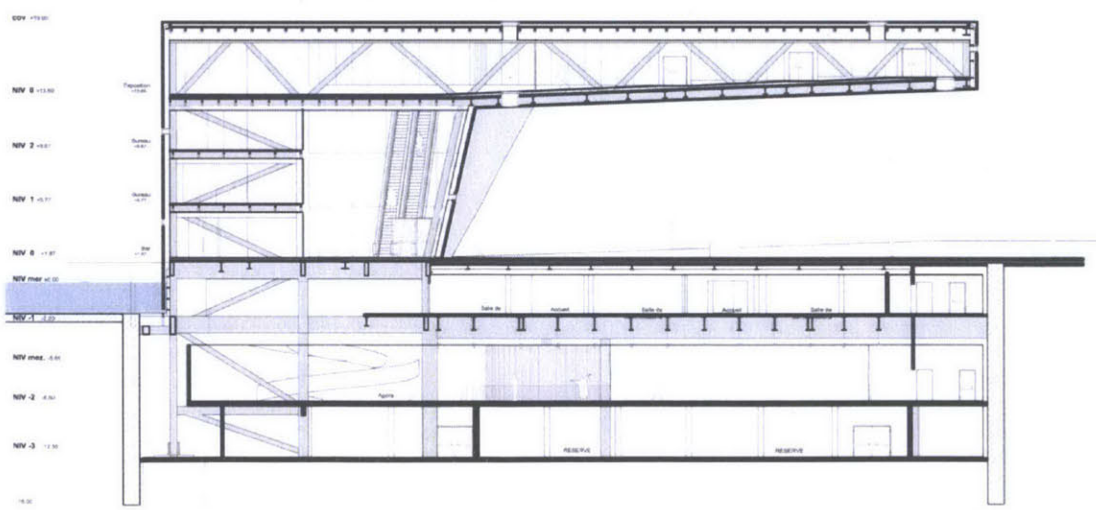
http://schwartzsilver.com/

Villa Méditerranée

Architect	Stefano Boeri Architetti
Structural Engineer	AR&C
Location	Marseille, France
Year	2004-2013
Cantilever Length	131 ft

http://www.stefanoboeriarchitetti.net/en/

http://www.stefanoboeriarchitetti.net/en/



http://www.stefanoboeriarchitetti.net/en/

Nest and Cave House

Architect	Idis Turato
Structural Engineer	Ivan Arbanas
Location	Opatija, Croatia
Year	2012
Cantilever Length	55 ft

http://www.archdaily.com/297648/nest-cave-house-idis-turato/

http://www.archdaily.com/297648/nest-cave-house-idis-turato/

http://www.archdaily.com/297648/nest-cave-house-idis-turato/

Krishna P. Singh Center for Nanotechnology	
Architect	Weiss/Manfredi
Structural Engineer	Severud Associates
Location	Philadelphia, USA
Year	2013
Cantilever Length	68 ft

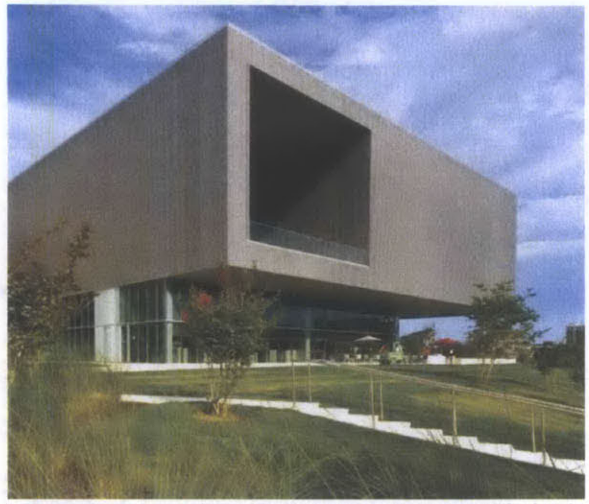
http://www.weissmanfredi.com/

http://www.weissmanfredi.com/

http://www.weissmanfredi.com/

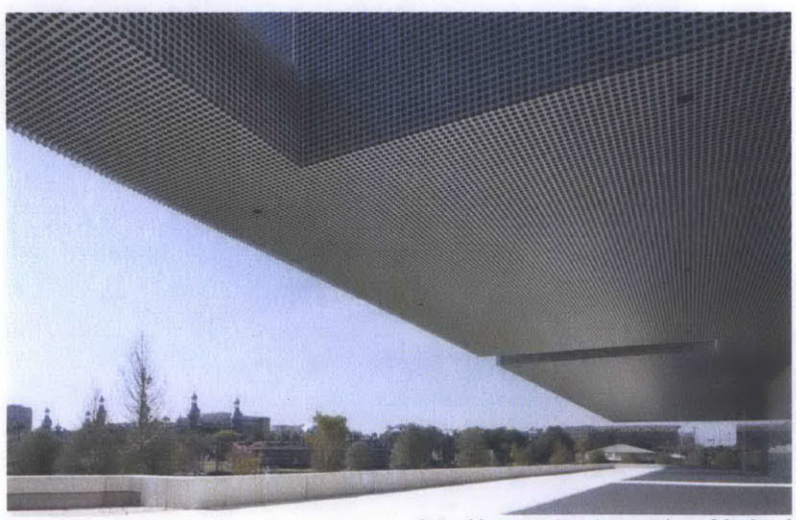
Tampa Museum of Art

Architect	Stanley Saitowitz Office / Natoma Architects
Structural Engineer	Walter P Moore
Location	Tampa, Florida
Year	2010
Cantilever Length	40 ft



http://www.saitowitz.com/portfolio.html

Structural Engineering & Design (Aug 2010)



http://www.saitowitz.com/portfolio.html

1.2 Model of Steel Braced Truss

The steel braced truss is a very common strategy for cantilever design. The members act in either tension or compression and the members can be an integral part of the cantilever's aesthetic (both interior and exterior). Since member sizes are thin compared to the overall size of the cantilever, this scheme allows for glazing and views in between the diagonal members. The geometry of a steel truss structural system can be simplified to the following model (shown in Figure 3) with the variables used listed in the table below.

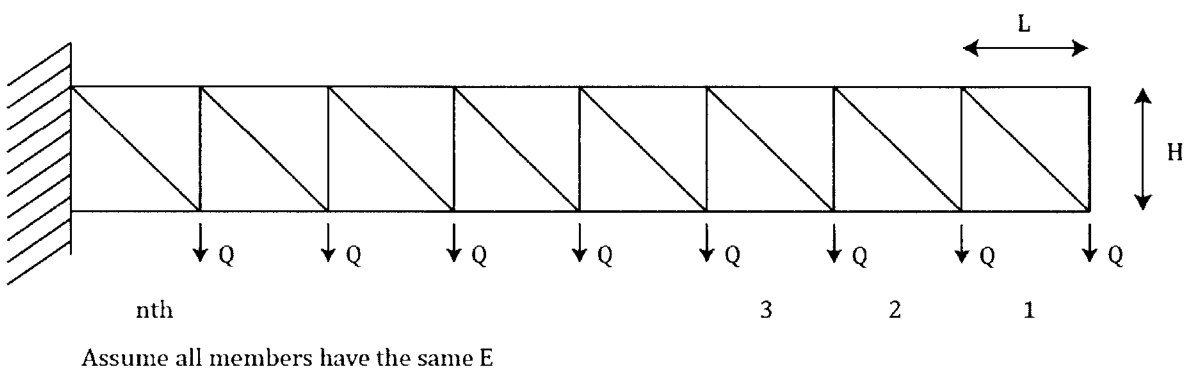


Figure 3: Model of a Steel Braced Truss

Variables	Description	Units
H	Height	Inches
L	Length of one bay	Inches
n	Number of bays	
Q	Applied nodal load (width of truss determines the magnitude but is kept to 1 Kip in examples)	Kips
A_i	Cross sectional area of a particular member in the i^{th} bay	in^2
$A_{Diagonal}$	Area of all diagonal members in the truss	in^2
A_{Top}	Area of all top members in the truss	in^2
A_{Bottom}	Area of all bottom members in the truss	in^2
$A_{Vertical}$	Area of all vertical members in the truss	in^2
	$\sigma_Y = 50$ Ksi	
s	Diagonal length (s) $= \sqrt{H^2 + L^2}$	in
ρ	Density of steel $= 0.26 \frac{lbs}{in^3}$	

The forces in each member of the truss can be solved using the method of joints and written in terms of the i^th^ number of bays counting from the tip of the cantilever.

$$\text{Force\_Diagonal member}_i = \frac{iQs}{H}$$

$$\text{Force\_Top member}_i = \frac{(i^2 - i)QL}{2H}$$

$$\text{Force\_Bottom member}_i = \frac{-(i^2 + i)QL}{2H}$$

$$\text{Force\_Vertical}_i = -(i - 1)Q$$

1.2.1 Deflection Criteria

The deflection caused by each member force is governed by

$$u = \frac{1}{Q}\sum_i \frac{F_i^2 L_i}{EA_i} \qquad \text{where } i = \text{for each bay of the truss}$$

$$F_i = \text{force of each member}$$

Thus, the total deflection is the summation of the deflection of each member for all members of the truss:

$$u_{total} = \frac{Q}{E}\left[\sum_i^n \left(\frac{i^2 s^3}{H^2 A_{Diagonal}} + \frac{L^3(i^2 - i)^2}{4H^2 A_{Top}} + \frac{L^3(i^2 + i)^2}{4H^2 A_{Bottom}} + \frac{H(i-1)^2}{A_{Vertical}}\right)\right]$$

Assume a deflection limit of $\frac{nL}{360}$

Total weight of the truss is the volume of steel multiplied by the density of steel:

$$W = \left[snA_{Diagonal} + LnA_{Top} + LnA_{Bottom} + HnA_{Vertical}\right] \times \rho$$

The total Cost function (C) to be optimized against member areas is not a monetary dollar amount but more of a sum of weight and deflection, factors that are important considerations to the design of a cantilever. A constant α is introduced as an adjustment factor for both consistency of units and for control on deflection. For larger α values, the cantilever is more deflection sensitive and thus Cost is more deflection governed than weight governed. The numerical values of α appear in the design example in the 1.2.2 Design Example and is a constant selected by the designer to achieve a target deflection. In this study, the target deflection is the deflection limit of L/360.

The Cost equation is

$$\text{Cost} = \text{Weight} + \alpha \times u_{total}$$

For a generic steel truss braced frame, an optimal area of diagonal, top, bottom and vertical members can be found while minimizing overall cost. The variables height (H) and number of bays (n) will be chosen as they are the most interesting to analyze. The length of each bay (L) remains constant but the total span increases with (n). Point load Q can be assumed to be 20 Kips and can be scaled up or down for other loading cases, depending on the tributary width of the truss.

Optimizing Cost with respect to different member areas give:

$$\frac{dC}{dA_{Diagonal}} = 0 \quad \rightarrow \quad A_{Diagonal\_optimal} = \sqrt{\frac{\alpha Q s^2}{\rho E n H^2}\sum_{i}^{n} i^2}$$

$$\frac{dC}{dA_{Top}} = 0 \quad \rightarrow \quad A_{Top\_optimal} = \sqrt{\frac{\alpha Q L^2}{4\rho E n H^2}\sum_{i}^{n} (i^2 - i)^2}$$

$$\frac{dC}{dA_{Bottom}} = 0 \quad \rightarrow \quad A_{Bottom\_optimal} = \sqrt{\frac{\alpha Q L^2}{4\rho E n H^2}\sum_{i}^{n} (i^2 + i)^2}$$

$$\frac{dC}{dA_{Vertical}} = 0 \quad \rightarrow \quad A_{Vertical\_optimal} = \sqrt{\frac{\alpha Q}{\rho E n}\sum_{i}^{n} (i - 1)^2}$$

It is important to note that the optimization of each member type (Diagonal, Top, Bottom, Vertical chords of the truss) can be done independent of other members. This is because the truss is statically determinate, so the member forces do not depend on the member areas. Thus, the optimal truss overall is the synthesis of the areas found from each optimization.

1.2.2 Design Example

An optimization procedure for height (H) is computed to illustrate how this optimization process would be done. The following parameters are chosen as dimensions:

H = (10:30) * 12	Height of each bay (in)
L = 20*12	Length of each bay (in)
α =39500	Weighing constant in Cost function
Deflection limit = 3.33 in	
n = 5	number of bays
E = 29000	Ksi
Q = 20	Nodal Load (Kips)

The range of feasible heights for a truss of this size (see category 1 for general dimensions) are H = 10 ft to 30 ft. The total length of this truss is 20 ft per bay with 5 bays. For each value of H (done in 1 ft increments), an optimum area of the diagonal member is found and plotted. The relationship between Area and Height is derived:

$$A_{\mathrm{Diagonal\_optimal}} = \sqrt{\frac{\alpha Q s^2}{\rho E n H^2}\sum_{i}^{n} i^2} \quad \rightarrow \quad A_{Diagonal\_optimal} \propto \frac{1}{H}$$

Similarly, for each value of H, an optimum area of the top and bottom members is found:

$$A_{Top\_optimal} = \sqrt{\frac{\alpha Q L^2}{4EnH^2}\sum_{i}^{n}(i^2 - i)^2} \quad \rightarrow \quad A_{Top\_optimal} \propto \frac{1}{H}$$

$$A_{Bottom\_optimal} = \sqrt{\frac{\alpha Q L^2}{4EnH^2}\sum_{i}^{n}(i^2 + i)^2} \quad \rightarrow \quad A_{Bottom\_optimal} \propto \frac{1}{H}$$

The optimal area of the vertical members do not depend on H, and thus there is no optimal solution for this design example. The area of the vertical member does not change with a changing height of the truss and the relationship is a horizontal line for different heights. These area and height relationships are shown in Figure 4, Figure 5 and Figure 6.

Each point on the following plots represents an alternative truss design, with the points from the left starting at truss height = 10 ft to height = 30 ft. Each of these heights has one corresponding optimal area for each group of members.

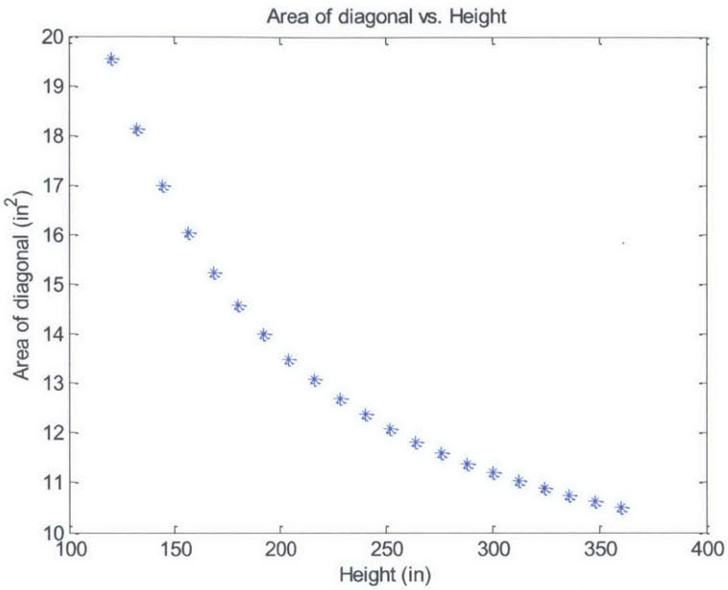


Figure 4: Area of optimal diagonal member vs. Height

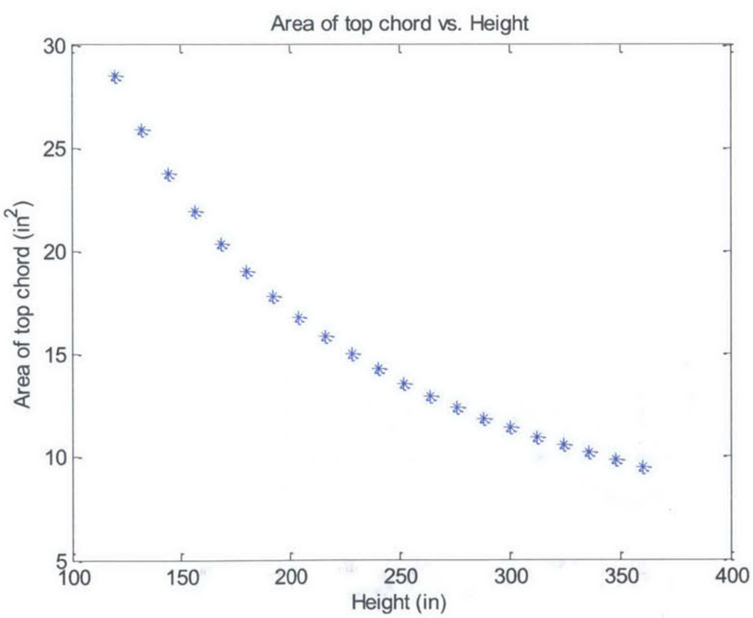


Figure 5: Area of optimal top member vs. Height

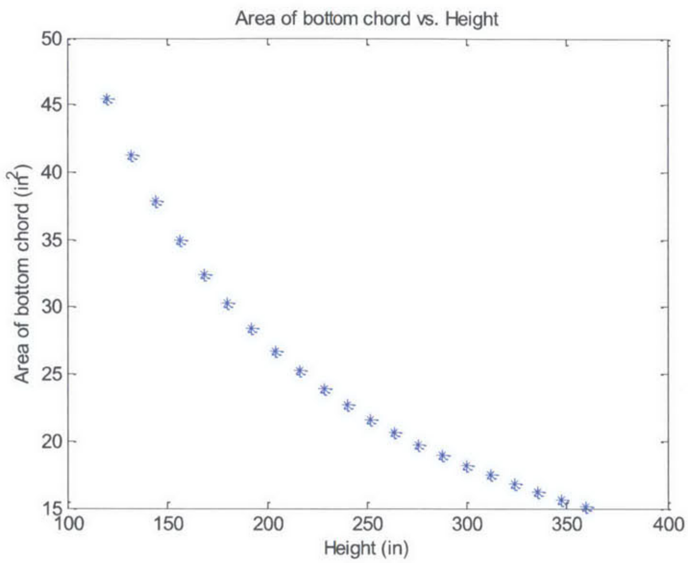


Figure 6: Area of optimal bottom member vs. Height

Weight = Area × Member length, and both area and length are increasing with increasing truss height. This is because as the height of the truss increases it also elongates the diagonal and vertical members (but not the top or bottom members). An optimal area is found for when every member of the truss has the same area. For each optimal area, a plot of it against weight shows that as area of the members increase, the weight of the structure (which is Area × Length) also increases in an almost linear relationship.

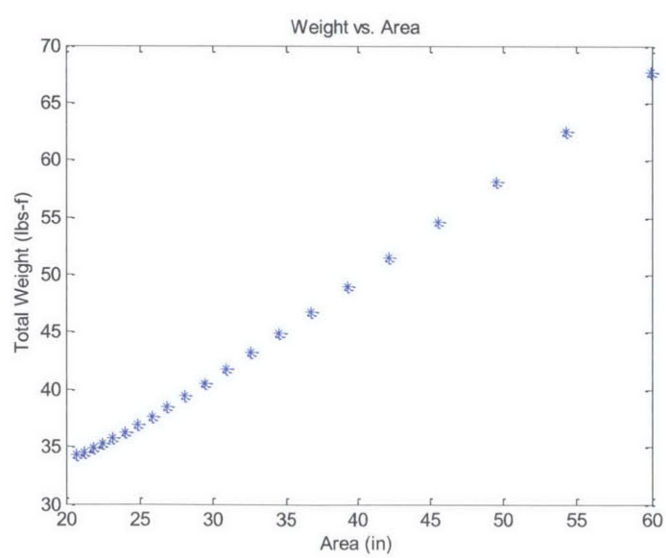


Figure 7: Optimal Area vs. Weight

Similarly, a plot of weight against the changing variable height shows the same relationship. From above we know that Area of top, bottom and diagonals are inversely proportional to Height Area $\propto \frac{1}{H}$ (again, area of vertical members is not related to height). Increasing the height of the truss also lengthens the diagonal chord s $(s = \sqrt{H^2 + L^2})$. This graph shows that as height of the truss increases, there is greater bending capacity and the weight of the truss reduces in a W $\propto \frac{1}{H}$ relationship.

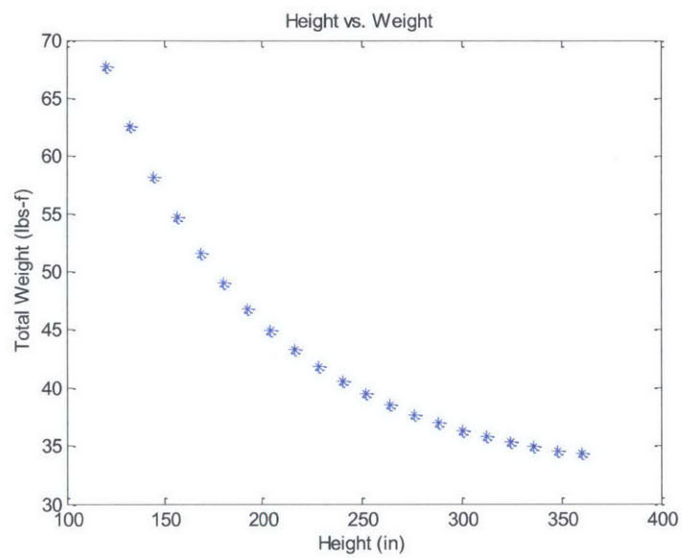


Figure 8: Height of truss vs. Weight

For each truss height ranging from 10 to 30 ft, an optimal diagonal member area can be found. This optimal area is then used to find a corresponding deflection due to diagonal members.

$$u_{Diagonal} \propto \left[\frac{\left(\sqrt{H^2 + L^2}\right)^3}{H^2 A_{Diagonal}}\right]$$

Substituting in $A_{Diagonal} \propto \frac{1}{H}$ gives

$$u_{Diagonal} \propto H^2$$

The relationship between height and deflection is plotted, and a parabolic relationship with a minimum deflection is present (see Figure 9).

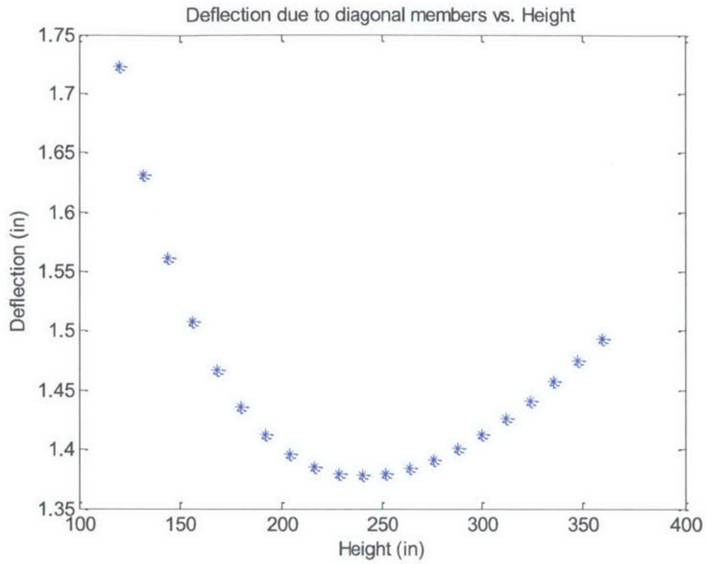


Figure 9: Deflection due to diagonal members vs. changing truss height

Similarly, by varying the truss height H, optimal areas of top and bottom members produce deflections in an inverse relationship:

$$u_{Top} \propto \left[\frac{L^3}{4H^2 A_{Top}}\right]$$

Substitute in $A_{Top} \propto \frac{1}{H}$

Gives $\boldsymbol{u_{Top}} \propto \boldsymbol{\frac{1}{H}}$

$$u_{total} \propto \left[\frac{L^3}{4H^2 A_{Bottom}}\right]$$

Substitute in $A_{Bottom} \propto \frac{1}{H}$

Gives $\boldsymbol{u_{Bottom}} \propto \boldsymbol{\frac{1}{H}}$

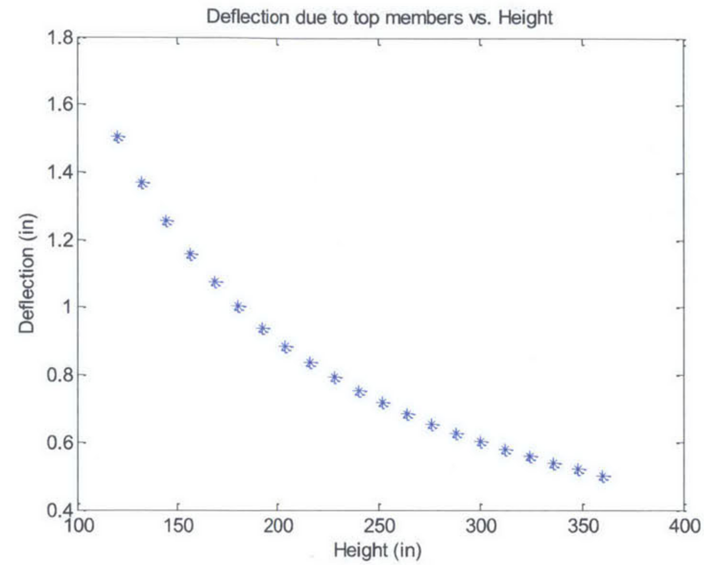


Figure 10: Deflection due to top members vs. height

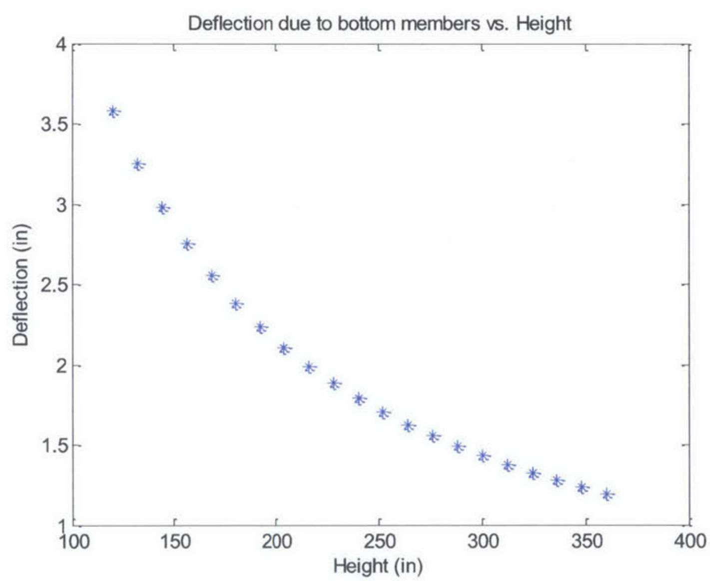


Figure 11: Deflection due to bottom members vs. height

Height does not factor into the optimal area of the vertical members, so a constant value of optimal vertical area is used to calculate the deflection. The relationship looks like this:

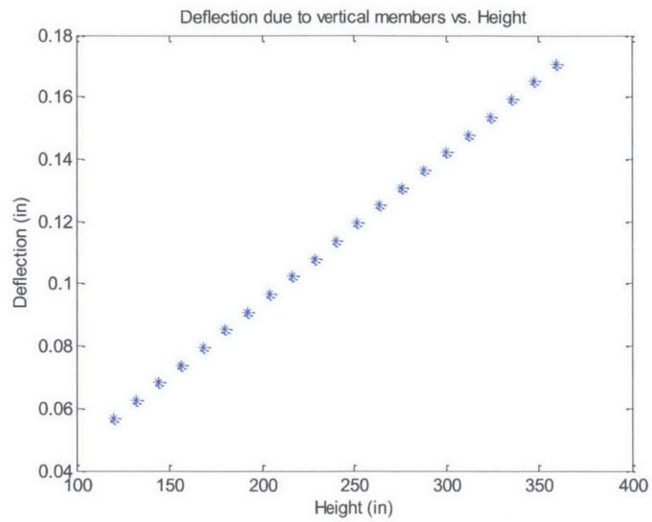


Figure 12: Deflection due to vertical members vs. height

Combining Figure 9, Figure 10, Figure 11, and Figure 12 gives the total deflection due to all members when height is varied. This is seen in Figure 13.

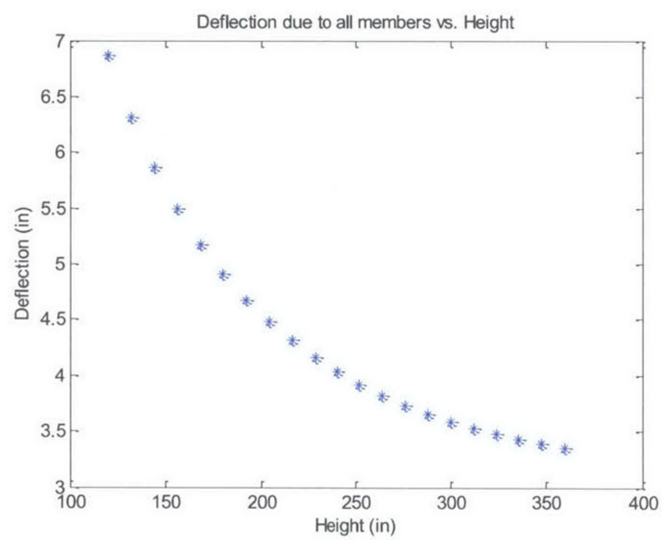


Figure 13: Total deflection vs. height

Lastly, of concern to the designer is the overall cost of the structural system. A plot of height against Cost shows that it follows a similar relationship to the Height vs. Weight plot. Here, the curve is the summation of Figure 13 and Figure 8, with the deflection having a greater influence on Cost (due to a large α value).

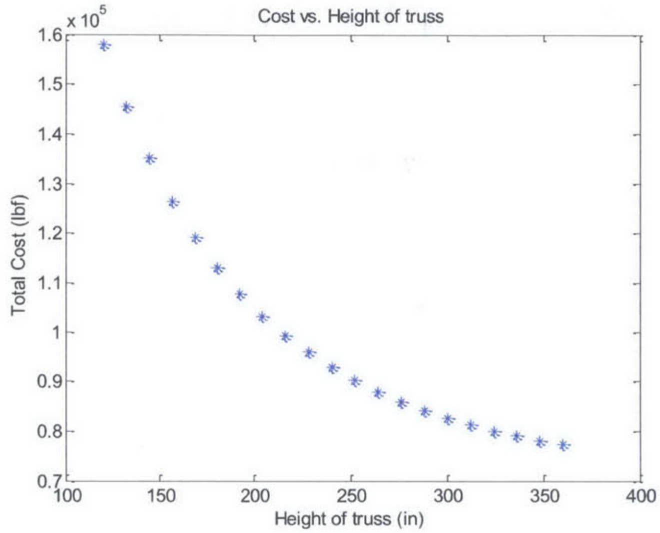


Figure 14: Cost vs. height

1.2.3 Varying the number of bays

Changing the variable n (number of bays) and thus changing the span also produces interesting results. From the design example, it is evident that weight and therefore cost is a cubic relationship to the number of bays (and thus the length of the cantilever). This relationship is proved with the analytical equations derived in 1.2.2 Design Example.

$$A_{Diagonal\_optimal} \propto \sqrt{\frac{1}{n}\sum_{i}^{n} i^2}$$

Using sum of series

$$\sum_{i}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$$

Substitute in to get

$$\boldsymbol{A_{Diagonal\_optimal} \propto n}$$

$$A_{Top_{optimal}} = \sqrt{\frac{1}{n}\sum_{i}^{n}(i^2 - i)^2}$$

Using sum of series

$$\sum_{i}^{n}(i^2 - i)^2 = \frac{1}{15}n(n-1)(n+1)(3n^2-2)$$

Substitute in to get

$$\boldsymbol{A_{Top\_optimal} \propto n^2}$$

$$A_{Bottom\_optimal} = \sqrt{\frac{1}{n}\sum_{i}^{n}(i^2 + i)^2}$$

Using sum of series

$$\sum_{i}^{n}(i^2 + i)^2 = \frac{1}{15}n(n+1)(n+2)(3n^2+6n+1)$$

Substitute in to get

$$\boldsymbol{A_{Bottom\_optimal} \propto n^2}$$

$$A_{Vertical\_optimal} = \sqrt{\frac{1}{n}\sum_{i}^{n}(i - 1)^2}$$

Using sum of series

$$\sum_{i}^{n}(i - 1)^2 = \frac{1}{6}n(2n^2-3n+1)$$

Substitute in to get

$$\boldsymbol{A_{Vertical\_optimal} \propto n}$$

$$Weight = 0.26\,\frac{lbs}{in^3} \times [snA_{Diagonal} + LnA_{Top} + LnA_{Bottom} + LnA_{Vertical}]$$

$$\boldsymbol{Weight \propto n^3}$$

1.3 Analysis

1.3.1 Steel Truss Optimization Comparison

The following example is shown to indicate the differences between precision in optimization of member areas and its impact on self-weight. The first analysis is referred to as "crude" because it is the simplest optimization scheme. Crude refers to the assumption that all the areas in the truss for all members are the same area (A) and thus only one area A is optimized. This produces the heaviest cantilever.

Next, "medium" refers to the assumption that each type of truss member (diagonal, top, bottom and verticals) is the same size. These areas A_d, A_t, A_b, A_v are then optimized and a total weight is found. "Fine" refers to the assumption that all members are a different size $A_{di}, A_{ti}, A_{bi}, A_{vi}$ where $i = 1$ to n bays. Each area is then optimized and a total weight is calculated. This produces the lightest cantilever.

The percentage differences in weight for each of the three options are shown in Table 1. It can be seen that the degree of precision to which a designer optimizes the truss does affect the structure's weight. Though this is only computed for a specific truss of dimensions shown in the table, the differences between crude, medium and fine will still be on the same order of magnitude. For designers who wish to cut down on self-weight and thus reduce deflection, being more precise with the member sizes could be enough of a gain to justify the extra effort in customizing individual member areas.

The optimization is done for different spans (the number of bays n increases) but each design satisfies its corresponding L/360 deflection limit by adjusting the value of α. As span increases, the L/360 deflection limit becomes harder to satisfy causing α to increase parabolically. The Cost plot also increases as span is increased, even when optimized areas are used.

L	20 ft
H	26 ft
Q	20 Kip
n	2 to 8 spans of length L = 20 ft

n	Weight crude vs. medium %	Weight medium vs. fine %	Weight crude vs. fine %
2	21.2%	14.3%	32.4%
3	13.4%	8.7%	21.0%
4	9.5%	10.8%	19.3%
5	8.1%	9.6%	16.9%
6	9.0%	24.7%	31.4%
7	9.2%	21.8%	29.0%
8	17.5%	13.5%	28.7%

Table 1: Weight comparisons between optimization schemes

Crude Optimization	Truss has one optimized area A
Medium Optimization	All diagonal members has an optimized area A_d All top members has an optimized area A_t All bottom members has an optimized area A_b All vertical members has an optimized area A_v
Fine Optimization	Every single member has a different optimized area

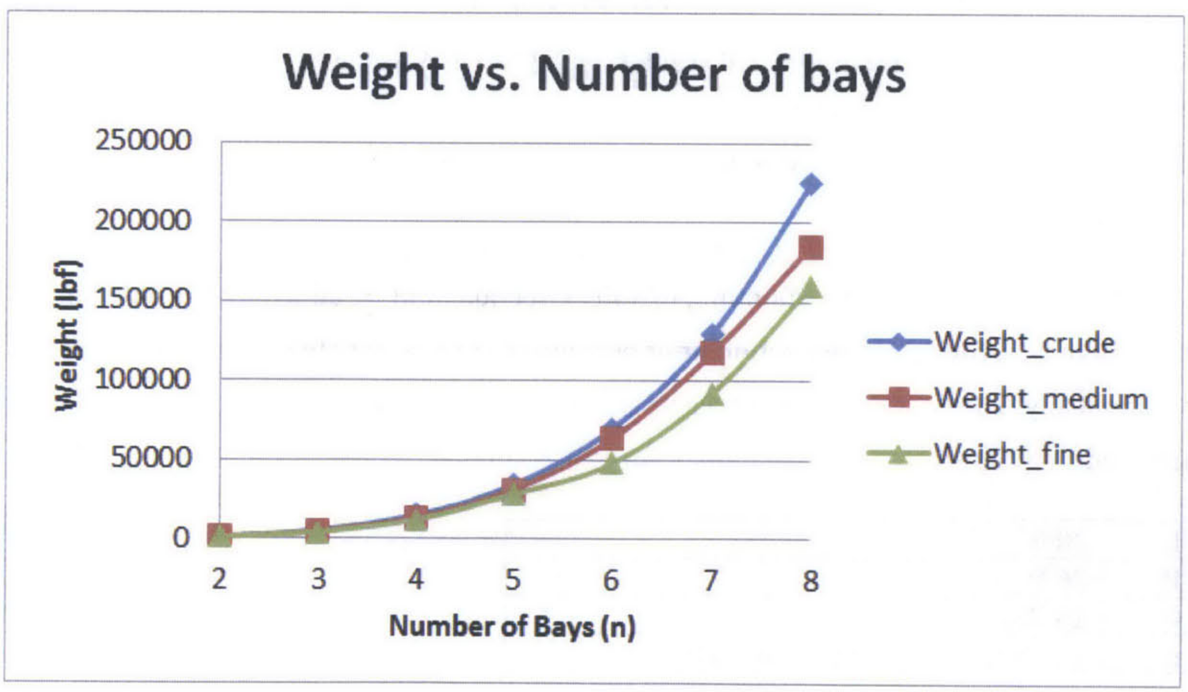


Figure 15: Weight vs. number of bays with optimized area

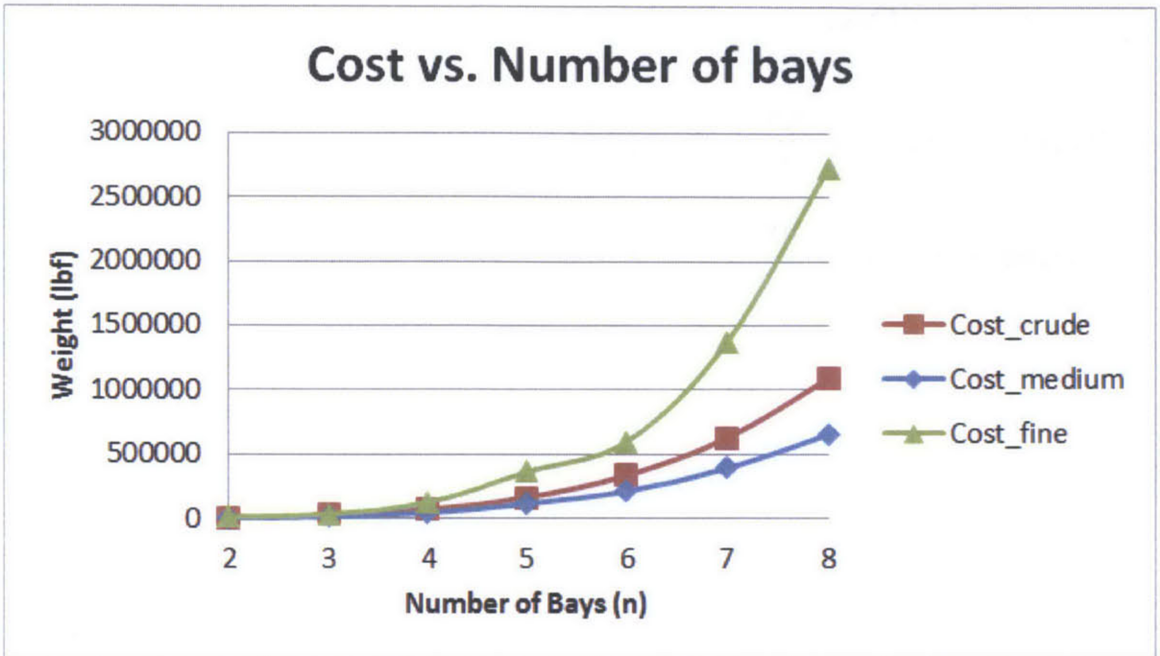


Figure 16: Cost vs. number of bays with optimized area

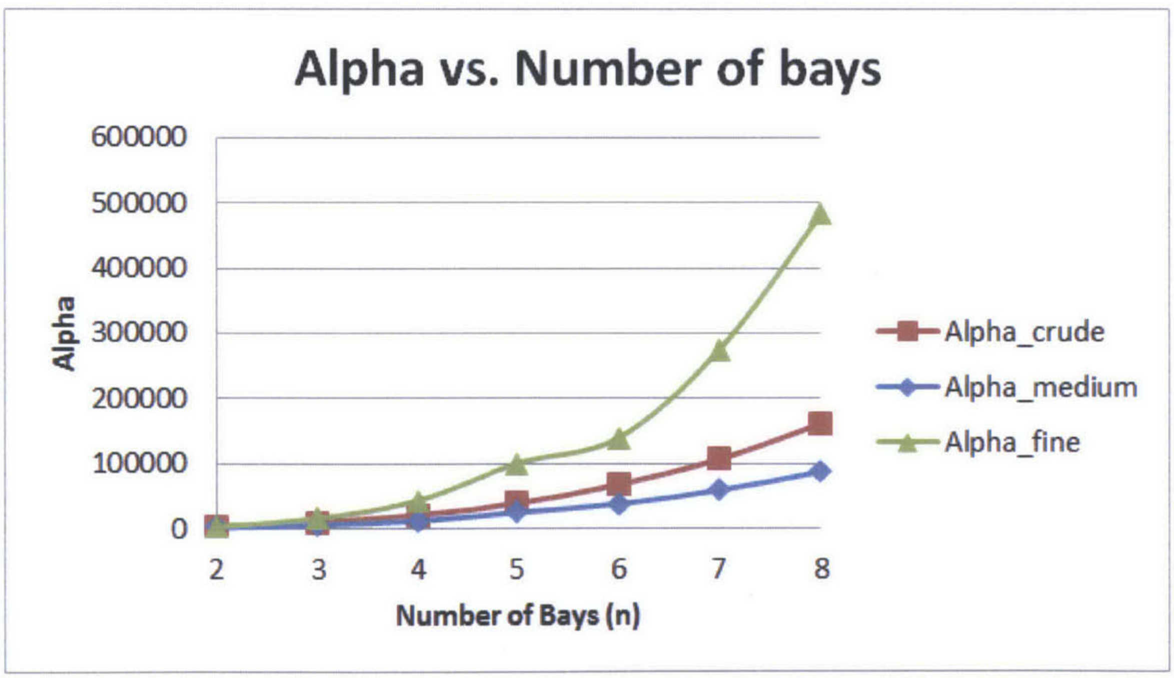


Figure 17: Alpha vs. span

1.3.2 Strength Criteria Check

$$\text{Force\_Diagonal member}_n = \frac{nQs}{H} = \frac{8 \times 20 \times 32.8}{26} = 202 \text{ Kips}$$

$$\text{Force}_{\text{Top}}\text{member}_n = \frac{(n^2 - n)QL}{2H} = \frac{(8^2 - 8) \times 1 \times 20 \times 20}{2 \times 26} = 430 \text{ Kips}$$

$$\text{Force\_Bottom member}_n = \frac{-(n^2 + n)QL}{2H} = \frac{-(8^2 + 8) \times 20 \times 20}{2 \times 26} = -554 \text{ Kips}$$

$$\text{Force\_Vertical member}_n = -(n-1)Q = -(8-1) \times 20 = -140 \text{ Kips}$$

$A_D = 47 \quad \rightarrow \quad \sigma_D = 4.3$ Ksi

$A_T = 73.35 \text{ in}^2 \quad \rightarrow \quad \sigma_T = 5.9$ Ksi

$A_B = 98.7 \text{ in}^2 \quad \rightarrow \quad \sigma_T = 5.6$ Ksi

$A_V = 98.7 \text{ in}^2 \quad \rightarrow \quad \sigma_T = 1.4$ Ksi

Thus the strength criteria of members are met. Buckling is not considered.

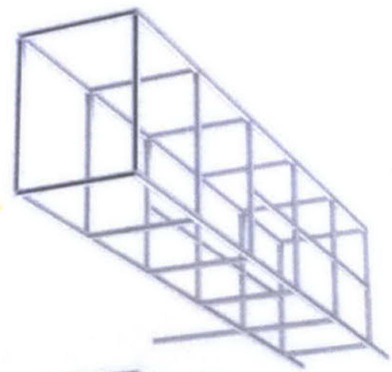
Category 2: Steel Vierendeel Truss

2.1 Case studies

The following case studies presented are built examples of a Vierendeel truss structural system. The beams act in bending, so they are larger than the axial members of the Steel Braced Truss. There are no diagonal members which offer more area for glazing.

Cantilever House

Architect	Anderson Anderson Architecture
Location	Granite Falls, Washington
Year	Prototype
Cantilever Length	Up to 32 ft



http://andersonanderson.com/

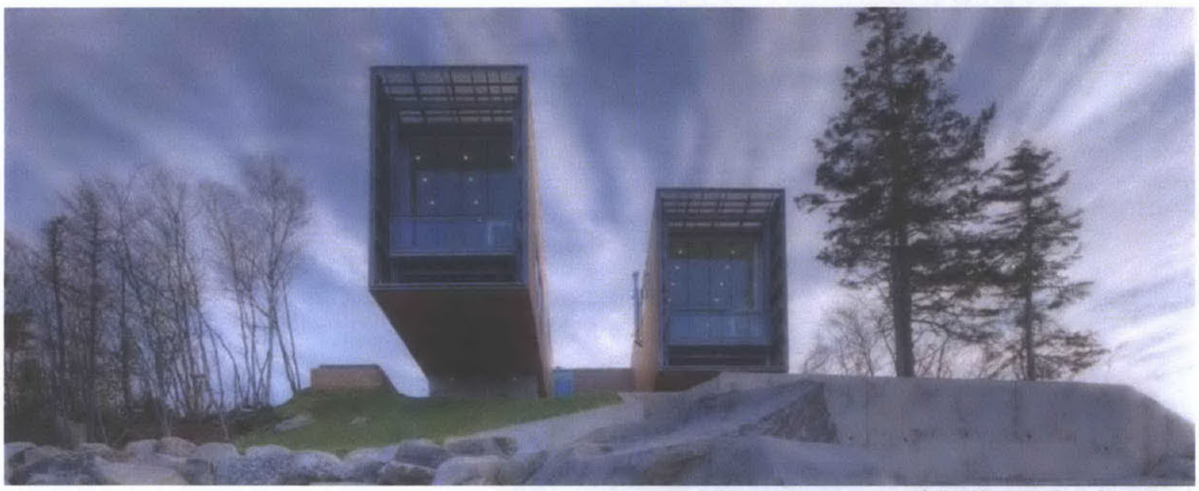
http://andersonanderson.com/

http://andersonanderson.com/

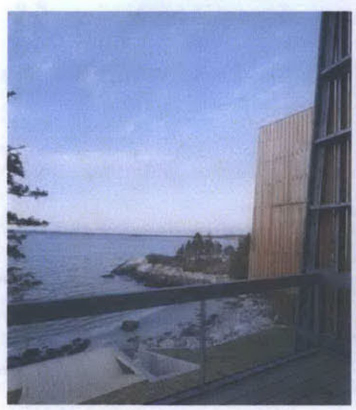
Two Hulls House

Architect	MacKay-Lyons Sweetapple Architects
Structural Engineer	Campbell Comeau Engineering Limited
Location	Nova Scotia, Canada
Year	2011
Cantilever Length	32 ft

http://www.mlsarchitects.ca/



http://www.mlsarchitects.ca/




http://www.mlsarchitects.ca/

2.2 Model of Steel Vierendeel Truss

This structural system is another common strategy for cantilever design and has the benefit of not having diagonal bracing, thereby freeing up the sides of the cantilever for views with floor to ceiling windows. The vertical members can be hidden from view to produce the thinnest cantilevers. The geometry of a Vierendeel truss can be simplified to the following model (shown in Figure 18) with the variables used listed in the table below.

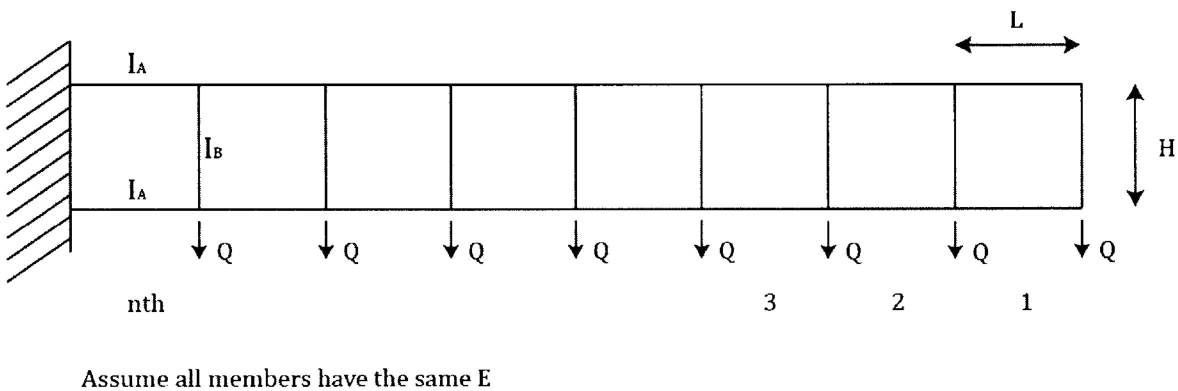


Figure 18: Simplified model of a Vierendeel Truss

Variables	Description	Units
H	Height	In
L	Length of one bay	In
n	Number of bays	
Q	Applied nodal load (width of truss determines the magnitude but is kept to 1 Kip in examples)	Kips
A	Cross sectional area	In^2
I	Moment of inertia in the strong axis - could also be I(x)	
s	Section modulus of bending members	In^3
	$\sigma_Y = 50\ Ksi$	
	$u_{limit} = \frac{nL}{360}$	

Modeling Assumptions

- Assume an I shape for bending members A and B
- Assume all members have the same Elastic Modulus
- Approximate the area that is acting in bending to be only the flanges of total area A. For simplicity of calculations, the contribution from the web is assumed negligible. Appendix A provides a worked example showing the percentage difference between moment of inertia with and without the web.
- $\frac{d}{2}$ is the distance from the centroid of top flange to centroid of bottom flange. The moment of inertia equations are therefore:

$$I_A = \frac{A_A {d_A}^2}{4} \qquad I_B = \frac{A_B {d_B}^2}{4}$$

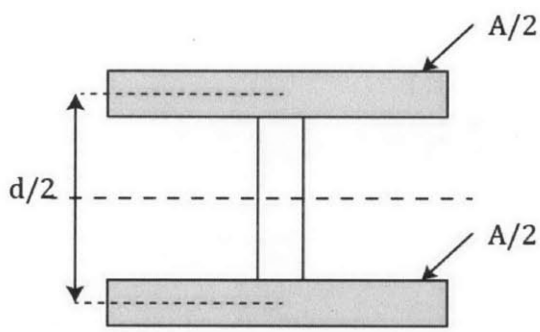


Figure 19: Cross section of bending member

2.2.1 Strength Criteria

Maximum Moment in members A at the support

$$M = \sum_{i}^{n} i \times \frac{Q}{2} \times L$$

Moment of inertia

$$I_A = \frac{A_A d_A^2}{4} \quad or \quad A_A = \frac{4 I_A}{d_A^2}$$

$$I_B = \frac{A_B d_B^2}{4} \quad or \quad A_B = \frac{4 I_B}{d_B^2}$$

Distance from top flange to neutral axis

$$y = \frac{d_A}{2}$$

Section Modulus

$$z = \frac{I}{y}$$

Substitute in for I and y gives:

$$z = \frac{A_A d_A}{2}$$

2.2.2 Deflection Criteria
Derivation of Model 1

To find the deflection equation of one bay, assume infinitely stiff B elements. Thus, vertical elements resist load but this model assumes no deformation of vertical elements. There are two bending members that share load Q.

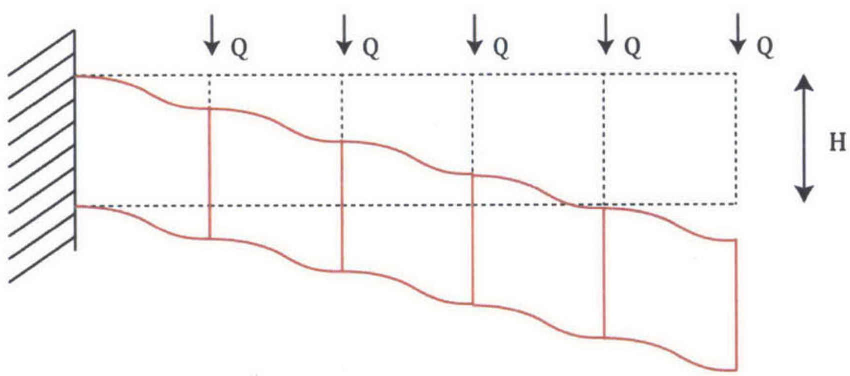


Figure 20: Deflected shape of model 1

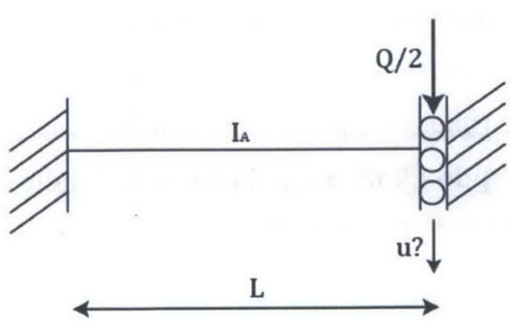


Figure 21: Model of a single bending member

Approximate the bending action of element A to be a cantilever with a rotation constraint, as shown in Figure 21. One bay can be modeled as shown in Figure 21; however, the parameter H (height of truss) is not present in the deflection equation. Slope deflection equations (equivalent to the exterior column of a moment frame) are applied:

$$V_L = \frac{6EI}{L^3}(-2u_0 - L\theta_0 + 2u_L - L\theta_L) + V_0^F$$

Boundary Conditions:

$$V_L^F = \frac{Q}{2} \qquad I = I_B \qquad L = \frac{H}{2} \qquad u_0 = 0 \qquad u_L = -u \qquad \theta_0 = \theta_L = 0$$

Substitute into slope deflection equations to get

$$\frac{Q}{2} = \frac{12EI_A u}{L^3}$$

$$k = \frac{24EI}{L^3}$$

$$u_{1Bay} = \frac{QL^3}{24EI_A}$$

The deflection of the entire truss is the summation of the deflection of each bay. The bay closest to the support carries the most point loads and thus experiences the greatest deflection.

$$U_{Total} = \sum_i^n i\left(\frac{QL^3}{24EI_A}\right)$$

Derivation of Model 2

Assume that both horizontal and vertical elements (A and B) act in bending, such that the model acts as a moment frame. The inflection points occur at midpoint of members, which can be represented as rollers supports. This model is does take into account of the dimension H and is a better approximation to the behavior of the actual truss.

Assumptions:

- Horizontal bending members have moment of inertia I_A
- Assume all A elements have the same bending rigidity EI_A
- Assume all B elements have the same bending rigidity EI_B
- There is an infection point at the center of every A and B element

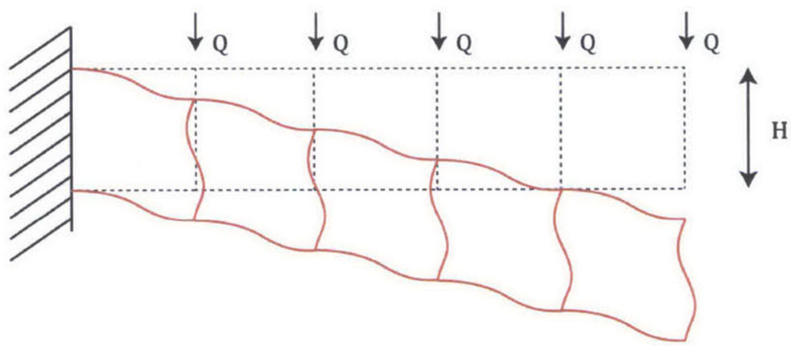


Figure 22: Deflected Shape of model 2

Apply slope deflection equations:

U = top displacement (from symmetry, central node displaces by 0.5u)

θ = rotation of central node

θ_B = rotation at center of B element

θ_A = rotation at center of A element

M_B = bending moment acting on B elements at central node

M_A = bending moment acting on A elements at central node

The slope deflection equation for the bending moment at the start of a beam is

$$M_0 = \frac{2EI}{L^3}\left(3Lu_0 + 2L^2\theta_0 - 3Lu_L + L^2\theta_L\right) + M_0^F$$

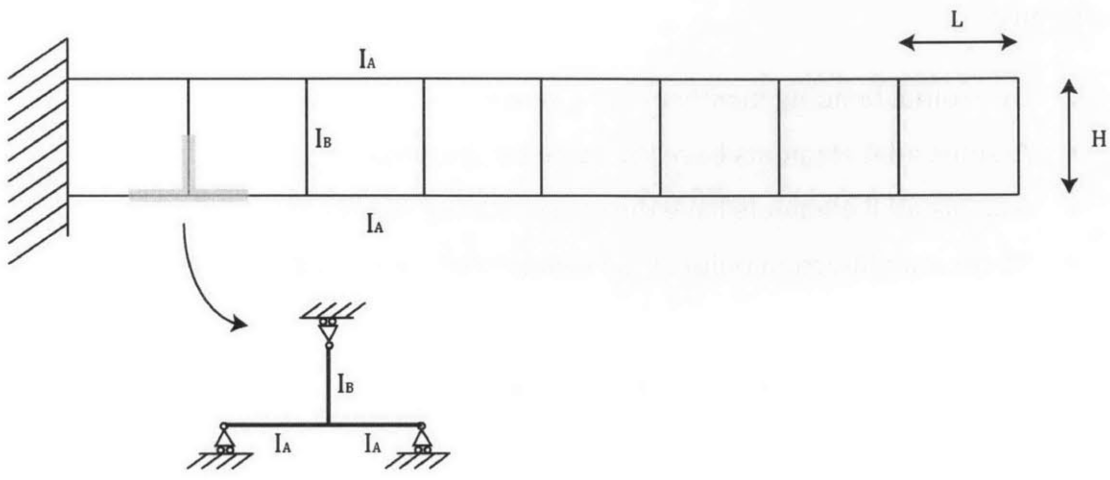


Figure 23: Simplified model of the Vierendeel truss

For the half element B to the left of the central node, the boundary conditions are:

$$M_0 = 0 \qquad I = I_B \qquad L = \frac{H}{2} \qquad u_0 = u_L = 0 \qquad \theta_0 = \theta_B \qquad \theta_L = 0$$

Substituting into the slope deflection equation and taking the sum of moments at the central node to equal zero gives $M_B + 2M_A = 0$

$$Q = K_{EQ} \text{ x } u$$

$$K_{EQ} = \frac{12EI_A}{L^3(1 + \frac{HI_A}{LI_B})}$$

For this Vierendeel truss, there are two bending members in each bay, thus

$$K_{total} = 2 \text{ x } K_{EQ}$$

$$K_{Total} = \frac{24EI_A}{L^3(1 + \frac{HI_A}{LI_B})}$$

$$U_{1Bay} = \frac{QL^3(1 + \frac{HI_A}{LI_B})}{24EI_A}$$

Again, the deflection of the entire truss is the summation of the deflection of each bay. The bay closest to the support carries the most point loads and thus experiences the greatest deflection. Comparing this equation to the deflection equation for model 1 shows that they are similar except for this term:

$$1 + \frac{\mathrm{HI_A}}{\mathrm{LI_B}}$$

This is a ratio of the stiffness of the horizontal members to that of the vertical members.

2.2.3 Cost Analysis

The cost analysis will use model 2's deflection equation since the height parameter (H) is an important design criteria and infinite stiffness of vertical members cannot be achieved in real life.

$$U_{Total} = \sum_{i}^{n} i \left(\frac{QL^3(1 + \frac{HI_A}{LI_B})}{24EI_A} \right)$$

The total weight of the truss is given by:

$$W = (Area\ of\ cross\ section \times Length\ of\ each\ member\ ) \times \rho$$

$$W = (\mathrm{H}A_B + 2LA_A\ ) \times n\ \times \rho$$

The total cost is given by the summation of weight and deflection.

$$Cost = W + (\alpha\ \times U_{Total})$$

Again, α is a control factor for deflection.

To optimize cost, differentiate with respect to the moment of inertia of the horizontal and vertical members.

$$\frac{dC}{dI_A} = 0 \quad \rightarrow \quad I_{A\_optimal} = \sqrt[2]{\frac{(\sum_i^n i)\alpha QL^2 d_A^2}{192\rho En}}$$

$$\frac{dC}{dI_B} = 0 \quad \rightarrow \quad I_{B\_optimal} = \sqrt[2]{\frac{(\sum_i^n i)\alpha QL^2 d_B^2}{96\rho En}}$$

2.3 Analysis

2.3.1 Design Example

An optimization procedure is done to illustrate an optimization scheme with different lengths of each bay. The following dimensions are chosen:

L = (10:30) * 12	Length of each bay (in)
H = 26*12	Height of beam (in)
α =11	Weighing constant in Cost function
Deflection limit = 2.0 in	
n = 6	number of bays
E = 29000	Ksi
Q = 10	Nodal Load (Kips)

The range of feasible lengths for a truss of this size (see category 2 for general dimensions) was L = 10 ft to 30 ft. The total height of this truss was 26 ft with 6 bays. For each value of L (analyzed in 1 ft increments), an optimum moment of inertia for both the horizontal and vertical members was found and plotted. This graph shows that

$$I_{A\_optimal} \propto \text{L}$$

$$I_{B_{optimal}} \propto \text{L}$$

which reflects the relationship found in. Figure 24: Optimal moment of inertia vs. Height

$$I_{A\_optimal} = 742.6\ in^4$$

$$I_{B\_optimal} = 1050.2\ in^4$$

The next relationship is how deflection changes with an increasing moment of inertia. From the deflection equation, taking the other parameters as constants give

$$u \propto \frac{1}{I_A} + \frac{1}{I_B}$$

which is seen in the plots in Figure 25.

$$Weight \propto A \propto I$$

thus the relationship is a linear one, where increasing the area of the cross sections increases its bending capacity but also the self-weight of the truss. This relationship can be seen in Figure 26.

The total cost curve is shown in Figure 27 is the sum of weight and deflection. Since α is small in this example, the contribution of deflection to cost is much smaller than weight, so the Cost curve looks very similar to the weight plots in Figure 26 and is therefore linear. When larger α values are used the contribution to the cost curve is more pronounced.

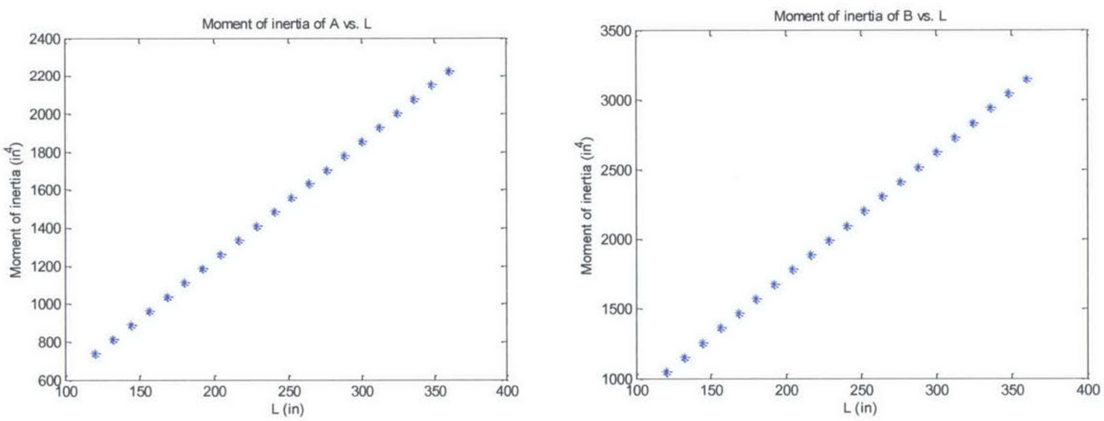


Figure 24: Optimal moment of inertia vs. Height

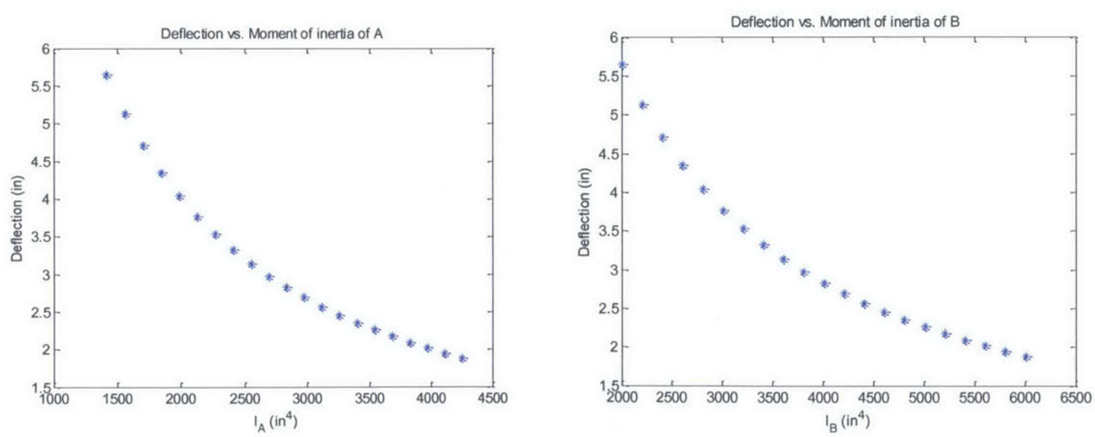


Figure 25: Moment of inertia vs. Truss Deflection

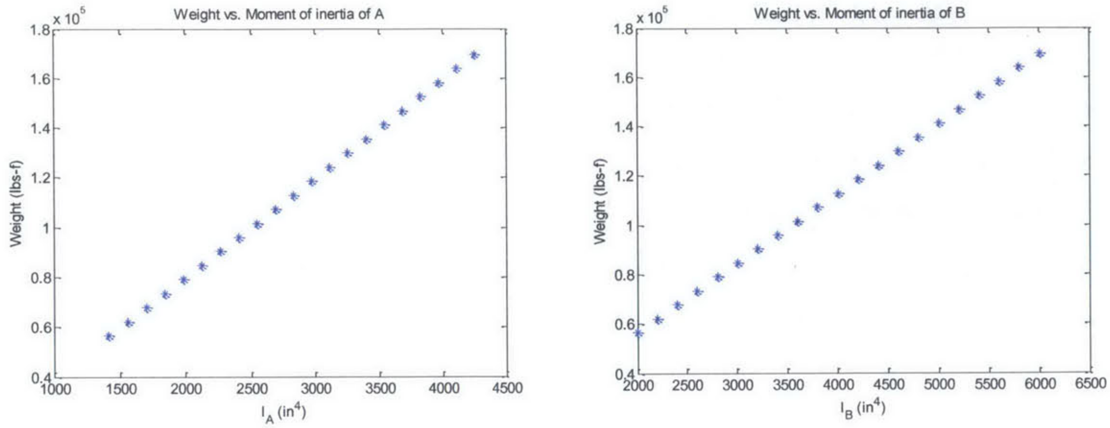


Figure 26: Moment of inertia vs. Weight

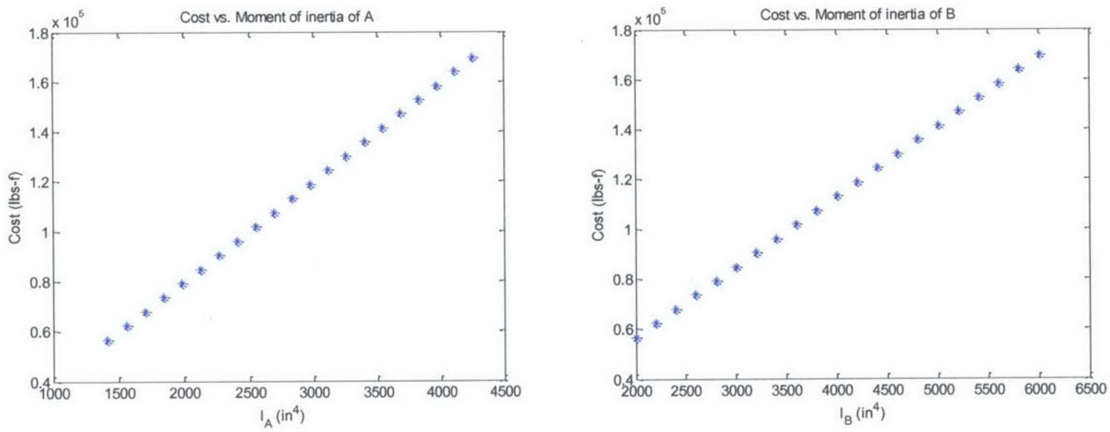


Figure 27: Moment of inertia vs. Cost

2.3.2 Category 2 Optimization comparison

The following example is shown to indicate the differences between precision in optimization of member areas and its impact on self-weight. The first analysis is referred to as "crude" because it is the simplest optimization scheme. Crude refers to the assumption that all horizontal members have the same optimized moment of inertia and all verticals also have the same optimized moment of inertia. This scheme produces the heavier cantilever. Next, "medium" refers to the assumption that each horizontal member has an optimized moment of inertia I_{Ai}, and each vertical member has an optimized moment of inertia, I_{Bi}. As the span increases, I_{Ai}, and I_{Bi} increase. This makes sense to have greater bending capacity closer to the support. This produces the lighter cantilever.

The percentage differences in weight for these two options are shown in Table 2. It shows that the degree of precision to which a designer optimizes the truss does affect the structure's weight. For designers who wish to cut down on self-weight and thus reduce deflection, being more precise with the member sizes could be enough of a gain to justify the extra effort in customizing member areas.

The optimization is done for different spans (the number of bays n increases) but each design satisfies its corresponding L/360 deflection limit. Figure 29 shows that the Cost plot increases as span is increased, even when optimized areas are used.

L	20 ft
H	26 ft
Q/2	10 Kip

	Weight crude vs. medium	Cost crude vs. medium
n	%	%
2	2.0%	1.9%
3	4.4%	4.2%
4	5.6%	5.3%
5	6.6%	6.3%
6	6.4%	5.9%
7	7.4%	6.8%
8	7.5%	6.9%

Table 2: percentage differences in alternative optimization schemes

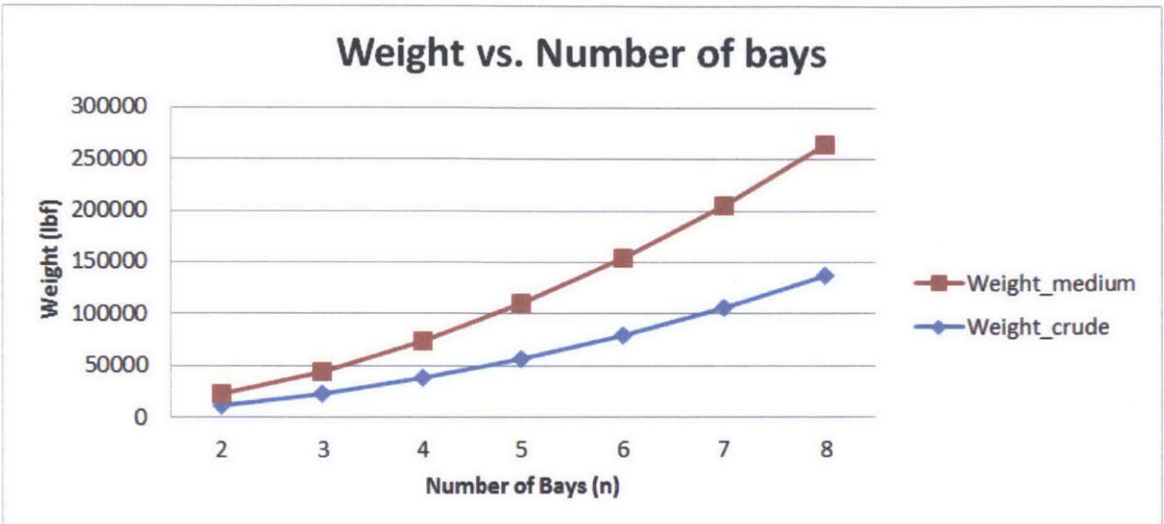


Figure 28: Weight vs. number of bays

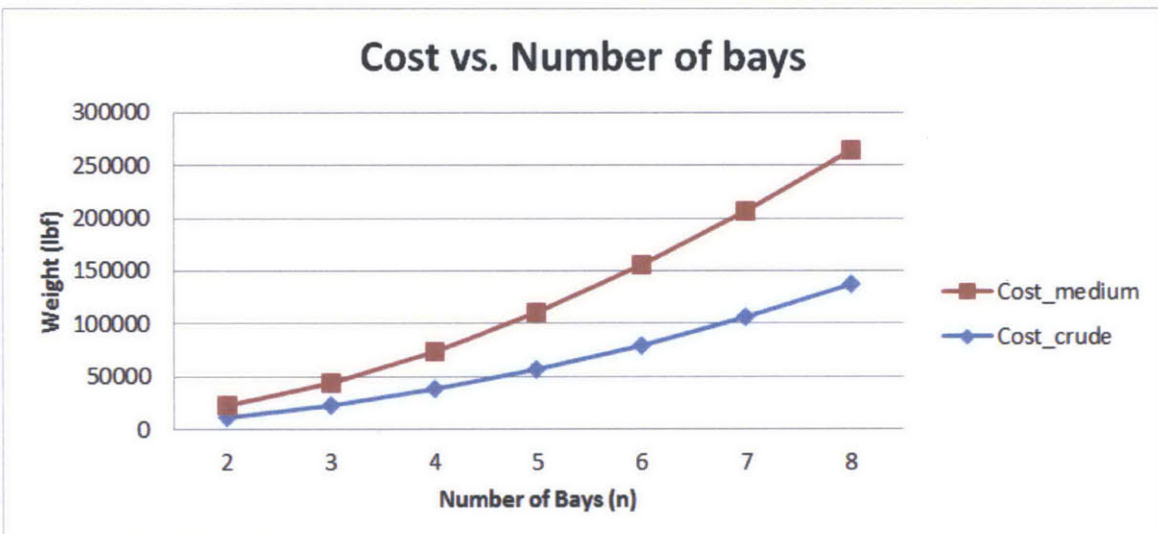


Figure 29: Cost vs. number of bays

Strength Criteria check

$$M = \frac{QL}{2} = \frac{10\text{ Kip} \times 20\text{ ft}}{2} = 100\text{ Kip} - \text{ft}$$

$$\text{Moment couple} = F \times d_A$$

$$F = \frac{100\text{ Kip} - \text{ft}}{12\text{ in}} = 100\text{ Kips}$$

Compare with $\sigma_Y = 50$ Ksi $\rightarrow$ $\sigma = \frac{F}{A_A} = \frac{100}{71.5} = 1.4$ Ksi

Category 3: Deep Beam

3.1 Case Studies

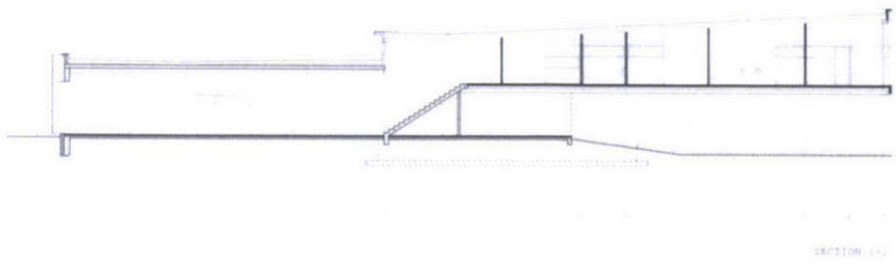
The following case studies presented are built examples of a deep beam structural system. The beams almost always vary in depth, with the deeper section greater at the support where it takes more loads. This variation in bending capacity along the beam is taken into account in the structural model in the next section. The structure is entirely below the floor slab allowing the designer freedom to do as they please with the interior and exterior.

Utriai Residence

Architect	Natkevicius & Partners
Structural Engineer	V.P.Čiras
Location	Klaipeda County, Lithuania
Year	2006
Cantilever Length	50 ft

http://www.archdaily.com/78438/utriai-residence-architectural-bureau-g-natkevicius-partners/

http://www.archdaily.com/78438/utriai-residence-architectural-bureau-g-natkevicius-partners/

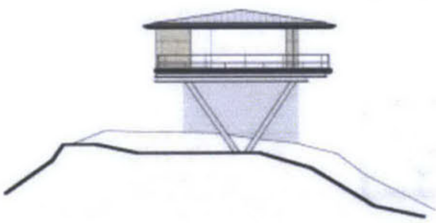


http://www.archdaily.com/78438/utriai-residence-architectural-bureau-g-natkevicius-partners/

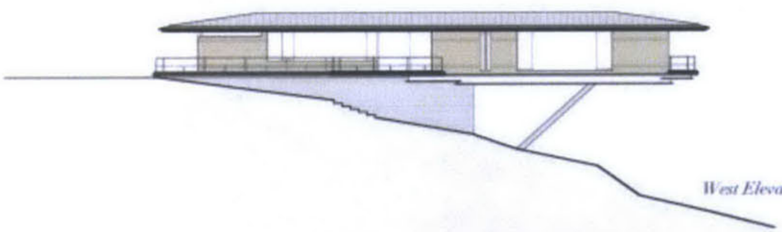
House in Yatsugatake
Note: with the supporting struts, this is not a true cantilever. If the struts were removed then it could be modeled as a deep beam.

Architect	Kidosaki Architects Studio
Structural Engineer	Toyohito Shibamura Structure Design, Takashi Manda, Mitsuru Kobayashi
Location	Yatsugatake, Japan
Year	2012

http://www.kidosaki.com/



South Elevation S=1:300



West Elevation S=1:300

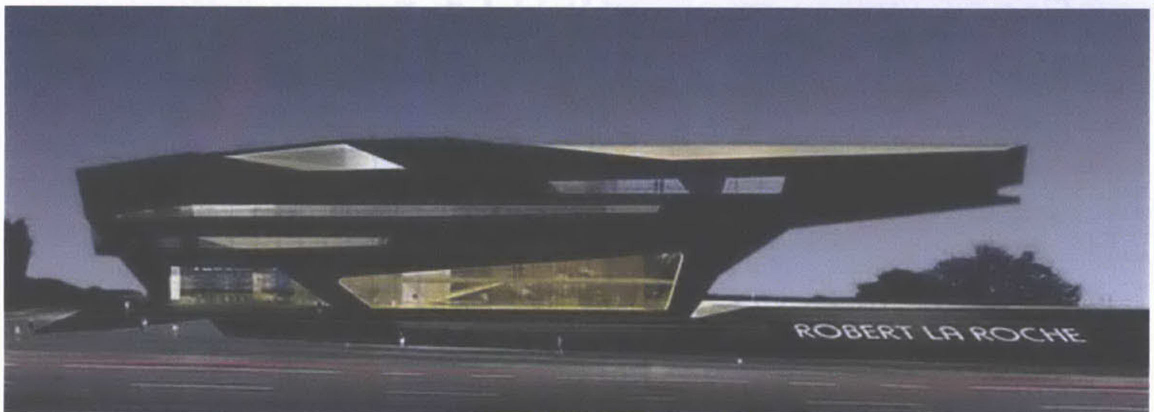
http://www.kidosaki.com/

http://www.kidosaki.com/

MP09 Headquarters

Architect	GS Architects
Structural Engineer	Wendl
Location	Graz, Austria
Year	2007-2009
Cantilever Length	Approx. 57 ft

http://www.gsarchitects.at/



http://www.gsarchitects.at/

http://www.gsarchitects.at/

3.2 Model of Deep Beam

This structural system is a common strategy for Reinforced Concrete. The depth of the concrete generally increases as it gets closer to the support so this model takes into account of a linearly varying height H(x). This system allows the cantilever space above to be free of structure, giving the designer more freedom to design the interior space (and thus allows unobstructed views). Compared to categories 1 and 2 this structural system weighs more because there is continuous material throughout rather than material where it is needed. The geometry of a deep beam structural system can be simplified to the following model (shown in Figure 30) with the variables used listed in the table below.

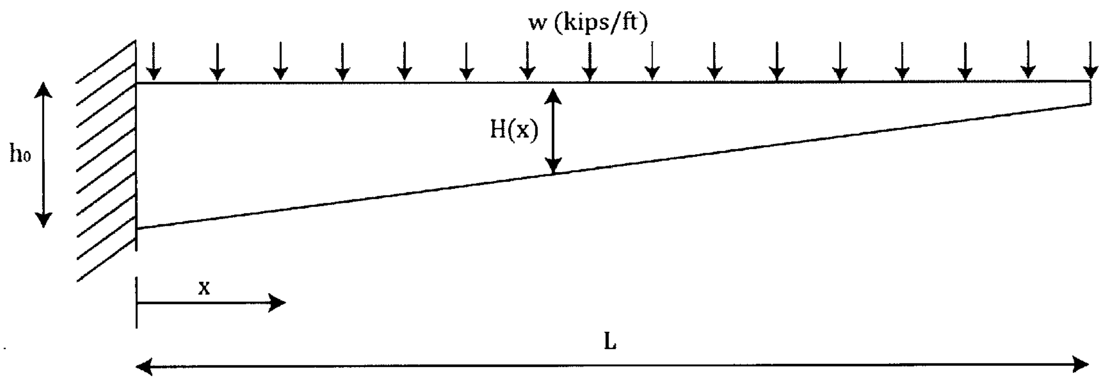


Figure 30: Model of Deep beam

Variables	Description	Units
H(x)	Height of Beam (can vary with x)	In
L	Length of one bay	In
b	Width of beam	
w	Applied distributed load on beam	Kips/ft
A	Cross sectional area	In^2
I(x)	Moment of inertia of the beam (can vary with x)	
s	Section modulus of bending members	In^3
h_0	Greatest depth of beam at the support	
	$\sigma_Y = 50\,Ksi$	

Note: Ignore slab thickness h_L in calculations to simplify the model, but the depth of the beam cannot be zero at the tip.

Modeling Assumptions

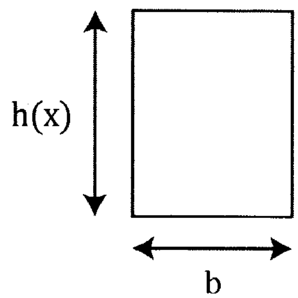


Figure 31: Cross Section of Beam

Assume a rectangular cross section of concrete where the top acts in tension (steel reinforcement is required) and the bottom of the cross section acts in compression. The area in compression needs to be checked that the concrete will not crush and fail.

3.2.1 Deflection Criteria

Height of the beam's cross section as it varies along the span can be written as:

$$H(x) = \frac{h_0}{L}(L - x)$$

Substituting this into the moment of inertia equation

$$I = \frac{bh^3}{12}$$

Gives $I(x) = \frac{b(h_0)^3}{12L^3}(L - x)^3$

Using the method of virtual forces to find the displacement at the tip of the beam:

$$\delta P \times u = \int_0^L \frac{M(x) \times \delta M(x)}{EI(x)} dx$$

Apply a δP force at the tip of the beam where maximum deflection occurs and find the corresponding moment equations under the real load and the virtual load.

$$M(x) = \frac{w(L - x)^2}{2}$$

$$\delta M(x) = \delta P(L - x)$$

$$\boldsymbol{u_{total} = \frac{6wL^4}{Ebh_0^3}}$$

Deflection Limit

$$u_{limit} = \frac{L}{360}$$

Design criteria

$$u \leq u_{limit}$$

$$Cost = Weight + \alpha \times u_{total}$$

$$Volume = \frac{bLh_0}{2}$$

$$Weight = Volume \times 0.26 \frac{lbs}{in^3}$$

$$\boldsymbol{Cost = 0.13bLh_0 + \frac{\alpha 6wL^4}{Ebh_0^3}}$$

To optimize the cost with respect to the maximum depth needed in the beam:

$$\frac{dC}{dh_0} = 0 \quad \rightarrow \quad h_0 = \sqrt[4]{\frac{80\alpha wL^3}{Eb^2}}$$

To optimize the cost with respect to the span of the beam L:

$$\frac{dC}{d\mathrm{L}} = 0 \quad \rightarrow \quad \mathrm{L} = \sqrt[3]{\frac{-0.13Eh_0{}^4b^2}{24\alpha w}}$$

This procedure can be used where a site constraint has a maximum depth the concrete can reach and the designer wants to find the longest span that still meets deflection criteria.

3.3 Analysis

Design Example

The following parameters are chosen to illustrate the optimization process:

w	1	Kip/ft
L	120 to 360	in
E	29000	Ksi
b	120	in
a	11000000	
u max	1	in

Table 3

For every length L ranging from 10 ft to 30 ft, an optimal depth h_0 is found. This relationship was plotted (see Figure 32). The graph indicates that the relationship is nonlinear, and this is proven by the equation:

$$h_0 = \sqrt[4]{\frac{80\alpha w L^3}{Eb^2}} \quad \rightarrow \quad \boldsymbol{h_0 \propto L^{\frac{3}{4}}}$$

Next, for each optimal beam depth, the deflection is found. Since each optimal depth increases with increasing span, the deflection grows faster than a linear relationship. The graph looks to be a polynomial relationship and is proven by the equation:

$$u_{total} = \frac{6wL^4}{Ebh_0^3} \quad \rightarrow \quad \boldsymbol{u_{total} \propto h_0^{\frac{7}{3}}}$$

Weight of the concrete structure is Area × Length. As the span increases, the optimal beam depth increases, thereby increasing the cross sectional area. The width of the beam stays constant. Since both length and Area are increasing, the Weight increases at a faster rate than a linear function. The graph looks to be a polynomial, and is proven in the equation below.

$$\text{Weight} = \rho b L h_0 \quad \rightarrow \quad \text{From the previous equation } h_0 \propto L^{\frac{3}{4}}$$

$$\boldsymbol{W \propto h_0^{\frac{7}{3}}}$$

Total Cost is a function of weight and α × deflection. Since weight and deflection increase in a polynomial manner, the Cost also looks like a parabolic relationship.

$$Cost = 0.13bLh_0 + \frac{\alpha 6wL^4}{Ebh_0^3} \quad \rightarrow \quad C \propto h_0^{\frac{7}{3}}$$

Another strategy to achieving longer spans in addition to optimizing dimensions is to Pre/Post tension (PT) the beam. This method uses tensioned steel rods (which are larger than the steel reinforcement typically used in beams or slabs) that put the concrete into compression. The tensioned rods increase the slab stiffness, which means it has greater bending capacity and thus spans longer distances. The use of PT slabs can also be done to control deflections; this is often done to camber the beam/slab under dead load. The tensioned steel rods need to be placed above the center of gravity of the concrete, such that the resultant is a force pulling it upwards. [2]

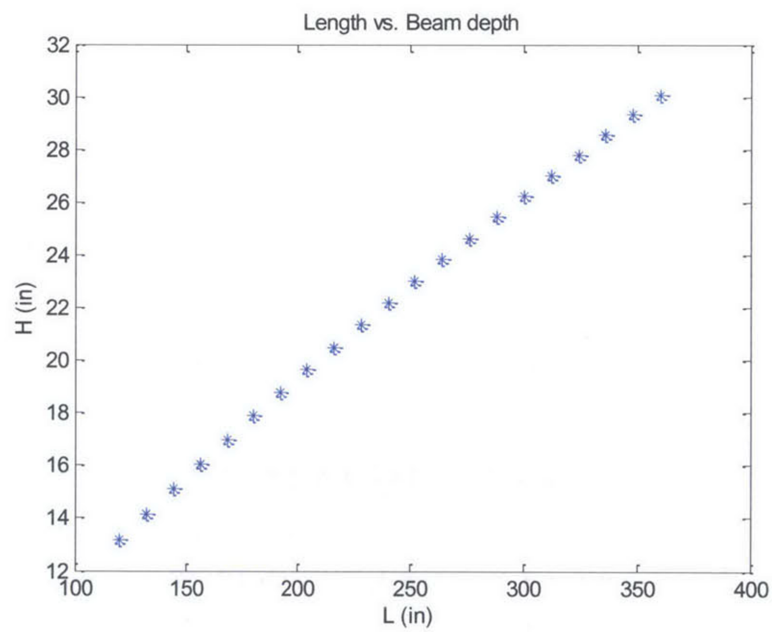


Figure 32: Length vs. Beam depth

[2] O'Brien, Eugene J, and Andrew S Dixon. *Reinforced And Prestressed Concrete Design*. 1st ed. Harlow, England: Longman Scientific & Technical, 1995. Print.

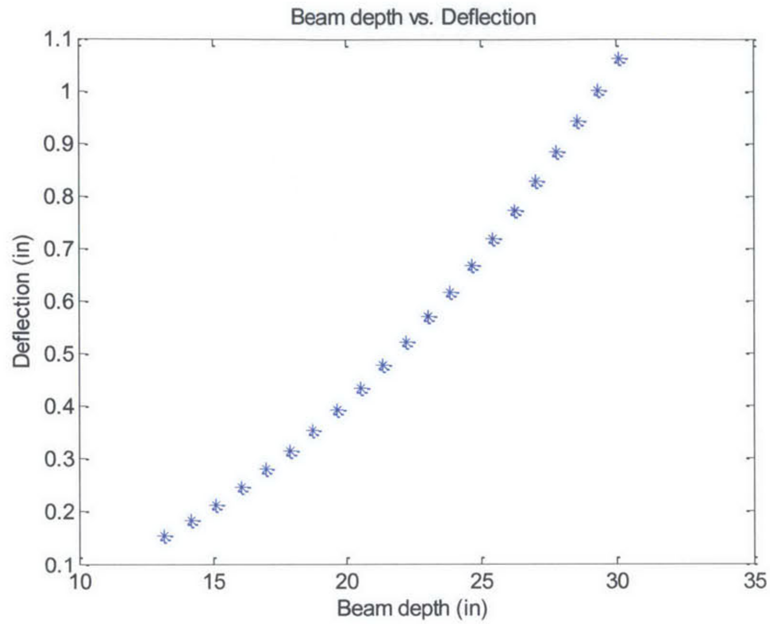


Figure 33: Deflection vs. Beam depth

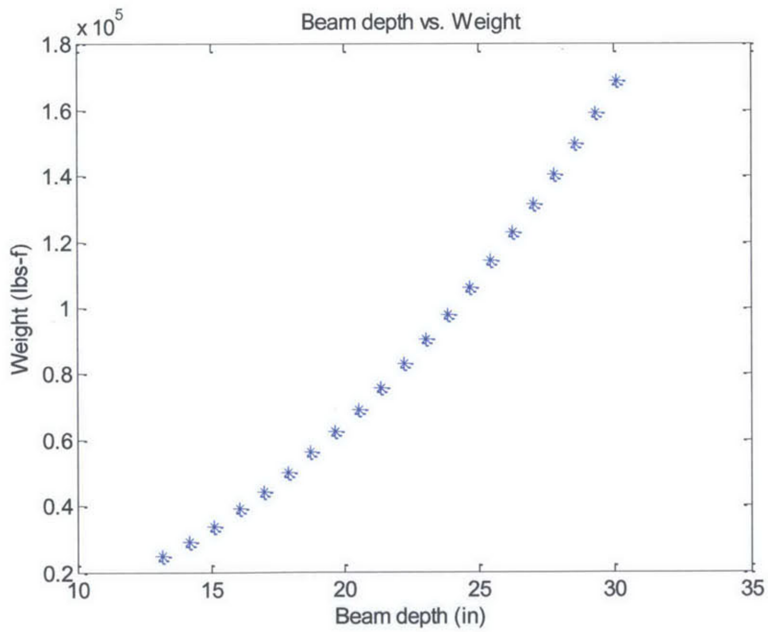


Figure 34: Weight vs. Beam depth

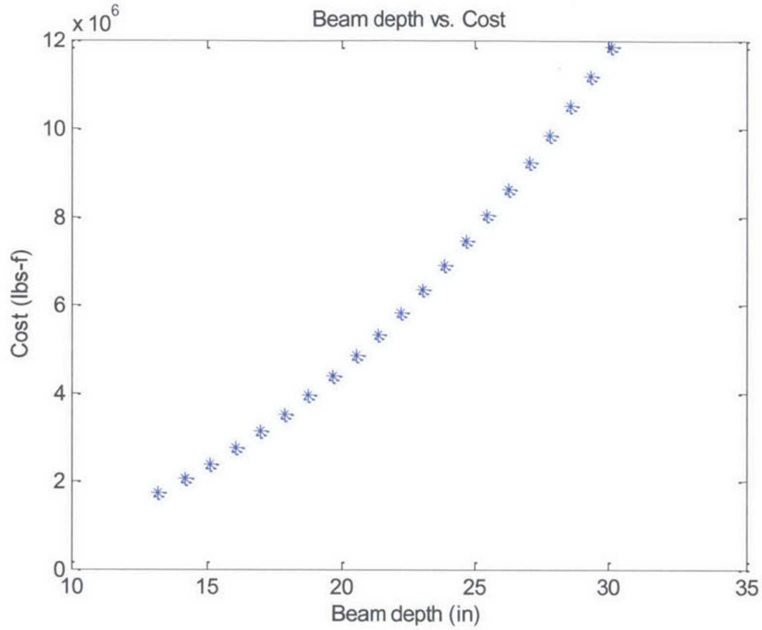


Figure 35: Cost vs. Beam depth

Category 4: Bending Tube

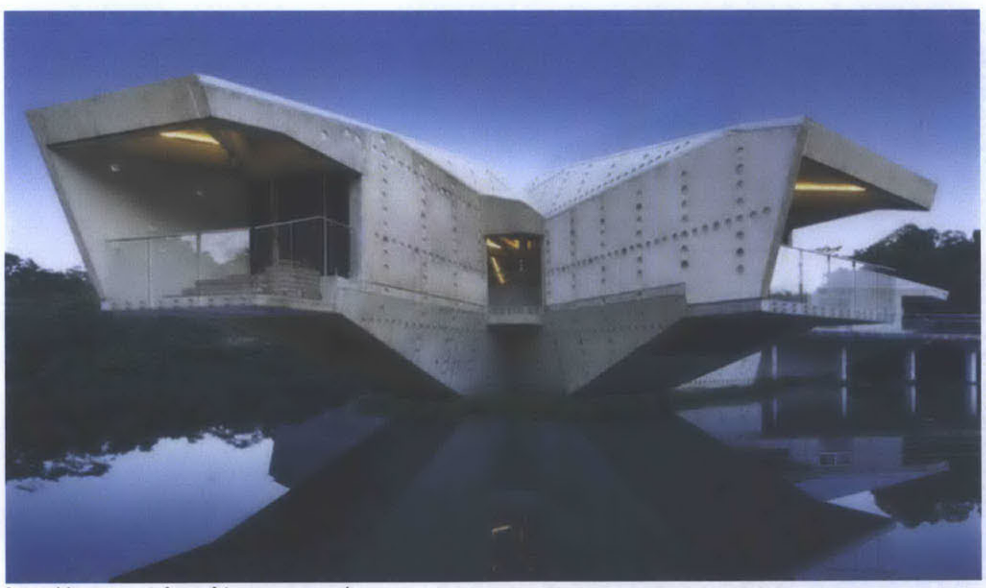
4.1 Case studies

The following case studies presented are built examples of a bending tube structural system. Using a tube as a structural system envelopes the interior space and can be sized to function as the side walls and floor slab. The example below incorporates a variation in beam depth, which results in a hybrid of categories 3 and 4.

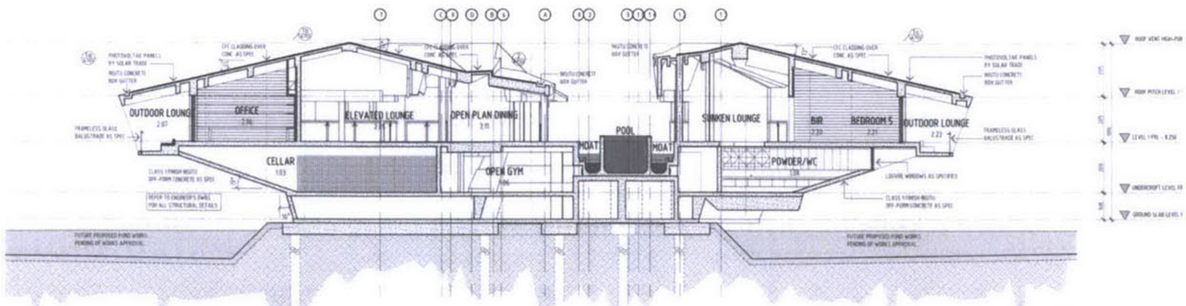
Off-Grid Stamp House

Architect	Charles Wright Architects
Structural Engineer	G&A Consultants Pty Ltd
Location	Queensland, Australia
Year	2013
Cantilever Length	Approx. 44 ft

http://www.wrightarchitects.com.au/



http://www.wrightarchitects.com.au/



http://www.wrightarchitects.com.au/

Academie Music Word and Dance

Architect	Carlos Arroyo
Structural Engineer	Norbert Provoost, Ghent
Location	Dilbeek, Brussels, Belgium
Year	2007-2012
Cantilever Length	60 ft

http://www.carlosarroyo.net/eng/proyectos/Dilbeek/00.htm

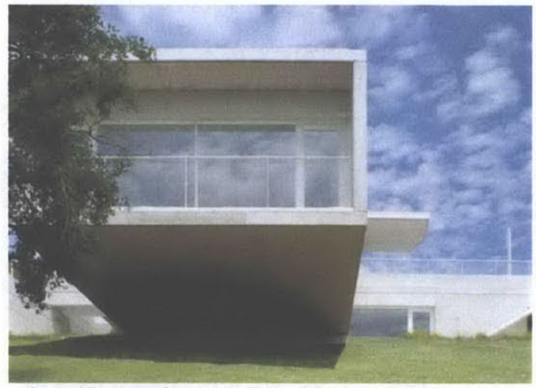
http://www.carlosarroyo.net/eng/proyectos/Dilbeek/00.htm

http://www.metalocus.es/content/en/blog/academie-mwd

http://www.carlosarroyo.net/eng/proyectos/Dilbeek/00.htm

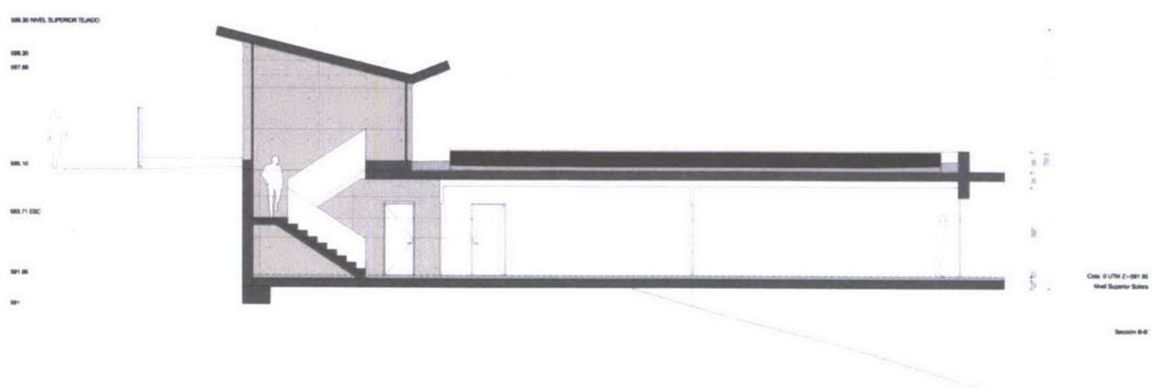
Dwelling Etura

Architect	Roberto Ercilla Arquitectura
Structural Engineer	Amaia Vasallo
Location	Etura - Álava, Spain
Year	2011
Cantilever Length	49 ft



http://www.robertoercilla.com/index.php?idioma=en

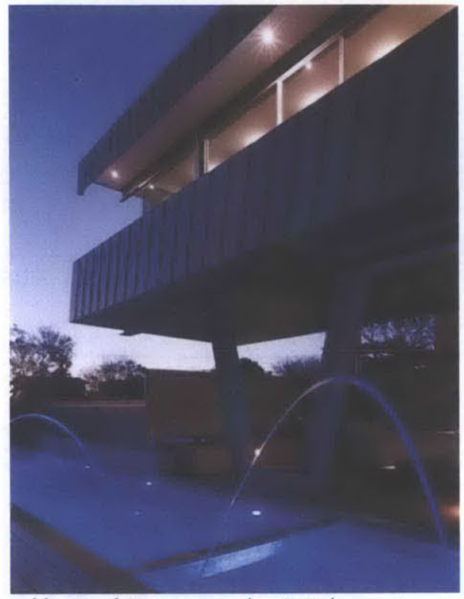
http://www.robertoercilla.com/index.php?idioma=en



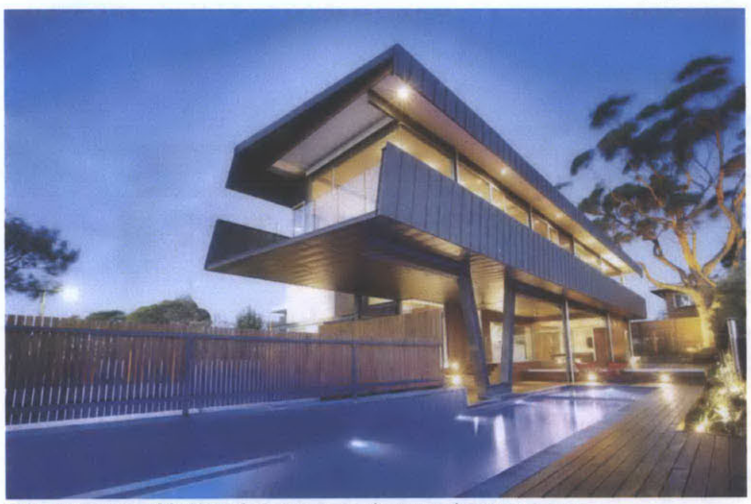
http://www.robertoercilla.com/index.php?idioma=en

Coronet Grove

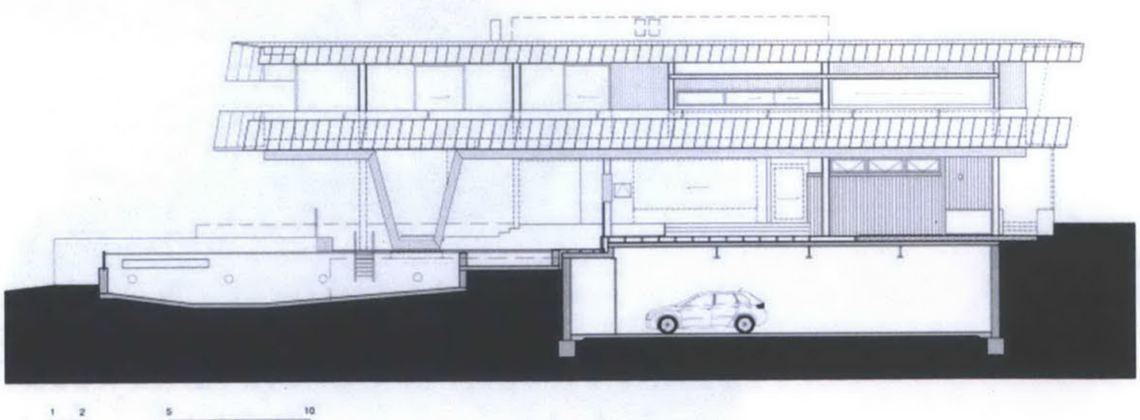
Architect	Maddison Architects
Structural Engineer	Ainley Engineering
Location	Melbourne, Australia
Year	2007
Cantilever Length	Approx. 20 ft



http://www.maddisonarchitects.com.au/projects/coronet-grove



http://www.maddisonarchitects.com.au/projects/coronet-grove

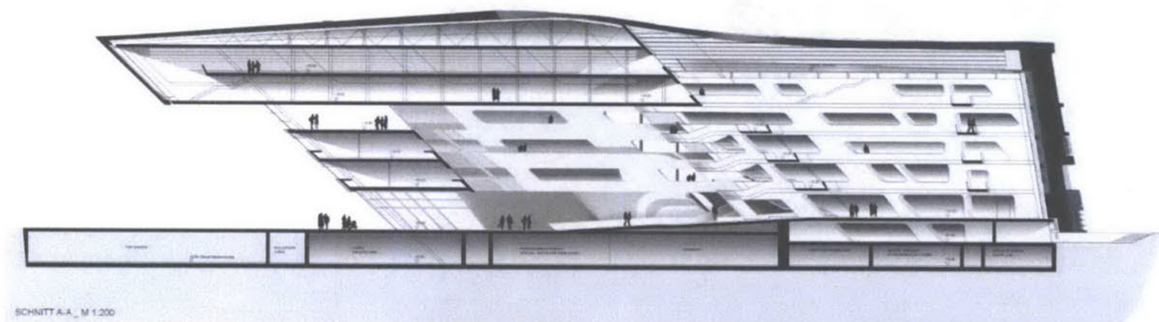


http://www.maddisonarchitects.com.au/projects/coronet-grove

Library & Learning Centre – University of Economics Vienna

Architect	Zaha Hadid Architects
Structural Engineer	Arup Berlin
Location	Vienna, Austria
Year	2013

http://www.zaha-hadid.com/



http://buildipedia.com/aec-pros/featured-architecture/zaha-hadids-library-and-learning-center

http://www.zaha-hadid.com/

4.2 Model of a Bending Tube

This type of structural system is popular for residential use and both the inside and the top of the tube can be inhabitable spaces, as seen in the Roberto Ercilla project. The dimensions of this system can be chosen by the designer, however, scaling up this system can cause local effects to govern the bahavior, rather than function as an entity. Local steel reinforcing and stiffening can be done to ensure that this bending tube does behave in a cohesive manner. The geometry of the tube structural system can be simplified to the following model (shown in Figure 36) with the variables used listed in the table below.

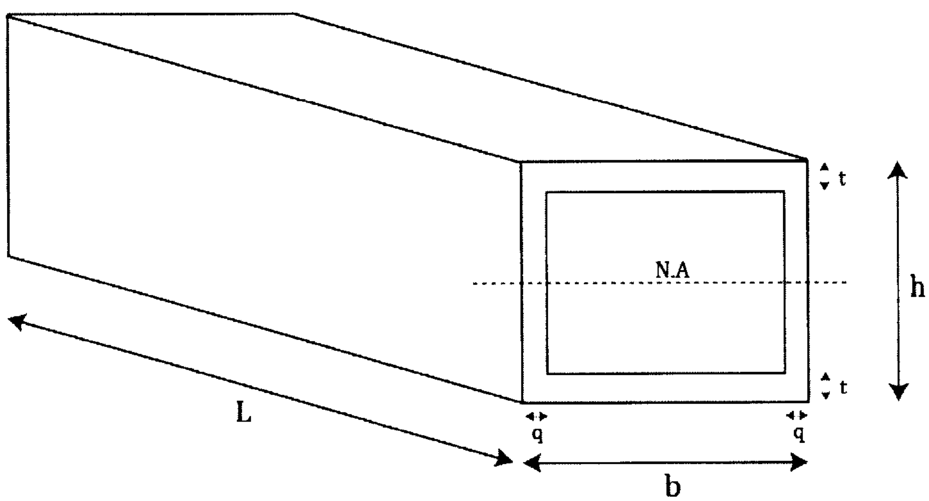


Figure 36: Model of the bending tube

Variables	Description	Units
h	Height of tube	In
L	Length of tube	In
b	Width of tube	In
t	Thickness of top and bottom flanges	In
q	Thickness of side walls	In
w	Applied distributed load on tube	Kips/ft
A	Cross sectional area	In^2
I	Moment of inertia of the beam	In^4
s	Section modulus of bending members	In^3
G	Shear modulus	Ksi
	$\sigma_Y = 50\ Ksi$	

The analytical solutions below assume the Neutral Axis (N.A) to be halfway between the top and bottom surface of the tube (the amount of steel rebars in the tension zone can be sized such that the location of the neutral axis is halfway).

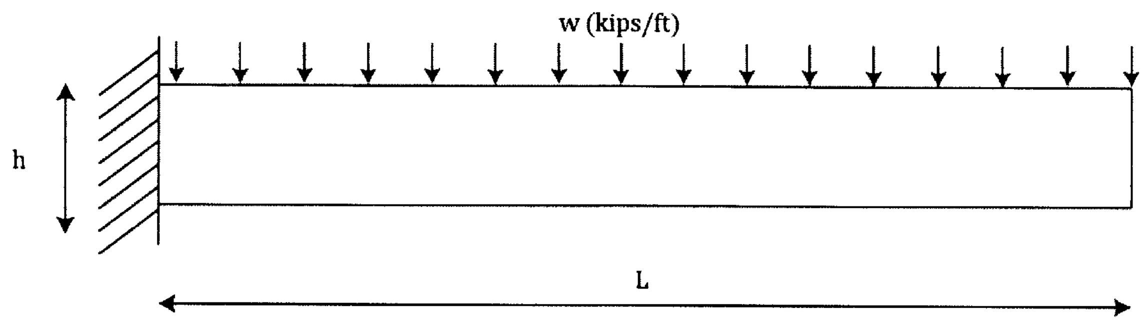


Figure 37: Distributed loading along the bending tube

4.2.1 Strength Criteria

Maximum moment

$$\frac{wL^2}{2}$$

Maximum shear

$$wL$$

Force in top or bottom flange

$$F_T = F_C = \frac{bL^2}{20h}$$

4.2.2 Deflection Criteria

Area of cross section

$$A = 2bt + 2q(h - 2t)$$

I_{total} is the moment of inertia of the entire cross section.

I_{flange} is the moment of inertia of only the flanges (neglect the web's contribution to bending.

A comparison between I_{total} and I_{flange} shows that the latter is a good approximation (see Appendix 1 for a numerical comparison). All calculations will use I_{flange}

$$I_{total} = 2\left[\frac{bt^3}{12} + bt(\frac{h}{2} - \frac{t}{2})^2\right] + 4\left[\frac{q\left(\frac{h}{2} - \frac{t}{2}\right)^3}{12} + \frac{q}{2}(\frac{h}{2} - \frac{t}{2})^2\right]$$

$$I_{flanges} = \frac{2bt^3}{3} + \frac{bth^2}{2} - bht^2$$

For a cantilever with a uniform moment of inertia, I acting in Bending, the deflection equation is:

$$u_{Bending} = \frac{wL^4}{8EI} = \frac{wL^4}{8E\left(\frac{2bt^3}{3} + \frac{bth^2}{2} - bht^2\right)}$$

For a cantilever acting in Shear, the deflection equation is:

$$u_{Shear} = \frac{\alpha wL^2}{2GA}$$

Where α is a variable called the form factor, with which the average shear stress τ_{av} must be multiplied in order to obtain the maximum shearing stress τ_{max} at the centroid of the cross section.

For this geometry:

$$\alpha = \frac{A}{16(Iq)}(bh^2 - bd^2 + 2qd^2)$$

Shear deformation cannot be ignored in the deflection equation because the side walls of the tube contribute to the overall behavior of the tube.

$$u_{Shear} = \frac{wL^2}{32IqG}(bh^2 - b(h - 2t)^2 + 2q(h - 2t)^2)$$

$$u_{Total} = u_{Shear} + u_{Bending}$$

$$\boldsymbol{u_{Total} = \frac{wL^2(bh^2 - b(h - 2t)^2 + 2q(h - 2t)^2)}{32qG\left(\frac{2bt^3}{3} + \frac{bth^2}{2} - bht^2\right)} + \frac{wL^4}{8E\left(\frac{2bt^3}{3} + \frac{bth^2}{2} - bht^2\right)}}$$

$$Cost = L[2bt + 2q(h-2t)] + \alpha\left[\frac{0.1bL^2(bh^2 - b(h-2t)^2 + 2q(h-2t)^2)}{32qG\left(\frac{2bt^3}{3} + \frac{bth^2}{2} - bht^2\right)} + \frac{0.1bL^4}{8E\left(\frac{2bt^3}{3} + \frac{bth^2}{2} - bht^2\right)}\right]$$

4.3 Analysis

To optimize the thickness of the flanges of this bending tube, choose dimensions for the variables L (length), h ((height of tube), q (thickness of side walls), b (width of tube).

$$\frac{dC}{dt} = 0 \rightarrow \text{Find the optimal thickness such that deflection is satisfied } (u < \frac{L}{360})$$

Numerical methods are used to find $h_{optimal}$ for the design example.

Also of interest to designers is to optimize the height of the beam: Choose dimensions for the variables L (length), t ((thickness of flanges), q (thickness of side walls), b (width of tube).

$$\frac{dC}{dh} = 0 \rightarrow \text{Find the optimal height of the tube such that deflection is satisfied } (u < \frac{L}{360})$$

Numerical methods are used to find $h_{optimal}$ for the design example.

4.3.1 Design example

Choose to range over a span of L = 40 ft to 160 ft in 20 ft increments.

Procedure: For every length L shown in Table 4 , a corresponding optimal flange thickness is found. A value for the thickness of the side walls (q) is also chosen but it is taken as a ratio of the thickness of the flange (or a minimum of 2 inches is assumed). Other parameters used are:

b	120	in
h	312	in
E	29000	Ksi
G	11154	Ksi

L (in)	Area (in2)	Weight (lbs-f)	Cost (lbs-f)
480	1390	58026	2.72E+05
720	1642	102880	1.13E+06
960	2716	226807	3.30E+06
1200	4064	424282	7.72E+06
1440	6228	780244	1.46E+07
1680	8840	1292054	2.54E+07
1920	11226	1875191	3.79E+07

Table 4: design example results

As length of the structural system increases (maintaining the same height of truss) a greater area is required to achieve the target deflection criteria of L/360. Both the thickness of the web and the flange increase, (with the flange area contributing more to resisting deflection) and the net effect is combined into the cross sectional area, shown in Table 4. Figure 38 shows that as length increases, the required area needed increases in a polynomial fashion. Similarly, as length increases, the weight of the bending tube also increases in a polynomial relationship (see Figure 39). Since

$$W = (Area\ of\ cross\ section \times Length\ of\ each\ member\ ) \times \rho$$

Both area and length are increasing so

$$\mathbf{W} \propto L^3$$

Overall deflection and length are linearly related (since we designed for a desired deflection of L/360).

$$W \propto L^3, \text{so } \mathbf{C} \propto \mathbf{L^3}$$

This can be seen in Figure 40.

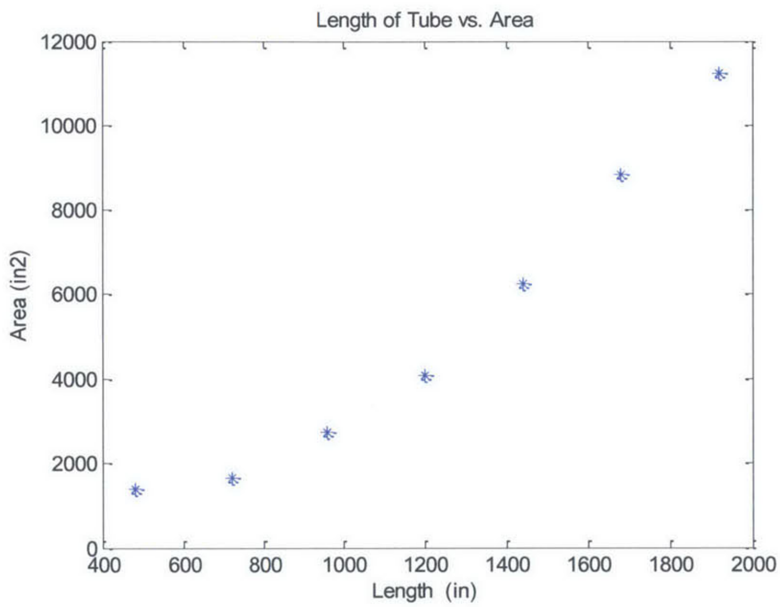


Figure 38: Length of tube vs. area for optimized flange thickness

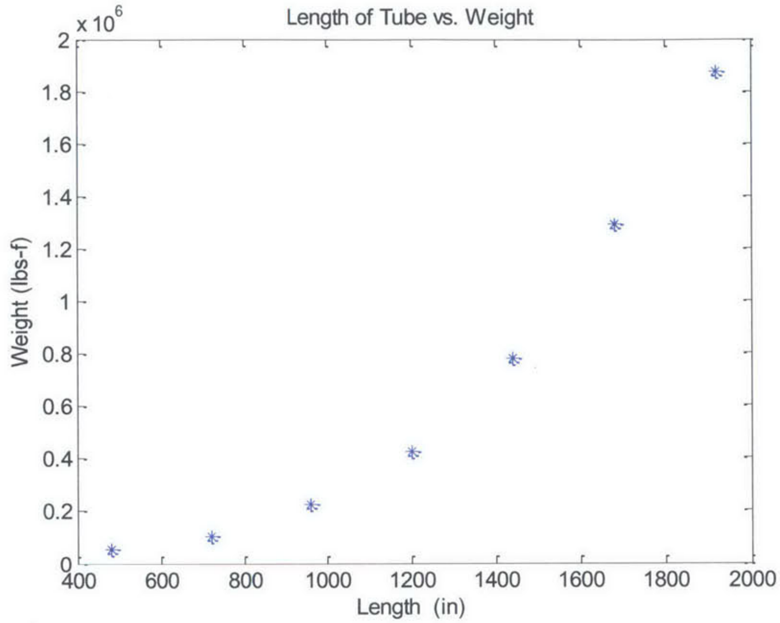


Figure 39: Length of tube vs. Weight for optimized flange thickness

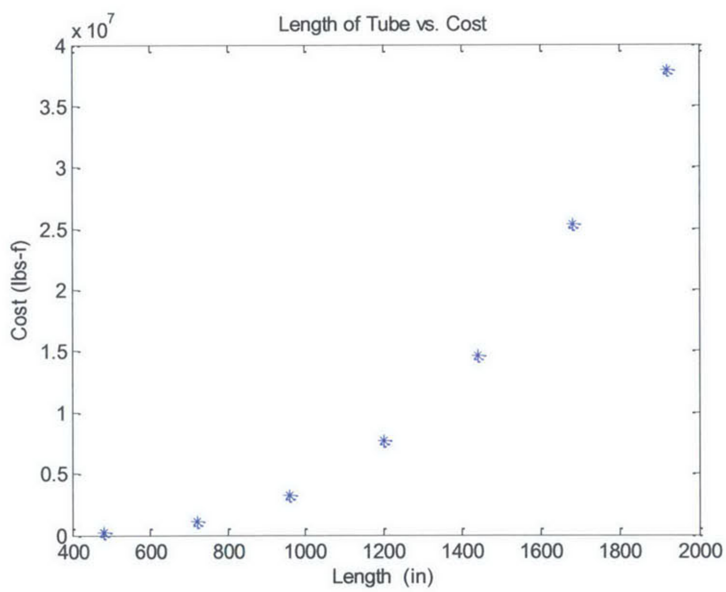


Figure 40: Length of tube vs. Cost for optimized flange thickness

4.3.2 Design example

Choose spans to range over L = 40 ft to 160 ft in 20 feet increments.

Procedure: For every length L shown in Table 5, a corresponding optimal height is found. Other parameters used are:

b	120	in
t	3	in
q	3	in
E	29000	Ksi
G	11153.85	Ksi

L (in)	h_optimal (in)	Area (in2)	Weight (lbs-f)	Cost (lbs-f)
480	120	1404	58631	2.52E+05
720	228	2052	128537	7.04E+05
960	348	2772	231517	1.44E+06
1200	492	3636	379598	2.59E+06
1440	648	4572	572780	4.18E+06
1680	840	5724	836620	6.54E+06
1920	1044	6948	1160594	9.60E+06

Table 5: L and Optimal height

As length increases , the optimal height required also increases in a non linear relationship. The deflection criteria of L/360 still governs. The thickness of the web and the flange are chosen to be 3 inches, which is not much cross sectional area but the key relationship derived is the impact of optimal height on the bending tube. Note that this assumes the tube will function as a cohesive system without local effects taking over its behavior. Figure 41 below shows that as length increases, the optimal height needed to acheive the deflection criteria increases in a polynomial fashion.

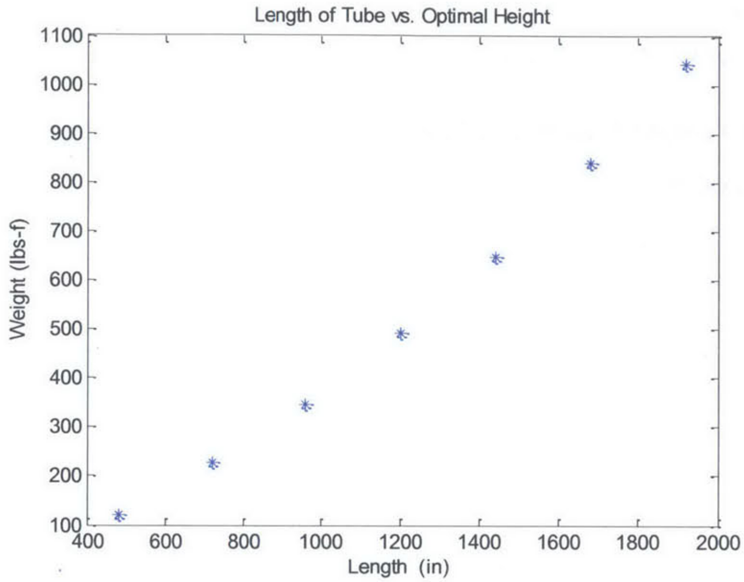


Figure 41: Length of tube vs. Optimal Height

The cross sectional area is plotted against length in Figure 42 because the change in height of the tube is essentially an increase in area since the top/bottom and sides stay the same thickness. The area equation is

$$A = 2bt + 2q(h - 2t)$$

$$A \propto h \qquad \rightarrow \qquad \boldsymbol{L} \propto \boldsymbol{A}^2$$

$$\text{Weight} = (\text{Area of cross section} \times \text{Length of each member}) \times \rho$$

Since both Area and Length are increasing,

$$\mathbf{W} \propto \boldsymbol{L}^3$$

This can be seen in Figure 43.

Lastly, of concern to the designer is the balance of self-weight of the structure and the sensitivity to deflection. Since deflection and length of bending tube are linearly related (this was the deflection criteria):

$$\mathrm{W} \propto \mathrm{L}^3, \text{thus } \mathbf{C} \propto \mathbf{L}^3 .$$

This relationship can be seen in Figure 44.

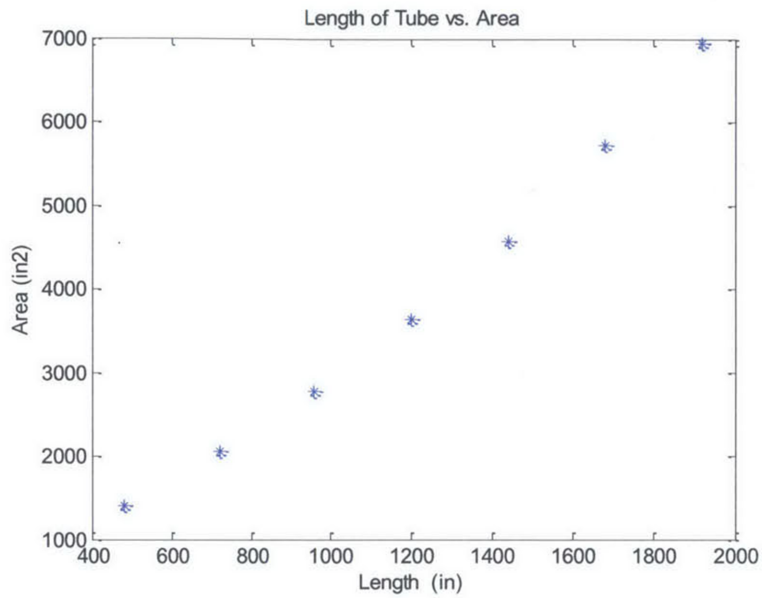


Figure 42: Length of tube vs. Area for optimized height

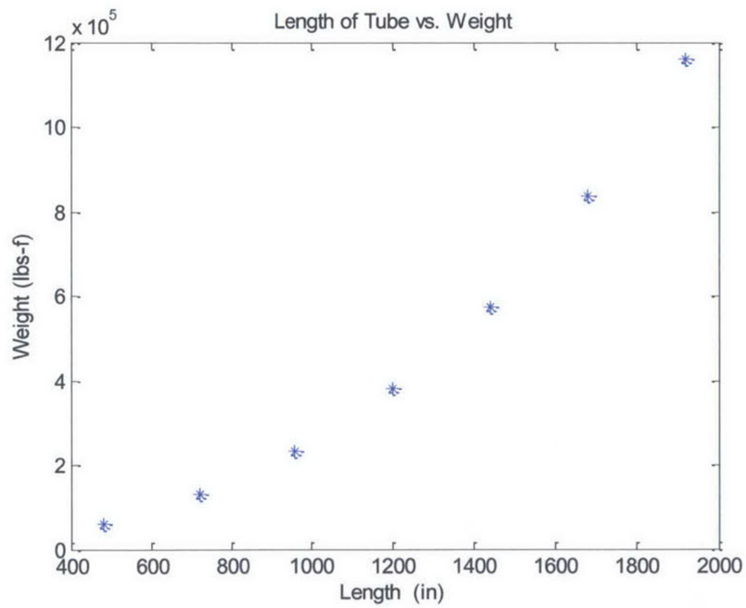


Figure 43: Length of tube vs. Weight for optimized height

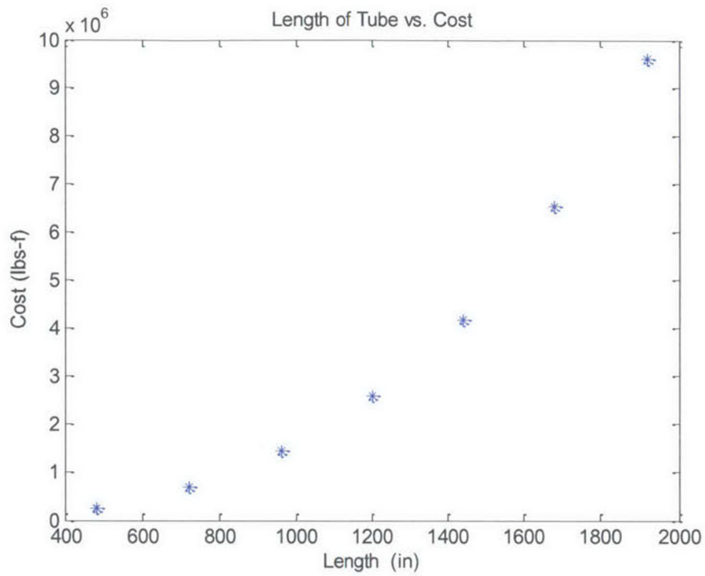


Figure 44: Length of tube vs. Cost for optimized height

The sensitivity of deflection on the overall cost does increase nonlinearly with span, and this polynomial relaionship between span and α is shown below. This is because as span increases, the bending tube needs to provide greater bending capacity and α needs to be much greater to achieve the same L/360 deflection criteria.

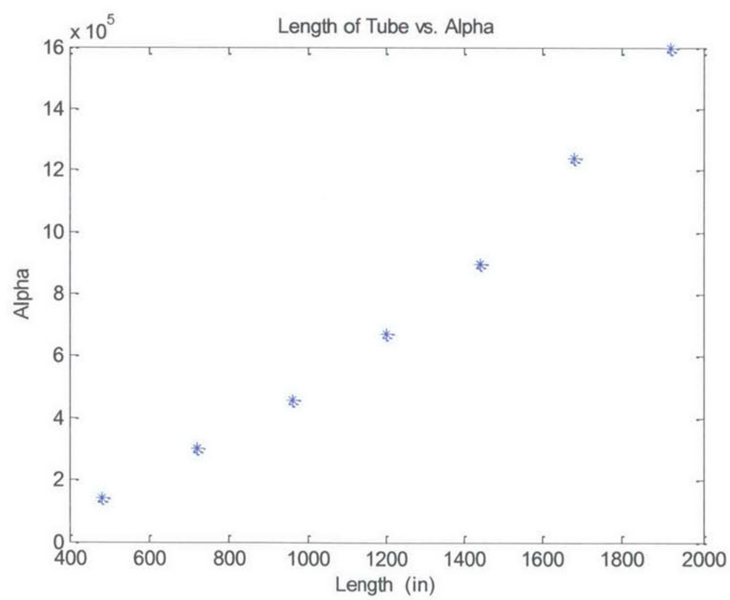


Figure 45: Length of tube vs. alpha for optimized height

Category 5: Suspended Beam

5.1 Case Studies

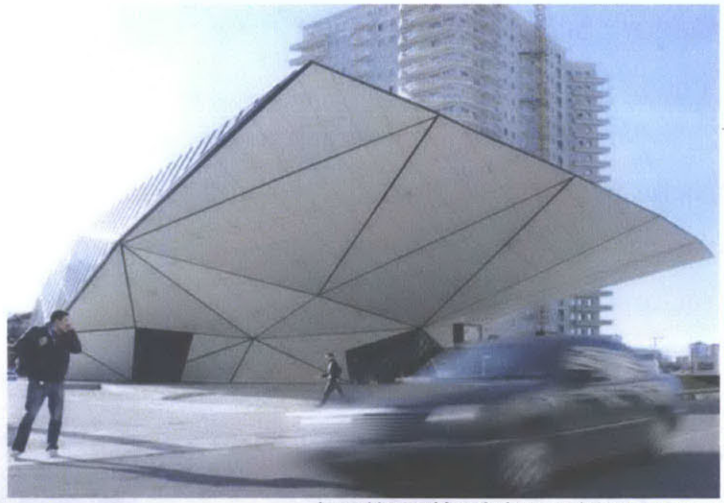
The following case studies presented are built examples of a suspended beam structural system. This scheme provides column free interiors, though the beam is tilted at an angle. The examples below illustrate the variety of programmatic use inside these structures.

Fuel Station + McDonalds

Architect	Giorgi Khmaladze
Structural Engineer	Capiteli
Location	Batumi, Georgia
Year	2013
Cantilever Length	600 square meter (6458 sq. ft) green roof system

http://giorgikhmaladze.com/#projects_view

http://giorgikhmaladze.com/#projects_view



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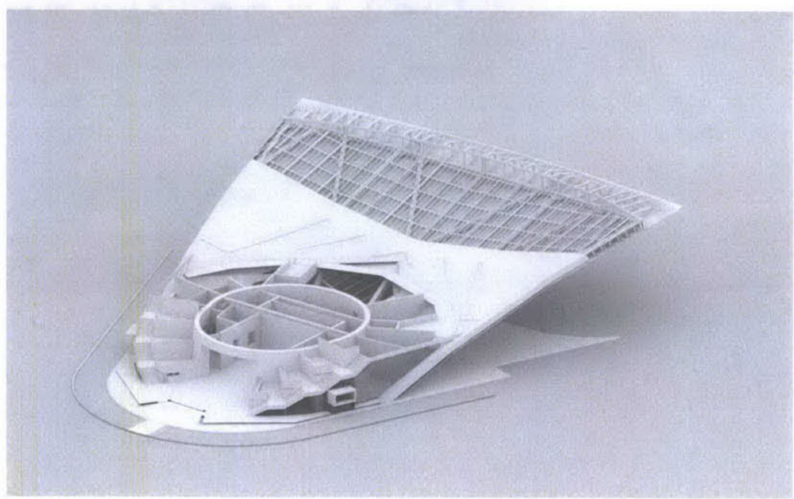


http://giorgikhmaladze.com/#projects_view

Voestalpine Stahl GmbH Sales & Financial Headquarters

Architect	Feichtinger Architectes
Structural Engineer	Schindelar ZT_GmbH
Location	Linz, Austria
Year	2009
Cantilever Length	140 ft

http://www.feichtingerarchitectes.com/

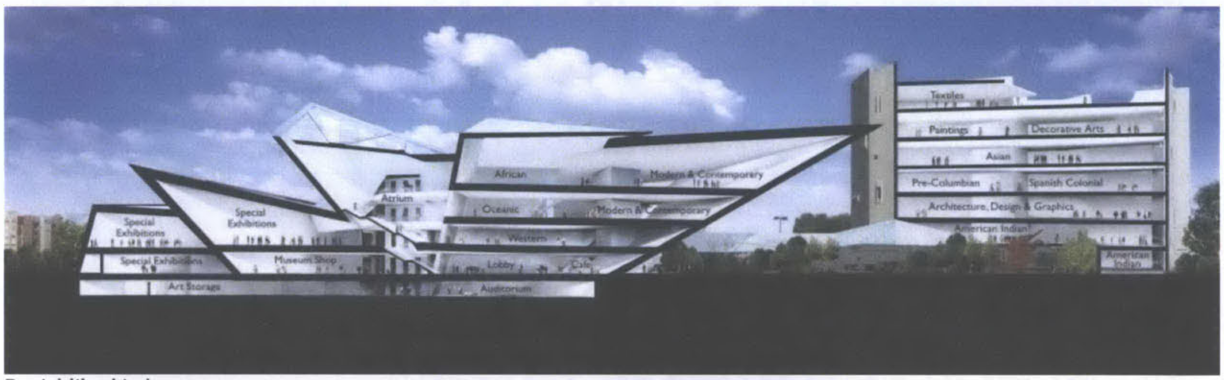
http://www.feichtingerarchitectes.com/

http://www.feichtingerarchitectes.com/

Denver Art Museum

Architect	Studio Daniel Libeskind
Structural Engineer	Arup
Location	Denver, Colorado
Year	2006

Daniel-libeskind.com



Daniel-libeskind.com

Daniel-libeskind.com

5.2 Model of Suspended Beam

This structural system is fairly uncommon but can be a good strategy for auditoriums, which have a shorter span and tiered seating. This system could be replicated as many times as needed in width, allowing a structure free interior. The element A shown in Figure 46 acts as a beam column so it is less efficient than a pure bending member or a purely axial member.

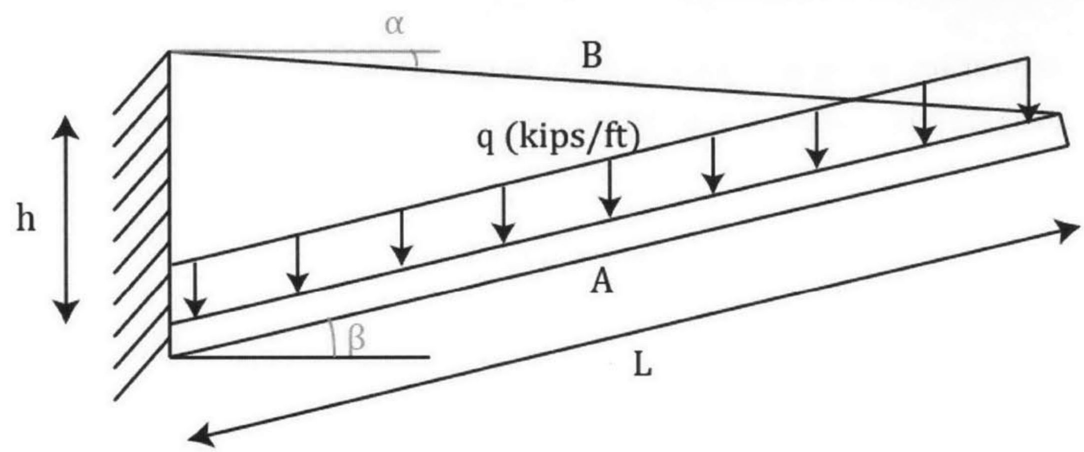


Figure 46: Model of a suspended beam with vertical loads

Variables	Description	Units
h	Height of Beam	In
L_A	Length of Beam A	In
b	Width of beam (not taken into account in the calculations because the analysis is done in 2D)	In
q	Applied distributed vertical load	Kips/ft
w	Applied distributed transverse load on beam assume 1Kip/ft	Kips/ft
A_A	Cross sectional area of bending member	In^2
A_B	Cross sectional area of tension member	In^2
I_A	Moment of inertia of the beam A	In^4
F	Tension in member B	
	$\sigma_Y = 50\ Ksi$	

5.2.1 Strength Criteria

Loads: q is the self-weight of applied vertical forces e.g. from people, auditorium seating etc. w is the transverse load that the beam feels once the load q is resolved using the angle β. Assume a rectangular cross section of concrete where the top acts in tension (steel reinforcement is required) and the bottom of the cross section acts in compression, alternatively, a steel beam can be used.

The total vertical load acting on Beam B is $q \times L$. To convert this into a transverse load w that acts along the length of Beam B, multiply it by $\cos \beta$. This gives a distributed load over beam

$$B = w = q \times \cos \beta$$

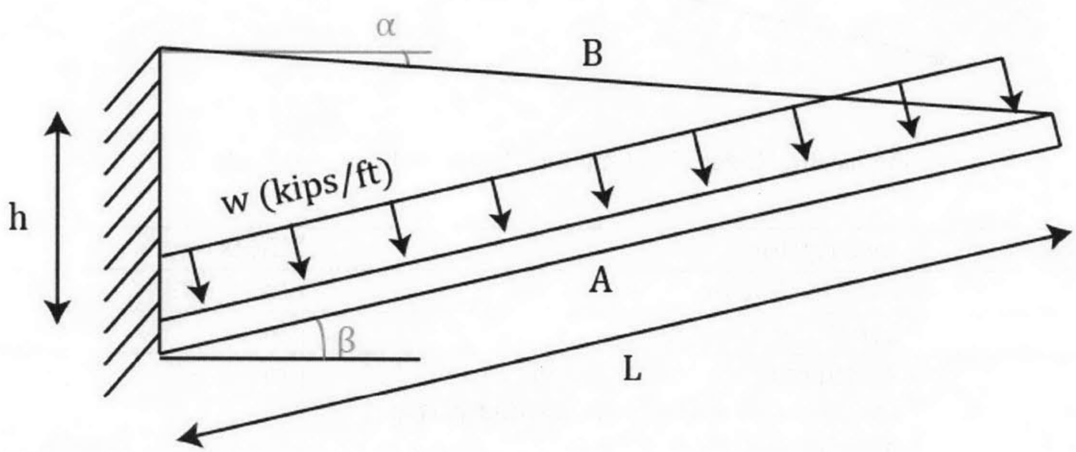


Figure 47: Model of suspended beam with transverse loads

Assuming an I shape for the bending member A such as the one shown in Figure 48.

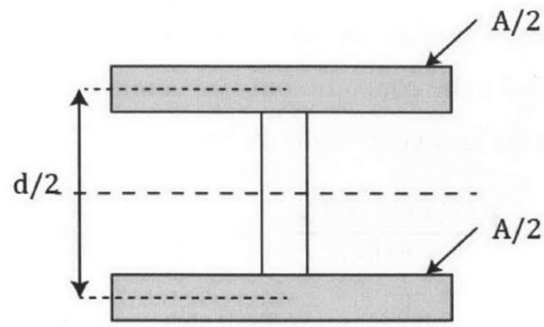


Figure 48: Cross section of member A

Moment of inertia:

$$I = \frac{bh^3}{12}$$

Force components:

$$F_y = Fsin\alpha$$

$$F_x = Fcos\alpha$$

Sum the moments about the bottom left of the beam:

$$F = \frac{qL^2cos\beta}{2(Lsin\alpha cos\beta + hcos\alpha)}$$

For the tension member:

$$\sigma_Y = \frac{F}{A}$$

Area of the cable (element B) in Figure 47 can be sized according to this minimum area:

$$A_{minimum} = \frac{qL^2cos\beta}{2(Lsin\alpha cos\beta + hcos\alpha)\sigma_Y}$$

5.2.2 Deflection Criteria

The method of virtual forces is used to find the displacement at the tip of the beam. Moment of inertia is assumed to be constant, but I(x) could also be used if the moment of inertia of element A decreases as it gets closer to the tip.

$$\delta P \times u = \int_{0}^{L} \frac{M(x) \times \delta M(x)}{EI(x)} dx + \int_{0}^{L} \frac{F(x) \times \delta F(x)}{EA} dx$$

Apply a δP force at the tip of the beam where maximum deflection occurs and find the corresponding moment and axial force equations under the real load and the virtual load.

$$M(x) = \frac{wx(L-x)}{2}$$

$$\delta M(x) = 0$$

$$F(x) = \frac{qL^2 cos\beta}{2(Lsin\alpha cos\beta + hcos\alpha)}$$

$$\delta F(x) = \frac{\delta P}{tan\beta}$$

$$\boldsymbol{u_{total}} = \frac{\boldsymbol{qL^3 cos\beta}}{\boldsymbol{2EA_A tan\beta(Lsin\alpha cos\beta + hcos\alpha)}}$$

Deflection Limit

$$u_{limit} = \frac{nL}{360}$$

Design criteria

$$u \leq u_{limit}$$

To optimize the cost with respect to the maximum depth needed:

$$Cost = Weight + \alpha \times u_{total}$$

$$Volume = A_A \times \sqrt{L^2 - h^2} + A_B \times L$$

$$Weight = Volume \times \rho$$

$$Cost = \rho \times \left(A_A \times \sqrt{L^2 - h^2} + A_B \times L\right) + \frac{\alpha qL^3 cos\beta}{2EA_A tan\beta(Lsin\alpha cos\beta + hcos\alpha)}$$

To optimize the area of the beam A with respect to Cost:

$$\frac{dC}{dA_A} = 0 \quad \rightarrow \quad A_A = \sqrt{\frac{\alpha q L^3 cos\beta}{2\rho E tan\beta (L sin\alpha cos\beta + h cos a)(\sqrt{L^2 - h^2})}}$$

5.3 Analysis

5.3.1 Design Example

For a range of lengths (30 to 150 ft) in 10 feet increments, an optimal area of element A is found. The value α is estimated and checked against the deflection until a suitable value is found that meets the deflection criteria. Once the optimal area is found, the weight and cost can then be computed.

L (in)	alpha	A_optimal (in2)	Weight (lbs-f)	Cost (lbs-ft)
360	52	1.1	398	449
480	135	1.9	909	1089
600	230	2.9	1724	2107
720	340	4.0	2898	3577
840	460	5.3	4459	5535
960	600	6.8	6504	8103
1080	750	8.4	9029	11276
1200	910	10.1	12073	15104
1320	1080	11.9	15671	19630
1440	1260	13.8	19857	24897
1560	1450	15.8	24663	30946
1680	1650	17.9	30122	37818
1800	1860	20.1	36263	45549

Table 6: design example

The graph (Figure 49) shows that as the length of element A increases, the optimal area required increases at a parabolic rate. The equation

$$A_A = \sqrt{\frac{\alpha q L^3 cos\beta}{2\rho E tan\beta (L sin\alpha cos\beta + h cos\alpha)(\sqrt{L^2 - h^2})}}$$

proves this relationship as well, and can be simplified to

$$L \propto A_A^2$$

With the optimal area, a total weight of the system can then be found. Since Weight = Area × Length, the above relationship gives

$$W \propto L^{1.5}$$

which is reflected in Figure 50.

Since $u_{total} \propto L^2$, Cost is also a parabolic relationship to the length of the system:

$$\boldsymbol{Cost \propto L^2}$$

The deflection weighting of α is less than those for other structural systems, which means that weight has a greater impact on the optimization than deflection. However, this design example was done for shorter spans (L) because the application of beam columns makes this system behave differently than the other deflection governed structural systems.

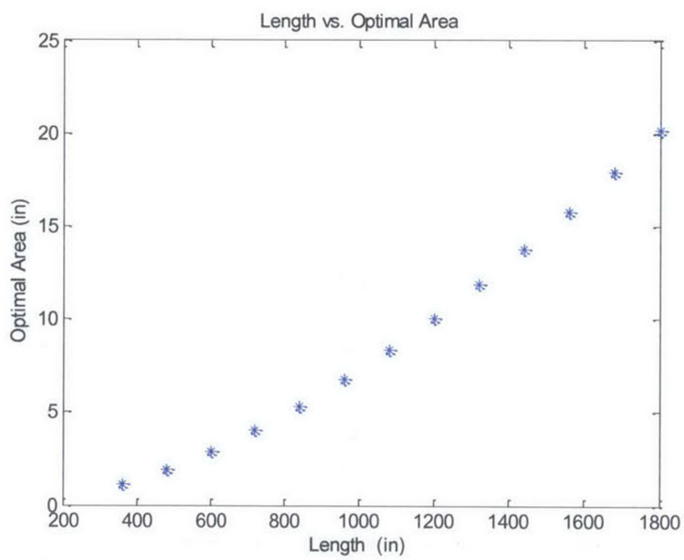


Figure 49: Length of beam vs. Optimal Area

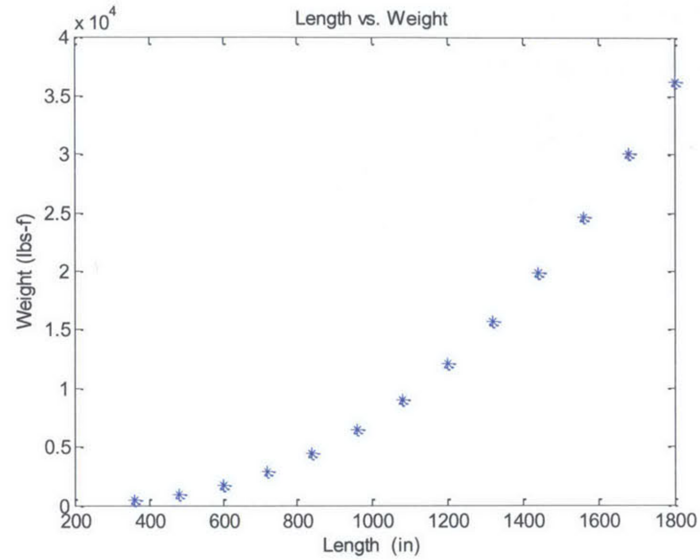


Figure 50: Length of beam vs. Weight

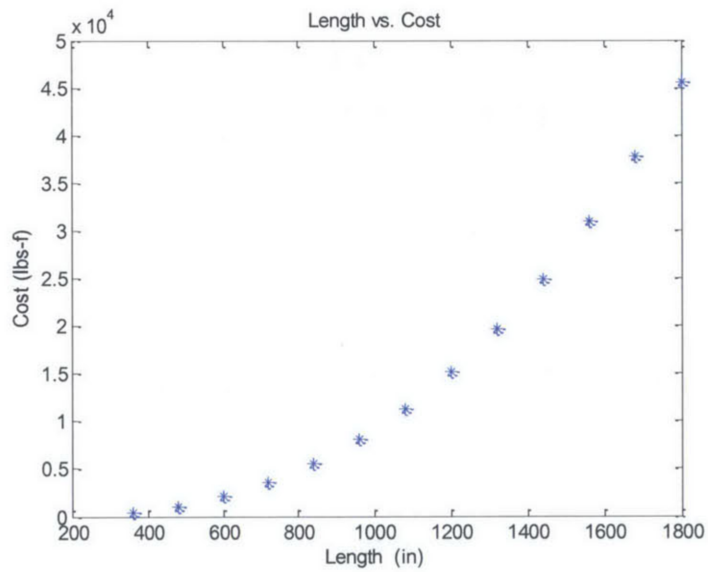


Figure 51: Length of beam vs. Cost

Though deflection is the governing issue in most cantilevers, this structural system relies on beam columns, which can be governed by a number of local issues such as buckling and deflection mid-span of the member itself. To prevent buckling of beam columns, stiffeners may need to be provided along the beam in the plane into the page of Figure 47. Strength calculations of beam columns will not be included since they are specific to dimensions, but the interaction equation of beams in compression and bending would be the next step in sizing members:

(a) For $\frac{P_r}{P_c} \geq 0.2$

$$\frac{P_r}{P_c} + \frac{8}{9}\left(\frac{M_{rx}}{M_{cx}} + \frac{M_{ry}}{M_{cy}}\right) \leq 1.0$$

(b) For $\frac{P_r}{P_c} < 0.2$

$$\frac{P_r}{2P_c} + \left(\frac{M_{rx}}{M_{cx}} + \frac{M_{ry}}{M_{cy}}\right) \leq 1.0$$

where

P_r = required axial compressive strength, kips (N)
P_c = *available axial compressive strength*, kips (N)
M_r = *required flexural strength*, kip-in. (N-mm) [3]

Having the repeating units close enough to each other with bracing in between can reduce the unbraced length of the beam column and thus reduce the loss of efficiency by using a beam column compared to a member in pure compression or pure bending. See the Giorgi Khmaladze for cantilever project for how bracing was added in this structural system and strengthen it laterally.

The optimal areas that the design example produces, for example, when L = 1200 inches, $\boldsymbol{A_{optimal} = 10.1\ in^2}$

In the AISC manual, a wide flange size that has this area could be a W18 x 35, however, it is slender for compression of F_y =50 Ksi. The next biggest size that is not slender would be an area almost double of the optimal. The same applies to the majority of optimal areas, leading to the conclusion that the beam column in this structural system is strength governed rather than deflection governed.

Similarly, the structural system can be optimized for height for a given span. This would have to be done numerically, and the same results for optimal area, weight and cost can be found. This optimization could be used where a site constraint has a maximum span the

[3] *Steel Construction Manual.* Chicago, IL: American Institute of Steel Construction, 2005. Print. Chapter H, page1550

beam can reach and the designer wants to find the depth needed that optimizes the cross sectional area of the beam column.

$$\frac{dC}{dh} = 0 \quad \rightarrow \quad \frac{A_A h \rho}{\sqrt{L^2 - h^2}} + \frac{L^3 q(\cos\alpha \times \cos\beta)}{EA_A \tan\beta(2h\cos\alpha + L\cos\beta \times \sin\alpha)^2} = 0$$

Weight Comparison between Structural Systems

A comparison of weight can be done between Categories 1, 2, 3, and 4. (Category 5 is not included since it is a strength governed design process and thus not comparable on the same terms as the rest). This data comes from the design examples produced in the respective categories; however they are for the following parameters:

H	26.00	ft
n	5	
L	20	ft
Total span	100	ft
Live load	100	psf
u_max	3.33	in
Q	20	Kips

The weights are normalized for ease of comparison between the structural systems. Overall, Category 1 (steel truss) which carries loads axially produces the lightest truss. The Vierendeel truss which takes loads in bending and is thus less efficient is heavier. In this specific example is almost twice the weight of the lightest steel truss. The degree of optimization is noted as "crude", "med" and "fine", and the greater the number of optimized member areas, the less the truss weighs overall. This is true for both Category 1 and 2.

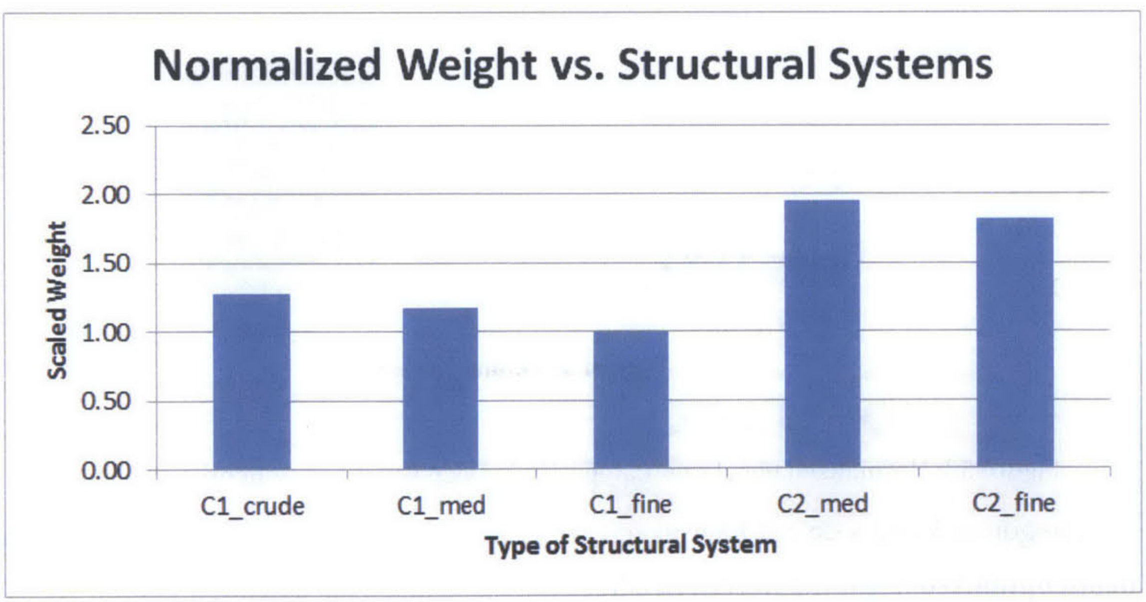


Figure 52: Normalized weight vs. Categories 1: Steel Braced Truss and Category 2: Steel Vierendeel Truss

The same comparison can be done for the concrete structural systems. Overall, they weigh more than the steel ones. Category 3: Deep Beam has the following chosen parameters:

b	120	in
H opt	44.72	in
L	100	ft
u_max	3.33	in
Live load	100	psf
w	0.0833	Kip/in

Category 4: Bending Tube parameters:

b	120	in
H	312.00	in
L	100	ft
u_max	3.33	in
Live load	100	psf
w	0.0833	Kip/in

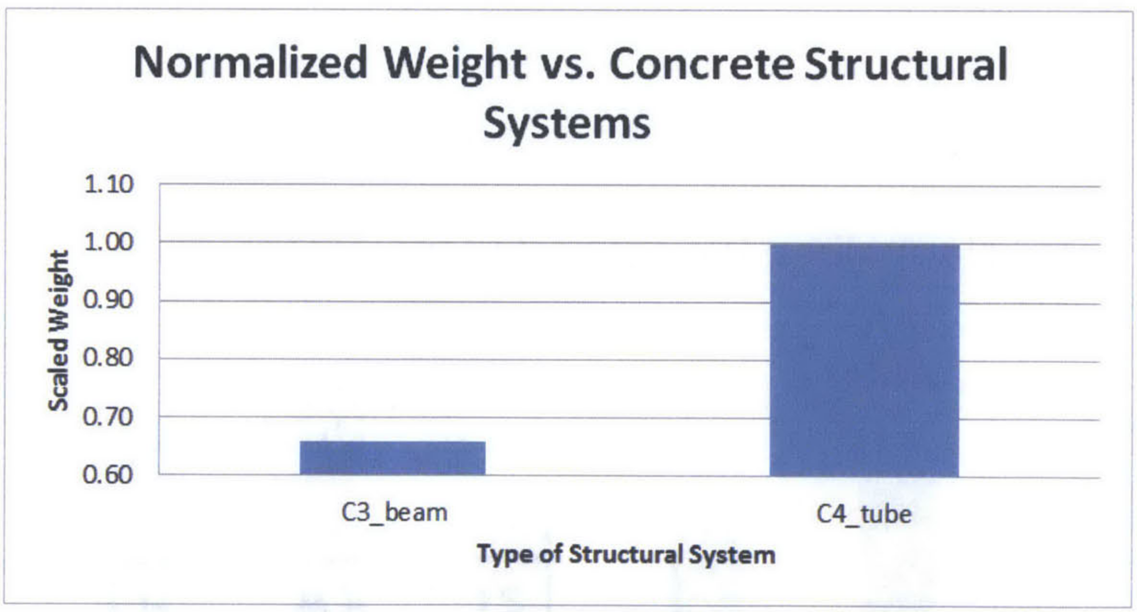


Figure 53: Normalized weight vs. Categories 3: Deep Beam and 4: Bending Tube

Categories 3 and 4 do not have exactly the same constraints, since for the deep beam the depth optimized h_{opt} is a different height to the one chosen for Category 4 (floor to ceiling). Nevertheless, they do provide a good approximation for the weights of the different systems.

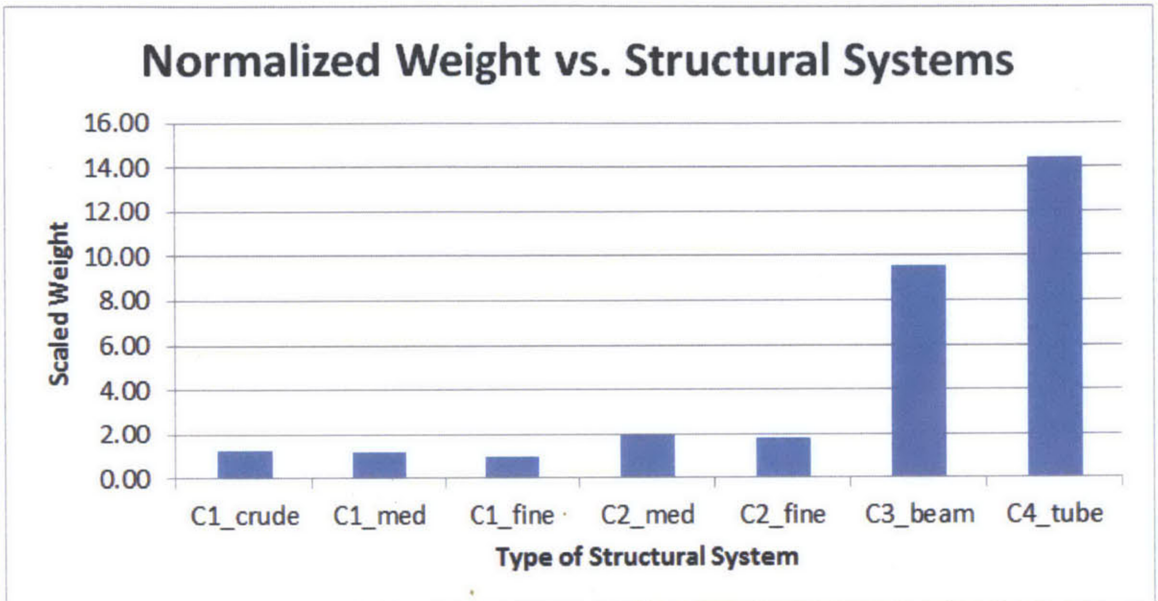


Figure 54: Normalized weight vs. Structural systems

For this design example, categories 3 and 4 after optimization turned out to weigh at least 10 times heavier than the steel trusses. Though this result is not applicable for all trusses of each structural system, a similar procedure can be done by the designer as a way to compare the efficiency of the systems.

Appendix

Category 4: Bending tube

This is a numerical comparison of I_{total}vs $I_{flanges}$ to find the percentage difference between the two approximations.

$$I_{total} = 2\left[\frac{bt^3}{12} + bt(\frac{h}{2} - \frac{t}{2})^2\right] + 4\left[\frac{q\left(\frac{h}{2} - \frac{t}{2}\right)^3}{12} + \frac{q}{2}(\frac{h}{2} - \frac{t}{2})^2\right]$$

$$I_{flanges} = \frac{2bt^3}{3} + \frac{bth^2}{2} - bht^2$$

Choosing the following dimensions which are similar to those in a one story, 30 ft wide space:

b	360	in
h	120	in
t	6	in
q	3	in

$I_{total} = 25472880\ in^4$

$I_{flanges} = 25297920\ in^4$

Percentage difference = 0.7 % which is a small enough error for the approximation of $I_{flanges}$ to be a good substitute for I_{total}.

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