The Structure of Auctions: Optimality and Efficiency

by

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B.S., Massachusetts Institute of Technology (2009)

Submitted to the Department of Mathematics
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Abstract

The problem of constructing auctions to maximize expected revenue is central to mechanism design and to algorithmic game theory. While the special case of selling a single item has been well understood since the work of Myerson, progress on the multi-item case has been sporadic over the past three decades. In the first part of this thesis we develop a mathematical framework for finding and characterizing optimal single-bidder multi-item mechanisms by establishing that revenue maximization has a tight dual minimization problem. This approach reduces mechanism design to a measure-theoretic question involving transport maps and stochastic dominance relations. As an important application, we prove that a grand bundling mechanism is optimal if and only if two particular measure-theoretic inequalities are satisfied. We also provide several new examples of optimal mechanisms and we prove that the optimal mechanism design problem in general is computationally intractable, even in the most basic multi-item setting, unless \( \text{ZPP} \) contains \( \text{P}^\# \).

Another key problem in mechanism design is how to efficiently allocate a collection of goods amongst multiple bidders. In the second part of the thesis, we study the problem of welfare maximization in the presence of unrestricted rational collusion. We generalize the notion of dominant-strategy mechanisms to collusive contexts, construct a highly practical such mechanism for multi-unit auctions, and prove that no such mechanism (practical or not) exists for unrestricted combinatorial auctions. Our results explore the power and limitations of enlarging strategy spaces to incentivize agents to reveal information about their collusive behavior.

Thesis Supervisor: Constantinos Daskalakis
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Part I

Introduction
Chapter 1

Overview

1.1 The Need for Mechanism Design

*How should we design economic systems?* We have the unprecedented opportunity, spurred largely by the development of the Internet and the boom of electronic commerce, to not only tweak existing infrastructures but to design entirely novel ways of facilitating economic interactions. As new applications, ranging from online advertisement auctions to electronic trading systems to internet routing protocols, continue to develop, it becomes increasingly important to develop a robust theory and understanding of the tasks faced by a modern mechanism designer. Mechanism design has become a progressively more challenging undertaking in recent years, due to factors such as the following:

**Increased size of classical systems.** The emergence of the Internet has transformed the way in which economic interactions occur. Stemming from applications such as online auction houses and electronic markets, systems now involve large numbers of agents interacting simultaneously in a variety of transactions and at rapid speeds. Due to the scale and complexity of these systems, classical approaches to their design and analysis may not be well-suited to handle the new algorithmic and dynamic concerns. Even classical problems such as revenue-maximization pose serious computational hurdles when the size of these systems becomes large.
New applications. In addition to expanding the size and scope of existing systems, the Internet has spurred the need for mechanisms in brand new markets. As a result of new design problems such as allocating online advertisement space and selling digital goods, we face the challenge of facilitating agent interactions in a wide variety of new scenarios. Mechanism designers are now forced to tackle problems that might not have even existed several years ago.

New algorithmic and robustness concerns. As economic systems become more complex, it is important to robustly model agent behavior and to design mechanisms with well-rooted performance guarantees. While related problems have long been studied in classical economic settings, the size and computational nature of the Internet have placed increased importance on algorithmic issues and robustness concerns. It is important, for example, that theoretical performance guarantees hold not only for idealized agents, but for mechanism participants with real-world computational limitations. The global nature of interactions has also rendered ineffective many classical techniques of guaranteeing mechanism outcomes. As a concrete example, later in this thesis we consider the problem of collusion in auctions. While classical approaches of detecting and punishing collusive participants may be practical in small-scale local interactions, such criminalization is unenforceable when agents are spread across the world, and it may be exceedingly difficult even to detect a group of collusive players hidden within a large system.

The fundamental question of this thesis, broadly stated, is the following:

- How can we design economic mechanisms for desired objectives, and what are the corresponding algorithmic implications?

Answering this question requires the combination of tools from theoretical computer science, economics, and mathematics. We stress also that mechanism design is not an isolated endeavor: we fully expect that the techniques developed to solve concrete mechanism design problems will find further applications in algorithms, optimization theory, complexity theory, and a range of other fields.
1.2 Our Mechanism Design Goals

Mechanism design is a multi-faceted field, and a designer’s goals vary significantly from one application to the next. A mechanism which performs very well for one task may be ill-suited for another. In this thesis we focus on two of the most important goals: optimality and efficiency.

After this introductory section, this thesis is divided into two independent parts: The bulk of the thesis, Chapters 3 through 7, focuses on the problem of revenue maximization, while Chapters 8 and 9 are concerned with efficient resource allocation. While the problems and the techniques used in these two sections are fairly distinct, they both have the same underlying theme: understanding mechanism structure. By gaining a deep knowledge of the problem at hand, we aim to understand the structural properties that a well-suited mechanism should have, and to use this knowledge for our design task.

1.2.1 Optimal Multi-Item Mechanisms

The bulk of this thesis studies mechanism optimality: the maximization of an auctioneer's expected revenue. We consider the situation where a seller has only probabilistic information about an additive\(^1\) quasilinear\(^2\) buyer’s values for each item. Myerson’s influential work provides an elegant and computationally efficient optimal single-item auction, and Myerson’s approach even applies to settings of multiple competing buyers. Generalizing this result to multi-item scenarios has been a central question in both economics and algorithmic game theory; see Section 2 as well as [40] and its references for additional prior work on the problem.

Maximizing revenue in multi-item auctions is one of the most important and natural mechanism design tasks, as many real-world scenarios are inherently multi-item. In applications as diverse as allocating sponsored search advertisements to selling financial instruments or distributing frequencies in the wireless spectrum, a

---

\(^1\)An additive buyer values a set of items equal to the sum of his values for each item in the set.

\(^2\)A quasilinear buyer’s utility for a set of items at price \(\tau\) is equal to his value for the set minus \(\tau\).
vendor has a variety of goods he wishes to sell to the consumers. Even though a plethora of real-world scenarios are well-suited for the use of multi-item auctions, the current state of economic and mathematical understanding of these situations is severely lacking. Unlike the single-item case, not only does the economic literature lack a general solution to the multi-item problem, but it does not even provide a framework for systematically approaching such situations.

Even in the single-buyer setting, very little is known about how to generalize Myerson's work to the multi-item case. Our central question is as follows:

- What is the revenue-optimal mechanism for selling multiple items to a single additive buyer, when the seller knows the probability distributions from which the buyer's values for the items are drawn?

It is worth mentioning that the optimal multi-item mechanism might not be comprised of optimal single-item mechanisms, even when the values of the items are distributed independently. Consider, for example, the following example from Hart and Nisan [33]: A single additive bidder values two items independently, with each item valued at $1 or $2 with equal probability. The optimal expected revenue for a single such item is $1, achieved by either setting a price of $1 for the item (in which case it would always be purchased) or setting a price of $2 (in which case it would be purchased half the time). In the two-item setting, however, the mechanism which offers just the bundle of both items for $3 achieves expected revenue of $3 \cdot 0.75 = $2.25, which is strictly more than the $1 + $1 = $2 expected revenue achieved by running two optimal single-item auctions in parallel.

In general, optimal mechanisms may have structure significantly more complex than either selling independently or bundling items together. For example, not only might optimal mechanisms assign prices to sets of items, but they might also assign prices to lotteries which allocate items with specified probabilities. In Section 6.2.2 we provide an example in which not only is the optimal mechanism randomized, but it offers infinitely many such lotteries. See Section 2 for a broader overview of optimal multi-item auctions.
1.2.2 Collusion-Resilient Efficient Mechanisms

Another primary goal of mechanism design is that of efficiency. The central question is how to allocate a collection of resources to maximize social welfare, the sum amount that participants value the allocation. Efficient allocation problems differ markedly from optimal auctions in that the auctioneer does not care about the amount of revenue received. Rather, monetary payments are leveraged to incentivize players to reveal information about their true values for the resources, thereby enabling the auctioneer to choose the socially optimal allocation.

One of the most well-known efficient auctions is the Vickrey-Clarke-Groves (VCG) mechanism \[63, 18, 32\] for optimizing welfare in a wide variety of multi-item settings. However, its practical applicability is hindered by the fact (as first demonstrated by Ausubel and Milgrom [4]) that it is vulnerable to player collusion. In real-world applications such as online auctions, it is nearly impossible to detect and punish participants who collude, and thus this vulnerability can destroy the theoretical efficiency guarantees. We therefore probe whether we can design mechanisms with performance guarantees that hold even in the presence of undetected collusion. Such mechanisms would have important real-world impact, as they eliminate the need to detect collusive participants. That is, we aim to deal with collusion using mechanism design instead of through external policing. We wish to achieve a notion of resilience analogous to dominant-strategy implementation, one of the strongest ways to provide performance guarantees in non-collusive settings. Thus:

- **How should implementation in dominant strategies be generalized to guarantee efficiency against unrestricted rational collusion?**

We furthermore ask in which situations this level of resilience can be achieved.

- **When do dominant-strategy collusion-resilient mechanisms exist? When they do exist, can we construct them as simple and highly-practical mechanisms?**
1.3 Overview of Main Results

We present a brief overview of selected results from this thesis. This discussion is relatively informal, as rigorous statements and proofs appear later.

1.3.1 Strong Duality Theorem for an Optimal Monopolist

Progress in optimal mechanism design has been severely hampered by a lack of structural understanding of optimal mechanisms. One of the key contributions of this thesis is the result that single (additive) bidder revenue maximization has a tight dual problem. Our strong duality theorem, Theorem 2 in Section 4, states not only that mechanism optimality can be certified by providing a dual certificate but that such a certificate always exists.

We use our strong duality theorem to obtain several new results in optimal mechanism design. In Theorem 3 in Section 5 we prove that a mechanism which simply posts a take-it-or-leave-it price for the grand bundle of all goods is optimal if and only if two stochastic dominance relations hold between particular measures derived from the joint value distribution. This technical result is a nontrivial application of our strong duality theorem, and essentially answers a longstanding question of under which conditions grand bundling is optimal.

As an application of our grand bundling characterization we prove Example 2, which states that for every integer $n > 0$ there exists a $c_0$ such that for all $c > c_0$, the optimal mechanism for selling $n$ i.i.d. goods whose values are uniform on $[c, c+1]$ is a take-it-or-leave-it offer for the grand bundle. This result is one of the first non-trivial optimal mechanism design solutions for arbitrarily many items.

We use our duality framework to solve several concrete examples of optimal mechanism design problems. In Section 6.1 we prove Theorem 4, which provides sufficient conditions for the optimal mechanism to have a certain structure in two-item instances. In Section 6.2.2 we use this formulation to obtain solutions of several optimal mechanism design problems, including the first constructive example of an optimal mechanism offering a continuum of randomized bundles.
1.3.2 The Complexity of Optimal Mechanism Design

Multi-item mechanism design is key to real-world economic applications, and it is therefore essential to consider the computational issues involved both in finding and in implementing such mechanisms. In Chapter 7 we prove that there exists no efficient algorithm for finding optimal mechanisms, even restricted to a very simple case of a single additive buyer with the values of goods being distributed independently.

Theorem 5. There is no expected polynomial-time solution to the optimal mechanism design problem (formal definition in Section 7.2) unless \( \text{ZPP} \subseteq \text{P}^{\#P} \).

This is true even in the case of a single additive quasilinear bidder whose values for the items are independently distributed on two rational numbers with rational probabilities.

We remark that the complexity of Theorem 5 appears not because we utilize exotic value distributions or impose combinatorial constraints on mechanisms (such as considering only deterministic mechanisms) but rather because our structural understanding of optimal mechanism design unveils the complex structure inherent in any optimal mechanism. We note that this theorem does not preclude the existence of efficient approximation schemes for the mechanism design problem. This remains an open question even in the case of a single additive bidder.

1.3.3 Collusive Dominant-Strategy Efficiency

In Chapter 9, we introduce a notion of dominant-strategy mechanism which is robust against “unrestricted rational collusion.” We aim for our model to be as general as possible, so that we might be resilient against a wide range of real-world collusive behavior. After a mechanism is announced, we allow players to partition themselves arbitrarily into collusive groups. Each group has the ability to perfectly coordinate their strategies in an attempt to maximize the sum utility of the group, but we make no assumption about a player’s knowledge about participants outside of his own collusive group. The structure of the collusive partition is entirely unknown to the mechanism designer.
In Section 9.3 we formally specify our collusive model and define our notion of collusive dominant-strategy efficiency. We remark that this notion of resiliency does not make any presuppositions about the particular structure of a mechanism, but requires only that every collusive group has a strategy subprofile which is dominant with respect to the group's sum utility.

Our goal for the remainder of the chapter is to understand the power and limitations of collusive dominant-strategy efficiency. On the positive side, in Section 9.4 we present a highly practical and efficient collusion-resilient mechanism for selling multiple copies of a single good. As demonstrated by Chen and Micali [16], the standard Vickrey mechanism [63] loses its efficiency guarantee in this auction context, even in the case of only two collusive players holding incorrect beliefs about their opponents. While our auction is not the first resilient mechanism in this context (a mechanism of Chen and Micali [16] achieves a similar notion of resiliency), our mechanism is the most practical: we do not impose excessive fines on the players (a main critique of the applicability of Chen and Micali's mechanism) and we do not require that the mechanism designer knows an upper bound on the valuation functions. On the negative side, we prove in Theorem 7 of Section 9.6 that no collusive dominant-strategy efficient mechanism exists for settings of unrestricted (combinatorial) valuation functions, even in the simple case of three players and two heterogeneous goods.

1.4 Discussion of Techniques

We briefly discuss some of the technical ideas used in this paper. Sections 1.4.1 and 1.4.2 primarily relate to our work on revenue maximization, while Section 1.4.3 primarily relates to collusion-resilient welfare maximization.

\[3\text{We make the standard assumption that each player has non-increasing marginal value for each additional copy of the good he receives.}\]
1.4.1 Stochastic Dominance and Duality Theorems

Before establishing our strong duality theorem for optimal mechanism design, we first define and then reformulate the mechanism design problem. It is well-known (see Rochet [54], Rochet and Choné [55], and Manelli and Vincent [39]) that a “valid” mechanism can be fully characterized by specifying its utility function, the mapping of each possible bidder type (the point in \( \mathbb{R}^n \) specifying the bidder’s value for each of the \( n \) items) to the utility that the bidder would enjoy from participating in the mechanism. A mechanism is feasible if and only if its utility function \( u \) satisfies several constraints, namely being nonnegative, nondecreasing, convex, and 1-Lipschitz with respect to the \( \ell_1 \) norm. After applying a robust version of integration by parts (similarly to [39]), we restate the optimal mechanism design problem as finding the valid utility function \( u \) which maximizes the integral \( \int ud\mu \), where \( \mu \) is a signed measure derived from the probability distribution of bidder types. One key result of this thesis, Theorem 2, is proving that this optimization program has a tight dual minimization program.

Our dual program is to find a joint measure \( \gamma \) (transporting the type space into itself) minimizing the integral \( \int \|x - y\|_1d\gamma(x, y) \), where \( \gamma \) satisfies a particular stochastic dominance constraint over the measure \( \mu \). Roughly speaking, our dual problem is a classical optimal transport problem with a twist: Before finding a transport map to minimize the \( \ell_1 \) transport distance between two measures, we may first “shuffle” the measures in a certain manner. (There is very good reason for why our dual problem resembles an optimal transportation problem. See Section 1.4.2 and Section A.)

The structure of our strong duality proof resembles the proof of Monge-Kantorovich duality for optimal transport as presented in [64], although several technical details are different. The heart of the proof is the Fenchel-Rockafellar duality theorem, a classical result in convex optimization.

A useful consequence of formulating the dual problem in terms of stochastic dominance is that we have a wide range of classical tools (see [58]) at our disposal. When applying our duality theorem, for example, we make frequent use of results relat-
ing stochastic dominance to coupling of measures. Nonconstructive tools such as Strassen's theorem enable us to prove the existence of desired measures exist without defining them explicitly. Indeed, one of the main difficulties in applying our framework is verifying stochastic dominance between particular measures. We develop some new technical tools (see Section 6.2.1), specifically tailored to certain applications, which simplify this verification in some scenarios.

1.4.2 Constraint Relaxations

Instead of directly applying our strong duality framework, in some settings it is simpler to use an alternate dual program which, while lacking the theoretical guarantees of Theorem 2, is geometrically simpler. In particular, if we relax the constraint that a utility function $u$ be convex, then the dual program is a standard optimal transport problem asking for the minimum-cost method of transforming the positive part of the measure $\mu$ into the negative part (see Appendix A). As in our strong duality theorem, any feasible transport map yields an upper bound on the optimal mechanism's revenue. While it is not always the case that the minimum transport cost equals the maximum revenue (equality only holds when the convexity constraint on $u$ is not the tight constraint), in certain situations (such as all examples from Section 6.2) optimal transport can indeed certify mechanism optimality.

In Theorem 5 of Chapter 7, we apply a discrete analog of this idea to prove that optimal mechanism design is computationally intractable. The main challenge in this result is to identify a class of mechanism design instances which are sufficiently complex to embed a $\#P$-hard problem yet whose structure is sufficiently constrained that we may extract the $\#P$ solution if given the optimal mechanism. In essence, the difficulty is to understand mechanism structure: Even if we cannot efficiently compute the optimal mechanism, we must understand its structure so that, if we were given the mechanism, we could extract the $\#P$ solution. Our approach to understanding this structure is to relax the mechanism design program by removing the discrete analog of the convexity constraint on $u$, to interpret its dual as a min-cost flow problem (a discrete analog of optimal transport) and to use complementary slackness
to characterize pairs of optimal solutions to the relaxed mechanism design primal and its min-cost flow dual. (In particular, we prove that, in our class of instances, the optimal min-cost flow is the output of a simple exponential-time greedy procedure.) We then show that the utility function solving the relaxed primal is monotone and supermodular (the analog of convexity) and therefore solves the original optimal mechanism design instance. This approach allows us to characterize the structure of a class of optimal mechanisms well enough to complete the hardness reduction.

1.4.3 Strategy Spaces and the Revelation Principle

A novel feature of our definition of collusive dominant-strategy efficiency is that our notion is applicable to arbitrary mechanisms and not only mechanisms whose strategy space takes a particular form. Indeed, Green and Laffont [31] have shown that collusive efficiency in dominant strategies cannot be achieved in a variety of contexts (including multi-unit auctions) when each player's strategy space coincides precisely with the set of his possible valuation functions. Chen and Micali [16] have previously bypassed this impossibility result by using a mechanism with a larger strategy space, allowing a player to declare not only his type but also a set of his colluders.

In Chapter 9, we propose a notion of collusive dominant-strategy efficiency which applies to mechanisms with arbitrary strategy sets. Our multi-unit auction of Section 9.4 achieves its high level of practicality by employing a strategy space even larger than that envisaged by [16]: A player is asked to declare not only his own valuation function and the identity of his colluders, but has the ability (if he desires) to provide information about the valuation functions of other members of his collusive group. Our mechanism, very roughly speaking, is a variant of the Vickrey auction [63] in which a collusive player is allowed to purchase items on behalf of other members in his collusive group.

Next, we consider settings of general valuation functions for heterogeneous goods and provide an impossibility result in Theorem 7. To achieve the highest generality impossibility result, we consider mechanisms with arbitrary strategy spaces. We show in Section 9.5, by a simple generalization of Myerson's revelation principle [48],
that it suffices to prove impossibility for mechanisms wherein each player is asked to reveal his own valuation function as well as the valuation functions of all of his colluders. Once restricting our attention to mechanisms with such a strategy space, our impossibility proof is by contradiction: Starting from an example of Ausubel and Milgrom [4] (showing the vulnerability of the VCG mechanism to collusion), we consider a sequence of scenarios to arrive at the contradiction that any efficient collusive dominant-strategy mechanism must make infinite payout to two bidders who report that they all belong to the same coalition and value all items at zero.

We remark that using enlarged strategy spaces, as in our multi-unit auction mechanism, requires a philosophical shift on how to handle collusion. Instead of criminalizing collusion, we incentivize collusive participants to reveal information about their collusive group. Chen and Micali have shown that asking for some information about collusive behavior can be beneficial, and here we demonstrate that asking for more information can be even more beneficial. Thus, we handle collusion not by criminalization but rather by bringing collusion into the open, enabling a mechanism designer to leverage collusion for his own purposes.

1.5 Organization of the Dissertation

Following this introduction, the dissertation consists of two parts. Part II consists of Chapters 2 through 7, while Part III consists of Chapters 8 and 9. The dissertation is organized as follows:

- In Chapter 2 we provide a historical overview of prior work concerning revenue maximization in mechanism design.

- In Chapter 3 we formally introduce the optimal mechanism design problem for a multi-good monopolist. Then, in Theorem 1, we reformulate optimal mechanism design as an optimization program over convex, nondecreasing, Lipschitz continuous utility functions. This reformulation is central to Part II of this thesis. Lastly, we provide simple examples of applying Theorem 1 to basic
mechanism design instances.

- In Chapter 4 we present mathematical preliminaries on stochastic dominance as well as the statement and proof of Theorem 2, our strong duality theorem for optimal mechanism design. Additionally, we provide a simple example of applying our duality framework to solve an optimal mechanism design instance.

- In Chapter 5 we discuss Theorem 3, an equivalent condition for grand bundling optimality in terms of stochastic dominance relationships. In this chapter we present several necessary probabilistic lemmas, prove Theorem 3, and, in Example 2, apply this result to a scenario with large numbers of uniformly distributed items.

- In Chapter 6 we state Theorem 4, providing a sufficient condition for optimal two-item mechanisms to have a particular structure, and we discuss several example applications.

- In Chapter 7 we prove Theorem 5, which shows that the multi-item optimal mechanism design is computationally intractable even in one of the most basic single-buyer settings.

- In Chapter 8 we provide a historical overview of prior work concerning welfare maximization in mechanism design, focusing on work concerning collusion resiliency.

- In Chapter 9 we study the problem of designing collusion-resilient efficient mechanisms. In this chapter, we first explain our collusive model and define our notion of resiliency. We then present a collusion-resilient variant of the Vickrey mechanism for efficient multi-unit auctions. We prove in Theorem 6 that this mechanism has the desired properties. Finally, in Theorem 7 we prove that no collusive dominant-strategy mechanism can satisfy our robustness criteria in combinatorial auctions with at least three players and two goods.
In Appendix A we introduce an optimal transport problem which is a weak dual to optimal mechanism design. This weak duality framework, stated in Theorem 8, provides a simple method of solving certain optimal mechanism design instances and provides further geometric insight into optimal mechanism design.

In Appendix B we informally discuss how some of our optimal mechanism design results extend to scenarios with unbounded type spaces, and we provide two example applications.

In Appendix C we discuss the restriction made in Chapters 3 through 6 that, in optimal mechanism design instances, the joint probability distribution over the type space is differentiable. In Lemma 19 of this appendix, we show that for sufficiently “close” distributions $\alpha$ and $\beta$ on the bidder’s type space, an optimal mechanism for one of the distributions can be transformed into a nearly-optimal mechanism for the other distribution. Thus, since any distribution is “close” to a smooth distribution, our framework can be applied to approximately solve a range of instances with type distributions which are not differentiable.

Chapters 2–7, Appendix A, Appendix B, and Appendix C are joint work with Constantinos Daskalakis and Christos Tzamos. These sections are derived from [20], [23], and [22].

Chapters 8–9 are joint work with Silvio Micali and are derived from [25].

\footnote{In particular, we note that parts of Chapter 6, including the diagrams and much of the content of Section 6.2, appeared in [22], the content of Chapter 7 appeared previously in [23], Appendix A contains the main result of [22], and Appendices B.2 and B.3 essentially appeared in [22]. Much of the remaining technical work for optimal mechanism design appears in [20] (currently under preparation), and the discussion concerning optimal mechanism design is derived from [22], [23], and [20].}
Part II

Maximizing Revenue
Chapter 2

Historical Overview

Optimal mechanism design is the problem of designing a revenue-optimal auction for selling a collection of $n$ items to a group of $m$ bidders. In Part II of this thesis we study the Bayesian setting, where each bidder’s type (valuation function) is drawn from a known prior distribution. The special case of selling a single item is well-understood, going back to the work of Myerson [49] and Cremer and McLean [19]. The general ($n > 1$) case has been much more challenging, even in the setting of a single bidder.

In Section 1.2.1, we provided a simple example with a single additive bidder where, even though item valuations were uncorrelated, the optimal mechanism priced only the bundle of items instead of selling each item separately. The question of when such bundling is optimal is a central stepping stone to understanding the structure of optimal mechanisms more generally. Armstrong [3] showed that, when the number of items becomes large, such a bundling mechanism often achieves nearly-optimal revenue. More recently, Manelli and Vincent [39] have derived sufficient (but not necessary) conditions for bundling optimality. Their work obtains conditions for bundling to be the best deterministic mechanism as well as the best mechanism overall.

Optimal mechanisms are typically much more complex than selling each item separately or bundling them all together. Even in the two-item setting, McAfee, McMillan, and Whinston [43] show that a combination of bundling and separate selling strictly dominates independent selling whenever the item values are distributed inde-
pendently, subject to a mild continuity condition on the joint distribution. McAfee and McMillan [42] proposed conditions under which the optimal mechanism is deterministic, however these were found insufficient by [62] and [39]. The advantage of deterministic mechanisms is that they have a finite description, namely the price they charge for every possible bundle of the items. Hence looking for the optimal one is feasible, computational considerations aside. In contrast, randomization adds an extra layer of difficulty: it is even possible—we exhibit such an example in Section 6.2.2—that the optimal mechanism offers a continuum of lotteries. Hence it is a priori not clear whether one could hope for a concise (or even a finite) description of the optimal mechanism, and it is even less clear whether one can optimize over the corresponding space of (infinite-dimensional) mechanisms.

Randomization can be necessary even in simple settings [62, 39, 40, 35]. Here is a two-item example whose proof follows from the techniques of Chapter 7: Consider an additive bidder who values the first item at $1 or $2 with equal probability and values the second item independently at $1 or $3 with equal probability. The optimal mechanism here offers not only the bundle of both items (at price $4) but also offers for $2.50 a randomized bundle that provides the first item with probability 1 and the second item with probability 1/2.

Given the complex structure that optimal mechanisms might have, it is unclear how to begin tackling the optimal mechanism design problem. A useful first step, and the approach we take in Chapter 3, is to express revenue as particular integral of a mechanism’s utility function. Such a transformation typically uses ether integration by parts or the divergence theorem, and has been used by work such as McAfee and McMillan [42], Rochet and Choné [55], and Manelli and Vincent [39]. We remark that some prior work, including [42], [39], and [53] impose an additional “hazard rate condition” on the joint density function $f$ of item values, assuming (essentially) the inequality $(n + 1)f(x) + x \cdot \nabla f(x) \geq 0$. Such an assumption simplifies the structure and analysis of mechanism revenue, but imposes a significant restriction on the applicability of the results.

Manelli and Vincent [40] support the observation that optimal mechanisms may
have intricate structure. They prove that optimal mechanisms are "extreme points" of the space of feasible mechanisms, and that every "undominated" mechanism is optimal for some independent distributions of item values. Determining whether or not a particular mechanism is dominated, however, is not always a simple task.

We mention also the work of Pavlov [53] on optimal two-item single-bidder mechanisms. This work studies additive bidders (as in this thesis) as well as unit-demand bidders. Pavlov derives properties of optimal mechanisms subject to a hazard rate condition on the joint density function of the two goods. In particular, he solves for the optimal mechanism for two items distributed independently and uniformly on the real interval $[c, c+1]$, both in the case of an additive bidder and of a unit-demand bidder.

Recently, progress has been made in algorithmic solutions to optimal mechanism design problems. Cai et al. provide efficiently computable revenue-optimal [9, 10] or approximately optimal mechanisms [11] in very general settings, including where there are combinatorial constraints over which allocations of items to bidders are feasible. However, these results, as well as the more specialized ones of Alaei et al. [2] for service-constrained environments, apply to the explicit setting, i.e. when the distributions over bidders' valuations are given explicitly, by listing every valuation in their support together with the probability it appears. Such a manner of describing a joint distribution, however, is often unnatural. For example, in the case of independently distributed items, explicitly listing the joint distribution can be exponentially longer than listing each marginal distribution separately.

Essentially the only known positive computational results for additive bidders in the implicit setting are for when the values are drawn from Monotone Hazard Rate distributions where Bhattacharya et al. [7] obtain constant factor approximations to the optimal revenue, and Daskalakis and Weinberg [24] and Cai and Huang [12] obtain polynomial-time approximation schemes. For general distributions but a single buyer, Hart and Nisan [33] show that selling the items through separate auctions guarantees

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1A unit-demand bidder is described by a vector $(v_1, \ldots, v_n)$ of values. If the bidder receives item $i$, her value is $v_i$, while if she receives a set of more than one item, her value for that set is her value for her favorite item in the set. Pavlov [53] refers to this scenario as a case of "substitutable goods."
a $O\left(\frac{1}{n}\right)$-fraction of the optimal revenue, which can be improved to $\frac{1}{2}$ for 2 items, even if the number of buyers is arbitrary. They also show that in the single-buyer setting with identically distributed items, offering the grand bundle at some optimal price guarantees a $O\left(\frac{1}{\log n}\right)$-fraction of the optimal revenue.

There has been considerable effort towards computational lower bounds for optimal mechanism design. Nevertheless, all prior results are for either somewhat exotic families of valuation functions or distributions over valuations, or for computing the optimal deterministic, as opposed to general (possibly randomized) auction. In the first vein, Dobzinski et al. [26] show that optimal mechanism design for OXS bidders is $\mathbb{NP}$-hard via a reduction from the CLIQUE problem. OXS valuations are described implicitly via a graph, include additive, unit-demand and much broader classes of valuations, and are more amenable to lower bounds given the combinatorial nature of their definition. In joint work with Daskalakis and Tzamos [21], we prove $\#\mathbb{P}$-hardness of the optimal mechanism design problem for a single item and a single bidder whose value for the item is the sum of independently distributed attributes. Compared to these lower bounds, our goal in Chapter 7 is to prove intractability results for very simple valuations (namely additive) and simple distributions (namely the item values are independent and the distribution of each item is given explicitly), and which have no combinatorial structure incorporated in their definition.

On the complexity of optimal deterministic auctions, Briest [8] shows inapproximability results for selling multiple items to a single unit-demand bidder via an item-pricing auction,\(^2\) when the bidder’s values for the items are correlated according to some explicitly given distribution. More recently, Papadimitriou and Pierrakos [52] show $\text{APX}$-hardness results for the optimal, incentive compatible, deterministic auction when a single item is sold to multiple bidders whose values for the item are correlated according to some explicitly given distribution. (We note that the settings considered in both Briest [8] and Papadimitriou-Pierrakos [52] are polynomial-time solvable via linear programming when the determinism requirement is removed [26, 24].) Finally,

\(^2\)An item-pricing auction posts a price for each item and lets the bidder buy whatever item she wants.
in joint work with Daskalakis and Tzamos [21], we provide SQRTSUM-hardness results for the optimal item-pricing problem when there is a single unit-demand bidder whose values for the items are independent of support two, and when either the values or the probabilities may be irrational. Compared to these results, our lower bounds of this thesis apply to general (i.e. possibly randomized) auctions, the values of the items are distributed independently, and both supports and probabilities are rational numbers.
Chapter 3

The (Monopolistic) Optimal Mechanism Design Problem

Our goal is to find the revenue-optimal mechanism $M$ for selling $n$ goods to a single additive bidder. In Section 3.1 we formally define the mechanism design problem and in Section 3.2 we reformulate the problem as an optimization program over convex, nondecreasing, and Lipschitz continuous utility functions. The formulation, stated formally in Theorem 1, is central to our approach and is used throughout Part II of this thesis. In Section 3.3 we provide examples of applying Theorem 1 to reformulate specific mechanism design instances.

We remark that in this chapter we focus only on direct mechanisms. This is without loss of generality, by Myerson’s revelation principle [48].

3.1 Definition of the Problem

An additive bidder has a type $z$ specifying his value for each good, where $z$ is an element of a type space $Z \subseteq \mathbb{R}_{\geq 0}^n$. While the bidder knows his type with certainty, the mechanism designer knows only the probability distribution on $Z$ from which $z$ is drawn. The type of a bidder is sometimes called his valuation.

\footnote{When we study algorithmic issues later in this thesis, we must take more care when applying the revelation principle. See Section 7.2.}
A mechanism consists of two functions: (i) an allocation function $\mathcal{P} : Z \rightarrow [0, 1]^n$ specifying the probabilities, for each possible type declaration of the bidder, that the bidder will be allocated each good, and (ii) a price function $\mathcal{T} : Z \rightarrow \mathbb{R}$ specifying, for each declared type of the bidder, the price that he is charged. When an additive bidder of type $z$ declares himself to be of type $z' \in Z$, he receives net utility $z \cdot \mathcal{P}(z') - \mathcal{T}(z')$.

We restrict our attention to mechanisms that are incentive compatible, meaning that the bidder must have adequate incentives to reveal his values for the items truthfully, and individually rational, meaning that the bidder has an incentive to participate in the mechanism. The key result of the current section is Theorem 1, where, in a manner very similar to that of [39] and [22], we reduce the optimal mechanism design problem to an optimization problem over feasible utility functions.

**Definition 1.** Mechanism $\mathcal{M} = (\mathcal{P}, \mathcal{T})$ over type space $Z \subseteq \mathbb{R}_{\geq 0}^n$ is incentive compatible (IC) if and only if $z \cdot \mathcal{P}(z) - \mathcal{T}(z) \geq z \cdot \mathcal{P}(z') - \mathcal{T}(z')$ for all $z, z' \in Z$.

**Definition 2.** Mechanism $\mathcal{M} = (\mathcal{P}, \mathcal{T})$ over type space $Z \subseteq \mathbb{R}_{\geq 0}^n$ is individually rational (IR) if and only if $z \cdot \mathcal{P}(z) - \mathcal{T}(z) \geq 0$ for all $z \in Z$.

When the type $z$ of a bidder is drawn from a known distribution, the mechanism designer's goal is to find the IC and IR mechanism $\mathcal{M} = (\mathcal{P}, \mathcal{T})$ maximizing the expectation of $\mathcal{T}(z)$, the revenue received by the mechanism. We call this the optimal mechanism design problem.

### 3.2 Reformulation of the Problem

The goal of this section is to reformulate the optimal mechanism design problem as an optimization program of over feasible utility functions. This reformulation is stated in Theorem 1.

We begin by noting that while the above definitions of individual rationality and incentive compatibility apply to mechanisms over an arbitrary type space $Z \subseteq \mathbb{R}_{\geq 0}^n$, it

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2A bidder whose utility depends linearly on his payment is called *quasilinear*. 

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is well-known (see an analogous result in [29]) that we can extend a mechanism's type space to any superset of \( Z \) without violating the IC or IR properties. We state this property as Fact 1. The proof of this fact constructs a new mechanism \( M' \) providing the new types a choice from the (closure of the) of price-allocation pairs offered by \( M \).

**Fact 1** (Domain Extension). Let \( M = (P, T) \) be an IC and IR mechanism over type space \( Z \subseteq \mathbb{R}_{\geq 0}^n \), and let \( X \supseteq Z \) be any subset of \( \mathbb{R}_{\geq 0}^n \) containing \( Z \). Then there exists an IC and IR mechanism \( M' = (P', T') \) over type space \( X \) such that \( P(z) = P'(z) \) and \( T(z) = T'(z) \) for all \( z \in Z \).

**Proof of Fact 1:** Let \( \mathcal{O} = \{(P(z), T(z)) : z \in Z\} \cup \{(0^n, 0)\} \) be the set of all possible outcome-price pairs of the mechanism \( M \), along with the "opt-out" option which allocates no items and charges nothing. We define the set \( \tilde{\mathcal{O}} \) to be the closure of \( \mathcal{O} \), under the standard topology on \( \mathbb{R}^{n+1} \).

We define the mechanism \( M' \) as follows:

\[
(P'(z'), T'(z')) = \begin{cases} 
(P(z'), T(z')), & \text{if } z' \in Z \\
\arg\max_{(p,t) \in \tilde{\mathcal{O}}} (z' \cdot p - t), & \text{otherwise.}
\end{cases}
\]

where the above argmax is chosen arbitrary if it is not unique. Mechanism \( M' \) behaves as \( M \) on all declared types \( z' \in Z \), and allows declared types outside of \( Z \) to choose any outcome in \( \tilde{\mathcal{O}} \) which would maximize the bidder's utility. We argue first that the above argmax is never empty: Notice that the set \( \{T(z) : z \in Z\} \) is bounded from below by some value \(-b\), as otherwise \( M \) would make arbitrarily high payouts to the players in \( Z \), meaning that no player would have a best strategy in \( M \) (since, regardless of his type, he could always play a different strategy in which the mechanism pays him significantly more money) violating the IC property. Furthermore, for any \( z' \in X \setminus Z \) we need only consider allocations with \( t \leq \|z'\|_1 \). Thus we are optimizing a continuous function over a compact subset of \((p,t)\) values, namely \( \mathcal{O} \cap ([0,1] \times [-b, \|z'\|_1]) \). The argmax is therefore nonempty.

We now show that \( M' \) is individually rational. Since \( M \) is IR, every player in
Z receives non-negative utility in \( \mathcal{M}' \). Furthermore, since and the outcome \((0^n, 0)\) is in \( \bar{\mathcal{O}} \), the utility of a player in \( X \setminus Z \) is bounded below by zero. Therefore no type receives negative utility in \( \mathcal{M}' \).

Lastly, we argue that \( \mathcal{M}' \) is incentive compatible. By construction, every type \( X \setminus Z \) receives a utility-maximizing price-allocation pair in \( \bar{\mathcal{O}} \). Thus, to prove that \( \mathcal{M}' \) is IC, we need only show that for all types \( z \in Z \), the outcome \((\mathcal{P}(z), \mathcal{T}(z))\) maximizes \( z \cdot p - t \) over all \((p, t) \in \mathcal{O}\). This follows immediately from the observations that (i) since \( \mathcal{M} \) is IC, \((\mathcal{P}(z), \mathcal{T}(z))\) maximizes the utility \( z \cdot p - t \) over all \((p, t) \in \mathcal{O}\), (ii) the utility function \( z \cdot p - t \) is a continuous function of \( p \) and \( t \), and (iii) \( \mathcal{O} \) is dense in \( \bar{\mathcal{O}} \).

When a buyer truthfully reports his type to a mechanism \( \mathcal{M} = (\mathcal{P}, \mathcal{T}) \) (over type space \( Z \)), we denote by \( u : Z \rightarrow \mathbb{R} \) the function that maps the buyer's valuation to the utility he receives by \( \mathcal{M} \). It follows by the definitions of \( \mathcal{P} \) and \( \mathcal{T} \) that \( u(z) = z \cdot \mathcal{P}(z) - \mathcal{T}(z) \). Due to Fact 1, we can focus on mechanisms whose domain is convex by extending our mechanism to a convex type space \( X \supseteq Z \). Under such a space, it is well-known (see [54], [55], and [39]), that whether or not a mechanism is IC and IR is can be fully determined by inspecting its utility function.

**Lemma 1.** Let \( \mathcal{M} = (\mathcal{P}, \mathcal{T}) \) be a mechanism defined over a convex type space \( X \subseteq \mathbb{R}^n_{>0} \). Then \( \mathcal{M} \) is IC and IR if and only if the utility function \( u(z) = z \cdot \mathcal{P}(z) - \mathcal{T}(z) \) is convex, nonnegative, nondecreasing, and 1-Lipschitz with respect to the \( \ell_1 \) norm.

We clarify that a function \( u \) is 1-Lipschitz with respect to the \( \ell_1 \) norm if \( u(x) - u(y) \leq \|x - y\|_1 \) for all \( x, y \in X \). Given a function \( u \) satisfying the conditions of Lemma 1, we can construct an IC and IR mechanism with utility function \( u \), as shown by the following claim (originally from [54]).

**Claim 1.** Let \( X \subseteq \mathbb{R}^n_{>0} \) be convex with nonempty interior, and let \( u : X \rightarrow \mathbb{R} \) be a utility function satisfying the conditions of Lemma 1. Then we can construct an IC and IR mechanism with utility function \( u \) by setting \( \mathcal{P}(z) = \nabla u(z) \) and \( \mathcal{T}(z) = \nabla u(z) \cdot z - u(z) \) wherever \( \nabla u \) is defined. On the measure-0 set on which \( \nabla u \) is not
defined, we can use an analogous expression for $P$ and $T$ by choosing appropriate values of $\nabla u$ from the subgradient of $u$.

Wherever $\nabla u$ is defined, the components of $\nabla u$ specify the corresponding allocation probabilities of the mechanism. In Remark 1, we note that any $u$ satisfying the conditions of Lemma 1 indeed has $\nabla u$ defined almost everywhere, and thus the mechanism corresponding to a feasible $u$ is essentially unique.

Remark 1. Let $X$ be a nonempty subset of $\mathbb{R}^n$ whose boundary has Lebesgue measure 0. Then every non-decreasing function $u : X \to \mathbb{R}$ which is 1-Lipschitz with respect to the $\ell_1$ norm satisfies $\nabla u \in [0, 1]^n$ almost everywhere.

Proof. Suppose that $u$ is non-decreasing and 1-Lipschitz. Since the boundary of $X$ has measure 0, it suffices to show that $\nabla u \in [0, 1]^n$ almost everywhere on $\text{int}(X)$, the interior of $X$. We now apply Rademacher's Theorem, which states that a Lipschitz continuous function mapping an open subset of $\mathbb{R}^n$ into $\mathbb{R}$ is differentiable almost everywhere, to conclude that $\nabla u$ exists at almost all $x \in \text{int}(X)$. For any point $x$ where $\nabla u$ exists, and any coordinate vector $e_i$, we have $\partial u/\partial x_i (x) = \lim_{\varepsilon \to 0} u(x + \varepsilon e_i) - u(x) / \varepsilon$. We now apply the 1-Lipschitz condition and the fact that $u$ is non-decreasing to bound $0 \leq u(x + \varepsilon e_i) - u(x) \leq \varepsilon$ to conclude that $\nabla u \in [0, 1]^n$ almost everywhere. □

We will formulate the mechanism design problem as an optimization problem over feasible utility functions $u$. We first define the notation:

Definition 3. Let $X$ be a convex subset of $\mathbb{R}^n$. Then

- $U(X)$ is the set of all functions $u : X \to \mathbb{R}$ which are continuous, non-decreasing, and convex.

- $L_1(X)$ is the set of all functions $f : X \to \mathbb{R}$ which are 1-Lipschitz with respect to the $\ell_1$ norm. That is, $f(x) - f(y) \leq \|x - y\|_1$ for all $x, y \in X$.

\(^3\)Thus, the 1-Lipschitz condition on $u$ is a consequence of the fact that allocation probabilities are at most 1.
In this notation, Lemma 1 states that a mechanism $\mathcal{M}$ is IC and IR if and only if its utility function $u$ satisfies $u \geq 0$ and $u \in \mathcal{U}(X) \cap L_1(X)$.

In the optimal mechanism design problem, the distribution of bidder types is specified by a probability density function over the type space $Z$. In order to integrate over $Z$ and to use tools such as integration by parts, we must impose mild technical conditions on $Z$. We call a type space satisfying these conditions well-behaved. Well-behaved spaces are bounded, although we can sometimes circumvent this restriction—see Remark 2 and Appendix B.

We recommend the reader not get bogged down in the other details of the definition of a well-behaved type space at the moment, as except for highly correlated distributions the type space is typically an open box, which is well-behaved.

**Definition 4.** A type space $U \subseteq \mathbb{R}^n_+$ is well-behaved if it is a Jordan-measurable bounded Lipschitz domain. That is, $U$ is open, bounded, connected,\(^4\) and the boundary $\partial U$ both has Lebesgue measure 0 and is locally the graph of a Lipschitz continuous function.

The following claim generalizes the approach of [39] by writing the revenue maximization problem as an optimization problem over feasible utility functions $u$. The proof and applies an appropriate version of the divergence theorem (more precisely, a technical variant of integration by parts) to the expression for expected revenue.\(^5\)

**Claim 2.** Let $U \subseteq \mathbb{R}^n_+$ be a well-behaved type space and let $f : U \rightarrow \mathbb{R}$ be a (differentiable) probability density function with bounded partial derivatives. Let $X = [0, M]^n \supset U$. Then the supremum revenue of an IC and IR mechanism for goods whose values are distributed according to the joint distribution $f$ is given by

$$\sup_{u \in \mathcal{U}(X) \cap L_1(X)} \left\{ \int_{\partial U} u(z)f(z)(z \cdot \hat{n})dz - \int_U u(z)(\nabla f(z) \cdot z + (n + 1)f(z))dz \right\}$$

\(^4\)We could define a well-behaved type space to be any finite union of Jordan-measurable Lipschitz domains whose closures are pairwise disjoint. This would cause no additional technical difficulties, but is unnecessary for this paper.

\(^5\)This is where we use the condition that $U$ is a Lipschitz domain. Imposing such a condition on the boundary is necessary. Even the basic “textbook” version of the divergence theorem, for example, requires a piecewise smooth boundary.
Proof of Claim 2: Let $u$ be the utility function of an IC and IR mechanism $M = (P, T)$ over type space $U$, and let $P', T'$ be an extension of $M$ to $X$, as constructed in Fact 1. The expected revenue of $M$ is given by $\int_X f(z)T'(z)dz$. We now apply Claim 1 to write the expected revenue as

$$
\int_X T'(z)f(z)dz = \int_U T(z)f(z)dz = \int_U [\nabla u(z) \cdot z - u(z)]f(z)dz
$$

We now apply a high-dimensional analog of integration by parts (see Theorem 6.1 of Chapter 3 of [56]) to write $\int_U \left( \frac{\partial u}{\partial z_i} z_i f + u \frac{\partial z_i f}{\partial z_i} - u \frac{\partial_i f}{\partial z_i} \right)dz$ as a surface integral $\int_{\partial U} ufz_i \delta_i ds$. This theorem applies because $u(z)$ and $z_i f(z)$ are both Lipschitz functions and $U$ is a bounded Lipschitz domain. The result follows by noting that $\sum_i \int_{\partial U} ufz_i \delta_i ds = \int_{\partial U} (nz - (n+1)uf)ds$.

Since $\nabla f$ is bounded, the expression for expected revenue in Claim 2 is a bounded linear functional of $u$. By the Riesz representation theorem, we can define a (signed) Radon measure $\nu$ (supported within $U \cup \partial U$) such that

$$
\int h d\nu \triangleq \int_{\partial U} h(z)f(z)(z \cdot \hat{n})dz - \int_U h(z)(\nabla f(z) \cdot z + (n+1)f(z))dz
$$

for all continuous bounded $h : \mathbb{R}^n \to \mathbb{R}$. With this notation, the expected revenue of a mechanism with utility function $u$ is given by $\int u d\nu$. We notice that $\nu$ is supported within $X$ and, by setting $h$ to be the constant function 1, we observe that $\nu(X) = -1$. Since it will be more convenient later to deal with Radon measures of net zero mass, we make Claim 3, which says that we can add an appropriate mass to $\nu$ so that its

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$^6$A Radon measure is a locally-finite inner-regular Borel measure.
net mass becomes zero without significantly affecting our optimization problem.

**Claim 3.** Let $X = [0, M]^n$ and let $\nu$ be a Radon measure supported within a set $U \subset X$ such that $\nu(X) = -1$. Pick any $z_0 \in X$ such that $z_0$ is coordinate-wise less than every point in $U$, and define the measure $\mu = \nu + \delta_{z_0}$. Then $\mu(X) = 0$ and

$$\sup_{u \in U(X) \cap L^1(X)} \int_X u \, dv = \sup_{u \in U(X) \cap L^1(X)} \int_X u \, d\mu.$$ 

**Proof of Claim 3:** Clearly, any feasible $u : X \to \mathbb{R}$ for the left-hand side is feasible for the right-hand side with $\int_X u \, dv \leq \int_X u \, d\mu$ (since $u(z_0) \geq 0$). Thus the supremum value of the left-hand side is at most as large as the supremum value of the right-hand side.

For the reverse direction, let $u$ be feasible for the right-hand side, and define the function $\bar{u} : X \to \mathbb{R}$ by $\bar{u}(x) \triangleq \max\{u(x) - u(z_0), 0\}$. It is straightforward to show that $\bar{u}$ is feasible for the left-hand. Furthermore, since $u$ is non-decreasing and continuous, by our choice of $z_0$ we have $\bar{u}(z) = u(z) - u(z_0)$ for all $z \in U \cup \partial U \cup \{z_0\}$. Observing that $\mu$ and $\nu$ are both supported entirely within $U \cup \partial U \cup \{z_0\}$ and $U \cup \partial U$, respectively, we now compute

$$\int_X \bar{u} \, dv = -\bar{u}(z_0) + \int_X \bar{u} \, d\mu = 0 + \int_X (u(z) - u(z_0)) \, d\mu$$

$$= \int_X u \, d\mu - u(z_0)\mu(X) = \int_X u \, d\mu$$

and thus the supremum value of the left-hand side is at least as large as the supremum value of the right-hand side. \qed

Claim 3 simply allows us to consider measures $\mu$ with net zero mass, and also lets us disregard the non-negativity constraint on $u$.\footnote{That is, $\mu$ is $\nu$ with an added point mass of $+1$ at $z_0$.} We call such a measure $\mu$ a **transformed measure** of $f$.

**Definition 5 (Transformed measure).** Let $U \subset \mathbb{R}^n_{\geq 0}$ be a well-behaved type space, let $f : U \to \mathbb{R}_{\geq 0}$ be a probability density function with bounded partial derivatives, and $\nu$ a Radon measure supported within a set $U \subset X$ such that $\nu(X) = -1$. Pick any $z_0 \in X$ such that $z_0$ is coordinate-wise less than every point in $U$, and define the measure $\mu = \nu + \delta_{z_0}$. Then $\mu(X) = 0$ and

$$\sup_{u \in U(X) \cap L^1(X)} \int_X u \, dv = \sup_{u \in U(X) \cap L^1(X)} \int_X u \, d\mu.$$
let \( z_0 \in \mathbb{R}^n_0 \) be any point which is coordinate-wise less than all points in \( U \). We say that a signed Radon measure \( \mu \) is a transformed measure of \( f \) if the relation

\[
\int h d\mu = h(z_0) + \int_{\partial U} h(z)f(z)(z \cdot \tilde{n})dz - \int_U h(z)(\nabla f(z) \cdot z + (n+1)f(z))dz
\]

holds for all continuous bounded functions \( h: \mathbb{R}^n \rightarrow \mathbb{R} \).

A transformed measure \( \mu \) is supported within \( U \cup \partial U \) and \( \mu(U \cup \partial U) = 0 \). We provide examples of transformed measures in Section 3.3.

Theorem 1 below forms the basis for our approach to optimal mechanism design, and for the remainder of the paper we focus on problems of the form given in the theorem.

**Theorem 1.** Let \( U \subset \mathbb{R}^n_0 \) be a well-behaved type space and let \( f: U \rightarrow \mathbb{R}^n_0 \) be a probability density function with bounded partial derivatives. Then the problem of determining the optimal IC and IR mechanism for a single additive buyer whose values for \( n \) goods are distributed according to the joint distribution \( f \) is equivalent to solving the optimization problem

\[
\sup_{u \in \mathcal{U}(X) \cap L_1(X)} \int_X ud\mu
\]

where \( X = [0, M]^n \supset U \) and \( \mu \) is a transformed measure of \( f \).

**Remark 2.** While Theorem 1 only applies when \( U \) is bounded, we can often obtain an analogous result when \( U \) is unbounded, as long as the density function \( f \) decays sufficiently quickly (to ensure that appropriate integrals and the supremum revenue are finite.) Indeed, many results from this paper can be generalized to the unbounded case. See Appendix B for an informal discussion and examples of such extensions.\(^9\)

### 3.3 Examples

We present two concrete examples of transforming measures and applying Theorem 1.

\(^{9}\)Appendix B discusses extensions of results from throughout Part II of this thesis, and not just from this chapter.
3.3.1 Independent Uniform Items

Consider $n$ independently distributed items, where the value of each item $i$ is drawn uniformly from the bounded interval $(a_i, b_i)$ with $0 < a_i < b_i < \infty$. The joint distribution is given by a constant density function $f$ over the well-behaved set $U = \prod_i (a_i, b_i)$, and we apply Theorem 1 with $X = \prod_i [0, b_i] \supset U$ and $z_0 = (a_1, \ldots, a_n)$.

For notational convenience, define $v \triangleq \prod_i (b_i - a_i)$, the volume of $U$. The joint distribution of the items is given by the constant density function $f$ taking value $1/v$ throughout $U$. The transformed measure $\mu$ of $f$ is given by the relation

$$\int h d\mu = h(a_1, \ldots, a_n) - \frac{n+1}{v} \int_U h(z) dz + \frac{1}{v} \int_{\partial U} h(z)(z \cdot \hat{n}) ds$$

for all continuous and bounded $h : \mathbb{R}^2 \to \mathbb{R}$. Thus, by Theorem 1, the optimal revenue is $\sup_{\mu \in U(X) \cap C_1(X)} \int_X u d\mu$, where $\mu$ is the sum of:

- A point mass of $+1$ at the point $z_0 = (a_1, \ldots, a_n)$.

- A mass of $-(n + 1)$ distributed uniformly throughout the region $U$.

- A mass of $\frac{b_i}{b_i - a_i}$ distributed on each surface $\{z \in \partial U : z_i = b_i\}$.

- A mass of $-\frac{a_i}{b_i - a_i}$ distributed on each surface $\{z \in \partial U : z_i = a_i\}$.

3.3.2 Correlated Items with Non-Convex Support

Our framework applies even to joint distributions with non-convex supports. While work such as [55] assume that the joint density function $f$ is continuous on a convex support, this restriction is avoidable. Consider two items whose joint distribution is uniform on the set $U \subset \mathbb{R}^2$, the interior of the shaded region in the diagram:
The density function is $f(z) = \frac{1}{3} \mathbb{1}_{z \in U}$, which is continuous on $U$. We note, however, that $f$ is continuous neither on the convex hull of $U$ nor on the set $X = [0, 2]^2$. We can still use Theorem 1 to write the expected revenue achieved by utility function $u$ as $\int u \, d\mu$, where $\mu$ is the sum of:

- A point mass of $+1$ as the origin.
- A mass of $-3$ uniformly distributed throughout the region $U$.
- A mass of $+4/3$ uniformly distributed on the line $[0, 2] \times \{2\}$ and a mass of $+4/3$ uniformly distributed on the line $\{2\} \times [0, 2]$.
- A mass of $-1/3$ uniformly distributed on the line $[0, 1] \times \{1\}$ and a mass of $-1/3$ uniformly distributed on the lines $\{1\} \times [0, 1]$. 
Chapter 4

The Mechanism Design Duality
Theorem

In this chapter we present Theorem 2, our strong duality theorem. This theorem states that every optimization problem of the form of Theorem 1 has a corresponding dual minimization problem, both of these problems admit optimal solutions, and the optimal values of these problems are equal.

4.1 Mathematical Preliminaries

Our dual problem in Theorem 2 is an optimization problem over measures satisfying a certain property. We define the following notation.

Definition 6. Let $X$ be a subset of $\mathbb{R}^n$.

- $\text{Radon}_+(X)$ is the set of all positive Radon measures (locally finite and inner regular Borel measures) supported on $X$.

- For any $\gamma \in \text{Radon}_+(X \times X)$, the marginal measures, denoted $\gamma_1, \gamma_2 \in \text{Radon}_+(X)$, are defined by $\gamma_1(S) = \gamma(S, X)$ and $\gamma_2(S) = \gamma(X, S)$ for all measurable subsets $S$ of $X$. We note the equality $\gamma_1(X) = \gamma_2(X) = \gamma(X \times X)$.

- For any $\alpha, \beta \in \text{Radon}_+(X)$ such that $\alpha(X) = \beta(X)$, the set $\Gamma(\alpha, \beta)$ is the set of all $\gamma \in \text{Radon}_+(X \times X)$ with respective marginals $\alpha$ and $\beta$. 

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• Radon(X) is the set of all signed Radon measures supported on X. For a measure \( \mu \in \text{Radon}(X) \), we will denote by \( \mu_+ , \mu_- \in \text{Radon}^+(X) \) the positive and negative parts of \( \mu \), respectively. That is, \( \mu = \mu_+ - \mu_- \).

• For a measure \( \mu \in \text{Radon}(X) \) and a measurable \( A \subseteq X \), we define the restriction of \( \mu \) to \( A \), denoted \( \mu|_A \), by the property \( \mu|_A(S) = \mu(A \cap S) \) for all measurable \( S \).

Our dual problem optimizes over measures satisfying a certain stochastic dominance property. In particular,

**Definition 7.** Let \( X \) be a bounded convex subset of \( \mathbb{R}^n_{\geq 0} \) and \( \alpha, \beta \in \text{Radon}(X) \). We say that \( \alpha \) convexly dominates \( \beta \), denoted \( \alpha \succeq_{\text{conv}} \beta \), if for all (non-decreasing, convex) functions \( u \in \mathcal{U}(\mathbb{R}^n) \), \( \int u \alpha \geq \int u \beta \). Similarly, for vector random variables \( A \) and \( B \) with values in \( X \), we say that \( A \succeq_{\text{conv}} B \) if \( \mathbb{E}[u(A)] \geq \mathbb{E}[u(B)] \) for all \( u \in \mathcal{U}(\mathbb{R}^n) \).

Convex dominance is a weaker condition than standard first-order stochastic dominance, as first-order dominance requires that the inequality holds even for non-convex \( u \). For intuition, we note that \( A \succeq_{\text{conv}} B \) if and only if every risk-seeking person "prefers" \( A \) to \( B \), while \( A \) dominates \( B \) in the first order if and only if every risk-neutral person "prefers" \( A \) to \( B \). We point out that, since the constant functions \( \pm 1 \) are in \( \mathcal{U}(\mathbb{R}^n) \), the relationship \( \alpha \succeq_{\text{conv}} \beta \) for signed measures \( \alpha \) and \( \beta \) implies that \( \alpha(X) = \beta(X) \).

### 4.2 Statement of Mechanism Design Duality

A primary result of this thesis is that the mechanism design problem established in Theorem 1 has a dual problem, as follows:

**Theorem 2 (Strong Duality Theorem).** Let \( X = [0,M]^n \) where \( M < \infty \) and let \( \mu \in \text{Radon}(X) \) such that \( \mu(X) = 0 \). Then

\[
\sup_{u \in \mathcal{U}(X)} \int_X u d\mu = \inf_{\gamma \in \text{Radon}_+^+(X \times X)} \int_X \|x - y\|_1 d\gamma(x, y)
\]

\[
\sup_{\nu \in \mathcal{N}} \int_X \int_X \int_X u \gamma(x, y, z) d\gamma(x, y, z) = \inf_{\gamma \in \text{Radon}_+^+(X \times X \times X)} \int_X \int_X \|x - y\|_1 d\gamma(x, y, z)
\]
and both the supremum and infimum are achieved. Moreover, the infimum is achieved for some \( \gamma^* \) such that \( \gamma_1^*(X) = \gamma_2^*(X) = \mu_+(X) \), \( \gamma_1^* \succeq_{\text{ext}} \mu_+ \), and \( \gamma_2^* \succeq_{\text{ext}} \mu_- \).

The dual problem of minimizing \( \int \|x - y\|_1 d\gamma \) is an optimization problem that can be intuitively thought as a two step process:

- **Step 1:** Transform the original measure \( \mu \) into a new measure \( \mu' \) with \( \mu'(X) = 0 \) such that \( \mu' \succeq_{\text{ext}} \mu \). It suffices to pick a \( \mu' \) with the property \( |\mu'|(X) = |\mu|(X) \).

- **Step 2:** Find a joint measure \( \gamma \in \Gamma(\mu'_+, \mu'_-) \) such that \( \int \|x - y\|_1 d\gamma(x, y) \) is minimized. This is an optimal mass transportation problem where the cost of transporting a unit of mass from a point \( x \) to a point \( y \) is the \( \ell_1 \) distance \( \|x - y\|_1 \).\(^1\) Transportation problems of this form have been extensively studied in the mathematical literature. See [64].

By Theorem 1 we can rewrite a mechanism design problem as finding \( u \in \mathcal{U}(X) \cap \mathcal{L}_1(X) \) to maximize \( \int ud\mu \), and by Theorem 2 there always exists an appropriate \( \gamma \) for the dual problem which certifies optimality of the best \( u \). We prove Theorem 2 in Section 4.4. We remark that one direction of the duality theorem is easy. Proving the reverse direction in Section 4.4 is significantly more challenging, and relies on results such as the Fenchel-Rockafellar duality theorem.

**Lemma 2** (Weak Duality). Let \( X \) be a convex subset of \( \mathbb{R}^n_+ \) and let \( \mu \in \text{Radon}(X) \) with \( \mu(X) = 0 \). Then

\[
\sup_{u \in \mathcal{U}(X) \cap \mathcal{L}_1(X)} \int_X ud\mu \leq \inf_{\gamma \in \text{Radon}_+(X \times X)} \int_{X \times X} \|x - y\|_1 d\gamma.
\]

**Proof of Lemma 2:** For any feasible \( u \) for the left-hand side and feasible \( \gamma \) for the

\(^1\)We remark that the implicitly assumed equality

\[
\inf_{\gamma \in \text{Radon}_+(X \times X)} \int_X \|x - y\|_1 d\gamma = \inf_{\gamma' \in \text{Radon}_+(X \times X)} \int_X \|x - y\|_1 d\gamma'
\]

follows from Kantorovich-Rubinstein duality, as \( \|x - y\|_1 \) is a metric.
right-hand side, we have
\[ \int_X u \, d\mu \leq \int_X u \, d(\gamma_1 - \gamma_2) = \int_{X \times X} (u(x) - u(y)) \, d\gamma(x, y) \leq \int_{X \times X} \|x - y\|_1 \, d\gamma(x, y) \]
where the first inequality follows from \( \gamma_1 - \gamma_2 \succeq_{\text{cov}} \mu \) and the second inequality follows from the 1-Lipschitz condition on \( u \).

From the proof of Lemma 2, we note the following "complementary slackness" conditions that a tight pair of optimal solutions must satisfy.

**Corollary 1.** Let \( u^* \) and \( \gamma^* \) be feasible for their respective problems above. Then
\[ \int u^* \, d\mu = \int \|x - y\|_1 \, d\gamma^* \] if and only if both of these conditions hold:

1. \( \int u^* \, d(\gamma_1^* - \gamma_2^*) = \int u^* \, d\mu. \)
2. \( u^*(x) - u^*(y) = \|x - y\|_1, \gamma^*(x, y) \)-almost surely.

**Proof of Corollary 1:** The inequalities in the proof of Lemma 2 are tight precisely when both conditions hold.

**Remark 3.** It is useful to geometrically interpret Corollary 1. The first condition is intricate, but we provide very rough intuition. We view \( \gamma_1^* - \gamma_2^* \) (denote this by \( \mu' \)) as a "shuffled" \( \mu \). That is, (stemming from the \( \mu' \succeq_{\text{cov}} \mu \) constraint) we change \( \mu \) into \( \mu' \) by repeatedly (1) picking a positive point mass \( \delta_x \) from \( \mu_+ \), (2) splitting the point mass into several pieces, and (3) sending the pieces in multiple directions. We require that the center of mass never decreases in any iteration of this process.\(^2\) The constraint \( \int u^* \, d\mu' = \int u^* \, d\mu \) says that if the center of mass \( y \) of the split pieces is strictly larger than \( x \) in coordinate \( i \), then \( (\nabla u^*)_i = 0 \) at \( x \) and at \( y \).\(^3\) In addition, whenever a point mass \( \delta_x \) is split, we can send a piece to a location \( z \) only if \( u^* \) varies

\(^2\)This explanation follows from the interpretation of \( \mu' \succeq_{\text{cov}} \mu \) as saying that all risk-seekers "prefer" \( \mu' \) to \( \mu \). This explanation of "splitting" is an oversimplification: \( \mu \) can also "split" regions of mass (instead of just points of mass), can "merge" negative masses, and so forth.

\(^3\)Technically, we only need that \( 0 \) is the \( i \)-th coordinate of a subgradient in the case that \( u^* \) is not differentiable at \( x \) or at \( y \). Recall also that \( u^* \) is convex, and thus \( (\nabla u^*)_i = 0 \) also at points "in between" \( x \) and \( y \).
linearly along the path from $x$ to $z$.\footnote{This condition is a bit more complicated if $u^*$ is not differentiable at $x$, when we must ensure that $u^*$ varies linearly with the same slope between $x$ and all locations to which a piece is sent. That is, the locations to which pieces are sent share the same common subgradient with $x$.}

The second condition is more straightforward than the first. We view $\gamma^*$ as a "transport" map between its component measures $\gamma_1^*$ and $\gamma_2^*$. The condition states that if $\gamma^*$ transports from location $x$ to location $y$, then (since $\nabla u^* \in [0, 1]^n$) it must be the case that (1) $x$ is component-wise greater than or equal to $y$ and (2) if $x_i > y_i$ in coordinate $i$, then $\nabla u^*(x)_i = \nabla u^*(y)_i = 1$.\footnote{If $u^*$ is not differentiable at $x$ or at $y$, we require that $u^*$ has a subgradient with $i^{th}$ coordinate 1 at each of these points.} That is, the mechanism allocates item $i$ with probability 1 for bidders of type $x$ and for bidders of type $y$.

By Lemma 2 and Corollary 1, if we can find a "tight pair" of $u^*$ and $\gamma^*$, then they are optimal for their respective problems. Theorem 2 shows that this approach always works: for any optimal $u^*$ there always exists a $\gamma^*$ satisfying the conditions of Corollary 1.

\textit{Remark 4.} Our duality framework, by achieving strong duality, encompasses all prior duality-based frameworks for optimal mechanism design in our setting. In particular, if we tighten the $\gamma_1 - \gamma_2 \succeq_{\text{cov}} \mu$ constraint in the dual problem to a first-order stochastic dominance constraint (maintaining the weak duality property but creating a possible gap between optimal primal and dual values), we essentially recover the duality framework of [22], which used optimal transport to dualize a relaxed version of the mechanism design problem in which the convexity constraint on $u$ was dropped. See Appendix A.

\subsection*{4.3 A Simple Example}

We now give an example of using Theorem 2 to prove optimality of a particular mechanism for selling two uniformly distributed independent items. We note that while the items in this example are independent, their distributions are not identical, so in particular the characterization of [53] does not apply. In addition, this example is one in which the framework of [22] fails: if we were to relax the constraint that the
utility function $u$ be convex, the mechanism design program would have a solution obtaining greater revenue than is actually possible.

**Example 1.** The optimal IC and IR mechanism for selling two items whose values are distributed uniformly and independently on the intervals $(4,16)$ and $(4,7)$ is as follows:

- If the buyer’s declared type is in region $Z$, he receives no goods and pays nothing.
- If the buyer’s declared type is in region $Y$, he pays a price of 8 and receives the first good with probability 50% and the second good with probability 1.
- If the buyer’s declared type is in region $W$, he pays a price of 12 and receives both goods.

This example was constructed for ease of analysis. Drawing inspiration from the work of [53] on iid uniform items, the two-item characterization of [22], and our two-item characterization of Chapter 6, we expect for instances of two uniform items for types receiving zero utility ($Z$ in the diagram) to have a pentagonal shape. This example is a degenerate case in which only the top edge of the pentagon is non-trivial, resulting in the triangular shape of $Z$.

**Proof of Example 1:** It is straightforward to verify that the mechanism described above is IC and IR. All that remains is to prove that the utility function $u^*$ induced by the mechanism is optimal.

Applying Theorem 1, the transformed measure $\mu$ is composed of:

- A point mass of +1 at $(4,4)$.
- Mass $-3$ distributed throughout the rectangle (Density $-\frac{1}{12}$)

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• Mass $+\frac{3}{5}$ distributed on upper edge of rectangle (Linear density $+\frac{7}{36}$)

• Mass $-\frac{2}{3}$ distributed on lower edge of rectangle (Linear density $-\frac{1}{9}$)

• Mass $+\frac{2}{3}$ distributed on right edge of rectangle (Linear density $+\frac{2}{9}$)

• Mass $-\frac{1}{3}$ distributed on left edge of rectangle (Linear density $-\frac{1}{9}$)

We claim that $\mu(Z) = \mu(Y) = \mu(W) = 0$, which is straightforward to verify.

We will construct an optimal $\gamma^*$ for the dual program of Theorem 2, using the intuition of Remark 3. Our $\gamma^*$ will be decomposed into $\gamma^* = \gamma^Z + \gamma^Y + \gamma^W$ with $\gamma^Z \in \text{Radon}_+(Z \times Z)$, $\gamma^Y \in \text{Radon}_+(Y \times Y)$, and $\gamma^W \in \text{Radon}_+(W \times W)$. To ensure that $\gamma_1^Z - \gamma_2^Z \succeq_{\text{cux}} \mu$, we will show that

$$\gamma_1^Z - \gamma_2^Z \succeq_{\text{cux}} \mu|Z; \quad \gamma_1^Y - \gamma_2^Y \succeq_{\text{cux}} \mu|Y; \quad \gamma_1^W - \gamma_2^W \succeq_{\text{cux}} \mu|W.$$ 

We will also show that the conditions of Corollary 1 hold for each of the measures $\gamma^Z$, $\gamma^Y$, and $\gamma^W$ separately, namely $\int u^*d(\gamma_1^A - \gamma_2^A) = \int_A u^*d\mu$ and $u^*(x) - u^*(y) = \|x - y\|_1$ hold $\gamma^A$-almost surely for $A = Z, Y,$ and $W$.

• Construction of $\gamma^Z$. Since $\mu_+|Z$ is a point-mass at $(4,4)$ and $\mu_-|Z$ is distributed throughout a region which is coordinatewise greater than $(4,4)$, we notice that $\mu|Z \succeq_{\text{cux}} 0$. We therefore set $\gamma^Z$ to be the zero measure, and the relation $\gamma_1^Z - \gamma_2^Z = 0 \succeq_{\text{cux}} \mu|Z$, as well as the two necessary equalities from Corollary 1, are trivially satisfied.

• Construction of $\gamma^W$. We will construct $\gamma^W \in \Gamma(\mu_+|W, \mu_-|W)$ such that $x \geq y$ component-wise holds $\gamma^W(x, y)$ almost surely. Geometrically, we view this as "transporting" $\mu_+|W$ into $\mu_-|W$ by moving mass downwards and leftwards. Indeed, since both items are allocated with probability 1 in $W$, being able to transport both downwards and leftwards is in line with our interpretation of the second condition of Corollary 1, as explained in Remark 3.\footnote{To prove the existence of such a map, it is equivalent by Strassen's theorem to prove that $\mu_+|W$ stochastically dominates $\mu_-|W$ in the first order, but in this example we will directly define such a map.}
We notice that $\mu_+|_W$ consists of mass distributed on the top and right edges of $W$, while $\mu_-|_W$ consists of mass on the interior and bottom of $W$. We first match the $\mu_+$ mass on $[8, 16] \times \{7\}$ with the $\mu_-$ mass on $[8, 16] \times \left[\frac{14}{3}, 7\right]$ by moving mass downwards, then we match the $\mu_+$ mass on $\{16\} \times [4, \frac{14}{3}]$ with the $\mu_-$ mass on $[\frac{32}{3}, 16] \times (4, \frac{14}{3}]$ by moving mass to the left, and we finally match the $\mu_+$ mass on $\{16\} \times \left[\frac{14}{3}, 7\right]$ with the remaining negative mass arbitrarily. Noticing that $u^*(x) = \|x\|_1 - 12$ for all $x \in W$, it is straightforward to verify the desired properties from Corollary 1.

- Construction of $\gamma^Y$. This is the most involved step of the proof. Since item 2 is allocated with 100\% probability in region $Y$, by Remark 3 we would like to transport the positive mass $\mu_+|_Y$ into $\mu_-|_Y$ by moving mass straight downwards. However, this is impossible without first “shuffling” $\mu$, due to the negative mass on the left boundary of $Y$. Intuitively, we must first “shuffle” $\mu$ to push positive mass onto the point $(4, 7)$ (the top-left corner of $Y$), and only then do we transport the positive part of the shuffled measure into the negative part by sending mass downwards.

That is, instead of constructing $\gamma^Y$ with $\gamma_1^Y - \gamma_2^Y$ equal to $\mu|_Y$, we will have $\gamma_1^Y - \gamma_2^Y = \mu|_Y + \alpha$, where $\alpha = \alpha_+ - \alpha_-$ has density function

$$f_\alpha(z_1, z_2) = \delta(z_2 - 7) \cdot \left(\frac{1}{9}\delta(z_1 - 4) + \frac{1}{24}(z_1 - \frac{20}{3})\right) \cdot 1_{z_\in Y}.$$

The measure $\alpha$ is supported on the line $[4, 8] \times \{7\}$ and consists of a point mass of $\frac{1}{9}$ at $(4, 7)$ followed by allocating mass along the 1-dimensional upper boundary of $Y$ according to a density function which begins negative and increases linearly.

We chose $\alpha$ in this manner so that $\alpha(Y) = 0$, $\int_Y u^* d\alpha = 0$, and $\alpha \succeq 0.7$. (The

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\(^7\text{Since } \alpha \text{ is supported on a 1-dimensional line, this verification uses a property analogous to the standard characterization of one-dimensional second-order stochastic dominance via the cumulative density function. Informally, we can argue that } \alpha \succeq 0 \text{ by considering integrals of one-dimensional test functions (by restricting our attention to the line } z_2 = 7) \text{ and noticing that, since } \alpha(Y) = 0, \text{ we need only consider test functions } h \text{ which have value } 0 \text{ at } z_1 = 4. \text{ We then use the fact that all linear functions integrate to } 0 \text{ under } \alpha \text{ and that (ignoring the point mass at } z_1 = 4, \text{ since } h \text{ is } 0 \text{ at this point) the density of } \alpha \text{ is monotonically increasing.}\)
The key underlying reason for why such a shuffling works is that \( u^* \) is linear on the line \([4, 8] \times \{7\}\). We will define \( \gamma^Y \in \Gamma(\mu_+|_Y + \alpha_+, \mu_-|_Y + \alpha_-) \). We notice any such \( \gamma^Y \) will satisfy \( \gamma_1^Y - \gamma_2^Y \preceq_{cvx} \mu|_Y \), as desired. We claim that \( \gamma^Y \) can be constructed so that \( x_1 = y_1 \) and \( x_2 \geq y_2 \) hold \( \gamma^Y(x, y) \) almost surely. This verification follows, and simply checks that \( \mu_+|_Y + \alpha_+ \) and \( \mu_-|_Y + \alpha_- \) assign the same density to any vertical “strip” in \( Y \).

Since \( \mu_+|_Y + \alpha_+ \) only assigns mass to the upper boundary of \( Y \), to show that \( \gamma^Y \) can be constructed so that all mass is transported “vertically downwards” we need only verify that \( \mu_+|_Y + \alpha_+ \) and \( \mu_-|_Y + \alpha_- \) assign the same density to any vertical “strip” in \( Y \). Indeed,

\[
(\mu_-|_Y + \alpha_-)(\{4\} \times [6, 7]) = \mu_-|_Y(\{4\} \times [6, 7]) = \frac{1}{9} = \alpha_+(\{4\} \times [6, 7]) = (\mu_+|_Y + \alpha_+)(\{4\} \times [6, 7])
\]

and, for all \( z_1 \pm \epsilon \in (4, 8] \), we compute the following, using the fact that the surface area of \( Y \cap ([z_1 - \epsilon, z_1 + \epsilon] \times [4, 7]) \) is \( 2 \epsilon \left( \frac{z_1}{2} - 1 \right) \):

\[
(\mu_-|_Y - \alpha|_Y)([z_1 - \epsilon, z_1 + \epsilon] \times [4, 7])
= \frac{1}{12} \cdot \left( 2 \epsilon \cdot \left( \frac{z_1}{2} - 1 \right) \right) - \frac{1}{24} \int_{z_1 - \epsilon}^{z_1 + \epsilon} (z - \frac{20}{3}) dz
= \frac{\epsilon z_1}{12} - \frac{\epsilon}{6} \left( 2 \epsilon z_1 - \frac{40 \epsilon}{3} \right)
= \frac{7 \epsilon}{18} = \mu_+|_Y([z_1 - \epsilon, z_1 + \epsilon] \times [4, 7]).
\]

Since \( u^* \) has the property that \( u^*(z_1, a) - u^*(z_1, b) = a - b \) for all \((z_1, a), (z_1, b) \in Y \) (as the second good is received with probability 1), it follows that \( \gamma^Y \) satisfies the necessary conditions of Corollary 1.
4.4 Proof of Strong Mechanism Design Duality

In this section we prove Theorem 2, our strong duality theorem for optimal mechanism design.

4.4.1 A Strong Duality Lemma

The overall structure of our proof of Theorem 2 is roughly parallel to the proof of Monge-Kantorovich duality presented in [64], although some technical aspects are different, mainly due to the added convexity constraint on $u$. We begin by stating the Legendre-Fenchel transformation and the Fenchel-Rockafellar duality theorem.

**Definition 8 (Legendre-Fenchel Transform).** Let $E$ be a normed vector space and let $\Lambda : E \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The Legendre-Fenchel transform of $\Lambda$, denoted $\Lambda^*$, is a map from the topological dual $E^*$ of $E$ to $\mathbb{R} \cup \{+\infty\}$ given by

$$\Lambda^*(z^*) = \sup_{z \in E} \langle z^*, z \rangle - \Lambda(z).$$

**Claim 4 (Fenchel-Rockafellar duality).** Let $E$ be a normed vector space, $E^*$ its topological dual, and $\Theta, \Xi$ two convex functions on $E$ taking values in $\mathbb{R} \cup \{+\infty\}$. Let $\Theta^*, \Xi^*$ be the Legendre-Fenchel transforms of $\Theta$ and $\Xi$ respectively. Assume that there exists $z_0 \in E$ such that $\Theta(z_0) < +\infty$, $\Xi(z_0) < +\infty$ and $\Theta$ is continuous at $z_0$. Then

$$\inf_{z \in E} [\Theta(z) + \Xi(z)] = \max_{z^* \in E^*} [-\Theta^*(-z^*) - \Xi^*(z^*)].$$

**Lemma 3.** Let $X$ be a compact convex subset of $\mathbb{R}^n$, and let $\mu \in \text{Radon}(X)$ be such that $\mu(X) = 0$. Then

$$\inf_{\gamma \in \text{Radon}_+(X \times X)} \int_{X \times X} \|x - y\|_1 d\gamma(x, y) = \sup_{\phi, \psi \in \mathbb{U}(X)} \left( \int_X \phi d\mu_+ - \int_X \psi d\mu_- \right)$$

and the infimum on the left-hand side is achieved.

**Proof of Lemma 3:** We will apply Fenchel-Rockafellar duality with $E = CB(X \times X), \ldots$
the space of continuous (and bounded) functions on $X \times X$ equipped with the $\| \cdot \|_\infty$ norm. Since $X$ is compact, by the Riesz representation theorem $E^* = \text{Radon}(X \times X)$.

We now define functions $\Theta, \Xi$ mapping $CB(X \times X)$ to $\mathbb{R} \cup \{+\infty\}$ by

$$\Theta(f) = \begin{cases} 0 & \text{if } f(x, y) \geq -\|x - y\|_1 \text{ for all } x, y \in X \\ +\infty & \text{otherwise} \end{cases}$$

$$\Xi(f) = \begin{cases} \int_X \psi d\mu_+ - \int_X \phi d\mu_+ & \text{if } f(x, y) = \psi(y) - \phi(x) \text{ for some } \psi, \phi \in \mathcal{U}(X) \\ +\infty & \text{otherwise} \end{cases}$$

We note that $\Xi$ is well-defined: If $\psi(x) - \phi(y) = \psi'(x) - \phi'(y)$ for all $x, y \in X$, then $\psi(x) - \psi'(x) = \phi(y) - \phi'(y)$ for all $x, y \in X$. This means that $\psi'$ differs from $\psi$ only by an additive constant, and $\phi$ differs from $\phi'$ by the same additive constant, and therefore (since $\mu_+$ and $\mu_-$ have the same total mass) $\int_X \psi d\mu_- - \int_X \phi d\mu_+ = \int_X \psi' d\mu_- - \int_X \phi' d\mu_+$.

It is clear that $\Theta(f)$ is convex, since any convex combination two functions for which $f(x, y) \geq -\|x - y\|_1$ will yield another function for which the inequality is satisfied. It is furthermore clear that $\Xi$ is convex, since we can take convex combinations of the $\psi$ and $\phi$ functions as appropriate. (Notice that $\mathcal{U}(X)$ is closed under addition and positive scaling of functions.)

Consider the function $z_0 \in CB(X \times X)$ which takes the constant value of 1. It is clear that $\Theta(z_0) = 0$ and $\Xi(z_0) = \mu_-(X) < \infty$. Furthermore, $\Theta(z) = 0$ for any $z \in CB(X \times X)$ with $\|z - z_0\|_\infty < 1$, and therefore $\Theta$ is continuous at $z_0$. We can thus apply the Fenchel-Fockafellar duality theorem.

---

8This holds because any such $z$ must be strictly positive.
We compute, for any $\gamma \in \text{Radon}(X \times X)$:

$$
\Theta^*(-\gamma) = \sup_{f \in \mathcal{CB}(X \times X)} \left[ \int_{X \times X} f(x, y)d(-\gamma(x, y)) \right]
- \begin{cases} 
0 & \text{if } f(x, y) \geq -\|x - y\|_1 \forall x, y \in X \\
+\infty & \text{otherwise}
\end{cases}
\sup_{f \in \mathcal{CB}(X \times X)} \left( -\int_{X \times X} f(x, y)d\gamma(x, y) \right) = 
\sup_{f \in \mathcal{CB}(X \times X)} \left( \int_{X \times X} \tilde{f}(x, y)d\gamma(x, y) \right) .
$$

We claim therefore that

$$
\Theta^*(-\gamma) = \begin{cases} 
\int_{X \times X} \|x - y\|_1d\gamma(x, y) & \text{if } \gamma \in \text{Radon}_+(X \times X) \\
\infty & \text{otherwise} 
\end{cases}
$$

Indeed, if $\gamma$ is a positive linear functional, then the result follows from monotonicity, since $\|x - y\|_1$ is the pointwise greatest function $\tilde{f}$ satisfying the constraint $\tilde{f}(x, y) \leq \|x - y\|_1$, and $\|x - y\|_1$ is continuous. Suppose instead that $\gamma$ is a signed Radon measure which is not everywhere positive. Then there exists a continuous nonnegative function $g : X \times X \to \mathbb{R}$ such that $\int g d\gamma = -\epsilon$ for some $\epsilon > 0$. Since $g(x, y) \geq 0$, it follows that $-kg(x, y) \leq 0 \leq \|x - y\|_1$ for any $k \geq 0$. Therefore

$$
\sup_{f \in \mathcal{CB}(X \times X)} \left( \int_{X \times X} \tilde{f}(x, y)d\gamma(x, y) \right) \geq \int -kg(x, y)d\gamma(x, y) = k\epsilon .
$$

The claim follows, since $k > 0$ is arbitrary.

---

Formally, we have used Lusin’s theorem to find such a $g$ which is continuous, as opposed to merely measurable.
We similarly compute, for any $\gamma \in \text{Radon}(X \times X)$:

$$\Xi^*(\gamma) = \sup_{f \in C_B(X \times X)} \left[ \int_{X \times X} f(x, y) d\gamma(x, y) - \right.$$

$$- \left\{ \begin{array}{ll}
\int_X \psi d\mu_- - \int_X \phi d\mu_+ & \text{if } f(x, y) = \psi(y) - \phi(x) \text{ and } \psi, \phi \in \mathcal{U}(X)
+ \infty & \text{otherwise}
\end{array} \right]$$

$$= \sup_{\psi, \phi \in \mathcal{U}(X)} \left[ \int_{X \times X} (\psi(y) - \phi(x)) d\gamma(x, y) - \int_X \psi d\mu_- + \int_X \phi d\mu_+ \right]$$

We notice that $\Xi^*(\gamma) \geq 0$ for all $\gamma \in \text{Radon}(X \times X)$ by setting $\psi = \phi = 0$. In particular, we now compute for $\gamma \in \text{Radon}^+(X \times X)$:

$$\Xi^*(\gamma) = \sup_{\psi, \phi \in \mathcal{U}(X)} \left[ \int_{X \times X} (\psi(y) - \phi(x)) d\gamma(x, y) - \int_X \psi d\mu_- + \int_X \phi d\mu_+ \right]$$

$$= \sup_{\psi, \phi \in \mathcal{U}(X)} \left[ \int_X \psi d(\gamma_2 - \mu_-) + \int_X \phi d(\mu_+ - \gamma_1) \right]$$

$$= \left\{ \begin{array}{ll}
0 & \text{if } \gamma_1 \succeq_{\text{cvx}} \mu_+ \text{ and } \gamma_2 \succeq_{\text{cvx}} \mu_-
\infty & \text{otherwise}.
\end{array} \right.$$

We now apply Fenchel-Rockafellar duality:

$$\inf_{f \in C_B(X \times X)} [\Theta(f) + \Xi(f)] = \max_{\gamma \in \text{Radon}(X \times X)} [-\Theta^*(-\gamma) - \Xi^*(\gamma)]$$

$$\inf_{f(x, y) \geq -\|x-y\|_1, f(x, y) = \psi(y) - \phi(x)} \left( \int_X \psi d\mu_- - \int_X \phi d\mu_+ \right) = \max_{\gamma \in \text{Radon}^+(X \times X)} \left( -\int_{X \times X} \|x-y\|_1 d\gamma(x, y) - \Xi^*(\gamma) \right)$$

$$\inf_{\phi(x) - \psi(y) \leq \|x-y\|_1} \left( \int_X \psi d\mu_- - \int_X \phi d\mu_+ \right) = \max_{\gamma \in \text{Radon}^+(X \times X), \gamma_1 \succeq_{\text{cvx}} \mu_+, \gamma_2 \succeq_{\text{cvx}} \mu_-} \left( -\int_{X \times X} \|x-y\|_1 d\gamma(x, y) \right)$$

$$\sup_{\phi(x) - \psi(y) \leq \|x-y\|_1} \left( \int_X \phi d\mu_+ - \int_X \psi d\mu_- \right) = \min_{\gamma \in \text{Radon}^+(X \times X), \gamma_1 \succeq_{\text{cvx}} \mu_+, \gamma_2 \succeq_{\text{cvx}} \mu_-} \left( \int_{X \times X} \|x-y\|_1 d\gamma(x, y) \right).$$

$\square$
4.4.2 From Two Convex Functions to One

Lemma 4. Let \( X = [0, M]^n \) for some \( M \in \mathbb{R}_{\geq 0} \), and let \( \mu \in \text{Radon}(X) \) such that \( \mu(X) = 0 \). Then

\[
\sup_{\phi, \psi \in \mathcal{U}(X), \phi(x) - \psi(y) \leq \|x - y\|_1} \left(\int_X \phi d\mu_+ - \int_X \psi d\mu_-\right) = \sup_{u \in \mathcal{U}(X) \cap \mathcal{L}_1(X)} \left(\int_X ud\mu_+ - \int_X ud\mu_-\right).
\]

Furthermore, if the supremum of one side is achieved, then so is the supremum of the other side.

Proof of Lemma 4: Given any feasible \( u \) for the right-hand side of Lemma 4, we observe that \( \phi = \psi = u \) is feasible for the left-hand side, and therefore the left-hand side is at least as large as the right-hand side. It therefore suffices to prove the reverse direction of the inequality. Let \( \phi \) and \( \psi \) be feasible for the left-hand side. Given \( \phi \), it is clear that \( \psi \) must satisfy \( \psi(y) \geq \sup_x [\phi(x) - \|x - y\|_1] \).

Set \( \psi(y) = \sup_x [\phi(x) - \|x - y\|_1] \). Since \( \psi \) exists, this supremum indeed has finite value. Since \( \psi \leq \psi \) pointwise, it follows that \( \int_X \psi d\mu_- \leq \int_X \psi d\mu_- \). We must now prove that \( \psi \in \mathcal{U}(X) \), thereby showing that \( \phi, \psi \) is feasible for the left-hand side and that replacing \( \psi \) by \( \psi \) does not decrease the objective value.

Claim 5. \( \bar{\psi} \in \mathcal{U}(X) \)

Proof of Claim 5: We need to show continuity, monotonicity, and convexity.

- **Continuity.** Continuity of \( \bar{\psi} \) follows from uniform continuity of \( \phi \) and of \( \|\cdot\|_1 \).

- **Monotonicity.** Let \( y \leq y' \) coordinate-wise and let \( x \) be arbitrary. We must show that there exists an \( x' \) such that \( \phi(x) - \|x - y\|_1 \leq \phi(x') - \|x' - y'\|_1 \). Set \( x'_i = \max\{x_i, y'_i\} \). Since \( x \leq x' \), we have \( \phi(x) \leq \phi(x') \). We notice that if \( x_i \geq y'_i \) then \( x'_i = x_i \) and thus \( |x'_i - y'_i| \leq |x_i - y_i| \), while if \( x_i \leq y'_i \) then \( |x'_i - y'_i| = 0 \). Therefore, we have that \( \|x - y\|_1 \leq \|x' - y'\|_1 \) and thus \( \phi(x) - \|x - y\|_1 \leq \phi(x') - \|x' - y'\|_1 \), as desired.
- **Convexity.** Let \( y, y', y'' \) be collinear points in \( X \) such that \( y = \frac{y' + y''}{2} \). Then, given any \( x \), we must show that there exist \( x' \) and \( x'' \) such that

\[
\phi(x') - \|x' - y'\|_1 + \phi(x'') - \|x'' - y''\|_1 \geq 2\phi(x) - 2\|x - y\|_1.
\]

We define \( x'_i \) and \( x''_i \) as follows:

- If \( y'_i \geq y''_i \), set \( x'_i = \max\{x_i, y'_i\} \) and \( x''_i = \max\{2x_i - x'_i, y''_i\} \).
- If \( y'_i < y''_i \), set \( x''_i = \max\{x_i, y'_i\} \) and \( x'_i = \max\{2x_i - x''_i, y'_i\} \).

Notice that \( x' + x'' \geq 2x \), and thus (since \( \phi \) is convex and monotone) we have \( \phi(x') + \phi(x'') \geq 2\phi(x) \).

Suppose without loss of generality that \( y'_i \geq y''_i \). We now consider two cases:

- \( y'_i \geq x_i \). We then have \( x'_i = y'_i \) and \( x''_i = \max\{2x_i - x'_i, y''_i\} \). Therefore, \( |y'_i - x'_i| = 0 \) and \( |y''_i - x''_i| \leq |y'_i - 2x_i + y'_i| = 2|y_i - x_i| \) since \( y'_i + y''_i = 2y_i \).
- \( y'_i < x_i \). We now have \( x'_i = x_i \) and \( x''_i = \max\{x_i, y''_i\} = x_i \). Therefore \( |y''_i - x''_i| + |y'_i - x'_i| \) is equal to \( |y'_i + y''_i - 2x_i| \), which equals \( |2y_i - 2x_i| \).

Therefore, we have that \( |y'_i - x'_i| + |y''_i - x''_i| \leq |2y_i - 2x_i| \) for all \( i \), which implies that \( \|x' - y'\|_1 + \|x'' - y''\|_1 \leq 2\|x - y\|_1 \).

We have thus shown that \( \bar{\psi} \in \mathcal{U}(X) \). \( \square \)

**Claim 6.** \( \bar{\psi} \in \mathcal{L}_1(X) \).

**Proof of Claim 6:** We have

\[
\bar{\psi}(x) - \bar{\psi}(y) = \sup_{z,w} \inf_{z,w} (\phi(z) - \|z - x\|_1 - \phi(w) + \|w - y\|_1) \\
\leq \sup_{z} (\phi(z) - \|z - x\|_1 - \phi(z) + \|z - y\|_1) = \sup_{z} (\|z - y\|_1 - \|z - x\|_1) \leq \|x - y\|_1
\]

\( \square \)

Since \( \phi, \bar{\psi} \) are a feasible pair of functions for the left-hand side of Lemma 4, we know that \( \phi \) satisfies the inequality \( \phi(x) \leq \inf_y [\bar{\psi}(y) + \|x - y\|_1] \). We now set
\( \bar{\phi}(x) = \inf_y [\bar{\psi}(y) + \|x - y\|_1] \). It is clear that the value of the left-hand objective function under \( \bar{\phi}, \bar{\psi} \) is at least as large as its value under \( \phi, \psi \).

We claim that not only is \( \bar{\phi} \) continuous, monotonic, and convex, but in fact that \( \bar{\phi} = \bar{\psi} \). We notice that \( \bar{\phi}(x) \leq \bar{\psi}(x) + \|x - x\|_1 = \bar{\psi}(x) \). To prove the other direction of the inequality, we compute

\[
\bar{\phi}(x) = \inf_y \left[ \bar{\psi}(y) + \|x - y\|_1 \right] = \bar{\psi}(x) + \inf_y \left[ \bar{\psi}(y) - \bar{\psi}(x) + \|x - y\|_1 \right] \geq \bar{\psi}(x)
\]

where the last inequality holds since \( \bar{\psi}(x) - \bar{\psi}(y) \leq \|x - y\|_1 \). Therefore \( \bar{\phi} = \bar{\psi} \), and thus \( \bar{\phi} \in U(X) \). Since \( \bar{\phi} \) satisfies the inequality \( \bar{\phi}(x) - \bar{\phi}(y) \leq \|x - y\|_1 \) it is feasible for the right-hand side of Lemma 4, and the value of the right-hand objective under \( \bar{\phi} \) is at least as large the value of the left-hand objective under \( \phi, \psi \). We notice finally that if \( \phi, \psi \) are optimal for the left-hand side, then \( \bar{\phi} \) is optimal for the right-hand side.

\[ \square \]

### 4.4.3 Completing the Proof

By combining Lemma 2, Lemma 3, and Lemma 4, we have

\[
\inf_{\gamma \in \text{Radon}_+(X \times X)} \int_{X \times X} \|x - y\|_1 d\gamma \geq \sup_{u \in U(X) \cap C_1(X)} \int_X u \, d\mu
\]

\[
= \sup_{\phi, \psi \in U(X)} \left( \int_X \phi \, d\mu_+ - \int_X \psi \, d\mu_- \right) = \inf_{\gamma \in \text{Radon}_+(X \times X)} \int_{X \times X} \|x - y\|_1 d\gamma(x, y).
\]

By Lemma 3, the last minimization problem above achieves its infimum for some \( \gamma^* \). We notice that \( \gamma^* \) is also feasible for the first minimization problem above, and therefore the inequality is actually an equality and \( \gamma^* \) is optimal for the first minimization problem. In addition, since \( \gamma^* \) is feasible for the last minimization problem, it satisfies \( \gamma^*_*(X) = \gamma^*_*(X) = \mu_+(X) \).

All that remains is to prove that the supremum to the maximization problem is achieved for some \( u^* \). Consider a sequence of feasible functions \( u_1, u_2, \ldots \in U(X) \cap L_1(X) \) such that \( \int_X u_i \, d\mu \) converges monotonically to the supremum value \( V \), which
we have proven is finite.\footnote{Finiteness is also obvious because $X$ is bounded and the infimum problem is feasible.} Since $\mu(X) = 0$, we may without loss of generality assume that $u_i(0^n) = 0$ for all $u_i$.

Pick an ordering $x_1, x_2, \ldots$ of the rational points in $X$. We note that $0 \leq u_j(x_i) \leq \|x_i\|_1$ for all points $x_i$, by the 1-Lipschitz condition. We now define a function $u^*$ on the points $x_i$ as follows:

- Set $u^*(0^n) = 0$ and initialize $T$ to be the sequence $1, 2, 3, \ldots$.

- For $i = 1, 2, 3, \ldots$
  
  - Since $0 \leq u_j(x_i) \leq \|x_i\|_1$, there exists a subsequence $T'$ of $T$ such that $(u_j(x_i))_{j \in T}$ converges.
  
  - Set $u^*(x_i) = \lim_{j \in T'} u_j(x_i)$ and replace $T$ with $T'$.

It is straightforward to prove that the resulting function $u^*$ is uniformly continuous on the points with rational coordinates. Therefore, we can uniquely extend $u^*$ to a continuous function on all of $X$. It is furthermore simple to show that this function $u^*$ indeed is increasing, convex, and 1-Lipschitz with respect to the $\ell_1$ norm.

Finally, we argue that $\int_X u^* d\mu$ achieves the supremum value $V$. Let $\epsilon > 0$ be arbitrary. Since $X$ is compact, there exists a finite collection of rational points $y_1, \ldots, y_k$ such that every point in $X$ is within $\epsilon$ (in $\ell_1$ distance) of some $y_i$. Denote by $T^*$ the subsequence above immediately after $u^*(y_1), \ldots, y^*(y_k)$ have been defined.

By assumption, there exists an $j^*$ such that for all $j > j^*$ it holds that $\int_X u_j d\mu$ is within an additive $\epsilon \cdot \mu_+(X)$ of the supremum value. Furthermore, since $y_1, \ldots, y_k$ is a finite collection of points, there exists $\tilde{j} > j^*$ such that $|u_j(y_i) - u^*(y_i)| < \epsilon$ for all $i = 1, \ldots, k$ and all $j > \tilde{j}$ in $T^*$.

For each $x \in X$, denote by $y_x \in X$ a $y$ point which is closest to $x$ in $\ell_1$ distance. Since $\|x - y_x\|_1 < \epsilon$, we have $|u(x) - u(y_x)| < \epsilon$ for any feasible $u$. Therefore, we have

$$|u^*(x) - u_j(x)| \leq |u^*(x) - u^*(y_x)| + |u^*(y_x) - u_j(y_x)| + |u_j(y_x) - u_j(x)| < 3\epsilon.$$
Thus,

\[ \int_X u^* d\mu \geq \int_X (u_j - 3\epsilon) d\mu_+ - \int_X (u_j + 3\epsilon) d\mu_- \]

\[ = \int_X u_j d\mu - 6\epsilon \mu_+(X) \geq V - \epsilon \mu_+(X) - 6\epsilon \mu_+(X). \]

Since \( \epsilon \) was arbitrary and \( \mu_+(X) \) is finite, we conclude that \( \int u^* d\mu = V. \)
Chapter 5

A Complete Characterization of Grand Bundling Optimality

5.1 Overview

Identifying conditions under which the optimal mechanism is a simple take-it-or-leave-it offer of the "grand bundle" of all goods has been an important question in the economic literature (see [39]). While prior work has identified sufficient conditions for grand bundling optimality, here we do better: Using the framework of Theorem 2, we obtain conditions which are both necessary and sufficient for grand bundling optimality. In particular, we show that grand bundling is optimal if and only if two stochastic dominance relations hold between certain restrictions of the measure $\mu$.

Theorem 2 allows us to certify that a mechanism is optimal by providing a tight $\gamma$ for the dual problem. Thus, the utility function $u$ resulting from a grand bundling mechanism is optimal if and only if a $\gamma$ exists which satisfies the "complementary slackness" conditions of Corollary 1. Instead of directly providing a tight $\gamma$, we can obtain necessary and sufficient conditions on $\mu$ for non-constructively proving either that such a $\gamma$ exists or that no such $\gamma$ is possible. Such a result is powerful, as verifying properties of $\mu$ is typically a much easier task than solving an optimization problem over all $\gamma \in \text{Radon}_+(X \times X)$.

We stress the importance of this result in that we do not merely derive a sufficient
condition to certify optimality, as in [39] and [22], but rather we prove that grand bundling optimality is equivalent to two measure-theoretic inequalities. The proof of this result is intricate and requires several technical lemmas, which are presented in the appendix. The most important such lemma for our purposes is Lemma 10, and we expect that this lemma can be used to obtain analogous measure-theoretic conditions for optimality in many other classes of natural mechanisms.

5.2 Statement of Grand Bundling Theorem

Before we state Theorem 3, we define two additional types of stochastic domination.

**Definition 9.** Let $X$ be a convex subset of $\mathbb{R}^n_{\geq 0}$ and $\alpha, \beta \in \text{Radon}(X)$. We say that $\alpha$ dominates $\beta$ in the first order, denoted $\alpha \succeq_1 \beta$, if for all non-decreasing bounded $u : \mathbb{R}^n \to \mathbb{R}$:

$$\int u \alpha \geq \int u \beta.$$

Similarly, for vector random variables $A$ and $B$ with values in $X$, we say that $A \succeq_1 B$ if $E[u(A)] \geq E[u(B)]$ for all non-decreasing bounded functions $u : \mathbb{R}^n \to \mathbb{R}$.

When $X$ is bounded, we say that $\alpha$ dominates $\beta$ in the second order, denoted $\alpha \succeq_2 \beta$, if the above inequality holds for all non-decreasing, concave functions $u : \mathbb{R}^n \to \mathbb{R}$. We define $A \succeq_2 B$ analogously.

The definition for second-order dominance is very similar to that of convex dominance presented earlier, except that we test over concave functions instead of convex functions.

The heart of this section is Theorem 3. Before stating the theorem, we first define two additional properties that a measure $\mu$ can satisfy.

**Definition 10 (Avoidance of $p$).** Let $\mu$ be a Radon measure on $\mathbb{R}^n$ and let $p > 0$. We say that $\mu$ avoids $p$ if $\mu(\{x : \|x\|_1 = p\}) = 0$.

The property that $\mu$ avoids $p$ is incredibly mild in our applications, as discussed in Remark 5. It is included mainly for simplicity of notation and analysis, and we expect that this condition can be removed with additional work.
Definition 11 (Grand Bundling Conditions). Let $X = [0, M]^n$, let $\mu \in \text{Radon}(X)$ with $\mu(X) = 0$, and let $p \in (0, M]$ such that $\mu$ avoids $p$. We say that $\mu$ satisfies the grand bundling conditions with respect to $p$ if

$$\mu\{x \in X : \|x\|_1 \leq p\} \leq \text{const} \cdot 0 \quad \text{and} \quad \mu\{x \in X : \|x\|_1 \geq p\} \geq 2 \cdot 0.$$ 

Theorem 3 (Optimality of Grand Bundling). Let $U \subset \mathbb{R}_{\geq 0}^n$ be a well-behaved type space, $f : U \to \mathbb{R}$ be a probability density function with bounded partial derivatives, and $\mu$ be the transformed measure of $f$. Let $p > 0$ such that $\mu$ avoids $p$. Then the optimal IC and IR mechanism for a single additive buyer whose values for $n$ goods are distributed according to the joint distribution $f$ is a take-it-or-leave-it offer of the grand bundle at price $p$ if and only if $\mu$ satisfies the grand bundling conditions with respect to $p$.

Remark 5. The condition that $\mu$ avoids $p$ is very mild, and is essentially always satisfied in our mechanism design applications. Given a density function $f$ as in Theorem 1, the transformed measure $\mu$ only has "surface density" on the boundary $\partial U$ of $U$ and at a particular point $z_0$. The avoidance condition will be satisfied unless $\|z_0\|_1 = p$ (which is a degenerate case of grand bundling, and simple analysis yields a condition similar to Theorem 3) or if $\partial U$ intersects $\{x : \|x\|_1 = p\}$ on a region with non-zero $(n - 1)$-dimensional surface measure. This only occurs for highly contrived correlated distributions $f$. We expect that the avoidance condition can be removed entirely with additional work, although such an extension would only be necessary for analyzing very special correlated distributions.

The entire proof of Theorem 3 is contained Lemma 5, a full proof of which is presented in Section 5.3.

Lemma 5. Let $X = [0, M]^n$, let $\mu \in \text{Radon}(X)$ with $\mu(X) = 0$, and let $p \in (0, M]$. Define the function $u_p : X \to R_{\geq 0}$ by $u_p(x) = \max\{\|x\|_1 - p, 0\}$ and define the regions $Z, P, W \subset X$ by

$$Z = \{x \in X : \|x\|_1 \leq p\}; \quad P = \{x \in X : \|x\|_1 = p\}; \quad W = \{x \in X : \|x\|_1 \geq p\}.$$
Suppose that $\mu(P) = 0$. Then the following conditions are equivalent:

1. There exists $\gamma \in \text{Radon}_+(X \times X)$ such that:
   - $\gamma_1 - \gamma_2 \geq_{\text{cvx}} \mu$
   - $\int u_p d\gamma_1 - \int u_p d\gamma_2 = \int u_p d\mu$, and
   - $u_p(x) - u_p(y) = \|x - y\|_1, \gamma(x, y)$-almost surely.

2. $\mu_z \geq_{\text{cvx}} \mu_+, |z|$ and $\mu_+|w \geq 2 \mu_-|w$.

In particular, if either of the two above conditions are satisfied, then $u_p$ maximizes $\sup_{u \in U(X) \cap C^1(X)} \int_X ud\mu$.

The function $u_p$ in Lemma 5 is the utility function corresponding to the grand bundling mechanism with price $p$, and the conditions on the measure $\gamma \in \text{Radon}_+(X \times X)$ are precisely the conditions necessary to certify optimality by Corollary 1. Furthermore, by the mechanism design duality theorem, we know that grand bundling is optimal if and only if the such a measure $\gamma$ exists.

One direction of Lemma 5 is easier to prove than the other: given $\mu$ satisfying the stochastic domination properties of condition 2, it is not very difficult (albeit still technical) to construct an appropriate $\gamma$. The construction uses the stochastic dominance relations on $Z$ and $W$ to define $\gamma$ separately on $Z \times Z$ and $W \times W$. The difficulty is in proving that condition 2 is necessary: given an optimal $\gamma$, it is not obvious that the stochastic dominance relation $\gamma_1 - \gamma_2 \geq_{\text{cvx}} \mu$ holds when we restrict $\gamma_1, \gamma_2$, and $\mu$ to either of the regions $Z$ or $W$, and it is conceivable that $\gamma$ transports mass from one region to the other. We must prove that, whenever a certifying $\gamma$ exists, it can be appropriately decomposed. The proof requires several technical lemmas, and is presented in Section 5.3.

5.3 Proof of Grand Bundling Theorem

Our goal in this section is to prove Lemma 5 which immediately implies Theorem 3, our complete characterization of grand bundling optimality in terms of stochastic
5.3.1 Lemmas on Stochastic Dominance

Before we prove Lemma 5, we present several supplementary lemmas. The most important of these is Lemma 10 in Section 5.3.2, which we expect can be a key tool for adapting the characterization and proof of Theorem 3 to other classes of mechanisms. Before stating this lemma, we first show two simple results.

**Lemma 6.** Let $A$ and $B$ be vector random variables with values in $[0, M]^n$, and suppose that $E[\|A\|_1] = E[\|B\|_1]$. Then $A \succeq_{cvx} B$ if and only if $B \succeq_2 A$.

**Proof of Lemma 6:** Suppose that $A \succeq_{cvx} B$. Let $h : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary non-decreasing concave function, and let

$$m = \sup_{x \neq y \in [0, M]^n} \frac{h(x) - h(y)}{\|x - y\|_1}.$$

Since $h$ is defined on an open subset containing $[0, M]^n$, we know that $h$ is Lipschitz continuous on $[0, M]^n$, and therefore $m$ is finite. Thus, the function

$$g(x) = -h(x) + m\|x\|_1$$

is convex, continuous, and non-decreasing. Therefore

$$E[h(A) - h(B)] = E[m\|A\|_1 - m\|B\|_1] - E[g(A) - g(B)] \leq 0$$

and thus $A \succeq_2 B$. The proof of the other direction of the lemma is analogous. □

**Lemma 7.** Let $X$ be a convex bounded subset of $\mathbb{R}^n$ and let $\alpha, \beta \in \text{Radon}_+(X)$ such that $\alpha \succeq_2 \beta$. Then there exists $\theta \in \text{Radon}_+(X)$ such that $\alpha \succeq_1 \theta$, $\theta \succeq_2 \beta$, and $\int \|x\|_1 d\theta = \int \|x\|_1 d\beta$.

**Proof of Lemma 7:** This proof follows from Theorem 7.A.3 and Theorem 4.A.6 of [58] applied to second order dominance, which state that for vector-valued random
variables \( A \preceq_2 B \) there exists a vector valued random variable \( T \) such that \( A \preceq_1 T \preceq_{cv} B \), where \( T \preceq_{cv} B \) means that \( \mathbb{E}[g(T)] \leq \mathbb{E}[g(B)] \) for all concave (but not necessarily monotonic) functions \( g : \mathbb{R}^n \to \mathbb{R} \).

In particular, \( T \preceq_{cv} B \) implies both that \( T \preceq_2 B \) and that \( \mathbb{E}[\|T\|_1] = \mathbb{E}[\|B\|_1] \). We notice that, since the proof of this lemma defines \( T \) by taking conditional expectations of \( A \) and \( B \), the values of \( T \) always lie within the convex set \( X \). The result follows by taking an analogous statement for positive Radon measures. \( \square \)

5.3.2 Probabilistic Lemmas

The goal of this section is to prove Lemma 10, which will be very useful in our proof of Lemma 5. We first present a useful result about convex dominance of random variables. For more information about this result, see Theorem 7.A.2 of [58].

**Lemma 8.** Let \( X \) and \( Y \) be random vectors. Then \( X \preceq_{cv} Y \) if and only if there exist random vectors \( \tilde{X} \) and \( \tilde{Y} \), defined on the same probability space, such that \( \tilde{X} =_{st} X \), \( \tilde{Y} =_{st} Y \), and \( \mathbb{E}[\tilde{Y}|\tilde{X}] \geq \tilde{X} \) almost surely, where the final inequality is componentwise and where \( =_{st} \) denotes equality in distribution.

Similarly, \( X \preceq_2 Y \) if and only if there exist random vectors \( \tilde{X} \) and \( \tilde{Y} \), defined on the same probability space, such that \( \tilde{X} =_{st} X \), \( \tilde{Y} =_{st} Y \), and \( \mathbb{E}[\tilde{X}|\tilde{Y}] \leq \tilde{Y} \) almost surely.

We now state a standard multivariate variant of Jensen's inequality along with the necessary condition for equality to hold. The proof of this result is standard and straightforward, and thus is omitted.

**Claim 7** (Jensen's inequality). Let \( V \) be a vector-valued random variable with values in \([0, M]^n\) and let \( u \) be a convex Lipschitz-continuous function mapping \([0, M]^n \to \mathbb{R}\). Then \( \mathbb{E}[u(V)] \geq u(\mathbb{E}[V]) \). Furthermore, equality holds if and only if, for every \( a \) in the subdifferential of \( u \) at \( \mathbb{E}[V] \), the equality \( u(V) = a \cdot (V - \mathbb{E}[V]) + u(\mathbb{E}[V]) \) holds almost surely.

The following lemma is a conditional variant of Claim 7, based on the multivariate
conditional Jensen's inequality, as in Theorem 10.2.7 of [27]. This lemma is used as a tool for Lemma 10, the main result of this subsection.

**Lemma 9.** Let \((\Omega, \mathcal{A}, P)\) be a probability space, \(V\) be a random variable on \(\Omega\) with values in \([0, M]^n\), and \(u : [0, M]^n \to \mathbb{R}\) be convex and Lipschitz continuous. Let \(\mathcal{C}\) be any sub-\(\sigma\)-algebra of \(\mathcal{A}\) and suppose that \(E[u(V)|\mathcal{C}] = u(E[V|\mathcal{C}])\) almost-surely. Then for almost all \(x \in \Omega\) the equality \(u(y) = a_{y_x} \cdot (y - y_x) + u(y_x)\) holds almost surely with respect to the law \(P_{V|\mathcal{C}}(\cdot, x)\), where \(y_x\) is the expectation of the random variable with law \(P_{V|\mathcal{C}}(\cdot, x)\) and \(a_{y_x}\) is any subgradient of \(u\) at \(y_x\).

**Proof of Lemma 9:** The proof is based on the proof of the multivariate conditional Jensen's inequality, as in Theorem 10.2.7 of [27]. This theorem requires \(|V|\) and \(u \circ V\) to be integrable, which is true in our setting. We note that the theorem applies when \(u\) is defined in an open convex set, but because \(u\) is Lipschitz continuous we can extend it to a function with domain an open set containing \([0, M]^n\). The multivariate conditional Jensen's inequality states that, almost surely, \(E[V|\mathcal{C}] \in \mathcal{C}\) and \(E[u(V)|\mathcal{C}] \geq u(E[V|\mathcal{C}])\). The proof of Theorem 10.2.7 in [27] furthermore shows that the following two equalities hold:

\[
E[V|\mathcal{C}](x) = \int_{[0, M]^n} yP_{V|\mathcal{C}}(dy, x); \quad E[u(V)|\mathcal{C}](x) = \int_{[0, M]^n} u(y)P_{V|\mathcal{C}}(dy, x).
\]

Since \(E[u(V)|\mathcal{C}](x) = u(E[V|\mathcal{C}])(x)\) for almost all \(x\), we apply the unconditional Jensen inequality (Claim 7) to the laws \(P_{V|\mathcal{C}}(\cdot, x)\) to prove the lemma. \(\square\)

We now present Lemma 10. Very roughly, this lemma states that for random variables \(X\) and \(Y\) with \(X \preceq_{\text{cov}} Y\) if it holds that \(u(X) = u(Y)\) for some convex function \(u\), then there exists a coupling between \(X\) and \(Y\) with several desirable properties, including that points are only matched if \(u\) shares a subgradient at these points.

**Lemma 10.** Let \(X\) and \(Y\) be vector random variables with values in \([0, M]^n\) such that \(X \preceq_{\text{cov}} Y\), and let \(u : [0, M]^n \to \mathbb{R}\) be 1-Lipschitz with respect to the \(\ell_1\) norm, convex, and monotonically non-decreasing. Suppose that \(E[u(X)] = E[u(Y)]\) and that
$g : [0, M]^n \to [0, 1]^n$ is a measurable function such that for all $z \in [0, M]^n$, $g(z)$ is a subgradient of $u$ at $z$.

Then there exist random variables $\hat{X} =_{st} X$ and $\hat{Y} =_{st} Y$ such that, almost surely:

- $u(\hat{Y}) = u(\hat{X}) + g(\hat{X}) \cdot (\hat{Y} - \hat{X})$
- $g(\hat{X})$ is a subgradient of $u$ at $\hat{Y}$.
- $E[\hat{Y} | \hat{X}]$ is componentwise greater than $\hat{X}$
- $u(E[\hat{Y} | \hat{X}]) = u(\hat{X})$.

Proof of Lemma 10: By Lemma 8, there exist random variables $\hat{X} =_{st} X$ and $\hat{Y} =_{st} Y$ such that $E[\hat{Y} | \hat{X}]$ is componentwise greater than or equal to $\hat{X}$ almost surely. We have

$$0 = E[u(\hat{Y}) - u(\hat{X})] \geq E[u(\hat{Y}) - u(E[\hat{Y} | \hat{X}])] = E[E[u(\hat{Y}) | \hat{X}] - u(E[\hat{Y} | \hat{X}])] \geq 0$$

and therefore $E[E[u(\hat{Y}) | \hat{X}]] = E[u(E[\hat{Y} | \hat{X}])] = E[u(\hat{Y})] = E[u(\hat{X})]$.

Since $u$ is monotonic, $u(\hat{X}) \leq u(E[\hat{Y} | \hat{X}])$ almost surely. Since $E[u(\hat{X})] = E[u(E[\hat{Y} | \hat{X}])]$, it follows that $u(\hat{X}) = u(E[\hat{Y} | \hat{X}])$ almost surely.

Select any collection of random variables $\{\hat{Y}_k | \hat{X}_x\}$ corresponding to the laws $P_{\hat{Y}_k | \hat{X}_x}(\cdot, x)$. For almost all values $x$ of $\hat{X}$, $E[\hat{Y}_k | \hat{X}_x]$ is componentwise greater than $x$ and $u(x) = u(E[\hat{Y}_k | \hat{X}_x])$. We claim now that any subgradient $a_x$ of $u$ at $x$ is also a subgradient of $u$ at $E[\hat{Y}_k | \hat{X}_x]$. Indeed, choose such a subgradient $a_x$. We compute

$$u(E[\hat{Y}_k | \hat{X}_x]) \geq u(x) + a_x \cdot (E[\hat{Y}_k | \hat{X}_x] - x) = u(E[\hat{Y}_k | \hat{X}_x]) + a_x \cdot (E[\hat{Y}_k | \hat{X}_x] - x)$$

and therefore $a_x \cdot E[\hat{Y}_k | \hat{X}_x] = a_x \cdot x$, by non-negativity of the subgradient. Furthermore, for any point $z \in [0, M]^n$,

$$u(z) \geq u(x) + a_x \cdot (z - x) = u(E[\hat{Y}_k | \hat{X}_x]) + a_x \cdot (z - x)$$

$$= u(E[\hat{Y}_k | \hat{X}_x]) + a_x \cdot (z - E[\hat{Y}_k | \hat{X}_x])$$
and thus $a_x$ is a subgradient of $u$ at $E[\hat{Y}|\hat{X}].$

Since $E[E[u(\hat{Y})|\hat{X}]] = E[u(E[\hat{Y}|\hat{X}])]$, by Jensen’s inequality it follows that $E[u(\hat{Y})|\hat{X}] = u(E[\hat{Y}|\hat{X}])$ almost surely. By Lemma 9, it therefore holds for almost all values $x$ of $\hat{X}$ that the equality

$$u(y) = a_x \cdot (y - E[\hat{Y}|\hat{X}]) + u(E[\hat{Y}|\hat{X}]) = a_x \cdot (y - x) + u(E[\hat{Y}|\hat{X}])$$

holds $\hat{Y}|\hat{X} = x$ almost surely.

Lastly, we will show that, almost surely, $a_x$ is a subgradient of $u$ at $\hat{Y}|\hat{X} = x$. Indeed, for any $p \in [0, M]$, and almost all values of $x$ we have

$$u(p) \geq u(x) + a_x \cdot (p - x) = u(x) + a_x \cdot (\hat{Y}|\hat{X} = x - x) + a_x \cdot (p - \hat{Y}|\hat{X} = x)$$

$$= u(\hat{Y}|\hat{X} = x) + a_x \cdot (p - \hat{Y}|\hat{X} = x).$$

\[\square\]

5.3.3 Completing the Proof

We will now prove Lemma 5 which immediately implies Theorem 3. We begin by showing the following corollary of Lemma 10. Roughly speaking, this corollary gives a useful property of Radon measures: if one measure convexly dominates another and if the grand bundling utility function $u_p$ has the same expectation under both of these measures, then appropriate stochastic dominance relations hold when restricted to the regions $Z$ and $W$.

**Corollary 2.** Let $X = [0, M]^n$, and let $A, B$ be random variables with values in $[0, M]^n$ such that $A \preceq_{\text{exx}} B$.

For $p \in \mathbb{R}_{\geq 0}$, define the function $u_p : X \to R_{\geq 0}$ by $u_p(x) = \max\{\|x\|_1 - p, 0\}$ and define the regions $Z, P, W \subset X$ by

$$Z = \{x \in X : \|x\|_1 \leq p\}; \quad P = \{x \in X : \|x\|_1 = p\}; \quad W = \{x \in X : \|x\|_1 \geq p\}.$$
If \( \mathbb{E}[u_p(A)] = \mathbb{E}[u_p(B)] \), then there exist \( \hat{A} =_st A \) and \( \hat{B} =_st B \) such that \( \hat{A} \leq \mathbb{E}[\hat{B} | \hat{A}] \) componentwise holds almost surely and

\[
\left((\hat{A} \in P) \cap (\hat{B} \in P)\right) \cup \left((\hat{A} \in Z) \cap (\hat{B} \in Z)\right) \cup \left((\hat{A} \in W) \cap (\hat{B} \in W)\right)
\]

holds almost surely. Furthermore,

\[
A \cdot \mathbb{1}_{A \in Z} \preceq_{cvx} B \cdot \mathbb{1}_{B \in Z}, \quad A \cdot \mathbb{1}_{A \in P} \preceq_{cvx} B \cdot \mathbb{1}_{B \in P}, \quad \text{and} \quad A \cdot \mathbb{1}_{A \in W} \preceq_{cvx} B \cdot \mathbb{1}_{B \in W}.
\]

Proof of Corollary 2: Select \( \hat{A} \) and \( \hat{B} \) as in Lemma 10, taking \( u = u_p \) and \( g(x) \) to be \( 0^n \) if \( x \in Z \), \( (\frac{1}{2})^n \) if \( z \in P \), and \( 1^n \) if \( z \in W \). The result that \( \hat{B} \) lies in the same region as \( \hat{A} \) follows from the property that \( g(\hat{A}) \) is a subgradient of \( u_p \) at \( \hat{B} \) almost surely.

The convex dominance conditions follow from Strassen's theorem for convex dominance, by observing that, for any region \( R = Z, P, \) or \( W \), the coupling between \( \hat{A} \) and \( \hat{B} \) satisfies

\[
\mathbb{E}[\hat{B} \cdot \mathbb{1}_{\hat{A} \in R} | \hat{A} = \mathbb{1}_{\hat{A} \in R}] \geq \hat{A} \cdot \mathbb{1}_{\hat{A} \in R}
\]

almost surely. Finally, it is obvious that a relation such as \( A \cdot \mathbb{1}_{A \in Z} \preceq_{cvx} B \cdot \mathbb{1}_{B \in Z} \) holds if and only if \( \hat{A} \cdot \mathbb{1}_{\hat{A} \in Z} \preceq_{cvx} \hat{B} \cdot \mathbb{1}_{\hat{B} \in Z} \), as \( \hat{A} =_st A \) and \( \hat{B} =_st B \).

Remark 6. The above corollary can be extended to many functions \( u \) other than the one described above. Informally, Lemma 10 allows us to couple random variables \( A \preceq_{cvx} B \) into random variables \( \hat{A} \) and \( \hat{B} \) such that, almost surely, the gradients of \( u \) at \( \hat{A} \) and \( \hat{B} \) coincide. Such a result aids in the decomposition of a single stochastic dominance condition into several stochastic dominance conditions over regions in which the gradients of \( u \) are constant.

We will now prove Lemma 5 by showing that the first and second conditions are equivalent.

Suppose that \( \mu \) satisfies the second condition of the theorem. We will construct an appropriate measure \( \gamma \in \text{Radon}_+(W \times W) \). Since \( \mu_-|_W \preceq \mu_+|_W \), by Lemma 7
there exists a measure $\theta \in \operatorname{Radon_+}(W)$ such that

$$\mu_-|W \leq_1 \theta \leq_2 \mu_+|W \quad \text{and} \quad \int \|x\|_1d\theta = \int \|x\|_1d\mu_+|W.$$ 

Since $\theta \geq_1 \mu_-|W$, by Strassen’s theorem there exists $\gamma \in \Gamma(\theta, \mu_-|W)$ such that $x \geq y$ holds $\gamma(x, y)$ almost surely.

We now verify the following points

- We have $\gamma_1 - \gamma_2 = \theta - \mu_-|W \leq_2 \mu|W$. Furthermore, by Lemma 6, we know that $\gamma_1 - \gamma_2 \succeq_{cuz} \mu|W$. Since $0 \succeq_{cuz} \mu|Z = \mu|Z \setminus p$, we have $\gamma_1 - \gamma_2 \succeq_{cuz} \mu|Z \setminus p + \mu_W = \mu$.

- We first note that

$$\int \|x\|_1d(\gamma_1 - \gamma_2) = \int \|x\|_1d(\theta - \mu_-|W) = \int \|x\|_1d\mu|W.$$ 

Furthermore, since $(\gamma_1 - \gamma_2)(W) = \mu(W) = 0$, we have

$$\int (\|x\|_1 - p)d(\gamma_1 - \gamma_2) = \int (\|x\|_1 - p)d\mu|W.$$ 

Since $u_p(x) = 0$ off of $W$, $u_p(x) = \|x\|_1 - p$ on $W$, and $\gamma \in \operatorname{Radon_+}(W \times W)$,

$$\int (\|x\|_1 - p)d(\gamma_1 - \gamma_2) = \int u_p(\gamma_1 - \gamma_2) = \int u_p d\mu|W = \int u_p d\mu$$

- Finally, by construction, it holds $\gamma(x, y)$ almost surely that $x \geq y$, and therefore $u_p(x) - u_p(y) = \|x\|_1 - \|y\|_1 = \|x - y\|_1$ holds $\gamma(x, y)$ almost surely.

Now suppose that $\gamma \in \operatorname{Radon_+}(X \times X)$ satisfies condition 1. Since $\gamma_1 + \mu_- \succeq_{cuz} \gamma_2 + \mu_+$, by Corollary 2 we know that

$$(\gamma_1 + \mu_-)|Z \succeq_{cuz} (\gamma_2 + \mu_+)|Z \quad \text{and} \quad (\gamma_1 + \mu_-)|W \succeq_{cuz} (\gamma_2 + \mu_+)|W.$$ 

Since $u_p(x) - u_p(y) = \|x - y\|_1$ holds $\gamma(x, y)$-almost surely, we know that almost surely either (i) $x, y \in Z$ and $x = y$ or (ii) $x, y \in W$ with $x \geq y$ coordinatewise.
We can therefore (uniquely) decompose $\gamma = \zeta + \eta$, where $\zeta \in Radon_+(Z \setminus P, Z \setminus P)$ and $\eta \in Radon_+(W, W)$ such that $x = y$ holds $\zeta(x, y)$ almost surely and $x \geq y$ coordinatewise holds $\eta(x, y)$ almost surely. We now make several claims

- We will show first that $\mu(W) = 0$. Since $\gamma_1|_W = \eta_1$ and $\gamma_2|_W = \eta_2$:

$$
\gamma_1|_W + \mu_-|_W \succeq_{cvx} \gamma_2|_W + \mu_+|_W
$$

$$
\eta_1 + \mu_-|_W \succeq_{cvx} \eta_2 + \mu_+|_W.
$$

Since convex dominance only holds between measures with equal total mass, $\eta_1(W) - \eta_2(W) = \mu(W)$. But since $\eta \in Radon_+(W, W)$, we know that $\eta_1(W) = \eta_2(W)$, and therefore $\mu(W) = 0$.

- We claim next that $\eta_1|_P = \eta_2|_P$. In particular, since $\gamma_1|_Z = \zeta_1 + \eta_1|_P$ and $\gamma_2|_Z = \zeta_2 + \eta_2|_P$, we have

$$
\gamma_1|_Z + \mu_-|_Z \succeq_{cvx} \gamma_2|_Z + \mu_+|_Z
$$

$$
\zeta_1 + \eta_1|_P + \mu_-|_Z \succeq_{cvx} \zeta_2 + \eta_2|_P + \mu_+|_Z.
$$

Since $x = y$ holds $\zeta(x, y)$ almost surely, it follows that $\zeta_1 = \zeta_2$, and thus $\eta_1|_P + \mu_-|_Z \succeq_{cvx} \eta_2|_P + \mu_+|_Z$. Convex dominance only holds between two measures with equal total mass, and therefore

$$
\eta_1|_P(Z) - \eta_2|_P(Z) = \mu(Z) = \mu(X) + \mu(P) - \mu(W) = 0.
$$

Since $x \geq y$ coordinatewise holds $\eta(x, y)$ almost surely, and since $P$ does not contain any $x \neq y$ such that $x \geq y$ coordinatewise, if $\eta_1|_P \neq \eta_2|_P$ we would have $\eta_1|_P(Z) < \eta_2|_P(Z)$, which is a contradiction. Therefore $\eta_1|_P = \eta_2|_P$.

- We now claim that $\mu_-|_Z \succeq_{cvx} \mu_+|_Z$. We compute, using the fact that $\gamma_1|_Z = \eta_1|_P$:
\(\zeta_1 + \eta_1|_p\) and \(\gamma_2|_z = \zeta_2 + \eta_2|_p\).

\[
\gamma_1|_z + \mu_-|_z \succeq \begin{align*}
\zeta_2 + \eta_2|_p + \mu_+|_z
\zeta_1 + \eta_1|_p + \mu_-|_z \succeq \begin{align*}
\zeta_2 + \eta_2|_p + \mu_+|_z
\end{align*}
\end{align*}
\]

Since \(\eta_1|_p = \eta_2|_p\) and \(\zeta_1 = \zeta_2\), we have \(\mu_-|_z \succeq \mu_+|_z\).

- We claim finally that \(\mu_-|_w \succeq \mu_+|_w\). Since \(u = 0\) on \(Z\), we know

\[
\int_W (\|x\|_1 - p)d\gamma_1 - \int_W (\|x\|_1 - p)d\gamma_2 = \int_W (\|x\|_1 - p)d\mu.
\]

Since \(\gamma_1|_w = \eta_1, \gamma_2|_w = \eta_2, \eta_1(W) = \eta_2(W),\) and \(\mu(W) = 0\), we have

\[
\int_W \|x\|_1d(\eta_1 - \eta_2) = \int_W \|x\|_1d\mu.
\]

In addition, we have \(\eta_1 - \eta_2 \succeq \mu|_w\). This implies by Lemma 6 that \(\mu|_w \succeq \eta_1 - \eta_2\). Furthermore, since \(\eta_1 - \eta_2 \succeq 0\) (by Strassen’s theorem, as \(x \geq y\) holds \(\eta(x, y)\) almost surely) we have \(\mu|_w \succeq 0\), and thus \(\mu_+|_w \succeq \mu_-|_w\).

### 5.4 Example of Grand Bundling Optimality

We now present an example of applying our characterization of grand bundling optimality. We note that this result applies to settings with arbitrarily many items, which is relatively rare in the optimal mechanism design literature.

**Example 2.** For any integer \(n > 0\) there exists a \(c_0\) such that for all \(c > c_0\), the optimal mechanism for selling \(n\) iid goods whose values are uniform on \((c, c + 1)\) is a take-it-or-leave-it offer for the grand bundle.

**Remark 7.** Pavlov [53] proved the above result for two items, and explicitly solved for \(c_0 \approx 0.077\). In our proof, for simplicity of analysis, we do not attempt to exactly compute \(c_0\) as a function of \(n\).
Our proof of Example 2 uses the following lemma, which enables us to appropriately match regions on the surface of a hypercube.

**Lemma 11.** For $n \geq 2$ and $\rho > 1$, define the $(n - 1)$-dimensional subsets of $[0,1]^n$:

\[
A = \left\{ x : 1 = x_1 \geq x_2 \geq \cdots \geq x_n \text{ and } x_n \leq 1 - \left( \frac{\rho - 1}{\rho} \right)^{(n-1)} \right\}
\]

\[
B = \{ y : y_1 \geq \cdots \geq y_n = 0 \}.
\]

There exists a continuous bijective map $\varphi : A \to B$ such that

- For all $x \in A$, $x$ is componentwise greater than or equal to $\varphi(x)$

- For subsets $S \subseteq A$ which are measurable under the $(n-1)$-dimensional surface Lebesgue measure $v(\cdot)$, it holds that $\rho \cdot v(S) = v(\varphi(S))$.

- For all $\epsilon > 0$, if $(\varphi(x))_1 \leq \epsilon$ then $x_n \geq 1 - \left( \frac{\epsilon^{n-1} + (\rho - 1)}{\rho} \right)^{(n-1)}$.

![Figure 5-1: The regions of Lemma 11 for the case $n = 3$.](image)

The main difficulty to proving Example 2 is verifying the necessary stochastic dominance relations above the grand bundling hyperplane. Our proof appropriately partitions this part of the hypercube into $2(n! + 1)$ sections, and uses Lemma 11 to show that a desired stochastic dominance relation holds for an appropriate pairing of sections.

**Proof of Lemma 11:** We define the mapping $\varphi : A \to B$ by $\varphi(x) = y$, where

\[
y_1 = \left[ 1 - \rho \left( 1 - (1 - x_n)^{n-1} \right) \right]^{1/(n-1)}; \quad y_i = \frac{x_i - x_n}{1 - x_n} \cdot y_1 \text{ for } i > 1.
\]
We first claim that ϕ is a bijection. As \( x_n \) ranges from 0 to \( 1 - \left( \frac{e-1}{\rho} \right)^{1/(n-1)} \), we see that \( y_i \) ranges from 1 to 0, and thus there is a bijection between valid \( y_i \) values and valid \( x_n \) values. Furthermore, for any fixed \( y_1 \) and \( x_n \), there is a bijection between \( x_i \) and \( y_i \) for \( i = 2, \ldots, n-1 \). (By varying \( x_i \) between \( x_n \) and 1 we can achieve all values of \( y_i \) between 0 and \( y_1 \).) Furthermore, for any fixed \( y_1 \) and \( x_n \) the mapping from \( x_i \) to \( y_i \) is an increasing function of \( x_i \), and therefore for all \( x \in A \) we have \( y_1 \in [0, 1] \) and \( y_1 \geq y_2 \geq \cdots \geq y_n = 0 \). Thus, \( \varphi \) is a bijection between \( A \) and \( B \).

Next, we claim that for any \( x \in A \), it holds that \( x \) is componentwise at least as large as \( \varphi(x) \). Since \( x_1 = 1 \), it trivially holds that \( x_1 \geq (\varphi(x))_1 \). Fix a value of \( x_n \) (and hence of \( y_1 \)), and consider the bijection \( g : [x_n, 1] \to [0, y_1] \) given by \( g(z) = y_1 (z - x_n)/(1 - x_n) \). We must show that \( z - g(z) \geq 0 \) for all \( z \in [x_n, 1] \). This follows from noticing that \( z - g(z) \) is a linear function of \( z \) and both \( x_n - g(x_n) = x_n \) and \( 1 - g(1) = 1 - y_1 \) are nonnegative.

We now show that \( \varphi \) scales surface measure of every measurable \( S \subset A \) by a factor of \( 1/\rho \). Instead of directly analyzing surface measures, it suffices to prove that the function \( \varphi' : W \to W \) scales volumes by \( \rho \), where \( W \subset \mathbb{R}^{n-1} \) is the set \( \{ w : 1 \geq w_1 \geq \cdots \geq w_{n-1} \geq 0 \} \) and \( \varphi'(w) \) drops the last (constant) coordinate of \( \varphi(1, w_1, \ldots, w_{n-1}) \) and then (for notational convenience) permutes the first coordinate to the end. That is,

\[
\varphi'(w_1, \ldots, w_{n-1}) = \left( \frac{w_1 - w_{n-1}}{1 - w_{n-1}} z(w_{n-1}), \ldots, \frac{w_{n-2} - w_{n-1}}{1 - w_{n-1}} z(w_{n-1}), z(w_{n-1}) \right)
\]

where \( z(w_{n-1}) = [1 - \rho (1 - (1 - w_{n-1})^{n-1})]^{1/(n-1)} \).

We now analyze the determinant of the Jacobian matrix \( J \) of \( \varphi' \). We notice that the only non-zero entries of \( J \) are the diagonals and the rightmost column. In particular, \( J \) is upper triangular, and therefore its determinant is the product of its
diagonal entries. We therefore compute

\[
det(J) = \left( \frac{z(w_{n-1})}{1-w_{n-1}} \right)^{n-2} \cdot \frac{\partial}{\partial w_{n-1}} [1 - \rho (1 - (1 - w_{n-1})^{n-1})]^{1/(n-1)}
\]

\[
= \left( \frac{z(w_{n-1})}{1-w_{n-1}} \right)^{n-2} \cdot \frac{-1}{n-1} \left( z(w_{n-1})^{-(n-2)} \cdot \rho \cdot (n-1)(1 - w_{n-1})^{n-2} \right) = -\rho
\]
as desired.

Lastly, suppose \( y_1 \leq \epsilon \). Then \([1 - \rho (1 - (1 - x_n)^{n-1})]^{1/(n-1)} \leq \epsilon \) and thus \( x_n \geq 1 - \left( \frac{\epsilon^{(n-1)+\rho-1}}{\rho} \right)^{1/(n-1)} \).

\[
\text{Proof of Example 2: We now complete the proof of Example 2. Fix the dimension } n.
\]

For any value of \( c \), the transformed measure on the hypercube \([c, c+1]^n\) we obtain is as follows:

- A point mass of \(+1\) at \((c, c, \ldots, c)\).
- Mass of \(-(n+1)\) uniformly distributed throughout the interior.
- Mass of \(-c\) distributed on each surface \(x_i = c\) of the hypercube.
- Mass of \(c+1\) distributed on each surface \(x_i = c+1\) of the hypercube.

For notational convenience when checking the stochastic dominance properties of Lemma 5, we will shift the hypercube to the origin. That is, we will consider instead the measure \(\mu^c\) on \([0, 1]^n\) which has mass \(+1\) at the origin, mass of \(-c\) on each each surface \(x_i = 0\), et cetera. It is important to notice that the mass that \(\mu\) assigns to the interior of \([0, 1]^n\) and to the origin do not depend on \(c\), while the mass on each surface is a function of \(c\).

For any \(h \in (0, 1)\), define the region \(Z(h) = \{x \in [0, 1]^n : \|x\|_1 \leq h\}\). For any fixed \(c_0\), it holds that \(\mu^c_+(Z(h)) = 1\) for all \(h \in (0, 1)\) and there exists a small enough \(h' > 0\) such that \(\mu^c_+(Z(h')) < 1\). Since for this fixed \(h'\) it holds that \(\mu^c_+(Z(h'))\) increases with \(c\) (and becomes arbitrarily large as \(c\) becomes large), there must exist a \(c' > c_0\) such that \(\mu^c_+(Z(h')) = 1\), and thus \(\mu^c_+(Z(h')) = 0\). We can therefore pick a decreasing
function $p^*: \mathbb{R}_{\geq 0} \to (0, 1)$ such that, for all sufficiently large $c$, $\mu^c(Z(p^*(c))) = 0$.\(^1\) It follows that $p^*(c) \to 0$ as $c \to \infty$.

For all $c$, define the following subsets of $[0, 1]^n$:

$$Z_c = \{ x : \|x\|_1 \leq p^*(c) \}; \quad W_c = \{ x : \|x\|_1 \geq p^*(c) \}.$$  

We notice that $\mu_+(Z_c \cap W_c) = \mu_-(Z_c \cap W_c) = 0$. By construction, for large enough $c$ we have $\mu^c(Z_c) = 0$. In addition, the only positive mass in $Z_c$ is at the origin, and thus $\mu_-|Z_c \succeq \mu_+|Z_c$.

To apply Theorem 3, it remains to show that, for sufficiently large $c$, $\mu_+|W_c \succeq_2 \mu_-|W_c$. To prove this, we will partition $W_c$ into $2(n!+1)$ disjoint\(^2\) regions, $P_0, P_{\sigma_1}, \ldots, P_{\sigma_n}$ and $N_0, N_{\sigma_1}, \ldots, N_{\sigma_n}$, where $\sigma_j$ is a permutation of $1, \ldots, n$. This partition will be such that $\cup_j P_j$ contains the entire support of $\mu_+|W_c$ and $\cup_j N_j$ contains the entire support of $\mu_-|W_c$. We will show that $\mu_+|P_j \succeq_2 \mu_-|N_j$ for all $j$, thereby proving $\mu_+|W_c \succeq_2 \mu_-|W_c$.

For every permutation $\sigma$ of $1, \ldots, n$, define:

$$P_{\sigma}' = \left\{ x : 1 = x_{\sigma(1)} \geq x_{\sigma(2)} \geq \cdots \geq x_{\sigma(n)} \geq 0 \text{ and } x_{\sigma(n)} \leq 1 - \left( \frac{1}{c+1} \right)^{1/(n-1)} \right\}$$

$$N_{\sigma}' = \{ y : 1 \geq y_{\sigma(1)} \geq \cdots \geq y_{\sigma(n-1)} \geq y_{\sigma(n)} = 0 \}$$

Denote by $\rho \triangleq (c + 1)/c$ the ratio between the surface densities of $\mu_+^c$ and $\mu_-^c$ on $P_{\sigma}'$ and $N_{\sigma}'$, respectively, and let $\varphi_{\sigma} : P_{\sigma}' \to N_{\sigma}'$ be the bijection given by Lemma 11. By construction, $\mu_+^c(S) = \mu_-^c(\varphi_{\sigma}(S))$ for all measurable $S \subseteq P_{\sigma}'$.

Denote $N_{\sigma} \triangleq N_{\sigma}' \setminus Z_c$ and $P_{\sigma} \triangleq \varphi^{-1}(N_{\sigma})$. By construction, $\varphi$ is a bijection between $P_{\sigma}$ and $N_{\sigma}$, preserving the respective the measures $\mu_+^c$ and $\mu_-^c$, such that for all $x \in P_{\sigma}$, $x$ is componentwise at least as large as $\varphi(x)$. Therefore, by Strassen's

\(^1\)Our intention is to argue that for $c$ large enough, the optimal mechanism will be grand bundling for a price of $p^*(c) + c$, where the additive $+c$ term comes from our shift of the hypercube to the origin.

\(^2\)For notational simplicity, our regions overlap slightly, although the overlap always has zero mass under both $\mu_+^c$ and $\mu_-^c$. 

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theorem, \( \mu^c_+|_{P_0} \geq \mu^c_-|_{N_0} \). Lastly, we define

\[
P_0 = \{ x \in [0,1]^n : x_i = 1 \text{ for some } i \} \setminus \left( \bigcup_{\sigma} P_{\sigma} \right); \quad N_0 = (0,1)^n \setminus Z_c.
\]

\( P_0 \) consists of all points on the outer surface of the hypercube which have not yet been matched to any \( N_\sigma \), and \( N_0 \) consists of all points on which \( \mu^c_- \) is nontrivial which have not yet been matched.\(^3\) It therefore remains only to show that \( \mu^c_+|_{P_0} \geq \mu^c_-|_{N_0} \).

We claim that, for large enough \( c \), \( P_0 \) only contains points with all coordinates greater than 3/4. Indeed:

- Every \( x \) with \( x_i = 1 \) but some \( x_j < 1 - \left( \frac{1}{c+1} \right)^{1/(n-1)} \) is in some \( P_{\sigma} \).

- For large \( c \), every \( x \) with \( x_i = 1 \) but some \( x_j \leq 3/4 \) is in some \( P_{\sigma}' \).

- We claim that for large \( c \), every \( x \in P_{\sigma}' \setminus P_\sigma \) has all coordinates at least 3/4. Indeed, for every \( x \in P_{\sigma}' \setminus P_\sigma \), it must be that \( \phi(x) \in Z_c \), and thus \( \|\phi(x)\|_1 \leq \rho^*(c) \). By Lemma 11, we have \( x_{\sigma(n)} \geq 1 - \left( \frac{\rho^*(c)^{n-1} + \rho - 1}{\rho} \right)^{1/(n-1)} \). As \( c \) gets large, \( \rho \to 1 \) and \( \rho^*(c) \to 0 \). Thus, for sufficiently large \( c \), we have \( x \in P_{\sigma}' \setminus P_\sigma \) implies \( x_{\sigma(n)} \geq 3/4 \). Since \( x_{\sigma(n)} \) is the smallest coordinate of \( x \), it follows that all coordinates of any \( x \in P_{\sigma}' \setminus P_\sigma \) are greater than 3/4.

- Thus, for sufficiently large \( c \), every \( x \) with \( x_i = 1 \) but some \( x_j < 3/4 \) lies in some \( P_\sigma \), and hence does not lie in \( P_0 \).

By construction, \( \mu^c_-|_{N_0} \) and \( \mu^c_+|_{P_0} \) have the same total mass. Consider independent random variables \( X \) and \( Y \) corresponding to \( \mu^c_-|_{N_0} \) and \( \mu^c_+|_{P_0} \), respectively, where we scale both measures so that they are probability distributions. By Lemma 8, it suffices to show that for sufficiently large \( c \), \( Y \geq \mathbb{E}[X] \) almost surely.\(^4\) Since \( \mu^c_+|_{P_0} \) is supported on \( P_0 \), we need only show that all coordinates of \( \mathbb{E}[X] \) are less than 3/4. We recall that \( \mu^c_- \) assigns a total mass of \( n+1 \), distributed uniformly, to the interior of the hypercube.

\(^3\)All other points on which \( \mu^c_- \) is nontrivial have been matched either to the origin (if the point lies in \( Z_c \)), or to some point in \( P_\sigma \) (if the point lies in \( N_\sigma \setminus Z_c \)).

\(^4\)In general, to prove second order dominance we might need to nontrivially couple \( X \) and \( Y \). In this case, however, choosing independent random variables suffices.

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As $c$ gets large, $p^*(c)$ approaches 0, and thus $\mu_c^c(\mathbb{Z}_c \cap (0,1)^n)/\mu_c^c((0,1)^n) \to 0$. For large $c$, therefore, $\mathbb{E}[X]$ becomes arbitrarily close to the center of the hypercube, which is the point with all coordinates equal to $1/2$. Therefore we have $\mu_1^c|_\mathcal{B} \succeq_2 \mu_c^c|_\mathcal{N}_0$. \hfill $\Box$
Chapter 6

A Partial Characterization of Optimal Two-Item Mechanisms

In many two-items optimal mechanism design instances, we find that the optimal mechanisms have a common structure. Our goal in this chapter is to explain this structure and the geometric intuition behind it.\(^1\) We remark that while there exist two-item examples (in particular Example 1 of Section 4.3) for which the framework of this chapter does not immediately apply, a multitude of interesting non-trivial two-item mechanism design instances are indeed solvable by a direct application of this framework.

6.1 A Structural Result for Two Items

We begin by defining the notion of a zero set. Every zero set gives rise to a mechanism, where the utility of a bidder is equal to the \(\ell_1\) distance between the bidder’s type and the closest point in the zero set. (Bidder types within the zero set receive zero utility, and hence the name.)

**Definition 12** (Zero set). Let \(X = [0, M]^2 \subset \mathbb{R}_{\geq 0}^2\). A zero set \(Z\) of \(X\) is a nonempty, \(\ldots\)

\(^1\)The structural result of this chapter is an extension of the two-item result from [22]. The examples and supporting lemmas from Section 6.2 and Appendix B appeared in an earlier form in [22]. (The proceedings version of [22] has a minor error in Definition 10.2 which does not affect any of the results.)
compact, and decreasing\(^2\) subset of \(X\).

Every zero set \(Z\) of \(X\) corresponds to a particular mechanism:

**Definition 13** (Mechanism of a Zero Set). A zero set \(Z\) of \(X\) induces a mechanism whose utility function \(u_Z : X \rightarrow \mathbb{R}\) is defined by:

\[
u_Z(x) = \min_{z \in Z} \| z - x \|_1.
\]

Since a zero set \(Z\) is closed, for any \(x \in X\) there exists a \(z \in Z\) such that \(u_Z(x) = \| z - x \|_1\).

Any such utility function \(u_Z\) satisfies the constraints of the mechanism design problem. That is, the mechanism corresponding to \(u_Z\) is IC and IR.

**Claim 8.** Let \(Z\) be a zero set of \(X\). Then \(u_Z\) is non-negative, non-decreasing, convex, and has Lipschitz constant (with respect to the \(\ell_1\) norm) at most 1. In particular, \(u_Z\) is the utility function of an incentive compatible and individually rational mechanism.

**Proof.** It is obvious that \(u_Z\) is non-negative. To show that \(u_Z\) is non-decreasing, it suffices to prove that \(u_Z(x) \geq u_Z(y)\) for \(x, y \in X \setminus Z\) with \(x\) component-wise greater than or equal to \(y\). Let \(z_x \in Z\) be the closest point to \(x\). Denote by \(z_y\) the point with each coordinate being the component-wise minimum of \(z_x\) and \(y\). Since \(Z\) is decreasing, \(z_y \in Z\). We now compute

\[
u_Z(x) = \| z_x - x \|_1 = \sum_i |(z_x)_i - x_i| \geq \sum_i \min\{(z_x)_i, y_i\} - y_i = \| z_y - y \|_1 \geq u_Z(y)
\]

and thus \(u_Z\) is non-decreasing.

We will now show that \(u_Z\) is convex. Pick arbitrary \(x, y \in X\). Denote by \(z_x\) and \(z_y\) points in \(Z\) such that \(u_Z(x) = \| x - z_x \|_1\) and \(u_Z(y) = \| y - z_y \|_1\). Since \(Z\) is convex, the point \((z_x + z_y)/2\) is in \(Z\). Thus

\[
u_Z \left( \frac{x + y}{2} \right) \leq \left\| \frac{x + y}{2} - z_x + z_y \right\|_1 \leq \left\| \frac{x - z_x}{2} + \frac{y - z_y}{2} \right\|_1 = \frac{u_Z(x) + u_Z(y)}{2}
\]

\(^2\)A decreasing subset \(Z \subseteq X\) satisfies the property that for all \(a, b \in X\) such that \(a\) is component-wise less than or equal to \(b\), if \(b \in Z\) then \(a \in Z\) as well.
and therefore $u_Z$ is convex.

Lastly, we verify that $u_Z$ has Lipschitz constant at most 1. Indeed,

$$u_Z(x) - u_Z(y) \leq \|x - z\|_1 - u_Z(y) = \|x - y\|_1 - \|y - z\|_1 \leq \|x - y\|_1.$$

\[\square\]

To provide sufficient conditions for $u_Z$ to be optimal, we define the concept of a canonical partition. A canonical partition divides $X$ into regions such that the mechanism's allocation function within each region has a similar form. Roughly, the canonical partition separates $X$ based on which direction (either "down," "left," or "diagonally") one must travel to reach the closest point in $Z$. While the definition is involved, the geometric picture of Figure 6-1 is straightforward.

**Definition 14 (Outer boundary function of a zero set).** Let $X = [0, M]^2 \subseteq \mathbb{R}^2$ and let $Z$ be a zero set of $X$. Denote by $c_1 \in [0, M]$ the point $c_1 = \max c : (c, M) \in Z$. We define the outer boundary function of $Z$ to be the function $s : [0, c_1] \rightarrow [0, M]$ given by

$$s(y_1) = \max y_2 : (y_1, y_2) \in Z.$$

**Definition 15 (Critical value, Canonical partition).** Let $Z$ be a zero set of $X$ with outer boundary function $s : [0, c_1] \rightarrow [0, M]$, as above. Denote by $a_1, b_1 \in [0, c_1]$ the points such that

- $0 \geq s'(z_1) > -1$ for (almost) all $z_1 \in [0, a_1)$
- $s'(z_1) = -1$ for all $z_1 \in (a_1, b_1)$
- $s'(z_1) < -1$ for almost all $z_1 \in (b_1, c_1]$.

We call $b_1$ the critical value of $Z$. We define the canonical partition of $X$ to be the partition of $X$ into $Z \cup A \cup B \cup W$, where

$$A = [0, a_1) \times [0, M] \setminus Z; \quad B = [b_1, M] \times [0, s(b_1)) \setminus Z; \quad W = X \setminus (Z \cup A \cup B),$$

as shown in Figure 6-1.

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Note that the outer boundary function $s$ of a zero set $Z$ is concave and thus is differentiable almost everywhere on $[0, c_1]$ and has non-increasing derivative.

![Figure 6-1: A canonical partition of $[0, M]^2$](image)

We now restate the utility function $u_Z$ in terms of a canonical partition.

**Claim 9.** Let $Z$ be a zero set of $X$ with outer boundary function $s$, and let $Z \cup A \cup B \cup W$ be a canonical partition. Then for all $(v_1, v_2) \in X$:

$$u_Z(v_1, v_2) = \begin{cases} 
0 & \text{if } (v_1, v_2) \in Z \\
v_2 - s(v_1) & \text{if } (v_1, v_2) \in A \\
v_1 - s^{-1}(v_2) & \text{if } (v_1, v_2) \in B \\
v_1 + v_2 - (a_1 + s(a_1)) & \text{if } (v_1, v_2) \in W.
\end{cases}$$

**Proof.** The proof is fairly straightforward casework. We prove one of the cases here, and the remaining cases are similar.

Pick any $v = (v_1, v_2) \in A$. We will show that the closest $z \in Z$ is the point $z^* = (v_1, s(v_1))$. Pick $z' = (z'_1, z'_2) \in Z$ such that $u_Z(v) = ||v - z'||_1$. It must be the case that $z'_1 \leq v_1$, since otherwise $(v_1, z'_2)$ would be in $Z$ (as $Z$ is decreasing) and strictly closer to $v$. Furthermore, we know that $z'$ lies on the boundary of $Z$, and thus $z' = (v_1 - \delta, s(v_1 - \delta))$ for some $\delta \geq 0$. We may assume that $\delta \leq v_2 - s(v_1)$.

Since $s'(\cdot) \geq -1$ in the range $[a_1, v_1]$, we know that $s(v_1 - \delta) \leq s(v_1) + \delta \leq v_2$.
Therefore
\[ \|v - z\|_1 = \delta + |v_2 - s(v_1 - \delta)| \geq \delta + (v_2 - s(v_1) - \delta) = v_2 - s(v_1). \]

We now describe sufficient conditions under which \( u_Z \) is optimal.

**Definition 16** (Well-formed canonical partition). Let \( Z \cup A \cup B \cup W \) be a canonical partition of \( X = [0, M]^2 \) and let \( \mu \) be a signed Radon measure on \( X \) such that \( \mu(X) = 0 \). We say that the canonical partition is well-formed (with respect to \( \mu \)) if the following conditions are satisfied:

1. \( \mu(Z) = 0 \)
2. \( 0 \preceq \text{cox} \mu|_Z \)
3. \( \mu|_W \geq_2 0 \)
4. For all \( v \in X \) and all \( \epsilon > 0 \), \( \mu|_A ([v_1, v_1 + \epsilon] \times [v_2, M]) \geq 0 \), with equality whenever \( v_2 = 0 \).
5. For all \( v \in X \) and all \( \epsilon > 0 \), \( \mu|_B ([v_1, M] \times [v_2, v_2 + \epsilon]) \geq 0 \), with equality whenever \( v_1 = 0 \).

We point out the similarities between a well-formed canonical partition and the characterization of grand bundling optimality of Theorem 3. When \( Z \) is the zero-set of a bundling mechanism (so \( Z = \{ z : \|z\|_1 \leq p \} \), \( A \) and \( B \) are empty and the conditions of a well-formed canonical partition are essentially the same as those of Theorem 3 in the two-item case. We interpret conditions 4 and 5 as saying that \( \mu|_A \) (resp. \( \mu|_B \)) allows for the positive mass in any vertical (resp. horizontal) “strip” to be matched to the negative mass in the strip by only transporting “downwards” (resp. “leftwards”). In practice, when \( \mu \) is given by a density function, we verify these conditions by analyzing the integral of the density function along appropriate vertical or horizontal lines.
Theorem 4. Let $U \subset \mathbb{R}^2$ be a well-behaved type space and let $f$ be a probability density function on $U$ with bounded partial derivatives. Let $X = [0, M]^2 \supset U$ and let $\mu$ be the transformed measure of $f$. If there exists a canonical partition $Z \cup A \cup B \cup W$ of $X$ which is well-formed with respect to $\mu$, then the optimal IC and IR mechanism for a single additive buyer whose values for 2 goods are distributed according to the joint distribution $f$ has the following allocation and price for a bidder with declared type $(z_1, z_2) \in X$:

- If $(z_1, z_2) \in Z$, the bidder receives no goods and is charged 0.
- If $(z_1, z_2) \in A$, the bidder receives item 1 with probability $-s'(z_1)$, item 2 with probability 1, and is charged $s(z_1) - z_1 s'(z_1)$.
- If $(z_1, z_2) \in B$, the bidder receives item 1 with probability 1, item 2 with probability $-1/s'(s^{-1}(z_2))$, and is charged $s^{-1}(z_2) - z_2/s'(s^{-1}(z_2))$.
- If $(z_1, z_2) \in W$, the bidder receives both goods with probability 1 and is charged $b_1 + s(b_1)$

where $s$ is the outer boundary function of $Z$ and $b_1$ is the critical value of $Z$.

Refer back to Figure 6-1 to visualize such a mechanism.

Proof of Theorem 4:

We will show that $u_Z$ maximizes $\sup_{u \in U_1(X) \cap C_1(X)} \int_X u d\mu$. By Corollary 1, it suffices to provide a $\gamma \in \text{Radon}^+(X \times X)$ such that $\gamma_1 - \gamma_2 \succeq_{\text{exc}} \mu$, $\int u_2 d(\gamma_1 - \gamma_2) = \int u_2 d\mu$, and $u_2(x) - u_2(y) = \|x - y\|_1$ holds $\gamma$-almost surely. The $\gamma$ we construct will never transport mass between regions. That is, $\gamma = \gamma_Z + \gamma_W + \gamma_A + \gamma_B$ where

- $\gamma_Z = 0$. We notice that $(\gamma_Z)_1 - (\gamma_Z)_2 = 0 \succeq_{\text{exc}} \mu|Z$.
- $\gamma_W$ is constructed such that $\gamma_W \succeq_{\text{exc}} \mu|W$ and the component-wise inequality $x \geq y$ holds $\gamma_W(x, y)$ almost surely. As in our proof of Theorem 3, the existence of such a $\gamma_W$ is guaranteed by the techniques of Theorem 7.A.3

\footnote{As in our prior example and as discussed in Remark 3, we aim for $\gamma_W$ to transport "downwards and leftwards" since both items are allocated with probability 1 in $W$.}

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and Theorem 4.A.6 of [58] applied to second order dominance, combined with Strassen's theorem for first-order stochastic dominance. In particular, the condition $\mu|_W \succeq_2 0$ implies the existence of a signed measure $\theta$ on $W$ such that $\theta(W) = 0$, $\int \|x\|_1 d\theta = \int \|x\|_1 d\mu|_W$, and $0 \preceq_1 \theta \preceq_2 \mu|_W$, where $\preceq_1$ denotes first-order stochastic dominance. Since $\theta_+ \preceq_1 \theta_-$, we use Strassen's theorem on first-order stochastic dominance to construct an appropriate map $\gamma_W$ with respective marginals $\theta_+$ and $\theta_-$. We have that $(\gamma_W)_1 - (\gamma_W)_2 = \theta \preceq_2 \mu|_W$. Furthermore, $\int \|x\|_1 d\theta = \int \|x\|_1 d\mu|_W$. It follows by Lemma 6 in Section 5.3.1 (used in the proof of Theorem 3) that $\theta \succeq_{cuz} \mu|_W$.\footnote{This is the only part of the proof where we use the fact that $X$ is bounded.} In addition, since $\mu|_W(W) = \theta(W) = 0$ and since $u_Z(x)$ differs from $\|x\|_1$ by a constant on $W$, we have $\int u_Z d\mu|_W = \int u_Z d\theta$.

- $\gamma_A \in \text{Radon}_+(A \times A)$ will be constructed to have respective marginals $\mu_+|_A$ and $\mu_-|_A$. (Thus, $(\gamma_A)_1 - (\gamma_A)_2 = \mu|_A$.) We construct $\gamma_A$ to send positive mass “downwards.”\footnote{Once again, the intuition of this construction follows from Remark 3.} That is, $\gamma_A(x, y)$ almost surely, it holds that $x_1 = y_1$ and $x_2 \geq y_2$. We claim that such a map can indeed be constructed, by the one-dimensional equivalent condition for first-order stochastic dominance of comparing the cumulative density functions.\footnote{For any $\epsilon > 0$, it is immediately clear that we can construct a mapping such that $c(x, y) \leq u_Z(x) - u_Z(y) + \epsilon$ holds $\gamma_A(x, y)$ almost surely. Since $\epsilon$ is arbitrary, this is already sufficient to certify optimality of $u_Z$.}

The measure satisfies $x_1 = y_1$, $\gamma_A(x, y)$ almost surely and

$$u_Z(x) - u_Z(y) = (x_2 - s(x_1)) - (y_2 - s(y_1)) = x_2 - y_2 = \|x - y\|_1.$$  

- $\gamma_B \in \text{Radon}_+(B \times B)$ is constructed analogously to $\gamma_A$, except sending mass to “leftwards.” That is, $\gamma_B(x, y)$ almost surely, the relationships $x_1 \geq y_1$ and $x_2 = y_2$ hold.

It follows by our construction that $\gamma = \gamma_Z + \gamma_W + \gamma_A + \gamma_B$ satisfies all necessary properties to certify optimality of $u_Z$. \hfill $\Box$
6.2 Applying the Structural Result

The goal of this section is to give examples of applying Theorem 4. One major obstacle to applying Theorem 4 is in verifying the appropriate stochastic dominance relation \( \mu|_{W} \succeq_{2} 0 \). In these examples, we will have the stronger property \( \mu|_{W} \succeq_{1} 0 \), which is easier to verify, yet still imposes some practical difficulties. In Section 6.2.1 we develop a tool, Lemma 14, for verifying such first-order stochastic dominance. In Section 6.2.2 we give examples which use this tool, along with Theorem 4, to solve optimal mechanism design instances.

6.2.1 Verifying First-Order Stochastic Dominance

The primary result of this section is Lemma 14, which provides a sufficient condition for a measure to stochastically dominate another. Motivated by our intended applications, we focus only on measures which can be described by a bounded density function. Lemma 14 is an application of Claim 13 which states that an equivalent condition for first-order stochastic dominance is that one measure has more mass than the other on all sets which are the union of finitely many "increasing boxes." Roughly speaking, when the conditions of Lemma 14 are satisfied we are able to induct on the number of boxes by removing one box at a time. We note that Lemma 14 is applicable even to certain distributions with unbounded support, which will be useful for the applications in Appendix B.

Informally, Lemma 14 deals with the scenario where two density functions, \( g \) and \( h \), are both nonzero only on some set \( C \setminus R \), where \( R \) is a decreasing subset of \( C \). This setup is motivated by Figure 6-1, where \( C = [a_{1}, M] \times [s(b_{1}), M] \) and \( R = C \cap Z \). To prove that \( g \succeq_{1} h \), it suffices to verify that (1) \( g - h \) has an appropriate form (2) the integral of \( g - h \) on \( C \) is positive and (3) if we integrate \( g - h \) along either a vertical or horizontal line outwards starting from any point in \( R \), the result is negative.

Before we prove Lemma 14, we begin with the standard result that a sufficient condition for first-order stochastic dominance is that one measure assigns more mass than the other to all increasing sets.
Claim 10. Let $\alpha, \beta$ be positive finite Radon measures on $\mathbb{R}^n_{\geq 0}$ with $\alpha(\mathbb{R}^n_{\geq 0}) = \beta(\mathbb{R}^n_{\geq 0})$. A necessary and sufficient condition for $\alpha \succeq_1 \beta$ is that for all increasing\footnote{An increasing set $A \subset \mathbb{R}^n_{\geq 0}$ satisfies the property that for all $a, b \in \mathbb{R}^n_{\geq 0}$ such that $a$ is component-wise greater than or equal to $b$, if $b \in A$ then $a \in A$ as well.} measurable sets $A$, $\alpha(A) \geq \beta(A)$.

Proof of Claim 10: Without loss of generality assume that $\alpha(\mathbb{R}^n_{\geq 0}) = \beta(\mathbb{R}^n_{\geq 0}) = 1$.

It is obvious that the condition is necessary by considering the indicator function of any increasing set $A$. To prove sufficiency, suppose that the condition holds and that on the contrary, $\alpha$ does not stochastically dominate $\beta$. Then there exists an increasing, bounded, measurable function $f$ such that

$$
\int f \beta - \int f \alpha > 2^{-k+1}
$$

for some positive integer $k$. Without loss of generality, we may assume that $f$ is nonnegative, by adding the constant of $-f(0)$ to all values. We now define the function $\tilde{f}$ by point-wise rounding $f$ upwards to the nearest multiple of $2^{-k}$. Clearly $\tilde{f}$ is increasing, measurable, and bounded. Furthermore, we have

$$
\int \tilde{f} \beta - \int \tilde{f} \alpha \geq \int f \beta - \int f \alpha - 2^{-k} > 2^{-k+1} - 2^{-k} > 0.
$$

We notice, however, that $\tilde{f}$ can be decomposed into the weighted sum of indicator functions of increasing sets. Indeed, let $\{r_1, \ldots, r_m\}$ be the set of all values taken by $\tilde{f}$, where $r_1 > r_2 > \cdots > r_m$. We notice that, for any $s \in \{1, \ldots, m\}$, the set $A_s = \{z : \tilde{f}(z) \geq r_s\}$ is increasing and measurable. Therefore, we may write

$$
\tilde{f} = \sum_{s=1}^{m} (r_s - r_{s-1}) I_s
$$

where $I_s$ is the indicator function for $A_s$ and where we set $r_0 = 0$. We now compute

$$
\int \tilde{f} \beta = \sum_{s=1}^{m} (r_s - r_{s-1}) \beta(A_s) \leq \sum_{s=1}^{m} (r_s - r_{s-1}) \alpha(A_s) = \int \tilde{f} \alpha,
$$

where
contradicting the fact that $\int \tilde{f} d\beta > \int \tilde{f} d\alpha$. \qed

Due to Claim 10, to verify that a measure $\alpha$ stochastically dominates $\beta$ in the first order, we must ensure that $\alpha(A) \geq \beta(A)$ for all increasing measurable sets $A$. This verification might still be difficult, since an increasing set can have fairly unconstrained structure. In Lemma 13 we simplify this task by showing that we need not verify the inequality for all increasing $A$, but rather only for a special class of increasing subsets.

**Definition 17.** For any $z \in \mathbb{R}_{\geq 0}^{n}$, we define the base rooted at $z$ to be

$$B_z \triangleq \{ z' : z' \text{ is component-wise greater than or equal to } z \},$$

the minimal increasing set containing $z$.

We denote by $Q_k$ to be the set of points in $\mathbb{R}_{\geq 0}^{n}$ with all coordinates multiples of $2^{-k}$.

**Definition 18.** An increasing set $S$ is $k$- discretized if $S = \bigcup_{z \in S \cap Q_k} B_z$. A corner $c$ of a $k$- discretized set $S$ is a point $c \in S \cap Q_k$ such that there does not exist $z \in S \setminus \{ c \}$ with $z$ component-wise less than or equal to $c$.

**Lemma 12.** Every $k$- discretized set $S$ has only finitely many corners. Furthermore, $S = \bigcup_{c \in C} B_c$, where $C$ is the collection of corners of $S$.

**Proof of Lemma 12:** We prove that there are finitely many corners by induction on the dimension, $n$. In the case $n = 1$ the result is obvious, since if $S$ is nonempty it has exactly one corner. Now suppose $S$ has dimension $n$. Pick some corner $\hat{c} = (c_1, \ldots, c_n) \in S$. We know that any other corner must be strictly less than $\hat{c}$ in some coordinate. Therefore,

$$|C| \leq 1 + \sum_{i=1}^{n} |\{ c \in C \text{ s.t. } c_i < \hat{c}_i \}| = 1 + \sum_{i=1}^{n} \sum_{j=1}^{2^k \hat{c}_i} |\{ c \in C \text{ s.t. } c_i = \hat{c}_i - 2^{-k} j \}|.$$
By the inductive hypothesis, we know that each set \( \{ c \in C \text{ s.t. } c_i = \hat{c}_i - 2^{-kj} \} \) is finite, since it is contained in the set of corners of the \((n - 1)\)-dimensional subset of \( S \) whose points have \( i^{th} \) coordinate \( \hat{c}_i - 2^{-kj} \). Therefore, \(|C|\) is finite.

To show that \( S = \bigcup_{c \in \mathcal{C}} B_c \), pick any \( z \in S \). Since \( S \) is \( k \)-discretized, there exists a \( b \in S \cap Q_k \) such that \( z \in B_b \). If \( b \) is a corner, then \( z \) is clearly contained in \( \bigcup_{c \in \mathcal{C}} B_c \).

If \( b \) is not a corner, then there is some other point \( b' \in S \cap Q_k \) with \( b' \) component-wise less than or equal to \( b \). If \( b' \) is a corner, we’re done. Otherwise, we repeat this process at most \( 2^k \sum_j b_j \) times, after which time we will have reached a corner \( c \) of \( S \). By construction, we have \( z \in B_c \), as desired. \( \square \)

We now show that, to verify that one measure dominates another on all increasing sets, it suffices to verify that this holds for all sets that are the union of finitely many bases.

**Lemma 13.** Let \( g, h : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0} \) be bounded integrable functions such that \( \int_{\mathbb{R}^n_{\geq 0}} g(x)dx \) and \( \int_{\mathbb{R}^n_{\geq 0}} h(x)dx \) are finite. Suppose that, for all finite collections \( Z \) of points in \( \mathbb{R}^n_{\geq 0} \), we have

\[
\int_{\bigcup_{z \in Z} B_z} g(x)dx \geq \int_{\bigcup_{z \in Z} B_z} h(x)dx.
\]

Then for all increasing sets \( A \subseteq \mathbb{R}^n_{\geq 0} \),

\[
\int_A g(x)dx \geq \int_A h(x)dx.
\]

**Proof of Lemma 13:** Let \( A \) be an increasing set. We clearly have \( A = \bigcup_{z \in A} B_z \). For any point \( z \in \mathbb{R}^n_{\geq 0} \), denote by \( z^{n,k} \) the point in \( \mathbb{R}^n_{\geq 0} \) such that for each component \( i \), the \( i^{th} \) component of \( z^{n,k} \) is the maximum of 0 and \( z_i - 2^{-k} \).

We define the following two sets, which we think of as approximations of \( A \):

\[
A^l_k \triangleq \bigcup_{z \in A \cap Q_k} B_z; \quad A^u_k \triangleq \bigcup_{z \in A \cap Q_k} B_{z^{n,k}}.
\]

It is clear that both \( A^l_k \) and \( A^u_k \) are \( k \)-discretized. Furthermore, for any \( z \in A \) there exists a \( z' \in A \cap Q_k \) such that each component of \( z' \) is at most \( 2^{-k} \) more than the
corresponding component of $z$. Therefore $A_k^l \subseteq A \subseteq A_k^c$.

We now will bound
\[
\int_{A_k^c} g(x)dx - \int_{A_k^l} g(x)dx.
\]

Let
\[
W_k = \{ z \in \mathbb{R}^n_{\geq 0} : z_i > k \text{ for some } i \}; \quad W_k^c = \{ z \in \mathbb{R}^n_{\geq 0} : z_i \leq k \text{ for all } i \}.
\]

The set $W_k^c$ contains all points which are lie inside in a box of side length $k$ rooted at the origin, and $W_k$ contains all points outside of this box. We have the immediate (loose) bound that
\[
\int_{A_k^c \cap W_k} gdx - \int_{A_k^l \cap W_k} gdx \leq \int_{W_k} gdx.
\]

Furthermore, since $\lim_{k \to \infty} \int_{W_k^c} gdx = \int_{\mathbb{R}^n_{>0}} gdx$, we know that $\lim_{k \to \infty} \int_{W_k} gdx = 0$. Therefore,
\[
\lim_{k \to \infty} \left( \int_{A_k^c \cap W_k} gdx - \int_{A_k^l \cap W_k} gdx \right) = 0.
\]

Next, we bound
\[
\int_{A_k^c \cap W_k^c} gdx - \int_{A_k^l \cap W_k^c} gdx \leq |g|_{\text{sup}} \left( V(A_k^c \cap W_k^c) - V(A_k^l \cap W_k^c) \right)
\]
where $|g|_{\text{sup}} < \infty$ is the supremum of $g$, and $V(\cdot)$ denotes the Lebesgue measure.

For each $m \in \{1, \ldots, n+1\}$ and $z \in \mathbb{R}^n_{\geq 0}$, we define the point $z^{m,k}$ by:
\[
z^{m,k}_i = \begin{cases} 
\max\{0, z_i - 2^{-k}\} & \text{if } i < m \\
z_i & \text{otherwise}
\end{cases}
\]
and set
\[
A_k^m \triangleq \bigcup_{z \in A^\infty \cap Q_k} B_{z^{m,k}}.
\]
We have, by construction, $A^i_k = A^j_k$ and $A^u_k = A^{u+1}_k$. Therefore,

$$V(A^u_k \cap W^c_k) - V(A^i_k \cap W^c_k) = \sum_{m=1}^{n} (V(A^{m+1}_k \cap W^c_k) - V(A^m_k \cap W^c_k)).$$

We notice that, for any point $(z_1, z_2, \ldots, z_{m-1}, z_{m+1}, \ldots, z_n) \in [0, k]^{n-1}$, there is an interval $I$ of length at most $2^{-k}$ such that

$$(z_1, z_2, \ldots, z_{m-1}, w, z_{m+1}, \ldots, z_n) \in (A^{m+1}_k \setminus A^m_k) \cap W^c_k$$

if and only if $w \in I$. Therefore,

$$V(A^{m+1}_k \cap W^c_k) - V(A^m_k \cap W^c_k) \leq \int_0^k \cdots \int_0^k \cdots \int_0^k 2^{-k} dz_1 \cdots dz_{m-1} dz_{m+1} \cdots dz_n = 2^{-k} k^{n-1}.$$ 

We thus have the bound

$$|g|_{\sup} (V(A^u_k \cap W^c_k) - V(A^i_k \cap W^c_k)) \leq |g|_{\sup} \sum_{m=1}^{n} 2^{-k} k^{n-1} = n |g|_{\sup} 2^{-k} k^{n-1}$$

and therefore

$$\int_A g dx - \int_{A^i_k} g dx = \int_{A^u_k \setminus W_k} g dx - \int_{A^i_k \cap W_k} g dx + \int_{A^u_k \cap W^c_k} g dx - \int_{A^i_k \cap W^c_k} g dx \leq \left( \int_{A^u_k \cap W_k} g dx - \int_{A^i_k \cap W_k} g dx \right) + n |g|_{\sup} 2^{-k} k^{n-1}.$$ 

In particular, we have

$$\lim_{k \to \infty} \left( \int_{A^u_k} g dx - \int_{A^i_k} g dx \right) = 0.$$ 

Since $\int_{A^u_k} g dx \geq \int_A g dx \geq \int_{A^i_k} g dx$, we have

$$\lim_{k \to \infty} \int_{A^u_k} g dx = \int_A g dx = \lim_{k \to \infty} \int_{A^i_k} g dx.$$
Similarly, we have

$$\int_A h dx = \lim_{k \to \infty} \int_{A'_k} h dx$$

and thus

$$\int_A (g - h) dx = \lim_{k \to \infty} \left( \int_{A'_k} g dx - \int_{A'_k} h dx \right).$$

Since $A'_k$ is $k$-discretized, it has finitely many corners. Letting $Z_k$ denote the corners of $A'_k$, we have $A'_k = \bigcup_{z \in Z_k} B_z$, and thus by our assumption $\int_{A'_k} g dx - \int_{A'_k} h dx \geq 0$ for all $k$. Therefore $\int_A (g - h) dx \geq 0$, as desired. \qed

We are now ready to state and prove Lemma 14.

**Lemma 14.** Let $C = [p_1, q_1] \times [p_2, q_2] \subseteq \mathbb{R}^2_{\geq 0}$ where $q_1$ and $q_2$ are possibly infinite, let $R$ be a decreasing nonempty subset of $C$, and let $g, h : C \to \mathbb{R}_{\geq 0}$ be bounded integrable functions which are $0$ on $R$, satisfy $\int_C g(x, y) dxdy < \infty$ and $\int_C h(x, y) dxdy < \infty$, and satisfy

- $\int_C (g - h) dxdy \geq 0$.
- For any basis vector $e_i \in \{(0, 1), (1, 0)\}$ and any point $z^* \in R$:
  $$\int_0^{q_i - z_i^*} g(z^* + \tau e_i) - h(z^* + \tau e_i) d\tau \leq 0.$$
- There exist non-negative functions $\alpha : [p_1, q_1] \to \mathbb{R}_{\geq 0}$, $\beta : [p_2, q_2] \to \mathbb{R}_{\geq 0}$ and an increasing function $\eta : C \to \mathbb{R}$ such that
  $$g(z_1, z_2) - h(z_1, z_2) = \alpha(z_1) \cdot \beta(z_2) \cdot \eta(z_1, z_2)$$
  for all $(z_1, z_2) \in C \setminus R$.

Then $\kappa \succeq \lambda$, where $\kappa$ and $\lambda$ are the measures corresponding to the density functions $g$ and $h$ respectively.

**Proof of Lemma 14:**
We begin by defining, for any $a$ and $b$ with $p_1 \leq a \leq b \leq q_1$, the function $\zeta^b_a: [p_2, q_2] \to \mathbb{R}$ by

$$
\zeta^b_a(w_2) \triangleq \int_a^b (g(z_1, w_2) - h(z_1, w_2))dz_1.
$$

This function $\zeta^b_a(w_2)$ represents the integral of $g - h$ along the vertical line from $(a, w_2)$ to $(b, w_2)$.

**Claim 11.** If $(a, w_2) \in R$, then $\zeta^b_a(w_2) \leq 0$.

**Proof of Claim 11:** The inequality trivially holds unless there exists a $z_1 \in [a, b]$ such that $g(z_1, w_2) > h(z_1, w_2)$, so suppose such a $z_1$ exists. It must be that $(z_1, w_2) \notin R$, since both $g$ and $h$ are 0 in $R$. Indeed, because $R$ is a decreasing set it is also true that $(z_1, w_2) \notin R$ for all $z_1 \geq z_1$. This implies by our assumption that

$$
g(z_1, w_2) - h(z_1, w_2) = \alpha(z_1) \cdot \beta(w_2) \cdot \eta(z_1, w_2),
$$

for all $z_1 \geq z_1$. Given that $g(z_1, w_2) > h(z_1, w_2)$ and that $\eta(\cdot, w_2)$ is an increasing function, we know that $g(\tilde{z}_1, w_2) \geq h(\tilde{z}_1, w_2)$ for all $\tilde{z}_1 \geq z_1$. Therefore, we have

$$
\zeta^{z_1}_a(w_2) \leq \zeta^b_a(w_2) \leq \zeta^{q_1}_a(w_2).
$$

We notice, however, that $\zeta^{q_1}_a(w_2) \leq 0$ by assumption, and thus the claim is proven. 

We now claim the following:

**Claim 12.** Suppose that $\zeta^b_a(w^*_{2}) > 0$ for some $w^*_{2} \in [c_2, q_2)$. Then $\zeta^b_a(w_2) \geq 0$ for all $w_2 \in [w^*_2, q_2)$. 

**Proof of Claim 12:** Given that $\zeta^b_a(w^*_{2}) > 0$, our previous claim implies that $(a, w^*_2) \notin R$. Furthermore, since $R$ is a decreasing set and $w_2 \geq w^*_2$, follows that $(a, w_2) \notin R$, and furthermore that $(c, w_2) \notin R$ for any $c \geq a$ in $[c_1, q_1)$. Therefore, we may write

$$
\zeta^b_a(w_2) = \int_a^b (g(z_1, w_2) - h(z_1, w_2))dz_1 = \int_a^b (\alpha(z_1) \cdot \beta(w_2) \cdot \eta(z_1, w_2))dz_1.
$$
Similarly, \((c, w^*_2) \not\in R\) for any \(c \geq a\), so

\[
\zeta^b_a(w^*_2) = \int_a^b (\alpha(z_1) \cdot \beta(w^*_2) \cdot \eta(z_1, w^*_2)) dz_1.
\]

Note that, since \(\zeta^b_a(w^*_2) > 0\), we have \(\beta(w^*_2) > 0\). Thus, since \(\eta\) is increasing,

\[
\zeta^b_a(w^*_2) = \int_a^b (\alpha(z_1) \cdot \beta(w^*_2) \cdot \eta(z_1, w^*_2)) dz_1 = \frac{\beta(w^*_2)}{\beta(w^*_2)} \zeta^b_a(w^*_2) \geq 0,
\]

as desired. \(\square\)

We extend \(g\) and \(h\) to all of \(\mathbb{R}^2_0\) by setting them to be 0 outside of \(C\). By Claim 13, to prove that \(g \geq h\) it suffices to prove that \(\int_A g dx dy \geq \int_A h dx dy\) for all sets \(A\) which are the union of finitely many bases. Since \(g\) and \(h\) are 0 outside of \(C\), it suffices to consider only bases \(B_{z'}\) where \(z' \in C\), since otherwise we can either remove the base (if it is disjoint from \(C\)) or can increase the coordinates of \(z'\) moving it to \(C\) without affecting the value of either integral.

We now complete the proof of Lemma 14 by induction on the number of bases in the union.

- **Base Case.**

We aim to show \(\int_{B_r} (g - h) dx dy \geq 0\) for any \(r = (r_1, r_2) \in C\). We have

\[
\int_{B_r} (g - h) dx dy = \int_{r_2}^{q_2} \int_{r_1}^{q_1} (g - h) dz_1 dz_2 = \int_{r_2}^{q_2} \zeta^{q_1}_{r_1}(z_2) dz_2.
\]

By Claim 12, we know that either \(\zeta^{q_1}_{r_1}(z_2) \geq 0\) for all \(z_2 \geq r_2\), or \(\zeta^{q_1}_{r_1}(z_2) \leq 0\) for all \(z_2\) between \(p_2\) and \(r_2\). In the first case, the integral is clearly nonnegative, so we may assume that we are in the second case. We then have

\[
\int_{r_2}^{q_2} \zeta^{q_1}_{r_1}(z_2) dz_2 \geq \int_{p_2}^{q_2} \zeta^{q_1}_{r_1}(z_2) dz_2 = \int_{p_2}^{q_2} \int_{r_1}^{q_1} (g - h) dx dy dz_2 = \int_{r_1}^{q_1} \int_{p_2}^{q_2} (g - h) dx dy dz_2 dz_1.
\]

By an analogous argument to that above, we know that either \(\int_{p_2}^{q_2} (g-h)(z_1, z_2) dz_2\) is nonnegative for all \(z_1 \geq r_1\) (in which case the desired inequality holds triv-
ially) or is nonpositive for all \( z_1 \) between \( p_1 \) and \( r_1 \). We assume therefore that we are in the second case, and thus

\[
\int_{p_1}^{q_1} \int_{p_2}^{q_2} (g - h)dz_2dz_1 \geq \int_{p_1}^{q_1} \int_{p_2}^{q_2} (g - h)dz_2dz_1 = \int_c (g - h)dxdy,
\]

which is nonnegative by assumption.

- **Inductive Step.** Suppose that we have proven the result for all sets which are finite unions of at most \( k \) bases. Consider now a set

\[
A = \bigcup_{i=1}^{k+1} B_{z(i)}.
\]

We may assume that all \( z^{(i)} \) are distinct and that there do not exist distinct \( z^{(i)} \), \( z^{(j)} \) with \( z^{(i)} \) component-wise less than \( z^{(j)} \), since otherwise we could remove one such \( B_{z(i)} \) from the union without affecting the set \( A \) and the desired inequality would follow from the inductive hypothesis.

We may therefore order the \( z^{(i)} \) such that

\[
p_1 \leq z^{(k+1)}_1 < z^{(k)}_1 < z^{(k-1)}_1 < \cdots < z^{(1)}_1.
\]

\[
p_2 \leq z^{(1)}_2 < z^{(2)}_2 < z^{(3)}_2 < \cdots < z^{(k+1)}_2.
\]

By Claim 12, we know that one of the two following cases must hold:

- **Case 1:** \( \zeta^{(k)}_{z_1^{(k+1)}}(w_2) \leq 0 \) for all \( p_2 \leq w_2 \leq z_2^{(k+1)} \).

In this case, we see that

\[
\int_{z_2^{(k+1)}}^{z_2^{(k)}} \int_{z_1^{(k+1)}}^{z_1^{(k)}} (f - g)dz_1dz_2 = \int_{z_2^{(k+1)}}^{z_2^{(k+1)}} \zeta^{(k)}_{z_2^{(k+1)}}(w)dw \leq 0.
\]

For notational purposes, we denote here by \( (f - g)(S) \) the integral \( \int_S (f - g)(S) \).
Figure 6-2: We show that either decreasing $z_2^{(k+1)}$ to $z_2^{(k)}$ or removing $z^{(k+1)}$ entirely decreases the value of $\int_A (f-g)$. In either case, we can apply our inductive hypothesis.

\[ (f - g)(A) \geq (f - g)(A) \]
\[ + (f - g) \left( \left\{ z : z_1^{(k+1)} \leq z_1^{(k)} \leq z_2^{(k)} \leq z_2 \right\} \right) \]
\[ = (f - g) \left( \bigcup_{i=1}^{k} B_{z_1^{(i)}} \cup B_{(z_1^{(k+1)}, z_2^{(k)})} \right) \]
\[ = (f - g) \left( \bigcup_{i=1}^{k-1} B_{z_1^{(i)}} \cup B_{(z_1^{(k+1)}, z_2^{(k)})} \right) \]

where the last equality follows from $(z_1^{(k)}, z_2^{(k)})$ being component-wise greater than or equal to $(z_1^{(k+1)}, z_2^{(k)})$. The inductive hypothesis implies that the quantity in the last line of the above derivation is $\geq 0$.

- **Case 2:** $\zeta_{z_1^{(k+1)}}^{(k)}(w_2) \geq 0$ for all $w_2 \geq z_2^{(k+1)}$.

In this case, we have

\[ \int_{z_2^{(k+1)}}^{q_2} \int_{z_1^{(k+1)}}^{z_1^{(k)}} (f - g) dz_1 dz_2 = \int_{z_2^{(k+1)}}^{q_2} \zeta_{z_1^{(k+1)}}^{(k)}(w) dw \geq 0. \]
Therefore, it follows that

\[(f - g)(A) = (f - g)\left(\bigcup_{i=1}^{k} B_{z(i)}\right) + (f - g)\left(\left\{ z : z_{1}^{(k+1)} \leq z_{1}^{(k)} \leq z_{2}^{(k+1)} \leq z_{2}\right\}\right) \geq (f - g)\left(\bigcup_{i=1}^{k} B_{z(i)}\right) \geq 0,\]

where the final inequality follows from the inductive hypothesis.

\[\square\]

### 6.2.2 Examples

We apply Theorem 4 (combined with Lemma 14) to solve example instances of optimal mechanism design.

#### Examples with Unbounded Support

Theorem 1, Lemma 2, and Theorem 4 can be extended to some instances with unbounded type spaces. We present two such examples below, and provide a sketch of the ideas in Appendix B.

In Example 3, the optimal mechanism is a grand bundling mechanism. This uses an analog of Theorem 4 (for unbounded type spaces) in the degenerate case in which \(A\) and \(B\) are empty.\(^8\)

**Example 3.** The optimal IC and IR mechanism for selling two goods whose values are distributed independently according to the probability densities 
\[f_1(z_1) = \frac{5}{1+z_1}^6\]
and 
\[f_2(z_2) = \frac{6}{1+z_2}^7\] respectively is a take-it-or-leave-it offer of the bundle of the two goods for price \(p^* \approx .35725\).

Example 4 provides a complete solution for the optimal mechanism for two items which are distributed according to independent exponential distributions. The proof is in Appendix B and uses Lemma 14 and an analog of Theorem 4.

\(^8\)This can also be thought of as an unbounded analog of Theorem 3.
Example 4. For all $\lambda_1 \geq \lambda_2 > 0$, the optimal IC and IR mechanism for an additive bidder whose values for two items are distributed according to independent exponential distributions $f_1$ and $f_2$ with respective parameters $\lambda_1$ and $\lambda_2$ offers the following menu:

1. Receive nothing, pay 0.
2. Receive the first item with probability 1 and the second item with probability $\lambda_2/\lambda_1$, pay $2/\lambda_1$.
3. Receive both items, pay $p^*$.

where $p^*$ is the unique $0 < p^* \leq 2/\lambda_2$ such that

$$\mu(\{y \in \mathbb{R}_{\geq 0}^2 : y_1 + y_2 \leq p^* \text{ and } \lambda_1 y_1 + \lambda_2 y_2 \leq 2\}) = 0,$$

where $\mu$ is the transformed measure of the joint distribution.

Figure 6-3: The canonical partition of $\mathbb{R}_{\geq 0}^2$ for the proof of Example 4. In this diagram, $p^* > 2/\lambda_1$. If $p^* \leq 2/\lambda_1$, $B$ is empty.

An Optimal Auction with Infinite Menu Size

In this example we obtain a description of the optimal mechanism for two items distributed according to the beta distributions shown below. Our approach, incorpo-
rating numerical techniques similar to those used in our power law example, illustrates a general recipe for employing Theorem 4 to find closed-form descriptions of optimal mechanisms. Our approach contains the following steps: (i) definition of the sets $S_{\text{top}}$ and $S_{\text{right}}$, (ii) computation of a critical price $p^*$, (iii) definition of a canonical partition in terms of (i) and (ii), and (iv) application of Theorem 4.

It is noteworthy that the optimal auction in this example offers the bidder a choice of infinitely many randomized allocations. Our approach nevertheless provides us with a technique to efficiently compute and specify this mechanism. Our example consists of two items whose values are distributed independently according to the following two density functions:

$$f_1(z_1) = \frac{1}{B(3,3)} z_1^2 (1 - z_1)^2; \quad f_2(z_2) = \frac{1}{B(3,4)} z_2^3 (1 - z_2)^3$$

for all $z_i \in (0, 1)$, where $B(\cdot, \cdot)$ is the "beta function" and is used for normalization.

We calculate $-\nabla f(z) \cdot z - 3 f(z) = f_1(z_1) f_2(z_2) \left( \frac{2}{1 - z_1} + \frac{3}{1 - z_2} - 12 \right)$ and $f(z)z = 0$ at $z_i = 0$ or $z_i = 1$, and thus the transformed measure $\mu$ is comprised of:

- A point mass of +1 at the origin.
- Mass distributed on $[0, 1)^2$ according to the density function

$$f_1(x_1) f_2(x_2) \left( \frac{2}{1 - x_1} + \frac{3}{1 - x_2} - 12 \right).$$

Note that the density of $\mu$ is positive on the set $\mathcal{X} = \left\{ z \in [0, 1)^2 : \frac{2}{1 - z_1} + \frac{3}{1 - z_2} > 12 \right\}$ and non-positive on $\mathcal{Y} = \left\{ z \in [0, 1)^2 \setminus \{0\} : \frac{2}{1 - z_1} + \frac{3}{1 - z_2} \leq 12 \right\}$, and that $\mathcal{Y} \cup \{0\}$ is a decreasing set.

**Step (i).** We define the set $S_{\text{top}} \subset [0, 1)^2$ by the rule that $(z_1, z_2) \in S_{\text{top}}$ if

$$\int_{z_2}^{1} (-\nabla f(z_1, t) \cdot (z_1, t) - (n + 1) f(z_1, t)) \, dt = 0.$$  

That is, starting from any point in $z \in S_{\text{top}}$ and integrating the density of $\mu$ "upwards" from $t = z_2$ to $t = 1$ yields zero, while starting above $S_{\text{top}}$ and integrating upwards yields a positive quantity.\(^9\) Simi-

\(^9\)Indeed, observe that $S_{\text{top}} \subset \mathcal{Y}$.
larly, we say that \((z_1, z_2) \in S_{\text{right}}\) if \(\int_{z_1}^{1} (-\nabla f(t, z_2) \cdot (t, z_2) - (n + 1) f(t, z_2)) dt = 0\).\(^{10}\)

Since \(X\) is an increasing set, \(S_{\text{top}}\) and \(S_{\text{right}}\) are both subsets of \(Y\). We compute analytically that \((z_1, z_2) \in [0, 1]^2\) is in \(S_{\text{top}}\) if and only if

\[
z_1 = \frac{2(-1 - 3z_2 - 6z_2^2 + 25z_2^3)}{3(-1 - 3z_2 - 6z_2^2 + 20z_2^3)}.
\]

Similarly, \((z_1, z_2) \in [0, 1]^2\) is in \(S_{\text{right}}\) if and only if \(z_2 = \frac{2(-2 - 4z_2 - 52z_2^2 + 27z_2^3 + 6z_2^4)}{7 - 4z_2 - 21z_2^2 + 72z_2^3}.\)

In particular, for any \(z_1 \in [0, .63718]\) there exists a \(z_2\) such that \((z_1, z_2) \in S_{\text{right}}\), and there does not exist such a \(z_2\) if \(z_1 > .63718\). Furthermore, it is straightforward to verify (by computing second derivatives in the appropriate regime) that the region below \(S_{\text{top}}\) and the region below \(S_{\text{right}}\) are convex.

**Step (ii).** We now compute \(p^* = .71307\) (this is the \(z_2\)-intercept of the 45° line in Figure 6-4 which causes \(\mu(Z) = 0\)) and define the set \(L = \{z \in [0, 1]^2 : z_1 + z_2 = p^*\}\). We compute that \(L \cap S_{\text{top}}\) contains the point (.16016, .55291) and that \(L \cap S_{\text{right}}\) contains the point (.62307, 0.09). We now define the curve \(s : [0, .63718] \rightarrow [0, 1]\) by

\[
s(z_1) = \begin{cases} 
    z_2 \text{ such that } (z_1, z_2) \in S_{\text{top}} & \text{if } 0 \leq z_1 \leq .16016 \\
    .71307 - z_1 & \text{if } .16016 \leq z_1 \leq .62307 \\
    z_2 \text{ such that } (z_1, z_2) \in S_{\text{right}} & \text{if } .62307 \leq z_1 \leq .63718.
\end{cases}
\]

It is straightforward to verify that \(s\) is a concave, decreasing, continuous function, and is the outer boundary function of the zero set \(Z\) shown in Figure 6-4.

**Step (iii).** We decompose \([0, 1]^2\) into the following regions:

\[
Z = \{z : z_1 \leq 0.63718 \text{ and } z_2 \leq s(z_1)\}; \quad A = ([0, 0.16016] \times (0, 1)) \setminus Z
\]

\[
B = ((0, 1] \times [0, 0.09] \setminus Z; \quad W = [0, 1]^2 \setminus (Z \cup A \cup B)
\]

as illustrated in Figure 6-4. This is the canonical partition with zero set \(Z\).

**Step (iv).** We claim that \(Z \cup A \cup B \cup W\) is a canonical partition and is well-formed with respect to \(\mu\). Recall that \(S_{\text{top}}\) and \(S_{\text{right}}\) are subsets of \(Y\). That is, \(\mu\) has\(^{10}\)

\(^{10}\)Note the similarities between the definitions of \(S_{\text{top}}\) and \(S_{\text{right}}\) to the definition of the absorption hyperplane from the exponential item example.
negative density along these curves at all points below either curve, other than at the origin. The only non-trivial condition before we can apply Theorem 4 is to verify $\mu|_{\mathcal{W}} \succeq_2 0$. In fact, we can apply Lemma 14 to conclude the stronger dominance relation $\mu|_{\mathcal{W}} \succeq_1 0$.

We set $C = [.16, 1) \times [.09, 1)$ and $R = Z \cap C$, so that $\mathcal{W} = C \setminus R$. We let $g$ and $h$ being the positive and negative parts of the density function of $\mu|_{\mathcal{W}}$, respectively, so the density of $\mu|_{\mathcal{W}}$ is given by $g - h$. Since $Z$ lies below both curves $S_{\text{top}}$ and $S_{\text{right}}$, we know that integrating the density of $\mu$ along any horizontal or vertical line outwards starting anywhere on the boundary of $Z$ yields a non-positive quantity. In addition, on $\mathcal{W} = C \setminus R$, we have

$$g(x_1, x_2) - h(x_1, x_2) = f_1(x_1)f_2(x_2)\left(\frac{2}{1-x_1} + \frac{3}{1-x_2} - 12\right)$$

which satisfies the conditions of Lemma 14, as $2/(1-x_1) + 3/(1-x_2) - 12$ is increasing.

Example 5. The optimal mechanism for selling independent goods whose valuations

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Figure 6-4: The well-formed canonical partition for $f_1(z_1) = \frac{z_1^2(1-z_1)^2}{B(3,3)}$ and $f_2(z_2) = \frac{z_2^2(1-z_2)^2}{B(3,4)}$.
are distributed according to \( f_1(z_1) = \frac{z_1^2(1-z_1)^2}{B(3,3)} \) and for \( f_2(z_2) = \frac{z_2^2(1-z_2)^3}{B(3,4)} \) has the following outcome for a bidder of type \((z_1, z_2)\) in terms of the function \( s(\cdot) \) defined above:

- If \((z_1, z_2) \in Z\), the bidder receives no goods and is charged 0.

- If \((z_1, z_2) \in A\), the bidder receives item 1 with probability \(-s'(z_1)\), item 2 with probability 1, and is charged \( s(z_1) - z_1s'(z_1) \).

- If \((z_1, z_2) \in B\), the bidder receives item 1 with probability 1, item 2 with probability \(-1/s'(s^{-1}(z_2))\), and is charged \( s^{-1}(z_2) - z_2/s'(s^{-1}(z_2)) \).

- If \((z_1, z_2) \in W\), the bidder receives both goods with probability 1 and is charged .71307.

Since \( s(z_1) \) is not linear for \( z_1 \in [0, .16016]\) and \( z_1 \in [.62307, .63718]\), Example 5 shows that an optimal mechanism might offer a continuum of randomized outcomes. Indeed, while this auction has infinite menu-size complexity (see [34]), using our techniques we can obtain a succinct and easily-computable description of the mechanism.
Chapter 7

The Complexity of Optimal Mechanism Design

Our goal in this chapter is to study the algorithmic complexity of the optimal mechanism design problem. The main result is Theorem 5 where we prove that, as the number of items becomes large, optimal mechanism design becomes computationally intractable even in one of the most basic settings.

7.1 Computational Aspects of Mechanism Design

While Theorem 2 ensures that every optimal mechanism has a dual certificate proving its optimality, it does not address the computational task of actually finding the mechanism. The optimal mechanism design problem has received considerable attention in the computer science community, and algorithmic ideas have provided useful insights. Algorithmic results typically have come in two flavors: approximations [13, 14, 7, 1, 37, 26, 33, 11], guaranteeing a constant fraction of the optimal revenue, and exact solutions [24, 9, 2, 10], guaranteeing full revenue extraction.

These results, however, do not pin down the computational complexity of the mechanism design problem. In the discrete setting, all known algorithms for exact mechanisms can be computed and implemented in time polynomial in the total number of valuations a bidder may have—i.e. the size of the support of each bidder's
valuation-distribution; they are computationally efficient only when the valuation distributions are provided explicitly in the specification of the problem by listing their support together with the probability assigned to each valuation in the support. However, this is not always the computationally meaningful way to describe the problem. The most trivial setting where this issue arises is that of a single additive quasilinear bidder whose values for the items are independent of support two. In this case, the bidder may have one of $2^n$ possible valuations, where $n$ is the number of items, and explicitly describing her valuation distribution would require $\Omega(2^n)$ vectors and probabilities. However, a mechanism with complexity polynomial in $\Omega(2^n)$ is clearly inefficient, and one would like a mechanism to run in time polynomial in the distribution’s natural description complexity, i.e. the bits required to specify the distribution’s $n$ marginals over the items.

To sum up, previous algorithmic work has provided solutions to the optimal mechanism design problem in broad multi-item settings [9, 2, 10], but these solutions fall short of characterizing the computational complexity of the problem, as do existing lower bound approaches [8, 52, 26, 21]. In this chapter, we answer the question of the complexity of optimal mechanism design by showing that the problem is computationally intractable, even in the most basic setting.

**Theorem 5.** There is no expected polynomial-time solution to the optimal mechanism design problem (formal definition in Section 7.2) unless $\text{ZPP} \supseteq \text{P}^\#P$.

In particular, it is $\#P$-hard to determine whether every optimal mechanism assigns a specific item to a specific type of bidder with probability 0 or with probability 1 (at the Bayesian Nash equilibrium of the mechanism), given the promise that one of these two cases holds simultaneously for all optimal mechanisms.

The above is true even in the case of a single additive quasilinear bidder whose values for the items are independently distributed on two rational numbers with rational probabilities.

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1 In contrast to the earlier chapters of Part II of this thesis, here we focus on instances in which a player's type space is discrete. This avoids complications of how to efficiently specify continuous distributions.
We present an algorithmic definition of the optimal mechanism design problem in Section 7.2. We remark that our formulation of optimal mechanism design given in Section 3.1 hides the algorithmic aspects of actually computing the allocation and price functions $\mathcal{P}$ and $\mathcal{T}$ induced by a mechanism $\mathcal{M} = (\mathcal{P}, \mathcal{T})$ and considers only direct mechanisms. While this formulation is without loss of generality in a non-algorithmic sense (as all mechanisms can be reduced to direct mechanisms by Myerson’s revelation principle [48]), for a fully general algorithmic theory of mechanism design we must carefully consider the details of how $\mathcal{P}$ and $\mathcal{T}$ are implemented.

Our algorithmic definition, in the single bidder setting, implies that any efficient solution enables us to exactly sample from the appropriate allocation distribution $\mathcal{P}(z)$ and to sample a randomized payment with expected value $\mathcal{T}(z)$ for any bidder type $z$. A single sample from $\mathcal{P}(z)$ would suffice to distinguish whether the bidder should receive a certain item with probability 0 or with probability 1. We note that our proof is subject to the assumption $\text{ZPP} \not\subseteq \text{P}^{\#P}$ (rather than $\text{P} \not\subseteq \text{P}^{\#P}$) solely because we prove lower bounds for randomized mechanisms.

The contribution of our result is two-fold. First, it gives a definitive proof that approximation is necessary for revenue optimization beyond Myerson's single-item setting, due to computational considerations, even in the ideal scenario where the value distributions are perfectly known. Approximation has been heavily used in algorithmic work on the problem, but there has been no justification for its use, at least in simple settings that don't induce combinatorial structure in the valuations of the bidders (sub-modular, OXS, etc.), or on the allocation constraints of the setting. Second, our result represents advancement in our techniques for showing lower bounds for optimal mechanism design. Despite evidence that the structure of the optimal mechanism is complicated even in simple settings, previous work has not been able to harness this structure to obtain computational hardness results.
7.2 Algorithmic Formalization of Optimal Mechanism Design

We describe now an algorithmic formalization of the optimal mechanism design problem. As we are aiming for a broad computational lower bound, in this chapter we place no constraints on the particular form of a sought-after mechanism. That is, we ensure that our model allows for exotic extensive-form mechanisms as well as any other conceivable type of mechanism. We note again that our prior formalization in Section 3.1 of restricting only to direct mechanisms is without loss of generality when ignoring computational issues, by Myerson’s revelation principle [48].

7.2.1 Input and Output of the Problem

We now describe the input and output of the optimal mechanism design problem.

INPUT: The input consists of the names of the items and the bidders, the allocation constraints of the setting (specifying, e.g., that an item may be allocated to at most one bidder, etc.), and a probability distribution on the types of the bidders. The type of a bidder incorporates information about how much she values every subset of the items as well as what utility she derives for receiving a subset at a particular price. For example, the type of an additive quasilinear bidder can be encapsulated in a vector of values (one value per item). We won’t make any assumptions about how the allocation constraints are specified. In general, these could either be hard-wired to a family of instances of the mechanism design problem, or provided as part of the input in a computationally meaningful way. For the purposes of our intractability results, the allocation constraints will be trivial, enforcing that we can only allocate at most one copy of each item. We restrict our attention to instances with precisely these allocation constraints. As far as the type distribution is concerned, we restrict our attention to additive quasilinear bidders with independent values for the items. So, for our purposes, the type distribution of a bidder is specified by giving its marginal on each item. We assume that each marginal is given explicitly, as a list of the possible
values for the item as well as the probabilities assigned to each value.²

**Desired Output:** The goal is to compute a (possibly randomized) auction that optimizes, over all possible auctions, the expected revenue of the auctioneer, i.e. the expected sum of prices paid by the bidders at the Bayes Nash equilibrium of the auction,³ where the expectation is taken with respect to the bidders' types, the randomness in their strategies (if any) at the Bayes Nash equilibrium, as well as any internal randomness that the auction uses.

### 7.2.2 Efficient Auction Computation and Simulation

There is a large universe of possible auctions with widely varying formats, e.g. sealed envelope auctions, dynamic auctions, all-pay auctions, etc. Furthermore, there may be many different auctions with the same expected revenue. As our goal is to prove robust intractability results, we take a general approach imposing no restrictions on the format of the auction and no restrictions on the way the auction is encoded. The encoding should, however, specify in a computationally meaningful way what actions are available to the bidders, how the items are allocated depending on the actions taken by the bidders, and what prices are charged to them, where both allocation and prices could be outputs of a randomized function of the bidders’ actions. Furthermore, the bidders should be able to efficiently compute their best strategies. In particular, any computationally efficient solution to a family $\mathcal{I}$ of optimal mechanism design problems⁴ induces a pair of algorithms $C$ and $S$ such that:

1. **[Auction computation]** $C$ is an expected polynomial-time algorithm mapping instances $I \in \mathcal{I}$ of the mechanism design problem to mechanism encodings. For

²There are of course other ways to describe these marginals. For example, we may only have sample access to them, or we may be given a circuit that takes as input a value and outputs the probability assigned to that value. As our goal is to prove lower bounds, and in our hardness proof our marginal distributions have support of size only two, the assumption that the marginals are provided explicitly in the input only makes the lower bounds stronger.

³If an auction has multiple Bayesian Nash equilibria, its revenue is not well-defined, as it may depend on what Bayesian Nash equilibrium the bidders end up playing. This complication won’t be relevant for our results, as all auctions we construct in our hardness proofs will have a unique Bayes Nash equilibrium.

⁴A family of mechanism design problems could consist of, for example, all single-bidder instances with $n$ items in which each item’s value distribution has support two.
example, $C(I)$ may be "second price auction", or "English auction with reserve price $5.""

2. \textit{[Auction simulation]} $S$ is an expected polynomial-time algorithm mapping instances $I \in \mathcal{I}$ of the mechanism design problem, encodings $C(I)$ of the optimal auction for $I$, and realized types $z_1, \ldots, z_m$ for the bidders, to a sample from the (possibly randomized) allocation and price rule of the auction encoded by $C(I)$ at the Bayes Nash equilibrium of the auction when the types of the bidders are $z_1, \ldots, z_m$.

Intuitively, an efficient solution to the optimal mechanism design problem (i) allows the designer to efficiently determine the mechanism $C(I)$ to use for instance $I$, (ii) enables the players of $C(I)$ to efficiently determine their BNE strategies, and (iii) allows for $C(I)$ to be executed even when all interactions are expected polynomial time. Property (i) implies the existence of the algorithm $C$, while properties (ii) and (iii) imply the existence of the algorithm $S$.

Let us discuss in a bit more detail why we require there to exist a simulator $S$. When the auction $C(I)$ is executed, somebody (either the auctioneer, or the bidders, or both) needs to do computation: the bidders need to decide how to play the auction (i.e. what actions from among the available ones to use), and the auctioneer needs to allocate the items and charge the bidders. In the special case of a direct Bayesian Incentive Compatible (direct) mechanism, the bidders need not do any computation, as it is a Bayes Nash equilibrium strategy for each of them to truthfully report their type to the auctioneer. In this case, all the computation is done by the auctioneer, who needs to sample from the (possibly randomized) allocation and price rule of the mechanism given the bidders’ reported types. In general (possibly non-direct or multi-stage) mechanisms, both bidders and auctioneer may need to do some computation: the bidders need to compute their Bayes Nash equilibrium strategies given their types, and the auctioneer needs to sample from the (possibly random) allocation and price rule of the mechanism given the bidders’ strategies. These computations must all be computationally efficient, as otherwise the execution of the auction $C(I)$ would not
be computationally efficient. Hence an efficient solution induces an efficient simulator $S$.

Remark 8. A proper study of the computational complexity of optimal mechanism design cannot drop the requirement that auction simulation be efficient. In particular, we remark that placing no computational restrictions on the bidders leads to spurious "efficient" solutions, as discussed in Section 7.8.2.

In view of the above discussion, Theorem 5 establishes that, even in very simple families $\mathcal{I}$ of mechanism design instances, there does not exist any pair $(C, S)$ of efficient auction computation and simulation algorithms for the optimal mechanism. That is, the optimal auction cannot be found computationally efficiently, or cannot be executed efficiently, or both.

7.2.3 Reduction to Randomized Direct Mechanisms

We now focus our attention to instances of a single additive quasilinear bidder. Any mechanism specifies a set $\mathcal{A}$ of actions available to the bidder together with a rule mapping each action $\alpha \in \mathcal{A}$ to a (possibly randomized) allocation $P_\alpha \in \{0, 1\}^n$, determining which items the bidder gets, and a (possibly randomized) price $T_\alpha \in \mathbb{R}$ that the bidder pays, where $P_\alpha$ and $T_\alpha$ could be correlated. Facing this mechanism, a bidder whose values for the items are instantiated to some vector $v \in \mathbb{R}^n$ chooses any action in $\arg\max_{\alpha \in \mathcal{A}} \{v \cdot E(P_\alpha) - E(T_\alpha)\}$ or any distribution on these actions, as long as the maximum is non-negative, since $v \cdot E(P_\alpha) - E(T_\alpha)$ is the expected utility of the bidder for choosing action $\alpha$.\(^5\) In particular, any such choice is a Bayesian Nash equilibrium behavior for the bidder. If for all vectors $v$ there is a unique optimal action $\alpha_v \in \mathcal{A}$ in the above optimization problem, then the mechanism induces a mapping from valuations $v$ to (possibly randomized) allocation and price pairs $(P_{\alpha_v}, T_{\alpha_v})$. If there are $v$'s with non-unique maximizers, then we break ties in favor of the action

\(^5\)If the maximum utility under $v$ is negative, the bidder would "stay home." To ease notation, we can include in $\mathcal{A}$ a special action "stay home" that results in the bidder getting nothing and paying nothing. If all other actions give negative utility, the bidder can use this special action.
with the highest $E(T_a)$ and, if there are still ties, lexicographically beyond that.\footnote{We can enforce this tie-breaking with an arbitrarily small hit on revenue as follows: For all $\alpha$, we decrease the (possibly random price) $T_a$ output by $M$ by a well-chosen amount—think of it as a rebate—which gets larger as $E(T_a)$ gets larger. We can choose these rebates to be sufficiently small so that they only serve the purpose of tie-breaking. These rebates won’t affect our lower bounds.}

In an efficient optimal mechanism design solution, the algorithms $C$ and $S$ allow us to efficiently sample from the distributions $(P_{\alpha}, T_{\alpha})$ of an optimal mechanism for type $\tilde{v}$ in instance $I$. That is, if a family $\mathcal{I}$ of instances has any efficient solution (of mechanisms of any form), then there exist expected polynomial time algorithms which, given an instance $I \in \mathcal{I}$ and type $v$ of bidder, output sample from the optimal allocation and price distributions. In particular, any efficient solution to the family of instances $\mathcal{I}$ implies the existence of a (possibly randomized) efficient direct mechanism for $I$.\footnote{This argument is essentially a computational variant of Myerson’s revelation principle [48].}

7.2.4 Non-Existence of Efficient Solutions

The hard instances $\mathcal{I}$ we use for our proof\footnote{While our hardness result applies to the family of all instances of a single additive bidder having two possible values for each item, in our hardness proof we narrow into a much smaller subclass of instances.} have the property that there is a unique price and allocation rule\footnote{The rules are unique up to expectations. Thus, any optimal mechanism charges a bidder of type $v$ the same amount in expectation, although different mechanisms can draw randomized prices from different distributions with this expectation. An analogous property holds for allocation functions.} for the optimal BIC and IR direct mechanism. Furthermore, for every such instance $I \in \mathcal{I}$ there is a particular item $i^*$ and type $v$ such that either all optimal mechanisms allocate $i^*$ to type $v$ with probability 1 or all mechanisms allocate it with probability 0. We prove that it is $\#P$-hard to determine which of these two cases is true.

If there were any efficient solution to the optimal mechanism design problem for the family of instances $\mathcal{I}$, then by Section 7.2.3 there would exist an expected polynomial time algorithm enabling us to sample from the optimal allocation distribution for a bidder of type $v$ in instance $I$. Using a single such sample, however, we could check whether or not good $i^*$ were allocated, thereby solving a $\#P$-hard problem in expected polynomial time. This establishes Theorem 5.
Remark 9 (Hardness of BIC Mechanisms). A lot of research on optimal mechanism design has focused on finding optimal Bayesian Incentive Compatible (BIC) mechanisms, as focusing on such mechanisms costs nothing in revenue due to the direct revelation principle (see [50] and Section 7.4). As an immediate corollary of Theorem 5 we obtain that it is #P-hard to compute the (possibly randomized) allocation and price rule of the optimal BIC mechanism. However, Theorem 5 is much broader, in two respects: (i) in the definition of the optimal mechanism design problem we impose no constraints on what type of auction should be found, and (ii) we don't require an explicit computation of the (possibly randomized) allocation and price rule of the mechanism, but allow an expected polynomial-time algorithm that samples from the allocation and price rule.

Remark 10. For the single-bidder instances we consider in this paper, a direct mechanism that is Bayesian Incentive Compatible is also Incentive Compatible and vice versa. As all our hardness results are for single-bidder instances, they simultaneously show the intractability of computing optimal Bayesian Incentive Compatible as well as optimal Incentive Compatible mechanisms.

7.3 Overview of Approach

There are serious obstacles to establishing intractability results for optimal mechanism design, the main one being the necessity of understanding the structure of optimal mechanisms. To prove Theorem 5, we must find a family of mechanism design instances whose optimal solutions are sufficiently complex to enable reductions from a #P-hard problem, while at the same time are sufficiently restricted so that solutions to the #P-hard problem could actually be extracted from the solution of the optimal mechanism.

We follow a discrete analog of the weak duality formulation of Appendix A\textsuperscript{10}: We start with a folklore, albeit exponentially large, linear program for revenue optimiza-

\textsuperscript{10}Very roughly, this approach is a discrete analog of the continuous-distribution optimal mechanism design program where we relax the constraint that a utility function be convex.
tion in a discrete setting, relax this LP, and show that, in a suitable class of instances, the solution of the relaxed LP is also a solution to the original LP. This solution has rich enough structure to embed a \#P-hard problem. In more detail, our approach is the following:

- In Section 7.5.1 we present LP1, the folklore, albeit exponentially large, linear program for computing a revenue optimal auction.

- In Section 7.5.2 we relax the constraints of LP1 to construct a new, still exponentially large, linear program LP2. The solutions of the relaxed LP need not provide solutions to the original mechanism design problem. We prove however that an optimal LP2 solution is indeed an optimal LP1 solution if it happens to be monotone and supermodular.

- In Section 7.5.3 we take LP3, the dual program to LP2. We interpret its solutions as solutions to a minimum-cost flow problem on a lattice.

- In Section 7.6.1 we characterize a canonical solution to a specific subclass of LP3 instances. This solution requires ordering of the subset sums of an appropriate set of integers.

- In Section 7.6.2 we use duality to convert a canonical LP3 solution to a unique LP2 solution. We are therefore able to characterize the unique solution for a variety of LP2 instances.

- In Section 7.6.3 we show that the LP2 solutions obtained above are also feasible and optimal for the corresponding LP1 instance. Thus, we gain the ability to characterize unique optimal solutions of a class of LP1 instances.

- In Section 7.7 we show how to encode a \#P-hard problem into the class of LP1 instances that we have developed.

As discussed above, we will encode of a \#P-hard problem into a mechanism design instance so that it is \#P-hard to determine whether an optimal mechanism should allocation a particular item with probability zero or with probability one.
7.4 Preliminaries and Notation

We restrict our attention to instances of the mechanism design problem where a seller wishes to sell a set \( N = \{1, 2, \ldots, n\} \) of items to a single additive quasilinear bidder whose values for the items are independent of support 2. In this chapter we will use notation which is tailored to this particular class of instances, and differs slightly from the more general notation introduced in Section 3.1: A bidder values item \( i \) at \( a_i \) with probability \( 1 - p_i \), and at \( a_i + d_i \), with probability \( p_i \), independently of the other items, where \( a_i, d_i, \) and \( p_i \) are positive rational numbers. If she values \( i \) at \( a_i \), we say that her value for \( i \) is “low” and, if she values it at \( a_i + d_i \), we say that her value is “high.” The specification of the instance comprises the numbers \( \{a_i, d_i, p_i\}_{i=1}^n \).

As discussed above, it suffices for us to study the structure of direct mechanisms, where the action set available to the bidder coincides with her type space \( \times_i\{a_i, a_i + d_i\} \). We find it convenient in our particular family of instances, instead of viewing the type space as a collection of vectors over \( \mathbb{R}^n \) (as we have done up until this point) to instead view the type space as the collection of subsets \( S \subseteq N \). With this notation, the type \( S \) of a bidder is the set of items \( i \) which the bidder values at the “high” amount \( a_i + d_i \).

For a bidder of type \( S \), we denote the bidder’s type vector by \( \vartheta(S) \triangleq \sum_{i \in N} a_i e_i + \sum_{i \in S} d_i e_i \) where \( e_i \) is the unit vector in dimension \( i \).\(^{11}\) In a direct mechanism, a bidder is asked to declare a subset of \( N \), supposedly her true type. As in Section 3.1, for all actions \( S \subseteq N \) the mechanism induces a vector \( \mathcal{P}(S) \in [0, 1]^n \) of probabilities that the bidder receives each item and an expected price \( \mathcal{T}(S) \in \mathbb{R} \) that the bidder pays. The expected utility of a bidder of type \( S \) for choosing action \( T \) is given by \( \vartheta(S) \cdot \mathcal{P}(T) - \mathcal{T}(T) \). We denote by \( u(S) = \vartheta(S) \cdot \mathcal{P}(S) - \mathcal{T}(S) \) her expected utility for reporting her true type.

For clarity, we restate our definitions of incentive compatibility and individual rationality (see Section 3.1) using our new notation.

\textbf{Definition 19.} A direct mechanism for the family of single-bidder instances we con-
sider in this chapter is Bayesian Incentive Compatible (BIC) if the bidder cannot benefit by misreporting the set of items he values highly. Formally:

$$\forall S, T \subseteq N : \bar{v}(S) \cdot P(S) - T(S) \geq \bar{v}(S) \cdot P(T) - T(T).$$

Or, equivalently:

$$\forall S, T \subseteq N : u(S) \geq u(T) + (\bar{v}(S) - \bar{v}(T)) \cdot P(T).$$

**Definition 20.** The mechanism is individually rational (IR) if $$u(S) \geq 0$$ for all $$S \subseteq N$$.

To prove Theorem 5, we will narrow into a family of instances for which there is a unique optimal BIC, IR, direct mechanism, and this mechanism satisfies the following: for a special item $$i^*$$ and a special type $$S^*$$, we know that $$P_{i^*}(S^*) \in \{0, 1\}$$ but it is \#P-hard to decide whether $$P_{i^*}(S^*) = 0$$. Our approach was outlined in Section 7.3 and is provided in detail in Sections 7.5 through 7.7.

### 7.5 A Linear Programming Approach

Our goal in Sections 7.5 through 7.7 is to show computational intractability of computing a BIC, IR direct mechanism that maximizes the seller's expected revenue, even in the single-bidder setting introduced in Section 7.4. In this section, we define three exponential-size linear programs which are useful for zooming into a family of hard instances that are also amenable to analysis.

#### 7.5.1 Mechanism Design as a Linear Program

The optimal BIC and IR mechanism for the family of single-bidder instances introduced in Section 7.4 can be found by solving the following linear program, which we call LP1.

Notice that the expression $$\bar{v}(S) \cdot P(S) - u(S)$$ in the objective function equals the


\[
\begin{align*}
\max & \mathbb{E}_S[\bar{v}(S) \cdot \mathcal{P}(S) - u(S)] \\
\text{subject to:} & \quad \forall S, T \subseteq N : \quad u(S) \geq u(T) + (\bar{v}(S) - \bar{v}(T)) \cdot \mathcal{P}(T) \quad \text{(BIC)} \\
& \quad \forall S \subseteq N : \quad u(S) \geq 0 \quad \text{(IR)} \\
& \quad \forall S \subseteq N, i \in N : \quad 0 \leq \mathcal{P}_i(S) \leq 1 \quad \text{(PROB)}
\end{align*}
\]

Figure 7-1: LP1, the linear program for revenue maximization.

price \(T(S)\) that the bidder of type \(S\) pays when reporting \(S\) to the mechanism. The expectation is taken over all \(S \subseteq N\), where the probability of set \(S\) is given by

\[
p(S) \triangleq \prod_{i \in S} p_i \cdot \prod_{j \notin S} (1 - p_j).
\]

We note that this program has exponential size both in the number of variables and the number of constraints.

### 7.5.2 A Relaxed Linear Program

We now remove constraints from LP1 and perform further simplifications, making the program easier to analyze. Later on we identify a subclass of instances where optimal solutions to the relaxed program induce optimal solutions to the original program (see Lemma 17).

As a first step, we relax LP1 by considering only BIC constraints that correspond to neighboring types (types that differ in one element). That is, instead of requiring that no type of player can benefit by falsely declaring to be any other type, we only require that no type of player can benefit by falsely declaring to be a neighboring type. For simplicity in our analysis, we also drop the constraint that the probabilities \(\mathcal{P}_i(S)\) are non-negative:

\[
\max \mathbb{E}_S[\bar{v}(S) \cdot \mathcal{P}(S) - u(S)]
\]
subject to:

\begin{align*}
\forall S \subseteq N, i \notin S: & \quad u(S \cup \{i\}) \geq u(S) + d_i P_i(S) \quad \text{(BIC1)} \\
\forall S \subseteq N, i \notin S: & \quad u(S) \geq u(S \cup \{i\}) - d_i P_i(S \cup \{i\}) \quad \text{(BIC2)} \\
\forall S \subseteq N: & \quad u(S) \geq 0 \quad \text{(IR)} \\
\forall S \subseteq N, i \in N: & \quad P_i(S) \leq 1 \quad \text{(PROB')} \\
\end{align*}

Since the coefficient of every $P_i(S)$ in the objective is strictly positive, no $P_i(S)$ can be increased in any optimal solution without violating a constraint. We therefore conclude the following about $P_i(S)$:

- If $i \in S$, then $P_i(S)$ is only upper-bounded by constraint PROB', and thus $P_i(S) = 1$ in every optimal solution.

- If $i \notin S$, then $P_i(S) = \min\{1, \frac{u(S \cup \{i\}) - u(S)}{d_i}\}$ from (BIC1) and (PROB'). Furthermore, from (BIC2) we have $\frac{u(S \cup \{i\}) - u(S)}{d_i} \leq P_i(S \cup \{i\}) = 1$, and thus $P_i(S) = \frac{u(S \cup \{i\}) - u(S)}{d_i}$.

The program therefore becomes (after setting $P_i(S) = 1$ whenever $i \in S$, removing the constant terms from the objective, and tightening the constraints (BIC1) to equality):

$$\max \mathbb{E}_S \left[ \sum_{i \notin S} v_i(S) P_i(S) - u(S) \right]$$

subject to:

\begin{align*}
\forall S \subseteq N, i \notin S : & \quad P_i(S) = \frac{u(S \cup \{i\}) - u(S)}{d_i} \quad \text{(BIC1')} \\
\forall S \subseteq N, i \notin S : & \quad u(S \cup \{i\}) - u(S) \leq d_i \quad \text{(BIC2)} \\
\forall S \subseteq N : & \quad u(S) \geq 0 \quad \text{(IR)} \\
\forall S \subseteq N, i \notin S : & \quad P_i(S) \leq 1 \quad \text{(PROB')} \\
\end{align*}

Notice that the constraint (PROB') is trivially satisfied as a consequence of (BIC1') and (BIC2). We now rewrite the objective, substituting $P_i(S)$ by (BIC1') and noting that $v_i(S) = a_i$ for $i \notin S$:

$$\mathbb{E}_S \left[ \sum_{i \notin S} a_i \frac{u(S \cup \{i\}) - u(S)}{d_i} - u(S) \right] = \mathbb{E}_S \left[ u(S) \left( -1 - \sum_{i \notin S} \frac{a_i}{d_i} + \sum_{i \in S} \left( \frac{a_i}{d_i} \cdot \frac{p(S \setminus \{i\})}{p(S)} \right) \right) \right]$$

obtained by grouping together all coefficients of $u(S)$, adjusting by the appropriate proba-
bilities. We note that \( p(S \setminus \{i\}) = -1 + \frac{1}{p_i} \), and our objective becomes

\[
E_S \left[ u(S) \left( -1 - \sum_{i \in N} \frac{a_i}{d_i} + \sum_{i \in S} \frac{a_i}{p_i d_i} \right) \right].
\]

We now perform a change of notation so that the program takes a simpler form. We set \( B \leftarrow \kappa \left( 1 + \sum_{i \in N} \frac{a_i}{d_i} \right) \) and \( x_i \leftarrow \frac{\kappa a_i}{p_id_i} \), where \( \kappa \) is some positive constant. The objective becomes \( \frac{1}{\kappa} E_S[\left( \sum_{i \in S} x_i - B \right) u(S)] \). Since \( 1/\kappa \) is constant, we study the following program, LP2:

\[
\max_u E_S[\left( \sum_{i \in S} x_i - B \right) u(S)]
\]

subject to:

\[
\forall S \subseteq N, i \notin S : \ u(S \cup \{i\}) - u(S) \leq d_i \quad (\text{BIC2})
\]

\[
\forall S \subseteq N : \ u(S) \geq 0 \quad (\text{IR})
\]

Figure 7-2: LP2, the relaxed linear program

In constructing LP2, our constants \( B \) and \( x \) were a function of \( \bar{p}, \bar{a}, \bar{d} \), and a newly introduced constant \( \kappa \). We note that, by adjusting \( \kappa \), we can obtain a wide range of relevant \( B \) and \( x \) values.

Lemma 15. For any \( B, \bar{x}, \bar{p} \) and \( \bar{d} \) such that \( B > \sum_{i \in N} p_i x_i \), there exist (efficiently computable) \( \bar{a} \) and \( \kappa \) such that \( B = \kappa (1 + \sum_{i \in N} \frac{a_i}{d_i}) \) and \( x_i = \frac{\kappa a_i}{p_i d_i} \). If \( B, \bar{x}, \bar{p} \) and \( \bar{d} \) are rational, then \( \bar{a} \) and \( \kappa \) are rational as well.

Proof. We want that \( x_i = \frac{\kappa a_i}{p_i d_i} \) and \( B = \kappa + \sum_{i \in N} \frac{\kappa a_i}{d_i} = \kappa + \sum_{i \in N} p_i x_i \). Indeed, these equalities follow from setting \( \kappa \leftarrow B - \sum_{i \in N} p_i x_i \) and \( a_i \leftarrow \frac{p_i \kappa d_i x_i}{\kappa} \). \( \square \)

7.5.3 The Dual of the Relaxed Program, and its Min-Cost Flow Interpretation

To characterize the structure of optimal solutions to LP2, we use linear programming duality. Consider LP3, LP2's dual program, which has a (flow) variable \( f_{S \cup \{i\} \rightarrow S} \) for every set \( S \) and \( i \notin S \).
We interpret LP3 as a minimum-cost flow problem on a lattice. Every node on the lattice corresponds to a set $S \subseteq N$, and flow may move downwards from $S$ to $S \setminus \{i\}$ for each $i \in S$. The variable $f_{S \rightarrow S \setminus \{i\}}$ represents the amount of flow sent this way, and the cost of sending each unit of flow along this edge is $d_i$.

For nodes $S$ with $p(S) (\sum_{i \in S} x_i - B) \geq 0$, we have an external source supplying the node with at least this amount of flow. We call such a node “positive.” Nodes with $p(S) (\sum_{i \in S} x_i - B) < 0$, which we call “negative,” can deposit at most $|p(S) (\sum_{i \in S} x_i - B)|$ to an external sink. Since $d_i > 0$ for all $i$, an optimal solution to LP3 will have net imbalance exactly $p(S) (\sum_{i \in S} x_i - B)$ for each positive node $S$.

### 7.6 Characterizing the Linear Programming Solutions

For the remainder of the paper, we restrict our attention to the case where $N$ is the only positive node in LP3. We notice that there is a feasible solution if and only if $p(N) (\sum_{i \in N} x_i - B) \leq -\sum_{\emptyset \subseteq S \subseteq N} p(S) (\sum_{i \in S} x_i - B)$, which occurs when $E_{\emptyset} [\sum_{i \in S} x_i - B] \leq 0$. Since the components of $S$ are chosen independently, the program is feasible precisely when $\sum_{i \in N} d_i x_i - B \leq 0$.

#### 7.6.1 The Canonical Solution to LP3

When there is a single positive node, we can easily construct an optimal solution to LP3 as follows. Define the cost of each node $S$ to be $\text{cost}(S) = \sum_{i \in N \setminus S} d_i$, which corresponds to the cost of sending a unit of flow from $N$ to $S$ in LP3. (The flow can be sent along any path to the node, since all such paths have the same cost.) We order the negative nodes...
in increasing order of cost (and lexicographically if there are ties). We greedily send flow to the negative nodes in order, moving to the next node only when all previous nodes have been saturated. We stop when a net flow of $p(N)\left(\sum_{i \in N} x_i - B\right)$ has been absorbed by the negative nodes. We call this the canonical solution to LP3, and notice that the canonical solution is the unique optimal solution to LP3 up to the division of flow between equal cost nodes.

7.6.2 From LP3 to LP2 Solutions

We now show how to use a canonical solution to LP3 to construct a solution to LP2. In most instances, this solution is unique.

**Lemma 16.** Let $S^*$ be the highest-cost negative node which absorbs non-zero flow in the canonical solution $f$ of LP3, and suppose that $S^*$ is not fully saturated by $f$. Then the utility function $u(S) = \max\{\text{cost}(S^*) - \text{cost}(S), 0\}$ is the unique optimal solution to LP2.

**Proof.** Consider an arbitrary optimal LP2 solution $u$. We will use linear programming complementarity to prove that $u$ is uniquely determined by the canonical solution $f$.

For any node $S$ that receives nonzero flow in $f$, there is a path $N = S_0, S_1, S_2, \ldots, S_k = S$ from $N$ to $S$ that has positive flow along each edge. By complementarity, the (BIC2) inequalities corresponding to these edges in the primal program are tight in $u$. That is, for all $i = 1, \ldots, k$, we have $u(S_{i-1}) - u(S_i) = d_x$, where $x$ is the unique element of $S_{i-1} \setminus S_i$.

So, for any $S$ which receives nonzero flow in $f$:

$$u(N) - u(S) = \sum_{i \in N \setminus S} d_i = \text{cost}(S).$$

For all nodes $S'$ which are not fully saturated in $f$ (i.e. $S^*$ as well as all nodes which receive no flow), $u(S')$ must be 0 in $u$ by complementarity, since the corresponding LP3 constraints are not tight. In particular, since $S^*$ receives flow but is not fully saturated, we have $u(S^*) = 0$ and hence:

$$u(N) = u(N) - u(S^*) = \text{cost}(S^*).$$

Therefore, any node $S$ which receives flow in $f$ must have $u(S) = u(N) - \text{cost}(S) = \text{cost}(S^*) - \text{cost}(S)$. 

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Furthermore, a node $S$ always receives flow in $f$ if its cost is less than $\text{cost}(S^*)$, and receives no flow if its cost is greater than $\text{cost}(S^*)$. Moreover, those nodes $S$ with $\text{cost}(S) = \text{cost}(S^*)$ either receive no flow in which case $u(S) = 0$, or receive flow in which case $u(S) = \text{cost}(S^*) - \text{cost}(S) = 0$. Thus, we have shown that for any node $S$, $u(S) = \max\{\text{cost}(S^*) - \text{cost}(S), 0\}$. It is easy to verify that this utility function satisfies all the constraints of LP2.

If the highest cost node $S^*$ to receive flow in $f$ were fully saturated, then the utility function described in Lemma 16 would still be an optimal LP2 solution. However, in this case, if the cheapest unfilled node in $f$ had strictly greater cost than $S^*$, then the optimal primal solution would not be unique.

### 7.6.3 From LP2 to LP1 Solutions

We now show that, in certain cases, a solution to LP2 allows us to obtain a solution to LP1 where $\bar{p}$, $\bar{a}$ and $\bar{d}$ are as in Lemma 15. The proof of Lemma 17 sets $q$ as in Section 7.5.2 and then verifies that $(u, q)$ satisfies all the constraints of LP1.

**Lemma 17.** Suppose $B > \sum_{i \in N} p_i x_i$ and an optimal solution $u$ to LP2 is monotone and supermodular. Then there is some $q$ such that $(u, q)$ is an optimal solution to LP1 where $\bar{p}$, $\bar{a}$, $\bar{d}$ are as in Lemma 15. If $u$ is the unique optimal solution to LP2, then $(u, q)$ is the unique optimal LP1 solution.

**Proof.** We set

$$P_i(S) = \begin{cases} 1, & \text{if } i \in S; \\ \frac{u(S \cup \{i\}) - u(S)}{d_i}, & \text{otherwise.} \end{cases}$$

With this choice, as explained in Section 7.5.2, $(u, q)$ is an optimal solution to a relaxation of LP1. So to establish optimality of $(u, q)$ for LP1 it suffices to show that $(u, q)$ satisfies all the constraints of LP1.

We first notice that the (IR) constraints are satisfied, since $u(S) \geq 0$ for all $S$ in LP2.

We now show that the (PROB) constraints are satisfied. Indeed, if $i \in S$, then $P_i(S) = 1$. If $i \notin S$, then $P_i(S) \geq 0$ follows from monotonicity of $u$. The inequality $P_i(S) \leq 1$ follows from constraint (BIC2) of LP2.
Finally, we show that the (BIC) constraints of LP1 are satisfied. By supermodularity of $u$ we have that for all $S$, all $i \not\in S$ and all $j \neq i$:

$$u(S \cup \{i\} \cup \{j\}) - u(S \cup \{j\}) \geq u(S \cup \{i\}) - u(S).$$

Dividing by $d_i$, we obtain $P_i(S \cup \{j\}) \geq P_i(S)$ for all $i \not\in S$ and $j \neq i$. Since the inequality is trivially satisfied if $i \in S$ (since both sides are 1), or $j = i$ (since $P_i(S \cup \{i\}) = 1$) we conclude that $\tilde{q}$ is monotone.

Now pick any distinct subsets $S, T \subseteq N$. We must show that:

$$u(S) \geq u(T) + (\tilde{v}(S) - \tilde{v}(T)) \cdot \tilde{P}(T).$$

Consider an ordering $i_1, i_2, \ldots, i_k$ of the elements of $T \setminus S$ and an ordering $j_1, j_2, \ldots, j_\ell$ of the elements of $S \setminus T$.

By (BIC2), we know that, for all $r = 1, \ldots, k$:

$$u\left(S \cup \bigcup_{t=1}^{r} \{i_t\}\right) \leq u\left(S \cup \bigcup_{t=1}^{r-1} \{i_t\}\right) + d_{i_r}.$$

Summing over $r$ and canceling terms, we conclude $u(S \cup T) \leq u(S) + \sum_{r=1}^{k} d_{i_r}$.

From our definition of $\tilde{q}$ it follows that for all $r = 1, \ldots, \ell$:

$$u\left(T \cup \bigcup_{t=1}^{r} \{j_t\}\right) = u\left(T \cup \bigcup_{t=1}^{r-1} \{j_t\}\right) + d_{j_r} P_{j_r} \left(T \cup \bigcup_{t=1}^{r-1} \{j_t\}\right).$$

By monotonicity of $\tilde{q}$, it follows that

$$u\left(T \cup \bigcup_{t=1}^{r} \{j_t\}\right) \geq u\left(T \cup \bigcup_{t=1}^{r-1} \{j_t\}\right) + d_{j_r} P_{j_r}(T).$$

Summing over $r$, we conclude that $u(S \cup T) \geq u(T) + \sum_{r=1}^{\ell} d_{j_r} P_{j_r}(T)$.

Combining this with our earlier upper bound for $u(S \cup T)$, we conclude that

$$u(S) \geq u(T) + \sum_{r=1}^{\ell} d_{j_r} P_{j_r}(T) - \sum_{r=1}^{k} d_{i_r}. $$
Since $P_{i_r}(T) = 1$ for all $r$, we have

$$u(S) \geq u(T) + \sum_{j \in S \setminus T} d_j P_j(T) - \sum_{i \in T \setminus S} d_i P_i(T)$$

and thus the (BIC) constraint of LP1 is satisfied.

If $u$ is the unique optimal solution to LP2, then the $(u, q)$ constructed as above is the unique optimal solution to LP1, as it is the unique optimal solution of a relaxation of LP1.

\(\square\)

7.6.4 Putting it All Together

In summary, we have shown that if the canonical solution of LP3 has a partially saturated node $S^*$, then LP2 has a unique optimal solution, namely $u(S) = \max\{\text{cost}(S^*) - \text{cost}(S), 0\}$. Since this utility function is monotone and supermodular, it also defines a unique optimal solution of the corresponding LP1 instance.

**Corollary 3.** Let $S^*$ be the highest-cost negative node which absorbs non-zero flow in the canonical solution of LP3, and suppose that $S^*$ is not fully saturated. Then the original mechanism design problem with $\bar{p}$, $\bar{a}$ and $\bar{d}$ as in Lemma 15 has a unique optimal solution, and the utility of a player of type $N$ in this solution is $\text{cost}(S^*)$.

7.7 Proof of Theorem 5: Hardness Of Mechanism Design

We use the results of the previous section to establish the computational hardness of optimal mechanism design. Our reduction is from the lexicographic rank problem, which we show to be $\#P$-hard in Section 7.7.1.
7.7.1 \#P-Hardness of LExRANK

Definition 21 (LExRANK problem). Given a collection $\mathcal{C} = \{c_1, \ldots, c_n\}$ of positive integers and a subset $S \subseteq \{1, \ldots, n\}$, we define the lexicographic rank of $S$, denoted $\text{lexr}_\mathcal{C}(S)$, by

$$\text{lexr}_\mathcal{C}(S) \triangleq \left\{ \left. S' : |S'| = |S| \text{ and } \left( \sum_{i \in S'} c_i < \sum_{j \in S} c_j \right) \text{ or } \left( \sum_{i \in S'} c_i = \sum_{j \in S} c_j \text{ and } S' \leq_{\text{lex}} S \right) \right\}$$

where $S' \leq_{\text{lex}} S$ is with respect to the lexicographic ordering.\(^{12}\) The LExRANK problem is: Given $\mathcal{C}$, $S$, and an integer $k$, determine whether or not $\text{lexr}_\mathcal{C}(S) \leq k$.

We show that the LExRANK problem is \#P-hard by a reduction from \#-SUBSETSUM.

Definition 22 (\#-SUBSETSUM problem). Given a collection $\mathcal{W} = \{w_1, \ldots, w_n\}$ of positive integers and a target integer $T$, compute the number of subsets $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} w_i \leq T$.

The \#-SUBSETSUM problem is known to be \#P-hard. Indeed, the reduction from SAT to SUBSETSUM as presented in [60] is parsimonious.

Given an oracle for the LExRANK problem, it is straightforward to do binary search to compute the lexicographic rank of a set $S$. We will prove hardness of LExRANK by reducing the \#-SUBSETSUM problem to the computation of lexicographic ranks of a collection of sets.

Let $(\mathcal{W}, T)$ be an instance of \#-SUBSETSUM, where $\mathcal{W} = \{w_1, \ldots, w_n\}$ is a collection of positive integers and $T$ is a target integer. We begin by defining, for $m = 1, \ldots, n$:

$$\text{count}_{\mathcal{W}}(T, m) \triangleq \left\{ S \subseteq \{1, \ldots, n\} : |S| = m \text{ and } \sum_{i \in S} t_i \leq T \right\}$$

Note that the number of subsets of $\mathcal{W}$ which sum to at most $T$ is simply $\sum_{m=1}^n \text{count}_{\mathcal{W}}(T, m)$. So it suffices to compute $\text{count}_{\mathcal{W}}(T, m)$ for all $m$.

To do this, we define $n$ different collections $\mathcal{C}_1, \ldots, \mathcal{C}_n$, where $\mathcal{C}_\ell = \{c_1', \ldots, c_{n+\ell}'\}$ is

\(^{12}\)To be precise, we say that $S_1 \leq_{\text{lex}} S_2$ iff the largest element in the symmetric difference $S_1 \Delta S_2$ belongs to $S_2$. 

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given by:
\[ c_i^\ell = \begin{cases} 
4nw_i & \text{if } 1 \leq i \leq n \\
4nT + 2n & \text{if } i = n + 1 \\
1 & \text{if } n + 2 \leq i \leq n + \ell.
\end{cases} \]

We also define a special set \( S_\ell \triangleq \{n + 1, n + 2, \ldots, n + \ell\} \). Notice that \( \sum_{i \in S_\ell} c_i^\ell = 4nT + 2n + \ell - 1 \). Furthermore, for every subset \( S \subseteq \{1, \ldots, n\} \), we have \( \sum_{i \in S} c_i^\ell = 4n \sum_{i \in S} w_i \).

Hence, for all \( \emptyset \neq S \subseteq \{1, \ldots, n\} \):

1. if \( \sum_{i \in S} w_i > T \), then \( \sum_{i \in S} c_i^\ell > \sum_{j \in S_\ell} c_j^\ell \);
2. if \( \sum_{i \in S} w_i \leq T \), then for all \( U \subseteq \{n+2, n+3, \ldots, n+\ell\} \) we have \( \sum_{i \in S \cup U} c_i^\ell < \sum_{j \in S_\ell} c_j^\ell \);
3. for all \( U \subseteq \{n + 2, n + 3, \ldots, n + \ell\} \), \( \sum_{i \in S \cup U \cup \{n+1\}} c_i^\ell > \sum_{j \in S_\ell} c_j^\ell \).

Given that \( |S_\ell| = \ell \) the above imply
\[
\text{lexr}_\ell(S_\ell) = 1 + \sum_{m=1}^{\ell} \left( \text{count}_W(T, m) \cdot \left( \frac{\ell - 1}{\ell - m} \right) \right).
\]

Suppose we have an oracle which can compute the lexicographic rank of a given set. We first use this oracle to determine \( \text{lexr}_\ell(S_1) \) and thereby compute \( \text{count}_W(T, 1) \). Next, we use the oracle to determine \( \text{lexr}_\ell(S_2) \) and thereby compute \( \text{count}_W(T, 2) \), using the previously computed value of \( \text{count}_W(T, 1) \). Continuing this procedure \( n \) times, we can compute \( \text{count}_W(T, m) \) for all \( m = 1, \ldots, n \). This concludes the proof.

### 7.7.2 Proof of Theorem 5

We now prove Theorem 5. Let \((\mathcal{C}, S, k)\) be an instance of LEXRANK where \( C = \{c_1, \ldots, c_n\} \) is a collection of positive integers, \( S \subseteq \{1, \ldots, n\} \), and \( k \) is integer. We wish to determine whether \( \text{lexr}_\mathcal{C}(S) \leq k \). We assume that \( |S| \neq 0, n \) as otherwise the problem is trivial to solve.

We denote by \( [n] \) the set \( \{1, 2, \ldots, n\} \) and \( [n+1] = [n] \cup \{n+1\} \). We construct an OMD instance indirectly, by defining an instance of LP3 with the following parameters:

- \( d_i = 2^{n+1} \left( c_i + \sum_{j=1}^{n} c_j \right) + 2^i \), for \( i \in [n] \);
• \( d_{n+1} = 1; \)
• \( x_i = 2, \text{ for all } i; \)
• \( B = 2n + 1; \)
• \( p_i = p \text{ for all } i, \) where we leave \( p \in [0.5, 1 - \frac{1}{2n+2}) \) a parameter which we will set later.

We note that \( B > \sum x_ip_i, \) and thus Lemma 15 implies that, for all \( p, \) an instance of LP3 as above arises from some OMD instance \( \{a_i, d_i, p_i\}_{i=1}^{n+1}, \) in the notation of Section 7.4.

Denote by \( S^c \) the set \( [n] \setminus S. \)\(^{13}\) Suppose that, for some value \( p, \) there is a partially filled node \( T^* \) in the canonical LP3 solution such that \( T^* \subseteq [n] \) and \( |T^*| = n - |S|. \) Using Lemma 16 we have

\[
P_{n+1}(S^c) = \frac{u^*(S^c \cup \{n + 1\}) - u^*(S^c)}{1}
= \max\{\text{cost}(T^*) - \text{cost}(S^c \cup \{n + 1\}), 0\} - \max\{\text{cost}(T^*) - \text{cost}(S^c), 0\}
= \max\{\text{cost}(T^*) - \text{cost}(S^c) + 1, 0\} - \max\{\text{cost}(T^*) - \text{cost}(S^c), 0\}
\]

Therefore, since the cost of each set is an integer, \( P_{n+1}(S^c) = \begin{cases} 0 & \text{if } \text{cost}(S^c) > \text{cost}(T^*) \\ 1 & \text{if } \text{cost}(S^c) \leq \text{cost}(T^*) \end{cases}. \)

Since \( n + 1 \) is in neither \( S^c \) nor \( T^* \), \( P_{n+1}(S^c) = \begin{cases} 0 & \text{if } \sum_{i \in S} d_i > \sum_{j \in [n] \setminus T^*} d_j \\ 1 & \text{if } \sum_{i \in S} d_i \leq \sum_{j \in [n] \setminus T^*} d_j \end{cases}. \) By our construction of the \( d_i'\)s we can see that since \( |T^*| = n - |S|, \)

\[
P_{n+1}(S^c) = \begin{cases} 0 & \text{if } \begin{cases} \sum_{i \in S} c_i > \sum_{j \in [n] \setminus T^*} c_j \\ \sum_{i \in S} c_i = \sum_{j \in [n] \setminus T^*} c_j \text{ and} \\ S > \text{lex } ([n] \setminus T^*) \end{cases} \\ 1 & \text{if } \begin{cases} \sum_{i \in S} c_i < \sum_{j \in [n] \setminus T^*} c_j \\ \sum_{i \in S} c_i = \sum_{j \in [n] \setminus T^*} c_j \text{ and} \\ S \leq \text{lex } ([n] \setminus T^*) \end{cases} \end{cases}
\]

Therefore, \( P_{n+1}(S^c) = 1 \) if \( \text{lex}(S) \leq \text{lex}([n] \setminus T^*) \) and 0 otherwise.

\(^{13}\)Note that \( \{n + 1\} \) is in neither \( S \) nor \( S^c. \)
Thus, our next goal is to set the parameter $p$ such that there is a partially filled node $T^*$ in the canonical LP3 solution such that $\text{lexre}_C([n] \setminus T^*) = k$, along with the properties $T^* \subseteq [n]$ and $|T^*| = n - |S|$. With such a value for $p$, distinguishing between $P_{n+1}(S^c) = 0$ and $P_{n+1}(S^c) = 1$ would allow us to solve the LExRANK instance. The next lemma shows that a $p$ as required can be found in polynomial time.

**Lemma 18.** In polynomial time, we can identify $p \in [0.5, 1 - \frac{1}{2n+2}]$ with $O(n \log n)$ bits of precision such that the partially filled node in the canonical LP3 solution with parameter $p = \tilde{p}$ is a set $T^* \subseteq [n]$ of size $n - |S|$ and $\text{lexre}_C([n] \setminus T^*) = k$.

**Proof of Lemma 18:** In our construction for the proof of Theorem 5, the lowest cost negative node is $[n]$. Furthermore, the cost of every negative node is unique, and for any $T \subseteq [n]$ there is no node with cost between that of $T \cup \{n + 1\}$ and $T$. Also, if $T$ and $T'$ are proper subsets of $[n]$ and if $|T| > |T'|$, then $\text{cost}(T) < \text{cost}(T')$.

For each $i$ between 1 and $n - 1$, let $T^i_1, T^i_2, \ldots$ be the ordering of the size-$i$ subsets of $[n]$ in increasing order of cost. In the canonical LP3 solution, $[n]$ fills first, and $T^i_j$ fills before $T^i_{j'}$ if it has larger size ($i > i'$) or the same size but smaller cost ($i = i'$ and $j < j'$). Furthermore, each node $T^i_j \cup \{n + 1\}$ fills immediately before the node $T^i_j$.

Our goal is to choose $p$ so that $T^{|n-|S|}_{n-|S|}$ is partially filled. Indeed, the sets $[n] \setminus T^{|n-|S|}_1$ through $[n] \setminus T^{|n-|S|}_n$ are precisely the sets counted in the computation of $\text{lexre}_C([n] \setminus T^{|n-|S|}_k)$. Notice that lexicographic tie-breaking of $\text{lexr}$ is enforced by construction of adding an additional $2^i$ to $d_i$.

The only positive node, $[n + 1]$, emits a net flow of $p^{n+1}$, and the node $[n]$ absorbs at most $p^n(1 - p)$ flow. For each size $i$ between $n - 1$ and $n - |S| + 1$, there are $\binom{n}{i}$ sets $T \subseteq [n]$ of size $i$, each of which can absorb

$$|p(T)(2|T| - B)| = (2(n - i) + 1)p^i(1 - p)^{n+1-i}$$

flow. Furthermore, each set $T \cup \{n + 1\}$ can absorb

$$|p(T \cup \{n + 1\})(2|T| + 2 - B)| = (2(n - i) - 1)p^{i+1}(1 - p)^{n-i}.$$
Thus, in total, $T$ and $T \cup \{n + 1\}$ can absorb

$$p^i(1 - p)^{n-i}((1 - p)(2n - 2i + 1) + p(2n - 2i - 1))$$

$$= p^i(1 - p)^{n-i}(2(n - i - p) + 1).$$

Finally, we notice that $T$ is responsible for at least a $1/(2n + 2)$ fraction of the quantity above, since

$$\frac{(2(n - i) + 1)p^i(1 - p)^{n+1-i}}{(2(n - i - p) + 1)p^i(1 - p)^{n-i}} \geq 1 - p > \frac{1}{2n + 2}$$

using that $p < 1 - \frac{1}{2n+2}$.

If all nodes strictly preceding (i.e. with smaller cost than) $T_t^{n-|S|} \cup \{n + 1\}$ have been saturated, the amount of flow still unabsorbed is

$$p^{n+1} - p^n(1 - p) - \sum_{i=n-|S|+1}^{n-1} \binom{n}{i} p^i(1 - p)^{n-i}(2(n - i - p) + 1)$$

$$= \sum_{i=n-|S|+1}^{n} \binom{n}{i} p^i(1 - p)^{n-i}(2(i + p - n) - 1).$$

Therefore, a sufficient condition for $T_k^{n-|S|}$ to be partially filled in the canonical solution is

$$f(p) = \sum_{i=n-|S|+1}^{n} \binom{n}{i} p^i(1 - p)^{n-i}(2(i + p - n) - 1)$$

$$\frac{p^{n-|S|}(1 - p)^{|S|}(2|S| - 2p + 1)}{p^{n-|S|}(1 - p)^{|S|}(2|S| - 2p + 1)} \leq \left(k - \frac{1}{2n + 2}, k\right).$$

We claim that there is such a $p^* \in [0.5, 1 - \frac{1}{2n+2})$ such that $f(p^*) = k - \frac{1}{4n+4}$. Indeed, for $p < 0.5$, only $[n]$ will ever receive flow, so in this case, $f(p) < 0$. Furthermore, we can lower-bound $f(p)$ by the following ratio (where we have added a negative quantity to the numerator)

$$f(p) \geq \frac{\sum_{i=n-|S|+1}^{n} \binom{n}{i} p^i(1 - p)^{n-i}(2(i + p - n) - 1) + \sum_{i=0}^{n-|S|} \binom{n}{i} p^n-i(1 - p)^i(2(i + p - n) - 1)}{p^{n-|S|}(1 - p)^{|S|}(2|S| - 2p + 1)}$$

$$- \sum_{i=0}^{n} \binom{n}{i} p^n-i(1 - p)^i(2(i + p - n) - 1)$$

$$= \frac{\sum_{i=n-|S|+1}^{n} \binom{n}{i} p^i(1 - p)^{n-i}(2(i + p - n) - 1) + \sum_{i=0}^{n-|S|} \binom{n}{i} p^n-i(1 - p)^i(2(i + p - n) - 1) - \binom{n}{|S|} p^n(1 - p)^{|S|}(2|S| + 2p - 1)}{p^{n-|S|}(1 - p)^{|S|}(2|S| - 2p + 1)}.$$
and thus

\[
f(p) \geq \frac{\sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} (2n - 2i - 2p + 1)}{p^n |S| (1-p)^{|S|} (2|S| - 2p + 1)} = \frac{n}{|S|} - \frac{2n - 2p + 1 - 2 \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i}}{p^n |S| (1-p)^{|S|} (2|S| - 2p + 1)}.
\]

Hence, for \( p = 1 - \frac{1}{2n+2} \), we get \( f(p) \geq \binom{n}{|S|} \geq k \). Using this, the continuity of \( f \), and that \( f(p) < 0 \) for \( p < 0.5 \), we conclude that there is a \( p^* \in [0.5, 1 - \frac{1}{2n+2}] \) such that \( f(p^*) = k - \frac{1}{4n+4} \).

We now consider \( \tilde{p} = p^* \pm \epsilon \in [0.5, 1 - \frac{1}{2n+2}] \). We claim that \( f(\tilde{p}) \in (k - \frac{1}{2n+2}, k) \) as long as \( \epsilon = O\left(\frac{(4n)^{4n}}{4n+4}\right) \). To show this, we bound the absolute value of the derivative \( \frac{df}{dp} \) at all points in \([0.5, 1 - \frac{1}{2n+2}]\). The numerator of \( \frac{df}{dp} \) is a polynomial in \( p \) of degree \( 2n + 1 \), where the coefficient of each term is, in absolute value, \( O(2^{3n} \cdot \text{poly}(n)) \leq O(2^{4n}) \)—using the crude bound \( \binom{n}{i} \leq 2^n \). Furthermore, the denominator \( (p^n |S| (1-p)^{|S|} (2|S| - 2p + 1))^2 \) of \( \frac{df}{dp} \) is greater than \( \left(\frac{1}{2n+2}\right)^{2n} \), since \( p \in [0.5, 1 - \frac{1}{2n+2}] \). Therefore, we can bound the magnitude of the derivative by \( O((2n + 2)^2n (2n + 1)^{2n}) = O((4n)^{4n}) \). Since this bound holds for all points in \([0.5, 1 - \frac{1}{2n+2}]\), we conclude that

\[
f(\tilde{p}) \in (f(p^*) - \epsilon O((4n)^{4n}), f(p^*) + \epsilon O((4n)^{4n}))
\]

and thus, for \( \epsilon = O\left(\frac{(4n)^{4n}}{4n+4}\right) \) we have that \( f(\tilde{p}) \in (k - \frac{1}{2n+2}, k) \).

Thus, we can find the desired \( \tilde{p} \) in polynomial time via binary search on \( f(p) \), requiring \( O(n \log n) \) bits of precision in \( p \).

We conclude our reduction as follows: First, we compute \( \tilde{p} \) as in Lemma 18. Next, we solve the OMD instance resulting from this choice of parameter. The solution allows us to sample in expected polynomial-time from the allocation and price rule (see Appendix 7.2) of the optimal auction for type \( S^c \). We have proven that this type receives item \( n + 1 \) with probability 1 if \( (C, S, k) \in \text{LEXRANK} \) and with probability 0 otherwise. A single sample from the allocation rule of the optimal auction suffices to tell which one is the case, since if \( \mathcal{P}_{n+1}(S^c) = 1 \) then every sample from the allocation rule should allocate the item and if \( \mathcal{P}_{n+1}(S^c) = 0 \) then no sample should allocate the item. So if we can solve OMD in expected
polynomial-time, then we can solve LEXRANK in expected polynomial-time by a. finding \( \hat{\bar{p}} \) using Lemma 18; b. solving the resulting OMD instance; c. drawing a single sample from the allocation rule of the auction for type \( S^c \) and outputting "yes" if and only if item \( n+1 \) is allocated by the drawn sample.

7.8 Remark on Computationally Powerful Bidders

If bidders are assumed to have exceptional computational power, then it is (perhaps counterintuitively) easier for a mechanism designer to implement an optimal (albeit indirect) mechanism. The underlying idea is that the designer can shift the computation burden from himself onto the bidders. In this section, we demonstrate how powerful bidders (combined with the use of indirect mechanisms) trivialize much of the complexity of optimal mechanism design.

In Section 7.8.1 we present a observation that optimal mechanism design is computationally difficult for a certain type of bidder with combinatorial valuations, and then use this motivating example in Section 7.8.2 to show how this type of mechanism design problem becomes trivial when the bidders have \( \text{NP} \) power. We further expand on this idea in Section 7.8.2 to show that mechanism design becomes even simpler when bidders have the ability to solve \( \text{PSPACE} \)-complete problems. We note that the mechanisms discussed in this section are not intended to be practical, but rather to add further motivation to our requirement (see Section 7.2) that all mechanism participants run in probabilistic polynomial time.

7.8.1 Motivating Example: Budget-Additive Bidders

When bidder valuations have combinatorial structure, it becomes easier to embed computationally hard problems into the optimal mechanism design problem. In this spirit, Dobzinski et al. [26] show that optimal mechanism design for OXS bidders is \( \text{NP} \)-hard. While we won’t define OXS valuations, we illustrate the richness in their structure by recalling that in [26] the items are taken to be edges of a graph \( G = (V, E) \), and there is a single bidder whose valuation is drawn from a distribution that includes in its support the following valuation
for all $E' \subseteq E$:

$$f(E') = \max \left\{ |A| \left| \begin{array}{c} A \subseteq E', \text{ every connected component of} \\ G' = (V,A) \text{ is either acyclic or unicyclic} \end{array} \right. \right\}$$

While the valuations used in [26] are quite rich, we notice here that the techniques of [26] can be used to establish hardness of optimal mechanism design for a bidder with very mild combinatorial structure, namely budget-additivity. Indeed, consider the problem of selling a set $N = \{1, \ldots, n\}$ of items to a budget-additive bidder whose value $x_i$ for each item is some deterministically known integer, but whose budget is only probabilistically known. In particular, suppose that, with probability $(1 - \epsilon)$, the bidder is additive, in which case her value $v_a(S)$ for each subset $S$ is $\sum_{i \in S} x_i$. With probability $\epsilon$, however, she has a positive integer budget $B$: she values each subset $S$ at $v_b(S) = \min\{\sum_{i \in S} x_i, B\}$. That is, when she has a budget, she receives at most $B$ utility from any subset. We claim that the optimal mechanism satisfies:

**Claim 13.** Suppose $\epsilon < \frac{1}{1+\sum x_i}$. Then every optimal individually rational and Bayesian incentive compatible direct mechanism for the budget-additive bidder described above has the following form:

- If the bidder is unbudgeted, she receives all items and is charged $\sum_{i \in N} x_i$.
- If the bidder is budgeted, she receives a probability distribution over the subsets $T$ of items such that $\sum_{i \in T} x_i$ is as large a value as possible without exceeding $B$. She is charged that value.

This claim follows from a lemma of [26], showing that if the bidder has no budget she must receive her value maximizing bundle, while if she is budgeted she must receive some bundle $T$ maximizing $v_b(T) - (1 - \epsilon)v_a(T)$. Determining the optimal direct mechanism is clearly hard in this context, as it is NP-hard to compute a subset $T$ with the largest value that does not exceed $B$. Using the same arguments as in Section 7.4 we can extend this lower bound to all mechanisms, by noticing that any sample from the allocation to the budgeted bidder answers whether there exists some $T$ such that $\sum_{i \in T} x_i = B$.

**Remark 11.** There is a polynomial-time Karp reduction from the subset-sum problem to the optimal mechanism design problem of selling multiple items to a single budget-additive...
quasi-linear bidder whose values for the items are known rational numbers, and whose budget is equal to some finite rational value or $+\infty$ with rational probabilities.

### 7.8.2 Optimal Mechanism Design with Powerful Bidders

#### NP Power

We observe that if the budget-additive bidder described in the previous section has the ability to solve NP-hard problems, then there is a simple indirect mechanism which implements the above allocation rule: The seller offers each set $S$ at price $\sum_{i \in S} x_i$, and leaves the computation of the set $T$ to the bidder. (As remarked in Section 7.4, we can use small rewards to guarantee that the same allocation rule is implemented.) Such a mechanism is intuitively unsatisfactory, however, and violates our requirement (see Section 7.2) that a bidder be able to compute her strategy efficiently. While an optimal indirect mechanism is easy for the seller to construct and to implement, it is intractable for the bidder to determine her optimal strategy in such a mechanism. Conversely, implementing the direct mechanism requires the seller to solve a subset sum instance. Thus, shifting from a direct to an indirect mechanism allows for a shift of computational burden from the seller to the bidder.

#### PSPACE and Beyond

Indeed, we can generalize this observation to show that optimal mechanism design for a polynomial-time seller becomes much easier if the bidder is quasi-linear, computationally unbounded, and has a known finite maximum possible valuation for an allocation. The intuition is that, if some canonical optimal direct mechanism is computable and implementable in polynomial space, then we can construct an extensive form indirect mechanism whereby the bidder first declares her type and then convinces the seller, via an interactive proof with completeness 1 and low soundness, what the allocation and price would be for her type in the canonical optimal direct revelation BIC and IR mechanism.\(^\text{14}\) In the event of a failure, the bidder is charged a large fine, significantly greater than her maximum possible valuation of any subset. The proof that this protocol achieves, in a subgame perfect

\[^{14}\text{We have fixed a canonical direct mechanism to avoid complications arising when the optimal mechanism is not unique.}\]
equilibrium, expected revenue equal to the optimal direct BIC and IR revenue follows from \( \text{IP} = \text{PSPACE} \) [59]. In particular, we note that the canonical solution of the mechanism design instances in the hardness proof of Section 7.7 can indeed be found in \( \text{PSPACE} \), and thus we can construct an easily-implementable extensive form game which achieves optimal revenue in subgame perfect equilibrium. Reaching this equilibrium, however, requires the bidder to have \( \text{PSPACE} \) power.

Indeed, if there are two or more computationally-unbounded bidders, then we can replace the assumption that the optimal direct mechanism is computable in \( \text{PSPACE} \) with the even weaker assumption that it is computable in nondeterministic exponential time, using the result that \( \text{MIP} = \text{NEXP} \) [5] to achieve the optimal revenue in a perfect Bayesian equilibrium. The intuition is that our mechanism now asks all players to (simultaneously) declare their type. Two players are selected and required to declare the allocation and price which the canonical optimal direct mechanism would implement given this type profile, and they then perform a multiparty interactive proof to convince the seller of the correctness of the named allocation and price.

Since the optimal allocation might be randomized and the prices might require exponentially many bits to specify, we must modify the above outline. The rough ideas are the following. First, a price of exponential bit complexity but magnitude bounded by a given value can be obtained as the expectation of a distribution that can be sampled with polynomially many random bits in expectation. Now, after the bidders declare their types, the seller reveals a short random seed \( r \), and requires the bidders to draw the prices and allocation from the appropriate distribution, using \( r \) as a seed. The bidders must prove the actual deterministic allocation and prices that the optimal direct mechanism would have obtained when given seed \( r \) to sample exactly from the optimal distribution. If \( r \) does not have sufficient bits to determine the sample, then the bidders prove interactively that additional bits are needed, and further random bits are appended to \( r \). If the bidders ever fail in their proof, the mechanism terminates and they are charged a large fine. Regardless of whether or not the players ever attempt to prove an incorrect statement, the protocol terminates in expected polynomial time.

We omit a formal proof of the above observations. We also note that we do not present the mechanisms of this section as practical. But we want to point out that the complexity of the optimal mechanism design problem may become trivial if no computational assumptions
are placed on the bidders.
Part III

Maximizing Welfare
Chapter 8

Historical Overview

In absence of collusion, the VCG mechanism [63, 18, 32] guarantees efficiency in dominant strategies even in unrestricted combinatorial auctions. This guarantee, however, no longer holds in the presence of unrestricted rational collusion, that is, when the players may secretly partition themselves into an arbitrary number of coalitions and the members of each coalition can perfectly coordinate their strategies by entering secret binding agreements (and make secret side-payments to each other) so as to maximize the sum of their individual utilities. Ausubel and Milgrom [4] show that just two such players can destroy the VCG efficiency guarantee. Chen and Micali [16] point out that even the efficiency of the Vickrey mechanism [63] (essentially the VCG mechanism specialized to multi-unit auctions) is vulnerable to just two collusive players holding wrong beliefs about the valuations of their opponents.

To combat collusion, Green and Laffont [31] put forward the notion of coalition incentive compatibility. Focusing on auctions, their notion (i) envisages that every player $i$ bids a valuation $v_i$, and (ii) guarantees that, for every subset $C$ of players and each player $i$ in $C$, there exists a valuation $v_i$ such that bidding the subprofile $v_C$ is weakly dominant with respect to the sum of the individual utilities of the players in $C$. However, Green and Laffont themselves prove the non-existence of coalition incentive compatible mechanisms for a variety of tasks, including efficiency in auctions of a single good in limited supply.

Chen and Micali [16] bypass this impossibility result for auctions of a single good in limited supply via the notion of collusive dominant-strategy truthfulness. Here, the pure strategies available to a player $i$ are pairs $(\theta_i, C)$, where $\theta_i$ is a possible valuation of $i$
(allegedly his true valuation), and $C$ is a subset of the players (allegedly, the set of his colluders). Collusive dominant strategy truthfulness means that, for each coalition $C$, every member truthfully reporting his valuation and the set $C$ is very weakly dominant for maximizing the sum of the individual utilities of the players in $C$. Two drawbacks, however, limit the applicability of their mechanism for auctioning $m$ identical copies of the same good:

1. The mechanism must know in advance a value $V$ which upper-bounds the maximum possible value any player has for a copy of the good, and

2. The mechanism may impose large fines (specifically, larger than $mV$) even when all bids are low.

These features of Chen and Micali's mechanism are problematic because no such upper bound may exist, let alone be known to the mechanism. Moreover, to ensure that it exceeds any possible value a player may have for a copy of the good, $V$ might be astronomically high, and $mV$ even higher. Therefore, a player may not have sufficient funds to pay such a fine. If it is not credible that the envisaged fine is enforceable, then the whole dominant-strategy structure of their mechanism collapses.

Weaker notions of collusion resilience have also been considered. As put forward by Schummer [57], bribe proofness requires that no two players can benefit by colluding together in a particular manner. This notion too presupposes that a player's strategy space coincides with the set of his possible valuations, and several impossibility results have been shown in this framework: the first by Schummer himself, and others by Mizukami [44].

The notion of $c$-truthfulness, due to Goldberg and Hartline [30], assumes that each coalition has at most $c$ members. Other notions of collusion resiliency have been studied when the players are incapable of making side payments, in particular by [41, 46, 61, 45, 6, 36, 47, 28, 51]. Laffont and Martimort [38] and Che and Kim [15] also study collusion resiliency using equilibrium-based solution concepts. Collusion leveraging, as put forward by Chen, Micali, and Valiant [17], aims at leveraging the players' knowledge about the payoff types of their opponents.
Chapter 9

Collusion Resilience, Dominant Strategies, and Efficient Auctions

9.1 Informal Discussion

We first provide a brief overview of our notion of collusion-resiliency, a summary of our results, and a few remarks about our model.

9.1.1 Our Notion of Resiliency

Our notion of dominant-strategy implementation resilient to unrestricted rational collusion is more general than the ones cited in Chapter 8. As before, we demand that each independent player has a "best" strategy no matter what all other players do, and each coalition $C$ has a best strategy subprofile, no matter what the players outside $C$ may do. (Also as before, each player belongs to at most one coalition. Else, he may not always be able to act so as to simultaneously maximize the joint utility of each of his coalitions.)

What differentiates us from the above prior approaches is that we do not envisage any restrictions on the strategy spaces of the players. A player's strategy was assumed to coincide with reporting one of his possible types in Green and Laffont [31], and with reporting a possible type and a set of players (i.e., a possible coalition) in Chen and Micali [16]. Envisaging an unrestricted strategy space may seem a small point, but has significant implications on whether efficiency is possible or impossible to achieve in a sufficiently complex
Notice that giving a player the ability of reporting a coalition to which he belongs presupposes the “decriminalization” of collusion. Criminalizing collusion is far from necessary, and is in fact mostly due to our inability to design mechanisms that are collusion resilient. In fact, if we care about “guaranteeing efficiency in dominant strategies whether or not the players collude,” then we must incentivize the players to report some information about collusion.

9.1.2 Our Results

We prove two results about efficient auctions in the presence of unrestricted rational collusion. On the positive side, we prove that efficiency in dominant strategies (in our general sense) can be guaranteed for multi-unit auctions with decreasing marginal valuations by means of a very practical mechanism. Our mechanism avoids the two aforementioned drawbacks of [16]. Indeed, in our case, not only does the mechanism designer need no prior knowledge about who colludes with whom, but the designer need not know any bound on the players’ valuations. Furthermore, the mechanism does not impose on any player a fine larger than the sum of values reported by that player.

On the negative side, we prove that even under our exceedingly general notion of collusion resilience, there does not exist any dominant-strategy resilient mechanism for combinatorial auctions, practical or not, in the simple case of auctioning two goods to three players.

9.1.3 Remarks

- **Collusion Choice.** Let us emphasize that the coalitions are not fixed beforehand. Rather, after the designer announces the mechanism, the players are free to choose with whom to secretly collude. The mechanism has no information about which coalitions are formed, and no limitations are assumed on the number or size of the coalitions. The only assumption is that once formed, a coalition is rational: that is, its members act so as to maximize the sum of their individual utilities.

- **Hard Cases.** The hard case for efficiency in an unrestricted collusive setting is generally not when all players belong to the same coalition, the “grand coalition.” Indeed,
for (essentially) all mechanisms, the grand coalition always has a dominant strategy profile. A more difficult case is when there are multiple coalitions of cardinality greater than 1.

- **Full Collusion Revelation ("Hyper-truthfulness").** As we shall see, rather than disallowing or limiting the reporting of collusiveness, our notion allows for a mechanism to elicit from a player any information deemed useful about the coalition to which he belongs. Indeed, we call a mechanism hyper-truthful if it asks a player to report the “maximum amount of information” about his own coalition, and argue, in line with the revelation principle, that focusing on such mechanisms poses no restrictions. In a sense, rather than trying to ban collusion, we totally embrace it.

- **Power and Limitations.** Our general notion of collusion resilience in dominant strategies can help, as in the case of efficiency in multi-unit auctions, the design of collusion-resilient mechanisms, but is not a panacea. For such such a strong solution concept there are, as for the case of efficiency in combinatorial auctions, social goals unachievable in the presence of unrestricted collusion.

### 9.2 Preliminaries

**Auction Contexts** In an auction, a player’s type is also called a valuation.

- In a multi-good auction, a valuation is a function mapping the empty subset to 0, and every other subset of the goods to a non-negative real number.

The set of players is \( N = \{1, 2, \ldots, n\} \), the number of goods is \( m \), the set of all possible valuations of a player \( i \) is \( \Theta_i \), and the true valuation of player \( i \) is \( \theta^*_i \). An outcome \( \omega \) is a pair \((A, P)\), where \( A = (A_0, \ldots, A_n) \) is a vector of subsets of the goods and \( P \) is a profile of numbers. Vector \( A \) is referred to as the allocation of \( \omega \) and must be such that \( A_i \cap A_j = \emptyset \) whenever \( i \neq j \). Component \( A_0 \) represents the set of unallocated goods, and, for \( i > 0 \), \( A_i \) represents the subset of the goods allocated to player \( i \). Each \( P_i \) represents the price paid by player \( i \). For each player \( i \), \( i \)'s utility function \( u_i \) maps a valuation \( \theta_i \) and an outcome \( \omega = (A, P) \) to \( u_i(\theta_i, \omega) = \theta_i(A_i) - P_i \). For brevity, when the true valuation of a player \( i \) is clear, we may write \( u_i(\omega) \) instead of \( u_i(\theta^*_i, \omega) \). An allocation \( A \) is efficient if \( \sum_i \theta^*_i(A_i) \geq \sum_i \theta^*_i(A'_i) \) for all allocations \( A' \).
In a m-unit auction, there are m identical copies of the same good for sale.

A valuation of a player $i$ is a sequence of non-negative real numbers, $t_i = t_i^{(1)}, \ldots, t_i^{(m)}$, such that $t_i^{(1)} \geq \cdots \geq t_i^{(m)} \geq 0$. That is, $t_i^{(j)}$ represents $i$'s marginal value for a $j$-th copy of the good, and valuations have non-increasing marginals. An allocation $A$ is a sequence of $n + 1$ non-negative values whose sum is $m$, $A = A_0, A_1, \ldots, A_n$, where $A_0$ is the number of unallocated copies and, for $i > 0$, $A_i$ is the number of copies allocated to player $i$. An outcome $\omega$ is a pair $(A, P)$, where $A$ is an allocation and $P$ a profile of prices. The utility of a player $i$ with valuation $t_i$ for an outcome $\omega = (A, P)$ is $\sum_{j=1}^{A_i} t_i^{(j)} - P_i$. All other notions and notations for multi-good auctions (such as efficiency and $u_i(\omega)$) automatically extend to multi-unit ones.

**Auction Mechanisms**  A mechanism $M$ for an auction context with $n$ players specifies:

- For each player $i$, the set $S_i$ of pure strategies available to $i$, and

- A function (traditionally also denoted by $M$) mapping each strategy profile in $S = S_1 \times \cdots \times S_n$ to an outcome $(A, P)$.

For $s \in S$, we denote by $M(s)$ the outcome generated by $M$ and by $u_i(M(s))$ the corresponding utility of player $i$. If $M$ is probabilistic, $M(s)$ is a distribution over outcomes, and $u_i(M(s))$ is the corresponding expected utility of $i$. If the underlying mechanism $M$ is clear, we may write $u_i(s)$ instead of $u_i(M(s))$.

**Additional Notation**  Let $i$ be a player and $A$ a subset of players. Then,

- For every profile $x$, $x_A$ is the subprofile obtained by restricting $x$ to $A$.

- $S_A$ and $\Theta_A$ respectively are the Cartesian product $\Pi_{i \in A} S_i$ and $\Pi_{i \in A} \Theta_i$.

- $-i$ is the set $N \setminus \{i\}$, and $-A$ is the set $N \setminus A$.

### 9.3 Collusive Rationality

Recall that a partition of a set $T$ is a collection of subsets of $T$, $T_1, \ldots, T_k$, such that $\bigcup_{i=1}^{k} T_i = T$ and $T_i \cap T_j = \emptyset$ whenever $i \neq j$.\footnote{If $A_i = 0$, then, according to usual conventions, $\sum_{j=1}^{A_i} t_i^{(j)}$ is $0$.}
The Green-Laffont model  Before the mechanism is executed (but possibly after the mechanism has been announced), the players are free to form an arbitrary partition of \( N \). The formed partition, the **collusive partition**, is denoted by \( C \). A set in \( C \) is called a **coalition**. A player \( i \) is independent if \( \{i\} \in C \).

If \( C \subseteq C \), then \( C \) is common knowledge to its members, but a player in \( C \) need not have any additional information about \( C \). The mechanism has no information about \( C \).

Focussing on auctions, for each subset \( C \) of the players, the **collective utility function** of \( C \), \( u_C \), maps a valuation subprofile \( \theta_C \in \Theta_C \) and an outcome \( \omega \) to \( \sum_{i\in C} u_i(\theta_i, \omega) \). When the valuation subprofile \( \theta_C \) under consideration is clear, we may write \( u_C(\omega) \) instead of \( u_C(\theta_C, \omega) \). If \( C \) is a coalition, then its players can perfectly coordinate their actions,\(^2\) and act so as to maximize \( u_C \).

**Collusive solution concepts**  A pure strategy subprofile \( s_A \in S_A \) is **(weakly) dominant** for a subset of players \( A \) if \( u_A(s_A, s_{-A}) = \max_{s'_A \in S_A} u_A(s'_A, s_{-A}) \) for all \( s_{-A} \in S_{-A} \).

A mechanism is **collusive dominant-strategy** if for every subset of players \( A \) and every true valuation subprofile \( \theta_A^* \in \Theta_A \) there exists a pure strategy subprofile \( d_A \in S_A \) which is dominant for \( A \).

A collusive dominant strategy mechanism is (ex-post) **coalitionally rational** if for every subset of players \( A \), every true-valuation subprofile \( \theta_A^* \in \Theta_A \), and every strategy subprofile \( s_{-A} \in S_{-A} \), the sum utility of \( A \) is non-negative under \( d_A \). That is, \( u_A(d_A, s_{-A}) \geq 0 \).

**Efficiency in collusive dominant-strategies**  A collusive dominant-strategy auction mechanism \( M \) is **efficient** if, for every collusive partition \( C = C_1 \cup C_2 \cup \cdots \cup C_k \) and every true valuation profile \( \theta^* \), the outcome \( M(d_{C_1}, \ldots, d_{C_k}) \) is an efficient allocation for \( \theta^* \).

9.4  **A Practical Collusive Dominant-Strategy Mechanism for Multi-Unit Auctions**

Our multi-unit auction mechanism modifies the one of Vickrey [63].

\(^2\)E.g., the members of a coalition can enter binding agreements and make side payments to each other
9.4.1 The Standard Vickrey Mechanism

In the mechanism of Vickrey [63] for \( m \)-unit auctions, a strategy consists of reporting a single valuation. Given a profile \( t \) of reported valuations, the mechanism constructs a sequence of \( n \cdot m \) "value-owner" pairs \( \{(t_{i}^{(k)}, i) : i \in N, k = 1, \ldots, m\} \), ordered in decreasing order with respect to the first ("value") component. We call the first \( m \) pairs in the sequence the "winning pairs" and all other ones the "losing pairs". For each player \( i \), we let \( m_i \) be the number of winning pairs with owner \( i \). The mechanism allocates \( m_i \) copies of the good to \( i \), identifies the first \( m_i \) losing pairs whose owner is not \( i \), and charges \( i \) the sum of the values of these pairs.

The Vickrey mechanism is efficient in dominant strategies in absence of collusion, but not otherwise [16].

9.4.2 The Intuition Behind Our Mechanism \( \mathcal{M} \)

To describe our mechanism \( \mathcal{M} \), we find it useful to present first an 'auxiliary' mechanism \( \mathcal{M}' \) guaranteeing efficiency under a solution concept that is very strong, but weaker than dominant strategies. We then show how to modify \( \mathcal{M}' \) to obtain our collusive dominant-strategy, collusively rational, and efficient mechanism \( \mathcal{M} \).

Mechanism \( \mathcal{M}' \). In \( \mathcal{M}' \), each player reports a sequence of \( m \) "value-beneficiary" pairs, \((v_1, b_1), \ldots, (v_m, b_m)\), where \((v_c, b_c) \in \mathbb{R} \times N\) signifies that the bidding player is willing to pay an amount \( v_c \) in order for player \( b_c \) to receive an additional copy of the good. For instance, if \( n = 5, m = 3 \) and player 2 reports \((10, 3), (8, 2), (4, 3)\), then player 2 declares that he is willing to pay to the mechanism \$10 for the first copy of player 3, \$8 for his own first copy, and \$4 for the second copy of player 3.

Given all these reports, \( \mathcal{M}' \) transforms each "value-beneficiary" pair into a "value-beneficiary-owner" triplet. If, as in the example above, player 2 reports \((4, 3)\) as one of his pairs, then \( \mathcal{M}' \) constructs \((4, 3, 2)\) as the corresponding triplet. After that, \( \mathcal{M}' \) orders all \( n \cdot m \) triplets in decreasing order by their value component, breaking ties by the beneficiary component (in any fixed order), and then by the owner component (in any fixed order). We call the first \( m \) triplets in the sequence the "winning triplets" and all others the "losing triplets". For each player \( i \), let \( m_i \) be the number of winning triplets with beneficiary \( i \), and let \( m'_i \) be the number of winning triplets with owner \( i \). Then, \( \mathcal{M}' \) allocates \( m_i \) copies of the
good to \( i \), identifies the first \( m' \) losing triplets whose owner is not \( i \), and charges \( i \) the sum of the value components of these triplets.

**An Intermediate Solution Concept.** As \( M' \) is not our final mechanism, we do not define the solution concept under which \( M' \) can be proved to be efficient, nor do we discuss such a proof. We simply provide some intuition in a very informal manner.

Assume for a moment that each player only bids value-beneficiary pairs whose beneficiary belongs to his own coalition. Then, it can be proven that the following strategy subprofile is dominant for each coalition \( C \): one member of \( C \), \( i^* \), bids the \( m \) pairs \((v_1, b_1), \ldots , (v_m, b_m)\), corresponding to the \( m \) highest marginal valuations of players in \( C \), and all other members of \( C \) bid pairs with value 0. We refer to such a report as a “smart strategy subprofile”. Clearly, when all coalitions choose smart strategy subprofiles, the allocation returned by \( M' \) is efficient.

However, if some player \( j \not\in C \) bids a pair \((v, b)\) where \( b \in C \), then the above strategy subprofile need not be dominant for \( C \). For instance, suppose there are two copies of the same good, and two players, 1 and 2, both independent and both valuing $10 a first copy and $5 a second one. Assume that player 1 truthfully bids the sequence \(($10, 1), ($5, 1)\), but player 2, for whatever reason, bids \(($8, 1), ($7, 2)\). Under this bid profile, player 1 gets both goods (which of course is not an efficient allocation) and pays $7, so that his utility is $8. Player 1 would be better off, however, bidding \(($5, 1), ($0, 1)\), so as to get one copy for a price of zero and have a net utility of $10. When player 1 is truthful, player 2 gets no goods, pays $5, and his utility is $-5. Notice that reporting \(($8, 1), ($7, 2)\) is, for player 2, dominated by reporting \(($8, 2), ($7, 2)\). More generally, in \( M' \), it is always best for a player to name only beneficiaries in his own coalition.

In sum, reporting a smart strategy subprofile is certainly “smart” but not quite dominant.\(^3\)

**From \( M' \) to \( M \).** We modify \( M' \) by having each player \( i \) bid either (1) a “representative” player \( j \in N \) or (2) a sequence of \( m \) “value-beneficiary” pairs \((v_1, b_1), \ldots , (v_m, b_m)\). In the first case, only the representative \( j \) has permission to declare a value-beneficiary pair with \( i \) as a beneficiary, while in the second case no other player has permission to declare a pair.

\(^3\)Indeed, if every coalition \( C' \) eliminates all strategy subprofiles that are dominated for \( C' \), then for every coalition \( C \), a smart strategy subprofile is dominant for \( C \) with respect to all surviving strategies.
with \( i \) as a beneficiary. If any player declares a value-beneficiary pair with a disallowed beneficiary, that pair is discarded by the mechanism.

We must ensure, however, that \( i \) never has incentive to name a player outside of his collusive group as a representative. Namely: if any player \( k \) reports a pair with beneficiary \( i \), and if \( k \) is not \( i \)'s declared representative, then the mechanism not only discards the pair but also forces \( k \) to pay to \( i \) an amount of money equal to the value of the discarded pair. With this modification, \( i \) will never have incentive to name an outside player \( k \) as his representative: \( i \) would be better off discarding \( k \)'s reported pair, receiving the money from \( k \), and (if desired) reporting the pair himself.

### 9.4.3 Our Mechanism \( \mathcal{M} \)

**Strategies.** Our mechanism is of normal form. Every player \( i \), simultaneously with his opponents, reports a strategy \( s_i \in N \cup (\mathbb{R}^+ \times N)^m \). That is, for every strategy \( s_i \) of player \( i \), either

- \( s_i \) is a player, \( s_i \in N \), or
- \( s_i \) is a sequence of \( m \) pairs: \( s_i = (v_1, b_1), \ldots, (v_m, b_m) \in (\mathbb{R}^+ \times N)^m \).

A player \( i \) reporting \( s_i \in N \) is called passive. In this case, we refer to \( s_i \) as "\( i \)'s representative," denoted by \( Rep(i) \).

A player \( i \) reporting \( s_i = (v_1, b_1), \ldots, (v_m, b_m) \in (N \times \mathbb{R}^+)^m \) is called active. In this case we refer to each \( (v_j, b_j) \) as a value-beneficiary pair, with value \( v_j \) and beneficiary \( b_j \).

**Choosing the Outcome.** When the players report a strategy profile \( s \), the mechanism chooses to allocate the copies of the goods and the price charged to each player by means of the following steps:

1. Initialize \( L \) to be an empty list of "value-beneficiary-owner" triplets.

2. For each active player \( i \) and value-beneficiary pair \( (v, b) \) reported by \( i \):

   (a) If either (i) \( b = i \) or (ii) \( b \) is passive and \( Rep(b) = i \),

   then append the value-beneficiary-owner triplet \( (v, b, i) \) to \( L \).
(b) Otherwise, $i$ makes a payment of $v$ to $b$.

(Call $R_i$ the net payment of $i$ after all executions of step 2b.)

3. Sort the value-beneficiary-owner triplets of $L$ in decreasing order by their value component. (If needed, break ties first by beneficiary, and then by owner, in lexicographic order.)

(Call the first $m$ triplets “winning” and the remaining triplets “losing.”)

4. Each player $j$ receives $m_j$ copies of the good, where $m_j$ is the number of winning triplets with beneficiary $j$.\(^4\)

5. For each active player $i$:

(a) Let $m'_i$ be the number of winning triplets with owner $i$.

(b) Identify the first $m'_i$ losing triplets in $L$ whose owner is not $i$ and (in addition to any payments from step 2b) have player $i$ pay to the mechanism an amount $Q_i$ equal to the sum of the value components of these triplets.\(^5\)

Note that, in $\mathcal{M}$, the final price of a player $i$ is $P_i = R_i + Q_i$. (As usual, if $P_i$ is positive, then $i$ disburses money, and otherwise he receives money.)

### 9.4.4 Analysis

We now prove the following theorem:

**Theorem 6.** The mechanism $\mathcal{M}$ is collusive dominant-strategy, coalitionally rational, and efficient.

**Proof.** Consider an arbitrary collusive group $A$. We begin with three claims:

Claim 14. Arbitrarily fix a strategy subprofile $s_A$ for the players in $A$ and a strategy subprofile $s_{-A}$ for the other players. Then there exists a strategy subprofile $\bar{s}_A \in S_A$ such that

\[
(a) \quad u_A(\bar{s}_A, s_{-A}) \geq u_A(s_A, s_{-A}),
\]

\(^4\)If $L$ contains fewer than $m$ winning triplets, the remaining goods are unallocated.

\(^5\)If there are fewer than $m'_i$ such triplets, instead charge $i$ the sum of the values of all such triplets.
(b) for all \( i \in A \),

if \( i \) is passive in \( \tilde{s}_i \), then \( \tilde{s}_i \in A \)

(i.e., \( i \) chooses his representative in his own coalition)

if \( i \) is active in \( \tilde{s}_i \), then the beneficiaries of all pairs in \( \tilde{s}_i \) are in \( A \).

**Proof of Claim 14:** For all \( i \in A \), we construct \( \tilde{s}_i \) by modifying the original strategy \( s_i \) according to the following (exhaustive) three cases.

1. Player \( i \) is passive in \( s_i \) and nominates a player \( j \in A \) as his representative.

   Then \( \tilde{s}_i = s_i \).

2. Player \( i \) is active in \( s_i \) and \((v_1, j_1), \ldots, (v_m, j_m)\) are his value-beneficiary pairs.

   Then player \( i \) is active in \( \tilde{s}_i \). He bids all pairs \((v_i, j_i)\) from \( s_i \) for which \( j_i \in A \). For the remaining pairs, if any, he bids \((0, i)\).

3. Player \( i \) is passive in \( s_i \) but nominates a player \( j \notin A \) as his representative.

   Then player \( i \) is active in \( \tilde{s}_i \) and chooses his \( m \) value-beneficiary pairs as follows:

   - If \( j \) is passive in \( s_{-A} \), then \( \tilde{s}_i \) consists of \( m \) pairs \((0, i)\).

   - If \( j \) is active in \( s_{-A} \), letting \((v_1, i), \ldots, (v_k, i)\) denote all the value-beneficiary pairs of \( s_j \) with beneficiary \( i \), set \( \tilde{s}_i \) to consist of the \( k \) pairs \((v_1, i), \ldots, (v_k, i)\) and the \( m - k \) pairs \((0, i)\).

Let us first show that each player in \( A \) receives at least as many copies of the good under \((\tilde{s}_A, s_{-A})\) as he does under \((s_A, s_{-A})\). Indeed, we shall argue that the triplets with beneficiary in \( A \) inserted into \( L \) under these two strategy profiles differ only in their owner components, with the exception that, under \((\tilde{s}_A, s_{-A})\), additional triplets may be inserted into \( L \) with value 0 and beneficiary in \( A \). This holds because of the following observations:

- For any \( i \in A \) who is active in \( s_i \), we have only removed his pairs \((v, j)\) when \( j \notin A \). Furthermore, for any pair \((v, j)\) with \( j \in A \), this pair "survives" into \( L \) under \( s \) only if either \( j = i \) or \( j \) is passive with \( \text{Rep}(j) = i \). In both of these cases, the pair will also survive into \( L \) under \((\tilde{s}_A, s_{-A})\).
• For every \( i \in A \) who is passive with \( \text{Rep}(i) = j \not\in A \) in \( s_i \), there are up to \( m \) triplets \( (v_1, i, j), \ldots, (v_k, i, j) \) which are inserted into \( L \) under \( (s_A, s_{-A}) \) but not under \( (\bar{s}_A, s_{-A}) \). The triplets \( (v_1, i, i), \ldots, (v_k, i, i) \) are, however, appended to \( L \) under \( (s_A, s_{-A}) \) but not under \( (s_A, s_{-A}) \).

Therefore, changing from \( (s_A, s_{-A}) \) to \( (\bar{s}_A, s_{-A}) \) affects only the owner component of triplets inserted into \( L \) with beneficiary in \( A \), possibly appends additional triplets with value 0, and might remove triplets with beneficiary outside \( A \). Since \( \mathcal{M} \) breaks ties according to the beneficiary components of triplets before the owner components, every winning triplet under \( (s_A, s_{-A}) \) has a corresponding winning triplet under \( (\bar{s}_A, s_{-A}) \).

Finally, we claim that the net amount charged to \( A \) (i.e., \( \sum_{i \in A} P_i \)) under \( (\bar{s}_A, s_{-A}) \) is no more than the net amount charged to \( A \) under \( (s_A, s_{-A}) \). Indeed, if a player \( i \in A \) is passive in \( s_i \) with \( \text{Rep}(i) \in A \), then \( R_i \) is the same and \( Q_i = 0 \) in both scenarios. If \( i \) is passive in \( s_i \) with \( \text{Rep}(i) \not\in A \), then the amount he is newly charged in step 5b of \( \mathcal{M} \) is no more than the additional amount he receives in step 2b. That is, in the original scenario \( Q_i \) was 0, and in the new scenario has become positive, yet the increase in \( Q_i \) is no more than the decrease in \( R_i \). Thus, it is in \( i \)'s interest to "discard" triplets with owner outside of \( A \), receive payment equal to the discarded triplets’ values, and declare the appropriate pairs himself.

Finally, if \( i \) is active in \( s_i \), then \( R_i \) either stays the same or decreases in the second scenario, while \( Q_i \) may increase or decrease. However, denoting by \( A' \) the subset of the coalition \( A \) who are active in \( s_A \), a simple case analysis shows that \( \sum_{j \in A'} Q_j \) does not increase in \( (\bar{s}_A, s_{-A}) \). The only crucial case to consider is when, for some \( i, j \in A', \ell \in A, \) and \( k \not\in A \), a triple \( (x, k, i) \) was winning in \( (s_A, s_{-A}) \), but after replacing \( (x, k, i) \) with \( (0, i, i) \) a previously losing triple \( (x', \ell, j) \) now wins in \( (\bar{s}_A, s_{-A}) \). In this case, \( Q_j \) increases by at most \( x' \), while \( Q_i \) decreases by at least \( x' \).

\[ \square \]

**Claim 15.** Arbitrarily fix a strategy subprofile \( s_{-A} \) for \(-A\). Let \( \bar{s}_A \) be a strategy subprofile for \( A \) satisfying the second and third properties of Claim 1, namely:

• for \( i \in A \), if \( i \) is passive in \( \bar{s}_i \), then \( \bar{s}_i \in A \).

• for \( i \in A \), if \( i \) is active in \( \bar{s}_i \), then the beneficiaries of all pairs in \( \bar{s}_i \) are in \( A \).
Then there exists a subprofile \( s'_A \in S_A \) such that (i) a single player in \( A \), \( i' \), is active, (ii) all other \( j \in A \) are passive and \( s'_j = \text{Rep}(j) = i' \), and (iii) \( u_A(s'_A, s_{-A}) \geq u_A(\tilde{s}_A, s_{-A}) \).

**Proof of Claim 15:** Let \( (v_1, b_1, q_1), \ldots, (v_k, b_k, q_k) \), where \( k \leq m \), be the value-beneficiary-owner triplets which, under \((\tilde{s}_A, s_{-A})\), are "winning" in \( L \) and have beneficiary \( b_i \in A \).

Set \( s'_A \) to be the strategy subprofile where a single \( i' \in A \) bids the pairs \((v_1, b_1), \ldots, (v_k, b_k)\) along with \( m - k \) pairs \((0, i')\). Comparing the execution of step 2 of the mechanism under the strategy profiles \((s'_A, s_{-A})\) and \((s_A, s_{-A})\), we note the following differences: (i) the new triplets \((v_1, b_1, i'), \ldots, (v_k, b_k, i')\) and \( m - k \) triplets \((0, i', i')\) are appended to \( L \) under \((s'_A, s_{-A})\) but not under \((s_A, s_{-A})\), (ii) the triplets \((v_1, b_1, q_1), \ldots, (v_k, b_k, q_k)\) are appended to \( L \) under \((s_A, s_{-A})\) but not under \((s'_A, s_{-A})\), and (iii) the losing triplets with beneficiary in \( A \) which are appended to \( L \) under \((s_A, s_{-A})\) are not appended under \((s'_A, s_{-A})\).

Since for every winning triplet \((v_i, b_i, q_i)\) under \((s_A, s_{-A})\) with \( b_i \in A \) there is a corresponding triplet \((v_i, b_i, i')\) under \((s'_A, s_{-A})\) differing only in the owner component, and since the ordering on \( L \) breaks ties lexicographically by beneficiary before owner, it is clear that the allocation under \((s'_A, s_{-A})\) assigns every player in \( A \) at least as many copies of the good as under \((\tilde{s}_A, s_{-A})\). (We note \( i' \) might receive additional copies under \((s'_A, s_{-A})\) if any of the \( m - k \) new triplets \((0, i', i')\) are winning.) Furthermore, removing non-winning triplets from \( L \) does not affect the final allocation and does not increase the price \( Q_j \) charged to any player \( j \) in step 5b.

It is clear that the amount \( Q_{i'} \) charged to \( i' \) in step 5b under \((s'_A, s_{-A})\) is no more than the net amount \( \sum_{j \in A} Q_j \) charged to all of \( A \) in step 5b under \((\tilde{s}_A, s_{-A})\). Indeed, by having a single player declare all of the pairs with beneficiary in \( A \), more of the high-value triplets are skipped when computing the price in step 5b. In short, the subprofile \( s'_A \) avoids the scenario where a triplet owned by a player in \( A \) causes the price paid by a different player in \( A \) to increase. We note that any payments \( R_j \) received by players \( j \in A \) from step 2b are the same under the two strategy profiles.

**Claim 16.** Let \( d_A \) be the following strategy subprofile for \( A \):

- A single player \( i^* \in A \) is active. He declares the \( m \) triplets \((v_1, j_1), \ldots, (v_m, j_m)\) corresponding to the \( m \) highest marginal valuations amongst players in \( A \).

\[ ^6 \text{For example, suppose } A = \{1, 2\}, m = 3, \text{ player 1's value of obtaining a first, second, and third copy of the good are } \$10, \$6 \text{ and } \$2, \text{ respectively, and player 2's values of obtaining an additional} \]
• All other players $j \in A$ are passive and announce $d_j = \text{Rep}(j) = i^\ast$.

Then $d_A$ is a best response of $A$ against every $s_{-A} \in S_{-A}$.

Proof of Claim 16: By Claim 1 and Claim 2, it suffices to show that $u_A(d_A, s_{-A}) \geq u_A(s'_A, s_{-A})$, where $s'_A$ is any strategy subprofile such that a single $i' \in A$ is active and all other $j \in A$ declare $\text{Rep}(j) = i'$.

We notice that under all such strategies $s'_A$, the player $i'$ owns all the triplets in $L$ which have beneficiary in $A$. The only remaining relevant features of the strategy subprofile are the pairs which $i'$ bids. From the perspective of $i'$, he is playing a standard Vickrey auction of a single good of limited supply, where his marginal valuations are the maximum of the marginal valuations in $A$. Thus, truthfully declaring the $m$ highest marginal valuations is best for $i'$. A formal proof of this fact is nearly identical to the proof that the standard Vickrey auction is truthful.

\[\Box\]

Finally, notice that, for every profile of true valuations and every collusive partition $C = A_1 \cup A_2 \cup \cdots \cup A_k$, when all coalitions $A_i$ play the dominant subprofile $d_A$ as in Claim 3, the resulting allocation is efficient. This holds since, under this strategy profile, one representative from each $A_i$ bids truthfully on behalf of the $m$ highest marginal valuations of $A_i$, no pairs are discarded in step 2b of $\mathcal{M}$, and the triplets with the $m$ highest valuations are winners.

\[\Box\]

9.5 A Simple Generalization of the Revelation Principle

In the next section we prove that no coalitionally rational and collusive dominant-strategy mechanism exists for unrestricted combinatorial auctions. Our proof applies to mechanisms with arbitrary strategy sets. (Already for our multi-unit auction mechanism $\mathcal{M}$ the pure strategies of a player $i$ did not coincide with his set of possible valuations, $\Theta_i$.) To simplify copy of the good are $9$, $7$, and $5$. Then the active player would bid $(10, 1)$, $(9, 2)$, and $(7, 2)$. In the event of a tie among the marginal valuations, $i^\ast$ can break the tie lexicographically according to the beneficiary.
our analysis we argue, very much in the spirit of the revelation principle, that we need to consider only mechanisms with very specific strategy sets. Namely,

**Definition 23.** A collusive dominant-strategy mechanism $M$ is hyper-truthful if:

- For each player $i$, $S_i = \{(A, \theta_A) : A \subseteq N, i \in A$ and $\theta_A \in \Theta_A\}$ and

- For every $A \subseteq N$, the strategy subprofile $t_A$ where every $j \in A$ selects strategy $(A, \theta_A^*) \in S_j$ is dominant for $A$.

In every collusive dominant-strategy mechanism, the optimal strategy subprofile for a set $A$ of players depends only on the set $A$ and the true type subprofile $\theta_A^*$ of the group, and not on the strategies or collusive structure of the outside players. Therefore, hyper-truthful mechanisms suffice in view of the following fact, whose proof is nearly identical to that of the revelation principle of Myerson [48].

**Fact 2.** If there exists an efficient, coalitionally rational, collusive dominant-strategy mechanism $M$ for an auction context $C$, then there also exists an efficient, coalitionally rational, hyper-truthful mechanism $M'$ for context $C$.

For example, let us show how our mechanism $\mathcal{M}$ can be transformed into a hyper-truthful mechanism $\mathcal{M}^*$. In an execution of $\mathcal{M}^*$, denote by $(A_i, \theta_{A_i})$ the strategy selected by each player $i$. Then $\mathcal{M}^*$ works by simulating an execution of $\mathcal{M}$ as follows: If $i$ is the lexicographically first player in $A_1$, then he is made active and made to bid the $m$ highest marginal valuations in $\theta_{A_1}$. Otherwise, $i$ is made passive and made to declare his representative to be the lexicographically first player in $A_i$.

**Remark 12.** Fact 2 is not, strictly speaking, a corollary of the revelation principle. This is so because a player's information about his own coalition is not an "original type." Indeed, the players partition themselves arbitrarily into collusive groups after a mechanism is announced. In principle, for one mechanism the players might want to collude in pairs, while for another mechanism they might want to collude in triples, and so forth. Nevertheless, the proof of the above fact follows from applying the revelation principle "conditionally on the collusive partition."
9.6 An Impossibility Result for Combinatorial Auctions

Theorem 7. No collusive dominant-strategy mechanism can both be coalitionally rational and guarantee efficiency in combinatorial auctions with at least 3 players and 2 goods. 7

Proof. In light of Fact 2, it suffices to prove our thesis for mechanisms which guarantee efficiency in collusively dominant hyper-truthful strategies.

We proceed by contradiction. Assume the existence of an efficient, coalitionally rational, dominant-strategy hyper-truthful mechanism \( \mathcal{M} \) for combinatorial auctions with 3 players and 2 goods. We derive a contradiction by showing that, when no upper-bound exists for a player's value for a subset of the goods, then \( \mathcal{M} \) must pay an infinite amount of money to the players when they report a special valuation profile. More precisely, for every \( x > 0 \), as long as every player may value \( 5x \) a subset of the goods, then \( \mathcal{M} \) must return revenue \( < -x \) whenever all players report that they belong to the same coalition and that they all value 0 every subset of the goods. (This would already be a contradiction for the case of bounded valuations, if the mechanism were required never to lose money.)

We obtain this contradiction via a sequence of 7 scenarios. In the first scenario, all players are independent and the true context and the bids are as follows:

<table>
<thead>
<tr>
<th>Player</th>
<th>Coalition Type</th>
<th>Type</th>
<th>( \text{Truth} )</th>
<th>( \text{Bids} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( {1} )</td>
<td>( {a} )</td>
<td>( 5x )</td>
<td>( 1 {1} {5x, 0, 5x} )</td>
</tr>
<tr>
<td>2</td>
<td>( {2} )</td>
<td>( {b} )</td>
<td>( 5x )</td>
<td>( 2 {2} {0, 5x, 5x} )</td>
</tr>
<tr>
<td>3</td>
<td>( {3} )</td>
<td>( {a, b} )</td>
<td>( 2x )</td>
<td>( 3 {3} {0, 0, 2x} )</td>
</tr>
</tbody>
</table>

Scenario 1

Faced with these bids, \( \mathcal{M} \) must allocate good \( a \) to player 1, good \( b \) to player 2, and nothing to player 3, since it is efficient. Let us now argue that the prices charged in Scenario 1, \( P_1^{(1)} \), \( P_2^{(1)} \), and \( P_3^{(1)} \), are \( \leq 0 \). We note first that \( P_3^{(1)} \leq 0 \) follows immediately from individual rationality, since player 3 receives no goods. Suppose now that \( P_1^{(1)} = \epsilon > 0 \).

\(^7\)If \( n > 3 \) or \( m > 2 \), we can extend the proof below by making the extra players and goods "irrelevant".
Then player 1 would have incentive to change his bid to \((1, (\epsilon/2, 0, \epsilon/2))\). In fact, by changing his strategy in this manner, while the bids of the other two players are unchanged, he would still be allocated the good (because the so modified bid profile could have been truthful and \(\mathcal{M}\) is efficient), but would be charged at most \(\epsilon/2\) (because the so modified bid profile could have been truthful and \(\mathcal{M}\) is individual rational). This contradicts the fact that bidding truthfully is a dominant strategy for player 1 in Scenario 1. Therefore, \(P_1^{(1)} \leq 0\). A similar argument shows that \(P_2^{(1)} \leq 0\).

Consider now the following scenario, borrowed from Ausubel and Milgrom [4]:

<table>
<thead>
<tr>
<th>Truth</th>
<th>Bids</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player</td>
<td>Coalition</td>
</tr>
<tr>
<td></td>
<td>{a}</td>
</tr>
<tr>
<td>1</td>
<td>{1, 2}</td>
</tr>
<tr>
<td>2</td>
<td>{1, 2}</td>
</tr>
<tr>
<td>3</td>
<td>{3}</td>
</tr>
</tbody>
</table>

Scenario 2

That is, players 1 and 2 are collusive but not truthful, so that the bids are identical to those of Scenario 1. Accordingly, \(\mathcal{M}\) returns in Scenario 2 the same outcome as in Scenario 1. In particular, in Scenario 2 the collective utility of coalition \(\{1, 2\}\) is at least \(x\). Consider now the following scenario:

<table>
<thead>
<tr>
<th>Truth</th>
<th>Bids</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player</td>
<td>Coalition</td>
</tr>
<tr>
<td></td>
<td>{a}</td>
</tr>
<tr>
<td>1</td>
<td>{1, 2}</td>
</tr>
<tr>
<td>2</td>
<td>{1, 2}</td>
</tr>
<tr>
<td>3</td>
<td>{3}</td>
</tr>
</tbody>
</table>

Scenario 3

The true context of Scenario 3 is identical to that of Scenario 2, but now players 1 and 2 are also truthful. Accordingly, since \(\mathcal{M}\) is hyper-truthful, it must produce an outcome in which the collective utility of coalition \(\{1, 2\}\) is at least \(x\). (Else, players 1 and 2 could increase their collective utility by bidding as in Scenario 2.) Furthermore, because \(\mathcal{M}\) must
return an efficient allocation when the bids are truthful, it must allocate both goods to player 3. Thus, to ensure that the collective utility of \( \{1, 2\} \) is at least \( x \), \( A \) must pay to players 1 and 2 a total of at least \( x \) (i.e., \( P_1^{(3)} + P_2^{(3)} \leq -x \)). We now consider Scenario 4:

<table>
<thead>
<tr>
<th>Player</th>
<th>Coalition</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{1,2}</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>{1,2}</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>{3}</td>
<td>2x</td>
</tr>
</tbody>
</table>

Scenario 4

Since the bids of Scenario 4 are identical to those of Scenario 3, \( A \) must return the same outcome. That is, the coalition \( \{1, 2\} \) obtains no items but receives a total payment of at least \( x \). We now consider Scenario 5:

<table>
<thead>
<tr>
<th>Player</th>
<th>Coalition</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{1,2}</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>{1,2}</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>{3}</td>
<td>2x</td>
</tr>
</tbody>
</table>

Scenario 5

Under the bids of Scenario 5, since \( A \) is efficient, both goods must be allocation to player 3. Furthermore, since \( A \) is hyper-truthful, the coalition \( \{1, 2\} \) must collectively receive at least as much utility as in Scenario 4, and thus must collectively receive a net payment of at least \( x \). Finally, player 3 could only receive money from \( A \) (that is, \( P_3^{(5)} \leq 0 \)), since otherwise (analogously to the argument above) he would have incentive to lower his bid and still receive the goods. We now consider Scenario 6:
The bids of scenario 6 are identical to those of scenario 5. Thus, the collective utility of coalition \( \{1, 2, 3\} \) is \( -F_1^{(5)} - F_2^{(5)} - F_3^{(5)} \), which is at least \( x \). Finally, we consider Scenario 7:

Since \( \mathcal{M} \) is dominant-strategy hyper-truthful, the coalition \( \{1, 2, 3\} \) must obtain at least as much net utility in Scenario 7 as they do in Scenario 6. Thus, the three players together must receive a net payment of at least \( x \).

**Final Remarks**

Theorem 1 tells us that mechanism \( \mathcal{M} \) guarantees perfect efficiency in an \( m \)-unit auction, in a tough collusion model, if all players are perfectly rational. Note that \( \mathcal{M} \) does not require that the valuations of the members of a coalition \( C \) are common knowledge within \( C \). It only requires that the \( m \) value-beneficiary pairs announced by the representative of a coalition \( C \) are indeed ‘optimal’ for \( C \). Not even this value-beneficiary information, however, need be common knowledge within \( C \).\(^8\) Also note that \( \mathcal{M} \) has some form of robustness. Informally speaking, when the players are not perfectly rational, but are rational enough to avoid fines and to have the representative of each coalition \( C \) report \( m \) value-beneficiary pairs that are sufficiently good for \( C \), then \( \mathcal{M} \) generates a sufficiently good social welfare.

\(^8\)Possibly, the optimal \( m \) pairs could arise from some sort of iterative process within \( C \).
Theorem 2 holds because the mechanism must work properly no matter how high a player's value for a subset of the goods may be. However, if (1) an upper bound \( V \) to every possible value that a player may have for a subset of the goods exists, and (2) negative revenue is not a problem, then there is an hyper-truthful, collusively rational, and collusive dominant-strategy mechanism that guarantees efficiency in unrestricted combinatorial auctions by "generating very negative revenue."\(^9\)

As collusion is a major problem in mechanism design, it is useful to develop strong notions of collusion resilience. It is also useful to construct practical mechanisms that, like \( M \) for multi-unit auctions, guarantee such resilience in dominant strategies for practical applications. The fact that no such mechanisms exist for unrestricted combinatorial auctions is a 'fact of life' (but such combinatorial auctions are also problematic in many other ways). This fact too, however, is useful to know.

\(^9\)Very Informally, such a mechanism \( M' \) can be constructed by modifying the one of [16] as follows.

If the reports of all players are consistent (i.e., all declared coalitions and valuations subprofiles match), then \( M' \) (1) gives a reward of \( kV \) to each player reporting to belong to a coalition of size \( k \), and (2) "conceptually coalesces each coalition \( C \) to a single player, conceptually runs the VCG mechanism with these coalesced players so as to compute the subset of goods \( A_C \) and price \( P_C \) of each coalition \( C \), charges \( P_C \) to —say— the lexicographically first player of \( C \), and finally uses the declared valuation subprofile of \( C \) to allocate efficiently, and \textit{ad personam}, the goods in \( A_C \) within \( C \)."

If, instead, a player \( i \) claims to be colluding with \( j \) but either (a) \( j \)'s declared coalition differs from the coalition declared by \( i \), or (b) \( i \) declares a different valuation function for \( j \) than \( j \) declares for himself, then \( M' \) imposes to each such player \( i \) a fine of \( 2n^2V \) payable to the mechanism and a huge fine payable to \( j \), and no goods are allocated.
Appendix A

Optimal Mechanism Design via Optimal Transport

The strong duality theorem of Chapter 4 is very powerful, as it shows that every optimal mechanism has a certificate (in the form of a Radon measure $\gamma$) proving its optimality. In practice, however, finding such a certificate is not always easy. The constraint $\gamma_1 - \gamma_2 \succeq_{czx} \mu$ can sometimes be unwieldy, as there are few natural tools for verifying that a proposed measure $\gamma$ satisfies this stochastic dominance relation. In this appendix, we modify the approach of Chapter 4 by formulating a simple weak dual program to optimal mechanism design.\footnote{We remark for clarity that the weak dual problem discussed in this chapter is distinct from the "weak duality" result of Lemma 2 in Section 4.2.} While lacking the theoretical guarantees of a strong duality theorem, this method is useful in practice for solving a variety of optimal mechanism design instances. The approach we take, broadly speaking, is to relax the constraint that a utility function $u$ be convex before formulating a dual program. This weak dual problem has a simple geometric interpretation (in the form of an optimal transport problem) and, in several interesting instances, achieves the same optimal value as the strong dual of Theorem 2, and thus suffices for certifying mechanism optimality. In particular, all examples of Section 6.2.2 can be certified in this relaxed framework. An analogous relaxation-based approach is also used in Chapter 7 for studying the computational complexity of optimal mechanism design.

Informally, our weak dual problem asks for the "best" way of pushing particles of mass to transform the positive Radon measure $\mu_+$ into the positive Radon measure $\mu_-$. Before
we formally define this transport problem, we specify what we mean by the “cost” of transporting a unit of mass from one point to another.

**Definition 24.** We define the “degenerate cost function” \( c : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) by

\[
c(x, y) \triangleq \sum_{i=1}^{n} \max\{x_i - y_i, 0\}.
\]

The cost function \( c \) is a variant of the \( \ell_1 \) norm of \( x - y \) which treats all negative coordinates as zero. Next, we define what it means to transport a measure into another.

**Definition 25.** Let \( \alpha \) and \( \beta \) be positive Radon measures on \( \mathbb{R}^n \). We define the set \( \Gamma(\alpha, \beta) \) to be the set of all positive Radon measures \( \gamma \) on \( \mathbb{R}^n \times \mathbb{R}^n \) with respective marginals \( \alpha \) and \( \beta \).

We are now ready to state our weak duality theorem.

**Theorem 8** (Mechanism Design by Optimal Transport). Let \( X \) be a convex subset of \( \mathbb{R}_{\geq 0}^n \) and let \( \mu \in \text{Radon}(X) \) with \( \mu(X) = 0 \) and \( \int_{X} ||x||_1 d\mu < \infty \). Then

\[
\sup_{u \in U(X) \cap C_{1}(X)} \int_{X} u d\mu \leq \inf_{\gamma \in \Gamma(\mu_+, \mu_-)} \int_{X \times X} c(x, y) d\gamma(x, y).
\]

Furthermore, if there exist \( u^* \) and \( \gamma^* \) feasible for their respective problems such that \( u^*(x) - u^*(y) = c(x, y) \) holds \( \gamma^* \)-almost surely, then \( u^* \) and \( \gamma^* \) are optimal for their respective problems.

**Proof.** The proof is very similar to that of Lemma 2. Namely, for feasible \( u \) and \( \gamma \):

\[
\int_{X} u d\mu = \int_{X} u d(\mu_+ - \mu_-) = \int_{X \times X} u(x) d\gamma(x, y) - \int_{X \times X} u(y) d\gamma(x, y)
\]

\[
= \int_{X \times X} (u(x) - u(y)) d\gamma(x, y) \leq \int_{X \times X} c(x, y) d\gamma(x, y)
\]

where the inequality \( u(x) - u(y) \leq c(x, y) \) follows from the fact that \( u \) is 1-Lipschitz and non-decreasing. Equality holds whenever \( u(x) - u(y) = c(x, y) \) holds \( \gamma \)-almost surely. \( \square \)

We have no a priori reason to believe that the inequality in Theorem 8 will be tight in any interesting instances. Indeed, equality fails for the simple example in Section 4.3. To understand how Theorem 8 relates to our strong duality result, we note that the right-hand
side of Theorem 8 is dual to the optimal mechanism design primal after relaxing the primal’s convexity constraint. Namely,

$$\sup_{u \in \mathcal{I}(X) \cap \mathcal{C}_1(X)} \int_X u d\mu \leq \sup_{u \in \mathcal{I}(X) \cap \mathcal{C}_1(X)} \int_X u d\mu = \inf_{\gamma \in \Gamma(\mu_+, \mu_-)} \int_{X \times X} c(x, y) d\gamma(x, y)$$

where $\mathcal{I}(X)$ is the space of increasing functions $u : X \to \mathbb{R}$. The above equality follows from the proof technique of Lemma 4 of the current paper combined with the Kantorovich duality theorem (see Theorem 1.3 of [64]) and is a variant of the standard optimal transport duality. We note furthermore, although to avoid technical details we do not present a proof, that the right-hand side above is essentially equivalent to strengthening the $\gamma_1 - \gamma_2 \succeq_{\text{conv}} \mu$ constraint in Theorem 2 to the tighter constraint $\gamma_1 - \gamma_2 \succeq_1 \mu$, where $\succeq_1$ denotes first-order stochastic dominance. Since all the two-item examples of Section 6.2.2 and Appendix B use only first-order stochastic dominance relations in their proofs (instead of convex or second-order dominance), it is simple to translate all of these optimality proofs into the optimal transport framework.
Appendix B

Optimal Mechanism Design with Unbounded Type Spaces

Many of the optimal mechanism design results from this thesis extend to unbounded type spaces, although such extensions impose additional technical difficulties. In Appendix B.1 we very informally discuss how some of our results generalize, and in Appendix B.2 and Appendix B.3 we provide concrete examples of optimal mechanism design solutions with unbounded type spaces.

B.1 Extending our Approach to Unbounded Type Spaces

We can often obtained a “transformed measure” (analogous to Definition 5 even when type spaces are unbounded) using integration by parts. It is desirable to ensure, however, that the density function $f$ decays sufficiently quickly so that there is no “surface term at infinity.” For example, we may require that $\lim_{z_1 \to \infty} f_i(z_1)z_i^2 \to 0$, as in [22]. We note that without some conditions on the decay rate of $f$, it is possible that the supremum revenue achievable is infinite and thus no optimal mechanism exists.

Similar issues arise when integrating with respect to an unbounded measure $\mu$. It is helpful therefore when proving duality and structural results to consider only measures $\mu$ such that $\int \|x\|_1 d\mu < \infty$, to ensure that $\int ud\mu$ is finite for any utility function $u$. The
measures in the examples in Appendix B.2 and Appendix B.3 satisfy this property. There is a similar technical issue with our definition of convex dominance, in that the integrals in the definition might be infinite. We can (informally speaking) attempt to extend this definition to unbounded measures (with regularity conditions such as $\int \|x\|_1 d\mu < \infty$) by ensuring that whenever the "smaller" side has infinite value, so does the larger side.

Importantly, the calculations of Lemma 2 (weak duality) hold for unbounded $\mu$, provided $\int \|x\|_1 d\mu < \infty$. Thus, tight certificates still can certify optimality, even in the unbounded case. However, our strong duality proof relies on technical tools which require compact spaces, and we do not yet have a proof that tight dual certificates always exist even when $\mu$ is unbounded.

To summarize so far, we can often transform measures and obtain an analogue of Theorem 1 for unbounded distributions (provided the distributions decay sufficiently quickly), and we have a weak duality result for such unbounded measures, but we do not yet have a proof of strong duality.

We mention a technical issue with extending our two-item characterization of Theorem 4 to unbounded type spaces. The only real difficulty that arises with extending this result to unbounded type spaces is with the condition $\mu|_W \succeq 2 0$. In our proof of Theorem 4, we obtain measures $\gamma_W$ and $\theta$ and use Lemma 6 to deduce that $\mu|_W \succeq \varnothing \theta$, given the properties $\theta \preceq 2 \mu|_W$ and $\int \|x\|_1 d\theta = \int \|x\|_1 d\mu|_W$. Our current proof of Lemma 6, however, requires that $\theta$ and $\mu|_W$ have bounded support. While we expect that this issue can be resolved, the current proof of Theorem 4 does not apply for unbounded type spaces. It is important to note, however, that if we use the weaker condition $\mu|_W \succeq 1 0$, then the proof indeed holds. (This is essentially the two-item characterization of [22].) The examples of Appendix B.2 and B.3 satisfy this first-order dominance condition.

A similar issue arises with the second-order dominance constraint when extending our bundling theorem of Theorem 3 to unbounded type spaces. In addition, the "hard" direction of the proof (showing that the grand bundling conditions hold whenever bundling is optimal) requires technical lemmas which may not immediately apply without additional work.
B.2 Example 3: Bundling Two Power-Law Items

We now derive the optimal mechanism for an example of selling two goods which are distributed according to particular power-law distributions. In this example, the optimal mechanism is a take-it-or-leave-it offer of the grand bundle at some price $p^*$. Using the terminology of Theorem 4, our zero set $Z$ consists of all points in $\mathbb{R}_{\geq 0}^2$ whose $\ell_1$ norm is at most $p_*$, and the region $\mathcal{W}$ is $\mathbb{R}_{\geq 0}^2 \setminus Z$. This is a degenerate case of Theorem 4 where $\mathcal{A}$ and $\mathcal{B}$ are empty.

Unlike in Example 4, here we demonstrate how numerical computations can be used to prove optimality of grand bundling in a single instance. Since the integrals involved in applications may be complicated, we suspect that this numerical approach will frequently be useful.

Our goal here is to deduce the optimal mechanism for selling two goods where the distribution of each good is independent with probability density function $f_i(z_i) = \frac{c_i - 1}{(1 + z_i)^{c_i}}$ for $z_i \in [0, \infty)$. When both $c_i$ are strictly greater than 2, we apply an infinite analog of Theorem 1 to obtain a signed measure $\mu$ on $\mathbb{R}_{\geq 0}^2$ consisting of:

- A point mass of $+1$ at the origin.
- (Positive) mass distributed on the region

$$\mathcal{X} = \left\{ z : \sum_i \frac{c_i z_i}{1 + z_i} > n + 1 \right\}$$

according to the density function

$$\hat{g}(x) = \prod_i \frac{c_i - 1}{(1 + x_i)^{c_i}} \cdot \left( \sum_j \frac{c_j x_j}{1 + x_j} - n - 1 \right) \cdot 1_{x \in \mathcal{X}}$$

- (Negative) mass distributed on the region

$$\mathcal{Y} = \left\{ z : \sum_i \frac{c_i z_i}{1 + z_i} \leq n + 1 \right\} \setminus \{ \vec{0} \}$$

\footnote{While we have not formally proven the infinite variant of Theorem 1, in the simple infinite examples of this section it is straightforward to show (by integration by parts) that the reduction holds. Furthermore, since the optimal utility functions of our infinite examples are well-behaved, our weak-duality arguments can immediately be applied.}
according to the density function

$$
\hat{h}(y) = -\left[ \prod_i \frac{c_i - 1}{(1 + y_i)^{c_i}} \cdot \left( n + 1 - \sum_j \frac{c_j y_j}{1 + y_j} \right) \right] \cdot 1_{y \in Y}
$$

We note that $\hat{g}$ and $\hat{h}$ have essentially the same algebraic form, although we have factored out a -1 in writing $\hat{h}$ to emphasize that $\hat{h}(y) \leq 0$. In Example 3, we fix $c_1 = 6$ and $c_2 = 7$.

**Numerically Computing the Bundle Price.**

We begin by computing the bundle price $p^*$. This must be chosen such that $\mu(Z) = 0$, where $Z = \{ z \in \mathbb{R}^2_{\geq 0} : \|x\|_1 \leq p^* \}$. Therefore, we solve for $p^*$ such that

$$
\int_0^{p^*} \int_0^{p^* - z_2} \left( 3 - \frac{c_1 z_1}{1 + z_1} - \frac{c_2 z_2}{1 + z_2} \right) \frac{c_1 - 1}{(1 + z_1)^{c_1}} \cdot \frac{c_2 - 1}{(1 + z_2)^{c_2}} \, dz_1 \, dz_2 = 1
$$

and numerically determine that $p^* \approx 0.35725$.

**Verifying Stochastic Dominance in $Z$.**

Next, we will verify that $\mu|_Z = \mu_+|_Z - \mu_-|_Z \preceq_{crx} 0$. We will show the stronger property that the only positive mass of $\mu|_Z$ is at the origin. (Using the notation above, $Z \cap X = \emptyset$.) That is, we will show that for all $z$ with $z_1 + z_2 \leq p^*$, it holds that

$$
\frac{c_1 z_1}{1 + z_1} + \frac{c_2 z_2}{1 + z_2} \leq 3.
$$

Since the left-hand side of the inequality is an increasing function, it suffices to prove the inequality when $z_1 + z_2 = p^*$. Substituting for $z_2$, the left-hand side of the above inequality becomes $\frac{c_1 z_1}{1 + z_1} + \frac{c_2 p^* - c_2 z_1}{1 + p^* - z_1}$.

We numerically compute, after setting $c_1 = 6$ and $c_2 = 7$, that the expression is maximized by $z_1 = 0.133226$, achieving value 1.98654. Since 1.98654 is significantly less than 3, we conclude that $p^*$ is indeed a critical price, even taking into consideration possible errors of precision.

**Verifying Stochastic Dominance in $W$.**

All that remains to prove optimality of grand bundling by Theorem 4 is to verify that $\mu_+|_W \succeq_1 \mu_-|_W$. We prove this stochastic dominance using Lemma 14 with $g$ and $h$ being the respective density functions of $\mu_+|_W$ and $\mu_-|_W$. (That is, $g(x) = \tilde{g}(x)$ if $x \in W$ and
zero otherwise. We define \( h(x) \) to be \( -\tilde{h}(x) \) if \( x \in \mathcal{W} \) and zero otherwise.\(^2\)

We now verify that the conditions of Lemma 14 (with \( C = \mathbb{R}^2 \geq 0 \) and \( R = Z \)) apply in Example 3. The condition \( \int_{\mathbb{R}^2_0} (g - h)dydx = 0 \) is satisfied by construction of \( p^* \), since \( \mu(Z) = 0 \) implies that \( \mu(\mathbb{R}^2 \geq 0 \setminus Z) = 0 \).

Since \( g \) and \( h \) have disjoint supports, we notice that for any \( z \in \mathbb{R}^2_0 \setminus Z \) we have

\[
g(z) - h(z) = \left( \frac{c_1z_1}{1 + z_1} + \frac{c_2z_2}{1 + z_2} - 3 \right) f_1(z_1) \cdot f_2(z_2).
\]

Thus, the third condition of Lemma 14 is satisfied with \( \alpha(z_1) = f_1(z_1) \), \( \beta(z_2) = f_2(z_2) \), and \( \eta(z_1, z_2) = \frac{c_1z_1}{1 + z_1} + \frac{c_2z_2}{1 + z_2} - 3 \), noting that \( \eta \) is indeed an increasing function.

All that remains is to verify the second condition of Lemma 14. We break this verification into two parts, depending on whether we are integrating with respect to \( z_1 \) or \( z_2 \).

- We begin by considering integration with respect to \( z_1 \). That is, for any fixed \( 0 < z_2 \leq p^* \), we must prove that

\[
\int_{p^*-z_2}^\infty \left( \frac{c_1z_1}{1 + z_1} + \frac{c_2z_2}{1 + z_2} - 3 \right) \frac{c_1 - 1}{(1 + z_1)^{c_1}} \cdot \frac{c_2 - 1}{(1 + z_2)^{c_2}} dz_1 \leq 0.
\]

Since \( z_2 \) is fixed, it clearly suffices to prove that

\[
\int_{p^*-z_2}^\infty \left( \frac{6z_1}{1 + z_1} + \frac{7z_2}{1 + z_2} - 3 \right) \frac{1}{(1 + z_1)^6} dz_1 \leq 0.
\]

This integral evaluates to

\[
\frac{-0.18565 + 1.1145z_2 - 2z_2^2}{(1.35725 - z_2)^6(1 + z_2)^2}.
\]

Since the denominator is always positive, it suffices to prove that the numerator is negative. Indeed, the numerator is maximized at \( z_2 = .2786 \), in which case the numerator evaluates to \( -.0304 \).

- We now consider integration with respect to \( z_2 \). Analogously to the computation

\(^2\)We set \( h(x) = -\tilde{h}(x) \) so that \( h(x) \geq 0 \), as Lemma 14 applies to measures of the form \( h - g \) instead of \( h + g \).
above, for any fixed $0 \leq z_1 \leq p^*$ we must prove that

$$\int_{p^* - z_1}^{\infty} \left( \frac{6z_1}{1 + z_1} + \frac{7z_2}{1 + z_2} - 3 \right) \frac{1}{(1 + z_2)^7} dz_2 \leq 0.$$ 

This integral evaluates to

$$\frac{-0.0951667 + 0.595416z_1 - 1.66667z_2}{(1.35725 - z_1)^7(1 + z_1)}.$$ 

As before, it suffices to prove that the numerator is negative. We verify that, indeed, the numerator achieves its maximum at $z_1 = 0.178625$, in which case the numerator is $-0.0419886$.

**B.3 Example 4: Two Exponential Items**

In this section we provide a closed-form description of the optimal mechanism for two independent exponentially distributed items. In this example, the optimal mechanism has richer structure than only offering the grand bundle.

We suppose that the distribution of each item $i$ is given by the exponential density function $f_i(z_i) = \lambda_i e^{-\lambda_i z_i}$ for $z_i \in (0, \infty)$ and for some positive constant $\lambda_i$. We now apply an infinite analog of Theorem 1 to obtained a signed measure $\mu$ comprised of the following:

- A point mass of +1 at the origin.
- (Positive) mass distributed on the region

$$\mathcal{X} = \{ z : \sum \lambda_i z_i > n + 1 \}$$

according to the density function

$$\tilde{g}(x) = \left( \prod_j \lambda_j \right) \left( \sum_i \lambda_i x_i - n - 1 \right) e^{-\sum \lambda_i x_i} : 1_{x \in \mathcal{X}}$$

- (Negative) mass distributed on the region

$$\mathcal{Y} = \{ z : \sum \lambda_i z_i \leq n + 1 \}$$
according to the density function

\[ \tilde{h}(y) = - \left[ \prod_j \lambda_j \right] \left( n + 1 - \sum_i \lambda_i y_i \right) e^{-\sum \lambda_i y_i} \right] \cdot 1_{y \in \mathcal{Y}} \]

As in the previous example, \( \tilde{g} \) and \( \tilde{h} \) have essentially the same algebraic form, although we have factored out a -1 in writing \( \tilde{h} \) to emphasize that \( \tilde{h}(y) \leq 0 \).

In what follows, we denote by \( \lambda_{\min} = \min_i \lambda_i \) and \( \lambda_{\max} = \max_i \lambda_i \).

The Critical Price \( p^* \).

We now focus on the case \( n = 2 \). Similarly to our previous example, we aim first to find the set \( Z_{p^*} \) of points which receive zero utility under the optimal mechanism.

**Definition 26.** For any \( 0 < p \leq 2/\lambda_{\min} \), we define the zero set, \( Z_p \subset \mathcal{Y} \cup \{0\} \), to be

\[ Z_p \triangleq \{ y \in \mathbb{R}_+^2 : y_1 + y_2 \leq p \text{ and } \lambda_1 y_1 + \lambda_2 y_2 \leq 2 \} \].

See Figure 6-3 for an example of such a zero set.\(^3\) Our definition of \( Z_p \) is motivated by our geometric understanding of the absorption hyperplane, discussed later in this example. We will solve for the value of \( p \) under which \( \mu(Z_p) = 0 \).

**Definition 27.** For all \( \lambda_1, \lambda_2 > 0 \), the critical price \( p^* = p^*(\lambda_1, \lambda_2) \) is the unique \( 0 < p^* \leq 2/\lambda_{\min} \) such that \( \mu(Z_{p^*}) = 0 \).

We claim that such a critical price always exists.

**Claim 17.** For all \( \lambda_1, \lambda_2 > 0 \), there exists a unique critical price.

**Proof of Claim 17:**

Since \( \tilde{g}(x) \) is supported on the region \( \sum \lambda_i x_i > 3 \), and since \( Z_p \) only contains points with \( \sum \lambda_i x_i \leq 2 \), we notice that the support of \( \tilde{g}(x) \) never intersects \( Z_p \). Since the only positive mass in \( Z_p \) is at the origin and \( \tilde{h}(x) \) is nonzero on the interior of \( Z_p \), \( \mu(Z_p) \) is a strictly decreasing function of \( p \) for \( p \) in the range \([0, 2/\lambda_{\min}]\).

\(^3\)In this figure, the measure \( \mu_+ \) is supported on \( \mathcal{X} \cup \{0\} \) while \( \mu_- \) is supported on \( \mathcal{Y} \). (Note that \( Z_{p^*} \subset \mathcal{Y} \).)
We notice that $\mu(Z_0) = 1$, and therefore all that remains is to show that $\mu(Z_{2/\lambda_{\min}}) < 0$. We note that $Z_{2/\lambda_{\min}} = \{y : \sum \lambda_i y_i \leq 2\}$, and thus

$$
\mu(Z_{2/\lambda_{\min}}) = 1 + \int_{Z_{2/\lambda_{\min}}} \tilde{h}(y) \, dy
$$

$$
= 1 - \lambda_1 \lambda_2 \int_0^{\lambda_2 \lambda_1} \int_0^{\lambda_2 \lambda_1} (3 - \lambda_1 y_1 - \lambda_2 y_2) e^{-\lambda_1 y_1 - \lambda_2 y_2} \, dy_1 \, dy_2
$$

$$
= 1 - \lambda_1 \lambda_2 \int_0^{2/\lambda_2} e^{-\lambda_2 y_2 (2 - \lambda_2 y_2) \lambda_1} \, dy_2 = 1 - \lambda_1 \lambda_2 \cdot \frac{1 + \frac{1}{\lambda_1 \lambda_2}}{\lambda_1} < 0
$$

as desired.

Our goal is to prove Example 4. In particular, we will show that the optimal utility function in Example 4 is

$$
u^*(z_1, z_2) = \begin{cases} 
0 & \text{if } z_1 + z_2 \leq p^* \text{ and } \lambda_1 z_1 + \lambda_2 z_2 \leq 2; \\
\frac{z_1 + z_2}{\lambda_1} - \frac{2}{\lambda_1} & \text{if } z_2 \left(\frac{\lambda_1 - \lambda_2}{\lambda_1}\right) \leq p^* - \frac{2}{\lambda_1} \text{ and } \lambda_1 z_1 + \lambda_2 z_2 > 2; \\
z_1 + z_2 - p^* & \text{otherwise},
\end{cases}
$$

where $p^* = p^*(\lambda_1, \lambda_2)$ is the critical price of Claim 17.

Notice that the utility function $\nu^*$ of Example 4 is precisely the utility function $u_{Z^*}$ generated by the zero set $Z^*$. Example 4 is a degenerate case of a canonical partition in which the region $A$ is empty. To prove this result, we first take a slight detour to better understand the geometric structure of $\mu$.

The Absorption Hyperplane.

A useful feature of independent exponential distributions is that the resulting measure $\mu$ gives rise to a set $H \subset \mathbb{R}_{\geq 0}^n$ for which integrating the signed density $\tilde{g} + \tilde{h}$ outwards along any line starting from $H$ yields 0. This set $H$ provides useful geometric intuition behind the structure of the optimal mechanism. We prove the following claim in $n$ dimensions, since it is technically no more difficult than the proof for $n = 2$.

Claim 18. Suppose $z \in \mathbb{R}_{\geq 0}^n$ satisfies $\sum \lambda_j z_j = n$. Then, for any vector $\bar{v} \in \mathbb{R}_{\geq 0}^n$:

$$
\int_0^\infty \left( n + 1 - \sum_i \lambda_i (z_i + \tau v_i) \right) e^{-\sum \lambda_i (z_i + \tau v_i)} \, d\tau = 0.
$$
Proof. We have
\[
\int_0^\infty \left( n + 1 - \sum_i \lambda_i (z_i + \tau v_i) \right) e^{-\sum_i \lambda_i (z_i + \tau v_i)} d\tau \\
= \left. \frac{e^{-\sum_i \lambda_i (z_i + \tau v_i)} (\sum_i \lambda_i (z_i + \tau v_i) - n)}{\sum_i \lambda_i v_i} \right|_{\tau=0}^\infty = -\frac{e^{-\sum_i \lambda_i z_i} (\sum_i \lambda_i z_i - n)}{\sum_i \lambda_i v_i} = 0. 
\]
\hfill \square

We refer to the set \( \mathcal{H} = \{ z \in \mathbb{R}_+^n : \sum z_i \lambda_i = n \} \) as the absorption hyperplane. The absorption hyperplane \( \mathcal{H} \) will be useful in applying Theorem 4 and Lemma 14. We note the important property that \( Z_{p^*} \) always lies below the absorption hyperplane. In addition, since \( \mathcal{Y} \) lies "below" \( \mathcal{X} \), if we were to begin at a point outside the absorption hyperplane and integrate the density of \( \mu \) outwards along a line, we would obtain a strictly positive value.

Proof of Example 4: We will now prove Example 4, which fully specifies the optimal mechanism for two independent exponentially-distributed items. Using the zero set \( Z_{p^*} \) gives rise to the canonical partition illustrated in Figure 6-3 in which the partition region \( \mathcal{A} \) is empty. Recall that \( \mathcal{Y} \) denotes the support of the negative part of \( \mu \), while \( \mathcal{X} \) (along with the origin) denotes the support of the positive part of \( \mu \).

We now verify the conditions of Theorem 4. Our task is greatly simplified by the geometric structure of the absorption hyperplane, since integrating the density of \( \mu \) along any horizontal line in \( \mathcal{B} \) beginning at the boundary of \( Z_{p^*} \) and moving to the right will equal zero, as \( Z_{p^*} \) and \( \mathcal{B} \) meet along the absorption hyperplane. Thus, the only nontrivial verification is that \( \mu_+|_W \geq \mu_-|_W \).

To verify that \( \mu_+|_W \geq \mu_-|_W \) we will apply Lemma 14 with \( g \) and \( h \) being the restriction of \( \bar{g} \) and \( -\bar{h} \) to \( \mathcal{W} \), respectively.\(^4\) In our application of the Lemma, we let \( \mathcal{C} \) be the rectangle obtained by extending the boundary between \( \mathcal{W} \) and \( \mathcal{B} \) horizontally through the region \( Z_{p^*} \), so that \( \mathcal{C} \setminus Z_{p^*} = \mathcal{W} \) and \( R = \mathcal{C} \cap Z_{p^*} \).

We must now check that Lemma 14 applies. Our task is greatly simplified by our geometric understanding of the absorption hyperplane.

- We have \( \int_C (g - h) dx dy = \mu(\mathcal{W}) \), which we claim equals zero. In particular, we know \( \int_C (g - h) dx dy = \mu(\mathcal{W}) \).

\(^4\)As before, we set \( h = -\bar{h} \) so that \( h \) is a positive density function, as Lemma 14 applies to measures of the form \( f - g \).
that \( \mu(\mathbb{R}^2_{\geq 0}) = 0 \), \( \mu(Z_p^*) = 0 \) (by construction), and \( \mu(B) = 0 \) (since the integral of the density of \( \mu \) along any horizontal line is zero, by use of the absorption hyperplane) and \( \mu(\mathcal{W}) = 0 \) follows.

- Integrating the density of \( g - h \) along any horizontal or vertical line starting from the boundary between \( R \) and \( W \) is non-positive, as the boundary between these regions lies below the absorption hyperplane.

- For all \( z = (z_1, z_2) \in C \setminus R = \mathcal{W} \), we have

\[
g(z) - h(z) = \left( \prod_j \lambda_j \right) \left( \sum_i \lambda_i z_i - n - 1 \right) e^{-\sum \lambda_i z_i}
\]

\[
= \lambda_1 e^{-\lambda_1 z_1} \cdot \lambda_2 e^{-\lambda_2 z_2} \cdot (\lambda_1 z_1 + \lambda_2 z_2 - 3)
\]

\[
= f_1(z_1) \cdot f_2(z_2) \cdot (\lambda_1 z_1 + \lambda_2 z_2 - 3)
\]

and \( \lambda_1 z_2 + \lambda_2 z_2 - 3 \) is an increasing function.

Therefore \( \mu_+|_\mathcal{W} \preceq \mu_-|_\mathcal{W} \), and the proof is complete. \( \square \)
Appendix C

Near-Optimal Mechanism Design with Close Distributions

Most of our work thus far on optimal mechanism design (with the exception of Chapter 7) has required that the joint probability distribution of items be given by a differentiable density function \( f \) (typically with bounded support) with bounded partial derivatives. While this differentiability assumption is fairly mild, there are several important scenarios, in particular the case of discrete distributions, where it fails. In this chapter we show that any joint probability distribution function is arbitrarily close to particular smooth distributions such that solving the mechanism design problem on smooth instances yields nearly optimal mechanisms for the original instance. The techniques in this chapter are fairly elementary, and do not require use of our duality framework.

While the smoothing approach in this appendix is not always necessary (in Chapter 7 we consider the situation where every item has a discrete distribution of support two, and in that scenario we elect to use linear programming duality and mirror the optimal transport framework of Appendix A), it shows that our current framework is without (much) loss of generality. We do, however, remark that while the approach in this chapter yields mechanisms obtaining nearly optimal revenue, we make no claim that the structure of our mechanisms are similar to the structure of the exactly optimal mechanism.

We first define what it means for two probability measures to be close to each other. While there are several standard definitions for closeness of distributions, here we say that measures are \( \epsilon \) close if one can be transformed into the other by moving points of mass at

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most $\epsilon$ distance. This transport-based definition of closeness is motivated by the relationship between mechanism design problems and optimal transportation.\footnote{\textsuperscript{1}}

**Definition 28.** Let $\alpha$ and $\beta$ be two probability measures on $\mathbb{R}_{\geq 0}^n$. We say that $\alpha$ is $\epsilon$-close to $\beta$ if there exists a coupling $\gamma \in \Gamma(\alpha, \beta)$ such that, $\gamma(x,y)$-almost surely, $\|x - y\|_1 \leq \epsilon$.

We observe the useful property that any distribution is $\epsilon$-close to a smooth distribution.

**Remark 13.** Let $\alpha$ be a probability measure on $\mathbb{R}_{\geq 0}^n$ and let $\epsilon > 0$ be arbitrary. Then there exists a smooth probability density function $f_\beta : \mathbb{R}_{\geq 0}^n \to \mathbb{R}_{>0}$ such that the measure induced by $f_\beta$ is $\epsilon$-close to $\alpha$.

**Proof.** We partition $\mathbb{R}_{\geq 0}^n$ into (non-overlapping) cubes $C_1, C_2, \ldots$ each having side length $\epsilon/n$. We define $f_\beta$ to be the countable sum of one smooth bump functions on the interior of each cube, where the bump function supported on the interior of cube $C_j$ has total mass $\alpha(C_j)$. Since each bump function is zero on the boundary of its cube and its derivatives are all zero on the boundary as well, it follows that $f_\beta$ is smooth. Furthermore, since the maximum $l_1$ distance between any two points in a cube is $\epsilon$, it follows that there exists a transport map between $\alpha$ and $\beta$ transporting points a distance no more than $\epsilon$. \qed 

**Lemma 19.** Let $\alpha$ and $\beta$ be $\epsilon$-close probability measures on $\mathbb{R}_{\geq 0}^n$ with $0 < \epsilon < 1$, and suppose that a truthful, individually rational, direct mechanism $\mathcal{M}$ achieves expected revenue $r$ when the bidder's type is drawn from $\alpha$. Then there exists a truthful, individually rational direct mechanism $\mathcal{M}'$ achieving expected revenue at least $(1 - \sqrt{\epsilon})r - \sqrt{\epsilon}$ when the bidder's type is drawn from $\beta$.

**Proof of Lemma 19:** We first remark that setting $\mathcal{M}' = \mathcal{M}$ may not suffice. Indeed, suppose that $\alpha$ places all mass on $\$1$ while $\beta$ places all mass on $\$1 - \epsilon$. Then the mechanism $\mathcal{M}$ which is a take-it-or-leave-it offer of the item for $\$1$ achieves expected revenue $\$1$ under $\alpha$ but $\$0$ under $\beta$.

As in the proof of Fact 1, we recall that mechanism $\mathcal{M}$ induces a (possibly infinite) collection $\mathcal{O}$ of pairs $\{(P(z), T(z)) : z \in \mathbb{R}_{\geq 0}^n\}$ specifying the allocation probabilities and total variation distance is insufficient for our purposes. In particular, the distribution which is always 0 is $\epsilon$ away in total variation distance from the distribution which has value 0 with probability $1 - \epsilon$ and value $1/\epsilon^2$ with probability $\epsilon$, yet the optimal mechanism revenue against the first distribution is 0 while the optimal revenue against the second distribution is $1/\epsilon$. 

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prices possibly output by the mechanism. We will assume that \( T(z) \geq 0 \), since removing negative charges only increases the mechanism’s revenue.

We will specify \( \mathcal{M}' \) by giving the collection \( \mathcal{O}' \) of possible outcomes output by \( \mathcal{M}' \). Given \( \mathcal{O}' \) and a type declaration \( y \), the mechanism \( \mathcal{M}' \) simply chooses outcome from \( \mathcal{O}' \) which maximizes the utility under type \( y \), and breaks ties in favor of the highest-price outcomes. We will ensure that \((\vec{0}, $0)\) is an element of \( \mathcal{O}' \) in our construction, and therefore \( \mathcal{M}' \) will be incentive compatible and individually rational.

We define

\[
\mathcal{O}' = \{(p, (1 - \sqrt{\epsilon})r)\}_{(p, r) \in \mathcal{O}} \cup (\vec{0}, $0)
\]

and claim that \( \mathcal{M}' \) achieves the desired expected revenue. Intuitively, the mechanism \( \mathcal{M}' \) offers the same outcome pairs as \( \mathcal{M} \) but pays the bidder a discount proportional to the price of the bundle she selects.

We will actually show that for any types \( x, y \) such that \( \|x - y\|_1 \leq \epsilon \), the price \( \tau^M_z \) paid by a bidder of type \( x \) in \( \mathcal{M} \) and \( \tau^{'M}_y \) paid by a bidder of type \( y \) in \( \mathcal{M}' \) satisfy \( \tau^{'M}_y \geq (1 - \sqrt{\epsilon})\tau^M_z - \sqrt{\epsilon} \). The result follows from the fact that \( \alpha \) and \( \beta \) are \( \epsilon \)-close, and thus the distributions can be appropriately coupled.

For a bidder of type \( t \), let \( \tau^M_t \) and \( \tau^{'M}_t \) be the prices charged to a bidder of type \( t \) in \( \mathcal{M} \) and \( \mathcal{M}' \) respectively, and let \( p^M_t \) and \( p^{'M}_t \in [0, 1]^n \) be the respective allocation vectors.

If \( \tau^M_z \leq \sqrt{\epsilon} \), then the result is obvious, since we need only prove that \( \tau^{'M}_y \geq 0 \), which follows from individual rationality. Therefore, for the remainder of this proof we suppose that \( \tau^M_z > \sqrt{\epsilon} \).

Since \( \mathcal{M}' \) is truthful and since \( (p^M_z, (1 - \sqrt{\epsilon})\tau^M_z) \in \mathcal{O}' \), it follows that

\[
y \cdot p^{'M}_y - \tau^{'M}_y \geq y \cdot p^M_z - (1 - \sqrt{\epsilon})\tau^M_z = (x \cdot p^M_z - \tau^M_z) + (y - x) \cdot p^M_z + \sqrt{\epsilon} \tau^M_z \\
\geq (y - x) \cdot p^M_z + \sqrt{\epsilon} \tau^M_z \geq -\|y - x\|_1 + \sqrt{\epsilon} \tau^M_z \geq -\epsilon + \sqrt{\epsilon} \tau^M_z > 0.
\]

We have shown that \( y \) obtains strictly positive utility in \( \mathcal{M}' \), and therefore \( y \) does not “opt out” by selecting \((\vec{0}, $0)\). Therefore \((p^{'M}_y, \tau^{'M}_y) = (p^M_z, (1 - \sqrt{\epsilon})\tau^M_z) \) for some \( z \in \mathbb{R}^n_{\geq 0} \).

By the incentive compatibility property of \( \mathcal{M} \), a player of type \( x \) in \( \mathcal{M} \) has no incentive
to falsely declare herself to be type \( z \) in mechanism \( M \). Thus:

\[
\tau_z^M \geq \tau_x^M - x \cdot (p_{z}^M - p_{z}^M) \\
= \tau_x^M - y \cdot (p_{z}^M - p_{z}^M) + (y - x) \cdot (p_{z}^M - p_{z}^M) \\
\geq \tau_x^M + y \cdot (p_{z}^M - p_{z}^M) - \| y - x \|_1 \geq \tau_x^M + y \cdot (p_{z}^M - p_{z}^M) - \epsilon.
\]

Furthermore, by incentive compatibility of \( M' \), a bidder of type \( y \) never has incentive to falsely declare herself to be type \( x \) in mechanism \( M' \). Thus:

\[
y \cdot y_{y}^{M'} - \tau_{y}^{M'} = y \cdot p_{z}^M - (1 - \sqrt{\epsilon})\tau_z^M \geq y \cdot p_{z}^M - (1 - \sqrt{\epsilon})\tau_x^M \\
y \cdot (p_{z}^M - p_{z}^M) \geq (1 - \sqrt{\epsilon}) (\tau_z^M - \tau_x^M).
\]

Combining these inequalities, we have

\[
\tau_z^M \geq \tau_x^M + y \cdot (p_{z}^M - p_{z}^M) - \epsilon \geq \tau_x^M + (1 - \sqrt{\epsilon}) (\tau_z^M - \tau_x^M) - \epsilon \\
\sqrt{\epsilon} \tau_z^M \geq \sqrt{\epsilon} \tau_x^M - \epsilon \\
\tau_z^M \geq \tau_x^M - \sqrt{\epsilon}.
\]

Therefore,

\[
\tau_{y}^{M'} = (1 - \sqrt{\epsilon})\tau_z^M \geq (1 - \sqrt{\epsilon})\tau_x^M - (1 - \sqrt{\epsilon}) \sqrt{\epsilon} \geq (1 - \sqrt{\epsilon})\tau_x^M - \sqrt{\epsilon}
\]

as desired. \( \square \)

We remark that the proof technique of Lemma 19 explicitly constructs the mechanism \( M' \) based on \( M \), and the mechanisms have “similar” structure. For example:

**Remark 14.** Let \( 0 < \epsilon < 1 \), let \( \alpha \) and \( \beta \) be \( \epsilon \)-close probability distributions on \( \mathbb{R}_{\geq 0} \), and suppose that a take-it-or-leave-it offer of all goods for price \( p \) achieves expected revenue \( r \) when the bidder is drawn from \( \beta \). Then when the types are drawn from distribution \( \alpha \), a take-it-or-leave-it offer of all goods for price \( (1 - \sqrt{\epsilon})p \) gives expected revenue at least \( (1 - \sqrt{\epsilon})r - \sqrt{\epsilon} \).

It is also an immediate consequence of Lemma 19 that \( \epsilon \)-close value distributions have
similar optimal mechanism revenue.

**Corollary 4.** Let $0 < \epsilon < 1$ and let $\alpha$ and $\beta$ be $\epsilon$-close probability distributions on $\mathbb{R}_2^n$. Then optimal expected revenues $OPT(\alpha)$ and $OPT(\beta)$ for mechanisms with types drawn from $\alpha$ and $\beta$ respectively are related by:

$$(1 - \sqrt{\epsilon})OPT(\alpha) - \sqrt{\epsilon} \leq OPT(\beta) \leq \frac{OPT(\alpha)}{1 - \sqrt{\epsilon}} + \frac{\sqrt{\epsilon}}{1 - \sqrt{\epsilon}}$$
Bibliography


