On Planar Rational Cuspidal Curves

by

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Abstract
This thesis studies rational curves in the complex projective plane that are homeomorphic to their normalizations. We derive some combinatorial constraints on such curves from a result of Borodzik-Livingston in Heegaard-Floer homology. Using these constraints and other tools from algebraic geometry, we offer a solution to certain cases of the Coolidge-Nagata problem on the rectifiability of planar rational cuspidal curves, that is, their equivalence to lines under the Cremona group of birational automorphisms of the plane.

Thesis Supervisor: James McKernan, FRS
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Chapter 1

Introduction

As its title indicates, this thesis studies rational cuspidal curves in the complex projective plane \( \mathbb{P}^2 \).

Before we begin the formal discussions, perhaps a few words for the lay reader are in order, since the objects of study are sufficiently concrete (by the standards of modern mathematics) as to be accessible to many people with a scientific background, even quite far from algebraic geometry. The complex projective plane \( \mathbb{P}^2 \) may be thought of as the ordinary complex plane \( \mathbb{C}^2 \) completed with a “line at infinity” that contains one point corresponding to every direction in \( \mathbb{C}^2 \). It was realized centuries ago that many geometrical problems can be stated more elegantly in \( \mathbb{P}^2 \) as opposed to \( \mathbb{C}^2 \), and that is the case, in particular, for the problems we will be considering. That being said, readers who are not familiar with this notion will do reasonably well, for the time being, just to think in terms of the more familiar complex plane \( \mathbb{C}^2 \), or indeed even the real plane \( \mathbb{R}^2 \).

An algebraic curve (which is the only kind of curve we will be considering) is, more or less, the locus of points \((x, y)\) in the plane that satisfy an equation of the form \( f(x, y) = 0 \), for a specified polynomial \( f \) in two variables with complex coefficients. Such a curve is rational if it admits a parametrization by rational functions (ratios of polynomials in \( x \) and \( y \)). Thus, the line \( y = mx + b \) (or \( y - mx - b = 0 \)), which may be so parametrized as \((x, y) = (t, mt + b)\), is rational. Perhaps more surprisingly, the circle \( x^2 + y^2 = 1 \) is also rational, because it may be parametrized as \((x, y) = (\frac{2t}{t^2 + 1}, \frac{t^2 - 1}{t^2 + 1})\). Indeed, we can quickly verify that

\[
\left( \frac{2t}{t^2 + 1} \right)^2 + \left( \frac{t^2 - 1}{t^2 + 1} \right)^2 = \frac{4t^2}{t^4 + 2t^2 + 1} + \frac{t^4 - 2t^2 + 1}{t^4 + 2t^2 + 1} = 1.
\]

It turns out that not all curves are rational (and in some sense “most” curves are not), though perhaps this is far from obvious at first.

As is often the case, just by looking at the \( \mathbb{R} \)-valued points on a curve, it may not be so clear geometrically what rationality means. But if one is able to imagine the complex-valued points as well, then a complex “curve” really looks to us like a real surface, and from this point of view, “rationality” has a clear topological meaning, at least for smooth “curves.” The kinds of real surfaces that arise this way are classified topologically by a non-negative integer called their genus. Informally, a sphere (by which we always mean the surface only, not a solid ball) has genus 0, a torus (the surface of a bagel with one hole) has genus 1, and in general a torus with \( g \) holes has genus \( g \). Then it turns out that a smooth complex “curve” is rational if and only if it looks like a real surface of genus 0.

At any rate, all curves, rational or not, may have smooth points (points on the curve where the defining polynomial \( f \) has at least one non-zero partial derivative) and singular points, or singularities (points on the curve where both partial derivatives \( \partial f/\partial x \) and \( \partial f/\partial y \) are zero). The curve itself is called smooth if all its points are smooth points. Much research has been done on
classifying the singularities that can arise on curves. For our thesis, we will be concerned with one dichotomy in particular.

At some singularities, a curve may have at least two distinct "branches." For example, the origin on the curve \( y^2 = x^3(x+1) \) is a singularity known as "the ordinary node"; the curve is seen to have one branch tangent to the \( x \)-axis, and a different one tangent to the \( y \)-axis. If one zooms in close to this singularity, it looks like two separate lines meeting at a point. If we try to visualize the complex points, it looks like two separate real planes meeting at a point inside a four-real-dimensional space.

At other singularities, a curve may have only one "branch." For example, the origin on the curve \( y^2 = x^3 \) is a singularity known as "the ordinary cusp"; the curve has only one branch there. Here the real picture may be a bit misleading, since it looks as though one could draw a distinction between the points with positive versus negative \( y \). But one should keep in mind that in the world of complex numbers, such distinctions are usually not meaningful. If we zoom in on the ordinary cusp, while trying to visualize all the complex points, we see what looks like the cone over a trefoil knot in three-dimensional space. That is, just as in everyday language a "cone" is the result of taking every point on a circle in the plane, and joining that point by a straight line to a fixed cone point outside the plane in three-space, here we are asked to imagine the result of taking every point on a trefoil knot in three-space, and joining that point by a straight line to a fixed cone point outside three-space, in a larger, four-dimensional space. This cone point is precisely the point at the cusp itself. From this picture, I hope it is at least somewhat plausible that this singularity has only one branch: there is no natural way to partition the points on the trefoil knot into multiple distinct sets, in the way that we teased apart the two real planes in the example of the ordinary node above.

A curve is called cuspidal if its singularities (and it may not have any at all) are of the second kind, that is, they have only one branch. A rational cuspidal curve in the complex plane, then, looks like a topological sphere that, although it may be "pinched" at certain points, cannot "cross itself." In the language of topological, we say that the surface is homeomorphic to a sphere. Now, at least, I hope I have conveyed some sense of what the basic objects of this thesis — rational cuspidal curves in the plane — are.

Now let us begin the more formal introduction. All our varieties are defined over the field of complex numbers \( \mathbb{C} \). A projective curve is called cuspidal if is integral and all its singularities are unibranched (locally analytically irreducible). In other words, its normalization map is a homeomorphism; intuitively, the curve does not "cross itself" as it would at a node. A curve is rational if it is birational to \( \mathbb{P}^1 \), which is to say it admits a parametrization by rational functions. Thus, in the analytic topology, rational cuspidal curves are characterized by being homeomorphic to the 2-sphere \( S^2 \). Here are some examples of rational cuspidal curves:

1. A straight line is rational and has no singularities at all, so it is vacuously cuspidal.

2. A smooth conic is also a rational cuspidal curve.

3. The cuspidal cubic, given by the equation \( y^2 = x^3 \), is the most basic singular example. More generally, the curves \( y^a = x^b \), for relatively prime integers \( b > a > 1 \), are rational and each have one cusp at the origin \([0 : 0 : 1]\), and possibly a second cusp at \([0 : 1 : 0]\), if \( b-a > 1 \).

4. The tricuspidal quartic: imagine the curve \( T \) traced out in \( \mathbb{R}^2 \) by a point on the circumference of a wheel of radius \( \frac{1}{3} \) rolling along the inside of a fixed unit circle \([?]\). This turns out to be a cuspidal rational quartic curve. It has three \( A_1 \) cusps, namely, the real points where the curve meets the fixed unit circle.
5. Let $C$ be the image of the mapping $\mathbb{P}^1 \to \mathbb{P}^2$ given by

$$\iota(s) = [3s^4 : 2s^5 + s^2 : 4s^3 - 1].$$

It is a quintic with four cusps: in the ADE classification, it has one $A_5$ cusp at $\iota(0) = [0 : 0 : 1]$, and one $A_1$ cusp at $\iota(\zeta) = [\zeta : \zeta : 1]$ for each cubic root of unity $\zeta$. This example is discussed, for example, in [6].

Many more examples are known; see [10], example, or [3, 5, 4, 6, 12, 15].

The requirement that a cuspidal curve have no nodes is quite restrictive. Tono [16] proved that every rational cuspidal curve in the plane has at most 8 cusps, and, more generally, a planar cuspidal curve of geometric genus $g$ has at most $(21g + 17)/2$ cusps, regardless of its degree. In fact, no rational cuspidal curves are known with 5 or more cusps, and only one such curve (up to $PGL(3)$ actions) is known with 4 cusps, namely our fifth example above. See [15] for a conjectural classification of all rational cuspidal curves with 3 or more cusps. At any rate, the situation contrasts sharply with the abundance of nodal curves, since there is no upper bound on the number of nodes of a planar nodal curve of any fixed genus.

By using the methods of Heegaard–Floer homology, Borodzik–Livingston [1] produced a series of constraints on rational cuspidal curves, known as the “semigroup distribution property” (see Fact 2.2 below). In the case of a rational cuspidal curve of degree $d$ with exactly one cusp, these constraints say that, for $0 \leq j \leq d - 1$, the number of elements of the semigroup $W$ of the cusp in the interval $[0, jd]$ equals the triangular number $\binom{j+2}{2}$. We define a notion of triangulating sequences to study such semigroups, and calculate many constraints that they must satisfy. We show that these constraints are in fact sufficient to recover the classification in [4] of unicuspidal rational curves whose cusp is homeomorphic to one of the form $y^d = x^b$. We also prove the following theorem, in the next simplest case:

**Theorem 1.1.** Suppose $C \subset \mathbb{P}^2$ is a rational cuspidal curve of degree $d$ with a single cusp locally homeomorphic to the cusp parametrized by $(x, y) = (t^{mn_2}, t^{nm_2}(1 + t^{n_2}))$, for some positive integers where $\gcd(m, n) = \gcd(m_2, n_2) = 1$ and $m, n, m_2 \geq 2$. Then, up to interchanging the roles of $m$ and $n$, one of the following ten statements holds. (Here $F_i$ denotes the $i$th Fibonacci number; $F_0 = 0$, $F_1 = 1$, and $F_{k+1} = F_k + F_{k+1}$, for all $k \in \mathbb{Z}$.)

1. There exist $k, \ell \in \mathbb{Z}$ such that $d = F_{2k-1}F_{2k+1}m$, and $(m, n) = \ell(F_{2k-1}, F_{2k+1}) + (F_{2k-3}, F_{2k-1} + 2)$, and $(m_2, n_2) = (F_{2k-1}, m)$.

2. There exist $k, \ell \in \mathbb{Z}$ such that $d = F_{2k+1}m$, and $(m, n) = \ell(F_{2k-1}, F_{2k+1}) + (F_{2k-3}, F_{2k-1} + 2)$, and $(m_2, n_2) = (F_{2k-1}, \ell F_{2k-1} + F_{2k-5})$.

3. $d = 8n^2 + 4n + 1$, $(m, n) = (4n + 1, n)$, and $(m_2, n_2) = (4n + 1, (2n + 1)^2)$.

4. $d = nm_2$, $(m, n) = (n - 1, n)$, and $(m_2, n_2) = (m_2, nm_2 - 1)$.

5. $d = n^2 + 1$, $(m, n) = (n - 1, n)$, and $(m_2, n_2) = (n, (n + 1)^2)$.

6. $d = 20$, $(m, n) = (2, 3)$, and $(m_2, n_2) = (6, 31)$.

7. $d = 2nm_2$, $(m, n) = (m, 4m - 1)$, and $(m_2, n_2) = (m_2, 4m - 1)$.

8. $d = 17$, $(m, n) = (2, 7)$, and $(m_2, n_2) = (4, 17)$.

9. $d = F_{4k+2}$, $(m, n) = \left(\frac{F_{4k}}{3}, \frac{F_{4k+4}}{3}\right)$, and $(m_2, n_2) = (3, 1)$.

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It would be interesting to determine which of these invariants belong to actually existing curves. We know that at least the curves in case (4) exist by [3].

Let us turn now to a specific question that can be asked of planar rational curves. A rational curve $C \subset \mathbb{P}^2$ is called rectifiable if there is a birational automorphism of $\mathbb{P}^2$ that is defined at the generic point of $C$ and maps $C$ onto a line. To my knowledge, Matsuoka and Sakai [9] first stated the following conjecture:

**Conjecture 1.2** (Coolidge-Nagata problem). *Every rational cuspidal curve $C \subset \mathbb{P}^2$ is rectifiable.*

The problem is named after Coolidge, who studied the images of algebraic curves under Cremona transformations in his treatise [2], and Nagata, who reinterpreted classical problems in a modern language [11].

The conjecture has been verified for all known rational cuspidal curves. Palka [13] proved it for curves with 5 or more cusps, and later [14] for curves with 3 or more cusps, and in some other cases that we will describe shortly. He mentioned in private correspondence that he and Mariusz Koras have proved the full conjecture, but that paper seems not to be available at the time of this writing.

Let us show how to rectify the examples we listed above.

1. A straight line is by definition rational.

2. A smooth conic $C$ is rectified by a Cremona transformation centered at any three points of $C$. That is, blows up the three chosen points, and then contracts the proper transforms of the lines between them.

3. For $\gcd(a, b) = 1$, we may choose integers $s, t$ such that $as + bt = 1$. Define the birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ sending $(x, y)$ to $(u, v) = (x^by^{-a}, x^syt)$. (Its inverse sends $(u, v)$ to $(x, y) = (u^tv^a, u^{-s}v^b)$.) Then the curve $y^a = x^b$ becomes the line $u = 1$.

4. The tricuspidal quartic can be transformed into a smooth conic by a Cremona transformation centered at the three cusps. This reduces the problem to item 2 above.

5. The quintic with four cusps is a bit more difficult to rectify explicitly. Consider the conics $G_0 = \{y^2 = xz\}$ and $G^*_\zeta = \{4\zeta x^2 + 4\zeta xy + 3y^2 + xz = 0\}$. Note $G_0$ passes through all four cusps of $C$, and $G^*_\zeta$ passes through all but $\zeta(\zeta)$. Form a surface $S$ over $\mathbb{P}^2$ by blowing up three times along $C$ over $\zeta(0)$ with exceptional divisors $E_1, E_2, E_3$, and once more at each $\zeta(\zeta)$ with corresponding exceptional divisor $F_\zeta$: see Figure 1.0.1, in which $\omega = \frac{-1+\sqrt{5}}{2}$. In $S$, the proper transforms of $G_0$ and the three curves $G^*_\zeta$ are pairwise disjoint $(-1)$-curves. Contract them and the two curves $E_2$ and $E_1$ to obtain a morphism $S \to \mathbb{P}^2$ whose composition with $f^{-1} : \mathbb{P}^2 \dashrightarrow S$ is a birational automorphism of $\mathbb{P}^2$ rectifying $C$.

In this thesis, we prove the following:

**Theorem 1.3.** *Let $C \subset \mathbb{P}^2$ be a rational cuspidal curve of degree $d$ and $s$ cusps, such that the $i$th cusp has $k_i$ Newton pairs. Then $C$ is rectifiable if $s \geq 3$, or if $\sum_{i=1}^{s} k_i \leq 2$. (See section 2.1 for the definition of Newton pairs.) The proof is carried out in Lemmas 3.15, 3.16, 3.17, 3.18, and 3.20. We also prove a partial result, Lemma 3.19, toward the case of one cusp with at least 3 Newton pairs.*
Figure 1.0.1: Diagram in $S$ showing rectification of quintic rational curve with four cusps
Chapter 2

The topology of cusps

2.1 Topological invariants of cusps

We first recall the basic topological invariants of cusps. These facts are all standard; see [10, 5], for example. Let $P$ be a cusp of a curve $C$ in a smooth surface $X$. On a small analytic neighborhood of $P$, we can choose holomorphic coordinates $(x, y)$ centered at $P$ such that $C$ is described by the parametric equations

$$
x = t^a
\frac{y = c_1 t^{b_1} + c_2 t^{b_2} + \cdots}
$$
as $t$ ranges over a small neighborhood of 0 in $\mathbb{C}$. Here $1 < a < b_1 < b_2 < \cdots$ are integers such that $a \nmid b_1$ and $\gcd(a, b_1, b_2, \ldots) = 1$. The multiplicity\(^1\) of the cusp is $a$. The sequence of $b_i$'s may be finite or infinite, but we require all the $c_i$ to be non-zero constants. Such a choice of coordinates and parametrization is not uniquely determined, although the numbers $a$ and $b_1$ are. If we momentarily let $g_i$ denote $\gcd(a, b_1, b_2, \ldots, b_i)$, then there is a finite sequence $i_1 < i_2 < \cdots < i_k$ of indices at which $g_i$ decreases; that is, $i_1 = 1$, and

$$
g_{i_1} = \cdots > g_{i_2} > g_{i_3} = \cdots > g_{i_3} = \cdots > g_{i_k} = 1.
$$
It is convenient to set $i_0 = b_0 = 0$ and $g_0 = a$ as well. Let us define

$$
M_j = g_{i_j-1}, \quad (1 \leq j \leq k + 1),
$$

$$
N_j = b_{i_j} - b_{i_{j-1}}, \quad (1 \leq j \leq k).
$$

One associates a sequence of $k$ Newton pairs to the cusp:

$$(m_j, n_j) = (\frac{M_j}{M_{j+1}}, \frac{N_j}{M_{j+1}}) = (\frac{g_{i_{j-1}}}{g_{i_j}}, \frac{b_{i_j} - b_{i_{j-1}}}{g_{i_j}}), \quad (1 \leq j \leq k).
$$

The two numbers in any given Newton pair are relatively prime positive integers, and $m_1 < n_1$. The sequence of Newton pairs is an invariant of the cusp, independent of the choice of coordinates and parametrization. It is not a complete analytic set of invariants, but it does determine the topology

---

\(^1\)The multiplicity of a point $P$ on a planar curve $C$ is the coefficient $m$ such that $f^*C = f^{-1}_*C + mE$, where $f$ is a blow-up of the plane at $P$, with exceptional divisor $E$. In fact, the multiplicity can be defined intrinsically in terms of the Hilbert-Samuel polynomial of the local ring $\mathcal{O}_{P, C}$, but we will not need to use such a definition.
of cusp, in the classical topology. Some exponents of the parametrization can be recovered from the sequence of Newton pairs:

\[
M_j = m_j m_{j+1} \cdots m_k, \\
N_j = n_j n_{j+1} \cdots n_k, \\
a = M_1, \\
b_j = N_1 + N_2 + \cdots + N_j.
\]

The advantage of Newton pairs is that they are bound by no *a priori* relations. To facilitate comparison with other papers, however, let us mention some equivalent ways of presenting the same data. The *k Puiseux pairs* of the cusp are

\[
\left(\frac{a}{g_{i_1}}, \frac{b_{i_1}}{g_{i_1}}\right), \left(\frac{g_{i_1}}{g_{i_2}}, \frac{b_{i_2}}{g_{i_2}}\right), \ldots, \left(\frac{g_{i_{k-1}}}{g_{i_k}}, \frac{b_{i_k}}{g_{i_k}}\right),
\]

and the *characteristic sequence* is

\[
(a; b_{i_1}, \ldots, b_{i_k}).
\]

The *semigroup* of the cusp is the set

\[
W := \{\text{dim}_C \mathcal{O}_{P,C}/(f) | 0 \neq f \in \mathcal{O}_{P,C}\} \subset \mathbb{N} = \{0, 1, 2, \ldots\},
\]

where \(\mathcal{O}_{P,C}\) is the local ring of \(C\) at the cusp \(P\). In other words, \(W\) is the set of all possible local intersection multiplicities at \(P\) of \(C\) with other curves. The semigroup \(W\) has \(k + 1\) minimal generators \(w_1, \ldots, w_{k+1}\), given by

\[
w_1 = M_1, \\
w_2 = N_1, \\
w_j = m_{j-2} w_{j-1} + N_{j-1}, \quad (3 \leq j \leq k + 1).
\]

The semigroup encodes the same information as the sequence of Newton pairs, so it is also a complete topological invariant of the cusp.

The *\(\delta\)-invariant* of the cusp is

\[
\delta = \frac{1}{2} \left((M_1 - 1)(N_1 - 1) + \sum_{j=2}^{k} (M_j - 1)N_j\right),
\]

and it is the contribution of \(P\) to the difference between the arithmetic and geometric genera of \(C\). It equals the number of elements of \(\mathbb{N} \setminus W\), and \(2\delta - 1\) is the maximum element of \(\mathbb{Z} \setminus W\).

When we blow up the ambient surface \(X\) at the point \(P\), the strict transform of \(C\) has either a smooth point or another cusp lying over \(P\). We may, by repeated blow-ups, pass to a minimal log resolution \(\tilde{X} \rightarrow X\) of the cusp \(P\), for which the \(\pi\)-exceptional curves along with the strict transform \(\tilde{C} = \pi^{-1}(C)\) form a simple normal crossings divisor \(D\) on \(\tilde{X}\). Then two different irreducible components of \(D\) intersect each other either once transversely, or not at all. The *dual graph* for this resolution is a graph whose vertices are components of \(D\) and whose edges correspond to intersection points between two different components. Often the dual graph is considered along with the data of the self-intersection numbers of its various components.

The dual graph for a cusp is a tree whose structure that can be computed from the Newton pairs. To avoid confusion with the algebro-geometric notion of degrees of planar curves, we will say that the *branching number* of a component of \(D\) is the number of other components that it meets, or, as we will sometimes say, *is adjacent to*. Every \(\pi\)-exceptional curve has branching number at
Figure 2.1.1: Dual graph of minimum resolution for Example 2.1. The labels are the negatives of the self-intersection numbers.

most 3. The curve $\tilde{C}$ is adjacent to exactly one of the exceptional curves; that curve has branching number 3, and is the only exceptional curve with self-intersection number $-1$.

Exactly $k + 1$ exceptional curves are tips (i.e., they have branching number 1); let us number them $0, \ldots, k$ in order of their appearance (when we think of constructing $\tilde{X} \to X$ by a sequence of blow-ups). If, starting at one of these tips, we walk along components of branching number at most 2 as far as possible, then we trace out a maximal twig of $D$.

We now explain how to draw the dual graph for a cusp. To avoid a morass of notation, we trust that the algorithm will be clear enough from one example:

**Example 2.1.** Let $P$ be a cusp with three Newton pairs: $(m_1, n_1) = (37, 53)$, $(m_2, n_2) = (4, 1)$, and $(m_3, n_3) = (3, 11)$. Compute the continued-fraction expansions:

$$\frac{n_1}{m_1} = \frac{1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{5}}}}{\frac{1}{4}}, \quad \frac{n_2}{m_2} = \frac{0 + \frac{1}{4}}{1}, \quad \frac{n_3}{m_3} = \frac{3 + \frac{1}{1 + \frac{1}{2}}}{1}.$$

Let us agree to abbreviate this as

$$\frac{n_1}{m_1} = [1, 2, 3, 5], \quad \frac{n_2}{m_2} = [0, 4], \quad \frac{n_3}{m_3} = [3, 1, 2].$$

The log resolution of $C$ then has $(1 + 2 + 3 + 5) + (0 + 4) + (3 + 1 + 2) = 21$ exceptional curves lying over $P$. In Figure 2.1.1, the corresponding vertices of the dual graph are arranged, from left to right, in the order of their appearance when we resolve $C$ by repeated monoidal transformations.

Since the cusp has three Newton pairs, the resolution graph has three branching divisors, labeled $E_1$, $E_2$, and $E_3$ in the figure. We may think of each $E_i$ as corresponding to the $i$th Newton pair $(m_i, n_i)$. Reading from left to right, we see $1 + 2 + 3 + 5 = 11$ vertices up to and including $E_1$, then $0 + 4 = 4$ vertices up to and including $E_2$, and finally $3 + 1 + 2 = 6$ vertices up to and including $E_3$.

Let us first understand how the 11 vertices up to and including $E_1$ are arranged. Each one of them appears on one of two branches, which are drawn as the “lower” and “upper” branches; the two branches join together at $E_1$. Looking closely, we see that these 11 vertices are arranged into units of 1, 2, 3, and 5 — as in continued-fraction expansion of $n_1/m_1 = [1, 2, 3, 5]$ — which appear on alternating branches. That is, there is a unit comprising 1 vertex (the one labeled “4”) on the lower branch, then a unit comprising 2 vertices (the ones labeled “2” and “5”) on the upper branch, then a unit comprising 3 vertices (the ones labeled “2, 2, 6”) on the lower branch again, and finally a unit comprising 5 vertices (labeled “2, 2, 2, 2, 5,” including $E_1$) on the upper branch again. This process culminates in $E_1$, which is joined by an edge to the lower branch also. (The numerical labels in the figure are self-intersection numbers, which can be ignored for the time being."

Thus, the rule is to take the four numbers 1, 2, 3, 5 from the continued-fraction expansion of $n_1/m_1 = [1, 2, 3, 5]$, make four units with the corresponding numbers of vertices in each, distribute
the units alternately among the lower and upper branches (starting with the lower branch), and finally connect the two branches together at $E_1$.

Now let us understand how the next 4 vertices, up to and including $E_2$, are arranged. They, too, can be thought of as appearing on two branches: a “lower” one connected to $E_1$, and an “upper” one created anew from a tip of the tree. Recall the continued fraction expansion $n_2/m_2 = [0, 4]$. The rules above suggest that we should take the two numbers 0, 4 from this continued fraction expansion, make two units with the corresponding numbers of vertices in each, distribute the units alternately between the lower and upper branches (starting with the lower branch), and finally connect the two branches together at $E_2$. This is indeed the procedure. We have purposely drawn an extra-wide space to the right of $E_1$ to remind ourselves there is a unit of 0 vertices there.

Finally, let us how understand the last 6 vertices, up to and including $E_3$, are arranged, but there should be no surprises here. These 6 vertices also appear on two branches: a “lower” one connected to $E_2$, and an “upper” one created anew from a tip of the tree. In accordance with the continued fraction expansion $n_3/m_3 = [3, 1, 2]$, they occur in units of 3 vertices on the lower branch, then 1 vertex on the upper branch, and finally 2 vertices on the lower branch, ending on $E_3$, which is joined to the upper branch as well as the strict transform $C$ of the curve.

In sum, the tree is structured so that the 21 vertices naturally fall into units of 1, 2, 3, 5; then 0, 4; then 3, 1, and 2 vertices. The three branching divisors have been labeled $E_1$, $E_2$, and $E_3$, and each naturally belongs to the unit immediately to its left.\(^2\) We have drawn an extra-wide space to the right of $E_1$ to illustrate the “0” occurring in the continued-fraction expansion of $n_2/m_2$. The graph-theoretic significance of this “0” is that the exceptional curve appearing immediately after $E_1$ occurs on a maximal twig, and is not adjacent to $E_1$. Compare this to the case of the third Newton pair: there the continued-fraction expansion of $n_3/m_3$ begins with a positive number, so the exceptional curve appearing immediately after $E_2$ is adjacent to it, and not on a maximal twig.

The self-intersection numbers of these exceptional curves are all negative, and their absolute values are shown in the figure. To compute this value for any exceptional curve, except the $(-1)$-curve $E_k$ at the very end, count the number of vertices between it and its right-side neighbor, inclusive. We also have

$$\tilde{C}^2 = C^2 - \sum_{j=1}^{k} M_j N_j = C^2 - 2\delta + 1 - M_1 - \sum_{j=1}^{k} N_j.$$ (2.1.3)

2.2 Notation

Let us fix the notation for the rest of this chapter and the next. Let $C \subset \mathbb{P}^2$ be a rational cuspidal curve of degree $d$ with $s$ cusps $P_1, \ldots, P_s$, and let the cusp $P_i$ have $k_i$ Newton pairs, namely

$$(m_{i,1}, n_{i,1}), \ldots, (m_{i,k_i}, n_{i,k_i}).$$

Let $M_{ij} = \prod_{j' = j}^{k} m_{ij}$ and $N_{ij} = n_{ij} M_{i,j+1}$, so in particular $M_{i1}$ is the multiplicity of $P_i$ on $C$. Let $W_i$ be the semigroup of the cusp $P_i$, so in particular the lowest two elements of $W_i$ are 0 and $M_{i1}$.

\(^2\)Of course, we may choose to write the continued-fraction expansions as

$$\frac{n_1}{m_1} = 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4 + \cdots}}}, \quad \frac{n_2}{m_2} = 0 + \frac{1}{3 + \frac{1}{4}}, \quad \frac{n_3}{m_3} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}},$$

instead, in which case we may think of the three branching divisors as constituting their own units. It is hardly coincidental that there are two ways of writing the end of the continued-fraction expansion, just as the vertex for the branching divisor is adjacent to two vertices on its left.
Recall that the $\delta$-invariant for the cusp $P_i$ is
\[ \delta_i = \frac{1}{2} \left( (M_1 - 1)(N_1 - 1) + \sum_{j=2}^{k} (M_j - 1)N_j \right), \]
so
\[ \sum_{i=1}^{s} \delta_i = \frac{(d - 1)(d - 2)}{2}. \] (2.2.1)

Following [1], let
\[ R_i(a) = \# W_i \cap [0, a), \quad a \in \mathbb{N}, \]
and for functions $f, g : \mathbb{N} \to \mathbb{N}$ let $\circ$ denote the convolution product in the min-plus semiring:
\[ (f \circ g)(c) = \min_{a+b=c} (f(a) + g(b)). \]

Let $V \xrightarrow{\pi} \mathbb{P}^2$ be the minimal log resolution of $(\mathbb{P}^2, C)$. Let $\tilde{C}$ be the proper transform $\pi^{-1}^{-1}(C)$, and let $D$ be the sum of all irreducible components of $\pi^{-1}(C)$. Let the $k_i$ branching divisors lying over the $P_i$ be $E_{i,1}, \ldots, E_{i,k_i}$, so that $E_{i,k_i}$ is a $(-1)$-curve meeting $\tilde{C}$.

Let $\kappa$ be the Kodaira dimension of $K_V + D$, or in other words the logarithmic Kodaira dimension of the open surface $V \setminus D = \mathbb{P}^2 \setminus C$. If $\kappa = 2$, let $K_V + D = (K_V + D)^+ + (K_V + D)^-$ denote the Zariski decomposition of $K_V + D$.

### 2.3 The semigroup distribution property

By using the methods of Heegaard-Floer homology, Borodzik-Livingston [1] produced a series of constraints on rational cuspidal curves, known as the "semigroup distribution property":

**Fact 2.2** ([1], Theorem 6.5). *For natural numbers $j \leq d - 1$, we have*
\[ (R_1 \circ \cdots \circ R_s)(jd + 1) = \frac{(j + 1)(j + 2)}{2}. \]

We should note that, in the paper they state this for $j \leq d - 2$, but in fact the same equation holds follows for $j \leq d - 1$ as well, simply by the genus equation:
\[ (d - 2)d + 1 \geq (d - 1)(d - 2) = \sum_{i=1}^{s} (\max(N \setminus W_i) + 1), \]
so for all $j \geq d - 2$ we have
\[ (R_1 \circ \cdots \circ R_s)(jd + 1) = (R_1 \circ \cdots \circ R_s)((j - 1)d + 1) + d. \]

The inequality $\frac{(j + 1)(j + 2)}{2}$ follows from the "semigroup density property": for any $x_1, \ldots, x_s \in \mathbb{N}$ such that $x_1 + \cdots + x_s < \frac{(j + 1)(j + 2)}{2}$, there exists a curve in $\mathbb{P}^2$ of degree $j$ whose local intersection multiplicity with $C$ at the $i$th cusp is at least the $x_i$th non-zero element of the semigroup $W_i$, and Bézout's theorem ensures that these local intersection multiplicities sum to at most $jd$. The deeper result of Fact 2.2 is the $\leq$ direction. The proof involves calculating the "$d$-invariant" or "correction term" for the chain complex $CF^\infty$ associated to the boundary $\partial N$ of a
tubular neighborhood $N$ of $C$ in $\mathbb{CP}^2$, together with Spin$^c$ structures on $\partial N$ that extend over the complement $\mathbb{CP}^2 \setminus N$, which is a rational homology ball.

At any rate, the combinatorial implications of Fact 2.2 are quite intricate, and we have relegated the details of these calculations to Chapter 4. In Definition 4.1, we introduce a notion of a \textit{weakly triangulating sequence} $(\frac{d}{m_k}, \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k})$, which is a necessary condition for a rational cuspidal curve of degree $d$ and a single cusp with Newton pairs $(m_1, n_1), \ldots, (m_k, n_k)$ to satisfy the purely combinatorial constraints imposed by Fact 2.2. This notion is useful in cases of more than one cusp, also: if a rational cuspidal curve $C$ of degree $d$ has one cusp with Newton pairs $(m_1, n_1), \ldots, (m_k, n_k)$, and $m_{i+1} \cdots m_k$ exceeds the multiplicity all other cusps of $C$, then $(\frac{d}{m_1 \cdots m_k}, \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k})$ must also be weakly triangulating.

As a direct consequence of Proposition 4.25, we recover the classification in [4] of unicuspidal rational curves whose cusp has one Newton pair:

**Theorem 2.3.** Let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve of degree $d$ whose cusp has only one Newton pair $(m, n)$. Then one of six cases holds. (Here $F_i$ denotes the $i$th Fibonacci number; $F_0 = 0$, $F_1 = 1$, and $F_{k+1} = F_k + F_{k-1}$, for all $k \in \mathbb{Z}$.)

1. For some $k \geq 3$, we have $d = F_{2k-1}$ and $(m, n) = (F_{2k-3}, F_{2k+1})$. That is to say, $m, d, n$ are consecutive odd-indexed Fibonacci numbers with $2 \leq m < d < n$.

2. For some $k \geq 2$, we have $d = F_{2k-1}F_{2k+1} = F_{2k}^2 + 1$ and $(m, n) = (F_{2k-1}^2, F_{2k+1}^2)$.

3. For some $n \geq 3$, we have $d = n$ and $(m, n) = (n-1, n)$.

4. For some $m \geq 2$, we have $d = 2m$ and $(m, n) = (m, 4m - 1)$.

5. We have $d = 8$ and $(m, n) = (3, 22)$.

6. We have $d = 16$ and $(m, n) = (6, 43)$.

We also derive constraints on unicuspidal rational curves whose cusp has two Newton pair, namely Theorem 1.1 from the Introduction, which is a direct consequence of Theorem 4.26.
Chapter 3

The Coolidge–Nagata problem

In this chapter we discuss some results on the Coolidge–Nagata problem (Conjecture 1.2). As we mentioned in the Introduction, Karol Palka has mentioned in private correspondence that he and Mariusz Koras have proved the full conjecture, but that paper seems not to be available at the time of this writing. However, Palka [13, 14] has released some partial results, see Fact 3.5 below.

3.1 Conditions for rectifiability

In this section, we survey the previously known results on rational cuspidal curves that will be useful to us. The main tool for proving rectifiability of plane rational curves is the following theorem:

**Fact 3.1** (Coolidge [2], Kumar-Murthy [8]). A rational cuspidal curve $C \subset \mathbb{P}^2$ is rectifiable if and only if

$$h^0(V, 2K_V + \tilde{C}) = 0.$$ 

Coolidge [2, Book IV, page 396] defined a “special adjoint” to $C$ of index $i \geq 1$ to be a curve in the linear series $|iK_V + \tilde{C}|$. He showed that the property of having special adjoints is preserved under Cremona transformations, and that a singular curve with no special adjoints can be transformed into one with milder singularities. Since the straight line has no special adjoints, it follows that a rational curve is rectifiable if and only if it has no special adjoints. An argument with Riemann-Roch [8, Corollary 2.4] then yields Fact 3.1.

Actually, to be slightly less historically inaccurate, we should mention that Coolidge, writing in the 1930’s, worked on $\mathbb{P}^2$ without explicit reference to $V$. The point is that, given a divisor $L$ on $\mathbb{P}^2$ of degree

$$\deg \pi_*(iK_V + \tilde{C}) = -3i + \deg C,$$

one can determine whether $L$ is the push-forward (under $\pi$) of some effective divisor $\tilde{L} \sim iK_V + \tilde{C}$ on $V$: such $\tilde{L}$ exists if and only if, for every point $P$ of multiplicity $m$ on $C$ (possibly an “infinitely near point,” as the classical algebraic geometers called valuations), $L$ passes through $P$ with multiplicity at least $m - i$. Strictly speaking, it is $L$, rather than $\tilde{L}$, that Coolidge calls the “special adjoint,” but we may use the term more liberally.

The rational cuspidal curves $C$ for which $\kappa < 2$ have been classified [5], and the Coolidge--Nagata conjecture has been verified for them:

**Fact 3.2** ([5], [17]). If $\kappa < 2$, then $C$ is rectifiable. Moreover, $C$ has at most 2 cusps.

If $\kappa = 2$, then the Zariski decomposition of $K_V + D = (K_V + D)^+ + (K_V + D)^-$ can be computed:
Fact 3.3 ([16], [13]). Call a maximal twig of $D$ "admissible" if its components have self-intersection at most $-2$. If $\kappa = 2$, then $(K_V + D)^+$ has zero intersection with all components of admissible maximal twigs of $D$, and $(K_V + D)^-$ is supported on the union of these admissible maximal twigs.

Note that $\tilde{C}$ forms a maximal twig of $D$ if and only if $s = 1$. Since the intersection form on $V$ is negative-definite when restricted to the linear span of the components of admissible maximal twigs, Fact 3.3 specifies the Zariski decomposition of $K_V + D$ exactly. Thus, we are able to compute the volume of $K_V + D$. Our intention is to exploit the following:

Fact 3.4 (Logarithmic Bogomolov-Miyaoka-Yau inequality, [7]). If $\kappa = 2$, then
\[ \text{vol}(K_V + D) \leq 3 \cdot e(V \setminus D) = 3, \]
where $e$ denotes the topological Euler characteristic.

By exploiting the above facts, Palka [13, 14], proved the following:

Fact 3.5 ([13]). The curve $C$ is rectifiable if $s + k_1 + \cdots + k_s \geq 7$, and in particular if $s \geq 3$.

Matsuoka and Sakai [9] proved the following:

Fact 3.6 ([9]). We have
\[ d < 3 \max_i M_{i1}. \]

In particular, we immediately have the following:

Fact 3.7. The curve $C$ is rectifiable if $d \leq 6$.

Proof. Suppose that, in accordance with Fact 3.1, there were a curve $L \sim 2K_V + \tilde{C}$. Then
\[ d - 6 = \deg C + 2 \deg K_{p2} = \deg \pi_*(\tilde{C} + 2K_V) = \deg \pi_*L \geq 0, \]
so $d = 6$. Then some cusp $P_i$ has multiplicity $M_{i1} > d/3 = 2$ by Fact 3.6. From Coolidge's original point of view (of working on $\mathbb{P}^2$), the fact that $L$ is an adjoint of index 2 to $C$ implies that $\pi_*L$ must pass through $P_i$ with multiplicity at least $M_{i1} - 2 > 0$. But $\deg \pi_*L = 0$, so this is impossible. \qed

Orevkov [12] sharpened the Matsuoka-Sakai inequality to
\[ d < \varphi^2(1 + \max_i M_{i1}) + \frac{1}{\sqrt{5}}, \]
where $\varphi$ is the golden ratio, $\frac{1 + \sqrt{5}}{2}$. Moreover, Orevkov showed the following:

Fact 3.8 ([12]). If $\kappa = 2$, then
\[ d < \varphi^2(1 + \max_i M_{i1}) - \frac{1}{\sqrt{5}}. \]
3.2 An effective divisor and a volume computation

We begin with a simple observation.

**Lemma 3.9.** Fix \(1 < i < s\), and let \(x\) be the unique integer such that \(0 \leq 2x \leq n_{ki}\) and
\[m_{ki}x \equiv \pm 1 \pmod{n_{ki}}.\]

Consider the longest chain of \((-2)\)-curves in \(D\) that meets the \((-1)\)-curve \(E_{i,ki}\) and lies over the cusp \(P_i\); let \(\ell_i\) be the length of this chain. Then

\[
\ell_i = \begin{cases} 
  m_{ki} - 1 & \text{if } n_{ki} = 1, \\
  n_{ki} - 2 & \text{if } n_{ki} \geq 3 \text{ and } m_{ki} \equiv -1 \pmod{n_{ki}}, \\
  \left\lfloor \frac{n_{ki}}{x} \right\rfloor - 1 & \text{otherwise}.
\end{cases}
\]

**Proof.** For simplicity, let us drop the subscripts \(i\) and \(k_i\). Express \(\frac{n}{m}\) as a continued fraction
\[
\frac{n}{m} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ldots + \frac{1}{a_r}}}} = [a_0, \ldots, a_r]
\]
with \(a_r > 1\). (We require \(a_0, a_1, \ldots, a_r \in \mathbb{N}\) and \(a_1, \ldots, a_{r-1} > 0\).) The algorithm described in Example 2.1 shows that \(\ell = a_r - 1\). The theory of continued fractions tells us that
\[
[a_r, \ldots, a_0] = \frac{n}{x},
\]
provided we interpret \(1/0\) as \(\infty\) and \(1/\infty\) as \(0\). Let us consider the three cases described in the lemma:

**Case 1:** \(n = 1\). Then \(\frac{n}{m} = [0, m]\), so \(\ell = m - 1\), which agrees with the lemma.

**Case 2:** \(n \geq 3\) and \(m = np - 1\). Then
\[
\frac{n}{m} = \begin{cases} 
  [1, m] = [1, n - 1] & \text{if } p = 1, \\
  [0, p - 1, 1, n - 1] & \text{if } p \geq 2,
\end{cases}
\]
so \(\ell = n - 2\), and the lemma again yields the correct result.

**Case 3:** In all remaining cases, it remains to prove that \(a_r = [a_r, \ldots, a_0]\). This equation usually holds, with two exceptions (if we keep in mind that \(m \geq 2\) and \(\gcd(m, n) = 1\)). The first exception occurs when \(r = 1\) and \(a_0 \in \{0, 1\}\), for then \([a_1, 0] = \infty\) and \([a_1, 1] = a_1 + 1\). But then \(n \in \{1, m + 1\}\), and we should be in Case 1 or 2 above. The second exception occurs when \(r = 3\), \(a_0 = 0\), and \(a_2 = 1\), for then \([a_3, 1, a_1, 0] = a_3 + 1\). But then
\[
\frac{n}{m} = [0, a_1, 1, a_3] = \frac{a_3 + 1}{a_1 a_3 + a_1 + a_3},
\]
so we should be in Case 2. \(\square\)

With Lemma 3.9 in mind, and with \(0 \leq 2x \leq n_{ki}\) and \(m_{ki}x \equiv \pm 1 \pmod{m_{ki}}\) as above, let us define
\[
\varepsilon_i := \begin{cases} 
  m_{ki} & \text{if } n_{ki} = 1, \\
  n_{ki} - 1 & \text{if } n_{ki} \geq 3 \text{ and } m_{ki} \equiv -1 \pmod{n_{ki}}, \\
  \left\lfloor \frac{n_{ki}}{x} \right\rfloor & \text{otherwise}.
\end{cases}
\]

Then \(\varepsilon_i \geq 2\), and there is a length \(\varepsilon_i - 1\) chain of \((-2)\)-curves in \(D\) that meets \(E_{i,ki}\). A few values of \(\varepsilon_i\) are shown in Table 3.1.

We now have the following:
Lemma 3.10. If $C$ is not rectifiable, then in $V$ there is an effective divisor linearly equivalent to

$$L := 2K_V + \tilde{C} - \sum_{i=1}^{s} \varepsilon_i E_{i,k_i}.$$  

Proof. Fact 3.1 guarantees that there is an effective divisor

$$L' \sim 2K_V + \tilde{C}.$$  

Our aim is to prove that $E_{i,k_i}$ is a component of $L'$ appearing with coefficient at least $\varepsilon_i$. We can consider the various cusps separately, so fix a particular $i$. Let $F_1, \ldots, F_{\varepsilon_i}$ be the chain of $\pi$-exceptional divisors in $D$ such that $F_1^2 = \cdots = F_{\varepsilon_i-1}^2 = -2$ and $F_{\varepsilon_i} = E_{i,k_i}$, so that $F_{\varepsilon_i}^2 = -1$. Suppose that $F_j$ occurs with coefficient $a_j \geq 0$ in $L'$. Let $a_{-1} = 0$, for convenience. Then $L' \cdot F_1 = \cdots = L' \cdot F_{\varepsilon_i-1} = 0$ and $L' \cdot F_{\varepsilon_i} = 2(-1) + 1 = -1$, so for each $j$

$$0 \leq \left( L' - \sum_{j=1}^{\varepsilon_i} a_j F_j \right) \cdot F_j = \begin{cases} 2a_j - a_{j-1} - a_{j+1} & \text{if } 1 \leq j \leq \varepsilon_i - 1, \\ a_j - a_{j-1} - 1 & \text{if } j = \varepsilon_i. \end{cases}$$

Then

$$1 \leq a_{\varepsilon_i} - a_{\varepsilon_i-1} \leq a_{\varepsilon_i-1} - a_{\varepsilon_i-2} \leq \cdots \leq a_2 - a_1 \leq a_1,$$

so by induction on $j$ we see that $a_j \geq j$ for all $j$. In particular,

$$a_{\varepsilon_i} \geq \varepsilon_i,$$

as desired. \qed

Suppose we had a counterexample to the Coolidge–Nagata conjecture, so there is an effective divisor

$$L \sim 2K_V + \tilde{C} - \sum_{i=1}^{s} \varepsilon_i E_{i,k_i}.$$  

Our goal is to exploit the fact that $L$ must have non-negative intersection product with the nef divisor $(K_V + D)^+$. Palka [13] used the same strategy, but without the $\sum_{i=1}^{s} \varepsilon_i E_{i,k_i}$ term.

Lemma 3.11. If a rational cuspidal curve $C$ is not rectifiable, then

$$\tilde{C}^2 \leq -4 - \sum_{i=1}^{s} \varepsilon_i \leq -4 - 2s.$$  

Table 3.1: Values of $\varepsilon_i$

<table>
<thead>
<tr>
<th>$(m_{i,k_i}, n_{i,k_i} \mod m_{i,k_i})$</th>
<th>1</th>
<th>2</th>
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</table>
Proof. By assumption the effective divisor $L$ exists. Note that
\[ \deg \pi_* L = d - 6 < \deg \pi_* \tilde{C}, \]
so $\tilde{C}$ cannot be a component of $L$. It follows that
\[ 0 \leq L \cdot \tilde{C} = 2 (K_V + \tilde{C}) \cdot \tilde{C} - \tilde{C}^2 - \sum_{i=1}^s \varepsilon_i = -\tilde{C}^2 - 4 - \sum_{i=1}^s \varepsilon_i, \]
as claimed. \( \square \)

Thus, if $s = 1$, then $\tilde{C}$ appears with positive coefficient in $(K_V + D)^-$, according to Fact 3.3.

Let us make some relevant calculations. First note that $(K_V + D)^+$ has zero intersection product with all $\pi$-exceptional divisors that are not branching in the dual graph $D$. One can check that, if $s \geq 2$, then
\[
(K_V + D)^+ \cdot \tilde{C} = s - 2, \quad \quad (K_V + D)^+ \cdot E_{ij} = \begin{cases} 1 - \frac{1}{m_{1,j}} - \frac{1}{m_{1,i}} & \text{if } j = 1, \\ 1 - \frac{1}{m_{1,j}} & \text{if } 2 \leq j \leq k_i. \end{cases}
\]
When $s = 1$, the only difference is that $\tilde{C}$ appears in $(K_V + D)^-$ with coefficient $1/(-\tilde{C}^2) > 0$, so
\[
(K_V + D)^+ \cdot \tilde{C} = 0, \quad \quad (K_V + D)^+ \cdot E_{1,j} = \begin{cases} 1 - \frac{1}{m_{1,1}} - \frac{1}{m_{1,1}} & \text{if } 1 = j < k_1 \\ 1 - \frac{1}{m_{1,j}} & \text{if } 1 < j < k_i \\ 1 - \frac{1}{m_{1,j}} - \frac{1}{-\tilde{C}^2} & \text{if } 1 = j = k_i \\ 1 - \frac{1}{m_{1,j}} - \frac{1}{-\tilde{C}^2} & \text{if } 1 < j = k_i. \end{cases}
\]
It follows that
\[
(K_V + D)^+ \cdot (K_V + D - \frac{L}{2}) = (K_V + D)^+ \cdot \left[ \frac{\tilde{C}}{2} + \sum_{i=1}^s \left( \frac{\varepsilon_i}{2} E_{i,k_i} + \sum_{j=1}^{k_i} E_{ij} \right) \right].
\]
In light of Fact 3.4, one of the most basic inequalities at our disposal is this:

**Lemma 3.12.** If $C$ is not rectifiable, then
\[ \sum_{i=1}^s \gamma_i \leq \vol(K_V + D) + \min \left\{ \frac{s}{2}, 1 \right\} \leq \min \left\{ 3 + \frac{s}{2}, 4 \right\}, \]
where
\[ \gamma_i := \frac{1}{2} + (K_V + D)^+ \cdot \left( \frac{\varepsilon_i}{2} E_{i,k_i} + \sum_{j=1}^{k_i} E_{ij} \right). \]

If $s \geq 2$, then, since $\varepsilon_i \geq 2$, we have
\[
\gamma_i = \begin{cases} \frac{3 + \varepsilon_i}{2} - \frac{2 + \varepsilon_i}{2} \left( \frac{1}{n_{i,j}} + \frac{1}{m_{i,j}} \right) & \text{if } k_i = 1, \\ k_i + \frac{1 + \varepsilon_i}{2} - \frac{1}{m_{i,j}} + \sum_{j=1}^{k_i-1} \frac{k_i-1}{m_{i,j}} - \frac{2 + \varepsilon_i}{2} \frac{1}{m_{i,k}} & \text{if } k_i \geq 2, \end{cases}
\]
\[
\geq \begin{cases} \frac{5}{2} - \frac{2}{n_{i,j}} - \frac{2}{m_{i,j}} & \text{if } k_i = 1, \\ k_i + \frac{3}{2} - \frac{1}{n_{i,j}} - \frac{1}{m_{i,j}} - \cdots - \frac{1}{m_{i,k_i-1}} - \frac{2}{m_{i,k_i}} & \geq \frac{k_i}{2} + \frac{2}{3} & \text{if } k_i \geq 2. \end{cases}
\]
If \( s = 1 \), then, since \( \varepsilon_1 \geq 2 \), we have

\[
\gamma_1 = \begin{cases} 
\frac{3+\varepsilon_1}{2} - \frac{2+\varepsilon_1}{2} \left( \frac{1}{n_{1,1}} + \frac{1}{m_{1,1}} + \frac{1}{\mathcal{C}^2} \right) & \text{if } k_1 = 1, \\
k_1 + \frac{1+\varepsilon_1}{2} - \frac{1}{n_{1,1}} - \sum_{j=1}^{k_1} \frac{1}{m_{1,j}} - \frac{2+\varepsilon_1}{2} \left( \frac{1}{m_{1,k_1}} + \frac{1}{\mathcal{C}^2} \right) & \text{if } k_1 \geq 2, \\
\left( \frac{5}{2} - \frac{2}{n_{1,1}} - \frac{2}{m_{1,1}} - \frac{2}{\mathcal{C}^2} \right) - \frac{1}{n_{1,1}} - \sum_{j=1}^{k_1-1} \frac{1}{m_{1,j}} - \frac{2}{m_{1,k_1}} - \frac{2}{\mathcal{C}^2} & \text{if } k_1 = 1, \\
k_1 + \frac{3}{2} - \frac{1}{n_{1,1}} - \sum_{j=1}^{k_1-1} \frac{1}{m_{1,j}} - \frac{2}{m_{1,k_1}} - \frac{2}{\mathcal{C}^2} & \text{if } k_1 \geq 2.
\end{cases}
\]

**Proof.** The formulas for \( \gamma_i \) follow from our previous calculations. The inequality

\[
\sum_{i=1}^{s} \gamma_i \leq \text{vol}(K_V + D) + \min \left\{ \frac{s}{2}, 1 \right\}
\]

is nothing more than a restatement of the fact that

\[
(K_V + D)^+ \cdot \left( K_V + D - \frac{L}{2} \right).
\]

At this point, we are ready to deal with the case of four or more cusps, so the reader may skip ahead to Lemma 3.15 if desired.

From 2.1.3 and the genus equation, we find

\[
-\mathcal{C}^2 = \sum_{i,j} M_{i,j} N_{i,j} - d^2,
\]

\[
= \sum_{i=1}^{s} \left( M_{i,1} + \sum_{j=1}^{k_i} N_{i,j} - 1 \right) + 2 - 3d. \tag{3.2.1}
\]

For future reference, let us calculate the volume of \( K_V + D \).

**Lemma 3.13.** Assume that \( \kappa = 2 \). If \( s \geq 2 \), then

\[
\text{vol}(K_V + D) := ((K_V + D)^+)^2 = 7 - 3d + \sum_{i=1}^{s} \left[ -1 + M_{i,1} - \frac{m_{i1}}{n_{i1}} + \sum_{j=1}^{k_i} \left( N_{ij} - \frac{n_{ij}}{m_{ij}} \right) \right].
\]

If \( s = 1 \) and \( \mathcal{C}^2 \leq -2 \), then

\[
\text{vol}(K_V + D) = 5 + \frac{\left( -\mathcal{C}^2 - \frac{1}{\mathcal{C}^2} \right) - \frac{m_{11}}{n_{11}} - \sum_{j=1}^{k_1} \frac{n_{1j}}{m_{1j}}.}
\]

**Proof.** First suppose \( s \geq 2 \). One may check that

\[
K_V = \pi^* K_{p^2} + \sum_{i=1}^{s} \sum_{j=1}^{k_i} \left( M_{i,1} + N_{i,1} + N_{i,2} + \cdots + N_{i,j} \right) E_{ij} + (\text{non-branching exceptional divisors}).
\]
So
\[
\text{vol}(K_V + D) = (K_V + D)^+ \cdot (K_V + D) - (K_V + D)^+ \cdot (K_V + D)^-
\]
\[
= (K_V + D)^+ \cdot \left[ \pi^* K_{g2} + \tilde{C} + \sum_{i=1}^{k_i} \sum_{j=1}^{k_i} \left( \frac{M_{i,1} + N_{i,1} + N_{i,2} + \cdots + N_{i,j}}{M_{i,j+1}} \right) E_{ij} \right] - 0.
\]

Now
\[
(K_V + D)^+ \cdot (\pi^* K_{g2} + \tilde{C}) = (d - 3) \cdot (-3) + (s - 2) = 7 - 3d + s,
\]
and
\[
(K_V + D)^+ \cdot \sum_{i=1}^{s} \sum_{j=1}^{k_i} \left( \frac{M_{i,1} + N_{i,1} + N_{i,2} + \cdots + N_{i,j}}{M_{i,j+1}} \right) E_{ij}
\]
\[
= \sum_{i=1}^{s} \left[ \frac{-m_{i1}}{n_{i1}} - 1 + M_{i1} \left( 1 - \frac{1}{M_{i1}} \right) + \sum_{j' = 1}^{M_{i1}} N_{ij'} \left( 1 - \frac{1}{M_{ij'}} \right) \right].
\]

The sums over \( j \) telescope, so the expression equals
\[
\sum_{i=1}^{s} \left[ \frac{-m_{i1}}{n_{i1}} - 1 + M_{i1} \left( 1 - \frac{1}{M_{i1}} \right) + \sum_{j' = 1}^{M_{i1}} N_{ij'} \left( 1 - \frac{1}{M_{ij'}} \right) \right].
\]

Therefore,
\[
\text{vol}(K_V + D) = 7 - 3d + s + \sum_{i=1}^{s} \left[ \frac{-m_{i1}}{n_{i1}} - 1 + M_{i1} - 1 + \sum_{j' = 1}^{M_{i1}} N_{ij'} \left( 1 - \frac{1}{M_{ij'}} \right) \right],
\]
proving the lemma when \( s \geq 2 \).

When \( s = 1 \), the only difference is that \( \tilde{C} \) appears in \( (K_V + D)^- \) with coefficient \( 1/(-\tilde{C}^2) > 0 \). We should therefore subtract
\[
\frac{(K_V + D) \cdot \tilde{C}}{-\tilde{C}^2} = (K_V + \tilde{C}) \tilde{C} + E_{i,1,k_1} \cdot \tilde{C} = \frac{-2 + 1}{-\tilde{C}^2}
\]
from the formula of the \( s \geq 2 \) case. That is indeed what the lemma claims. \( \square \)

The following Corollary is sometimes useful in conjunction with the formula (3.2.1) for \( -\tilde{C}^2 \):

**Corollary 3.14.** We have (under our standing assumption that \( \kappa = 2 \))
\[
-\tilde{C}^2 \leq \sum_{i=1}^{s} \left( \frac{m_{i,1}}{n_{i,1}} + \sum_{j=1}^{k_i} \frac{n_{i,j}}{m_{i,j}} \right) - \begin{cases} 2 + \frac{1}{-\tilde{C}^2} & \text{if } s = 1 \\ 2 & \text{if } s \geq 2. \end{cases}
\]
If $C$ is not rectifiable, then in fact
\[
\sum_{i=1}^{s} \left( \frac{m_{i,1}}{n_{i,1}} + \frac{k_{i}}{\sum_{j=1}^{s} \frac{n_{i,j}}{m_{i,j}}} + \gamma_{i} \right) - \left\{ \frac{11}{6} + \frac{1}{C^{2}} \right\} \text{ if } s = 1 \\
\sum_{i=1}^{s} \left( \frac{m_{i,1}}{n_{i,1}} + \frac{k_{i}}{\sum_{j=1}^{s} \frac{n_{i,j}}{m_{i,j}}} + \gamma_{i} \right) - \left\{ \frac{1}{C^{2}} \right\} \text{ if } s \geq 2.
\]

Proof. The first follows from 3.4 and Lemma 3.12, which says
\[
-\bar{C}^{2} = \sum_{i=1}^{s} \left( \frac{m_{i,1}}{n_{i,1}} + \frac{k_{i}}{\sum_{j=1}^{s} \frac{n_{i,j}}{m_{i,j}}} \right) + \text{vol}(K_{D}) - \left\{ \frac{5 + \frac{1}{C^{2}}}{6} \right\} \text{ if } s = 1 \\
\text{vol}(K_{D}) - \left\{ \frac{5}{6} \right\} \text{ if } s \geq 2.
\]

If $C$ is not rectifiable, then by Lemma 3.12, the right-hand side of the above equation is at least
\[
\sum_{i=1}^{s} \left( \frac{m_{i,1}}{n_{i,1}} + \frac{k_{i}}{\sum_{j=1}^{s} \frac{n_{i,j}}{m_{i,j}}} + \gamma_{i} \right) - \left\{ \frac{11}{6} + \frac{1}{C^{2}} \right\} \text{ if } s = 1 \\
\sum_{i=1}^{s} \left( \frac{m_{i,1}}{n_{i,1}} + \frac{k_{i}}{\sum_{j=1}^{s} \frac{n_{i,j}}{m_{i,j}}} + \gamma_{i} \right) - \left\{ \frac{1}{C^{2}} \right\} \text{ if } s \geq 2. \]

\]

3.3 Three or more cusps

We can immediately deal with the case of at least four cusps:

Lemma 3.15. If $C$ has $s \geq 4$ cusps, then $C$ is rectifiable.

Proof. Suppose that $s \geq 4$, but $C$ is not rectifiable. We may assume $d \geq 7$, by Fact 3.7. If $s \geq 5$, then, in the notation of Lemma 3.12,
\[
\sum_{i=1}^{s} \gamma_{i} \geq \frac{5}{6} \geq \frac{25}{6} > 4,
\]
a contradiction. Likewise, if $s = 4$ but any $k_{i} \geq 2$, then
\[
\sum_{i=1}^{s} \gamma_{i} \geq 3 \cdot \frac{5}{6} + \frac{2}{3} = \frac{25}{6} > 4,
\]
a contradiction. Therefore, we must have $s = 4$ and $k_{1} = \cdots = k_{4} = 1$. For simplicity of notation, let just write $(m_{i}, n_{i})$ for $(m_{i,1}, n_{i,1})$. Then $\gamma_{i} \geq \frac{5}{2} - \frac{2}{m_{i}} - \frac{2}{n_{i}}$, so we have
\[
\sum_{i=1}^{4} \left( \frac{1}{m_{i}} + \frac{1}{n_{i}} \right) \geq 3.
\]
On the other hand, Lemma 3.6 says $\max m_{i} \geq 3$. Without loss of generality, assume $m_{1} \geq 3$. Since
\[
2 \left( \frac{1}{3} + \frac{1}{4} \right) + 2 \left( \frac{1}{2} + \frac{1}{3} \right) < \left( \frac{1}{4} + \frac{1}{5} \right) + 3 \left( \frac{1}{2} + \frac{1}{3} \right) < 3,
\]
it must be that $m_{1} = 3$ and $m_{2} = m_{3} = m_{4} = 2$. As
\[
\left( \frac{1}{3} + \frac{1}{4} \right) + 2 \left( \frac{1}{2} + \frac{1}{3} \right) + \left( \frac{1}{2} + \frac{1}{5} \right) < \left( \frac{1}{3} + \frac{1}{7} \right) + 3 \left( \frac{1}{2} + \frac{1}{3} \right) < 3,
\]

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we must have $n_1 \leq 5$ and $n_2 = n_3 = n_4 = 3$. Then the sum of the $\delta$-invariants of the four cusps is, by 2.2.1,

$$\sum_{i=1}^{4} \frac{(m_i - 1)(n_i - 1)}{2} \leq 4 + 1 + 1 + 1 = 7,$$

which contradicts the fact that $C$ has arithmetic genus

$$\frac{(d-1)(d-2)}{2} \geq 15.$$ 

\[\square\]

The case of three cusps is a bit more involved, and uses both Lemma 3.13 and, in one place, Fact 2.2.

**Lemma 3.16.** If $C$ has $s = 3$ cusps, then $C$ is rectifiable.

**Proof.** Suppose that $s = 3$, but $C$ is not rectifiable. Because

$$2 \cdot \frac{5}{6} + \left( \frac{4}{2} + \frac{3}{2} \right) > \frac{5}{6} + 2 \cdot \left( \frac{2}{2} + \frac{3}{3} \right) > 4,$$

we contradict Lemma 3.12 if any cusp has $k_i \geq 4$, or if there are at least two cusps with $k_i \geq 2$. Therefore, we may assume without loss of generality that $k_1 = k_2 = 1$, and $k_3 \leq 3$. Our goal now is to find an upper bound on $d$.

For simplicity of notation, let us write $(m_i, n_i)$ for $(m_{i,1}, n_{i,1})$ when $k_i = 1$, and let us write $(p_j, q_j)$ for $(m_{3,j}, n_{3,j})$.

If $k_3 = 1$, then since

$$\frac{5}{2} - \frac{1}{24} - \frac{1}{25} + 2 \cdot \frac{5}{6} > 4,$$

we must have $m_i \leq 23$ for all $i$, so Fact 3.8 implies that

$$d < 24\varphi^2 < 63.$$

If $k_3 = 2$, then since

$$4 \geq \sum_{i=1}^{3} \gamma_i \geq \left( \frac{5}{2} - \frac{2}{n_1} - \frac{2}{m_1} \right) + \left( \frac{5}{2} - \frac{2}{n_2} - \frac{2}{m_2} \right) + \left( 2 + \frac{3}{2} - \frac{1}{\mu_1} - \frac{1}{\mu_1} - \frac{2}{\mu_2} \right)
\geq \frac{5}{6} + \frac{5}{6} + \left( \frac{2}{2} + \frac{3}{3} \right) = \frac{10}{3},$$

it is easy to see that we must have

$$\frac{1}{m_i} + \frac{1}{n_i} \geq \frac{1}{2}, \quad \frac{1}{\mu_1} + \frac{1}{\nu_1} + \frac{2}{\mu_2} \geq \frac{7}{6},$$

which implies

$$m_1, m_2 \leq 3, \quad \mu_1\mu_2 \leq \max\{11 \cdot 2, 3 \cdot 3, 2 \cdot 6\} = 22.$$

Fact 3.8 implies that

$$d < 23\varphi^2 - \frac{1}{\sqrt{5}} < 60.$$
If $k_3 = 3$, then since
\[
4 \geq \sum_{i=1}^{3} \gamma_i \geq \left( \frac{5}{2} - \frac{2}{n_1} - \frac{2}{m_1} \right) + \left( \frac{5}{2} - \frac{2}{n_2} - \frac{2}{m_2} \right) + \left( 3 + \frac{3}{2} - \frac{1}{\nu_1} - \frac{1}{\mu_1} - \frac{1}{\mu_2} - \frac{2}{\mu_3} \right)
\]
\[
\geq \frac{5}{6} + \frac{5}{6} + \left( \frac{3}{2} + \frac{2}{3} \right) = \frac{23}{6},
\]
it is easy to see that we must have
\[
(m_1, n_1) = (m_2, n_2) = (2, 3), \quad \mu_1 = \mu_3 = 2,
\]
since these are lower bounds for the variables in question, and increasing any one of them would force a contradiction, if we bear in mind that $\gcd(m_i, n_i) = 1$ and $\nu_1 > \mu_1$. Moreover, $\mu_2 \leq 3$, so
\[
M_{1,1} = M_{1,2} = 2 < M_{3,1} = \mu_1 \mu_2 \mu_3 \leq 12.
\]
Fact 3.8 implies that
\[
d < 13\varphi^2 - \frac{1}{\sqrt{5}} < 34.
\]
So, in all three cases, we see that $d \leq 62$. Given this bound, there are finally many combinations of Newton pairs satisfying the genus equation (2.2.1). A computer program can enumerate them all, and, with the help of Lemma 3.13, check whether $\text{vol}(K_V + D) \leq 3$ (in accordance with Fact 3.4) and $(K_V + D)^{+} \cdot L \geq 0$.

It turns out that only one combination of numerical invariants passes all these tests: $d = 13$, $(m_1, n_1) = (m_2, n_2) = (2, 3), \ k_3 = 2, (\mu_1, \nu_1) = (3, 4), \text{ and } (\mu_2, \nu_2) = (3, 20)$. However, such a curve cannot exist. Indeed, it would have semigroups
\[
W_1 = W_2 = \{0, 2, 3, 4, \ldots \}
\]
and
\[
W_3 = \{0, 9, 12, 18, 21, 24, 27, 30, 33, 36, 39, \ldots \},
\]
so that
\[
(R_1 \circ R_2 \circ R_3)(3 \cdot 13 + 1) = R_3(40) = 11 > \frac{(3 + 1)(3 + 2)}{2},
\]
which violates Fact 2.2 for $j = 3$.

\[\square\]

### 3.4 One cusp

Since there is only one cusp, instead of writing $(m_{1j}, n_{1j})$, we will simply write $(m_j, n_j)$ for the Newton pairs, and so forth.

The unicuspidal curves whose cusp has only one Newton pair were classified in [4] (see Theorem 2.3 above), and they have been verified to be rectifiable. For the sake of completeness, let us reproduce the quick verification:

**Lemma 3.17.** Let $C$ be a rational cuspidal curve of degree $d$ with one cusp of one Newton pair $(m, n)$. Then $C$ is rectifiable.
Proof. Formula (3.2.1) and Lemma 3.11 together imply that
\[ m + n - 3d + 1 = \tilde{C}^2 \geq 6 \]
if \( C \) fails to be rectifiable. But if we go through the list in Theorem 2.3, we find this is never the case. (As before, \( F_i \) denotes the \( i \)th Fibonacci number. We will use some identities involving Fibonacci numbers without proof.)

1. If \( d = F_{2\ell-1} \) and \( (m, n) = (F_{2\ell-3}, F_{2\ell+1}) \), then we have
\[ -\tilde{C}^2 = F_{2\ell-3} + F_{2\ell+1} - 3F_{2\ell-1} + 1 = 1 < 6. \]

2. If \( d = F_{2\ell-1}F_{2\ell+1} \) and \( (m, n) = (F_{2\ell-1}^2, F_{2\ell+1}^2) \), then
\[ -\tilde{C}^2 = F_{2\ell-1}^2 + F_{2\ell+1}^2 - 3F_{2\ell-1}F_{2\ell+1} + 1 = -1 + 1 = 0 < 6. \]

3. If \( d = n \) and \( (m, n) = (n-1, n) \), then
\[ -\tilde{C}^2 = n - 1 + n - 3n + 1 = -n < 6. \]

4. If \( d = 2m \) and \( (m, n) = (m, 4m - 1) \), then
\[ -\tilde{C}^2 = m + 4m - 1 - 6m + 1 = -m < 6. \]

5. If \( d = 8 \) and \( (m, n) = (3, 22) \), then
\[ -\tilde{C}^2 = 3 + 22 - 24 + 1 = 2 < 6. \]

6. If \( d = 16 \) and \( (m, n) = (6, 43) \), then
\[ -\tilde{C}^2 = 6 + 43 - 48 + 1 = 2 < 6. \]

So in each case \( d \) is indeed rectifiable. \( \Box \)

Let us show the same in the case of two Newton pairs:

**Lemma 3.18.** Let \( C \) be a rational cuspidal curve with one cusp of 2 Newton pairs. Then \( C \) is rectifiable.

**Proof.** We try to follow the same strategy as in Lemma 3.17: formula (3.2.1) and Lemma 3.11 together imply that
\[ M_1 + N_1 + N_2 - 3d + 1 = -\tilde{C}^2 \geq 6 \]
if \( C \) fails to be rectifiable. Let us run through each of the ten cases enumerated in Theorem 1.1. As before, \( F_i \) denotes the \( i \)th Fibonacci number, and we will use identities involving Fibonacci numbers without proof. We will play a little fast and loose with the ordering of \( m, n \); besides the fact that the multiplicity of the cusp in \( \min\{m, n\}m_2 \) rather than \( \max\{m, n\}m_2 \), the situation between \( m \) and \( n \) is completely symmetric. So for the duration of this proof, we do not require \( m < n \). This allows us to avoid writing two sets of formulas in cases 1 and 2 below.
1. If for some \( i, \ell \in \mathbb{Z} \) we have \( d = F_{2i-1}F_{2i+1}m \), and \( (m, n) = \ell(F_{2i-1}, F_{2i+1}) + (F_{2i-3}, F_{2i-1} + 2) \), and \( (m_2, n_2) = (F_{2i-1}, m) \), then
\[
-\tilde{C}^2 = 0 < 6.
\]

2. If for some \( i, \ell \in \mathbb{Z} \) we have \( d = F_{2i+1}m \), and \( (m, n) = \ell(F_{2i-1}, F_{2i+1}) + (F_{2i-3}, F_{2i-1} + 2) \), and \( (m_2, n_2) = (F_{2i-1}, \ell F_{2i-1} + F_{2i-5}) \), then
\[
-\tilde{C}^2 = 1 < 6.
\]

3. If \( d = 8n^2 + 4n + 1 \), \( (m, n) = (4n + 1, n) \), and \( (m_2, n_2) = (4n + 1, 2n + 1)^2 \), then
\[
-\tilde{C}^2 = (4n + 1)(5n + 1) + (2n + 1)^2 - 3(8n^2 + 4n + 1) + 1 = n.
\]

4. If \( d = nm_2 \), \( (m, n) = (n - 1, n) \), and \( (m_2, n_2) = (m_2, nm_2 - 1) \), then
\[
-\tilde{C}^2 = m_2(2n - 1) + nm_2 - 1 - 3nm_2 + 1 = -m_2 < 6.
\]

5. If \( d = n^2 + 1 \), \( (m, n) = (n - 1, n) \), and \( (m_2, n_2) = (n, (n + 1)^2) \), then
\[
-\tilde{C}^2 = n(2n - 1) + (n + 1)^2 - 3(n^2 + 1) + 1 = n - 1.
\]

6. If \( d = 20 \), \( (m, n) = (2, 3) \), and \( (m_2, n_2) = (6, 31) \), then
\[
-\tilde{C}^2 = 5 \cdot 6 + 31 - 3 \cdot 20 + 1 = 2 < 6.
\]

7. If \( d = 2mm_2 \), \( (m, n) = (m, 4m - 1) \), and \( (m_2, n_2) = (m_2, mm_2 - 1) \), then
\[
-\tilde{C}^2 = (5m - 1)m_2 + mm_2 - 1 - 3 \cdot 2mm_2 + 1 = -m_2 < 6.
\]

8. If \( d = 17 \), \( (m, n) = (2, 7) \), and \( (m_2, n_2) = (4, 17) \), then
\[
-\tilde{C}^2 = 4 \cdot 9 + 17 - 3 \cdot 17 + 1 = 3 < 6.
\]

9. If \( d = F_{4k+2} \), \( (m, n) = \left( \frac{F_{4k}}{3}, \frac{F_{4k+4}}{3} \right) \), and \( (m_2, n_2) = (3, 1) \), then
\[
-\tilde{C}^2 = 3(m + n) + 1 - 3(m + n) + 1 = 2 < 6.
\]

10. If \( d = 2F_{4k+2} \), \( (m, n) = \left( \frac{F_{4k}}{3}, \frac{F_{4k+4}}{3} \right) \), and \( (m_2, n_2) = (6, 1) \), then
\[
-\tilde{C}^2 = 6(m + 1) + 1 - 3 \cdot 2(m + n) + 1 = 2 < 6.
\]

So by Lemma 3.11 we are already done in eight of the cases. In the remaining two, let us use Corollary 3.14. Assume for the sake of contradiction that \( C \) is not rectifiable.
3. If \( d = 8n^2 + 4n + 1 \), \((m, n) = (4n + 1, n)\), and \((m_2, n_2) = (4n + 1, (2n + 1)^2)\), then note that

\[
\frac{m_1 - 1}{n_1} + \sum_{j=1}^{k-1} \frac{n_j - 1}{m_j} + \frac{n_k - \frac{2k+1}{2}}{m_k} + k + \frac{1 + \varepsilon}{2} - \frac{4 + \varepsilon}{2(-C^2)} - 6
\]

\[
\geq 4 + n - 1 + \frac{(2n + 1)^2 - 2}{4n + 1} + 2 + \frac{3 - 6}{2} - 6
\]

\[
= 4 + \frac{1}{4} - \frac{5}{4n + 1} + n + 3 - \frac{7}{4(4n + 1)} + 2 + \frac{3 - 3}{2} - 6
\]

\[
= n + \frac{5}{2} - \frac{3}{4n + 1} - \frac{3}{n}
\]

\[
\geq n + \frac{5}{2} - \frac{3}{25} - \frac{3}{6} = n + \frac{47}{25} > n = -\hat{C}^2,
\]


5. If \( d = n^2 + 1 \), \((m, n) = (n - 1, n)\), and \((m_2, n_2) = (n, (n + 1)^2)\), then note that

\[
\frac{m_1 - 1}{n_1} + \sum_{j=1}^{k-1} \frac{n_j - 1}{m_j} + \frac{n_k - \frac{2k+1}{2}}{m_k} + k + \frac{1 + \varepsilon}{2} - \frac{4 + \varepsilon}{2(-C^2)} - 6
\]

\[
\geq \frac{n - 2}{n} + 1 + \left(\frac{(n + 1)^2 - 1}{n}\right) + 2 + \frac{3 - 6}{2} - 6 \frac{2(n - 1)}{2(n - 1)} - 6
\]

\[
= 1 + 1 - \frac{2}{n} + n + 2 + 2 + \frac{3}{2} - \frac{3}{n - 1} - 6
\]

\[
= n + \frac{3}{2} - \frac{2}{n - 1} - \frac{3}{n - 1}
\]

\[
\geq n + \frac{3}{2} - \frac{2}{7} - \frac{3}{6} = n + \frac{5}{7} > n - 1 = -\hat{C}^2,
\]

again contradicting 3.14.

This completes the proof. \(\square\)

In the cases of \( k \geq 3 \) cusps, we are able to prove a partial result:

**Lemma 3.19.** Let \( C \) be a rational cuspidal curve with one cusp of \( k \geq 3 \) Newton pairs \((m_1, n_1), \ldots, (m_k, n_k)\). Then \(-\hat{C}^2 \leq m_2 + 2\), and if \( m_k \leq 4 \), then \( C \) is rectifiable.

**Proof.** The semigroup \( W' \) generated by the \( k \) integers

\[
\frac{w_1}{m_k} = \frac{M_1}{m_k} = m_1m_2 \cdots m_{k-1}, \quad \frac{w_2}{m_k} = \frac{N_1}{m_k} = n_1m_2 \cdots m_{k-1}, \ldots, \quad \frac{w_k}{m_k}
\]

includes all but

\[
\frac{1}{2} \left[ \left( \frac{M_1}{m_k} - 1 \right) \left( \frac{N_1}{m_k} - 1 \right) + \sum_{i=2}^{k-1} \left( \frac{M_i}{m_k} - 1 \right) \left( \frac{N_i}{m_k} - 1 \right) \right]
\]

non-negative integers. (Let us not confuse \( W' \) with the semigroup \( W \), which is generated by \( w_1, \ldots, w_{k+1} \).) For every positive integer \( j \leq d - 2 \), the interval \([0, jd]\) contains \( \lfloor jd/m_k \rfloor \) multiples of \( m_k \), so, in light of the above observation, Fact 2.2 says

\[
1 + \left\lfloor \frac{jd}{m_k} \right\rfloor - \frac{1}{2} \left[ \left( \frac{M_1}{m_k} - 1 \right) \left( \frac{N_1}{m_k} - 1 \right) + \sum_{i=2}^{k-1} \left( \frac{M_i}{m_k} - 1 \right) \left( \frac{N_i}{m_k} - 1 \right) \right] \leq \frac{(j+1)(j+2)}{2}.
\]
Now plug in for $j$ the integer closest to $d/m_k - \frac{3}{2}$, to find that
\[
\frac{1}{2} \left[ \left( \frac{M_1}{m_k} - 1 \right) \left( \frac{N_1}{m_k} - 1 \right) + \sum_{i=2}^{k-1} \left( \frac{M_i}{m_k} - 1 \right) \frac{N_i}{m_k} \right] \geq 1 + \left\lfloor \frac{jd}{m_k} \right\rfloor - \frac{(j+1)(j+2)}{2} \geq \frac{jd}{m_k} - \frac{(j+1)(j+2)}{2} = \frac{1}{2} \left( \frac{d}{m_k} - \frac{3}{2} \right)^2 - 1 - \frac{1}{2} \left( \frac{d}{m_k} + \frac{3}{2} \right)^2 \geq \frac{1}{2} \left( \frac{d}{m_k} - \frac{3}{2} \right)^2 - 1 - \frac{1}{8}.
\]

Multiply both sides by $2m_k^2$ to find
\[
(M_1 - m_k)(N_1 - m_k) + \sum_{i=2}^{k-1} (M_i - m_k)N_i > d^2 - 3m_kd.
\]

Subtract this from the genus equation, which says
\[
(M_1 - 1)(N_1 - 1) + \sum_{i=2}^{k} (M_i - 1)N_i = d^2 - 3d + 2,
\]
to find that
\[
(m_k - 1)(M_1 + N_1 + N_2 + \cdots + N_k - m_k - 1) < 3(m_k - 1)d + 2.
\]
Because both sides are integers, we have in fact
\[
(m_k - 1)(M_1 + N_1 + N_2 + \cdots + N_k - m_k - 1) \leq 3(m_k - 1)d + 1,
\]
and therefore, by equation (3.2.1), we have
\[
\bar{C}^2 = 3d - 1 - (M_1 + N_1 + N_2 + \cdots + N_k) \geq -m_k - 2 - \frac{1}{m_k - 1}.
\]

If $m_k \in \{2, 3\}$, then
\[
\bar{C}^2 \geq -2 - 2 - \frac{1}{1} = -3 - 2 - \frac{2}{2} = -5,
\]
so $C$ is rectifiable, by Lemma 3.11. If $m_k = 4$, then note that $\varepsilon \geq 3$ (see Table 3.1), but $\bar{C}^2 \geq -4 - 2 - \frac{1}{3} = -6\frac{1}{3}$, so again $C$ is rectifiable, by Lemma 3.11. \hfill \Box

### 3.5 Two cusps, each with only one Newton pair

**Lemma 3.20.** Let $C$ be a rational cuspidal curve with two cusps, each with one Newton pair. Then $C$ is rectifiable.

**Proof.** Suppose $C$ is such a curve, but $C$ is not rectifiable. In light of Fact 3.2, we may assume $\kappa = 2$. Recall that $d$ is the degree of $C$. To simplify the notation, let the Newton pairs be $(m, n) = (m_{1,1}, n_{1,1})$ and $(m', n') = (m_{2,1}, n_{2,1})$, with $m \geq m'$. Given any combination of concrete numbers for $d$, $(m, n)$ and $(m', n')$, we may try to prove that $C$ cannot exist, by using the following tools:
• The genus equation, \((d - 1)(d - 2) = (m - 1)(n - 1) + (m' - 1)(n' - 1)\).
• The BMY equality (Fact 3.4) with the aid of the volume formula in Lemma 3.13.
• Fact 2.2, for \(0 \leq j \leq d - 2\).

If that fails, we may try to prove that \(C\) must be rectifiable. By Lemma 3.9, it suffices to show that no divisor

\[ L \sim 2K_V + \tilde{C} - \sum_{i=1}^{s} \varepsilon_i E_{i,k_i} \]

can be effective. Here we have the following tools (assuming \(C\) is not rectifiable):

• \(L \cdot \tilde{C} \geq 0\), since \(\tilde{C}\) is not a component of \(L\).
• \(L \cdot (K_V + D)^+ \geq 0\), since \((K_V + D)^+\) is by definition nef.

All the above tests are mechanical. Recall, too, that \(d \geq 7\) by Fact 3.7. We wrote a computer program to test all the finitely many combinations of numerical invariants with \(7 \leq d \leq 250\). (Even the genus equation alone shows that the number of combinations is finite, though it is hardly efficient to enumerate all the combinations from that angle. In practice, Lemma 3.12 serves as a useful first line of defense, cutting down the search space to something manageable.) Among all those possibilities, only one combination of invariants is not automatically eliminated by the tools we listed above:

\[ d = 8, \quad (m, n) = (3, 16), \quad (m', n') = (2, 13). \]

We prove in Lemma 3.21 that no curve \(C\) with these invariants can exist.

Having checked all curves of degrees up to 250, we may assume that \(d > 250\) from now on. By Fact 3.8, we have

\[ m = \max\{m, m'\} > 95. \]

By Lemma 3.12, we have

\[ \gamma_1 + \gamma_2 = \left( \frac{5}{2} - \frac{2}{96} - \frac{2}{97} \right) + \left( \frac{5}{2} - \frac{2}{m'} + \frac{2}{n'} \right) \leq 4, \]

which means either \(m' = 2\) or \((m', n') \in \{(3, 4), (3, 5)\}\). But if \((m', n') = (3, 4)\), then we can calculate \(\varepsilon_2 = 3\), so Lemma 3.12 in fact implies

\[ \left( \frac{5}{2} - \frac{2}{96} - \frac{2}{97} \right) + \left( \frac{1}{2} + \frac{5}{2} \cdot \left( \frac{1}{m'} + \frac{1}{n'} \right) \right) \leq 4, \]

which is (just barely) false. So we are left with two cases: either \(m' = 2\), or \((m', n') = (3, 5)\).

Before diving into a morass of technical details, let us informally describe our strategy for dealing with the remaining cases. We have at our disposal the following formulas, from equation (2.2.1) and Lemmas 3.12 and 3.13:

\[ (m - 1)(n - 1) + (m' - 1)(n' - 1) = (d - 1)(d - 2) \]
\[ m + n - \frac{m}{n} - \frac{n}{m'} + m' + n' - \frac{m'}{n'} - \frac{n'}{m'} = 3d + \text{vol}(K_V + D) - 5 \quad (3.5.1) \]
\[ 2 \left( \frac{1}{m} + \frac{1}{n} + \frac{1}{m'} + \frac{1}{n'} \right) \geq 4 - \text{vol}(K_V + D). \]
Pretend for a moment that $m'$, $n'$, and $\text{vol}(K_V + D)$ are small, in comparison to $m$, $n$, and $d$, which are comparable in size. Then the first two equations above imply that

$$mn - m - n \approx d^3 - 3d$$
$$m + n \approx 3d,$$

which, taken together show that

$$\varphi^2 m \approx d \approx \frac{n}{\varphi^2},$$

and in particular

$$\frac{m}{n} + \frac{n}{m} \approx 7.$$  \hfill (3.5.2)

This observation is the starting point of our proof. Let

$$\ell := (m + n - 3d) + m' + \frac{n' + 3}{2};$$

then $\ell$ must be an integer (since either $m' = 2$, in which case $n'$ must be odd, or else $n' = 5$). Then we will try to estimate the difference between $\ell$ and $\frac{m}{n} + \frac{n}{m}$, and show that it is far from being an integer, which contradicts (3.5.2).

Now let us continue with the formal proof.

**Case 1: $m' = 2$.** Then from (3.5.1) we have

$$\ell = m + n - 3d + \frac{n' + 7}{2} = \frac{m}{n} + \frac{n}{m} + 2 + \text{vol}(K_V + D) - \frac{7}{2},$$

and

$$m + n + \frac{n'}{2} = 3d + \ell - \frac{7}{2}$$

$$(m - 3)(n - 3) = d^2 - 9d + (18 - 2\ell)$$

$$\frac{m - 2}{n} + \frac{n - 2}{m} \leq \ell + \frac{1}{2} \leq \frac{m}{n} + \frac{n}{m} + \frac{2}{n'},$$

First we claim that $\ell \leq 7$. Indeed, Fact 3.8 says that

$$d < \varphi^2(1 + m) - \frac{1}{\sqrt{5}},$$

so if $\ell \geq 8$, then

$$n - 3 \leq \frac{d^2 - 9d + 2}{m - 3} < 7m - 3$$

the last inequality involves some tedious calculation and the fact that $d > 250$. But then

$$\ell + \frac{1}{2} \leq \frac{m}{n} + \frac{n}{m} + \frac{2}{n'} \leq \frac{1}{7} + 7 + \frac{2}{3},$$

contradicting the assumption that $\ell \geq 8$.

From the fact that $d > 250$ and $\ell \leq 7$, we easily see that

$$\sqrt{d^2 - 9d + 18 - 2\ell} > d - 4.6.$$
If $\alpha \geq 2$, then we would have

$$\frac{\sqrt{d^2 - 9d + 18 - 2\ell}}{m - 3} > \frac{d - 4.6}{m - 3} > \frac{d - 3\alpha}{m - 3} \geq \alpha,$$

and consequently

$$\frac{n - 3}{m - 3} = \frac{d^2 - 9d + 18 - 2\ell}{(m - 3)^2} > \alpha^2,$$

so that

$$\frac{m - 3}{n - 3} + \frac{n - 3}{m - 3} > \alpha^2 + \alpha^{-2}.$$

Since

$$\left(\frac{m - 2}{n} + \frac{n - 2}{m}\right) \geq \frac{m - 3}{m} \cdot \left(\frac{m - 3}{n} + \frac{n - 3}{m - 3}\right),$$

and $m > 95$, we conclude that

$$\alpha \geq 2 \implies \frac{m - 2}{n} + \frac{n - 2}{m} > 0.96(\alpha^2 + \alpha^{-2}). \quad (3.5.3)$$

On the other hand, we always have

$$\frac{\sqrt{d^2 - 9d + 18 - 2\ell}}{n - 3} > \frac{d - 4.6}{n - 3} = \frac{d - 3/\beta}{n - 3} \geq \frac{4.6 - 3/\beta}{n - 3} \geq \frac{1}{\beta} \geq \frac{1}{d} \frac{4.6 - 3/\beta}{n - 3} \geq \frac{1}{\beta} \left(1 - \frac{4.6 - 3/\beta}{d}\right) \frac{1}{\beta} \geq \frac{1}{\beta} \frac{1}{\beta} \frac{1}{\beta},$$

and consequently

$$\frac{m}{n} > \frac{m - 3}{n - 3} = \frac{d^2 - 9d + 18 - 2\ell}{(n - 3)^2} > \left(1 - \frac{4.6}{d}\right) \frac{1}{\beta} \frac{1}{\beta},$$

so that, since $d > 250$,

$$\frac{m}{n} + \frac{n}{m} < \left(\frac{\beta}{0.9816}\right)^2 + \left(\frac{0.9816}{\beta}\right)^2 < 1.04 \cdot \beta^2 + 0.97 \cdot \beta^{-2}. \quad (3.5.4)$$

More precisely, we could note that

$$\frac{m}{n} = \frac{m - 3}{n - 3} + \frac{3(n - m)}{n(n - 3)} > \frac{m - 3}{n - 3} + \frac{3(\beta - 1/\alpha)}{\beta^2d} > \left(1 - \frac{9.2 - 3\beta + 1/\alpha}{d}\right) \frac{1}{\beta^2},$$

so that in fact

$$\frac{m}{n} + \frac{n}{m} < \left(1 - \frac{9.2 - 3\beta + 1/\alpha}{d}\right)^{-1} \beta^2 + \left(1 - \frac{9.2 - 3\beta + 1/\alpha}{d}\right) \frac{1}{\beta^2}. \quad (3.5.5)$$

From Fact 2.2, applied to $j \in \{1, 2, 5\}$, we find the following:

$$\min\{2m, n\}, \min\left\{\frac{5m}{2}, \frac{n}{2}\right\}, \min\left\{\frac{13m}{5}, \frac{2n}{5}\right\} \leq d.$$
• $\beta \leq 1$. In this case, inequality (3.5.4) says that
\[
\frac{m}{n} + \frac{n}{m} < 2.01.
\]
Since evidently
\[
\frac{m-2}{n} + \frac{n-2}{m} \geq 2 - 2 \cdot \left( \frac{1}{n} + \frac{1}{m} \right) > \frac{3}{2},
\]
we must have $\ell = 2$ and $n' = 3$. But then
\[
m + n = 3d - 3,
\]
which is patently absurd when $m < n \leq d$ and $d > 250$.

• $\beta \leq 2 \leq \alpha$. In this case, inequalities (3.5.3) and (3.5.4) imply that
\[
4.08 < \frac{m-2}{n} + \frac{n-2}{m} < \frac{m}{n} + \frac{n}{m} < 4.4025,
\]
so we must have $\ell = 4$ and $n' < 21$. But then
\[
m + n > 3d - 10,
\]
which is patently absurd when $m \leq d/2$, $n < 2d$ and $d > 250$.

• $\beta \leq \frac{2}{3} \leq \alpha$. In this case, inequality (3.5.3) says that
\[
6.1536 < \frac{m-2}{n} + \frac{n-2}{m},
\]
so $\ell \geq 6$. On the other hand, inequality (3.5.5) says that
\[
\frac{m}{n} + \frac{n}{m} < \left( 1 - \frac{9.2 - 3\beta + 1/\alpha}{d} \right)^{-1} \frac{1}{\beta^2} + \left( 1 - \frac{9.2 - 3\beta + 1/\alpha}{d} \right) \frac{1}{\beta^2}.
\]
Note that the term $\left( 1 - \frac{9.2 - 3\beta + 1/\alpha}{d} \right) \frac{1}{\beta^2}$ is decreasing in $\alpha$ and $d$, and increasing in $\beta$, so
\[
\left( 1 - \frac{9.2 - 3\beta + 1/\alpha}{d} \right) \frac{1}{\beta^2} \geq \left( 1 - \frac{9.2 - 3(5/2) + 2/5}{251} \right) \frac{1}{(5/2)^2} > 0.15866,
\]
and consequently
\[
\frac{m}{n} + \frac{n}{m} < 6.462.
\]
So in fact we must have $\ell = 6$ and $n' < 53$. But then
\[
m + n > 3d - 24,
\]
which is just barely absurd when $m \leq 2d/5$, $n \leq 5d/2$, and $d > 250$.

• $\frac{13}{5} \leq \alpha$. In this case, inequality 3.5.3 says
\[
\frac{m-2}{n} + \frac{n-2}{m} > 6.63,
\]
so $\ell \geq 7$. Now by Fact 2.2 for $j = 3$, we have $\min\{8m, m+n\} \leq 3d$. Let us consider two cases:
If \( n' = 3 \), then we have
\[
\begin{align*}
m + n & = 3d + 2 \\
(m - 3)(n - 3) & = d^2 - 9d + 4 \\
mn & = d^2 + 1.
\end{align*}
\]
so it's of course impossible to have \( 3d \geq m + n \). It must, consequently, be the case that \( 3d \geq 8m \), so \( m \leq \frac{3}{8}d \) and \( n \geq \frac{24}{8}d + 2 \). Then
\[d^2 + 1 = mn \leq \frac{3}{8}d \left( \frac{21}{8}d + 2 \right) = \frac{63}{64}d^2 + \frac{3}{4}d.\]
This is impossible for \( d > 250 \).

If \( n' \geq 5 \), then \( \frac{m}{n} + \frac{n}{m} \geq \frac{21}{10} \). Then note that
\[m + n - 1 \leq 3d,
\]
but
\[(m - 3)(n - 3) = d^2 - 9d + 4,
\]
so some arithmetic shows
\[2(m + n) + 17 \geq m^2 + n^2 - 7mn\]
Divide both sides by \( mn \):
\[\frac{2(m + n) + 17}{mn} \geq \frac{m}{n} + \frac{n}{m} - 7 \geq \frac{1}{10},\]
and
\[(m - 20)(n - 20) \leq 570,\]
which is evidently absurd when \( n > m > 95 \).

**Case 2:** \((m', n') = (3, 5)\). This case is easier:
\[
l := m + n + 7 - 3d = \frac{m}{n} + \frac{n}{m} + l^2 - 3 - \frac{7}{24},\]
and
\[
m + n = 3d + l - 7 \\
mn = d^2 + l - 14,
\]
which together imply
\[
\frac{m}{n} + \frac{n}{m} = 7 + 2(l - 7) \left( \frac{1}{m} + \frac{1}{n} \right) + \frac{-l^2 + 5l + 77}{mn},
\]
and
\[
\frac{m - 2}{n} + \frac{n - 2}{m} - \frac{1}{15} \leq l + \frac{7}{24} \leq \frac{m}{n} + \frac{n}{m}.
\]
• If \( l \leq 6 \), then
\[
7 + 2(l - 8) \left( \frac{1}{m} + \frac{1}{n} \right) + \frac{-l^2 + 5l + 77}{mn} \leq l + \frac{7}{24} + \frac{1}{15} = l + \frac{43}{120},
\]

or in other words
\[
\left( 7 - \frac{43}{120} - l \right) mn \leq (16 - 2l)(m + n) + (l^2 - 5l - 77).
\]

Now
\[
l^2 - 5l - 77 \leq 36 - 30 - 77 = -71,
\]

so in fact
\[
\left( 7 - \frac{43}{120} - l \right) mn < (16 - 2l)(m + n)
\]

and
\[
\frac{1}{m} + \frac{1}{n} > 7 - \frac{43}{120} = \frac{67}{77},
\]

which is absurd when \( n > m > 95 \).

• If \( l \geq 7 \), then
\[
7 + 2(l - 7) \left( \frac{1}{m} + \frac{1}{n} \right) + \frac{-l^2 + 5l + 77}{mn} \geq l + \frac{7}{24}.
\]

Now
\[
-l^2 + 5l + 77 \leq -49 + 35 + 77 = 63,
\]

so
\[
\frac{63}{mn} \geq (l - 7) \left( 1 - \frac{2}{m} - \frac{2}{n} \right) + \frac{7}{24},
\]

which is absurd when \( n > m > 95 \).

Lemma 3.21. There is no rational cuspidal curve \( C \subset \mathbb{P}^2 \) of degree 8 with two cusps, the first with one Newton pair \( (3, 16) \), and the second with one Newton pair \( (2, 13) \).

Proof. Of course, an assertion of this type is expressible in the first-order theory of algebraically closed fields of characteristic 0, so it must be decidable, given sufficient computing power. More practically, in the notation of Fact 2.2, note that
\[
(R_1 \circ R_2)(2d + 1) = R_1(15) + R_2(2) = 5 + 1 = 6 = \frac{(2 + 1)(2 + 2)}{2},
\]

since
\[
W_1 = \{0, 3, 6, 9, 12, \ldots, 15, 16, 18, 19, 21, 22, 24, 25, 26, 27, 28, 30, 31, 32, \ldots \},
W_2 = \{0, 2, 4, 6, 8, 10, 12, 13, \ldots \}.
\]

By dimension-counting, there exist divisors \( A', B \subset \mathbb{P}^2 \) of degree 2 with the following local intersection multiplicities:
\[
(A' \cdot C)_{p_1} \geq 12, \quad (A' \cdot C)_{p_2} \geq 2,
\]
\[
(B \cdot C)_{p_1} \geq 15, \quad (B \cdot C)_{p_2} \geq 0.
\]
Let $A$ be the unique line through $P_1$ with maximal $(A \cdot C)_{P_1}$; that is, $A$ is "tangent" to $C$ at $P_1$ in the sense that its proper transform continues to intersect the proper transform of $C$ even after blowing up $P_1$. We have $(A \cdot C)_{P_1} \geq 6$.

We first claim that $B$ must be a smooth conic. If $B$ is a sum of two lines, then since we are only interested in maximizing $(B \cdot C)_{P_1}$, we might as well take $B = 2A$, so that $(A - C)_{P_1} = 8$. But 8 does not belong to $W_1$, so in fact $(A - C)_{P_1} > 9$. This is absurd, since $\deg A = 1$ and $\deg C = 8$.

Thus, $B$ is a smooth conic. Now, since $(B \cdot C)_{P_1} > 15$, the proper transforms of $B$ meet the proper transforms of $C$ over $P_1$ even after 4 blow-ups. (The reader may wish to bear in mind the resolution graph for $P_1$, which we explained how to calculate in Example 2.1.) Any other smooth conic $B'$ could say the same for only up to 3 blow-ups, since $B'$ can at most osculate to $C$ to order 3; it follows that $(B' \cdot C)_{P_1} < 9$. Likewise, the line $A$ can be at most tangent to $B$, so

$$(A \cdot C)_{P_1} = 6.$$ 

Note, by the way, that $B$ cannot pass through $P_2$, for otherwise $(B \cdot C)_{P_2}$ would be at least 2, and that would contradict the fact that $B \cdot C = 16$.

Now we claim that $A' = 2A$. Indeed, $A'$ passes through $P_2$, so $A' \neq B$. If $A'$ were smooth, then as we noted above, $(A' \cdot C)_{P_1}$ could be at most 9. Therefore, $A'$ is a sum of two lines. Now $A$ is the only line with $(A \cdot C)_{P_1} \geq 6$; all other lines have intersection multiplicity at most 3 there. So in fact $A'$ must be $2A$. It follows that $A$ passes through $P_2$.

We have arrived at a relatively concrete geometrical picture. By performing a suitable projective-linear transformation, we may assume $P_1 = [0 : 0 : 1]$, $P_2 = [1 : 0 : 0]$, $A$ is the $x$-axis $(Y = 0)$, and $B$ is the parabola $y = x^2$ (that is, $X^2 = YZ$). Choose a parametrization $[f(t) : g(t) : h(t)] : \mathbb{P}^1 \to C$ mapping $0 \mapsto P_1$ and $\infty \mapsto P_2$. The fact that $C$ has a double point at $P_2$ and is tangent to $A$ at $P_1$, where it has a triple point, means that

$$f(t) = -t^3 + ct^4 + dt^5 + et^6 + bt^7 + at^8, \quad (a \neq 0)$$
$$g(t) = -t^6,$$
$$h(t) = _+ + _t^1 + _t^2 + _t^3 + _t^4 + _t^5 + _t^6,$$

where _ stands for coefficients we have not bothered to name. We may rescale $(f, g, h)$ and $t$ separately to ensure that the coefficient of $t^3$ in $f(t)$ and that of $t^6$ in $g(t)$ are both equal to 1. Since $(B \cdot C)_{P_1} = 15$,

$$f(t)^2 \equiv g(t)h(t) \pmod{t^9}$$

in the polynomial ring $\mathbb{C}[t]$, so

$$h(t) \equiv (1 + et + dt^2 + et^3 + bt^4 + at^5)^2 \pmod{t^9}.$$ 

Because of the cusp $P_2$, we know that $h$ has no $t^7$ or $t^8$ term, so

$$2da + 2cb = 2ca + b^2 = 0, \quad (3.5.6)$$

and

$$h(t) = 1 + 2et + (2d + e^2)t^2 + (2c + 2ed)t^3 + (2b + 2ec + d^2)t^4 + (2a + 2eb + 2dc)t^5 + (2ea + 2db + c^2)t^6.$$
Now let us look at this from the perspective of $t = \infty$ and $P_2$. Let $s := \frac{1}{t}$, so that $F(s) := s^8 f(t) = s^8 f(\frac{1}{t})$, and so on, so that

$$F(s) = a + bs + cs^2 + ds^3 + es^4 + s^5,$$

$$G(s) = s^2,$$

$$H(s) = (2ea + 2db + c^2)s^2 + (2a + 2eb + 2dc)s^3 + (2b + 2ec + d^2)s^4$$
$$+ (2c + 2ed)s^5 + (2d + e^2)s^6 + 2es^7 + s^8.$$

The cusp $P_2$ is an $A_{12}$ cusp. In particular, it is not an $A_2$ cusp, so the coefficient of $s^3$ in $H(s)$ must vanish:

$$2a + 2eb + 2dc = 0. \quad (3.5.7)$$

Since $a \neq 0$, the equations (3.5.6) and (3.5.7) together imply that $b \neq 0$. Therefore, we can solve for $c, d, e$ in terms of $a, b$. The important thing is to consider the power series expansions of $\frac{G(s)}{F(s)}$ and $\frac{H(s)}{F(s)}$, which are the local affine coordinates of points on $C$. A little calculation shows that

$$\frac{H(s)}{F(s)} - \frac{7b^5 - 8a^4 G(s)}{4a^2 b} \frac{G(s)}{F(s)} - 3ab \left( \frac{G(s)}{F(s)} \right)^2 = \frac{b^5(4a^4 + b^5)}{4a^6} \cdot s^5 + O(s^6).$$

Thus, the fact that $P_2$ is not an $A_6$ cusp implies that the coefficient of $s^5$ in the above expression must vanish. Since $b \neq 0$, this implies $4a^4 + b^5 = 0$. But then

$$\frac{H(s)}{F(s)} - \frac{7b^5 - 8a^4 G(s)}{4a^2 b} \frac{G(s)}{F(s)} - 3ab \left( \frac{G(s)}{F(s)} \right)^2 + \frac{9b}{2} \left( \frac{G(s)}{F(s)} \right)^2 = \frac{2}{b} \cdot s^7 + O(s^8),$$

so $P_2$ is in fact an $A_8$ cusp, not an $A_{12}$ cusp as we supposed. This contradiction shows that the curve $C$ cannot exist. \[\square\]
Chapter 4

Triangulating sequences

In this Chapter, we define a notion of triangulating sequences, and prove some combinatorial results about them. The motivation, as we discussed in section 2.3, is to prove some restrictions on unicuspidal rational curves.

We abandon the notation from previous chapters. Let \( \mathbb{N} \) be the additive semigroup of non-negative integers. Let \( \alpha_1, \ldots, \alpha_k \) \( (k \geq 1) \) be a finite sequence of positive real numbers such that \( \alpha_i \) is rational whenever \( i < k \). We may write, in lowest terms,

\[
\alpha_i = \frac{n_i}{m_i}, \quad (m_i, n_i \in \mathbb{N}, \quad \gcd(m_i, n_i) = 1)
\]

for \( i < k \). Define \( g_1, \ldots, g_{k+1} \) by the following formulas:

\[
\begin{align*}
g_1 &= \prod_{j=1}^{k-1} m_j, \\
g_2 &= \alpha_1 \prod_{j=1}^{k-1} m_j, \\
g_i+1 &= m_{i-1}g_i + \alpha_i \prod_{j=i}^{k-1} m_j, \quad (\text{for } 2 \leq i \leq k).
\end{align*}
\]

An empty product is taken to be 1.)

Note that \( g_1, \ldots, g_k \in \mathbb{N} \), and, for \( 1 \leq i \leq k \),

\[
\gcd(g_1, \ldots, g_i) = \prod_{j=i}^{k-1} m_j,
\]

so let \( W_i \) be the semigroup

\[
W_i = \prod_{j=i}^{k-1} m_j^{-1}(g_1, \ldots, g_i) \subset (\mathbb{N}, +).
\]

Note that \( W_1 = \mathbb{N} \), and for \( 2 \leq i \leq k \),

\[
\max(\mathbb{N}\setminus W_i) = \left( \prod_{j=i}^{k-1} m_j^{-1} \right) \left( m_{i-1}g_i - \prod_{j=1}^{k-1} m_j - \sum_{i'=1}^{i-1} n_{i'} \prod_{j=i'+1}^{k-1} m_j \right),
\]

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We are mostly concerned with $W_k$, so write $W$ for $W_k$ when no confusion can arise.

In the case that $\alpha_k$ is also rational, we may write $\alpha_k = \frac{n_k}{m_k}$ in lowest terms, and define

$$W_{k+1} := m_k(g_1, \ldots, g_{k+1}) \subseteq (\mathbb{N}, +);$$

it, too, contains all but finitely many non-negative integers.

It makes no difference to the semigroups $W, \ldots, W_k$ if we replace $a_i$ by $a_j$, that is, if we swap $m_i$ and $n_i$. So we will sometimes prefer to deal with the case $n_i \geq m_i$, and at other times exploit the extra flexibility of swapping $m_i$ and $n_i$ to simplify our formulas.

**Definition 4.1.** Given a sequence $\alpha_1, \ldots, \alpha_k$ as above, a real number $\delta > 0$, and $j \in \mathbb{N}$, we say the $j$th triangulating set is

$$S_j(\delta; \alpha_1, \ldots, \alpha_k) := \{(w, z) \in W_k \times \mathbb{N} | w + zg_{k+1} \leq j\delta\}.$$

The $j$th triangulating condition, or triangulating condition $j$, for $(\delta; \alpha_1, \ldots, \alpha_k)$ is the statement that

$$\#S_j(\delta; \alpha_1, \ldots, \alpha_k) = \binom{j+2}{2}.$$

We say $(\delta; \alpha_1, \ldots, \alpha_k)$ is a strongly triangulating sequence of length $k$ if it satisfies the $j$th triangulating condition for all $j \in \mathbb{N}$. We say $(\delta; \alpha_1, \ldots, \alpha_k)$ is weakly triangulating if $\alpha_k = \frac{n_k}{m_k} \in \mathbb{Q}$ and $(\delta; \alpha_1, \ldots, \alpha_k)$ satisfies the $j$th triangulating condition for all $j \in \mathbb{N}$ such that both of the following conditions hold:

(i) $j < m_k\delta$, or $(j - 1)(j - 2) \leq \max(\mathbb{N}\setminus W_{k+1})$, or $m_k\delta(j - 3) < \max(\mathbb{N}\setminus W_{k+1}) - 1$.

(ii) $\delta j < g_{k+1}$, or $\#\{(w, z) \in W_k \times \mathbb{N} | w + zg_{k+1} < m_k g_{k+1}\} \geq \binom{j+2}{2}$.

We will often say a sequence $\alpha_1, \ldots, \alpha_k$ is strongly or weakly triangulating if there exists $\delta > 0$ making $(\delta; \alpha_1, \ldots, \alpha_k)$ so.

Note that, if $(\delta; \alpha_1, \ldots, \alpha_k)$ is weakly triangulating and $j$ is small enough to satisfy (ii) above, then in fact

$$\#\{(w, z) \in W_k \times \mathbb{N} | w + zg_{k+1} \leq j\delta\} = \#(W_{k+1} \cap [0, jm_k\delta]).$$

**Lemma 4.2.** (1) If $(\delta; \alpha_1, \ldots, \alpha_k)$ is weakly triangulating, then so is $(\frac{\delta}{m_1 \cdots m_{k-1}}; \alpha_1, \ldots, \alpha_i)$ for any $1 \leq i < k$. In fact, $(\frac{\delta}{m_1 \cdots m_{k-1}}; \alpha_1, \ldots, \alpha_i)$ satisfies the $j$th triangulating condition for all $j$ satisfying the equivalent of (ii) in the definition of "weakly triangulating," that is, $\delta j < g_{i+1}$ or $\#\{(w, z) \in m_i \cdots m_{k-1}W_i \times \mathbb{N} | w + zg_{i+1} < m_i g_{i+1}\} \geq \binom{j+2}{2}$.

(2) If $(\delta; \alpha_1, \ldots, \alpha_k)$ is weakly triangulating but, for some $j \in \mathbb{N}$, we have $\#S_j(\delta; \alpha_1, \ldots, \alpha_k) > \binom{j+2}{2}$, then $j$ must fail condition (ii) in the definition of "weakly triangulating."

**Proof.** Statement (1) holds because, for any $j$ small enough to satisfy (ii) relative to the sequence $(\frac{\delta}{m_1 \cdots m_{k-1}}; \alpha_1, \ldots, \alpha_i)$ is also small enough to satisfy (i) and (ii) relative to the sequence $(\delta; \alpha_1, \ldots, \alpha_k)$, and moreover the $j$th triangulating conditions for $(\frac{\delta}{m_1 \cdots m_{k-1}}; \alpha_1, \ldots, \alpha_i)$ and for $(\delta; \alpha_1, \ldots, \alpha_k)$ are equivalent.
As for (2), for such a \( j \), by definition, \( j \) must fail at least one of the conditions, (i) or (ii). We have to show that in fact \( j \) must fail condition (ii). Indeed, if \( j \) were the smallest counterexample, satisfying (ii) but failing (i), then we would have

\[
\#S_{j-1}(\delta; \alpha_1, \ldots, \alpha_k) < \binom{j+1}{2}, \quad \#S_j(\delta; \alpha_1, \ldots, \alpha_k) > \binom{j+2}{2},
\]

and so

\[
\# \{(w, z) \in W_k \times \mathbb{N} \mid (j - 1)m_k \delta < m_k w + m_k g_{k+1} z \leq j m_k \delta \} \geq j + 2.
\]

The coefficients \( m_k \) and \( m_k g_{k+1} \) are both integers, so \( m_k w + m_k g_{k+1} z \in \mathbb{N} \). The interval \( ((j - 1)m_k \delta, j m_k \delta] \) contains at least \( j + 2 \) integers, so we must have

\[
m_k \delta \geq j + 2,
\]

which means that (i) in fact holds, contrary to our hypothesis. \( \square \)

The triangulating conditions are readily seen to boil down to some counts of lattice points. So, in the limit as \( j \to \infty \), we are more or less computing areas of triangles. It follows that, if \( (\delta; \alpha_1, \ldots, \alpha_k) \) is strongly triangulating, then we must have

\[
\delta^2 = g_{k+1}.
\]

We are sometimes interested in weakly triangulating sequences \( (\delta; \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k}) \) for which \( m_k \delta \in \mathbb{N} \). The following lemma shows that we need only consider the minimum possible \( \delta \) for such questions.

**Lemma 4.3.** Suppose \( (\delta; \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k}) \) is a weakly triangulating sequence with \( \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k} \), all in lowest terms, \( n_1 \geq 2 \), and all \( m_i \geq 2 \). Let \( \delta_{\text{min}} \) be the minimal real number for which \( (\delta'; \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k}) \) is weakly triangulating. Then \( \delta_{\text{min}} \in \mathbb{Q} \), and if \( m_k \delta \in \mathbb{N} \), then \( \delta = \delta_{\text{min}} \).

**Proof.** Every triangulating condition for \( (\delta'; \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k}) \) imposes the constraint that \( \delta' \) lie inside some half-open interval depending on \( \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k} \). The number of triangulating conditions that \( (\delta'; \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k}) \) is required to satisfy (in order to be a weakly triangulating sequence) is a weakly increasing function of \( \delta' \) that jumps up only at certain rational values of \( \delta' \). This explains why \( \delta_{\text{min}} \in \mathbb{Q} \). Suppose that, according to the definition of “weakly triangulating,” \( (\delta_{\text{min}}; \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k}) \) is required to satisfy the \( j \)th triangulating condition for all \( j \leq j_{\text{max}} \), so in particular \( m_k \delta_{\text{min}}(j_{\text{max}} - 2) \geq \max(\mathbb{N}\setminus W_{k+1}) - 1 \) or \( m_k \delta_{\text{min}} j_{\text{max}} \geq m_k g_{k+1} \). In either case, \( m_k \delta_{\text{min}} j_{\text{max}} > \max(\mathbb{N}\setminus W_k) \).

For all \( j \leq j_{\text{max}} \), we must have \( S_{j}(\delta; \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k}) = S_{j}(\delta_{\text{min}}; \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k}) \), which is to say there cannot exist an element of \( W_{k+1} \) in the interval \( (m_k \delta_{\text{min}} j, m_k \delta_{\text{max}} j) \). Every integer exceeding \( j_{\text{max}} m_k \delta_{\text{min}} \) in fact belongs to \( W_{k+1} \), so there cannot be any integers at all in the interval \( (m_k \delta_{\text{min}} j_{\text{max}}, m_k \delta_{\text{max}} j_{\text{max}}) \). But if \( \delta > \delta_{\text{min}} \), then by assumption \( j_{\text{max}} m_k \delta \) would itself be such an integer. This cannot happen, so we must have \( \delta = \delta_{\text{min}} \). \( \square \)

Our motivation for studying weakly triangulating sequences comes from Fact 2.2 restricting the topological types of cusps of cuspidal rational curves in \( \mathbb{P}^2 \). For example, suppose we have such a curve, of degree \( d \), with only one cusp with Newton pairs \( (m_1, n_1), \ldots, (m_k, n_k) \). Then Fact 2.2 (the main result in [1]) says that \( (\frac{d}{m_k}, \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k}) \) satisfies the \( j \)th triangulating condition for \( 0 \leq j \leq d - 1 \). In sum, the \( j \)th triangulating condition is satisfied for all non-negative integers \( j < d \). From this it is not hard to see that in fact \( (\frac{d}{m_k}, \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k}) \) satisfies our definition of "weakly triangulating."

Our goal is to prove necessary conditions for a sequence to be weakly triangulating. In fact, many of the conditions we derive will also happen to be sufficient for this combinatorial question,
but as our primary purpose is to constrain the possible topological types of singularities on planar cuspidal rational curves, the sufficiency of conditions for a sequence to be weakly triangulating is of less interest to us.

As it turns out, we will often find it useful to relax the condition that the last element of a triangulating sequence be a real number, and tweak it by an infinitesimal $\varepsilon$:

**Definition 4.4.** Define the $j$th triangulating set to be

$$S_j(\delta; \alpha_1, \ldots, \alpha_k + \varepsilon) := \{(w, z) \in W_k \times \mathbb{N} | w + zg_{k+1} < j\delta \text{ or } (w, z\gamma_{k+1}) = (j\delta, 0)\},$$

$$S_j(\delta; \alpha_1, \ldots, \alpha_k - \varepsilon) := \{(w, z) \in W_k \times \mathbb{N} | w + zg_{k+1} < j\delta \text{ or } (w, z\gamma_{k+1}) = (0, j\delta)\}.$$  

We say $(\delta; \alpha_1, \ldots, \alpha_k \pm \varepsilon)$ is strongly pseudo-triangulating if it satisfies the $j$th triangulating condition:

$$\#S_j(\delta; \alpha_1, \ldots, \alpha_k \pm \varepsilon) = \binom{j + 2}{2}$$

for all $j \in \mathbb{N}$. (One could analogously define a notion of "weakly pseudo-triangulating," but we will not be using such a notion.)

### 4.1 Strongly triangulating and pseudo-triangulating sequences of length 1

In this section we give some examples of sequences $(\delta; \alpha)$ that are strongly triangulating. In each case, the proof is the same: we check $j$th triangulating condition for $0 \leq j \leq 2$ by hand, and then induct by 3's.

Let $\varphi := \frac{1 + \sqrt{5}}{2}$ be the golden ratio.

**Proposition 4.5.** The sequence $(\delta; \alpha) = (\varphi^2; \varphi^4)$ is strongly triangulating.

**Proof.** The $j$th triangulating set $S_j(\delta; \alpha)$ consists of pairs $(x, y) \in \mathbb{N}^2$ with $x + \varphi^4 y \leq \varphi^2 j$, or in other words

$$\varphi^{-2} x + \varphi^2 y \leq j.$$  

We will show $\#S_j(\delta; \alpha) = \binom{j + 2}{2}$ by induction on $j$. Three base cases, for $0 \leq j \leq 2$, can easily be checked.

Suppose we are given an arbitrary $j \geq 3$, and we know the claim holds for $j - 3$. The crucial point is that

$$\varphi^{-2} + \varphi^2 = 3.$$  

Consider any $(x, y) \in \mathbb{N}^2$ such that $\varphi^{-2} x + \varphi^2 y \leq j$, or, what is equivalent,

$$\varphi^{-2}(x - 1) + \varphi^2(y - 1) \leq j - 3. \quad (4.1.1)$$

The induction hypothesis says that (4.1.1) has $\frac{(j - 2)(j - 1)}{2}$ solutions with $x, y \geq 1$. Since $\varphi^2$ is irrational, there are $\lceil \varphi^{-2} j \rceil$ solutions with $x = 0$ and $\lceil \varphi^2 j \rceil$ solutions with $y = 0$. Since $(0, 0)$ is also a solution, we just have to show that

$$\frac{(j - 2)(j - 1)}{2} + \lceil \varphi^{-2} j \rceil + \lceil \varphi^2 j \rceil - 1 = \frac{(j + 1)(j + 2)}{2},$$

or in other words

$$\lceil \varphi^{-2} j \rceil + \lceil \varphi^2 j \rceil = 3j + 1.$$  

This is true because $\varphi^{-2} j$ and $\varphi^2 j$ sum to exactly $3j$, but neither one is an integer. \qed

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Let $F_n := \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$ be the Fibonacci numbers, which are integers. For example,

$F_{-1} = 1$, $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, ....

**Proposition 4.6.** For every $k \in \mathbb{N}$, the sequence $(6; a) = (2; (k+1)^2)$ is strongly triangulating. (Note: for $k = 0$, this gives $(1; 1)$.)

**Proof.** For notational simplicity, let $a := F_{2k-3}$, $b := F_{2k-1}$, $c := F_{2k+1}$, and $D := bc$. One can easily verify that

$$ac - b^2 = 1, \quad b^2 + c^2 = 3D - 1, \quad \text{and} \quad a + c = 3b.$$ 

The proof proceeds along the same lines as that of Proposition 4.5, but is slightly more complicated.

We will show by induction that, for all $j \in \mathbb{N}$, there are exactly \( \frac{(j+1)(j+2)}{2} \) pairs \((x, y) \in \mathbb{N}^2 \) with

$$b^2 x + c^2 y \leq Dj.$$  \hspace{1cm} (4.1.2)

Three base cases, for $0 \leq j \leq 2$, can easily be checked.

Suppose we made (4.1.2) into an equation:

$$b^2 x + c^2 y = bcj.$$  \hspace{1cm} (4.1.3)

Since $b$ and $c$ are relatively prime, any putative solution must have $c \mid x$ and $b \mid y$, so if we write $x = cx'$ and $y = by'$, we have

$$bx' + cy' = j$$

for $(x', y') \in \mathbb{N}^2$. Since $ac - b^2 = 1$, the solutions to this equation have the form

$$x' = cz - bj, \quad y' = aj - bz$$

for $z \in \mathbb{Z}$. In order for both $x'$ and $y'$ to be non-negative, we need

$$\frac{bj}{c} \leq z \leq \frac{aj}{b}.$$ 

Hence, the number of solutions to (4.1.3) is

$$\left\lfloor \frac{aj}{b} \right\rfloor - \left\lfloor \frac{bj}{c} \right\rfloor + 1.$$ 

Suppose we are now given an arbitrary $j \geq 3$, and we know the triangulating condition holds for $j - 3$. Consider any $(x, y) \in \mathbb{N}^2$ such that $b^2 x + c^2 y \leq Dj$. Either equality (4.1.3) holds, or else $b^2 x + c^2 y \leq Dj - 1$. In the latter case, since $b^2 + c^2 = 3D - 1$, we have

$$b^2(x - 1) + c^2(y - 1) \leq D(j - 3).$$  \hspace{1cm} (4.1.4)

So the number of solutions to (4.1.2) is equal to the sum of the number of solutions to (4.1.3) and the number of solutions to (4.1.4). The induction hypothesis says that (4.1.4) has \( \frac{(j-2)(j-1)}{2} \) solutions with $x, y \geq 1$. And (4.1.4) has

$$\left\lfloor \frac{Dj - 1}{b^2} \right\rfloor + \left\lfloor \frac{Dj - 1}{c^2} \right\rfloor + 1$$

solutions with $xy = 0$, if we remember to count $(0, 0)$ once.
Since $\frac{D_j - 1}{b^2} = \frac{bcj - 1}{b^2}$ and $\frac{D_j - 1}{c^2} = \frac{bcj - 1}{c^2}$ are never integers,

$$\left[ \frac{D_j - 1}{b^2} \right] + \left[ \frac{D_j - 1}{c^2} \right] + 1 = \left[ \frac{D_j - 1}{b^2} \right] + \left[ \frac{D_j - 1}{c^2} \right] - 1 = \left[ \frac{D_j b}{b^2} \right] + \left[ \frac{D_j c}{c^2} \right] - 1 = \left[ \frac{c j}{b} \right] + \left[ \frac{b j}{c} \right] - 1.$$

It now suffices to show that

$$\frac{(j + 1)(j + 2)}{2} = \left( \left\lfloor \frac{a_j}{b} \right\rfloor - \left\lfloor \frac{b j}{c} \right\rfloor + 1 \right) + \frac{(j - 1)(j - 2)}{2} + \left( \left\lfloor \frac{c j}{b} \right\rfloor + \left\lfloor \frac{b j}{c} \right\rfloor - 1 \right),$$

or in other words

$$3j = \left\lfloor \frac{a_j}{b} \right\rfloor + \left\lfloor \frac{c j}{b} \right\rfloor.$$

This is true because in fact

$$3j = \frac{a_j}{b} + \frac{c j}{b},$$

since $a + c = 3b$.

**Proposition 4.7.** The sequence $(\delta; \alpha + \epsilon) = (\frac{8}{3}; \frac{64}{3} + \epsilon)$ is strongly pseudo-triangulating.

**Proof.** The $j$th triangulating condition says that the number of pairs $(x, y) \in \mathbb{N}^2$ with

$$9x + 64y < 24j \quad \text{or} \quad (x, y) = \left( \frac{8j}{3}, 0 \right)$$

is $\binom{j+2}{2}$. This is easily verified for $0 \leq j \leq 2$. Now suppose the triangulating condition for $j - 3$ holds; we wish to prove the $j$th triangulating condition holds as well. To that end, let $N$ denote the number of pairs $(x, y) \in \mathbb{N}^2$ with

$$9x + 64y < 24j, \quad x < 8. \quad (4.1.5)$$

Then

$$N = \sum_{i=0}^{7} \left\lfloor \frac{24j - 9i}{64} \right\rfloor = \sum_{i=0}^{7} \left\lfloor \frac{3j - i}{8} - \frac{i}{64} \right\rfloor.$$

For $0 \leq i \leq 7$, we have $0 \leq i/64 < 1/8$, and the fractional part of $(3j - i)/8$ is a multiple of $1/8$, so indeed

$$\left\lfloor \frac{3j - i}{8} - \frac{i}{64} \right\rfloor = \left\lfloor \frac{3j - i}{8} \right\rfloor.$$

Now as $i$ runs from 0 to 7, the fractional part of $-(3j - i)/8$ runs from $0/8$ to $7/8$ in some order, with an average value of $7/16$, so

$$N = \sum_{i=0}^{7} \left( \frac{3j - i}{8} + \frac{7}{16} \right) = 3j + \sum_{i=0}^{7} \frac{7}{8} - \frac{i}{8} = 3j.$$

It suffices to note that $(x, y)$ satisfies the $j$th triangulating condition if and only if $(x, y)$ satisfies $(4.1.5)$ or $x \geq 8$ and $(x - 8, y)$ satisfies the $(j - 3)$rd triangulating condition. So we are done, by induction. \qed

**Proposition 4.8.** The sequence $(\delta; \alpha - \epsilon) = (3; 9 - \epsilon)$ is strongly pseudo-triangulating.
Proof. The $j$th triangulating condition says that the number of pairs $(x, y) \in \mathbb{N}^2$ with

$$x + 9y < 3j \quad \text{or} \quad (x, y) = \left(0, \frac{j}{3}\right)$$

is $(\frac{j+2}{2})$. This is easily verified for $0 \leq j \leq 2$. Now suppose the triangulating condition for $j - 3$ holds; we wish to prove the $j$th triangulating condition holds as well. The number of pairs $(x, 0)$ satisfying the $j$th triangulating condition is $3j$, and, if $y \geq 1$, then $(x, y)$ satisfies the $j$th triangulating condition if and only if $(x, y - 1)$ satisfies the $(j - 3)$rd triangulating condition. So we are done, by induction. \qed

4.2 Simplest fractions, fair approximations, and continued fractions

Lemma 4.9. Fix real numbers $\alpha, \beta$ such that $0 < \alpha < \beta$ and $\beta - \alpha \leq 1$. Then there is a unique fraction $n/d$ of minimal denominator in the interval $(\alpha, \beta)$.

Proof. If not, say the fractions $n/d$ and $N/d$ both fit the criteria, but $n < N$. Then $d > 1$, since

$$\frac{1}{d} \leq \frac{N}{d} - \frac{n}{d} < \beta - \alpha \leq 1.$$

Let $x := \lfloor \frac{n}{d} \rfloor$. Then $xd \leq n \leq xd + (d - 1)$, so

$$\alpha < \frac{n}{d} \leq \frac{n - x}{d - 1} \leq \frac{n + 1}{d} \leq \frac{N}{d} < \beta.$$

In particular,

$$\frac{n - x}{d - 1} \in (\alpha, \beta),$$

contradicting the minimality of $d$. \qed

Lemma 4.10. Fix real numbers $\alpha, \beta$ such that $0 < \alpha < \beta$ and

$$\alpha^{-1} - \beta^{-1}, \beta - \alpha \leq 1.$$

Let $n/D$ be a fraction of minimal numerator in the interval $(\alpha, \beta)$, and let $N/d$ be a fraction of minimal denominator in $(\alpha, \beta)$. (Here we require $n, D, N, d \in \mathbb{N}$.) Then we claim in fact $n = N$ and $d = D$.

Proof. By minimality, we have $n \leq N$ and $d \leq D$, and so

$$\alpha < \frac{n}{D} \leq \frac{n}{d} \leq \frac{N}{d} < \beta.$$

By the previous lemma, we have $n = N$. Taking reciprocals, the previous lemma also shows that $d = D$. \qed

Definition. Fix distinct real numbers $\alpha, \beta$ of the same sign such that $\alpha < \beta$ and

$$\alpha^{-1} - \beta^{-1}, \beta - \alpha \leq 1.$$

Then the simplest fraction in the interval $(\alpha, \beta)$ is the unique fraction $p/q$ in that interval whose numerator and denominator have minimal absolute values; such a fraction exists by the above
lemma. We similarly define the simplest fractions in half-open and closed intervals the same way; the only caveat is that, in the case of a closed interval \([\alpha, \beta]\), we require both \(\alpha^{-1} - \beta^{-1}\) and \(\beta - \alpha\) to be strictly less than 1, so we can avoid ties for the lowest numerator or denominator.

We say a fraction \(\frac{n}{d}\) is a fair lower approximation to a real number \(a\) if \(\frac{n}{d} < a\) and \(\frac{n}{d}\) is itself the simplest fraction in the half-open interval \([\frac{n}{d}, a)\). We define fair upper approximations similarly. Note that, in particular, we require \(\frac{n}{d} - a\) and \(\frac{n}{d} - a^{-1}\) to be strictly less than 1. For example, \(\frac{1}{2}\) is a fair approximation to \(\frac{2}{3}\), and vice versa, although in general “fair approximation” is not a symmetric notion.

Every real number \(a\) has an increasing sequence of fair lower approximations converging to \(a\) and a decreasing sequence of fair upper approximations converging to \(a\).

One way of computing simplest fractions and fair approximations involves continued fractions. Let us follow the standard notation:

\[
[a_0, a_1, a_2, \ldots] := a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}.
\]

We require \(a_i > 0\) for \(i > 0\). Recall the following standard fact:

**Fact 4.11.** Fix any real number \(\alpha = [a_0, a_1, \ldots] > 2\), and define the following two sequences recursively:

\[
p_{-1} = 0, \quad p_0 = 1, \quad p_{i+1} = a_i p_i + p_{i-1}, \quad q_{-1} = 1, \quad q_0 = 0, \quad q_{i+1} = a_i q_i + q_{i-1}.
\]

Then \(p_i q_{i+1} - p_{i+1} q_i = (-1)^i\) for all \(i\), and \(p_i/q_i = [a_0, \ldots, a_{i-1}]\) for \(i \geq 1\). In particular,

\[
p_1 < p_2 < p_3 < \cdots, \quad q_1 < q_2 < q_3 < \cdots,
\]

and \(p_i\) and \(q_i\) are relatively prime for all \(i\).

**Lemma 4.12.** If \(\alpha = [a_0, a_1, a_2, \ldots] > 1\) is irrational, then its sequence of fair lower approximations is

\[
a_0 < \underbrace{[a_0, a_1, 1]}_{a_2 \text{ terms}} < \underbrace{[a_0, a_1, 2]}_{a_3 \text{ terms}} < \cdots < \underbrace{[a_0, a_1, a_2, 1]}_{a_4 \text{ terms}} < \cdots < \underbrace{[a_0, a_1, a_2, a_3, 1]}_{a_5 \text{ terms}} < \cdots \quad (*)
\]

and its sequence of fair upper approximations is

\[
\underbrace{[a_0, 1]}_{a_1 \text{ terms}} > \underbrace{[a_0, 2]}_{a_2 \text{ terms}} > \cdots > \underbrace{[a_0, a_1]}_{a_3 \text{ terms}} > \underbrace{[a_0, a_1, a_2, 1]}_{a_4 \text{ terms}} > \cdots > \underbrace{[a_0, a_1, a_2, a_3]}_{a_5 \text{ terms}} > \cdots \quad (\dagger)
\]

**Proof.** One can easily verify that the sequences of inequalities in (*) and (\dagger) are true, and their terms converge to \(\alpha\). If we can show that every fair approximation to \(\alpha\) appears in one of those sequences, then it follows that every term is in fact a fair approximation to \(\alpha\).

Suppose that \(\beta \in \mathbb{Q}\) is a fair lower approximation to \(\alpha\). Then \(\beta \geq a_0\). If \(\beta \in \mathbb{Z}\), then evidently \(\beta = a_0\), so let’s ignore that case and assume \(\beta \notin \mathbb{Z}\). Every rational number has two continued-fraction expansions, so we may choose a continued-fraction expansion of \(\beta\) with an odd number of terms:

\[
\beta = [b_0, b_1, \ldots, b_{2k}].
\]

Now if \(b_i = a_i\) for all \(i \leq 2k\), then \(\beta\) appears in (*), as desired. So assume there is a smallest index \(i\) such that \(b_i \neq a_i\). Let’s consider two cases, based on the parity of \(i\):

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• If $i$ is even, then $b_i < a_i$, since $\beta < \alpha$. If $i = 2k$, then $\beta$ appears in (*), as desired; note here that the case $i = 0$ is a bit special, but it creates no problems. If $i < 2k$, then since $[b_{i+1}, \ldots, b_{2k}] > 1$ and $b_i + 1 \leq a_i < [a_i, a_{i+1}, \ldots]$, we have

$$\beta < [b_0, \ldots, b_i, 1] = [b_0, \ldots, b_{i-1}, b_i + 1] < \alpha.$$ 

Since moreover $\beta$ has a numerator greater than that of $[b_0, \ldots, b_i, 1]$, we conclude that $\beta$ cannot be a fair approximation to $\alpha$.

• If $i$ is odd, then $b_i > a_i$, again because $\beta < \alpha$. Then because $[b_i, \ldots, b_{2k}] > b_i > [a_i, a_{i+1}, \ldots]$, we have

$$\beta < [b_0, \ldots, b_i] < \alpha.$$ 

Since moreover $\beta$ has a denominator greater than that of $[b_0, b_i, \ldots, b_i]$, we conclude that $\beta$ cannot be a fair approximation to $\alpha$.

The proof for fair upper approximations is similar.

**Lemma 4.13.** The fair approximations to a rational number $\alpha \geq 1$ may be found by pretending has a continued fraction expansion of the form

$$\alpha = [a_0, a_1, \ldots, a_k, 1, \infty] \text{ or } [a_0, a_1, \ldots, a_k + 1, \infty];$$

the expansion with an even number of finite terms produces the fair lower approximations, while the expansion with an odd number of finite terms produces the fair upper approximations.

For example, the fair lower approximations to $\alpha = \frac{7}{3} = [2, 3, \infty]$ are

$$2 < [2, 3, 1] < [2, 3, 2] < [2, 3, 3] < \cdots < [2, 3, k] < \cdots,$$

and the fair upper approximations to $\alpha = \frac{7}{3} = [2, 2, 1, \infty]$ are

$$[2, 1] > [2, 2] > [2, 2, 1, 1] > [2, 2, 1, 2] > [2, 2, 1, 3] > \cdots > [2, 2, 1, k] > \cdots.$$

We omit the proof is of Lemma 4.13, which is similar to the one for 4.12, only slightly more tedious.

**Corollary 4.14.** Let $\beta_1 = \frac{p_1}{q_1}$ and $\beta_2 = \frac{p_2}{q_2}$ be two consecutive fair approximations to a number $\alpha > 0$, with $\beta_2$ lying between $\beta_1$ and $\alpha$ (i.e., either $\beta_1 < \beta_2 < \alpha$ or $\alpha < \beta_2 < \beta_1$). Then $|p_1 q_2 - p_2 q_1| = 1$, and the simplest fraction strictly between $\beta_1$ and $\beta_2$ is $\frac{p_1 + p_2}{q_1 + q_2}$.

**Proof.** Write $\alpha = [a_0, a_1 \ldots]$ and $\beta_1 = [b_0, \ldots, b_i]$ according to Lemma 4.12 or 4.13. If $b_i < a_i$, then

$$\beta_2 = [b_0, \ldots, b_{i-1}, b_i + 1] = [b_0, \ldots, b_i, 1].$$

If $b_i = a_i$, then

$$\beta_2 = [b_0, \ldots, b_i, a_{i+1}, 1] = [b_0, \ldots, b_i, a_{i+1} + 1].$$

In either case, Fact 4.11 shows that $|p_1 q_2 - p_2 q_1| = 1$. By Lemma 4.13, the simplest fraction between $\beta_1$ and $\beta_2$ is either $[b_0, \ldots, b_i, 1, 1]$, if $b_i < a_i$, or $[b_0, \ldots, b_i, a_{i+1} + 1, 1]$, if $b_i = a_i$. In both cases, the fraction in question equals $\frac{p_1 + p_2}{q_1 + q_2}$ by Fact 4.11.

The following lemma relates simplest fractions to the triangulating conditions.
Lemma 4.15. Let \((\sqrt{\alpha}; \alpha)\) be strongly triangulating with \(\alpha \geq 1\), and let \(\beta \geq 1\) be a real number with \(0 < |\alpha - \beta| \leq 1\). Let \(q/p\) be the simplest fraction in the interval \((\alpha, \beta)\) or \((\beta, \alpha)\), depending on whether \(\alpha\) or \(\beta\) is lesser. Let \(\delta > 0\) and \(j \in \mathbb{N}\).

1. Suppose that \((\delta; \beta)\) satisfies the \(j\)th triangulating condition. Then

\[
x + \alpha y \leq j \sqrt{\alpha} \iff x + \beta y \leq j \delta \quad \text{whenever } (x, y) \in \mathbb{N}^2 \text{ and } x < q \text{ and } y < p.
\]

(4.2.1)

2. Conversely, if

\[
j < \min \left\{ \frac{q}{\sqrt{\alpha}}, \frac{q}{\delta}, \frac{p \beta}{\delta} \right\}
\]

and (4.2.1) holds, then \((\delta; \beta)\) satisfies the \(j\)th triangulating condition.

Proof. (1) Assume for the sake of contradiction that \((x, y)\) is a counterexample, with \(x < q\) and \(y < p\). Assume also that

\[
\frac{x + \alpha y}{\sqrt{\alpha}} \leq j < \frac{x + \beta y}{\delta}.
\]

(The other case, with inequalities going the other way, is similar.) The triangulating condition implies there must then exist some other \((x', y') \in \mathbb{N}^2\) such that

\[
\frac{x' + \beta y'}{\delta} \leq j < \frac{x' + \alpha y'}{\sqrt{\alpha}}.
\]

This means that

\[
\frac{x' + \beta y'}{\delta} < \frac{x + \beta y}{\delta} \quad \text{and} \quad \frac{x + \alpha y}{\sqrt{\alpha}} < \frac{x' + \alpha y'}{\sqrt{\alpha}},
\]

so

\[
\alpha(y' - y) < x' - x < \beta(y - y').
\]

(4.2.2)

It follows that \(y - y' \neq 0\) and \(\frac{x' - x}{y' - y}\) is strictly between \(\alpha\) and \(\beta\). Because \(q/p\) is the simplest fraction between \(\alpha\) and \(\beta\), we have \(|x' - x| \geq q\) and \(|y - y'| \geq p\). But this is impossible when \(x < q\) and \(y < p\), since \(x' - x\) and \(y - y'\) have the same sign, in light of (4.2.2).

(2) is obvious. \(\square\)

4.3 Perturbations of strongly triangulating sequences

The following are two theorems that allow us to construct new triangulating sequences out of old ones.

Proposition 4.16. Let \(m_1, \ldots, m_k, n_1, \ldots, n_k\) be positive integers, with \(\gcd(m_i, n_i) = 1\) for all \(i\), and let \(\delta\) be a rational number such that \(m_k \delta \in \mathbb{N}\). Suppose that \(m', n' \in \mathbb{N}\) and \(n_k m' - m_k n' = 1\). Then \(A := \left(\delta; \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k}\right)\) is strongly triangulating if and only if

\[
A' := \left(m' \delta; \frac{n_1}{m_1}, \ldots, \frac{n_{k-1}}{m_{k-1}}, \frac{n'}{m'}, \frac{m'}{m_k}\right)
\]

is strongly triangulating. (In particular, if these conditions hold, then \(\left(\delta; \frac{n_1}{m_1}, \ldots, \frac{n_{k-1}}{m_{k-1}}\right)\) is weakly triangulating.)
Proof. As a matter of notation, when we write $g_i$ or $W_i$ in this proof, we always refer to the $A$ sequence. Whenever $m_0$ appears below, we take it to equal 0. Note that
\[ m_{k-1}g_k + \frac{n_k}{m_k} = g_{k+1} = \delta^2, \]
so
\[ m'm_{k-1}g_k + n' = m'\delta^2 + \frac{n_km' - m_kn'}{m_k} = m'\delta^2 - \frac{1}{m_k}. \]

Let
\[ S := \{(X, Y, Z) \in W_k \times \mathbb{N} \times \mathbb{N} \mid Y < m'\}. \]
The $j$th triangulating condition for $A$ deals with the number of triples $(X, Y, Z) \in S$ such that
\[ X + \delta^2(Y + m'Z) \leq \delta_j, \]
(This comes from the $j$th triangulating condition, where we have — seemingly arbitrarily — broken up the integer $Y + m'Z$ into its quotient $Z$ and remainder $Y$ upon division by $m'$. If we multiply through by $m_k$, we get an inequality
\[ m_kX + m_k\delta^2(Y + m'Z) \leq m_k\delta_j \]
in which all the coefficients, including $m_k\delta^2$ and $m_k\delta$, are integers.

The $j$th triangulating condition for $A'$, meanwhile, deals with the number of triples $(X, Y, Z) \in S$ such that
\[ m'X + \left(m'\delta^2 - \frac{1}{m_k}\right)Y + m'\delta^2Z = m'X + (m'm_{k-1}g_k + n')Y + \left(m'^{2}m_{k-1}g_k + m'n' + \frac{m'}{m_k}\right)Z \leq m'\delta_j. \]
(Here the stipulation that $Y < m'$ is necessary to prevent the expression $m'X + (m'm_{k-1}g_k + n')Y$ from double-counting elements of the relevant semigroup generated by $m'W$ and $m'm_{k-1}g_k + n'$.) We may multiply through by $m_k/m'$ to rewrite this inequality as
\[ m_kX + m_k\delta^2(Y + m'Z) - \frac{Y}{m'} \leq m_k\delta_j, \]
and here all the coefficients, except \( \frac{1}{m'} \), are integers.

For all $(X, Y, Z) \in S$, we have $0 \leq \frac{Y}{m'} < 1$, so, since $m_k\delta^2$, $m_k\delta \in \mathbb{N}$,
\[ m_kX + m_k\delta^2(Y + m'Z) \leq m_k\delta_j \iff m_kX + m_k\delta^2(Y + m'Z) - \frac{Y}{m'} \leq m_k\delta_j, \]
as desired. \( \square \)

**Proposition 4.17.** Let $m_1, \ldots, m_k, n_1, \ldots, n_k$ be positive integers, with $\gcd(m_i, n_i) = 1$ for all $i$, and let $\delta$ be a rational number such that $m_k\delta \in \mathbb{N}$. Then the following are true:

(a) Suppose that $m', n' \in \mathbb{N}$ and $m_km' - m_kn' = 1$. If $A^- = \left(\delta; \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k} - \varepsilon\right)$ is strongly pseudo-triangulating, then
\[ B^- = \left(\frac{m_{k-1}g_k + n'/m'}{m_{k-1}g_k + n'/m_k}; \delta; \frac{n_1}{m_1}, \ldots, \frac{n_{k-1}}{m_{k-1}}, \frac{n'}{m'}\right) \]
is weakly triangulating.
(b) Suppose that $m', n' \in \mathbb{N}$ and $n_km' - m_kn' = -1$ instead. If $A^+ = \left( \delta; \frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k} + \varepsilon \right)$ is strongly triangulating, then

$$B^+ = \left( \delta; \frac{n_1}{m_1}, \ldots, \frac{n_{k-1}}{m_{k-1}}, \frac{n'}{m'} \right)$$

is weakly triangulating.

**Proof.** Imitate the proof of Proposition 4.16. Let

$$T := \{(X, Y) \in W_k \times \mathbb{N} \mid X \leq m'm_{k-1}g_k + n', \quad Y \leq m'\}.$$

For all $j$ in the range we need to consider, the $j$th triangulating set for $A^\pm$ consists of pairs $(X, Y) \in T$ with

$$m_kX + (m_{k-1}m_kg_k + n_k)Y \leq m_k\delta j,$$

subject to the additional condition that, if equality holds, then $X = 0$ (for $A^-$) or $Y = 0$ (for $A^+$).

With some arithmetic, we can see that the $j$th triangulating set for $B^-$ consists of pairs $(X, Y) \in T$ with

$$\left( \frac{1}{m'm_{k-1}g_k + n'} \right)X + (m_km_{k-1}g_k + n_k)Y \leq m_k\delta j.$$

The $j$th triangulating set for $B^+$ consists of pairs $(X, Y) \in T$ with

$$m_kX + \left( m_km_{k-1}g_k + n_k + \frac{1}{m'} \right)Y \leq m_k\delta j.$$

It is not difficult to see that the $j$th triangulating sets for $A^\pm$ and $B^\pm$ are the same. \qed

Now we prove a result restricting the set of weakly triangulating sequences that are too close to a strongly triangulating one.

**Lemma 4.18.** Let $\alpha_i = \frac{n_i}{m_i}$ (for $1 \leq i \leq k - 1$) and $\alpha' = \frac{n'}{m'}$ be positive fractions in lowest terms. Let $\alpha$ be either a positive real $\alpha_k$, or an infinitesimal perturbation thereof, that is, $\alpha_k \pm \varepsilon$. Let $\frac{a}{p}$ be the simplest fraction in the "open interval" between $\alpha$ and $\alpha'$, where this "open interval" includes $\alpha_k$ if $\alpha$ is an infinitesimal perturbation of $\alpha_k$ away from $\alpha'$. Let $\delta, \delta' > 0$, $A = (\delta; \alpha_1, \ldots, \alpha_{k-1}, \alpha)$, and $A' = (\delta'; \alpha_1, \ldots, \alpha_{k-1}, \alpha')$. Suppose $j \in \mathbb{N}$ is small enough such that

$$(m_{k-1}gkp + q, 0), \quad (0, p) \notin S_j(A),$$

and both $A$ and $A'$ both satisfy the $j$th triangulating condition. Then $S_j(A) = S_j(A')$.

**Proof.** Suppose not; then there must exist $(w, z) \in S_j(A) \setminus S_j(A')$ and $(w', z') \in S_j(A') \setminus S_j(A)$. But then it is a simple matter to see that $w - w'$ and $z - z'$ are both non-zero and have opposite signs, and the ratio $\left| \frac{w - w'}{z - z'} \right|$ must lie strictly between $m_k - 1g_k + \alpha$ and $m_k - 1g_k + \alpha'$. But $m_k - 1g_k + \frac{a}{p}$ is the simplest fraction in the open interval between $m_k - 1g_k + \alpha$ and $m_k - 1g_k + \alpha'$, so we must have $|w - w'| \geq m_k - 1g_kp + q$ and $|z - z'| \geq p$. Thus, if $w > w'$, then $w \geq m_k - 1g_kp + q$, and if $z > z'$, then $z \geq p$. In either case, one of $(m_k - 1g_kp + q, 0)$ and $(0, p)$ must belong to $S_j(A)$, contrary to our hypotheses. \qed
Proposition 4.19. Suppose $\delta = \frac{\alpha}{b}$ and $\alpha_i = \frac{n_i}{m_i}$, for $1 \leq i \leq k$, are positive fractions in lowest terms, and $A = (\delta; \alpha_1, \ldots, \alpha_k)$ is strongly triangulating. Suppose, too, that

$$\frac{q_i}{p_i} < \alpha' < \frac{q_{i+1}}{p_{i+1}}$$

for some fair lower approximation $\frac{q_i}{p_i}$ to $\alpha_k$ and some fair upper approximation $\frac{q_{i+1}}{p_{i+1}}$ to $\alpha_k$, such that

$$p_{i+1} m_{i+1} - q_{i+1} > \max(\mathbb{Z}\setminus W_k) + c + 2$$

and $p_{i+1} > b$.

Suppose that $\alpha' \neq \alpha_k$, but $A' = (\delta'; \alpha_1, \ldots, \alpha_{k-1}, \alpha')$ is weakly triangulating with $S_j(A) \subseteq S_j(A')$ for all $j \in \mathbb{N}$. Then, we claim,

i. We must have $\delta' \geq \delta$, and $S_j(A) = S_j(A')$ for all $j$ such that

$$j < \min \left\{ \frac{cm'}{b}, \frac{b(m'm_{k-1}g_k + n')}{c} \right\}.$$

ii. Moreover, $\alpha'$ must be a fair approximation to $\alpha_k$.

Proof. i. To prove that $\delta' \geq \delta$, note that, for all sufficiently large positive integers $i$, we have $ic \in W_k$ and, consequently,

$$(ic, 0) \in S_{ib}(A) \subseteq S_{ib}(A').$$

This implies that

$$\delta' \geq \frac{ic}{ib} = \frac{c}{b} = \delta.$$

Then for

$$j < \min \left\{ \frac{cm'}{b}, \frac{b(m'm_{k-1}g_k + n')}{c} \right\},$$

we have

$$j < m' \delta \leq m' \delta'.$$

Moreover, the fact that $\delta^2 = g_{k+1} = m_{k-1}g_k + \frac{n_k}{m_k}$ implies that $m_k = b^2$ and $b^2 g_{k+1} = c^2$. For any $j$ satisfying the given inequality in the lemma statement, and for any $(w, z) \in S_j(A)$, we have

$$b^2 w + c^2 z = b^2 (w + g_k z) < bc j < \min \left\{ c^2 m', b^2 (m'm_{k-1}g_k + n') \right\},$$

so

$$m'w + (m'm_{k-1}g_k + n')z \leq \max \left\{ \frac{m' m_{k-1}g_k + n'}{b^2}, \frac{m' m_{k-1}g_k + n'}{c^2} \right\} \cdot (b^2 w + c^2 z) < m'(m'm_{k-1}g_k + n'),$$

and

$$\binom{j + 2}{2} = \#S_j(A) \leq \# \{(w, z) \in W_k \times \mathbb{N} \mid m'w + z(m'm_{k-1}g_k + n') < m_k(m'm_{k-1}g_k + n') \}.$$

Thus, $A'$ is required to satisfy the $j$th triangulating condition, so

$$\#S_j(A') = \binom{j + 2}{2} = \#S_j(A).$$

Since $S_j(A) \subseteq S_j(A')$, this implies $S_j(A) = S_j(A')$. 55
ii. Suppose for the sake of contradiction that \( n'/m' \) is not a fair approximation to \( n_k/m_k \). Let \( \frac{q}{p} \) be the simplest fraction strictly between them. Let us consider two cases, depending on which of \( \alpha' \) or \( \alpha_k \) is larger.

Case 1: If \( \alpha' < \alpha_k \), then write

\[
\frac{q_0}{p_0} \leq \frac{q'}{p'} < \frac{n'}{m'} < \frac{q}{p} < \alpha_k,
\]

where \( \frac{q'}{p'} \) and \( \frac{q}{p} \) are consecutive fair lower approximations to \( \alpha_k \); then by Corollary 4.14,

\[
m' \geq p' + p \geq p_0 + p \quad \text{and} \quad n' \geq q' + q \geq q_0 + q,
\]

so

\[
(m'mk_{-1}g_k + n') - (pmk_{-1}g_k + q + \max(Z \setminus W_k)) = (m' - p)mk_{-1}g_k + (n' - q) - \max(Z \setminus W_k) \\
\geq p_0 mk_{-1}g_k + q_0 - \max(Z \setminus W_k) \geq c + 2.
\]

Therefore, we may choose \( x' \in \mathbb{N} \) such that \( pmk_{-1}g_k + q + \max(Z \setminus W) < cx' < m'mk_{-1}g_k + n' \).

Then, by (i), we know that \( A' \) is required to satisfy the \( j \)th triangulating condition, when

\[
j := bx' < \frac{b(m'mk_{-1}g_k + n')}{c} < \frac{cm'}{b}.
\]

Note that

\[
(cx' - (pmk_{-1}g_k + q) > \max(Z \setminus W_k),
\]

so we can be assured that

\[
(cx' - (pmk_{-1}g_k + q), cx' \in W_k.
\]

Thus, \((cx', 0) \in S_j(A)\), so we must have \((cx', 0) \in S_j(A')\) also. Then a little calculation shows

\[
(cx' - (pmk_{-1}g_k + q), p) \in S_j(A') \setminus S_j(A),
\]

so \( A' \) fails the \( j \)th triangulating condition, a contradiction.

Case 2: If, on the other hand, \( \alpha' > \alpha_k \), then write

\[
\frac{q_0}{p_0} > \frac{q'}{p'} > \frac{n'}{m'} > \frac{q}{p} > \alpha_k,
\]

where \( \frac{q'}{p'} \) and \( \frac{q}{p} \) are consecutive fair upper approximations to \( \alpha_k \); then by Corollary 4.14,

\[
m' \geq p' + p \geq p_0 + p \geq b + p.
\]

Therefore, we may choose \( x' \in \mathbb{N} \) such that \( p \leq by' < m' \). Then, by (i), we know \( A' \) is required to satisfy the \( j \)th triangulating condition, when

\[
j = cy' < \frac{cm'}{b} < \frac{b(m'mk_{-1}g_k + n')}{c}.
\]

Now \((0, by') \in S_j(A)\), so we must have \((0, by') \in S_j(A')\) also. Note that

\[
pmk_{-1}g_k + q \geq mk_{-1}g_k > \max(Z \setminus W_k),
\]

so \( pmk_{-1}g_k + q \in W_k \). Then a little calculation shows

\[
(pm_{k-1}g_k + q, by' - p) \in S_j(A') \setminus S_j(A),
\]

so \( A' \) fails the \( j \)th triangulating condition, a contradiction. □
Proposition 4.20. Let \( \delta = \frac{\alpha}{\beta} > 1 \) and \( \alpha_i = \frac{m_i}{m} \), for \( 1 \leq i \leq k \), be positive fractions in lowest terms, and suppose that \( A^+ = (\delta; \alpha_1, \ldots, \alpha_k + \varepsilon) \) is strongly pseudo-triangulating. Let \( \alpha' = \frac{\alpha'_p}{m} \) be a rational number strictly within 1 of \( \alpha_k \) such that \( A' = (\delta'; \alpha_1, \ldots, \alpha_{k-1}, \alpha') \) is weakly triangulating, and let \( \frac{a}{p} \) be the simplest fraction in \([\alpha', \alpha_k]\) if \( \alpha' < \alpha_k \), or in \((\alpha_k, \alpha']\) if \( \alpha' > \alpha_k \).

Then, we claim,

i. \( A' \) must satisfy the \( j \)th triangulating condition for all \( j \) such that

\[
(j - 1)(j - 2) \leq (m' - 1)(m'm_{k-1}g_k + n') + m' \max(\mathbb{N}\backslash W_k)
\]

and

\[
j < \min \left\{ \frac{cm'}{b}, \frac{b(m'm_{k-1}g_k + n')}{c} \right\}.
\]

If in fact

\[
j < \min \left\{ \frac{cp}{b}, \frac{b(m_{k-1}g_k p + q)}{c} \right\},
\]

then \( S_j(A') = S_j(A^+) \).

ii. If \( p > b \), then \( S_j(A') = S_j(A^+) \) for all \( j \leq c, \delta' \geq \delta \), and \( \alpha' > \alpha_k \).

iii. If \( \alpha_k < \alpha' < \frac{a_k}{p_h} \), for some fair upper approximation \( \frac{a_k}{p_h} \) to \( \alpha_k \) such that

\[
p_h > 2b + \frac{b^2(c+1)}{c^2},
\]

then \( \alpha' \) must be a fair upper approximation to \( \alpha_k \).

Proof. As in the proof of Proposition 4.19, the fact that \( \delta^2 = g_{k+1} = m_k - g_k + \frac{n_k}{m_k} \) implies that \( m_k = b^2 \) and \( b^2 g_k + 1 = c^2 \).

i. The semigroup \( W' \) generated by \( m'W_k \cup \{m'm_{k-1}g_k + n'\} \) is such that

\[
\max(\mathbb{N}\backslash W') = (m' - 1)(m'm_{k-1}g_k + n') + m' \max(\mathbb{N}\backslash W_k).
\]

For any \( j < \min \left\{ \frac{cm'}{b}, \frac{b(m'm_{k-1}g_k + n')}{c} \right\} \) and any \((w, z) \in S_j(A^+)\), we have, just as in Proposition 4.19,

\[
b^2 w + c^2 z = b^2(w + g_k z) \leq bcj < \min \left\{ c^2 m', b^2(m'm_{k-1}g_k + n') \right\},
\]

so

\[
m' w + (m'm_{k-1}g_k + n') z \leq \max \left\{ \frac{m'}{b^2}, \frac{m'm_{k-1}g_k + n'}{c^2} \right\} \cdot (b^2 w + c^2 z) < m'(m'm_{k-1}g_k + n'),
\]

and

\[
\left( \frac{j + 2}{2} \right) = \# S_j(A^+) \leq \# \{(w, z) \in W \times \mathbb{N} \mid m' w + z(m'm_{k-1}g_k + n') < m_k(m'm_{k-1}g_k + n') \}.
\]

Thus, \( A' \) is required to satisfy the \( j \)th triangulating condition if \( j \) satisfies the first two inequalities listed in (i). If moreover \( j < \min \left\{ \frac{cp}{b}, \frac{b(m_{k-1}g_k p + q)}{c} \right\} \), then by Lemma 4.18, we must have \( S_j(A^+) = S_j(A') \).
ii. Note that \( m' \geq p \geq b + 1 \) and
\[
m_{k-1}g_k + \frac{q+1}{p} \geq m_{k-1}g_k + \alpha_k = \frac{c^2}{b^2},
\]
so
\[
pm_{k-1}g_k + q \geq \frac{c^2p}{b^2} - 1.
\]
To show that \( S_j(A') = S_j(A^+) \) for all \( j \leq c \), it suffices to note that
\[
(m' - 1)(m'm_{k-1}g_k + n') + m' \max(N \setminus W_k) \geq (p - 1)(pm_{k-1}g_k + q) - m'
\]
\[
\geq (p - 1) \left( \frac{c^2p}{b^2} - 1 \right) - p \geq b \left( \frac{c^2(b+1)}{b^2} - 1 \right) - b - 1 = c^2 + \frac{c^2}{b} - 2b - 1
\]
\[
\geq c^2 - 3c + 2 \geq (j - 1)(j - 2)
\]
and
\[
\frac{cp}{b} > \frac{b(m_{k-1}g_kp + q)}{c} > \frac{b}{c} \left( \frac{c^2}{b^2}p - 1 \right) > \frac{b}{c} \left( \frac{c^2}{b^2}(b+1) - 1 \right) = c + \frac{c}{b} - \frac{b}{c} > j,
\]
and then use part (i).

Let \( x = \left[ \frac{c}{b^2} \right] \); note that \( bx \leq c \). Thus, \( S_{bx}(A') = S_{bx}(A^+) \), and in particular, as
\[
cx \geq \frac{c^2}{b^2} > \max(N \setminus W_k),
\]
we have
\[
(cx, 0) \in S_{bx}(A^+) = S_{bx}(A'),
\]
which shows that \( \delta' \geq \frac{c}{bx} = \delta \). Likewise, we have \( S_c(A') = S_c(A^+) \), and in particular
\[
(0, b) \notin S_c(A^+) = S_c(A').
\]
Thus,
\[
\frac{cx}{bx} \leq \delta' < \frac{b \cdot (m_{k-1}g_k + \alpha')}{c},
\]
and in particular
\[
\alpha' > \frac{c^2}{b^2} - m_{k-1}g_k = \alpha_k,
\]
as claimed.

iii. Suppose for the sake of contradiction that \( \alpha' \) is not a fair upper approximation to \( \alpha_k \). Then write
\[
\alpha_k < \frac{q}{p} < \alpha' = \frac{n'}{m'} < \frac{q'}{p'}
\]
for consecutive fair upper approximations \( \frac{q}{p}, \frac{q}{p} \) to \( \alpha_k \), with \( p' \geq p_{hi} > 2b + \frac{b^2(c+1)}{c^2} \). Then \( m' \geq p + p' \), so
\[
\frac{c(p + p')}{b} = (p + p')\delta \leq m'\delta = \frac{cm'}{b} < \frac{b(m'm_{k-1}g_k + n')}{c},
\]
and recall from (ii) that \( \delta \leq \delta' \). Thus, \( A' \) must satisfy the \( j \)th triangulating condition for all
\[
j < \frac{c(p + p')}{b}.
\]
Let $x := \left\lfloor \frac{mk-1gkp+q}{c} \right\rfloor - 1$, and note that
\[
3 \leq \left\lfloor \frac{3c}{b} \right\rfloor - 1 \leq \left\lfloor \frac{cp}{b^2} \right\rfloor - 1 \leq \left\lfloor \frac{1}{c} \left( \frac{c^2}{b^2} p \right) \right\rfloor - 1 \leq x \leq \left\lfloor \frac{1}{c} \left( \frac{c^2}{b^2} p + 1 \right) \right\rfloor = \left\lfloor \frac{cp}{b^2} + \frac{1}{c} \right\rfloor.
\]
First,
\[
3c > \frac{3}{4}c(x+1) > \frac{3}{4} (mk-1gkp+q) > \frac{3pc^2}{4b^2} > \frac{c^2}{b^2} > \text{max}(N\backslash W),
\]
so $cx \in W$. Second,
\[
bx + c < b \cdot \left( \frac{cp}{b^2} + \frac{1}{c} \right) + c = \frac{c}{b} \left( p + \frac{b^2}{c} + b \right) < \frac{c(p+2b)}{b} < \frac{c(p+p')}{b},
\]
so $A'$ must satisfy the jth triangulating condition for $j = bx + c$. Now $(cx, b)$ just barely fails to belong to $S_{bx+c}(A^+)$; we claim that $(cx, b) \notin S_{bx+c}(A')$ also. If not, there would have to exist some $(w, z) \in S_{bx+c}(A^+) \backslash S_{bx+c}(A')$. Since $\alpha' > \alpha_k$, we must have $w < cx$ and $z > b$, and
\[
mx-1gk + \alpha_k < \frac{cx-w}{z-b} < mx-1gk + \alpha',
\]
so
\[
mx > cx - w > mx-1gkp + q,
\]
which is evidently false, by the way we defined $x$. So indeed $(cx, b) \notin S_{bx+c}(A')$.

Now let
\[
y := \left\lfloor \frac{mx-1gk(p+1)+q}{c} \right\rfloor.
\]
First,
\[
cy - (mx-1gkp + q) \geq mx-1gk > \text{max}(N\backslash W),
\]
so $cy - (mx-1gkp + q) \in W$. Second,
\[
by < b \left( \frac{mx-1gk(p+1)+q}{c} + 1 \right) < b \left( \frac{c^2}{b^2} p + 1 \right) + b = \frac{pc}{b} + \frac{b}{c} + b,
\]
so
\[
by + 2c < \frac{c}{b} \left( p + \frac{b^2}{c} + \frac{b^2}{c} + 2b \right) < \frac{c(p+p')}{b} < \frac{cm'}{b} < \frac{b(m'mx-1gk + n')}{c},
\]
so by (i), $A'$ must satisfy the jth triangulating condition for $j = by + 2c$. Note that $(cy, 2b)$ just barely fails to belong to $S_{by+2c}(A^+)$. We claim that $(cy, 2b) \in S_{by+2c}(A')$, however. Otherwise, since
\[
mx-1gk + \alpha_k < \frac{mx-1gkp + q}{p} < mx-1gk + \alpha',
\]
we would have
\[
(cy - (mx-1gkp + q), 2b + p) \in S_{by+2c}(A^+) \backslash S_{by+2c}(A'),
\]
so there must exist $(w', z') \in S_{by+2c}(A') \backslash S_{by+2c}(A^+)$. Since $\alpha' > \alpha_k$, we must have $w' > cy$ and $z' < 2b$, and
\[
mx-1gk + \alpha_k < \frac{w' - cx}{2b - z'} < mx-1gk + \alpha',
\]
so that
\[
2b > 2b - z' > p,
\]
which is evidently false. So indeed \((cx, 2b) \in S_{6x+2c}(A')\).

It follows that
\[
\frac{cy + 2b(m_{k-1}g_k + \alpha')}{by + 2c} \leq \delta < \frac{cx + b(m_{k-1}g_k + \alpha')}{bx + c}
\]
Both bounds are weighted averages of \(\frac{\alpha}{b}\) and \(\frac{b(m_{k-1}g_k + \alpha')}{c}\). The right-hand side weights \(\frac{\alpha}{b}\) more heavily, because
\[
2x - y > 2 \left( \frac{m_{k-1}g_k + q}{c} - 1 \right) - \left( \frac{m_{k-1}g_k(p + 1) + q}{c} + 1 \right)
= \frac{m_{k-1}g_k(p - 1) + q - 3}{c} > \frac{c^2}{b^2}(p - 1) - 3 > 0,
\]
so it must be the case that
\[
\frac{b(m_{k-1}g_k + \alpha')}{c} < \frac{c}{b},
\]
which is to say
\[
\alpha' < \frac{c^2}{b^2} - m_{k-1}g_k = \alpha_k,
\]
a contradiction. \(\square\)

### 4.4 Weakly triangulating sequences of length 1

In this section, we classify all weakly triangulating sequences of the form \((\delta; \alpha = \frac{n}{m})\), with \(m, n > 2\).

#### 4.4.1 Weakly triangulating sequences \((\delta; \alpha)\) with \(1 \leq \alpha < \varphi^4\)

In this section, we will use Proposition 4.6 to find constraints on weakly triangulating sequences of the form \((\delta; \alpha)\) with \(1 \leq \alpha < \varphi^4\).

When dealing with Fibonacci numbers, it is helpful to introduce a bit of notation for continued fractions: when we write "\(a, b, c^k\)" within a continued fraction, we mean imagine the sequence "\(a, b, c\)" is written out \(k\) times. If the superscript \(k\) is missing, the sequence is meant to be repeated ad infinitum, in accordance with standard notation.

So, for example, \(\varphi^4 = [6, 1, 5]\), and for \(k \geq 1\),
\[
\left(\frac{F_{4k+1}}{F_{4k-1}}\right)^2 = [6, 1, 5^{k-1}, 3, 1, 5^{k-1}, 1], \quad \left(\frac{F_{4k+3}}{F_{4k+1}}\right)^2 = [6, 1, 5^{k-1}, 3, 5, 1^k],
\]
\[
\frac{F_{4k+3}}{F_{4k-1}} = [6, 1, 5^{k-1}, 2], \quad \frac{F_{4k+5}}{F_{4k+1}} = [6, 1, 5^{k-1}, 4].
\]

It follows that
\[
\left(\frac{F_1}{F_0}\right)^2 < \frac{F_3}{F_1} < \left(\frac{F_3}{F_1}\right)^2 < \frac{F_5}{F_3} < \left(\frac{F_5}{F_3}\right)^2 < \frac{F_7}{F_5} < \left(\frac{F_7}{F_5}\right)^2 < \frac{F_9}{F_7} < \cdots < \varphi^4. \quad (*)
\]

For convenience, let's write out the first few terms:
\[
\left(\frac{1}{1}\right)^2 < \frac{2}{1} < \left(\frac{2}{1}\right)^2 < \frac{5}{1} < \left(\frac{5}{2}\right)^2 < \frac{13}{2} < \left(\frac{13}{5}\right)^2 < \frac{34}{5} < \cdots < \varphi^4.
\]
Comparing the continued-fraction expansions with Lemma 4.13, we see that \( \frac{F_{2k+1}}{F_{2k-1}} \) is a fair upper approximation to \( \left( \frac{F_{2k+1}}{F_{2k-1}} \right)^2 \) for all \( k \in \mathbb{N} \), and \( \frac{F_{2k+1}}{F_{2k-3}} \) is a fair lower approximation to \( \left( \frac{F_{2k+1}}{F_{2k-1}} \right)^2 \) for all \( k \in \mathbb{N} \setminus \{1, 2\} \).

**Lemma 4.21.** Fix \( k \in \mathbb{N}, a = F_{2k-3}, b = F_{2k-1}, c = F_{2k+1}, \) and \( d = F_{2k+3} \). Let \( n > m > 0 \) be relatively prime integers such that \( n/m \) is a fair approximation to \( (c/b)^2 \) in the interval \( (c/a, d/b) \).

(i0) If \( \frac{c}{a} < \frac{n}{m} < (\frac{c}{b})^2 \), then either there is an integer \( i \equiv k \) (mod 2) with \( -1 \leq i < k \) such that

\[
(m, n) = \frac{F_{2i-1}(a, c) + F_{2i+1}(b, d)}{3},
\]

or else there is an integer \( \ell \geq 0 \) such that

\[
(m, n) = \ell(b^2, c^2) + (a^2, b^2 + 2).
\]

(The two expressions are equal when \( i = k - 2 \) and \( \ell = 0 \).)

(ii) If \( (\frac{c}{b})^2 < \frac{n}{m} \leq \frac{d}{b} \), then either there are integers \( i \equiv k \) (mod 2) and \( \epsilon \) with \( 2 \leq i \leq k \) and \( 1 \leq \epsilon \leq \min\{5, 3i - 4\} \) such that

\[
(m, n) = \frac{(F_{2i-1} - \epsilon F_{2i-5})(a, c) + (F_{2i+1} - \epsilon F_{2i-3})(b, d)}{3},
\]

or else there is an integer \( \ell \geq 1 \) such that

\[
(m, n) = \ell(b^2, c^2) - (a^2, b^2 + 2).
\]

(The two expressions are equal when \( i = k \) and \( \epsilon = \ell = 1 \).)

**Sketch of proof.** Use the continued-fraction expansion of \( (c/b)^2 \) and Lemma 4.13. The pairs \((m, n)\) of the form \( \ell(b^2, c^2) \pm (a^2, b^2 + 2) \) come from changing the "oo" in the continued-fraction expansions of \( (c/b)^2 \) Lemma 4.13 to a positive integer. The other pairs \((m, n)\) come from truncations of these expansions (along with, optionally, decreasing the last number of the expansion that remains). \( \square \)

**Proposition 4.22.** Fix \( k \in \mathbb{N}, a = F_{2k-3}, b = F_{2k-1}, c = F_{2k+1}, \) and \( d = F_{2k+3} \). Let \( (\delta; \alpha = \frac{n}{m}) \) be weakly triangulating with \( m > 1 \) and \( \frac{c}{a} \leq \alpha \leq \frac{d}{b} \), and let \( D = m\delta \). Then the following statements hold:

i. If \( \alpha \notin \{\frac{c}{a}, \frac{d}{b}\} \), then we have \( D \geq \max\{\frac{cm}{b}, \frac{bn}{c}\} \). The same inequality holds if \( \alpha = \frac{c}{a} \) and \( (\delta; \alpha) \) satisfies the \( b \)th triangulating condition, or if \( \alpha = \frac{d}{b} \) and \( (\delta; \alpha) \) satisfies the \( c \)th triangulating condition, even though these triangulating conditions are not required by the definition of "weakly triangulating."

ii. If \( \alpha \notin \{\frac{c}{a}, \frac{d}{b}\} \), then for all \( j \in \mathbb{N} \), we have \( S_j((c/b), (c/b)^2) \subset S_j(\delta; \alpha) \), i.e.,

\[
b^2x + c^2y \leq bcj \quad \implies \quad mx + ny \leq Dj, \quad (4.4.1)
\]

iii. The fraction \( n/m \) is a fair approximation to \( (c/b)^2 \). If \( \alpha \notin \{\frac{c}{a}, \frac{d}{b}\} \), then for \( (x, y) \in \mathbb{N} \)

\[
b^2x + c^2y \leq bcj \iff mx + ny \leq Dj. \quad (4.4.2)
\]

whenever \( bj < cm \) and \( cj < bn \). 61
iv. Either

\[ m \leq b \quad \text{and} \quad n \leq d, \quad \text{(small)} \]

or

\[ |c^2m - b^2n| \leq 1. \quad \text{(large)} \]

v. In case (small) above, \((m, n)\) is one of

\((a, c), \quad \left( \frac{a + b}{3}, \frac{c + d}{3} \right), \quad \left( \frac{a + 2b}{3}, \frac{c + 2d}{3} \right), \quad \text{and} \quad (b, d), \]

unless \((m, n) = (3, 20)\). In case (large) above, \((m, n)\) is either \((b^2, c^2)\) or of the form

\[ \ell(b^2, c^2) \pm (a^2, b^2 + 2), \quad \ell \in \mathbb{N}. \]

Proof. It is useful to distinguish between two cases:

\[ \frac{c}{a} \leq \frac{n}{m} < \left( \frac{c}{b} \right)^2, \quad \text{(lo)} \]

and

\[ \left( \frac{c}{b} \right)^2 < \frac{n}{m} \leq \frac{d}{b}. \quad \text{(hi)} \]

Always keep in mind that \((c/b; (c/b)^2)\) is strongly triangulating, by Proposition 4.6.

i. In case (lo), \( \frac{cm}{b} > \frac{bn}{c} \), so we just have to prove \( cm \leq bD \). The simplest fraction in \((\alpha, (c/b)^2)\) has a numerator exceeding \(c\) and a denominator not less than \(a\). (This holds even the cases \(k \in \{1, 2\}\).) Note that

\[ (b - 1)(b - 2) = (a - 1)(c - 1) < (m - 1)(n - 1), \]

and recall that \(ac - b^2 = 1\), so in particular \(ac > b^2\), and \((\delta; \alpha)\) must satisfy the \(b\)th triangulating condition. By Lemma 4.15 for \(j = b\), we have for all \((x, y) \in \mathbb{N}^2\),

\[ b^2x + c^2y \leq b^2c \implies x + ay \leq b \sqrt{\alpha}. \]

In particular, set \((x, y) = (c, 0)\) to find that \(cm \leq bD\).

In case (hi), \( \frac{cm}{b} < \frac{bn}{c} \), so we just have to prove \(bn \leq cD\). The simplest fraction in \(((c/b)^2, \alpha)\) has a numerator exceeding \(d\) and a denominator exceeding \(b\). Note that

\[ (c - 1)(c - 2) = (b - 1)(d - 1) < (m - 1)(n - 1), \]

and recall that \(bd - c^2 = 1\), so in particular \(bd > c^2\), and and \((\delta; \alpha)\) must satisfy the \(c\)th triangulating condition. By Lemma 4.15 for \(j = c\), we have for all \((x, y) \in \mathbb{N}^2\),

\[ b^2x + c^2y \leq bc^2 \implies x + ay \leq c \sqrt{\alpha}. \]

In particular, set \((x, y) = (0, b)\) to find that \(bn \leq cD\).

ii. Suppose \(b^2x + c^2y \leq bcj\). In case (lo), \(b^2n < c^2m\), so

\[ mx + ny \leq \frac{m}{b^2}(b^2x + c^2y) \leq \frac{m \cdot cj}{b} \leq Dj, \]

and in case (hi), \(c^2m < b^2n\), so

\[ mx + ny \leq \frac{n}{c^2}(b^2x + c^2y) \leq \frac{n \cdot bj}{c} \leq Dj. \]
iii. Note that $\frac{e}{a}$ and $\frac{f}{b}$ are already fair approximations to $\frac{c^2}{b^2}$. If $\alpha \notin \{\frac{e}{a}, \frac{f}{b}\}$, then (iii) follows from (ii) by Proposition 4.19.

iv. Assume that either $m > b$ or $n > d$. Also assume that

$$\Delta := |b^2n - c^2m| > 1,$$

so that $(m, n)$ is not of the form

$$\ell(b^2, c^2) \pm (a^2, b^2 + 2)$$

for any $\ell \in \mathbb{Z}$.

In case (lo), Lemma 4.21 implies that there is an integer $i \equiv k \pmod{2}$ with $2 \leq i \leq k - 4$ such that

$$(m, n) = \frac{F_{2i-1}(a, c) + F_{2i+1}(b, d)}{3},$$

since any smaller value of $i$ would entail $m \leq b$ and $n \leq d$. Let

$$y' = F_{2i-1} - 1, \quad j = cy' + b - 3.$$

We claim that $j$ is small enough that (4.4.2) must hold, and yet we will exhibit a particular pair $(x, y)$ for which (4.4.2) fails.

First we claim that

$$c(F_{2i-3} - 3) < b(F_{2i-1} - 3).$$

Indeed, if $2 \leq i \leq 3$ then this is clear. If $i \geq 4$, then both sides are positive, and since $c/b < \varphi^2$ it suffices to show that

$$\frac{F_{2i-1} - 3}{F_{2i-3} - 3} > \varphi^2,$$

or, in other words,

$$\frac{F_{2i-1} - 3}{F_{2i-3} - 3} + \frac{F_{2i-3} - 3}{F_{2i-1} - 3} > \varphi^2 + \varphi^{-2} = 3.$$

To that end, note that $F_{2i}^2 - 3F_{2i-1}F_{2i-3} + F_{2i-3}^2 = -1$, so

$$(F_{2i-1} - 3)^2 + (F_{2i-3} - 3)^2 - 3(F_{2i-1} - 3)(F_{2i-3} - 3) = 3(F_{2i-1} + F_{2i-3}) - 10 > 0.$$

Since $F_{2i-3} + F_{2i+1} = 3F_{2i-1}$, it follows that

$$3y' = F_{2i-3} - 3 + F_{2i+1} < \frac{b}{c}(F_{2i-1} - 3) + F_{2i+1}.$$

Thus,

$$cj = c(cy' + b - 3)$$

$$< bc \cdot \frac{F_{2i-1} - 3}{3} + c^2 \cdot \frac{F_{2i+1}}{3} + bc - 3c$$

$$< bc \cdot \frac{F_{2i-1}}{3} + (c^2 + 1) \cdot \frac{F_{2i+1}}{3}$$

$$= bc \cdot \frac{F_{2i-1}}{3} + bd \cdot \frac{F_{2i+1}}{3}$$

$$= bn,$$

so $j < \frac{bn}{c} < \frac{cm}{b}$, and (4.4.2) holds.
And yet, let 
\[(x, y) = (c - 1, by' - 1).\]

Then 
\[dm - bn = \frac{F_{2n-1}(da - bc)}{3} = F_{2n-1} = y' + 1\]
so by our assumption that \(\Delta > 1,\)
\[(c^2m - b^2n)y' = \Delta y' \geq 2y' \geq y' + 1 = dm - bn = (3c - b)m - bn.\]

This means that 
\[b(mx + ny) = bm(c - 1) + bn(by' - 1) \leq cm(cy' + b - 3) = cmj,\]
so that, in light of (i),
\[mx + ny \leq \frac{cm}{b} \cdot j \leq Dj.\]

On the other hand,
\[b^2x + c^2y = b^2(c - 1) + c^2(by' - 1) = bcj + 3bc - b^2 - c^2 = bcj + 1 > bcj,\]
so (4.4.2) fails. This concludes the proof of (iv) in the (lo) case.

Case (hi) is similar, only slightly more tedious. Lemma 4.21 implies that there is an integer 
\(i \equiv k \pmod{2}\) with \(2 \leq i \leq k\) and a positive integer \(\epsilon \leq \min\{5, 3i - 4\}\) such that
\[(m, n) = \frac{(F_{2i-1} - \epsilon F_{2i-5})(a, c) + (F_{2i+1} - \epsilon F_{2i-3})(b, d)}{3}.\]

If \(i = \epsilon = 2,\) or if \(i = 3\) and \(\epsilon = 5,\) then \((m, n) = (b, d),\) and if \(i = k\) and \(\epsilon = 1,\) then \(\Delta = 1;\) by assumption, we are not dealing with these cases, so in particular, \(k \geq 3.\)

Let 
\[x' = F_{2i+1} - \epsilon F_{2i-3} - 3, \quad j = bx' + c - 3.\]

We claim that \(j\) is small enough that (4.4.2) must hold, and yet we exhibit a particular pair \((x, y)\)
for which (4.4.2) fails.

First we claim that
\[\frac{F_{2i+1} - \epsilon F_{2i-3} - 3}{F_{2i-1} - \epsilon F_{2i-5}} < \frac{F_{2k+1}}{F_{2k-1}} = \frac{c}{b}.\]

We fix \(k,\) and prove this by induction on \(i.\)

- If \(i = 2,\) then the only case to consider is \(\epsilon = 1:\) since \(k \geq 2,\)
  \[\frac{F_{2i+1} - \epsilon F_{2i-3} - 3}{F_{2i-1} - \epsilon F_{2i-5}} = \frac{F_5 - 1 \cdot F_1 - 3}{F_3 - 1 \cdot F_1} = \frac{5 - 1 - 3}{2} = 1 < \frac{F_{2k+1}}{F_{2k-1}} = \frac{c}{b}.\]

- If \(i = 3,\) then we only need to consider the cases where \(1 \leq \epsilon \leq 4: \) since \(k \geq 3,\)
  \[\frac{F_{2i+1} - \epsilon F_{2i-3} - 3}{F_{2i-1} - \epsilon F_{2i-5}} = \frac{13 - 2\epsilon - 3}{5 - 2\epsilon} = 2 < \frac{F_{2k+1}}{F_{2k-1}} = \frac{c}{b}.\]
If \( i \geq 4 \), then note that \( F_{2i+1} - 6F_{2i-3} = F_{2i-3} - F_{2i-7} \). Since

\[
\frac{F_{2i+1} - \epsilon F_{2i-3} - 3}{F_{2i-1} - \epsilon F_{2i-5}} = \frac{(F_{2i+1} - 6F_{2i-3} - 3) + (6 - \epsilon)F_{2i-3}}{(F_{2i-1} - 6F_{2i-5}) + (6 - \epsilon)F_{2i-5}} = \frac{(F_{2i-3} - F_{2i-7} - 3) + (6 - \epsilon)F_{2i-3}}{(F_{2i-5} - F_{2i-9}) + (6 - \epsilon)F_{2i-5}},
\]

the quantity in question is a weighted average of

\[
\frac{F_{2i-3} - F_{2i-7} - 3}{F_{2i-5} - F_{2i-9}} \quad \text{and} \quad \frac{F_{2i-3}}{F_{2i-5}}.
\]

The first of these is less than \( b/a \), by the induction hypothesis (with \( i - 2 \) replacing \( i \) and \( 1 \) replacing \( \epsilon \)). The second of these is an increasing function of \( i \geq 4 \) and so, since \( i \leq k \), we have

\[
\frac{F_{2i-3}}{F_{2i-5}} \leq \frac{F_{2k-3}}{F_{2k-5}} < \frac{F_{2k+1}}{F_{2k-1}} = \frac{c}{b}
\]
as well.

Thus,

\[
bj = b(bx' + c - 3) < b \left( \frac{c(x' + 3)}{3} + \frac{(3b - c)x'}{3} \right) = bc \cdot \frac{F_{2i+1} - \epsilon F_{2i-3}}{3} + ba \cdot \frac{F_{2i+1} - \epsilon F_{2i-3} - 3}{3} < bc \cdot \frac{F_{2i+1} - \epsilon F_{2i-3}}{3} + ca \cdot \frac{F_{2i-1} - \epsilon F_{2i-5}}{3} = cm,
\]

so \( j < \frac{cm}{b} < \frac{bn}{c} \), and (4.4.2) holds.

And yet, let

\[
(x, y) = (cx' - 1, b - 1).
\]

Then

\[
an - cm = \frac{(F_{2i+1} - \epsilon F_{2i-3})(ad - cb)}{3} = F_{2i+1} - \epsilon F_{2i-3} = x' + 3.
\]

Also note that

\[
\Delta = b^2n - c^2m = (F_{2i-1} - \epsilon F_{2i-5})(b^2c - c^2a) + (F_{2i+1} - \epsilon F_{2i-3})(b^2d - c^2b) = -(F_{2i-1} - \epsilon F_{2i-5})c + (F_{2i+1} - \epsilon F_{2i-3})b.
\]

We claim that \( \Delta \cdot x' \geq x' + 3 \). If \( i = 2 \) and \( \epsilon = 1 \), then \( x' = 1 \) and

\[
\Delta = 4b - c \geq 7
\]
since \( k \geq 3 \), so indeed \( \Delta \cdot x' \geq x' + 3 \). If \( i = 3 \) and \( \epsilon = 4 \), then \( x' = 2 \) and \( \Delta = 5b - c \geq 12 \), so again \( \Delta \cdot x' \geq x' + 3 \). Otherwise, \( x' \geq 3 \), so by our assumption that \( \Delta > 1 \),

\[
(b^2n - c^2m)x' = \Delta \cdot x' \geq 2x' \geq x' + 3 = an - cm = (3b - c)n - cm.
\]
This means that
\[ c(mx + ny) = cm(cx' - 1) + cn(b - 1) \leq bn(bx' + c - 3) = bnj, \]
so that, in light of (i),
\[ mx + ny \leq \frac{bn}{c} \cdot j \leq D j. \]

On the other hand,
\[ b^2 x + c^2 y = b^2(cx' - 1) + c^2(b - 1) = bcj + 3bc - b^2 - c^2 = bcj + 1 > bcj, \]
so (4.4.2) fails. This concludes the proof of (iv).

\( v \). This follows from (iv) if we can eliminate the case where \( k > 4 \) is odd and
\[ (m, n) = \frac{(a, c) + (b, d)}{3}. \]
In this case, (4.4.2) holds, so since \( b^2 c = b \cdot bc \), we must have
\[ cm \leq bD. \]

Choose \( \eta \in \{4, 5\} \) such that \( \eta \equiv a \pmod{3} \). Since \( k \) is odd, one can check the numbers
\[ x := \frac{7c - b - \eta}{9} = \frac{4c + d - \eta}{9}, \quad y := \frac{8b - 2c - \eta}{9} = \frac{2a + 2b - \eta}{9}, \quad \text{and} \quad j := \frac{2c + \eta}{3} \]
are all positive integers. Since \( ac = b^2 + 1 \), \( bd = c^2 + 1 \), and \( b^2 + c^2 + 1 = 3bc \), we have
\[ b^2 x + c^2 y = \frac{b^2(4c + d - \eta) + c^2(2a + 2b - \eta)}{9} = \frac{3bc(2b + c - \eta) + b + 2c + \eta}{9} > bcj, \]
and
\[ \frac{(m + 1)(n + 1)}{2} = \frac{(2a + b + 3)(2c + d + 3)}{18} = \frac{(7b - 2c + 3)(5c - b + 3)}{18}, \]
so that one can check
\[ \frac{(m + 1)(n + 1)}{2} - \frac{(j + 1)(j + 2)}{2} = \frac{(4b + 2c + 9 - \eta)\eta + 2}{18} > 0, \]
and similarly
\[ (j - 1)(j - 2) < (m - 1)(n - 1). \]

Therefore, by (4.4.2), we must have
\[ mx + ny > D j \geq \frac{cm}{b} \cdot j. \]

Since
\[ mx + ny = \frac{c(16b - 3c) - (2b + c)\eta + 5}{9}, \]
\[ mj = \frac{b(16b - 3c) - (7b - 2c)\eta + 2}{9}, \]
This means
\[ 0 > cmj - b(mx + ny) = \frac{2c - 5b - 2\eta}{9} = \frac{b - 2a - 2\eta}{9} = \frac{F_{2k-4} - 2\eta}{9}. \]

But if \( k = 5 \), then \( F_{2k-4} = 8 \) and \( \eta = 4 \), so \( cmj - b(mx + ny) = 0 \). If \( k \geq 7 \), then \( F_{2k-4} \geq 55 > 2\eta \), so \( cmj - b(mx + ny) > 0 \). This is a contradiction. \( \square \)

In summary, we showed that, in many cases, if a fraction \( n/m \) is sufficiently close to a strongly triangulating ratio, then in fact \( n/m \) must be a fair approximation thereto. For fractions less than \( \varphi^4 \), we were able to use strongly triangulating ratios of the form \( (F_{2k+1}/F_{2k-1})^2 \).

### 4.4.2 Weakly triangulating sequences \( (\delta; \alpha) \) with \( \varphi^4 < \alpha \leq 9 \)

**Proposition 4.23.** Let \( (\delta; \alpha = \frac{n}{m}) \) be weakly triangulating with \( m \geq 2 \) and \( \varphi^4 < \alpha \leq 9 \). Then

i. If \( \varphi^4 < \alpha < 7 \), then \( \alpha \) is a fair upper approximation to \( \varphi^4 = [6, 1, 5] \), i.e., for some integer \( k \geq 2 \), we have
\[ \alpha = \frac{n}{m} = [6, 1, 5^{k-1}, 1] = \frac{F_{4k+4}/3}{F_{4k}/3}. \]

ii. If \( 7 < \alpha < 8 \), then \( \alpha \) is a fair upper approximation to \( \frac{\delta}{\alpha} = [7, 8, 1] \), i.e., either \( \frac{n}{m} = 7 + \frac{1}{m} \) for some \( 2 \leq m \leq 8 \), or \( \frac{n}{m} = 7, 8, 1, k \) is \( \frac{64k+57}{9k+8} \) for some integer \( k \geq 1 \).

iii. If \( 8 < \alpha < 9 \), then either \( \alpha = 9 \) or else \( \alpha \) is a fair lower approximation to \( 9 \), i.e., \( \frac{n}{m} = 9 - \frac{1}{m} \).

**Proof.** Let \( D = m\delta \). The cases where \( \alpha \in \mathbb{N} \) have been dealt with before, so let us assume \( m > 1 \). It is not hard to see that
\[ \frac{(m+1)(n+1)}{2} \geq \frac{(2+1)(15+1)}{2} > \frac{(3+1)(3+2)}{2}, \]
so the third triangulating condition applies. It follows that

\[ \frac{n}{m} < 7 \implies m + n \leq 3D < 8m, \quad (4.4.3) \]
\[ \frac{n}{m} > 7 \implies 8m \leq 3D < m + n. \quad (4.4.4) \]

i. Suppose \( \varphi^4 < \frac{n}{m} < 7 \) but \( \frac{n}{m} \) is not a fair upper approximation to \( \varphi^4 \). Let \( \frac{a}{p} \) be the simplest fraction in \( (\varphi^4, \alpha) \). Then \( \frac{a}{p} \) must be a fair upper approximation to \( \varphi^4 \), so let \( k \) be the positive integer such that
\[ (p, q) = \left( \frac{F_{4k}}{3}, \frac{F_{4k+4}}{3} \right). \]

Let \( a := F_{4k-2}, b := F_{4k} = 3p, c := F_{4k+2}, \) and \( d := F_{4k+4} = 3q. \) Note that \( ac + 1 = b^2, a + c = 3b, \) and
\[ 3a - b = F_{4k-4}. \]

We have
\[ \varphi^4 < \frac{d/3}{b/3} < \frac{n}{m} < \frac{b/3}{(3a - b)/3}. \]
and \( \frac{b/3}{(3a-b)/3} \) and \( \frac{d/3}{b/3} \) are consecutive fair upper approximations to \( \varphi^4 \). By Corollary 4.14, there are two cases; either

\[
(m, n) = \left( \frac{3a-b}{3}, \frac{b}{3} \right) + \left( \frac{b}{3}, \frac{d}{3} \right) = (a, c)
\]

or \( m > a \) and \( n > c \).

First suppose \( (m, n) = (a, c) \). By (4.4.3), we have

\[
D \geq \frac{m+n}{3} = \frac{a+c}{3} = b,
\]

as one can easily check. Since

\[
(a-1)(a-2) \leq (a-1)(c-1) - 1
\]

and

\[
\frac{(a+1)(c+1)}{2} > \frac{(a+1)(a+2)}{2},
\]

the \( a \)th triangulating condition must hold. Yet we claim that \( S_a(\varphi^2; \varphi^4) \subsetneq S_a(\delta; \alpha) \), so that \( |S_a(\delta; \alpha)| > |S_a(\varphi^2; \varphi^4)| = \left( \frac{a+2}{2} \right) \). Indeed, note that

\[
ab \leq aD \quad \text{and} \quad \frac{b}{a} > \varphi^2,
\]

so \((b, 0) \in S_a(\delta; \alpha) \setminus S_a(\varphi^2; \varphi^4) \). If \((x, y) \in S_a(\varphi^2; \varphi^4) \setminus S_a(\delta; \alpha) \), then the above considerations show that \( x < b \) and \( y > 0 \), so that

\[
cy > a(D - x) \geq a(b - x)
\]

and

\[
\frac{\varphi^4}{y} < \frac{\varphi^2 a - x}{y} < \frac{b - x}{y} < \frac{c}{a} = \frac{n}{m},
\]

which contradicts the fact that \( \frac{q}{p} \) is the simplest fraction in \((\varphi^4, \alpha) \), since \( b - x < q \).

Now suppose

\[
m > a, \quad n > c.
\]

Since

\[
(b-1)(b-2) = (a-1)(c-1) + 2 \leq (m-1)(n-1)
\]

and

\[
\frac{(m+1)(n+1)}{2} \geq \frac{(a+2)(c+2)}{2} = \frac{ac + 2(a+c) + 4}{2} = \frac{(b^2 - 1) + 6b + 4}{2} > \frac{(b+1)(b+2)}{2},
\]

the \( b \)th triangulating condition must hold. Yet we claim that \( S_b(\varphi^2; \varphi^4) \subsetneq S_b(\delta; \alpha) \), so that \( |S_b(\delta; \alpha)| > |S_b(\varphi^2; \varphi^4)| = \left( \frac{b+2}{2} \right) \). Indeed,

\[
\frac{m+n}{m} > \frac{b+d}{b} = \frac{3c}{b} \quad \text{and} \quad \frac{c}{b} > \varphi^2,
\]

so \((c, 0) \in S_b(\delta; \alpha) \setminus S_b(\varphi^2; \varphi^4) \). If \((x, y) \in S_b(\varphi^2; \varphi^4) \setminus S_b(\delta; \alpha) \), then

\[
m(x - p) + n(y - p) > bD - (m+n)p \geq 0
\]

while

\[
\frac{x-p}{\varphi^2} + \varphi^2(y - p) \leq b - 3p = 0.
\]
Since \( n/m \geq \varphi^4 \), we must have \( x < p < y \) and

\[
\varphi^4 \leq \frac{p-x}{y-p} < \frac{n}{m},
\]

which contradicts the fact that \( \frac{q}{p} \) is the simplest fraction in \( (\varphi^4, \alpha) \), since \( p-x < q \).

ii. Suppose \( 7 < \alpha < 8 \) but \( \alpha \) is a not fair upper approximation to \( \frac{64}{9} \).

First we claim that \( \alpha > \frac{64}{9} \). If not, then

\[
(8-1)(8-2) < (9-1)(64-1) \leq (m-1)(n-1)
\]

and

\[
\frac{(m+1)(n+1)}{2} \geq \frac{(9+1)(64+1)}{2} = 325 > \frac{(8+1)(8+2)}{2},
\]

so the 8th triangulating condition must hold. But in fact \( S_8(\delta; \alpha) = S_8(\frac{8}{9}; \frac{64}{9} + \varepsilon) \cup \{(0, 3)\} \) because, by (4.4.4),

\[
21m < 14m + n < 7m + 2n < 3n \leq \frac{8}{3} \cdot 8m \leq 8D \quad \text{but} \quad 3 \cdot \frac{8}{3} \not< 8,
\]

Thus, we must indeed have \( \alpha > \frac{64}{9} \).

So \( \frac{64}{9} < \frac{n}{m} < 8 \). Let \( \frac{q}{p} \) be the simplest fraction in \( (\frac{64}{9}, \frac{n}{m}) \). Table 4.1 lists the possible values of \( \frac{q}{p} \) and the corresponding ranges for \( \frac{n}{m} \). The third column, "Max \( j \)," lists the maximum \( j \) for which the \( j \)th triangulating condition is required to hold; it is calculated with the aid of Corollary 4.14. For example, in the last row, if

\[
\frac{64k + 57}{9k + 8} < \frac{n}{m} < \frac{64k - 7}{9k - 1},
\]

then Corollary 4.14 tells us that

\[
m \geq (9k + 8) + (9k - 1) = 18k + 7
\]

\[
n \geq (64k + 57) + (64k - 7) = 128k + 50,
\]
so that

\[
\frac{(m + 1)(n + 1)}{2} \geq \frac{(18k + 8)(128k + 51)}{2} = \frac{(48k + 19)(48k + 20) + 70k + 28}{2} > \frac{(j + 1)(j + 2)}{2}
\]

as long as \(j \leq 48k + 18\). Note that the coefficients of \(D\) in the fourth and fifth columns never exceed the "Max \(j\)" value in the second. We’ll show that, if in the fourth column a lower bound on \(jD\) is violated, then the triangulating condition fails because \(S_j(\delta; \alpha) \subseteq S_j(\frac{8}{3}; \frac{64}{9} + \varepsilon)\). Similarly, if in the fifth column an upper bound on \(jD\) is violated, then the triangulating condition fails because \(S_j(\frac{8}{3}; \frac{64}{9} + \varepsilon) \subseteq S_j(\delta; \alpha)\). Then the last column follows immediately from the fourth and fifth columns and contradicts the second one. So, without further ado, let us prove the bounds on \(D\) in the fourth and fifth columns.

Lower bound for \(p = 2\): Suppose \(22m > 8D\). Then \(7m + 2n > 22m > 8D\), so \(S_8(\delta; \alpha) \cap \{(7, 2)\} \subset S_8(\frac{8}{3}; \frac{64}{9} + \varepsilon)\).

Lower bound for \(3 \leq p \leq 4\): Suppose \(7m + 2n > 8D\). Then \((7, 2) \in S_8(\frac{8}{3}; \frac{64}{9} + \varepsilon) \setminus S_8(\delta; \alpha)\). If \((x, y) \in S_8(\delta; \alpha) \setminus S_8(\frac{8}{3}; \frac{64}{9} + \varepsilon)\), then it is not hard to see that \(x > 7, y < 2\), and

\[
\frac{64}{9} < \frac{x - 7}{2 - y} < \frac{n}{m},
\]

contradicting the fact that \(\frac{8}{p}\) is the simplest fraction in \((\frac{64}{9}, \frac{m}{n})\), since \(2 - y \leq 2 < p\).

Lower bound for \(5 \leq p \leq 6\): Suppose the lower bound given in Table 4.1 does not hold. We claim that \((7, 5) \in S_{16}(\frac{8}{3}; \frac{64}{9} + \varepsilon) \setminus S_{16}(\delta; \alpha)\): In the \(p = 5\) case, we have \(7m + 5n > 43m > 16D\), whereas in the \(p = 6\) case, this is by assumption. If \((x, y) \in S_{16}(\delta; \alpha) \setminus S_{16}(\frac{8}{3}; \frac{64}{9} + \varepsilon)\), then it is not hard to see that \(x > 7, y < 5\), and

\[
\frac{64}{9} < \frac{x - 7}{5 - y} < \frac{n}{m},
\]

Since \(\frac{8}{p}\) is the simplest fraction in \((\frac{64}{9}, \frac{m}{n})\), we must have \(p = 5\) and \((x, y) = (0, 43)\), but then the inequality \(43m \leq 16D\) actually holds.

Lower bound for \(p = 7\): Suppose \(57m + n > 24D\). Then \(7m + 8n > 57m + n > 24D\), so \((7, 8) \in S_{24}(\frac{8}{3}; \frac{64}{9} + \varepsilon) \setminus S_{24}(\delta; \alpha)\). If \((x, y) \in S_{24}(\delta; \alpha) \setminus S_{24}(\frac{8}{3}; \frac{64}{9} + \varepsilon)\), then it is not hard to see that \(x > 7, y < 8\), and

\[
\frac{64}{9} < \frac{x - 7}{8 - y} < \frac{n}{m},
\]

Besides \(\frac{50}{7}\) and \(\frac{57}{8}\), all fractions in the interval \((\frac{64}{9}, \frac{n}{m})\) have denominator at least 13. Therefore, either \((x, y) = (57, 1)\) or \((64, 0)\). But by assumption \((57, 1) \notin S_{24}(\delta; \alpha)\), and \((64, 0) \in S_{24}(\frac{8}{3}; \frac{64}{9} + \varepsilon)\), so neither of these pairs lies in \(S_{24}(\delta; \alpha) \setminus S_{24}(\frac{8}{3}; \frac{64}{9} + \varepsilon)\).

Lower bound for \(p = 9k + 8\), where \(k \in \mathbb{N}\) (note \(p = 8\) when \(k = 0\)): Let \(j := 24k + 40\), and suppose \((64k + 64)m + 6n > jD\). Then

\[
7m + (9k + 14)n > (64k + 64)m + 6n > (24k + 40)D,
\]

so \((7, 9k + 14) \in S_j(\delta; \alpha) \setminus S_j(\delta; \alpha)\). If \((x, y) \in S_j(\delta; \alpha) \setminus S_j(\frac{8}{3}; \frac{64}{9} + \varepsilon)\), then it is not hard to see that \(x > 7, y < 9k + 14\), and

\[
\frac{64}{9} < \frac{x - 7}{(9k + 14) - y} < \frac{n}{m}.
\]

Besides \(\frac{8}{p}\), all fractions in \((\frac{64}{9}, \frac{n}{m})\) have denominator at least \(p + 7 > 9k + 14\), so

\[
(x, y) = (q + 7, 9k + 14 - p) = (64k + 64, 6) \in S_j(\delta; \alpha),
\]

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contrary to our assumption that \((64k + 64)m + 6n > jD\).

Upper bound for \(p = 2\): Suppose \(11m \leq 4D\). Then \(3m + n < 11m \leq 4D\), so \(S_4(\frac{8}{3}, \frac{64}{9} + \epsilon) \sqcup \{(11,0)\} \subset S_4(\delta; \alpha)\).

Upper bound for \(3 \leq p \leq 6\): Suppose \((8p-7)m+n \leq 3pD\). Then \((8p-7, 1) \in S_{3p}(\delta; \alpha) \setminus S_{3p}(\frac{8}{3}, \frac{64}{9} + \epsilon)\). If \((x, y) \in S_{3p}(\frac{8}{3}, \frac{64}{9} + \epsilon) \setminus S_{3p}(\delta; \alpha)\), then it is not hard to see that \(x < 8p - 7, y > 1\), and

\[
\frac{64}{9} < \frac{8p - 7 - x}{y - 1} < \frac{n}{m},
\]

contradicting the fact that \(\frac{8}{p} \) is the simplest fraction in \((\frac{64}{9}, \frac{n}{m})\), since \(8p - 7 - x \leq 8p - 7 < q\).

Upper bound for \(p = 7\): Suppose \(48m + 3n \leq 26D\). Then \((48, 3) \in S_{26}(\delta; \alpha) \setminus S_{26}(\frac{8}{3}, \frac{64}{9} + \epsilon)\). If \((x, y) \in S_{26}(\frac{8}{3}, \frac{64}{9} + \epsilon) \setminus S_{26}(\delta; \alpha)\), then it is not hard to see that \(x < 48, y > 3\), and

\[
\frac{64}{9} < \frac{48 - x}{y - 3} < \frac{n}{m},
\]

contradicting the fact that \(\frac{50}{7} \) is the simplest fraction in \((\frac{64}{9}, \frac{n}{m})\), since \(48 - x \leq 48 < 50\).

Upper bound for \(p = 9k+8\), where \(k \in \mathbb{N}\): Let \(j := 24k + 29\), and suppose \((64k + 56)m + 3n \leq jD\). Then \((64k + 56, 3) \in S_j(\delta; \alpha) \setminus S_j(\frac{8}{3}, \frac{64}{9} + \epsilon)\). If \((x, y) \in S_j(\frac{8}{3}, \frac{64}{9} + \epsilon) \setminus S_j(\delta; \alpha)\), then it is not hard to see that \(x < 64k + 56, y > 3\), and

\[
\frac{64}{9} < \frac{64k + 56 - x}{y - 3} < \frac{n}{m},
\]

contradicting the fact that \(\frac{9}{7} \) is the simplest fraction in \((\frac{64}{9}, \frac{n}{m})\), since \(64k + 56 - x \leq 64k + 56 < 64k + 56 < q\).

iii. Suppose \(8 < \frac{n}{m} < 9\), but \(\frac{n}{m}\) is not a fair lower approximation to \(9\). Let \(\frac{q}{p}\) be the simplest fraction in \((\frac{n}{m}, 9)\); then \(p \geq 2\) and \(q = 9p - 1 \geq 17\). Then by Corollary 4.14, we have \(m \geq 2p + 1\) and \(n \geq 2q + 9\), so

\[
(m - 1)(n - 1) \geq 2p(18p + 6) = 6p(6p + 2) > (6p + 1)(6p)
\]

and

\[
\frac{(m + 1)(n + 1)}{2} \geq \frac{(2p + 2)(18p + 8)}{2} = \frac{(6p + 3)(6p + 4) + 10p + 4}{2}.
\]

Thus, the triangulating conditions must hold for all \(j \leq 6p + 2\).

We claim that, if \(6m + p n > (3p + 2)D\), then \(S_{3p+2}(\delta; \alpha) \subset S_{3p+2}(3; 9 - \epsilon)\). Indeed, under this assumption, we have

\[
(q + 6)m > 6m + pn > (3p + 2),
\]

so \((q + 6, 0) \in S_{3p+2}(3; 9 - \epsilon) \setminus S_{3p+2}(\delta; \alpha)\). If \((x, y) \in S_{3p+2}(\delta; \alpha) \setminus S_{3p+2}(3; 9 - \epsilon)\), then it is not hard to see that \(x < q + 6, y > 0\), and

\[
\frac{n}{m} < \frac{q + 6 - x}{y} < 9.
\]

Besides \(\frac{q}{p}\), all fractions in \((\frac{n}{m}, 9)\) have numerator at least \(2q - 9 > q + 6\), so we must have \((x, y) = (6, p) \in S_{3p+2}(\delta; \alpha)\), contradicting our assumption that \(6m + pn > (3p + 2)D\).

We claim that, if \((9p + 2)m \leq (3p + 1)D\), then \(S_{3p+1}(3; 9 - \epsilon) \subset S_{3p+1}\). Indeed, under this assumption, we have \((3, p) \in S_{3p+1}(\delta; \alpha) \setminus S_{3p+1}(3; 9 - \epsilon)\). If \((x, y) \in S_{3p+1}(3; 9 - \epsilon) \setminus S_{3p+2}(\delta; \alpha)\), then it is not hard to see that \(x > 3, y < p\), and

\[
\frac{n}{m} < \frac{x - 3}{p - y} < 9,
\]

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which contradicts the fact that \( \frac{9}{3} \) is the simplest fraction in \( \left( \frac{a}{3}, 9 \right) \).

Therefore, by the \( j \)th triangulating conditions for \( j = 3p + 2 \) and \( j = 3p + 1 \), we have

\[
\frac{6m + pn}{3p + 2} \leq D < \frac{(9p + 2)m}{3p + 1},
\]

and

\[
9p + 5 = q + 6 < \frac{6m + pn}{m} < \frac{(9p + 2)(3p + 2)}{3p + 1} = 9p + 5 - \frac{1}{3p + 1},
\]

which is absurd. \( \Box \)

### 4.4.3 Classification of weakly triangulating sequences of length 1

If \( (\delta = \frac{D}{m}; \alpha = \frac{a}{m}) \) is weakly triangulating and \( n > m \geq 2 \), then \((m, n)\) must appear somewhere on the following list, according to the results of previous sections. We also give some bounds on \( D \) in each case. In cases involving only finitely many pairs \((m, n)\), these bounds on \( D \) can be checked easily and straightforwardly, so we omit the verification. But in cases involving an infinite sequence of pairs \((m, n)\), we sketch the proof.

Let \( z, a, b, c, d \) denote consecutive odd-indexed Fibonacci numbers; more precisely, 
\[ z = F_{2k-5}, \quad a = F_{2k-3}, \quad b = F_{2k-1}, \quad c = F_{2k+1}, \quad d = F_{2k+3} \]
for some \( k \in \mathbb{N} \).

We claim that, if \( m \geq 2 \) and \( \alpha > 3 \), then \((\delta; \alpha)\) must satisfy the \( j \)th triangulating condition as long as \( j \leq D - 1 \). Suppose that were not the case, let \( j \) be the integer such that

\[
\left( \begin{array}{c} j + 2 \\ 2 \end{array} \right) < \frac{(m + 1)(n + 1)}{2} \leq \left( \begin{array}{c} j + 3 \\ 2 \end{array} \right);
\]

by assumption, we have \( j \leq D - 2 \). The fact that \((\delta; \alpha)\) satisfies both the \( j \)th and \((j - 1)\)st triangulating conditions means that there are \( j + 1 \leq D - 1 \) non-negative linear combinations of \( m \) and \( n \) in the interval \((j - 1)D, jD]\), an interval which contains at least \( |D| > D - 1 \) integers. Now every integer that is at least \((m - 1)(n - 1)\) is a non-negative linear combination of \( m \) and \( n \), so it must be the case that

\[
j^2 + j - 2 = (j - 1)(j + 2) \leq (j - 1)D < (m - 1)(n - 1) - 1 = mn - (m + n)
\]

And yet

\[
j^2 + 5j + 6 = (j + 2)(j + 3) \geq (m + 1)(n + 1) = mn + (m + n) + 1
\]

so we must have

\[
4D > 4j + 8 \geq 2(m + n) + 1.
\]

But then \( m + n < 2D \), and, under the assumption that \( 3m \leq n \), we see that \((x, 0) \in S_2(\delta; \alpha)\) for \( 0 \leq x \leq 4 \), and \((1, 1), (2, 1) \in S_2(\delta; \alpha)\), so we know that \((\delta; \alpha)\) violates the second triangulating condition, which is a contradiction because \((m + 1)(n + 1) \geq 3 \cdot 4 > 6 \).

- \((m, n) = (3, 20), \) and \( \frac{30}{5} \leq D < \frac{47}{6} \).
- \((m, n) = (4, 27), \) and \( \frac{52}{6} \leq D < \frac{94}{9} \).
- \((m, n) = (a, c), \) and \( b \leq D < \frac{a + b}{b - 1} \) (assuming \( a \geq 2 \)). The lower bound follows immediately from the third triangulating condition, and can be boosted up to \( \frac{a + b}{b - 1} \) if \((\frac{D}{m}; \frac{a}{m})\) satisfies the \( b \)th triangulating condition. To establish the upper bound, let \( j = b - 1 \). Note that \( j < D \),

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\[m \geq 2, \text{ and } \alpha > 3, \text{ so } (\delta; \alpha) \text{ is required to satisfy the } j\text{th triangulating condition, which in light of Proposition 4.22, part (ii), means that } S_j(\delta; \alpha) = S_j(\xi, (\xi)^2). \] Let

\[(x, y) := \begin{cases} 
\left(\frac{\alpha^2 - a^2}{3}, \frac{2a - 1}{3}\right) \text{ if } 2a \equiv c \equiv 1 \pmod{3}, \\
\left(\frac{2c^2 - a^2}{3}, \frac{a - 1}{3}\right) \text{ if } a \equiv 2c \equiv 1 \pmod{3}.
\end{cases}
\]

Recall that \(b^2 + 1 = ac\) and \(b^2 + c^2 + 1 = 3bc\), so \(c^2 a \geq b^2 c\), and

\[
b^2 x + c^2 y \geq \frac{b^2(2c - 1) + c^2(a - 1)}{3} = \frac{2b^2 c - b^2 + c(b^2 + 1) - c^2}{3} = \frac{3b^2 c - b^2 c + 1 + c}{3} > bc(b - 1),
\]

so

\[(x, y) \notin S_j\left(\frac{c}{b}, \left(\frac{c}{b}\right)^2\right) = S_j(\delta; \alpha),\]

which implies that

\[D < \frac{ax + cy}{b - 1} = \frac{1}{b - 1} \left(\frac{ac - a}{3} + \frac{2ac - c}{3}\right) = \frac{1}{b - 1} \frac{3ac - 3b + c}{3} = \frac{ac - b}{b - 1}.\]

- \((m, n) = (\frac{a + \epsilon b}{3}, \frac{c + \epsilon d}{3})\) and \(\frac{cm}{b} \leq D < \frac{m+n}{3} + \frac{\epsilon + 1}{m+n-\eta}\), where \(\epsilon \in \{1, 2\}\) and \(\eta \in \{4, 5\}\), with \(a + \epsilon b \equiv m + n - \eta \equiv 0 \pmod{3}\), assuming \(a \geq 2\). The lower bound was proved in Proposition 4.22, part (i). To establish the upper bound, let \(j = \frac{m+n-\eta}{3}\). Note that \(\alpha > \frac{\xi}{\alpha} > 3\), and

\[j = \frac{b(m + n - \eta)}{3b} < \frac{bm + dm - b\eta}{3b} = \frac{3cm - b\eta}{3b} < D - 1,
\]

so \((\delta; \alpha)\) is required to satisfy the \(j\)th triangulating condition, which in light of Proposition 4.22, part (ii), means that \(S_j(\delta; \alpha) = S_j(\xi, (\xi)^2)\). Note that \(m + n = b + \epsilon c\), and let

\[(x, y) := \begin{pmatrix} j, j \end{pmatrix} + (d - c, a - b) = \begin{cases} 
\left(\frac{c+2d-\eta}{3}, \frac{2a+b-\eta}{3}\right) \text{ if } \epsilon = 1,
\\
\left(\frac{2c+2d-\eta}{3}, \frac{a+4b-\eta}{3}\right) \text{ if } \epsilon = 2.
\end{cases}
\]

Since the Fibonacci numbers are periodic modulo 9, it is a simple matter to verify that \(x, y \in \mathbb{N}\). Then

\[3(b^2 x + c^2 y) = (b^2 + c^2)j + b^2(d - c) + c^2(a - b)
\]

\[= (3bc - 1)j + b(c^2 + 1) + b^2c + (b^2 + 1)c - c^2b
\]

\[= 3bcj - \frac{b + \epsilon c - \eta}{3} + b + c
\]

\[= 3bcj + \frac{2b + (3 - \epsilon c) + \eta}{3}
\]

\[> 3bcj,
\]

so \((x, y) \notin S_j\left(\frac{\xi}{\alpha}, (\frac{\xi}{\alpha})^2\right) = S_j(\delta; \alpha)\), which implies that

\[D < \frac{mx + ny}{j} = \frac{m + n}{3} + \frac{m(d - c) + n(a - b)}{3j} = \frac{m + n}{3} + \frac{(a + \epsilon b)(d - c) + (c + \epsilon d)(a - b)}{9j}
\]

\[= \frac{m + n}{3} + \frac{3(1 + \epsilon)(ad - bc)}{9j} = \frac{m + n}{3} + \frac{\epsilon + 1}{m + n - \eta}.
\]
\( (m, n) = (b^2, c^2) \), and \( bc \leq D < bc + \frac{1}{bc-1} \), assuming \( b \geq 2 \). The lower bound was proved in Proposition 4.22. To establish the upper bound, let \( j = bc - 1 \leq D - 1 \). Since \( \alpha \geq (\frac{5}{2})^2 > 3 \), we know \( (\delta; \alpha) \) is required to satisfy the \( j \)th triangulating condition, so \( S_j(\delta; \alpha) = S_j(\frac{\delta}{b}, (\frac{\delta}{b})^2) \).

Let
\[
(x, y) = (c(b - a) - 1, 2ab - 1),
\]
and note that
\[
b^2x + c^2y = b^2c(b - a) + 2abc - (b^2 + c^2) = b^2c(b - a) + 2(b^2 + 1)bc - (3bc - 1) = b^2c(b - a + 2b) - bc + 1 = bc(b - c + 1),
\]
so indeed
\[
D < \frac{bc(b - c + 1) + 1}{bc - 1} = bc + \frac{1}{bc - 1}.
\]

\( (m, n) = \ell(b^2, c^2) + (a^2, b^2 + 2) \), and \( \frac{cm}{b} \leq D < \frac{(c\ell + a + 2)(cm - bn)}{(c\ell + a + 2)b - c} \), provided \( b \geq 2 \), and \( (m, n) \neq (4, 27) \) — the \((4, 27)\) case will be treated separately. The lower bound was proved in Proposition 4.22. To establish the upper bound, let \( j = (c\ell + a + 2)b - c \). Note that \( \alpha \geq \frac{b^2 + 2}{\alpha^2} \geq 3 \) and
\[
D \geq \frac{cm}{b} = b(\ell + a) + \frac{a}{b} = j + (c - 2b) + \frac{a}{b} > j + 1,
\]
so \( (\delta; \alpha) \) must satisfy the \( j \)th triangulating condition, so \( S_j(\delta; \alpha) = S_j(\frac{\delta}{b}, (\frac{\delta}{b})^2) \). Since we have specifically excluded the case \( (m, n) = (4, 27) \) from consideration, we have \( m \geq b \), and so we may let
\[
(x, y) = (2c - 1, m - b) \in \mathbb{N}^2.
\]
Then
\[
c^2m - b^2 = c^2(b^2 + a^2) - b^2 = b^2c^2\ell + ac(b^2 + 1) - (ac - 1) = b^2c(\ell + a) + 1,
\]
and
\[
b^2x + c^2y = 2b^2c - b^2 + c^2m - bc^2 = bc(2b - c) + b^2c(\ell + a) + 1 = bcj + 1,
\]
so \((x, y) \notin S_j(\frac{\ell}{b}, (\frac{\ell}{b})^2) = S_j(\delta; \alpha) \), which implies that
\[
D < \frac{mx + ny}{j} = \frac{(2c - 1)m + (m - b)n}{j} = \frac{(n - 1 + 2c)m - bn}{(c\ell + a + 2)b - c} = \frac{(c\ell + a + 2)m - bn}{(c\ell + a + 2)b - c}.
\]

\( (m, n) = \ell(b^2, c^2) - (a^2, b^2 + 2) \) with \( b = 1 \), and \( \frac{mn}{c} \leq D < \frac{m(n - c)}{cm - b} \). Note that \((m, n) = (m, c^2m + 1) \). The lower bound was proved in Proposition 4.22. To establish the upper bound, let \( j = cm - b \). Then
\[
(j - 1)(j - 2) = (cm - 2)(cm - 3) = c^2m^2 - 5cm + 6 < c^2m(m - 1) = (m - 1)(n - 1)
\]
and
\[
\binom{j + 2}{2} = \frac{(cm)(cm + 1)}{2} = \frac{c^2m^2 + cm}{2} < \frac{(m + 1)(c^2m + 1)}{2} = \frac{(m + 1)(n + 1)}{2},
\]
so \( (\delta; \alpha) \) must satisfy the \( j \)th triangulating condition, which in light of Proposition 4.22, part (ii), means that \( S_j(\delta; \alpha) = S_j(1, 1) \). Now let \((x, y) = (n - c, 0) \). We have
\[
b^2(n - c) = c^2m + 1 - c > c(cm - b) = bcj,
\]
and
so we must have \((x, y) \not\in S_j\left(\frac{c}{b}, \left(\frac{c}{b}\right)^2\right)\), which implies that
\[
D < \frac{mx + ny}{j} = \frac{m(n - c)}{cm - b}.
\]

- \((m, n) = \ell(4, 25) - (1, 6)\), and \(\frac{2n}{5} \leq D < \frac{2n - 5m}{5\ell - 2}\). The lower bound was proved in Proposition 4.22, with \((b, c) = (2, 5)\). To establish the upper bound, let \(j = 10\ell - 4\). Note that \(\alpha > 3\) and
\[
D \geq \frac{2n}{5} = 10\ell - \frac{12}{5} > j + 1,
\]
so \((\delta; \alpha)\) must satisfy the \(j\)th triangulating condition, so \(S_j(\delta; \alpha) = S_j\left(\frac{c}{b}, \left(\frac{c}{b}\right)^2\right)\). Let \((x, y) = (25\ell - 16, 1)\). Then
\[
(b^2x + c^2y = 100\ell - 64 + 25 = 10(10\ell - 4) + 1 > bcj,
\]
so \((x, y) \not\in S_j\left(\frac{c}{b}, \left(\frac{c}{b}\right)^2\right)\), which implies that
\[
D < \frac{mx + ny}{j} = \frac{(25\ell - 16)(4\ell - 1) + (25\ell - 6)}{10\ell - 4} = \frac{2\ell n - 5m}{5\ell - 2}.
\]

- \((m, n) = \ell(b^2, c^2) - (a^2, b^2 + 2)\) with \(b \geq 5\), and \(\frac{bn}{c} \leq D < \frac{(\ell - z + 1)bn - 3cm}{(\ell - z + 1)c - 3b}\). The lower bound was proved in Proposition 4.22. To establish the upper bound, let
\[
j = (\ell - z + 1)c - 3b = bcl - ab - 3 + c - 3b = b(c\ell - a) - (a + 3).
\]
Note that \(\alpha > 3\) and
\[
D \geq \frac{bn}{c} = bcl - ab - \frac{b}{c} > j + 1,
\]
so \((\delta; \alpha)\) must satisfy the \(j\)th triangulating condition, so \(S_j(\delta; \alpha) = S_j\left(\frac{c}{b}, \left(\frac{c}{b}\right)^2\right)\). Let \((x, y) = (n - 3c, b - 1)\). Then
\[
b^2x + c^2y = b^2(c\ell - a - 1 - 3c) + c^2(b - 1)
\]
\[
= b^2c(\ell - a - 3) - b^2 - c^2 + bc^2
\]
\[
= b^2c(\ell - a - 3) - 3bc + 1 + bc^2
\]
\[
= bc(b(c\ell - a - 3) - 3 + c) + 1 = bcj + 1 > bcj,
\]
so \((x, y) \not\in S_j\left(\frac{c}{b}, \left(\frac{c}{b}\right)^2\right)\), which implies that
\[
D < \frac{mx + ny}{j} = \frac{m(n - 3c) + n(b - 1)}{j} = \frac{(m - 1 + b)n - 3cm}{j} = \frac{(\ell - z + 1)bn - 3cm}{(\ell - z + 1)c - 3b}.
\]

- If \((m, n) = \left(\frac{F_{4k}}{3}, \frac{F_{4k+4}}{3}\right)\), then \(\frac{F_{4k+2}}{3} \leq D < \frac{F_{4k+2}}{3} + \frac{1}{m+n-\eta}\), where \(\eta \in \{4, 5\}\) and \(\eta \equiv m + n \pmod{3}\). To establish the lower bound, it suffices to use the third triangulating condition, noting that \(m + n = F_{4k+2}\). To establish the upper bound, let \(j = \frac{m+n-n}{3} < D - 1\), so that \((\delta; \alpha)\) must satisfy the \(j\)th triangulating condition, which means \(S_j(\delta; \alpha) = S_j(\varphi^2; \varphi^4)\), since \(\frac{m}{n}\) is a fair approximation to the strongly triangulating ratio \(\varphi^4\). One may check that
\[
(x, y) := \frac{(j, j) + (m + n, n - 8m)}{3} = \left(\frac{4m + 4n - \eta}{9}, \frac{4n - 23m - \eta}{9}\right) \in \mathbb{N}^2.
\]
2
15 16
8 D < \frac{11m}{4} = \frac{11}{2}
3
22
8 = \frac{8m}{3} \leq D < \frac{19m}{7} = \frac{57}{7}
4
29
\frac{43}{4} = \frac{7m+2n}{8} \leq D < \frac{17m+n}{9} = \frac{97}{9}
5
36
\frac{107}{8} = \frac{7m+2n}{8} \leq D < \frac{25m+n}{12} = \frac{161}{12}
6
43
\frac{16}{3} = \frac{8m}{3} \leq D < \frac{33m+n}{15} = \frac{241}{15}
7
50
\frac{299}{16} = \frac{7m+5n}{16} \leq D < \frac{24m+3n}{17} = \frac{318}{17}
9k + 8 | 64k + 57
\frac{24k + \frac{64}{3}}{12k + 20} \leq D < \frac{32(2k+1)m+3n}{24k+20} = 24k + \frac{64}{3} + \frac{1}{12(6k+5)}
Table 4.2: Bounds on D-values for fair upper approximations to \( \frac{64}{9} \)

Now \( 1 + \varphi^4 = 3\varphi^2 \) and

\[ 3\varphi^6 = 3\varphi^2(3\varphi^2 - 1) = 9\varphi^4 - 3\varphi^2 = 8\varphi^4 - 1, \]

so

\[ x + \varphi y = \frac{(1 + \varphi^4)j + (m + n) + \varphi^4(n - 8m)}{3} = \varphi^2(j + n - \varphi^4m) > \varphi^2j, \]

so \((x, y) \notin S_j(\varphi^2; \varphi^4) = S_j(\delta; \alpha)\). Therefore,

\[ D < \frac{mx + ny}{j} = \frac{m + n}{3} + \frac{m(m + n) + n(n - 8m)}{m + n - \eta} = \frac{m + n}{3} + \frac{1}{m + n - \eta}, \]

since \(m(m + n) + n(n - 8m) = m^2 - 7mn + n^2 = 1\).

- \( \frac{n}{m} \) is a fair upper approximation to \( \frac{64}{9} \), and \( D \) is bounded as shown in Table 4.2. The table also explains which triangulating conditions to apply, in order to establish these bounds. For example, let us establish the upper bound in the generic case \((m, n) = (9k + 8, 64k + 57)\). Let \( j = 24k + 20 < D - 1 \); we know that \((\delta; \alpha)\) must satisfy the \( j \)th triangulating condition, and, since \( \frac{n}{m} \) is a fair approximation upper to a strongly pseudo-triangulating pair, we know that \( S_j(\delta; \alpha) = S_j\left(\frac{8}{3}; \frac{64}{9} + \varepsilon\right)\). Now let \((x, y) = (32(2k + 1), 3)\), and note that

\[ 32(2k + 1) \cdot 3^2 + 3 \cdot 8^2 = 3 \cdot 8 \cdot j, \]

so that \((x, y) \notin S_j\left(\frac{8}{3}; \frac{64}{9} + \varepsilon\right) = S_j(\delta; \alpha)\). Thus,

\[ D > \frac{mx + ny}{j} = \frac{32(2k + 1)m + 3n}{24k + 20}, \]

as claimed in the Table.

- \((m, n) = (m, 9m - 1)\), and \( \frac{8}{3} \leq D < \frac{mn - 6m + 1}{3m - 2} \). To establish the lower bound, use the third triangulating condition. To establish the upper bound, let \( j = 3m - 2 < (3m - \frac{1}{3}) - 1 = D - 1 \); so that \((\delta; \alpha)\) must satisfy the \( j \)th triangulating condition. Since \( \frac{n}{m} \) is a fair lower approximation to a strongly pseudo-triangulating pair, we must have \( S_j(\delta; \alpha) = S_j(3, 9 - \varepsilon)\). Let \((x, y) = (3, m - 1)\). Note that

\[ 3 + 9(m - 1) = 3(3m - 2) = 3j, \]

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so $(x, y) \notin S_j(3, 9 - \varepsilon) = S_j(\delta; \alpha)$. So

$$D < \frac{mx + ny}{j} = \frac{3m + (m - 1)n}{j} = \frac{mn - n + 3m}{3m - 2},$$

as desired.

It may be helpful to restate the bounds on $D$ in a more uniform way, in the cases where $|c^2m - b^2n| = 1$ and $m, n \geq 2$. Instead of requiring that $b \leq c$ and $m < n$, let us reverse the roles of $m$ and $n$, and of $b$ and $c$, and stipulate that $c^2m - b^2n = 1$. The key fact that makes this possible is the equation

$$\ell(b^2, c^2) - (a^2, b^2 + 2) = (\ell - 7)(b^2, c^2) + (c^2 + 2, d^2).$$

Rewriting things in this way, we have the following lemma:

**Lemma 4.24.** Suppose $a, b, c$ are consecutive odd-indexed Fibonacci numbers, with possibly negative indices. Let $(m, n) = \ell(b^2, c^2) + (a^2, b^2 + 2)$, where $\ell$ is any integer, as long as $m, n \geq 2$. Suppose $(D; \frac{n}{m})$ is weakly triangulating. Then

$$\frac{bn}{c} < \frac{cm}{b} \leq D < \frac{(\ell + a + \alpha)cm - \beta bn}{(\ell + a + \alpha)b - \beta c} = \frac{cm}{b} + \frac{1}{b} \cdot \frac{\beta}{(\ell + a + \alpha)b - \beta c},$$

where

$$(\alpha, \beta) = \begin{cases} (1, 3) & \text{if } a > b > c \geq 5, \\ (1, 2) & \text{if } (a,b,c) = (13,5,2), \\ (1,1) & \text{if } a > b \geq c = 1, \\ (5,2) & \text{if } (m,n) = (4,27), \\ (2,1) & \text{if } a \leq b \text{ and } (m,n) \neq (4,27). \end{cases}$$

If $D$ is rational, then its denominator is at least $b$.

**Proof.** The lower and upper bounds on $D$ itself are proved directly above. To prove the statement about the denominator of $D$, assume $b \geq 2$, since this statement is vacuous if $b = 1$. It suffices to note that there is no fraction of denominator less than $b$ in the interval

$$\left[ \frac{cm}{b}, \frac{cm}{b} + \frac{1}{b} \cdot \frac{\beta}{(\ell + a + \alpha)b - \beta c} \right].$$

This is obvious if

$$\frac{1}{b} \cdot \frac{\beta}{(\ell + a + \alpha)b - \beta c} \leq \frac{1}{b \cdot (b - 1)},$$

which (one can somewhat laboriously check) is the case in every instance except when $(z,a,b,c) = (13,5,2)$ and $\ell = -6$. But in that case, our bounds are $7\frac{1}{2} \leq D < 7\frac{7}{2}$, and there are no fractions of denominator less than $5$ in that interval anyway. \[\square\]

**Proposition 4.25.** If $(\delta; \frac{n}{m})$ is weakly triangulating for an integer $m\delta \in \mathbb{N}$, and $m, n \geq 2$, then $(\delta; \frac{m\delta}{n})$ is in the following list, with one caveat: recall that if $(\delta; \frac{n}{m})$ is weakly triangulating, then so is $(\frac{m\delta}{n}, \frac{m}{n})$; the list below may list just one of these two. Let $z, a, b, c$ be consecutive odd-indexed Fibonacci numbers, possibly with negative indices.

1. $(\delta; \frac{n}{m}) = (\delta; \frac{\delta^2}{b^2})$, provided that $b, c \geq 2,$
2. \((\delta; \frac{n}{m}) = (\frac{b}{a}, \frac{c}{a})\), provided that \(a, c \geq 2\),

3. \((\delta; \frac{n}{m}) = (1; \frac{m-1}{m})\),

4. \((\delta; \frac{n}{m}) = (2; \frac{4m-1}{m})\),

5. \((\delta; \frac{n}{m}) = (\frac{8}{3}; \frac{22}{3})\),

6. \((\delta; \frac{n}{m}) = (\frac{8}{3}; \frac{43}{6})\).

**Proof.** This follows directly from the results outlined in this section. Lemma 4.3 may be useful in the verification process. \(\square\)

### 4.5 Weakly triangulating sequences of length 2

In this section we will prove some necessary conditions for a sequence \((\delta; \frac{n_1}{m_1}, \frac{n_2}{m_2})\) to be weakly triangulating. The main results are Proposition 4.33 and Lemmas 4.34, 4.35, and 4.39. In fact, computer tests seem to suggest that we do not have too many false positives, so our enumeration can be considered as a reasonable step towards a full classification, if one is interested in classifying these sequences. We will write \(\frac{n}{m} = \frac{n_1}{m_1}\) and \(g_3 = mn + \frac{n_2}{m_2}\) throughout. Evidently, one necessary condition is that \((\frac{\delta}{m}, \frac{n}{m})\) be weakly triangulating, so the results of the previous section will be our starting point. If one is interested in strongly triangulating or pseudo-triangulating sequences, then the main results in this section are Lemmas 4.27, 4.28, and 4.37.

Here is one consequence of the results we are about to prove:

**Proposition 4.26.** If \((\delta; \frac{n_1}{m_1}, \frac{n_2}{m_2})\) is weakly triangulating for an integer \(m_2 \delta \in \mathbb{N}\), and \(m, n, m_2 \geq 2\), then \((\delta; \frac{n_1}{m_1}, \frac{n_2}{m_2})\) is in the following list. Let \(z, a, b, c\) be consecutive odd-indexed Fibonacci numbers, possibly with negative indices, and write \((\frac{n}{m}) = (m_1, n_1)\) for convenience.

1. \((\delta; \frac{n}{m}, \frac{n_2}{m_2}) = (\frac{cm}{b}, \frac{c^2 + b^2 + 2}{b^2 + d^2}, \frac{n_2}{m_2})\), for some \(\ell \in \mathbb{Z}\), as long as \(m, n \geq 2\).

2. \((\delta; \frac{n}{m}, \frac{n_2}{m_2}) = (\frac{cm}{b}, \frac{c^2 + b^2 + 2}{b^2 + d^2}, \frac{b^2 + z}{b})\), for some \(\ell \in \mathbb{Z}\), as long as \(m, n \geq 2\).

3. \((\delta; \frac{n}{m}, \frac{n_2}{m_2}) = (\frac{8n^2 + 4n + 1}{4n + 1}, \frac{n}{4n + 1}, \frac{(2n + 1)^2}{4n + 1})\), as long as \(n \geq 2\).

4. \((\delta; \frac{n}{m}, \frac{n_2}{m_2}) = (m; \frac{m - 1}{m}, \frac{mm^2 - 1}{m^2})\), as long as \(m \geq 3\) and \(m_2 \geq 2\).

5. \((\delta; \frac{n}{m}, \frac{n_2}{m_2}) = (\frac{m^2 + 1}{m}, \frac{m - 1}{m}, \frac{(m + 1)^2}{m})\), as long as \(m \geq 3\).

6. \((\delta; \frac{n}{m}, \frac{n_2}{m_2}) = (\frac{10}{3}; \frac{2}{3}, \frac{31}{6})\).

7. \((\delta; \frac{n}{m}, \frac{n_2}{m_2}) = (2m; \frac{4m - 1}{m}, \frac{mm^2 - 1}{m^2})\), as long as \(m, m_2 \geq 2\).

8. \((\delta; \frac{n}{m}, \frac{n_2}{m_2}) = (\frac{17}{4}; \frac{7}{2}, \frac{17}{4})\).

9. \((\delta; \frac{n}{m}, \frac{n_2}{m_2}) = (\frac{F_{4k+2}}{3}; \frac{F_{4k+4}/3}{4}, \frac{1}{3})\) for some \(k \geq 2\).

10. \((\delta; \frac{n}{m}, \frac{n_2}{m_2}) = (\frac{2F_{4k+2}}{3}; \frac{F_{4k+4}/3}{4}, \frac{1}{6})\) for some \(k \geq 2\).
Proof. We go through the lists in Proposition 4.33 and Lemmas 4.34, 4.35, and 4.39, and look for examples where \( m_2 \delta \) is an integer. This procedure is finite and trivial in the case of the three lemmas, among which only Lemma 4.39 yields any solutions, namely (9)-(10) in our list above. It remains to consider the list in Proposition 4.33, so let \((m, n) = \ell(b^2, c^2) + (a^2, b^2 + 2)\). By Lemma 4.24, we know that \( \delta \geq \frac{cm}{b} \), and \( \delta \) has denominator at least \( b \), so \( m_2 \geq b \).

Consider item (a) of the list in Proposition 4.33. Recall from Lemma 4.27 that \( S_j\left(\frac{cm}{b}; \frac{n}{m}, \frac{m_2}{m_2}\right) \) is strongly triangulating, so, by Lemma 4.3, if \( \frac{m_2}{m} = \frac{n}{m} \), then we must be in case (1).

If \( \frac{m_2}{m} \) is a fair approximation to \( \frac{n}{m} \), then it must be that \( S_j(\delta; \frac{n}{m}, \frac{m_2}{m}) = S_j\left(\frac{cm}{b}; \frac{n}{m}, \frac{m_2}{m}\right) \) whenever \( (\delta; \frac{n}{m}, \frac{m_2}{m}) \) satisfies the jth triangulating condition. If \( \frac{m_2}{m} = \frac{b + z}{b} \), then we must be in case (2) above. Otherwise, \( m_2 \geq b + 1 \) and \( \frac{m_2}{m} \neq \frac{n}{m} \). Let us consider two cases:

- If \( \frac{m_2}{m} < \frac{n}{m} \), then \( S_j\left(\frac{cm}{b}; \frac{n}{m}, \frac{m_2}{m}\right) \subset S_j\left(\frac{cm}{b}; \frac{n}{m}, \frac{m_2}{m}\right) \subset S_j(\delta; \frac{n}{m}, \frac{m_2}{m}) \) for all \( j \), so whenever \( (\delta; \frac{n}{m}, \frac{m_2}{m}) \) satisfies the jth triangulating condition, so does \( \left(\frac{cm}{b}; \frac{n}{m}, \frac{m_2}{m}\right) \). By Lemma 4.3, we must in fact have \( \delta = \frac{cm}{b} \), so \( b \mid m_2 \).

If \( b = 1 \), then we are in case (4) or (7).

If \( b > 1 \), then we claim there are in fact no fair lower approximations to \( \frac{m_2}{m} = \ell + \frac{a^2}{b^2} \) with denominator divisible by \( b \). It suffices to prove that \( \frac{a^2}{b^2} \) has no fair lower approximation of the form \( \frac{a}{b} \). For \( \frac{a}{c} < \frac{a}{b} < \frac{a^2}{b^2} \), this follows from Lemma 4.21; all the fair approximations to \( \frac{a^2}{b} \) in that interval are well understood. If \( c = 1 \), then nothing remains to be done. Otherwise, to eliminate the possibility that \( \frac{a}{b} < \frac{a}{c} \), let us calculate (from the continued fraction expansions) some consecutive fair lower approximations to \( \frac{a^2}{b^2} \) below \( \frac{a}{c} \). Let \( \epsilon \) be the unique number in \( \{1, 2\} \) such that \( 3 \mid z + \epsilon a, b + \epsilon c \). If \( a > b > c \), then

\[
\frac{(z + \epsilon a)/3}{(b + \epsilon c)/3} < \frac{a}{c}
\]

are two consecutive fair lower approximations to \( \frac{a^2}{b^2} \). If \( a < b < c \) and \( \epsilon = 2 \), then

\[
\frac{(a - z)/3}{(c - b)/3} < \frac{a}{c}
\]

are two consecutive fair lower approximations to \( \frac{a^2}{b^2} \). If \( a < b < c \) and \( \epsilon = 1 \), then

\[
\frac{(a - 2z)/3}{(c - 2b)/3} < \frac{(2a - z)/3}{(2c - b)/3} < \frac{a}{c}
\]

are three consecutive fair lower approximations to \( \frac{a^2}{b^2} \). In the three inequalities above, the smallest denominators (that is, \( \frac{b + \epsilon c}{3}, \frac{c - b}{3}, \) and \( \frac{c - 2b}{3} \)) are all less than \( b \), and none of the denominators is a multiple of \( b \). Therefore, \( \frac{a^2}{b^2} \) admits no fair lower approximations with denominator divisible by \( b \), as claimed.

- If \( \frac{m_2}{m} > \frac{n}{m} \), then since \( m_2 > b \), \( (\delta; \frac{n}{m}, \frac{m_2}{m}) \) must satisfy the triangulating condition \( cm \), so \( bg_3 \leq cm \), where \( g_3 = mn + \frac{m_2}{m_2} \) as usual. Then \( S_j\left(\frac{cm}{b}; \frac{n}{m}, \frac{m_2}{m}\right) \subset S_j\left(\frac{bg_3}{cm}; \frac{n}{m}, \frac{m_2}{m}\right) \subset S_j(\delta; \frac{n}{m}, \frac{m_2}{m}) \) for all \( j \), so whenever \( (\delta; \frac{n}{m}, \frac{m_2}{m}) \) satisfies the jth triangulating condition, so does \( \left(\frac{bg_3}{cm}; \frac{n}{m}, \frac{m_2}{m}\right) \).

By Lemma 4.3, we must in fact have

\[
\delta = \frac{bg_3}{cm} = \frac{b(mnm_2 + n_2)}{cmn_2},
\]

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so \( cm \mid b(mn_2 + n_2) \), and in particular \( m \divides n_2 \), so \( n_2 \geq m \). But if \( \frac{n_2}{m^2} \) is a fair upper approximation to \( \frac{n}{m} \) with \( n_2 \geq m \), then it must be the case that \( b^2n_2 - mn_2 = 1 \), and so \( m \nmid n_2 \) after all, a contradiction.

Now let us consider item (b) of the list in Proposition 4.33. If \( \frac{n}{m} = \frac{(b^2 + \epsilon)^2}{m} \), then by Lemma 4.3 we must have \( m_2\delta = \frac{bmn + b\epsilon + c}{c} \) and \( c = 1 \), and we are in case (3) or (5) above. If \( \frac{n}{m} \neq \frac{(b^2 + \epsilon)^2}{m} \), then by inspection, only two of the sporadic examples has integral \( m_2\delta \), namely cases (6) and (8) above.

Finally, item (c) of the list in Proposition 4.33 is easily eliminated by inspection.

4.5.1 Cases of the form \( \frac{n_2}{m_1} = \frac{\ell F_{k+1}^2 + F_{k+2}^2 + \ell^2}{\ell F_{k-1}^2 + F_{k-2}^2 + \ell^2} \)

Lemma 4.27. If \( a, b, c \) are consecutive odd-indexed Fibonacci numbers, then \( \left( \frac{a}{b} ; \frac{n}{m} ; \frac{m_2}{n_2} \right) \) is strongly triangulating when \( (m, n) = \ell(b^2, c^2) + (a^2, b^2 + 2) \) for some \( \ell \in \mathbb{Z} \) (provided that \( m, n \geq 2 \)). Moreover, if \( (D; \frac{n}{m}; \frac{m_2}{n_2}) \) is weakly triangulating with \( m_2 \geq 2 \), then \( \frac{n_2}{m_2} > \left\lfloor \frac{m}{b^2} \right\rfloor - 1 \).

Proof. Proposition 4.6 says \( \left( \frac{a}{b} ; c^2 ; \frac{m_2}{n_2} \right) \) is strongly triangulating. It follows from Proposition 4.16 that \( \left( \frac{a}{b} ; \frac{n}{m} ; \frac{m_2}{n_2} \right) \) is also strongly triangulating, because \( c^2m - b^2n = 1 \).

Now suppose for the sake of contradiction that \( (D; \frac{n}{m}; \frac{m_2}{n_2}) \) is weakly triangulating, with \( m_2 \geq 2 \), but there is an integer \( x \) such that

\[
\frac{n_2}{m_2} < x < \frac{m}{b^2} = \ell + \frac{a^2}{b^2}.
\]

Let us note that

\[
x - \ell = \left\lfloor \frac{a^2}{b^2} \right\rfloor - 1 = \begin{cases} 6 & \text{if } a > b > c, \\ 3 & \text{if } (a, b, c) = (2, 1, 1), \\ 0 & \text{if } a \leq b < c.
\end{cases}
\]

We may check that, in each of the three cases of \( (a, b, c) \) appearing in the above equation, we have

\[
x < \ell + \frac{a(b^2 + 2) - b^2}{b^2c} = \frac{2cm - b^2(\ell c + a + 1)}{b^2c}
\]

and

\[
x \leq \ell + \frac{b^2 - c + 3}{c^2} = \frac{n - c + 1}{c^2}.
\]

Let

\[
i := cx + \ell c + a + 1 = cx + \frac{n + c - 1}{c}
\]

and

\[
j := bi = bcx + b(\ell c + a + 1) < \frac{2cm}{b}.
\]

Note, by Lemma 4.27, that \( D \geq \frac{cm}{b} \), so

\[
j < 2D \leq m_2D,
\]

and

\[
2g_3 - jD > 2(mn + x) - cm \left( cx + \frac{n + c - 1}{c} \right)
\]

\[
= 2mn - m(n + c - 1) + (2 - c^2m)x
\]

\[
= m(n - c + 1 - c^2x) + 2x
\]

\[
\geq 2x > 0.
\]
Thus, \( \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) \) is required to satisfy the \( j \)th triangulating condition.

On the other hand, we claim that

\[
(b^2x(n + (c - 1)m), 1) \in S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) \setminus S_j \left( \frac{cm}{b}; \frac{n}{m}, \frac{m}{b^2} \right).
\]

Indeed,

\[
cmi = cm \left( cx + \frac{n + c - 1}{c} \right) = c^2mx + m(n + c - 1) = b^2nx + (c - 1)m + (mn + x),
\]

so

\[
b^2nx + (c - 1)m + g_3 \leq cmi \leq jD
\]

and

\[
b^2nx + (c - 1)m + \left( mn + \frac{m}{b^2} \right) > cmi = j \cdot \frac{cm}{b}.
\]

So we conclude that \( \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) \) cannot be triangulating, a contradiction. \( \square \)

The case \((m, n) = (2, 3)\) is special:

**Lemma 4.28.** The sequence \( \left( \frac{10}{3}; \frac{3}{2}, \frac{46}{9} + \varepsilon \right) \) is strongly pseudo-triangulating.

**Proof.** Note that

\[
S_j \left( \frac{10}{3}; \frac{3}{2}, \frac{46}{9} + \varepsilon \right) = \{(w, z) \in \mathbb{N} \times \mathbb{N} \mid w \neq 1 \text{ and } (9w + 100z < 30j \text{ or } 3w = 10j)\};
\]

we have to show that

\[
\#S_j \left( \frac{10}{3}; \frac{3}{2}, \frac{46}{9} + \varepsilon \right) = \binom{j + 2}{2}
\]

for all \( j \in \mathbb{N}. \) Let us induct by 10's. The base cases \( 0 \leq j < 10 \) can be checked by hand. If it holds for \( j - 10, \) then since

\[
z \geq 3 \text{ and } (w, z) \in S_j \left( \frac{10}{3}; \frac{3}{2}, \frac{46}{9} + \varepsilon \right) \iff (w, z - 3) \in S_{j-10} \left( \frac{10}{3}; \frac{3}{2}, \frac{46}{9} + \varepsilon \right) \text{ and } (w, z) \neq \left( \frac{10(j - 10)}{3}, 0 \right),
\]

there are

\[
\begin{pmatrix} (j - 10) + 2 \\ 10 \end{pmatrix} - \begin{pmatrix} 1 \text{ if } j \equiv 1 \pmod{3}, \\ 0 \text{ else} \end{pmatrix}
\]

elements \((w, z) \in S_j \left( \frac{10}{3}; \frac{3}{2}, \frac{46}{9} + \varepsilon \right) \) with \( z \geq 3. \) Meanwhile, there are

\[
\sum_{z=0}^{2} \left\lfloor \frac{30j - 100z}{9} \right\rfloor = \begin{cases} \frac{10j}{3} + \frac{(10j - 12)}{3} + \frac{(10j - 23)}{3} = 10j - 35 & \text{if } j \equiv 0 \pmod{3}, \\ \frac{10j - 1}{3} + \frac{(10j - 11)}{3} + \frac{(10j - 22)}{3} = 10j - 34 & \text{if } j \equiv 1 \pmod{3}, \\ \frac{10j - 2}{3} + \frac{(10j - 11)}{3} + \frac{(10j - 22)}{3} = 10j - 35 & \text{if } j \equiv 2 \pmod{3}. \end{cases}
\]

elements \((w, z) \in S_j \left( \frac{10}{3}; \frac{3}{2}, \frac{46}{9} + \varepsilon \right) \) with \( z < 3. \) So, in all cases, we see that the total cardinality of \( S_j \left( \frac{10}{3}; \frac{3}{2}, \frac{46}{9} + \varepsilon \right) \) is

\[
\begin{pmatrix} (j - 10) + 2 \\ 10 \end{pmatrix} + 10j - 35 = \begin{pmatrix} (j - 10) + 2 \\ 10 \end{pmatrix} - 1 + 10j - 34 = \begin{pmatrix} j + 2 \\ 10 \end{pmatrix},
\]

and induction is complete. \( \square \)
Corollary 4.29. If \((D; \frac{3}{2}; \frac{n_2}{m_2})\) or, equivalently, \((D; \frac{\varphi}{2}; \frac{n_2}{m_2})\), is weakly triangulating and \(m_2 \geq 2\), then \(\frac{n_2}{m_2}\) is either a fair approximation to \(3\) or a fair upper approximation to \(\frac{46}{9}\).

Proof. We know from Lemma 4.27 that \((3; \frac{\varphi}{2}, 3)\) is strongly triangulating, but that is the same thing as saying \((3; \frac{3}{2}, 3)\) is strongly triangulating.

If \(m_2 < 20\), then a computer test verifies that the corollary is true. If \(m_2 > 20\), then \(30D < 30 \cdot 4 < 20 \cdot 6 < 20g_3 \leq m_2g_3\), we have \((0, m_2) \notin S_{30}(D; \frac{3}{2}; \frac{n_2}{m_2})\), so \((D; \frac{3}{2}; \frac{n_2}{m_2})\) satisfies the \(j\)th triangulating conditions for all \(j \leq 30\). Then a computer test shows that either \(\frac{26}{9} < \frac{n_2}{m_2} < \frac{31}{10}\) or \(\frac{46}{9} < \frac{n_2}{m_2} < \frac{344}{67}\). In the first case, we have a fortiori \(\frac{5}{2} < \frac{n_2}{m_2} < 4\), and \(\frac{n_2}{m_2}\) must be a fair approximation to \(3\), by Proposition 4.19 and the fact that \((3; \frac{3}{2}, 3)\) is strongly triangulating. In the second case, we have a fortiori \(5 < \frac{n_2}{m_2} < \frac{36}{7}\), and \(\frac{n_2}{m_2}\) must be a fair upper approximation to \(\frac{46}{9}\), by the Lemma and Proposition 4.20.

Lemma 4.30. Let \(z, a, b, c\) be consecutive odd-indexed Fibonacci numbers, and let \(\ell \in \mathbb{Z}\). Let \((m, n) = (m_1, n_1) = (\ell(b^2, c^2) + (a^2, b^2 + 2))\) and \((m_2, n_2) = (m, (b\ell + z)^2) = (m, (\ell + 7)m - n)\) and

\[
D = \frac{bmn + b\ell + z}{cm} = \frac{g_3 + n}{c(b\ell + z)}.
\]

Suppose that \(m, n \geq 2\). Then:

i. \((D; \frac{n}{m}, \frac{n_2}{m_2})\) satisfies the \(j\)th triangulating condition for all

\[
j \leq \frac{cm^2}{b} = c \left( (b\ell + z)(m - 1) + \frac{1}{b} \right) < \frac{mg_3}{D}.
\]

ii. In fact, \((D; \frac{n}{m}, \frac{n_2}{m_2})\) satisfies the \(j\)th triangulating condition for all

\[
j \leq c(b\ell + z)m - bi,
\]

except for those \(j\) of the form

\[
j = c(b\ell + z)m - bi,
\]

for integers \(i\) such that \(1 \leq i \leq \frac{n}{c} = cl + a + \frac{1}{c}\), in which case

\[
S_j(D; \frac{n}{m}, \frac{n_2}{m_2}) = \left( \frac{j + 2}{2} \right) - 1.
\]

Proof. Note that \(b\ell + z = \frac{m_1 - 1}{b} > 0\), and \(cl + a = \frac{n - 1}{c} \geq 0\). Note also that \(-7b^2 + a^2 = -(c^2 + 2) < 0\), so \(\ell \geq -6\) in all cases, and

\[
m \geq -6b^2 + a^2 = b(z - 6b) - 1.
\]

i. First we claim that (i) holds when \(m = 1\). In that case, we can easily see that \(a = 1\) and \(\ell = 0\). (The cases where \(c = 1\) and \(\ell = -6\) fail because they force \(n\) to be 0, which we are assuming is not the case.) If \((z, a, b, c) = (2, 1, 1, 2)\), then \(n = 3\), \(g_3 = 7\), \(D = 5/2\), and \(cm^2/b = 2\). If \((z, a, b, c) = (1, 1, 2, 5)\), then \(n = 6\), \(g_3 = 7\), \(D = 13/5\), and \(cm^2/b = 5/2\). In both cases, we can directly verify that the \(j\)th triangulating condition holds for all \(j \leq 2\), which is enough.

So we may assume that \(m > 1\) below.
Step 1: under the assumption that \( m > 1 \), we claim that the \( j \)th triangulating condition for \((D; \frac{n}{m}, \frac{m_2}{m})\) holds for any given \( j < c(b\ell + z) \). Recall that \((\frac{cm}{b}; \frac{n}{m}, \frac{m}{b^2})\) is strongly triangulating and, therefore, satisfies the \( j \)th triangulating condition. It suffices to show, therefore, that \( \#S_j = \#T_j \), where

\[
S_j := \{ (X, Y, Z) \in \mathbb{N}^3 \mid Y < m \text{ and } mX + nY + g_3 Z \leq jD \},
\]
\[
T_j := \{ (X, Y, Z) \in \mathbb{N}^3 \mid Y < m \text{ and } mX + nY + (mn + \frac{m}{b^2}) Z \leq \frac{cmj}{b} \}.
\]

The equality of cardinalities obviously holds if the sets themselves are equal, so let us investigate the set differences. Suppose that \((X, Y, Z) \in S_j \setminus T_j \) or \((X, Y, Z) \in T_j \setminus S_j \).

Case 1: \( Z = 0 \). Then, since \( D > \frac{cm}{b} \), it must be the case that \((X, Y, 0) \in S_j \setminus T_j \), and

\[
\frac{cmj}{b} < mX + nY \leq \left( \frac{cm}{b} + \frac{1}{bcm} \right) j = jD,
\]

so\[
\frac{cm}{b} < \frac{mX + nY}{j} \leq \frac{cm}{b} + \frac{1}{bcm}.
\]

But \( \frac{cm}{b} + \frac{1}{bcm} = \frac{bmn + b\ell + z}{cm} \) is a fair upper approximation to \( \frac{cm}{b} \), so it must be that \( j \geq cm \). Since we're assuming that \( j < c(b\ell + z) \), this implies

\[
b^2\ell + a^2 = m < b\ell + z.
\]

We claim this implies \( b = 1 \). Indeed, if \( b > a \geq 1 \), then \( a^2 \geq z \) and, since \( m > 0 \), we must have \( \ell > 0 \), so \( b^2\ell + a^2 \geq b\ell + z \). And if \( a > b > 1 \), then one can easily check \( z > 6b \); then, since \( \ell \geq -6 \), we have

\[
-6b^2 + a^2 = b(z - 6b) - 1 \geq z - 6b,
\]

so \( -6b^2 + a^2 \geq -6b + z \), and \textit{a fortiori} we have \( m \geq b\ell + z \). So indeed this proves that we must have \( b = 1 \), which is to say \( (z, a, b, c) \in \{(5, 2, 1, 1), (2, 1, 1, 2)\} \).

If \((z, a, b, c) = (5, 2, 1, 1)\), then we have \( j < c(b\ell + z) = m + 1 \) and \( n = m - 1 \) and

\[
m < \frac{mX + nY}{j} \leq m + \frac{1}{m},
\]

so there is just one possibility:

\[
(X, Y, Z; j) = (2, m - 1, 0; m).
\]

If \((z, a, b, c) = (2, 1, 1, 2)\), then we have \( j < c(b\ell + z) = 2m + 2 \) and

\[
2m = \frac{cm}{b} < \frac{mX + nY}{j} \leq \frac{cm}{b} + \frac{1}{bcm} = 2m + \frac{1}{2m},
\]

so there are two possibilities:

\[
(X, Y, Z; j) \in \{(5, m - 1, 0; 2m), (7, m - 1, 0; 2m + 1)\}.
\]

Case 2: \( Z \geq 1 \). Then the mere fact that \((X, Y, 1) \in S_j \cup T_j \) implies that \((0, 0, 1) \in S_j \cup T_j \), so at the very least

\[
mn < jD.
\]

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Since \( c(b + z) - 3 = b(c + a) \geq 0 \), a little calculation shows

\[
mn - (c(b + z) - 3)D = \frac{(b^2m - 1)(c + a) + am}{cm} > 0.
\]

So it must be that \( j = c(b + z) - \epsilon \), with \( \epsilon \in \{1, 2\} \). Note that

\[
mX + nY + g_3Z \leq jD \iff mX + nY \leq (g_3 + n) - \epsilon D - Zg_3 = n - \epsilon D - (Z - 1)g_3,
\]

\[
mX + nY + \left( \frac{mn + \frac{m}{b^2}}{b} \right) Z \leq \frac{cmj}{b} \iff mX + nY \leq n - \frac{\epsilon cm}{b} - \frac{1}{b^2} - (Z - 1) \left( \frac{mn + \frac{m}{b^2}}{b} \right) < n - (Z - 1) \left( \frac{mn + \frac{m}{b^2}}{b} \right),
\]

so \( Z = 1 \) and \( Y = 0 \), and we can only have \((X, Y, Z) \in T_j \setminus S_j\) and

\[
\frac{\epsilon cm}{b} - \frac{1}{b^2} \leq n - mX < \epsilon D = \frac{\epsilon cm}{b} + \frac{\epsilon}{bcm},
\]

so

\[
\left\lfloor \frac{\epsilon cm}{b} - \frac{1}{b^2} \right\rfloor < \left\lfloor \frac{\epsilon cm}{b} + \frac{\epsilon}{bcm} \right\rfloor.
\]

We claim that \( b \mid \epsilon \). If not, then as \( \gcd(b, cm) = 1 \), we also have \( b \nmid \epsilon cm \), and \( b > 1 \), so

\[
\left\lfloor \frac{\epsilon cm}{b} - \frac{1}{b^2} \right\rfloor = \left\lfloor \frac{\epsilon cm}{b} \right\rfloor
\]

and

\[
\frac{\epsilon}{bcm} > \frac{1}{b},
\]

which is to say \( 1 \leq cm < \epsilon \). But then \( \epsilon = 2 \) and \( c = m = 1 \), so the properties of Fibonacci numbers are such that \( b \in \{2, 1\} \), and in both cases we indeed have \( b \mid \epsilon \mid 2 \), as claimed.

If \( b = 2 \), then \( \epsilon = 2 \), and we have two cases for \((z, a, b, c)\). If \((z, a, b, c) = (13, 5, 2, 1)\), then

\[
m - \frac{1}{4} \leq n - mX < m + \frac{1}{m}.
\]

If, on the other hand, \((z, a, b, c) = (1, 1, 2, 5)\), then

\[
5m - \frac{1}{4} \leq n - mX < 5m + \frac{1}{5m}.
\]

Both cases are impossible under our assumption that \( m > 1 \), since \( \gcd(m, n) = 1 \).

If \( b = 1 \), then again we have two cases for \((z, a, b, c)\). If \((z, a, b, c) = (5, 2, 1, 1)\), then \( n = m - 1 \) and

\[
\epsilon m - 1 \leq n - mX = m(1 - X) - 1,
\]

which implies \( \epsilon = 1 \) and \( X = 0 \):

\[(X, Y, Z; j) = (0, 0, 1; m)\).
\]

If, on the other hand, \((z, a, b, c) = (2, 1, 1, 2)\), then

\[
2\epsilon m - 1 \leq 4m - 1 - mX < 2\epsilon m + \frac{\epsilon}{2m},
\]

or in other words

\[
0 \leq (4 - 2\epsilon - X)m < 1 + \frac{\epsilon}{2m} \leq 2.
\]

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Since $m > 1$, we must have $X = 4 - 2\epsilon$, which implies

$$(X, Y, Z; j) \in \{(0, 0, 1; 2m), (2, 0, 1; 2m + 1)\}.$$  

At any rate, we have $\#(S_j \setminus T_j) = \#(T_j \setminus S_j)$ in all cases, so $\#S_j = \#T_j$, as desired.

Step 2. Having established that the $j$th triangulating condition holds for all $0 \leq j < c(b\ell + z)$, we induct in increments of $c(b\ell + z)$. Suppose that $c(b\ell + z) \leq j < \frac{cm^2}{b}$, and the $j$th triangulating condition holds for $j - c(b\ell + z)$. Note that for $Y, Z \geq 1$, we have

$$mX + nY + g_3Z \leq jD \iff mX + n(Y - 1) + g_3(Z - 1) \leq (j - c(b\ell + z))D.$$  

To ward off duplications, let us stipulate that $0 \leq X < n$. To show that the $j$th triangulating condition holds, it remains to show that the number of integer solutions $(X, Y, Z)$ to

$$0 \leq X < n, \quad \min\{Y, Z\} = 0, \quad mX + nY + g_3Z \leq jD$$  

equals

$$(j - c(b\ell + z) + 2) + \cdots + (j + 1) = 1 + jD - \frac{(m - 1)(n - 1)}{2} + \frac{bn}{cm}j - n.$$  

To that end, note that there are $n$ solutions to (4.5.1) with $Y = Z = 0$; namely, $X$ could be any integer from 0 to $n - 1$. So it suffices to prove two things: First, the number of solutions to (4.5.1) with $Z = 0$ is $1 + \lfloor jD \rfloor - \frac{(m - 1)(n - 1)}{2}$, and second, the number of solutions to (4.5.1) with $Y = 0$ is $\left\lfloor \frac{bn}{cm}j \right\rfloor$. These are steps 3 and 4 below.

Step 3: we claim the number of solutions to (4.5.1) with $Z = 0$ is $1 + \lfloor jD \rfloor - \frac{(m - 1)(n - 1)}{2}$. Indeed, these solutions are in bijection with the elements of the semigroup generated by $\{m, n\}$ within the interval $[0, jD]$; since

$$jD \geq c(b\ell + z)D = g_3 + n > (m - 1)(n - 1),$$

exactly $\frac{(m - 1)(n - 1)}{2}$ integers in that interval are missing from the semigroup, as required.

Step 4: we claim the number of solutions to (4.5.1) with $Y = 0$ is $\left\lfloor \frac{bn}{cm}j \right\rfloor$. This is a bit more subtle. It suffices to prove, for $0 \leq X < n$ and $Z \in \mathbb{N}$, that

$$X + nZ < \frac{bn}{cm}j \iff mX + g_3Z \leq Dj.$$  

Let $r = bnj - cm(X + nZ)$, and substitute $j = \frac{cm(X + nZ) + r}{bn}$ to obtain (after some computation)

$$bn(mX + g_3Z - Dj) = n(bg_3 - cmD)Z - m(cD - bn)X - Dr = \frac{(b\ell + z)n}{m}Z - (b\ell + z)X - Dr,$$

and note that

$$cm = \frac{mD}{b\ell + z} + \frac{bn}{c(b\ell + z)},$$

so our task is to prove

$$r > 0 \iff nZ \leq mX + \left(\frac{cm}{c(b\ell + z)} - \frac{bn}{c(b\ell + z)}\right)r.$$  

(4.5.2)

Note that $r \equiv -cmX \pmod{n}$, so

$$X + cr \equiv (b^2n + 1)X + cr \equiv c^2mX + cr \equiv 0 \pmod{n},$$

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and in particular, the key point is that

\[ X + cr = 0 \quad \text{or} \quad |X + cr| \geq n. \]

Let us consider two cases depending on \( r \).

Case 1: \( r > 0 \). In this case evidently \( X + cr \geq n \), and (since we are assuming \( j \leq \frac{cm^2}{b} \))

\[ cm(X + nZ) < bnj \leq cm \cdot mn, \]

so \( Z < m \). Since

\[ bn < bn + 3c - b = c^2(b + z), \]

we have

\[ nZ \leq n(m - 1) \leq \left( m - \frac{bn}{c^2(b + z)} \right) n \]

\[ \leq \left( m - \frac{bn}{c^2(b + z)} \right) (X + cr) \leq mX + \left( cm - \frac{bn}{c(b + z)} \right) r, \]

so (4.5.2) holds.

Case 2: \( r \leq 0 \). Thanks to our lower bound on \( j \) and upper bound on \( X \), we have

\[ Z = \frac{bnj - cmX - r}{cmn} \geq \frac{bn \cdot c(b + z) - cm(m - 1) - 0}{cmn} = \frac{2cn}{cmn} > 0, \]

so \( Z \geq 1 \). We distinguish between two sub-cases:

Case 2a: \( \frac{bn}{c} < r \leq 0 \). Then \( -n \leq X - n < X + cr \leq X < n \), and \( X + cr \equiv 0 \pmod{n} \), so we must have \( X = -cr \). Since \( Z \geq 1 \), we have

\[ nZ - mX - \left( cm - \frac{bn}{c(b + z)} \right) r \geq n + cmr - cmr + \frac{bn}{c(b + z)} r \]

\[ \geq n - \frac{bn^2}{bn + (3c - b)} > 0, \]

so (4.5.2) holds, as required.

Case 2b: \( r \leq -\frac{bn}{c} \). Then the mere fact that \( X < n \) implies that

\[ nZ - mX - \left( cm - \frac{bn}{c(b + z)} \right) r \geq n - m(n - 1) + \left( cm - \frac{bn}{c(b + z)} \right) \frac{n}{c} \]

\[ \geq n - mn + m + mn - \frac{bn^2}{c^2(b + z)} \]

\[ = n + m - \frac{bn^2}{bn + (3c - b)} > 0, \]

so (4.5.2) holds, as required.

ii. In light of (i), we may assume \( j = c(b + z)m - i \), for some \( 0 \leq i \leq c(b + z) \). We know that

\[ j - c(b + z) \leq c(b + z)(m - 1) < \frac{cm^2}{b}, \]

so by (i) we have

\[ S_{j-c(b+z)}(D; \frac{n}{m}, \frac{n_2}{m_2}) = \left( j - c(b + z) + 2 \right). \]
In the proof of (i) above, the only place we used the upper bound on \( j \) was step 4, case 1, where we needed to show that \( Z < m \). So, in the case at hand, it suffices to find all pairs \((X, Z) \in \mathbb{N}^2\) with \( 0 \leq X < n \) and \( Z \geq m \), such that \((4.5.2)\) fails, which is to say, more precisely,

\[
 r := bnj - cm(X + nZ) > 0 \quad \text{and} \quad nZ > mX + \left( cm - \frac{bn}{c(bl + z)} \right) r, \quad (4.5.3)
\]

Well evidently, for this to be true, we must have

\[
 Z < \frac{bj}{cm} = b(bl + z) - \frac{b^2}{m} = m + 1 - \frac{b^2}{m},
\]

so we must have \( Z = m \) exactly. Plugging in for \( Z \) and \( j \), we have

\[
 r = cm(n - X) - bni
\]

and

\[
 nZ - mX - \left( cm - \frac{bn}{c(bl + z)} \right) r = \frac{bn}{c(bl + z)} \cdot \left( \left( c^2m(bl + z) - bn \right) i - cm^2(n - X) \right),
\]

so we are just looking for cases where

\[
 \frac{c}{b} - \frac{1}{bcm} = \frac{bn}{cm} < \frac{n - X}{i} < \frac{c^2m(bl + z) - bn}{cm^2} = \frac{c}{b} + \frac{1}{bcm^2}.
\]

Now note that \( \frac{c}{b} - \frac{1}{bcm} \) is a fair lower approximation to \( \frac{c}{b} \), and \( \frac{c}{b} + \frac{1}{bcm^2} \) is a fair upper approximation to \( \frac{c}{b} \). Therefore, \( \frac{c}{b} \) is the only fraction in the interval

\[
 \left( \frac{c}{b} - \frac{1}{bcm}, \frac{c}{b} + \frac{1}{bcm^2} \right)
\]

with numerator (when put in lowest terms) at most \( bn \). Since evidently

\[
 n - X \leq n \leq bn,
\]

we must have

\[
 \frac{n - X}{i} = \frac{c}{b},
\]

which is to say, \( X = n - \frac{ci}{b} \). So indeed \((X, Z) = (n - \frac{ci}{b}, m)\) is the unique pair satisfying \((4.5.3)\), as required.

**Lemma 4.31.** Let \( z, a, b, c \) be consecutive odd-indexed Fibonacci numbers, and let \( \ell \in \mathbb{Z} \). Let \((m, n) = (m_1, n_1) = \ell(b^2, c^2) + (a^2, b^2 + 2)\), and suppose that \( m, n \geq 2 \). Suppose that \((D; \frac{n}{m_1}, \frac{n}{m_2})\) is a weakly triangulating sequence with \( m_2 \geq 2 \) and

\[
 \max \left\{ \frac{bmn + bl + z}{cm}, \frac{g_3 + n}{c(bl + z)} \right\} \leq D < \frac{bg_3}{cm},
\]

where \( g_3 = mn + \frac{n}{m_2} \). Then of course \( \frac{n}{m_2} > \frac{bmn + bl + z}{b} - mn = \frac{bl + z}{b} \), and the following are true. (Parts (i)–(ix) are needed for the proof only. See (x)–(xiv) for the results.)

i. For every integer \( j \) with \( 0 \leq j \leq \frac{cmn^2}{b} \), we have \( \#S_j \left( D; \frac{n}{m}, \frac{n}{m_2} \right) \leq \binom{j+2}{2} \). That is to say, if \((D; \frac{n}{m}, \frac{n}{m_2})\) fails the \( j \)th triangulating condition, it can only be because \( S_j \left( D; \frac{n}{m}, \frac{n}{m_2} \right) \) is too small, not too large.
ii. The number \( \frac{m^2 \ell + \ell + b + 1}{b^2} = \ell + 7 - \frac{c^2}{b} \) is a fair lower approximation to \( \frac{(b \ell + z)^2}{m} \), and \( \frac{b^2 + \ell + b + 1}{b^2} \) is a fair upper approximation to \( \frac{m^2 \ell + \ell + b}{b^2} \). Moreover, \( \frac{(b \ell + z)^2}{m} \) has a fair lower approximation with denominator \( b \), namely \( \frac{b \ell + z}{b} \) if \( b > 1 \), or \( \ell + z + 1 \) if \( b = 1 \).

iii. There is no fraction \( \frac{a}{p} \) strictly between \( \frac{a}{m^2} \) and \( \frac{(b \ell + z)^2}{m} \) with \( p < \min\{m, m^2\} \).

iv. If \( m^2 \leq m \) and \( \frac{a}{m^2} < \frac{(b \ell + z)^2}{m} \), then \( m^2 \leq b^2 \), and \( \frac{a}{m^2} \) is either \( \frac{m^2 \ell + \ell + b}{b^2} = \frac{b^2 \ell + \ell + b + 1}{b^2} \) or a fair lower approximation thereto. If \( m^2 \leq m \) and \( \frac{a}{m^2} \geq \frac{(b \ell + z)^2}{m} \), then \( \frac{a}{m^2} \) is a fair upper approximation to \( \frac{m^2 \ell + \ell + b}{b^2} \).

v. If \( m > b \), then for each integer \( i \) with \( \max\{c, \frac{m}{c}\} < i < \min\{\frac{c(mn + b)}{b^2}, c(b \ell + z) - 1\} \), we have

\[
(\ell + 7 - \frac{c^2}{b})D.
\]

In particular

\[
(\ell + 7 - \frac{c^2}{b})D.
\]

vi. For each integer \( i \) with \( 1 \leq i \leq \min\{b \ell + z - 1, \frac{n^2}{m} + \frac{1}{cm}\} \), we have

\[
i(mn + b \ell + z) - cm > (icm - b)D.
\]

In particular,

\[
\frac{n^2}{m^2} - \frac{(b \ell + z)^2}{m} < \frac{b \ell + z}{b(mb - c)}.
\]

vii. Suppose \( i \) is an integer with \( 0 \leq i \leq \min\{m - 2, m - b, m + b - 1\} \) and, if \( a = 1 \), in fact \( i < m^2 - b - 1 \). Then

\[
i(g + 3 + n) + bmn + b \ell + z - cm > [ic(b \ell + z) + cm - b]D.
\]

In particular,

\[
\frac{(b \ell + z)^2}{m} - \frac{n^2}{m^2} < \frac{1}{icm}.
\]

viii. Assume that \( m^2 > m \) and \( i \in \mathbb{N} \) with \( 1 \leq i \leq \frac{m}{c} \). Then there exists exactly one pair \((W, Z) \in S_{\frac{c(b \ell + z)m - b}{m}}\) with

\[
0 \leq m^2n + (b \ell + z)^2 + m(n - c) - W - \left( m^2n + (b \ell + z)^2 \right) Z < \frac{i}{cm}.
\]

If \( i \leq c \), then moreover

\[
(W, Z) \in \{(m(n - c), m), (m(n - c) + m^2n + (b \ell + z)^2, 0)\}.
\]

ix. If \( m^2 > m \) and \( b \ell + z > 2 \) and \( m > \max\{b, 2\} \), then

\[
(m(n - c), m), (m(n - c) + m^2n + (b \ell + z)^2, 0) \notin S_{\frac{c(b \ell + z)m - b}{m}}\left( D; \frac{n}{m}, \frac{n^2}{m^2}\right)
\]

for each \( i \in \mathbb{N} \) such that

\[
\frac{m}{m - \max\{b, 2\}} < i < \frac{(b \ell + z)cm}{b}.
\]
If \( m_2 < b \), then \( \frac{n_2}{m_2} = \frac{b \ell + z}{m} + \frac{1}{bm_2} \), which implies \( m_2 \) is the unique integer in the interval \((0, b)\) with \( 3m_2 \equiv a \pmod{b} \).

xi. If \( b \leq m_2 \leq m \) and \( \frac{n_2}{m_2} < \frac{(b \ell + z)^2}{m} \), then these are the only possibilities:

\[-(z, a, b, c) = (34, 13, 5, 2), \quad \text{with } \ell \in \{-6, -5\}, \quad \text{and } \frac{n_2}{m_2} = \frac{6 \ell + 41}{6}.\]

\[-(z, a, b, c) = (13, 5, 2, 1) \quad \text{and } \frac{n_2}{m_2} = \frac{4 \ell + 27}{4}.\]

xii. If \( b \leq m_2 \leq m \) and \( \frac{n_2}{m_2} > \frac{(b \ell + z)^2}{m} \), then \( c = 1 \) and \( \frac{n_2}{m_2} = \frac{(b \ell + z)^2 - (b^2 \ell + bz + 1)}{m - b^2} \).

xiii. If \( m_2 > m \) and \( c > 1 \), then these are the only possibilities:

\[-(z, a, b, c) = (34, 13, 5, 2), \quad \ell = -6, \quad \frac{n}{m} = \frac{3}{19} \quad \text{and } \frac{n_2}{m_2} = \frac{27}{32}.\]

\[-(z, a, b, c) = (2, 1, 1, 2), \quad \ell = 1, \quad \frac{n}{m} = \frac{5}{2} \quad \text{and } \frac{n_2}{m_2} \in \left\{ \frac{13}{6}, \frac{17}{6}, \frac{21}{6}, \frac{25}{6} \right\}.\]

\[-(z, a, b, c) = (1, 2, 5, 13), \quad \ell = 0, \quad \frac{n}{m} = \frac{27}{4}, \quad \text{and } \frac{n_2}{m_2} \in \left\{ \frac{5}{19}, \frac{6}{23} \right\}.\]

xiv. If \( m_2 > m \) and \( c = 1 \), then these are the only possibilities:

\[-\frac{n}{m} = \frac{2}{3} \quad \text{and } \frac{n_2}{m_2} \text{ is a fair upper approximation to } \frac{46}{9}.\]

\[-\frac{n}{m} = \frac{3}{4} \quad \text{and } \frac{n_2}{m_2} = \frac{341}{5}.\]

**Proof.** i. Suppose for the sake of contradiction that, for such \( j \), we had

\[\#S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) > \binom{j + 2}{2}.\]

Then, since \( (D; \frac{n}{m}, \frac{n_2}{m_2}) \) is weakly triangulating, and

\[D \geq \frac{bm_2 + b \ell + z}{cm} = \frac{cm}{b} + \frac{1}{bcm} > \frac{j}{m_2},\]

it must be the case that

\[\{(W, Z) \in W_2 \times \mathbb{N} \mid W + Zg_3 < m_2g_3\} < \binom{j + 2}{2}.\]

In particular,

\[(0, m_2) \in S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right).\]

Now the assumption that \( cmD < bg_3 \) tells us that

\[\frac{cmD}{b} \cdot m_2 < m_2g_3 \leq JD,\]

which contradicts our assumption that \( j \leq \frac{cmD}{b} \).

ii. Note that

\[
\frac{(b \ell + z)^2}{m} = \ell + \frac{(bz + 1) \ell + z^2}{b^2 \ell + a^2},
\]

so

\[
\frac{(b \ell + z)^2}{m} - \left( \ell + \frac{b \ell + 1}{b^2} \right) = \frac{(bz + 1) \ell + z^2}{b^2 \ell + a^2} - \frac{b \ell + 1}{b^2} = \frac{(bz)^2 - a^2(bz + 1)}{b^2 m} = \frac{1}{b^2 m},
\]
so $\frac{b^2t+2z+1}{b^2}$ and $\frac{(b^2t+z)^2}{m^2}$ are fair approximations to each other. If $b = 1$, then this is all there was to prove for part (ii). Otherwise, note that in all the cases we have to consider $m \geq b - 1$, so if $b > 1$, then

$$\frac{(bl+z)^2}{m} - \frac{bl+z}{b} = \frac{bl+z}{bn} \leq \frac{m+1}{b^2m} \leq \frac{1}{b(b-1)},$$

which easily implies that $\frac{bl+z}{b}$ is a fair lower approximation to $\frac{(bl+z)^2}{m}$. Indeed, if there existed some fraction $\frac{q}{p} < \frac{bl+z}{b}$ and $p < b$, then

$$\frac{1}{bp} \leq \frac{q}{p} - \frac{bl+z}{b} < \frac{(bl+z)^2}{bm} - \frac{bl+z}{b} \leq \frac{1}{b(b-1)},$$

which is a contradiction.

iii. Suppose for the sake of contradiction that such a fraction $\frac{q}{p}$ exists; without loss of generality we may take $p$ minimal, and $\frac{q}{p}$ to be a fair approximation to $\frac{(bl+z)^2}{m}$. Let us consider two cases, depending on whether $\frac{n_2}{m_2}$ is less than or greater than $\frac{(bl+z)^2}{m}$.

Case 1: We have

$$\frac{n_2}{m_2} < \frac{q}{p} < \frac{(bl+z)^2}{m}.$$  

This inequality, along with our assumption that $\frac{bmn+bl+z}{cm} \leq D$, implies that

$$S_j\left(\frac{bmn+bl+z}{cm}; \frac{n}{m}, \frac{(bl+z)^2}{m}\right) \subseteq S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right)$$

for all $j \in \mathbb{N}$. Let $i \in \mathbb{N}$ be smallest integer such that $W := i(bmn+bl+z) \geq mnp+q$; that is to say,

$$i = \left[\frac{mnp+q}{bmn+bl+z}\right] \leq \left[\frac{(mn+(bl+z)^2)p}{bmn+bl+z}\right] = \left[p \left(\frac{1}{b} + \frac{bl+z}{bm(bmn+bl+z)}\right)\right] \leq \left[p + \frac{1}{b}\right] \leq \left[\frac{m}{b}\right] \leq bl+z.$$

Then let $j = icm \leq (bl+z)cm$. Note that $cm = \frac{b^2n+1}{c} > \frac{n}{c}$, so

$$\frac{(bl+z)cm-j}{b} = \frac{(bl+z-i)cm}{b}$$

cannot be a positive integer less than or equal to $\frac{n}{c}$. It follows from Lemma 4.30, part (ii), that the $j$th triangulating condition holds for $\left(\frac{bmn+bl+z}{cm}; \frac{n}{m}, \frac{(bl+z)^2}{m}\right)$. Now we have

$$W = j \cdot \frac{bmn+bl+z}{cm} \leq jD,$$

so

$$(W,0) \in S_j\left(\frac{bmn+bl+z}{cm}; \frac{n}{m}, \frac{(bl+z)^2}{m}\right) \subseteq S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right),$$

and, due to our assumption that $\frac{n_2}{m_2} < \frac{q}{p} < \frac{(bl+z)^2}{m}$,

$$(W-(mnp+q),p) \in S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right) \setminus S_j\left(\frac{bmn+bl+z}{cm}; \frac{n}{m}, \frac{(bl+z)^2}{m}\right).$$
Thus, \( \# \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) > \binom{j+2}{2} \), so by part (i) we must have

\[
m_2 < \frac{bj}{cm} = bi.
\]

Since \( \frac{bf+z}{b} < \frac{n_2}{m_2} < \frac{q}{p} \), it follows from part (ii) that \( \frac{n_2}{m_2} \) lies between \( \frac{q}{p} \) and some other fair lower approximation to \( \frac{(bf+z)^2}{m} \) having denominator at least \( b \), so by Corollary 4.14,

\[
m_2 \geq b + p.
\]

Thus, \( b + p \leq m_2 < bi \), or in other words

\[
\frac{p}{b} + 1 < i \leq \left\lfloor \frac{p+1}{b} \right\rfloor,
\]

which is patently absurd.

Case 2: We have

\[
\frac{(bl+z)^2}{m} < \frac{q}{p} < \frac{n_2}{m_2}.
\]

Let

\[
j = pc(bl+z) \leq (m-1)c(bl+z) < \frac{cm^2}{b},
\]

so \( \left( \frac{bmn+bl+z}{cm}, \frac{n}{m}, \frac{(bl+z)^2}{m} \right) \) satisfies the \( j \)th triangulating condition, by Lemma 4.30, part (i). Note that

\[
np + p \left( mn + \frac{(bl+z)^2}{m} \right) = p(bl+z) \cdot \frac{bmn + bl+z}{m} = j \cdot \frac{bmn + bl+z}{cm},
\]

and \( np + pg3 \leq jD \) by assumption, so

\[
(np,p) \in S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) \cap S_j \left( \frac{bmn + bl+z}{cm}; \frac{n}{m}, \frac{(bl+z)^2}{m} \right).
\]

Now \( p < m \), so

\[
(p+1) \left( mn + \frac{(bl+z)^2}{m} \right) = j \cdot \frac{bmn + bl+z}{m} - np + mn + \frac{(bl+z)^2}{m} > j \cdot \frac{bmn + bl+z}{m},
\]

so

\[
(0, p+1) \notin S_j \left( \frac{bmn + bl+z}{cm}; \frac{n}{m}, \frac{(bl+z)^2}{m} \right).
\]

A little thought shows that, due to the assumption that \( \frac{(bl+z)^2}{m} < \frac{n_2}{m_2} \), we have

\[
S_j \left( \frac{bmn + bl+z}{cm}; \frac{n}{m}, \frac{(bl+z)^2}{m} \right) \subset S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right).
\]

But since \( \frac{(bl+z)^2}{m} < \frac{q}{p} < \frac{n_2}{m_2} \), we have

\[
((m+1)p + q, 0) = (mnp + q + np, 0) \in S_j \left( D; \frac{n_2}{m_2} \right) \setminus S_j \left( \frac{bmn + bl+z}{cm}; \frac{n}{m}, \frac{(bl+z)^2}{m} \right).
\]
Thus, \( \#(D; \frac{n}{m}, \frac{n_2}{m_2}) > \binom{j+2}{2} \), so by part (i) we have
\[
m_2 < \frac{b_j}{cm} = \frac{pb(b\ell + z)}{m} = \frac{p(m + 1)}{m} = p + \frac{p}{m} < p + 1,
\]
which contradicts our assumptions.

iv. Assume \( m_2 \leq m \). The case where \( \frac{n_2}{m_2} > \frac{(b\ell + z)^2}{m} \) follows directly from parts (ii) and (iii), so suppose \( \frac{n_2}{m_2} < \frac{(b\ell + z)^2}{m} \). Then by (ii), \( \frac{(b\ell + z)^2}{m} \) is a fair upper approximation to \( \frac{b^2\ell + bz + 1}{b^2} \), so, as \( m_2 \leq m \), we must have
\[
\frac{n_2}{m_2} \leq \frac{b^2\ell + bz + 1}{b^2},
\]
By (iii), then, we must have \( m_2 = \min\{m, m_2\} \leq b^2 \), and \( \frac{n_2}{m_2} \) is either \( \frac{b^2\ell + bz + 1}{b^2} \) or a fair approximation thereto.

v. Let \( j := ib - c \). Note that
\[
j = ib - c \leq \frac{c(mn_2 + b)}{b} - c = \frac{cmn_2}{b} \leq \frac{cm^2}{b},
\]
so Lemma 4.30 tells us that \( \left( \frac{bmn + b\ell + z}{cm} ; \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right) \) satisfies the \( j \)th triangulating condition. Moreover,
\[
j \cdot \frac{bmn + b\ell + z}{cm} = \left( \frac{c^2m^2 + 1}{cm} - bn - \frac{c(b\ell + z)}{cm} = icm + \frac{i - c(b\ell + z)}{cm} - bn < icm - bn,
\]
and
\[
j \cdot \frac{bmn + b\ell + z}{cm} \leq \frac{cmn_2}{b},
\]
so that
\[
(i cm - bn, 0), (0, m_2) \notin S_j \left( \frac{bmn + b\ell + z}{cm} ; \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right),
\]
even though
\[
icm - bn = (ic - n)m + (m - b)n \in W_2.
\]
Let us suppose for the sake of contradiction that \( icm - bn \leq (ib - c)D \), so that
\[
(i cm - bn, 0) \in S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) \backslash S_j \left( \frac{bmn + b\ell + z}{cm} ; \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right).
\]
Then part (iii) implies that
\[
S_j \left( \frac{bmn + b\ell + z}{cm} ; \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right) \subseteq S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right),
\]
which contradicts part (i).

So indeed \( icm - bn \leq (ib - c)D \), so
\[
\frac{n_2}{m_2} < c(b\ell + z)D - n - mn < c(b\ell + z) \cdot \frac{icm - bn}{ib - c} - bn(b\ell + z) = \frac{i(b\ell + z)}{ib - c} = \frac{b\ell + z}{b} + \frac{c(b\ell + z)}{b(ib - c)},
\]
as desired.
vi. Let \( j := icm - b \), and note that

\[
j \leq \min \left\{ \frac{(b\ell + z - 1)cm - b}{b}, \frac{cmm2}{b} \right\}
\]

\[
= \min \left\{ \frac{cm^2 - [cm(b - 1) + b^2]}{b}, \frac{cmm2}{b} \right\} \leq \frac{cm \cdot \min\{m,m2\}}{b},
\]

so Lemma 4.30 tells us that \( \left( \frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right) \) satisfies the \( j \)th triangulating condition. Note, too, that

\[
b(bmn + b\ell + z) = (b^2m)n + b(b\ell + z) = (c^2m - 1)m + (m + 1) = c^2m^2 + 1,
\]

so

\[
\frac{j \cdot bmn + b\ell + z}{cm} = i(bmn + b\ell + z) - \frac{b(bmn + b\ell + z)}{cm} < i(bmn + b\ell + z) - cm,
\]

and

\[
(i(bmn + b\ell + z) - cm, 0) \notin S_j \left( \frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right).
\]

Likewise,

\[
j \cdot \frac{bmn + b\ell + z}{cm} \leq \min\{m,m2\} \cdot \left( \frac{mn + b\ell + z}{b} \right) < \min\{m,m2\} \cdot \left( \frac{mn + (b\ell + z)^2}{m} \right),
\]

so

\[
(0, n), (0, m2) \notin S_j \left( \frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right).
\]

Let us suppose for the sake of contradiction that

\[
i(bmn + b\ell + z) - cm \leq (icm - b)D.
\]

Now \( bn + b\ell + z = dm \), so

\[
i(bmn + b\ell + z) - cm = [(b - 1)(n - 1) + 2c - 1 + (i - 1)(bn + d)]m + (m - bi)n \in W_2,
\]

and

\[
(i(bmn + b\ell + z) - cm, 0) \in S_j \left( \frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right).
\]

Then part (iii) implies that

\[
S_j \left( \frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right) \subseteq S_j \left( \frac{n}{m}, \frac{m2}{m} \right),
\]

which contradicts part (i).

So indeed

\[
i(bmn + b\ell + z) - cm > (icm - b)D > (icm - b) \cdot \frac{g_3 + n}{c(b\ell + z)},
\]

so

\[
(icm - b) \left( \frac{n2}{m2} - \frac{(b\ell + z)^2}{m} \right) < b \left( (m + 1)n + \frac{(b\ell + z)^2}{m} \right) - c^2(b\ell + z)m
\]

\[
= b(m + 1)n + \frac{(b\ell + z)(m + 1)}{m} - b(m + 1)n - (b\ell + z)
\]

\[
= \frac{b\ell + z}{m}.
\]
vii. Note the claim is vacuous if $m_2 < b$, so we may assume that $m_2 \geq b$.
Let $j := ic(b\ell + z) + cm - b$, and note that if $i < m - b$ then

\[ j \leq c(b\ell + z)(m - b - 1) + cm - b \]
\[ = c(b\ell + z)(m - 1) - c - b < c \left( (b\ell + z)(m - 1) + \frac{1}{b} \right) = \frac{cm^2}{b}, \]

whereas if $i = m - b$, then (by assumption) $b > 1$ and

\[ j = ic(b\ell + z) + cm - b = c(b\ell + z)m - c - b, \]

which is not of the form $c(b\ell + z)m - i'b$ for any $i' \in \mathbb{N}$. In either case, Lemma 4.30 says that

\[ \left( \frac{bmn+b\ell+z}{cm}, \frac{n}{m}, \frac{(b+z)^2}{m} \right) \]

satisfies the $j$th triangulating condition.

Note that

\[ \frac{bmn+b\ell+z}{cm} = \left( b(b\ell+z)n + \frac{(b+z)^2}{m} \right) + bmn + b\ell + z - \frac{b(bmn+b\ell+z)}{cm}, \]
\[ = i \left( mn + \frac{(b+z)^2}{m} + n \right) + bmn + b\ell + z - cm - \frac{1}{cm} \]
\[ = (i + b + 1) \left( mn + \frac{(b+z)^2}{m} \right) - (m - i)n - \frac{(b+z)^2}{m} - \frac{b\ell + z - cm}{cm} - \frac{1}{cm} \]
\[ < (i + b + 1) \left( mn + \frac{(b+z)^2}{m} \right) \leq (m + 1) \left( mn + \frac{(b+z)^2}{m} \right), \]

so

\[ (in + bmn + b\ell + z - cm, i), (0, m_2 + 1) \notin S_j \left( \frac{bmn+b\ell+z}{cm}, \frac{n}{m}, \frac{(b+z)^2}{m} \right), \]

and a fortiori $(0, m_2 + i) \notin S_j \left( \frac{bmn+b\ell+z}{cm}, \frac{n}{m}, \frac{(b+z)^2}{m} \right)$.

Suppose for the sake of contradiction that the inequality

\[ i(g_3 + n) + bmn + b\ell + z - cm > [ic(b\ell + z) + cm - b]D \]

fails for some $i \leq \min\{m - 2, m - b, m_2 - b\}$, and let $i$ be the minimal number for which it fails; thanks to part (vi), we know that $i \geq 1$. As

\[ in + bmn + b\ell + z - cm = ((b - 1)(n - 2) + 2(c - 1))m + (m - b + i)n \in W_2, \]

we have

\[ (in + bmn + b\ell + z - cm, i) \in S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) \setminus S_j \left( \frac{bmn+b\ell+z}{cm}, \frac{n}{m}, \frac{(b+z)^2}{m} \right). \]

Then part (iii), along with the fact that $(0, m_2 + i) \notin S_j \left( \frac{bmn+b\ell+z}{cm}, \frac{n}{m}, \frac{(b+z)^2}{m} \right)$, implies that

\[ S_j \left( \frac{bmn+b\ell+z}{cm}, \frac{n}{m}, \frac{(b+z)^2}{m} \right) \subseteq S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right). \]

Then, since $(D; \frac{n}{m}, \frac{n_2}{m_2})$ is weakly triangulating, by Lemma 4.2, it must be the case that

\[ \{ (W, Z) \in W_2 \times \mathbb{N} \mid W + Zg_3 < m_2g_3 \} < \left( \frac{j + 2}{2} \right), \]

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and in particular
\[(0, m_2) \in S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right).\]

Thus,
\[m_2 g_3 \leq jD = [ic(b\ell + z) + cm - b]D.\]

By minimality of \(i\), we may assume
\[(i - 1)g_3 + bmn + b\ell + z - cm > [(i - 1)c(b\ell + z) + cm - b]D,\]

and recall that
\[b g_3 = bmn + b \cdot \frac{n_2}{m_2} > bmn + b\ell + z.\]

So it must be that
\[(m_2 - b - i + 1)g_3 + cm = m_2 g_3 + (bmn + b\ell + z - bg_3) - [(i - 1)g_3 + bmn + b\ell + z - cm] \leq jD - [(i - 1)c(b\ell + z) + cm - b]D + cm = c(b\ell + z)D,

which, given that \(i \leq m_2 - b\), implies
\[ (cm, 1) \in S_{c(b\ell+z)} \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right), \]

and if \(a = 1\) then in fact \(i \leq m_2 - b - 1\), so
\[ (cm, 2) \in S_{c(b\ell+z)} \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right). \]

Recall that, under the assumption that \(0 \leq i \leq m_2 - b\), we have \(m_2 \geq \max\{m_2, b\}\), so
\[ \frac{cmn_2}{b} \geq cm \cdot \max\{1, \frac{2}{b}\} \geq c(b\ell + z)D, \]

and it is easy to see from our original assumptions on \(D\) that
\[ S_{c(b\ell+z)} \left( \frac{bmn + b\ell + z}{cm}, \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right) \subseteq S_{c(b\ell+z)} \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right), \]

so part (i) says we must in fact have
\[ S_{c(b\ell+z)} \left( \frac{bmn + b\ell + z}{cm}, \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right) = S_{c(b\ell+z)} \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right). \]

If \(a \geq 2\), then it is not hard to see that
\[ cm = \frac{b^2n + 1}{c} = \frac{(ac - 1)n + 1}{c} > \left( a - \frac{1}{c} \right) n \geq n, \]

so we must have
\[ cm + mn + \frac{(b\ell + z)^2}{m} > n + mn + \frac{(b\ell + z)^2}{m} = c(b\ell + z) \cdot \frac{bmn + b\ell + z}{cm}, \]
so
\[(cm, 1) \notin \mathcal{S}_{c(b\ell+z)} \left( \frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right) = \mathcal{S}_{c(b\ell+z)} \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right),\]

which means
\[cm + g_3 > c(b\ell + z)D,\]
a contradiction. And even if \(a = 1\), it is clear that
\[cm + 2 \left( mn + \frac{(b\ell + z)^2}{m} \right) > n + mn + \frac{(b\ell + z)^2}{m},\]
so
\[(cm, 2) \notin \mathcal{S}_{c(b\ell+z)} \left( \frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right) = \mathcal{S}_{c(b\ell+z)} \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right),\]
again a contradiction.
So indeed we know that
\[i(g_3 + n) + bmn + b\ell + z - cm > [ic(b\ell + z) + cm - b]D,\]
and recall that
\[D > \frac{bmn + b\ell + z}{cm},\]
so we have
\[i(g_3 + n) > [ic(b\ell + z) + cm - b] \cdot \frac{bmn + b\ell + z}{cm} - (bmn + b\ell + z - cm)\]
\[= i \left( mn + \frac{(b\ell + z)^2}{m} + n \right) - \frac{1}{cm},\]
which implies
\[\frac{(b\ell + z)^2}{m} - \frac{n_2}{m_2} < \frac{1}{icm}.\]

**viii.** Let’s write
\[Q := m^2n + (b\ell + z)^2 + m(n - ci) - W - \left( mn + \frac{(b\ell + z)^2}{m} \right) Z,\]
and let \(j := (b\ell + z)cm - bi\). Since \(m_2 > m\), we have
\[j < \frac{cmn_2}{b} < Dm_2,\]
and
\[jD < m_2 \cdot \frac{cmD}{b} < m_2g_3,\]
so \(D; \frac{n}{m}, \frac{n_2}{m_2}\) must satisfy the \(j\)th triangulating condition. Recall that \(b(b\ell + z) = m + 1\) and \(b^2n + 1 = c^2m\), so
\[(b\ell + z)cm \cdot \frac{bmn + b\ell + z}{cm} = (m + 1)mn + (b\ell + z)^2\]
\[b \cdot \frac{bmn + b\ell + z}{cm} = \frac{c^2m^2 + 1}{cm} = cm + \frac{1}{cm},\]
and consequently

\[ j \cdot \frac{bmn + b\ell + z}{cm} = m^2n + (b\ell + z)^2 + m(n - ci) + \frac{i}{cm}. \]

Thus, for any \((W, Z) \in S_j \left(D; \frac{n}{m}, \frac{n_2}{m_2}\right)\), to say that \(Q < \frac{1}{cm}\) is the same as saying

\[ (W, Z) \notin S_j \left( \frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right), \]

and by Lemma 4.30, part (ii), there is precisely one pair with this property. Fix \((W, Z)\) to be this particular pair.

If \(W \in \{(m(n - ci), m), (m(n - ci) + m^2n + (b\ell + z)^2, 0)\}\), then we have \(Q = 0\) exactly, as desired. Otherwise, by linearity we must have \(Q > 0\). Now \(Q\) is always an integer multiple of \(\frac{1}{m}\), so if \(Q > 0\), then \(\frac{1}{cm} > Q > \frac{1}{m}\), which is to say \(i > e\).

\[ i. \text{ Note that} \]

\[ \frac{m_2}{b} + \frac{b}{cm} > \frac{m + 1}{b} = b\ell + z, \]

so part (vi) must hold with \(i = b\ell + z - 1\), which means

\[ \frac{cm}{b} < \frac{bmn + b\ell + z}{cm} \leq D < \frac{(b\ell + z - 1)(bmn + b\ell + z) - cm}{(b\ell + z - 1)cm - b}, \]

so

\[ (\lambda cm - \mu b)D < \lambda(bmn + b\ell + z) - \mu cm \]

whenever

\[ \lambda \leq (b\ell + z - 1)\mu. \]

In particular, if \(b\ell + z \geq 2\), then \((\lambda, \mu) = (b\ell + z, i)\) satisfies the above inequality when \(i \geq 2\). We conclude that, for \(i, b\ell + z \geq 2\),

\[ D < \frac{(b\ell + z)(bmn + b\ell + z) - icm}{(b\ell + z)cm - ib} = \frac{(m + 1)mn + (b\ell + z)^2 - icm}{(b\ell + z)cm - ib}, \]

provided the denominator is positive. In particular,

\[ (m(n - ci) + m^2n + (b\ell + z)^2, 0) \notin S_{(b\ell + z)cm - ib} \left(D; \frac{n}{m}, \frac{n_2}{m_2}\right). \]

We also know that part (vii) must hold with \(i = m - \max\{b, 2\}\), so we have

\[ \frac{cm}{b} < \frac{bmn + b\ell + z}{cm} < \frac{g_3 + n}{c(b\ell + z)} \leq D < \frac{(m - \max\{b, 2\})(g_3 + n) + bmn + b\ell + z - cm}{(m - \max\{b, 2\})c(b\ell + z) + cm - b} < \frac{(m - \max\{b, 2\})(g_3 + n) - cm}{(m - \max\{b, 2\})c(b\ell + z) - b}, \]

and therefore

\[ (\lambda c(b\ell + z) - \mu b)D < \lambda(g_3 + n) - \mu cm \]

whenever

\[ \lambda < (m - \max\{b, 2\})\mu. \]

Therefore, as long as \(m > 2\) and \(i > \frac{m}{m - \max\{b, 2\}}\), we have

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\[ D < \frac{m(g_3 + n - ci)}{(b\ell + z)cm - ib}, \]

provided the denominator is positive. In particular,

\[ (m(n - ci), m) \not\in S_{(b\ell+z)cm-ib} \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right). \]

\[ x. \] The case \( \frac{n}{m} = \frac{4}{27} \) is the only case where \( m \leq b \), and must be dealt with separately. However, the problem is finite, and a computer check readily shows that \( \frac{n_2}{m_2} = \frac{1}{4} \) is the only possibility here, which agrees with the lemma as stated.

So assume \( m_2 < b < m \). We first claim that \( m > a \):

- If \( a > b > c > 1 \), then \( m \geq -6b^2 + a^2 \geq -6 \cdot (\frac{9a}{13})^2 + a^2 = \frac{10a_4}{169} \cdot a \geq \frac{10}{13} a > a \).
- If \( c = 1 \), then \( m = b^2n + 1 \geq 2b^2 + 1 > a \).
- If \( a \leq b \), then \( m \geq \max\{2, a^2\} > a \).

Thus, as \( m_2 \geq 2 \), we have

\[ [2m - (m + 1)]m_2 = 2(mm_2 + b) - b[(b\ell + z)m_2 + 2] = m_2(m - 1) > 2 \left( a - \frac{1}{c} \right) = \frac{2b^2}{c}, \]

so

\[ \frac{c(mm_2 + b)}{b^2} - \frac{c[(b\ell+z)m_2 + 2]}{2b} \geq 1. \]

Now let

\[ i := \left\lfloor \frac{c(mm_2 + b)}{b^2} \right\rfloor \leq \frac{c(mm_2 + b)}{b^2} < \frac{c(mb + b)}{b^2} = c(b\ell + z). \]

Note that

\[ c^2(b\ell + z + 1) = bc^2\ell + c(ab + 3) + c^2 = bc^2\ell + b(ac + 1) + (3c - b) + c^2 = bn + d + c^2, \]

so we have

\[ i \geq \frac{c[(b\ell + z)m_2 + 2]}{2b} \geq \frac{c(b\ell + z + 1)}{b} = \frac{bn + d + c^2}{bc} > \frac{n + c}{b} \]

and

\[ ib - c > \frac{c(b\ell + z)m_2}{2}, \]

so, by (v) we have

\[ \frac{n_2}{m_2} - \frac{b\ell + z}{b} < \frac{c(b\ell + z)}{b(ib - c)} < \frac{2}{bm_2}. \]

Recall that \( \frac{n_2}{m_2} > \frac{b\ell + z}{b} \), so we must have

\[ \frac{n_2}{m_2} - \frac{b\ell + z}{b} = \frac{1}{bm_2}, \]

as claimed. Thus, \( bm_2 - (b\ell + z)m_2 = 1 \), and in particular \( zm_2 \equiv -1 \pmod{b} \). If we multiply both sides by \(-a\) (which is relatively prime to \( b \)), then we have

\[ 3m_2 \equiv -3(bz - 1)m_2 \equiv -3a^2m_2 \equiv -a(3a - b)m_2 \equiv -azm_2 \equiv a \pmod{b}, \]

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xi. We first claim that $m_2 > b$. If $b = 1$, then we explicitly assumed $m_2 \geq 2$, and if $b > 1$, then $\frac{b_2 + z}{b}$ is a fair lower approximation to $\frac{(b_2 + z)^2}{m}$, by (ii), so as $\frac{b_2 + z}{b} < \frac{n_2}{m_2} < \frac{(b_2 + z)^2}{m}$, we must have $m_2 > b$.

Suppose for the moment that $c > 2$ and $m_2 \geq b + \frac{b}{c-1}$, and that either $a > 1$ or $m_2 \geq b + 1 + \frac{b+1}{c-1}$. Then $m_2 \leq c(m_2 - b)$, and either $a > 1$ or $m_2 \leq c(m_2 - b - 1)$. Since $m_2 \leq m$, we can find $i \in \mathbb{N}$ such that

$$\frac{m_2}{c} \leq i \leq m_2 - b,$$

and either $a > 1$ or $i \leq m_2 - b - 1$. In either case, the hypotheses of part (vii) are satisfied, so it must be that

$$\frac{1}{m_2} \leq \frac{(bl + z)^2}{m} - \frac{n_2}{m_2} < \frac{1}{icm},$$

a contradiction.

So one of three things must happen: $c = 1$, or $b < m_2 < b + \frac{b}{c-1}$, or both $a = 1$ and $b < m_2 < b + 1 + \frac{b+1}{c-1}$.

- If $z > a > b > c > 5$, then note that $m_2 < b + \frac{b}{c-1} \leq b + \max\{\frac{13}{4}, \frac{34}{12}, \frac{89}{33}, \ldots\}$, so it is not hard to see that, at any rate,

$$b + 1 \leq m_2 \leq b + 3.$$

Let $y := 3z - a$, and let $\varepsilon \in \{1, 2\}$ be such that $3 | a + \varepsilon b$, and note that $3 | y + \varepsilon z$ as well. We claim $\ell + \frac{(y + \varepsilon z)/3}{(a + \varepsilon b)/3}$ is the simplest fraction in the open interval $\left(\frac{b_2 + z}{b}, \frac{(b_2 + z)^2}{m}\right)$. Indeed,

$$\frac{y + \varepsilon z}{a + \varepsilon b} - \frac{z}{b} = \frac{1}{b(a + \varepsilon b)/3},$$

so $\frac{(y + \varepsilon z)/3}{(a + \varepsilon b)/3}$ is a fair upper approximation to $\frac{z}{b}$. Moreover,

$$8b^2 - (eb - c)(a + \varepsilon b) \geq 8b^2 - (2b - c)(a + 2b) = b(c + d) + 1 > 0,$$

so

$$\frac{b_2 + 1}{b^2} - \frac{y + \varepsilon z}{a + \varepsilon b} = \frac{eb - c}{b^2(a + \varepsilon b)} < \frac{1}{[(a + \varepsilon b)/3]^2},$$

and $\frac{(y + \varepsilon z)/3}{(a + \varepsilon b)/3}$ is a fair lower approximation to $\frac{b_2 + 1}{b^2}$. So, since $\frac{a + \varepsilon b}{3} < b^2$, we know from (ii) that $\ell + \frac{(y + \varepsilon z)/3}{(a + \varepsilon b)/3}$ is indeed the simplest fraction in the open interval $\left(\frac{b_2 + z}{b}, \frac{(b_2 + z)^2}{m}\right)$. It follows that in fact

$$m_2 - b \geq \frac{a + \varepsilon b}{3} - b = \frac{eb - c}{3} > 3,$$

a contradiction.

- If $(z, a, b, c) = (34, 13, 5, 2)$, note that $\ell \geq -6$. Part (iv) tells us that $\frac{n_2}{m_2}$ is either $\frac{b_2 + z}{b^2} = \ell + 6 + \frac{21}{25}$ or a fair lower approximation thereto. Now $\ell + 6 + \frac{21}{25}$ has these consecutive fair lower approximations:

$$\ell + 6 + \frac{4}{5} < \ell + 6 + \frac{5}{6} < \ell + 6 + \frac{21}{25}.$$
Recall that we are assuming that $b = 5 < m_2 < b + \frac{b}{c-1} = 10$, and $\frac{n_2}{m_2} > \frac{b_0 + z}{b} = \ell + 6 + \frac{4}{5}$, so the only possibility is $\frac{n_2}{m_2} = \ell + 6 + \frac{5}{6}$. But in this case, note that

$$cm + c(b_0 + z) - b = 60\ell + 401 < m_2 \frac{bmn + b_0 + z}{cm} \leq m_2 D,$$

so by Lemma 4.18, we must have

$$\frac{bmn + b_0 + z}{cm} \leq D < \frac{(bmn + b_0 + z) + (g_3 + n) - cm}{cm + c(b_0 + z) - b}.$$

Thus,

$$\frac{bmn + b_0 + z}{cm} < \frac{(g_3 + n) - cm}{c(b_0 + z) - b},$$

which can be simplified to

$$\frac{(b_0 + z)^2}{m} - \frac{1}{cm} < g_3 - mn = \frac{n_2}{m_2} = \ell + \frac{41}{6}$$

If we subtract $\ell$ from both sides, then multiply by $m$, we find that

$$171\ell + 1156 - \frac{1}{2} = (bz + 1)\ell + z^2 - \frac{1}{c} < \frac{41m}{6} = \left(171 - \frac{1}{6}\right)\ell + 1156 - \frac{7}{6},$$

which implies $\ell < -4$, as desired.

- If $(z, a, b, c) = (13, 5, 2, 1)$, then $\ell \geq -4$ and by part (iv), $m_2 \leq b^2 = 4$, and $\frac{n_2}{m_2}$ is either $\ell + \frac{27}{4}$ or a fair approximation thereto. Since we are assuming $m_2 > b$, we have $\frac{n_2}{m_2} \in \{\ell + \frac{20}{3}, \ell + \frac{27}{4}\}$.

But if $\frac{n_2}{m_2} = \ell + \frac{20}{3}$, then part (vii) must hold with $i = 1$, so

$$\frac{n + 1}{3m} = \frac{m - 3n}{3m} = \left(\ell + 7 - \frac{n}{m}\right) - \left(\ell + \frac{20}{3}\right) = \frac{(b_0 + z)^2}{m} - \frac{n_2}{m_2} < \frac{1}{cm} = \frac{1}{m},$$

which is a contradiction, as $n \geq 2$. So, in fact, we must have $\frac{n_2}{m_2} = \ell + \frac{27}{4}$, as claimed.

- If $b = 1$, then, by part (iv), $m_2 \leq b^2 = 1$, which is absurd.

- If $(z, a, b, c) = (1, 1, 2, 5)$, note that $\ell \geq 1$. Part (iv) tells us that $m_2 \leq b^2 = 4$ and $\frac{n_2}{m_2}$ is either $\ell + \frac{b^2 + b + 1}{b^2} = \ell + \frac{3}{2}$ or a fair lower approximation thereto, which implies $\frac{n_2}{m_2} = \ell + \frac{m_2 - 1}{m_2}$. Recall that we are assuming that $b < m_2 < b + 1 + \frac{b + 1}{c-1} = 3 + \frac{3}{4} < 4$, so we must have $\frac{n_2}{m_2} = \frac{3\ell + 2}{3}$. In that case, part (vii) must hold with $i = 1$, so

$$\frac{\ell + 1}{3m} = \left(\ell + 7 - \frac{n}{m}\right) - \left(\ell + \frac{2}{3}\right) = \frac{(b_0 + z)^2}{m} - \frac{n_2}{m_2} < \frac{1}{cm} = \frac{1}{5m},$$

which is impossible. So this case does not arise.

- If $z < a < b < c$, then we have $b < m_2 < b + \frac{b}{c-1} < b + 1$, which is absurd.
We consider three cases.

Case 1: Suppose \( c > 1 \), and that there exists an integer \( i \) with

\[
\frac{(b\ell + z)m_2 + b}{cm} \leq i \leq \frac{m_2}{b} + \frac{b}{cm}.
\]

Since \( b \leq m_2 \leq m \), we have \( b\ell + z = \frac{m+1}{b} > 1 \). Note that

\[
cm(b\ell + z - 1) - [(b\ell + z)m + b] = m((c - 1)(b\ell + z - 1) - 1) - b.
\]

We claim the right-hand side is non-negative. The only way it could fail to be so, under our assumption that \( c > 1 \), is if \( c = b\ell + z = 2 \), in which case \( m = b(b\ell + z) - 1 = 2b - 1 \). The fact that \( m \geq 2 \) implies that \( b > 1 \), so \((z,a,b,c) = (34,13,5,2)\) and \( \ell = -\frac{32}{9} \notin \mathbb{Z} \), which is impossible. So indeed, we always have

\[
\frac{(b\ell + z)m_2 + b}{cm} \leq \frac{(b\ell + z)m + b}{cm} \leq b\ell + z - 1,
\]

so we may assume

\[
\frac{(b\ell + z)m_2 + b}{cm} \leq i \leq \min \left\{ \frac{m_2}{b} + \frac{b}{cm}, b\ell + z - 1 \right\}.
\]

Then, by part (vi), we have

\[
\frac{1}{mm_2} \leq \frac{n_2}{m_2} - \frac{(b\ell + z)^2}{m} \leq \frac{b\ell + z}{m(icm - b)} \leq \frac{b\ell + z}{m(b\ell + z)m_2},
\]

which is absurd.

Case 2: Suppose \( c > 1 \), but there is no integer \( i \) with

\[
\frac{(b\ell + z)m_2 + b}{cm} \leq i \leq \frac{m_2}{b} + \frac{b}{cm}.
\]

Let

\[
i := \left\lfloor \frac{m_2}{b} + \frac{b}{cm} \right\rfloor \geq 1,
\]

so that

\[
b_i - \frac{b^2}{cm} \leq m_2 \leq \left\lfloor b(i + 1) - \frac{b^2}{cm} \right\rfloor \leq b(i + 1) - 1.
\]

We also must have

\[
\frac{(b\ell + z)m_2 + b}{cm} > i,
\]

so that

\[
icm - b < m_2 \leq b(i + 1) - 1,
\]

\[
icm - b < (b\ell + z)m_2 \leq (m + 1)(i + 1) - (b\ell + z),
\]

\[
i(c - 1)m - i < m + 1 + b - (b\ell + z) = \left( 1 - \frac{1}{b} \right) m + b + 1 - \frac{1}{b}.
\]

Since \( i \geq 1 \), we have

\[
\left( c - 2 + \frac{1}{b} \right) m < b + 2 - \frac{1}{b},
\]

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which, since \( m \geq \max\{b, 2\} \), implies \( c < 4 \), and so in fact \( c = 2 \), and

\[
i(m - 1) < \left(1 - \frac{1}{b}\right) m + b + 1 - \frac{1}{b}.
\]

If \((z, a, b, c) = (2, 1, 1, 2)\), then \( i = \left\lfloor \frac{m^2}{b} + \frac{b}{cm} \right\rfloor \geq m_z \geq 2 \), so

\[
2(m - 1) \leq 1,
\]

which is impossible. If, on the other hand, \((z, a, b, c) = (34, 13, 5, 2)\), then \( m \geq 19 \) and

\[
i < \frac{4m + 29}{5(m - 1)} < 2,
\]

so \( i = 1 \) and \( m < 34 \), so in fact \( \ell = -6 \) and \( \frac{m}{m} = \frac{3}{19} \). In this case we have

\[
8 < \frac{38 - 5}{4} = \frac{cm - b}{bl + z} < m_2 < b(i + 1) - \frac{b^2}{cm} = 10 - \frac{25}{38} < 10
\]

so \( m_2 = 9 \). But we know from parts (iii) and (iv) that \( \frac{n}{m} \) is a fair upper approximation to both \( \frac{(b^2 + z)^2}{m} \) and \( \ell + \frac{171}{29} \). Since \( m_2 \leq m \), this means in fact \( m_2 < m \) and \( \frac{n}{m} \) is among

\[
\frac{171}{25} < \frac{130}{19} < \frac{89}{13} < \frac{48}{7} < \frac{7}{1}.
\]

But evidently this is not the case, if \( m_2 = 9 \).

Case 3: Suppose that \( c = 1 \). By (iv), \( \frac{n}{m} \) is a fair upper approximation to

\[
\ell + 7 - \frac{c^2}{b^2} = \ell + 7 - \frac{1}{b^2},
\]

so it is not hard to see that \( m_2 \equiv 1 \equiv m \mod b^2 \) and

\[
\frac{n}{m} = \frac{b^2 \ell + bz + 1}{b^2} + \frac{1}{b^2 m_2},
\]

and consequently

\[
\frac{n_2}{m_2} - \frac{(b^2 + z)^2}{m} = \left( \frac{b^2 \ell + bz + 1}{b^2} + \frac{1}{b^2 m_2} \right) - \left( \frac{b^2 \ell + bz + 1}{b^2} + \frac{1}{b^2 m} \right) = \frac{1}{b^2} \left( \frac{1}{m_2} - \frac{1}{m} \right) = \frac{m - m_2}{b^2 m_2 m}.
\]

Let

\[
\ell := \frac{m_2 + 1}{b} - 1,
\]

and note that \( \ell \in \mathbb{N} \) and

\[
1 \leq \ell \leq \min \left\{ \frac{m + 1}{b} - 1, \frac{m_2}{b} + \frac{b}{m} \right\},
\]

so by part (vi) we have

\[
\frac{m - m_2}{b^2 m_2 m} = \frac{n_2}{m_2} - \frac{(b^2 + z)^2}{m} < \frac{b^2 + z}{m (im - b)} = \frac{b^2 + z}{m (im - b)}
\]

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which is to say
\[ \frac{im - b}{m - m_2} < \frac{b^2(b\ell + z)m_2}{m - m_2} \]

Thus, if \( m_2 = m - b^2x \), then
\[
0 < \frac{x}{m} \left( \frac{b^2(b\ell + z)m_2}{m - m_2} - (im - b) \right) = \frac{(b\ell + z)(m - b^2x)}{m} + \frac{bx}{m} - \frac{(m_2 + 1/b - 1)x}{m} \\
= b\ell + z - \frac{b(m + 1)x - bx}{m} - \frac{m + 1}{b} - bx^2 + x \\
= b\ell + z - bx - (b\ell + z)x + bx^2 + x \\
= (b\ell + z - bx)(1 - x) + x \\
= 1 - (b\ell + z - 1 - bx)(x - 1),
\]
so
\[
(b\ell + z - 1 - bx)(x - 1) < 1.
\]

Now we are assuming \( m_2 \geq b \), so
\[
b\ell + z - 1 - bx = b\ell + z - 1 + \frac{m_2 - m}{b} \geq b\ell + z - \frac{m}{b} = \frac{1}{b} > 0.
\]

Therefore, it must be that \( x = 1 \). Then \( m_2 = m - b^2 \) and
\[
\frac{m_2}{m} = \frac{b^2\ell + bz + 1}{b^2} + \frac{1}{b^2(m - b^2)} = \frac{(b\ell + z)^2 - (b^2\ell + bz + 1)}{m - b^2},
\]
as claimed.

**xiii. If** \( b\ell + z \geq 2, m \geq 3 \), and there is an integer \( i \) with
\[
\frac{m}{m - \max\{b, 2\}} < i \leq \min\{c, c\ell + a\},
\]
then we immediately have a contradiction between parts (viii) and (ix).

So one of three things must hold: \( b\ell + z = 1 \), or \( m = 2 \), or there is no integer \( i \) with
\[
\frac{m}{m - \max\{b, 2\}} < i \leq \min\{c, c\ell + a\}.
\]

If the last of these is true, then, under our assumption that \( c > 1 \), we must have either \( c\ell + a = 1 \) or \( m \leq \max\{2b, 4\} \).

- If \( z > a > b > c > 5 \), then (by plugging in \(-6\) for \( \ell \)) we easily see that \( b\ell + z \geq c\ell + a \geq 4 \) and \( m > 2b > 4 \), which we have shown to be impossible.

- If \((z, a, b, c) = (34, 13, 5, 2)\), then (by plugging in \(-6\) for \( \ell \)) we easily see that \( b\ell + z \geq 4 \) and \( m > 2b > 4 \). Thus, we must have \( c\ell + a \leq 1 \), which implies \( \ell = -6 \) and \( c\ell + a = 1 \). In that case \( \frac{m}{m} = \frac{3}{19} \) and \( b\ell + z = 4 \). A computer check shows that \( \frac{m}{m_2} = \frac{27}{32} \) is the only possibility.

- If \((z, a, b, c) = (2, 1, 1, 2)\), then \( b\ell + z = m + 1 \geq 3 \) and \( c\ell + a = 2m - 1 \geq 3 \). Let us consider four sub-cases:
  - If \( m = 2 \), then \( \frac{m}{m} = \frac{7}{2} \). A computer check shows that \( \frac{m}{m_2} \in \{13/3, 17/4, 21/5, 25/6\} \).
- If \( m = 3 \), then \( \frac{n}{m} = \frac{11}{3} \). A computer check shows this is impossible.
- If \( m = 4 \), then \( \frac{n}{m} = \frac{15}{4} \). A computer check shows this is impossible.
- If \( m \geq 5 \), then in fact \( m > 4 > 2b \), which we have shown to be impossible.

- If \((z, a, b, c) = (1, 1, 2, 5)\), then \( \ell = \frac{m-1}{2} > 0 \), so in fact \( \ell \geq 1 \). Then \( b\ell + z \geq 3 \), \( c\ell + z \geq 6 \), and \( m > 4 > 2b \), which we have shown to be impossible.

- If \((z, a, b, c) = (1, 2, 5, 13)\), then \( c\ell + a \geq 2 \), so either \( b\ell + z = 1 \) or \( m \leq 2b = 10 \). Both of these imply \( \ell = 0 \), in which case \( \frac{n}{m} = \frac{17}{4} \). A computer check shows that \( \frac{na}{m^2} \in \{ \frac{5}{19}, \frac{6}{23} \} \).

- If \( 2 \leq z < a < b < c \), then \( \ell \geq 0 \), so \( b\ell + z \geq 2 \), \( m \geq 25 \), and \( \min\{c\ell + a, c\} \geq a \geq 5 \), so we must have

\[
\frac{m}{m - \max\{b, 2\}} > 5,
\]

which is to say

\[
5b > 4m \geq 4a^2,
\]

which is clearly false.

xiv. The case \( \frac{n}{m} = \frac{5}{3} \) was dealt with in Corollary 4.29. Let us, therefore, assume that \( \frac{n}{m} \neq \frac{5}{3} \); then we have \( \frac{m}{m-2} < n \), and by part (viii) there's a unique pair \((W, Z) \in S_{c(b\ell+z)m-bn}(D; \frac{n}{m}, \frac{na}{m^2})\) with

\[
0 \leq m^2n + (b\ell + z)^2 - W - \left( mn + \frac{(b\ell + z)^2}{m} \right) Z < \frac{n}{m},
\]

so that in particular \( 0 \leq Z \leq m \). By part (ix) we know \((W, Z) \notin \{(m(n - ci), m), (m(n - ci) + m^2n + (b\ell + z)^2, 0)\}\), so that

\[
0 < m^2n + (b\ell + z)^2 - W - \left( mn + \frac{(b\ell + z)^2}{m} \right) Z < \frac{n}{m} = \frac{n}{b^2n + 1} < \frac{1}{b^2}
\]

and \( 0 < Z < m \). Recall that \( mn + \frac{(b\ell+z)^2}{m} = mn + \ell + 7 - \frac{n}{m} \), so

\[
W + \left( mn + \frac{(b\ell + z)^2}{m} \right) Z = m^2n + (b\ell + z)^2 - \left\{ \frac{nZ}{m} \right\},
\]

where \( \{ \frac{nZ}{m} \} \) denotes the fractional part of \( \frac{nZ}{m} \), so \( 0 < \{ \frac{nZ}{m} \} < \frac{n}{m} \). In particular, \( 1 < Z < m \), so

\[
\frac{Z - 1}{b^2} < \frac{nZ}{m} = Z - 1 + \frac{1}{b^2} \left( \frac{m - Z}{m} \right) < \frac{Z}{b^2}.
\]

Thus, in order for \( \frac{nZ}{m} \) to have fractional part less than \( \frac{1}{b^2} \), we must have \( Z \equiv 1 \pmod{b^2} \), so we may write \( Z = b^2Y + 1 \), with \( 1 \leq Y \leq n - 1 \). Then

\[
W + \left( mn + \frac{(b\ell + z)^2}{m} \right) Z = m^2n + (b\ell + z)^2 - \frac{n - Y}{m}.
\]

Let

\[
(W_1, Z_1) = ((b\ell + z - 3)(bnm + b\ell + z) - m + (b^2 + 1)n, b^2 + 1);
\]

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note that this satisfies

\[ m^2n + (b\ell + z)^2 - W_1 - \left( mn + \frac{(b\ell + z)^2}{m} \right) Z_1 = \frac{n - 1}{m}. \]

We claim that in fact \((W, Z) = (W_1, Z_1)\). If not, then we have \(Z > Z_1\), \(Y > 1\), and, by (vii),

\[
(W + g_3Z) - (W_1 + g_3Z_1) = -\frac{n - Y}{m} + \frac{n - 1}{m} + \left( \frac{n_2}{m_2} - \frac{(b\ell + z)^2}{m} \right) (Z - Z_1)
\]

\[
> \frac{Y - 1}{m} - \frac{b^2(Y - 1)}{(m - 2)m} = \frac{Y - 1}{m} \left( 1 - \frac{b^2}{b^2n - 1} \right) \geq 0.
\]

But then \((W_1, Z_1) \in S_{c_i(b\ell + z)m - bn} (D; \frac{n}{m}, \frac{n_2}{m_2})\) also, contradictory to the uniqueness of \((W, Z)\). Thus, we conclude that \((W, Z) = (W_1, Z_1)\), so

\[
D \geq \frac{W_1 + g_3Z_1}{c_i(b\ell + z)m - bn} = \frac{(b^2 + 1)(g_3 + n) + (b\ell + z - 3)(bmn + b\ell + z) - cm}{(b^2 + 1)c_i(b\ell + z) + (b\ell + z - 3)cm - b},
\]

according to a tedious calculation.

Meanwhile, note that

\[
((b\ell + z)(bm\ell + b\ell + z) - c^2m^2 - 2, b^2) \notin S_{c_i(b\ell + z)c_m} \left( \frac{bmn + b\ell + z}{cm}, \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right) = S_{c_i(b\ell + z)c_m} \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right)
\]

so after another tedious calculation

\[
D < \frac{(b\ell + z)(bm\ell + b\ell + z) - c^2m^2 - 2 + b^2g_3}{(b\ell + z)cm} = \frac{b^2(g_3 + n) + (b\ell + z - b)(bmn + b\ell + z) - cm}{b^2c_i(b\ell + z) + (b\ell + z - b)cm - b}.
\]

Recall from part (vii) that

\[
D < \frac{(m - 2)(g_3 + n) + bmn + b\ell + z - cm}{(m - 2)c_i(b\ell + z) + cm - b}.
\]

The time has come for us to consider the set \(R \subset \mathbb{R}^2\) of pairs \((\alpha, \beta)\) such that

\[
[\alpha c_i(b\ell + z) + \beta cm - b]D < \alpha (g_3 + n) + \beta (bmn + b\ell + z) - cm.
\]

According to the three inequalities, \(R\) supposedly includes \((b^2, b\ell + z - b)\) and \((m - 2, 1)\), but not \((b^2 + 1, b\ell + z - 3)\). Now \(R\) also includes \((0, 0)\) and is by nature convex, so we can obtain a contradiction by showing that \((b^2 + 1, b\ell + z - 3)\) lies in the convex hull of \((0, 0), (b^2, b\ell + z - b)\) and \((m - 2, 1)\). By Cramer’s rule,

\[
(b^2 + 1, b\ell + z - 3) = \frac{(m - 2)(b\ell + z - 3) - (b^2 + 1)}{(m - 2)(b\ell + z - b) - b^2}(b^2, b\ell + z - b) + \frac{(b^2 + 1)(b\ell + z - b) - b^2(b\ell + z - 3)}{(m - 2)(b\ell + z - b) - b^2}(m - 2, 1).
\]

- If \((z, a, b, c) = (5, 2, 1, 1)\) and \(n \geq 4\), then we have \(m = n + 1\) and \(b\ell + z = n + 2\), so

\[
(a, b\ell + z - 3) = \frac{n^2 - 2n - 1}{n^2 - 2}(b^2, b\ell + z - b) + \frac{n + 3}{n^2 - 2}(m - 2, 1).
\]

Since we’re assuming \(n \geq 4\), we have \(n^2 - 2n - 1 = (n - 1)^2 - 2 > 0\) and

\[
(n^2 - 2n - 1) + (n + 3) = n^2 - n + 2 \leq n^2 - 2,
\]

so \((a, b\ell + z - 3)\) lies in the convex hull of \((b^2, b\ell + z - b), (m - 2, 1),\) and \((0, 0)\), a contradiction.
If \((z, a, b, c) = (5, 2, 1, 1)\) and \(\frac{n}{m} = \frac{3}{4}\), then a computer test shows that \(\frac{n}{m^2}\) can be \(\frac{31}{8}\), but this is the only exception.

If \((z, a, b, c) = (13, 5, 2, 1)\), then we have \(m = 4n + 1\) and \(b\ell + z = 2n + 1\), so

\[
(a, b\ell + z - 3) = \frac{(4n - 1)(2n - 2) - 5}{(4n - 1)(2n - 1) - 4} (b^2, b\ell + z - b) + \frac{5(2n - 1) - 4(2n - 2)}{(4n - 1)(2n - 1) - 4} (m - 2, 1)
\]

\[
= \frac{8n^2 - 10n - 3}{8n^2 - 6n - 3} (b^2, b\ell + z - b) + \frac{2n + 3}{8n^2 - 6n - 3} (m - 2, 1).
\]

Now \((4n - 1)(2n - 2) - 5 > 7 \cdot 2 - 5 > 0\) and

\[
(8n^2 - 10n - 3) + (2n + 3) = 8n^2 - 8n < 8n^2 - 6n - 3,
\]

so \((a, b\ell + z - 3)\) lies in the convex hull of \((b^2, b\ell + z - b)\), \((m - 2, 1)\), and \((0, 0)\), a contradiction.

This completes the proof. \(\Box\)

**Lemma 4.32.** Let \(z, a, b, c, d\) be consecutive odd-indexed Fibonacci numbers, and let \(\ell \in \mathbb{Z}\). Let \((m, n) = (m_1, n_1) = \ell(b^2, c^2) + (a^2, b^2 + 2)\), and suppose that \(m, n \geq 2\). Suppose that \((D, \frac{m_1}{m_2}, \frac{n_2}{m_2})\) is a weakly triangulating sequence with \(m_2 \geq 2\) and

\[
D < \frac{g_3 + n}{c(b\ell + z)},
\]

where \(g_3 = mn + \frac{n_2}{m_2}\). Then here are the possibilities:

- \((z, a, b, c) = (34, 13, 5, 2)\), \(\ell = -6\), \(\frac{n}{m} = \frac{3}{19}\), and \(\frac{n}{m^2} = \frac{5}{4}\), with \(\frac{61}{8} \leq D < \frac{885}{116}\).

- \((z, a, b, c) = (13, 5, 2, 1)\) and \(\frac{n}{m^2} = \frac{2n + 3}{2}\). In the special case \(\ell = -4\), \(\frac{n}{m} = \frac{2}{9}\), there are two additional possibilities: \(\frac{n}{m^2} = \frac{10}{3}\) with \(\frac{152}{33} \leq D < \frac{84}{15}\), or \(\frac{n}{m^2} = \frac{13}{4}\) with \(\frac{21}{5} \leq D < \frac{313}{68}\).

- \((z, a, b, c) = (1, 1, 2, 5)\), \(\ell = 1\), \(\frac{n}{m} = \frac{31}{5}\), and \(\frac{n}{m^2} = \frac{7}{3}\), with \(37\frac{3}{5} \leq m_2D < 37\frac{17}{28}\).

**Proof.** Let us remark from the outset that a few cases pose certain technical difficulties, but the lemma can be checked by computer in these cases:

- If \((z, a, b, c) = (89, 34, 13, 5)\) and \(-6 \leq \ell \leq -2\), then there are no solutions.

- If \((z, a, b, c) = (34, 13, 5, 2)\) and \(-6 \leq \ell \leq -3\), then the only solution is the one described in the lemma statement.

- If \((z, a, b, c) = (13, 5, 2, 1)\) and \(\ell = -4\), then the only solutions are the ones described in the lemma statement.

- If \((z, a, b, c) = (5, 2, 1, 1)\) and \(-1 \leq \ell \leq 0\), then there are no solutions.

- If \((z, a, b, c) = (1, 1, 2, 5)\) and \(\ell = 1\), then the only solution is the one described in the lemma statement.
Henceforth assume we are not in any of these exceptional cases. That is, if \((z, a, b, c)\), then \(\ell \geq -1\), and so on.

Let \(\alpha, \beta\) be as in Lemma 4.24, and let \(\gamma := \left| 8 - \frac{8}{m} \right| \geq 1\). Note that

\[
bn - \gamma - z \geq 7b - \left( \frac{bn}{m} \right) - z = \frac{dn - bn}{m} > 0.
\]

For \(j \leq 2c(b + z - \frac{1}{b}) = 2(b(c\ell + a) + \frac{a}{b})\), we have

\[
2 \left( \frac{mn + (b + z)^2}{m} + n \right) - j \left( \frac{bmn + b + z}{m} \right) = (2c(b + z) - j) \frac{bmn + b + z}{cm} \geq \frac{2c}{b} \cdot \frac{bmn + b + z}{cm} > 2n,
\]

so

\[
(0, 2) \notin S_j \left( \frac{bmn + b + z}{cm}; \frac{n}{m}; \frac{(b + z)^2}{m} \right),
\]

and \(S_j \left( \frac{bmn + b + z}{cm}; \frac{n}{m}; \frac{(b + z)^2}{m} \right)\) contains only pairs \((x, y)\) with \(y \in \{0, 1\}\).

The proof proceeds in eleven steps:

i. We show

\[
D \geq \frac{mn + n + \ell + \gamma}{c(b + z)} = \frac{c(\ell + \gamma) \cdot cm - (b\gamma - z) \cdot bn}{c(\ell + \gamma) \cdot b - (b\gamma - z) \cdot c}
\]

and

\[
\frac{mn + (b + z)^2}{m} < mn + \ell + \gamma < g_3.
\]

For every \(j \in \mathbb{N}\), if \(S_j(D; \frac{n}{m}, \frac{n_2}{m_2}) \notin S_j \left( \frac{bmn + b + z}{cm}; \frac{n}{m}; \frac{(b + z)^2}{m} \right)\), we show there must exist \(x \in \mathbb{N}\) such that

\[
(x, 0) \in S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) \setminus S_j \left( \frac{bmn + b + z}{cm}; \frac{n}{m}; \frac{(b + z)^2}{m} \right).
\]

By the way, \((D; \frac{n}{m}, \frac{n_2}{m_2})\) must satisfy the \(j\)th triangulating condition for all \(0 \leq j \leq \frac{cmn_2}{b}\).

ii. If \(\lambda > 0\) and \(\mu \geq 0\) are such that

\[
\frac{\mu}{\lambda} \leq \frac{1}{c(b + z)} < \frac{\beta}{c\ell + a + \alpha} < \frac{\mu + b}{\lambda + c},
\]

then we show that

\[
S_{b\lambda - c\mu} \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) \subseteq S_{b\lambda - c\mu} \left( \frac{bmn + b + z}{cm}; \frac{n}{m}; \frac{(b + z)^2}{m} \right).
\]

iii. We eliminate all but four cases: \((z, a, b, c) \in \{(89, 34, 13, 5), (34, 13, 5, 2), (13, 5, 2, 1), (1, 1, 2, 5)\}\).

After this point we assume \((z, a, b, c) \in \{(89, 34, 13, 5), (34, 13, 5, 2), (13, 5, 2, 1), (1, 1, 2, 5)\}\).

iv. We show that

\[
D \geq \frac{(g_3 + n_3) + (b\gamma - z) \cdot bn}{c(b + z) + (b\gamma - z) \cdot c},
\]

v. We eliminate the case \((z, a, b, c) = (89, 34, 13, 5)\).
After this point we assume \((z, a, b, c) \in \{(34, 13, 5, 2), (13, 5, 2, 1), (1, 1, 2, 5)\}\). Note that, in these cases, \(\beta = b\gamma - z = 1\).

vi. Let

\[
i := \min \left\{ \left\lfloor \frac{c\ell + a + \frac{a - c}{b^2}}{c(b - 1)\ell + z - \gamma} \right\rfloor, c((b - 1)\ell + z - \gamma) - 1 \right\} = \begin{cases} 
   2\ell + 13 & \text{if } (z, a, b, c) = (34, 13, 5, 2), \\
   \ell + 5 & \text{if } (z, a, b, c) = (13, 5, 2, 1), \\
   5\ell - 1 & \text{if } (z, a, b, c) = (1, 1, 2, 5).
\end{cases}
\]

We show that

\[
D < \frac{g_3 + n + \lambda cm + \mu bn}{c(b\ell + z) + \lambda b + \mu c}
\]

as long as \(\lambda, \mu \geq 0\) and \(\lambda + c(\ell + \gamma)\mu \leq i\). Moreover,

\[
g_3 > mn + \ell + \gamma + \frac{i}{c(b\ell + z)}
\]

\[
= mn + \ell + \gamma + \begin{cases} 
   2\ell + 13 & \text{if } (z, a, b, c) = (34, 13, 5, 2), \\
   \ell + 5 & \text{if } (z, a, b, c) = (13, 5, 2, 1), \\
   5\ell - 1 & \text{if } (z, a, b, c) = (1, 1, 2, 5).
\end{cases}
\]

vii. We show

\[
g_3 + (1 + b)n < \frac{g_3 + (1 + b)n}{bc(\ell + \gamma)} \leq D,
\]

\[
g_3 < mn + \ell + \gamma + \frac{c(\ell + \gamma)}{(c\ell + a + \alpha) \cdot b - c}.
\]

We note for future reference that when \((z, a, b, c) = (13, 5, 2, 1)\), this bound says \(g_3 < mn + \ell + \gamma + \frac{\ell + 7}{2\ell + 11}\).

viii. In the case \((z, a, b, c) = (34, 13, 5, 2)\), we show that

\[
D < \frac{(2(c\ell + a) + 1) \cdot cm - 2bn}{(2(c\ell + a) + 1) \cdot b - 2c},
\]

\[
g_3 < mn + \ell + \gamma + \frac{4\ell + 28}{20\ell + 131},
\]

and \(m_2 \geq 5\).

After this point, we know \(m_2 \geq b\). (In the cases where \(b = 2\), this is given.)

ix. In the case \((z, a, b, c) = (1, 1, 2, 5)\), we show that

\[
D < \frac{(c\ell + a + 3) \cdot cm - bn}{(c\ell + a + 3) \cdot b - c}.
\]

x. By considering the \(j\)th triangulating condition for \(j = cm\), we eliminate the case \((z, a, b, c) = (34, 13, 5, 2)\) and show that \(m_2 \geq 3\) if \((z, a, b, c) = (1, 1, 2, 5)\).

xi. We finish the proof in the cases \((z, a, b, c) \in \{(13, 5, 2, 1), (1, 1, 2, 5)\}\), by considering the \(j\)th triangulating condition for \(j = c(m + b)\).
Without further ado, let us prove each of these in turn.

i. Our strategy is to compare $S_j(D; \frac{n}{m}, \frac{n_2}{m_2})$ to $S_j\left(\frac{bnm + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right)$. For example, note that

$$\left[\frac{c(b\ell + z)(bnm + b\ell + z)}{cm} + 1\right] = mn + n + \ell + \gamma,$$

so $mn + n + \ell + \gamma$ is the smallest integer $x$ for which $x \in W$ but $(x, 0) \not\in S_j\left(\frac{bnm + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right)$. When $j = c(b\ell + z)$, we have

$$j < \frac{2cm}{b} \leq \frac{cmn_2}{m} \leq Dm_2$$

and

$$jD < \frac{j(g_3 + n)}{c(b\ell + z)} \leq g_3 + n < mg_3,$$

so $(D; \frac{n}{m}, \frac{n_2}{m_2})$ must satisfy the $j$th triangulating condition. We already know

$$(n, 1) \in S_j\left(\frac{bnm + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right) \setminus S_j(D; \frac{n}{m}, \frac{n_2}{m_2}),$$

so we must have $(mn + n + \ell + \gamma, 0) \in S_j(D; \frac{n}{m}, \frac{n_2}{m_2}) \setminus S_j\left(\frac{bnm + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right)$ in return, which implies

$$D \geq \frac{mn + n + \ell + \gamma}{c(b\ell + z)} = \frac{bn}{c} + \frac{\ell + \gamma}{c(b\ell + z)} > \frac{bn}{c} + \frac{1}{bc} = \frac{cm}{b},$$

since, $b\gamma - z > 0$, as we noted above. Incidentally, this implies that

$$g_3 > c(b\ell + z)D - n \geq mn + \ell + \gamma > mn + \ell + 7 - \frac{n}{m} = mn + \frac{(b\ell + z)^2}{m}.$$

For $0 \leq j \leq \frac{cmn_2}{b} < m_2D$, we have

$$jD < \frac{j(g_3 + n)}{c(b\ell + z)} \leq \frac{mn_2(g_3 + n)}{m + 1} = mg_3 + \frac{m - g_3}{m + 1} < mg_3,$$

so $(D; \frac{n}{m}, \frac{n_2}{m_2})$ must satisfy the $j$th triangulating condition. Now if there is any

$$(x, y) \in S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right) \setminus S_j\left(\frac{bnm + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right),$$

then the above inequality ensures that

$$(x + y(mn + \ell + \gamma), 0) \in S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right) \setminus S_j\left(\frac{bnm + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right)$$

as well.

ii. The main time-saving calculation principle is that $\frac{cm}{b} > \frac{bn}{c}$, so two weighted averages of $\frac{cm}{b}$ and $\frac{bn}{c}$ can be compared on the basis of weights alone. (Later on in this proof, we will also use this principle with other ratios that are just as well understood.) For example, it helps to write

$$D \geq \frac{mn + n + \ell + \gamma}{c(b\ell + z)} = \frac{c(\ell + \gamma) \cdot cm - (b\gamma - z) \cdot bn}{c(\ell + \gamma) \cdot b - (b\gamma - z) \cdot c}.$$
This is not quite a “weighted average” in the ordinary sense of the term, for the weight \(-(b \gamma - z)\) is negative:

\[
b \gamma - z > (7b - z) - \frac{bn}{m} = \frac{dm - bn}{m} = \frac{bl + z}{m} > 0.
\]

Nonetheless, the principle remains valid.

As a first application of this principle, we have, by the assumptions on \(\lambda, \mu\), that

\[
\frac{\lambda \cdot cm - \mu \cdot bn}{\lambda \cdot b - \mu \cdot c} \leq \frac{c(b \gamma + z) \cdot cm - bn}{c(b \gamma + z) \cdot b - c} = \frac{bmn + bl + z}{cm},
\]

whereas

\[
\frac{\lambda \cdot cm - \mu \cdot bn + 1}{\lambda \cdot b - \mu \cdot c} = \frac{(\lambda + c) \cdot cm - (\mu + b) \cdot bn}{(\lambda + c) \cdot b - (\mu + b) \cdot c} > \frac{(c + a + \alpha) \cdot cm - \beta \cdot bn}{(c + a + \alpha) \cdot b - \beta \cdot c} > D,
\]

so there cannot possibly exist any integer \(x\) with \((x, 0) \in S_f\left(D, \frac{n}{m}, \frac{n_2}{m_2}\right) \setminus S_f\left(\frac{bmn + bl + z}{cm}, \frac{n}{m}, \frac{(b + z)^2}{m}\right)\).

It follows from part (i) that

\[
S_{b \lambda - c \mu}(D; n, n_2, m, m_2) \subseteq S_{b \lambda - c \mu}\left(\frac{bmn + bl + z}{cm}, \frac{n}{m}, \frac{(b + z)^2}{m}\right).
\]

iii. As another application of the same principle, perhaps even more direct than the previous one, recall from part (i) and Lemma 4.24 that

\[
\frac{c(\ell + \gamma) \cdot cm - (b \gamma - z) \cdot bn}{c(\ell + \gamma) \cdot b - (b \gamma - z) \cdot c} \leq D < \frac{(c + a + \alpha) \cdot cm - \beta \cdot bn}{(c + a + \alpha) \cdot b - \beta \cdot c}.
\]

Then the principle says it must be the case that

\[
\frac{c(\ell + \gamma)}{b \gamma - z} > \frac{c(\ell + a + \alpha)}{\beta}.
\]

This allows us to rule out most cases immediately:

If \(z > a > b > c \geq 13\), then \((\alpha, \beta, \gamma) = (1, 3, 7)\), so

\[
\frac{c(\ell + 7)}{d} = \frac{c(\ell + 7)}{7b - z} = \frac{c(\ell + \gamma)}{b \gamma - z} > \frac{c(\ell + a + \alpha)}{\beta} = \frac{c(\ell + a + 1)}{3},
\]

and \(3c(\ell + 7) > d(c\ell + a + 1)\). As \(d \geq 5 > 3\), this inequality becomes more and more stringent, the larger \(\ell\) is. Since \(\ell \geq -6\), we must have at the very least \(3c > d(-6c + a + 1) \geq 5(-6c + a + 1)\), or \(33c > 5a + 5\). But this is absurd because \(\frac{\ell}{c} \geq \frac{89}{13} > \frac{33}{5}\).

If \((z, a, b, c) = (5, 2, 1, 1)\), then \((\alpha, \beta, \gamma) = (1, 1, 7)\), so

\[
\frac{\ell + 7}{2} = \frac{c(\ell + \gamma)}{b \gamma - z} > \frac{c(\ell + a + \alpha)}{\beta} = \frac{\ell + 3}{1},
\]

and \(\ell < 1\). That leaves just \(\ell \in \{-1, 0\}\) as possibilities, which were discussed at the beginning of this proof.

If \((z, a, b, c) = (2, 1, 1, 2)\), then \((\alpha, \beta, \gamma) = (2, 1, 4)\), so

\[
\frac{2(\ell + 4)}{2} = \frac{c(\ell + \gamma)}{b \gamma - z} > \frac{c(\ell + a + \alpha)}{\beta} = 2\ell + 3,
\]

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so \( \ell < 1 \) and \( m = 1 \), which we are assuming not to be the case.

If \( (m, n) = (4, 27) \), then \((\alpha, \beta, \gamma) = (5, 2, 1) \) and \( \ell = 0 \), so

\[
\frac{13}{4} = \frac{13(\ell + 1)}{5 \cdot 1 - 1} = \frac{c(\ell + \gamma)}{b\gamma - z} > \frac{\ell + a + \alpha}{\beta} = \frac{13\ell + 2 + 5}{2} = \frac{7}{2},
\]

which is simply false.

If \((m, n) = (4, 27)\), then \((a, /, y) = (5, 2, 1)\) and \( f = 0 \), so

\[
\frac{c(\ell + 1)}{b - z} = \frac{c(\ell + \gamma)}{b\gamma - z} > \frac{\ell + a + \alpha}{\beta} = \frac{\ell + a + 2}{1}.
\]

Since \( b - z \geq 5 - 1 > 1 \) and \( \ell \geq 0 \), this implies at the very least

\[
c > (b - z)(a + 2).
\]

If \((z, a, b, c) = (1, 2, 5, 13)\), then in fact \( 13 < 4 \cdot 4 \), and otherwise, \( b - z \geq 13 - 2 = 11 \), but \( c < 7a \). So this is absurd.

iv. The remaining cases, where \((z, a, b, c) \in \{(89, 34, 13, 5), (34, 13, 5, 2), (13, 5, 2, 1), (1, 1, 2, 5)\}\), are a bit more involved. Note that, barring the exceptions we listed above, we have

\[
\frac{0}{c(\ell + \gamma)} < \frac{1}{c(b\ell + z)} < \frac{\beta}{c\ell + a + \alpha} < \frac{0 + b}{c(\ell + \gamma) + c}.
\]

Only the past part requires any explanation:

\[
b(\ell + a + \alpha) - \beta[c(\ell + \gamma) + c] = (b - \beta)c(\ell + \gamma - 1) + b(a + \alpha - c(\gamma - 1)) - 2\beta c = \begin{cases} 50(\ell + 6) + 35 & \text{if } (z, a, b, c) = (89, 34, 13, 5), \\ 6(\ell + 6) + 2 & \text{if } (z, a, b, c) = (34, 13, 5, 2), \\ (\ell + 6) - 2 & \text{if } (z, a, b, c) = (13, 5, 2, 1), \\ 5(\ell + 0) - 4 & \text{if } (z, a, b, c) = (1, 1, 2, 5). \end{cases}
\]

So, in order for this quantity to be positive, we need to have

\[
\ell \geq \begin{cases} -6 & \text{if } (z, a, b, c) = (89, 34, 13, 5), \\ -6 & \text{if } (z, a, b, c) = (34, 13, 5, 2), \\ -3 & \text{if } (z, a, b, c) = (13, 5, 2, 1), \\ 1 & \text{if } (z, a, b, c) = (1, 1, 2, 5). \end{cases}
\]

The first, second, and fourth of these are implied by the fact that \( m \geq 2 \). The third fails only in the exception we listed at the beginning of this proof.

Note that \( b\gamma - z \leq 2 \) in all four cases, so

\[
bc(\ell + \gamma) \leq c(b\ell + z + 2) \leq \frac{c(m + 1 + 2b)}{b} \leq \frac{cmn_2}{b}
\]
given the bounds on \( \ell \) above, so, by (i), \( \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) \) must satisfy the \( j \)th triangulating condition for \( j = bc(\ell + \gamma) \). It follows from part (ii) and the triangulating condition that

\[
S_{bc(\ell + \gamma)} \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) = S_{bc(\ell + \gamma)} \left( \frac{bmn + b\ell + z}{cm} \cdot \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right).
\]

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Now write $bc(\ell + \gamma)$ as $(b\gamma - z) \cdot c + c(\ell + \gamma)$, to see that

$$((b\gamma - z) \cdot bn + n, 1) \in S_{(b\gamma - z) \cdot c + c(\ell + \gamma)} \left( \frac{bmn + b\ell + z}{cm}, \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right),$$

so we must likewise have

$$((b(b\gamma - z) + 1)n, 1) \in S_{(b\gamma - z) \cdot c + c(\ell + \gamma)} \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right),$$

which implies

$$\frac{(g_3 + n) + (b\gamma - z) \cdot cn}{c(b\ell + z) + (b\gamma - z) \cdot b} \leq D,$$

as claimed.

v. Let us now consider the case where $(z, a, b, c) = (89, 34, 13, 5)$. Part (iv) tells us

$$D \geq \frac{d \cdot bn + (n + g_3)}{d \cdot c + c(b\ell + z)},$$

so, as $d < 12$, we have a fortiori

$$D \geq \frac{(3\ell + 13)cm + 12bn + (n + g_3)}{(3\ell + 13)b + 12c + c(b\ell + z)} = \frac{(15\ell + 65)m + 157n + g_3}{104\ell + 674},$$

and

$$((15\ell + 65)m + 157n, 1) \in S_{104\ell + 674} \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right).$$

On the other hand, since we are assuming $\ell \geq -1$, we have $\frac{8\ell + 53}{3} \geq \frac{5\ell + 35}{2} = \frac{c(\ell + 7)}{d}$, so that

$$\frac{(8\ell + 53) \cdot cm - 3 \cdot bn}{(8\ell + 53) \cdot b - 3 \cdot c} \leq \frac{c(\ell + 7) \cdot cm - d \cdot bn}{c(\ell + 7) \cdot b - d \cdot c} = \frac{mn + n + \ell + 7}{c(b\ell + z)} \leq D.$$

But we also have $\frac{8\ell + 53}{3} < \frac{5(13\ell + 89)}{1} = \frac{c(b\ell + z)}{1}$, so

$$\frac{(8\ell + 53) \cdot cm - 3 \cdot bn}{(8\ell + 53) \cdot b - 3 \cdot c} \geq \frac{c(b\ell + z) \cdot cm - bn}{c(b\ell + z) \cdot b - c} = \frac{bmn + b\ell + z}{cm}.$$

Since $(8\ell + 53)b - 3c = 104\ell + 674$, this means that

$$((8\ell + 53)cm - 3bn, 0) \in S_{104\ell + 674} \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) \setminus S_{104\ell + 674} \left( \frac{bmn + b\ell + z}{cm}, \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right).$$

We conclude that $S_{104\ell + 674}(D; \frac{n}{m}, \frac{n_2}{m_2}) \supset S_{104\ell + 674} \left( \frac{bmn + b\ell + z}{cm}, \frac{n}{m}, \frac{(b\ell + z)^2}{m} \right)$, a contradiction, if we note that $(D; \frac{n}{m}, \frac{n_2}{m_2})$ must satisfy the $j$th triangulating condition for $j = 104\ell + 674$, by (i) and the fact that

$$104\ell + 674 < \frac{10(169\ell + 1158)}{13} = \frac{2cm}{b} \leq \frac{cmn_2}{b}.$$

vi. Now three cases remain, namely

$$(z, a, b, c) \in \{(34, 13, 5, 2), (13, 5, 2, 1), (1, 1, 2, 5)\},$$

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and in each of them we have, as if by coincidence, $\beta = b\gamma - z = 1$. By assumption $i < c(b\ell + z) - c(\ell + \gamma)$, so we have

$$\frac{c(\ell + \gamma)}{b\gamma - z} \leq \frac{i}{b\gamma - z} < \frac{c(b\ell + z)}{1}$$

so

$$\frac{bmn + b\ell + z}{cm} = \frac{(c(b\ell + z) \cdot cm - bn)}{(c(b\ell + z) \cdot b - c)} < \frac{(c(\ell + \gamma) + i) \cdot cm - (b\gamma - z) \cdot bn}{(c(\ell + \gamma) + i) \cdot b - (b\gamma - z) \cdot c} \leq \frac{c(\ell + \gamma) \cdot cm - (b\gamma - z) \cdot bn}{c(\ell + \gamma) \cdot b - (b\gamma - z) \cdot c} \leq D.$$ 

Thus, if we let

$$j = c(b\ell + z) + bi = (c(\ell + \gamma) + i) \cdot b - (b\gamma - z) \cdot c,$$

then

$$((c(\ell + \gamma) + i) \cdot cm - (b\gamma - z) \cdot bn, 0) \in S_j(D; \frac{n}{m}, \frac{n_2}{m_2}) \setminus S_j\left(\frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right).$$

By the way, we have

$$j \leq c(b\ell + z) + b\left(c\ell + a + \frac{a - c}{b^2}\right) = c(b\ell + z) + b(c\ell + a) + 3 + \frac{a - 3b - c}{b} = 2c\left(b\ell + z - \frac{1}{b}\right),$$

so $(0, 2) \notin S_j\left(\frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right)$. By Lemma 4.2, then, there must be some element $(x, 1) \in S_j\left(\frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right) \setminus S_j(D; \frac{n}{m}, \frac{n_2}{m_2})$. Note that the bounds we are assuming on $\ell$ imply that

$$\ell \geq \frac{z - \gamma - a^2}{(b^2 - b + 1)} = \begin{cases} \frac{34 - 7 - 13^2}{5^2 - 5^2 + 1} = \frac{-142}{21} & \text{if } (z, a, b, c) = (34, 13, 5, 2), \\ \frac{13 - 7 - 5^2}{2^2 - 2 - 1} = \frac{-19}{3} & \text{if } (z, a, b, c) = (13, 5, 2, 1), \\ \frac{1 - 1 - 1^2}{2^2 - 2 - 1} = \frac{-1}{3} & \text{if } (z, a, b, c) = (1, 1, 2, 5), \end{cases}$$

so $(b - 1)\ell + z - \gamma \leq b^2\ell + a^2 = m$ and

$$i < c((b-1)\ell + z - \gamma) \leq cm,$$

so

$$\frac{c + i}{b} < \frac{c(b\ell + z)}{1},$$

and

$$\frac{i}ib = \frac{(c + i) \cdot cm - b \cdot bn}{(c + i) \cdot b - b \cdot c} > \frac{c(b\ell + z) \cdot cm - bn}{c(b\ell + z) \cdot b - c} = \frac{bmn + b\ell + z}{cm},$$

while

$$\frac{n + mn + (b\ell + z)^2/m}{c(b\ell + z)} = \frac{bmn + b\ell + z}{cm},$$

so $x \leq icm + n$. Thus, we must have $(icm + n, 1) \notin S_j(D; \frac{n}{m}, \frac{n_2}{m_2})$. The upshot is

$$D < \frac{icm + n + g_3}{ib + c(b\ell + z)},$$

which is a special case of our desired result, when $\mu = 0.$
Now if we recall from part (i) that
\[
D > \frac{c(\ell + \gamma) \cdot cm - (b \gamma - z) \cdot bn}{c(\ell + \gamma) \cdot b - (b \gamma - z) \cdot c} = \frac{c(\ell + \gamma) \cdot cm - bn}{c(\ell + \gamma) \cdot b - c},
\]
we find that
\[
D < \frac{n + g_3 + i \cdot cm - \mu(c(\ell + \gamma) \cdot cm - bn)}{c(bl + z) + i \cdot b - \mu(c(\ell + \gamma) \cdot b - c)}
\]
for any \(\mu \geq 0\), as long as the denominator is positive. Set \(\lambda = i - c(\ell + \gamma)\mu\) to recover the result we claimed.

Note that
\[
b \cdot \frac{mn + n + \ell + \gamma}{c(bl + z)} - cm = \frac{b(m + 1)n + bl + b \gamma - (b^2 n + 1)(bl + z)}{c(bl + z)} = \frac{br - z}{c(bl + z)} = \frac{1}{c(bl + z)}
\]
As a consequence of the above inequalities and part (i), we have
\[
g_3 > (ib + c(bl + z)) \cdot D - icm - n
\]
\[
\geq (ib + c(bl + z)) \cdot \frac{mn + n + \ell + \gamma}{c(bl + z)} - icm - n
\]
\[
= mn + \ell + \gamma + \frac{i}{c(bl + z)}.
\]

**vii.** The bound on \(D\) follows directly from part (iv) and the fact that \(b \gamma - z = 1\) now. The upper bound on \(g_3\) then follows from Lemma 4.24, after some calculation:
\[
g_3 < bc(\ell + \gamma) \cdot \frac{(cl + a + \alpha) \cdot cm - \beta \cdot bn}{(cl + a + \alpha) \cdot b - \beta \cdot c} - (1 + b)n
\]
\[
= mn + \ell + \gamma + \frac{c(\ell + \gamma)}{(cl + a + \alpha) \cdot b - c}.
\]

**viii.** In the case \((z, a, b, c) = (34, 13, 5, 2)\), we in fact claim that
\[
D < \frac{(cl + a + i)cm - 2bn}{(cl + a + i)b - 2c}
\]
for \(2 \leq i \leq cl + a + 1\). Obviously, \(i = cl + a + 1\) provides the tightest bound, so that is the only case of any importance. At any rate, let \(j := (cl + a + i)b - 2c\). Note that \(j \leq (cl + a + i)b - 2c < 2 \left(c(bl + z) - \frac{\gamma}{b}\right)\), so \((0, 2) \notin S_j \left(\frac{bmn + bl + z}{cm} ; \frac{n}{m}, \frac{(bl + z)^2}{m}\right)\), and in particular, by Lemma 4.2,
\[
\#S_j \left(D; \frac{n}{m}, \frac{n}{m_2}\right) \leq \left(\frac{j + 2}{2}\right).
\]
We claim that every \((x, y) \in S_j \left(\frac{bmn + bl + z}{cm} ; \frac{n}{m}, \frac{(bl + z)^2}{m}\right)\) also belongs to \(S_j \left(D; \frac{n}{m}, \frac{n}{m_2}\right)\). If \(y = 0\), this is obvious, so it remains to consider the case \(y = 1\). Now note that
\[
(cl + a + i)b - 2c = c(bl + z + 1) + (i - 3)b + 3c,
\]
and
\[
\frac{(b + 1)n + g_3 + (i - 3)cm + 3bn}{c(bl + z + 1) + (i - 3)b + 3c} \leq D,
\]
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because \( \frac{(b+1)n+g_1}{c(bl+z+1)} \leq D \) by part (vii), \( \frac{bn}{c} < \frac{cm}{b} \leq D \), and \((i-3)b+3c \geq (2-3)b+3c = 1 > 0 \). Note that

\[
\frac{(i-3)cm + 4bn + 1}{(i-3)b + 4c} = \frac{(i-3+c) \cdot cm - (b-4)bn}{(i-3+c) \cdot b - (b-4) \cdot c} > \frac{c(bl+z) \cdot cm - bn}{c(bl+z) \cdot b - c} = \frac{bmn + bl + z}{cm},
\]

because

\[
\frac{i-3+c}{b-4} = i - 1 \leq 5 \ell + 2 < 10 \ell + 68 = \frac{c(bl+z)}{1}.
\]

So every \((x,1) \in S_j\left(\frac{bmn+bl+z}{cm^2}, \frac{b}{m}, \frac{(bl+z)^2}{m^2}\right)\) satisfies \(x < n + (i-3)cm + 4bn + 1\), and consequently \((x,1) \in S_j(D; \frac{n}{m}, \frac{n_2}{m_2})\) also.

So \(#S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right) \geq #S_j\left(\frac{bmn+bl+z}{cm^2}, \frac{b}{m}, \frac{(bl+z)^2}{m^2}\right) = \binom{i+2}{2}\), and we noted the reverse inequality earlier, so equality must hold, and we must have \(S_j(D; \frac{n}{m}, \frac{n_2}{m_2}) = S_j\left(\frac{bmn+bl+z}{cm^2}, \frac{b}{m}, \frac{(bl+z)^2}{m^2}\right)\). Now note that

\[
\frac{2\ell + 13 + i}{2} = \frac{cl + a + i}{2} \leq \frac{cl + a + 1}{2} = 5\ell + 2 + \frac{1}{2} < 10\ell + 68 = \frac{c(bl+z)}{1},
\]

so that

\[
\frac{(cl + a + i) \cdot cm - 2bn}{(cl + a + i) \cdot b - 2c} > \frac{c(bl+z) \cdot cm - bn}{c(bl+z) \cdot b - c} = \frac{bmn + bl + z}{cm},
\]

which means

\[
((cl + a + i)cm - 2bn, 0) \notin S_{(cl+a+i)b-2c}\left(\frac{bmn+bl+z}{cm}, \frac{b}{m}, \frac{(bl+z)^2}{m}\right) = S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right),
\]

and consequently

\[
D < \frac{(cl + a + i)cm - 2bn}{(cl + a + i) - 2c},
\]

as claimed.

Consequently,

\[
g_3 < c(bl+z+1) \cdot \frac{(cl + a + i) \cdot cm - 2bn}{(cl + a + i) \cdot b - 2c} - (1+b)n
\]

\[
= mn + \ell + \gamma + \frac{2c(\ell + \gamma)}{(cl + a + i) \cdot b - 2c}
\]

We may as well let \(i = cl + a + 1\) for the best bound, so in light of (i)

\[
0 < g_3 - (mn + \ell + \gamma) = \frac{n_2}{m_2} - (\ell + \gamma) < \frac{2c(\ell + \gamma)}{(2(cl + a) + 1) \cdot b - 2c} = \frac{4\ell + 28}{20\ell + 131} < \frac{1}{5},
\]

since we are assuming \(\ell \geq -4\) at least. This implies \(m_2 \geq 5\).

\textbf{ix.} In the case at hand, namely \((x,a,b,c) = (1,1,2,5)\), let \(j := (cl + a + 3)b - c = 10\ell + 7\). Since \(\ell' \geq 1\), we have \(j < 20\ell + 5 = 2(c(bl+z) - \frac{c}{\ell})\), so \((0,2) \notin S_j\left(\frac{bmn+bl+z}{cm^2}, \frac{b}{m}, \frac{(bl+z)^2}{m^2}\right)\), and in particular, by Lemma 4.2, \(#S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right) \leq \binom{i+2}{2}\). Moreover,

\[
\frac{cl + a}{bl - 1} = \frac{5\ell + 1}{2\ell - 1} < \frac{5\ell + 3}{1} = \frac{cl + a + \alpha}{\beta},
\]

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\[
\frac{3n - m}{7} = \frac{(n-1)m - (m-3)n}{7} = \frac{(c\ell + a) \cdot cm - (b\ell - 1) \cdot bn}{(c\ell + a) \cdot b - (b\ell - 1) \cdot c} > \frac{(c\ell + a + \alpha) \cdot cm - \beta \cdot bn}{(c\ell + a + \alpha) \cdot b - \beta \cdot c} > D,
\]

and therefore, by part (vii),

\[
D \geq \frac{g_3 + (b+1)n - (c\ell + a) \cdot cm + (b\ell - 1) \cdot bn}{c(b\ell + z + 1) - (c\ell + a) \cdot b + (b\ell - 1) \cdot c} = \frac{g_3 + m}{j}.
\]

We claim that every \((x, y) \in S_j\left(\frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right)\) also belongs to \(S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right)\). If \(y = 0\), this is obvious, so it remains to consider the case \(y = 1\). Now \(n < 7m\), so

\[
\frac{n - 2m}{2} < \frac{5m}{2} = \frac{cm + b\ell + z}{cm},
\]

and consequently

\[
\frac{(mn + \frac{(b\ell + z)^2}{m}) + 2m}{j} = \frac{(mn + \frac{(b\ell + z)^2}{m} + n) - (n - 2m)}{c(b\ell + z) - 2} > \frac{bmn + b\ell + z}{cm},
\]

so \((2m, 1) \notin S_j\left(\frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right)\). Thus, for every \((x, 1) \in S_j\left(\frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right)\), we know \(x\) is in the semigroup generated by \(m\) and \(n\), and \(x < 2m\), so in fact \(x \leq m\), and \((x, 1) \in S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right)\) also.

So \(#S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right) > #S_j\left(\frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right) = \binom{j+2}{2}\), and we noted the reverse inequality earlier, so equality must hold, and we must have \(S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right) = S_j\left(\frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right)\). Now note that

\[
(c\ell + a + 3) = 5\ell + 4 < 10\ell + 5 = c(b\ell + z),
\]

so that

\[
\frac{(c\ell + a + 3) \cdot cm - bn}{(c\ell + a + 3) \cdot b - c} > \frac{c(b\ell + z) \cdot cm - bn}{c(b\ell + z) \cdot b - c} = \frac{bmn + b\ell + z}{cm},
\]

which means

\[
((c\ell + a + 3)cm - bn, 0) \notin S_{(c\ell + a + 3)cm - bn}\left(\frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right) = S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right),
\]

and consequently

\[
D < \frac{(c\ell + a + 3)cm - bn}{(c\ell + a + 3) - c},
\]

as claimed.

**x.** We first claim that, given the lower bounds on \(\ell\) we are assuming, we have

\[
S_{cm}\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right) \subseteq S_{cm}\left(\frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right).
\]

If \((z, a, b, c) \in \{(13, 5, 2, 1), (1, 1, 2, 5)\}\), this follows from part (iv) if we verify that \(bc(b\ell + z) - c = cm\) and

\[
\frac{1}{c(b\ell + z)} \leq \frac{1}{c(b\ell + z)} < \frac{\beta}{c\ell + a + \alpha} < \frac{1 + b}{c(b\ell + z) + c},
\]
as usual, only the last part requires any explanation:

\[(b + 1)(c\ell + a + \alpha) - \beta[c(b\ell + z) + c] = c\ell + (b + 1)(a + \alpha) - c(z + 1)\]

\[= \begin{cases} 
\ell + 4 & \text{if } (z, a, b, c) = (13, 5, 2, 1), \\
5\ell - 1 & \text{if } (z, a, b, c) = (1, 1, 2, 5).
\end{cases}\]

In the case \((z, a, b, c) = (34, 13, 5, 2)\), we have

\[\frac{2}{2(c\ell + a) + 1} = \frac{2}{4\ell + 2} < \frac{6}{10\ell + 70} = \frac{1 + b}{c(b\ell + z) + c}\]

as long as \(\ell \geq -5\) (which is part of our assumption). Then by using the upper bound on \(D\) we described in part (viii) in lieu of the weaker bound of \((\ell + a + \alpha)cm - \beta bn\), the same reasoning as in part (ii) shows that we must have \(S_{cm}(D; \frac{n}{m}, \frac{n_2}{m_2}) = S_{cm}\left(\frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right)\) anyway.

Provided that \(m_2 \geq b\), the triangulating condition \(cm\) must hold, by (i), so

\[S_{cm}(D; \frac{n}{m}, \frac{n_2}{m_2}) = S_{cm}\left(\frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right).\]

Now

\[bmn + b\ell + z - (b - 1)(mn + \ell + \gamma) = mn + \ell + \gamma - (b\gamma - z) = mn + \ell + \gamma - 1,\]

and

\[mn + \frac{(b\ell + z)^2}{m} < mn + \ell + \gamma,\]

so we have

\[(mn + \ell + \gamma - 1, b - 1) \in S_{cm}\left(\frac{bmn + b\ell + z}{cm}; \frac{n}{m}, \frac{(b\ell + z)^2}{m}\right) = S_{cm}\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right),\]

which implies

\[\frac{(b - 1)g_3 + mn + \ell + \gamma - 1}{cm} \leq D.\]

If \((z, a, b, c) = (34, 13, 5, 2)\), this means, in light of part (viii), and after some tedious computation,

\[(b - 1)g_3 < cm \cdot \frac{(2(c\ell + a) + 1) \cdot cm - 2bn}{(2(c\ell + a) + 1) \cdot b - 2c} - (mn + \ell + \gamma - 1)\]

\[= (b - 1)(mn + \ell + \gamma) + \frac{16\ell + 109}{20\ell + 125},\]

which implies

\[g_3 < mn + \ell + \gamma + \frac{16\ell + 109}{80\ell + 625},\]

and by part (vi), we have

\[\frac{1}{5} - \frac{3}{10\ell + 68} < \frac{2\ell + 13}{10\ell + 68} < g_3 - (mn + \ell + \gamma) < \frac{16\ell + 109}{80\ell + 625} = \frac{1}{5} - \frac{80}{80\ell + 68}.\]
but this is blatantly false.

If \((z, a, b, c) = (1, 1, 2, 5)\), this implies by part (ix), after much calculation,

\[
g_3 = (b - 1)g_3 < cm \cdot \frac{(c\ell + a + 3) \cdot cm - bn}{(c\ell + a + 3) \cdot b - c} - (mn + \ell + \gamma - 1)
= (b - 1)(mn + \ell + \gamma) + \frac{5\ell + 1}{10\ell + 3}.
\]

Together with part (vi), we have

\[
\frac{5\ell - 1}{10\ell + 5} < g_3 - (mn + \ell + \gamma) = \frac{n_2}{m_2} - (\ell + \gamma) < \frac{5\ell + 1}{10\ell + 3}.
\]

So, as long as \(\ell \geq 1\), this implies \(m_2 \geq 3\).

xi. Assume now that \((z, a, b, c) \in \{(13, 5, 2, 1), (1, 1, 2, 5)\}\), with \(\ell \geq -3\) if \((z, a, b, c) = (13, 5, 2, 1)\), and \(\ell \geq 2\), if \((z, a, b, c) = (1, 1, 2, 5)\). With this bounds, it is straightforward to verify that

\[
\frac{1}{c(b\ell + z + 1)} \leq \frac{1}{c(b\ell + z)} < \frac{\beta}{c\ell + a + \alpha} < \frac{1 + b}{c(b\ell + z + 1) + c},
\]

so we know from part (ii) that

\[
S_{c(m+b)} \left( D ; \frac{n}{m} , \frac{n_2}{m_2} \right) \subseteq S_{c(m+b)} \left( \frac{bmn + b\ell + z}{cm} , \frac{n}{m} , \frac{(b\ell + z)^2}{m} \right),
\]

since \(bc(b\ell + z + 1) - c = c(m + b)\).

With \(i\) as in part (vi), note that \(c(\ell + \gamma) \cdot \frac{b - 1}{b} \leq i\), when \(\ell\) has the bounds we mentioned. Then part (vi) tells us

\[
\frac{b^2 n}{c(m + b)} - \frac{b \cdot (g_3 + n) + (b - 1) \cdot bn}{b \cdot c(b\ell + z) + (b - 1) \cdot c} > D,
\]

so

\[
(b^2 n, b) \notin S_{c(m+b)} \left( D ; \frac{n}{m} , \frac{n_2}{m_2} \right).
\]

On the other hand, we know \((bn, b) \in S_{bc(b\ell + z)} \left( \frac{bmn + b\ell + z}{cm} , \frac{n}{m} , \frac{(b\ell + z)^2}{m} \right)\) and \((bn, 0) \in S_{c} \left( \frac{bmn + b\ell + z}{cm} , \frac{n}{m} , \frac{(b\ell + z)^2}{m} \right)\), so, by addition,

\[
(b^2 n, b) \in S_{c(m+b)} \left( \frac{bmn + b\ell + z}{cm} , \frac{n}{m} , \frac{(b\ell + z)^2}{m} \right) \setminus S_{c(m+2)} \left( D ; \frac{n}{m} , \frac{n_2}{m_2} \right).
\]

Hence,

\[
S_{cm} \left( D ; \frac{n}{m} , \frac{n_2}{m_2} \right) \subseteq S_{cm} \left( \frac{bmn + b\ell + z}{cm} , \frac{n}{m} , \frac{(b\ell + z)^2}{m} \right),
\]

and \((D; \frac{n}{m}, \frac{n_2}{m_2})\) fails the triangulating condition for \(c(m + b)\).

It follows from (i) that we must have \(\frac{bmn}{b} < c(m + b)\), which is to say \(m_2 < \frac{b(m+b)}{m} = b + \frac{b^2}{m} = 2 + \frac{4}{m}\), which implies \(m_2 = 2\). If \((z, a, b, c) = (1, 1, 2, 5)\) and \(\ell \geq 1\), this contradicts part (x). If \((z, a, b, c) = (13, 5, 2, 1)\), we have from parts (i) and (vii) that

\[
0 < g_3 - (mn + \ell + \gamma) < \frac{\ell + 7}{2\ell + 11},
\]

so the only possibility is

\[
\frac{n_2}{m_2} = g_3 - mn = \ell + 7 + \frac{1}{2} = \frac{2\ell + 15}{2} = \frac{2n + 3}{2},
\]

as claimed. \qed
Putting it all together, we have the following.

**Proposition 4.33.** Let \( z, a, b, c \) be consecutive odd-indexed Fibonacci numbers, \( \ell \in \mathbb{Z}, (m, n) = (b^2 \ell + a^2, c^2 \ell + b^2 + 2), \) and \((D; \frac{n}{m}, \frac{n}{m_2})\) a weakly triangulating sequence with \( m, n, m_2 \geq 2 \). Then \( m_2 > \left[ \frac{m}{b^2} \right] - 1 \). Moreover, at least one of three statements must hold:

(a) \( \frac{n_2}{m_2} \) equals \( \frac{n}{b} \) or a fair approximation to it, not exceeding \( \frac{b^2 + z}{b} \).

(b) \( \frac{n_2}{m_2} \) equals \( \frac{(b^2 + z)^2}{c^2} \), with \( \frac{bmn + b^2 + z}{m_2} \leq m_2D \), or we have one of the cases listed in Lemma 4.30, parts (x)-(xiv), which we repeat for convenience:

x. If \( m_2 < b \), then \( \frac{n_2}{m_2} = \frac{b^2 + z}{b} + \frac{1}{b} \), which implies \( m_2 \) is the unique integer in the interval \((0, b)\) with \( 3m_2 \equiv a \pmod{b} \).

xi. If \( b \leq m_2 \leq m \) and \( \frac{n_2}{m_2} < \frac{(b^2 + z)^2}{m} \), then these are the only possibilities:

* \((z, a, b, c) = (34, 13, 5, 2)\), with \( \ell \in \{-6, -5\} \), and \( \frac{n_1}{m_1} = \frac{66 + 41}{8} \). If \( \ell = -6 \), then we have \( \frac{n}{m} = \frac{19}{6} \) and \( \frac{887}{16} \leq m_2D < \frac{1871}{41} \). If \( \ell = -5 \), then we have \( \frac{n}{m} = \frac{44}{7} \), \( \frac{n_2}{m_2} = \frac{11}{6} \) and \( \frac{667}{44} \leq m_2D < \frac{1066}{101} \).

* \((z, a, b, c) = (13, 5, 2, 1)\) and \( \frac{n_2}{m_2} = \frac{4\ell + 27}{4} \), with \( 2m + \frac{2}{m} \leq 2m_2D < 2m + \frac{4}{2m-1} \).

xii. If \( b \leq m_2 \leq m \) and \( \frac{n_2}{m_2} > \frac{(b^2 + z)^2}{m} \), then \( c = 1 \) and \( \frac{n_2}{m_2} = \frac{(b^2 + z)^2 - (m + 2)}{m - b^2} \). If \((z, a, b, c) = (13, 5, 2, 1)\), then
\[
\frac{16n^3 - 6n^2 - 2}{2n + 1} = 8n^2 - 4n - 1 - \frac{1}{2n + 1} \leq m_2D < 8n^2 - 4n - 1 - \frac{4n - 4}{8n^2 - 4n - 3}.
\]

If \((z, a, b, c) = (5, 2, 1, 1)\), then
\[
\frac{m^3}{m + 1} = m^2 - m + 1 - \frac{1}{m + 1} \leq m_2D < \frac{m_2(m^3 - m^2 - 1)}{m^2 - m - 1}.
\]

xiii. If \( m_2 > m \) and \( c > 1 \), then these are the only possibilities:

* \((z, a, b, c) = (34, 13, 5, 2)\), \( \ell = -6 \), \( \frac{n}{m} = \frac{3}{19} \) and \( \frac{n_2}{m_2} = \frac{27}{32} \), with \( \frac{1947}{8} \leq m_2D < \frac{57080}{237} \).

* \((z, a, b, c) = (2, 1, 1, 2)\), \( \ell = 1 \), and \( \frac{n}{m} = \frac{7}{2} \) and \( \frac{n_2}{m_2} \in \{\frac{13}{3}, \frac{17}{4}, \frac{21}{5}, \frac{25}{6}\} \), with \( m_2D \) in the intervals
\[
\begin{align*}
\left[\frac{64}{5}, \frac{141}{11}\right], \\
\left[\frac{17}{10}, \frac{273}{16}\right], \\
\left[\frac{362}{10}, \frac{213}{16}\right], \\
\left[\frac{434}{17}, \frac{332}{13}\right],
\end{align*}
\]
respectively.

* \((z, a, b, c) = (1, 2, 5, 13)\), \( \ell = 0 \), \( \frac{n}{m} = \frac{27}{4} \), and \( \frac{n_2}{m_2} \in \{\frac{5}{19}, \frac{6}{23}\} \), with \( m_2D \) in the intervals
\[
\begin{align*}
\left[\frac{2570}{13}, \frac{37364}{189}\right], \\
\left[\frac{23931}{100}, \frac{54802}{229}\right],
\end{align*}
\]
respectively.

xiv. If \( m_2 > m \) and \( c = 1 \), then these are the only possibilities:

* \( \frac{n}{m} = \frac{2}{3} \) and \( \frac{n_2}{m_2} \) is a fair upper approximation to \( \frac{46}{9} \). If \( \frac{n_2}{m_2} = \frac{46k - 5}{9k - 1} \) for some \( k \geq 1 \), then \( \frac{10km_2}{3} \leq m_2D < \frac{900k^2 - 250k + 17}{30k - 5} \).
\[ \frac{n}{m} = \frac{3}{4} \quad \text{and} \quad \frac{n_2}{m_2} = \frac{31}{5}, \quad \text{with} \quad \frac{302}{17} \leq m_2 D < \frac{213}{10}. \]

(c) One of the cases listed in Lemma 4.32, which we repeat for convenience:

- \((z, a, b, c) = (34, 13, 5, 2), \ell = -6, \quad \frac{n}{m} = \frac{3}{10}, \quad \text{and} \quad \frac{n_2}{m_2} = \frac{5}{4}, \quad \text{with} \quad \frac{61}{2} \leq m_2 D < \frac{885}{20}.\)

- \((z, a, b, c) = (13, 5, 2, 1) \quad \text{and} \quad \frac{n_2}{m_2} = \frac{2n+3}{2}, \quad \text{with} \quad 4n + 1 + \frac{1}{2m+1} \leq m_2 D < 4n + 1 + \frac{1}{2m}.

In the special case \(\ell = -4, \quad \frac{n}{m} = \frac{2}{5}, \quad \text{with} \quad \frac{152}{11} \leq m_2 D < \frac{83}{17}, \quad \text{or} \quad \frac{n_2}{m_2} = \frac{13}{4}, \quad \text{with} \quad \frac{92}{5} \leq m_2 D < \frac{313}{37}.\)

- \((z, a, b, c) = (1, 1, 2, 5), \ell = 1, \quad \frac{n}{m} = \frac{31}{5}, \quad \text{and} \quad \frac{n_2}{m_2} = \frac{7}{3}, \quad \text{with} \quad \frac{188}{5} \leq m_2 D < \frac{1053}{28}.\)

Note that we are not claiming the converse. This is a necessary, but not sufficient, condition on the pair \((m_2, n_2).\)

Proof. We omit the proof of the bounds on \(D.\) Let us establish, however, that these are the only \(\frac{n_2}{m_2}\) that arise.

Note that

\[ \begin{align*}
\left\lfloor \frac{m}{b^2} \right\rfloor - 1 &< \frac{m}{b^2} < \frac{b\ell + z}{b} < \frac{(b\ell + z)^2}{m}.
\end{align*} \]

Obviously, on the low side, \(\left\lfloor \frac{m}{b^2} \right\rfloor - 1\) is a fair lower approximation to \(\frac{m}{b^2}.\) In the middle, we claim \(\frac{b\ell + z}{b} = \ell + \frac{z}{b}\) is a fair approximation to \(\frac{m}{b^2} = \ell + \frac{a^2}{b^2},\) because \(\frac{a^2}{b^2} - \frac{b^2}{b^2} = \frac{b^2 - a^2}{b^2} = \frac{1}{b^2},\) and no fraction of denominator less than \(b\) can be within \(\frac{1}{b^2}\) of \(\frac{z}{b}.\)

Similarly, this claim is vacuous of \(b = 1,\) and if \(b > 2,\) then it suffices to note that

\[ \frac{(b\ell + z)^2}{m} - \frac{(b\ell + z)}{b} = \frac{(b\ell + z)}{b} \leq \frac{1}{b(b-1)}, \]

which holds because \(m = b(b\ell + z) - 1 \geq (b-1)(b\ell + z).\) In the \(b > 2\) case, we can conclude that \(\frac{b\ell + z}{b}\) is in fact a fair lower approximation to \(\frac{(b\ell + z)^2}{m}.\) (If \(b = 1,\) there might be another integer strictly between them.)

In summary, no fraction in the interval \((\frac{m}{b^2}, \frac{(b\ell + z)^2}{m})\) has denominator less than \(b,\) and if \(b > 2,\) then \(\frac{b\ell + z}{b}\) is the only fraction in that interval with denominator \(b.\) Since we are assuming that \(m_2 \geq 2,\) it follows that, if

\[ \frac{m}{b^2} \leq \frac{n_2}{m_2} < \frac{(b\ell + z)^2}{m}, \]

then either \(\frac{n_2}{m_2} = \frac{b\ell + z}{b}\) or \(m > b.\) In the former case, \(\frac{n_2}{m_2}\) is indeed a fair approximation to \(\frac{m}{b^2},\) which falls under case (a) of our classification. In the latter case, \(\frac{n_2}{m_2}\) must satisfy the \(j\)th triangulating condition for \(j = cm.\) Comparing with \(S_{cm}(\frac{cm}{b}; \frac{n}{m}, \frac{n_2}{m_2})\), we see that

\[ \min\{bg_3, bmn + b\ell + z\} \leq cmD < \max\{bg_3, bmn + b\ell + z\}. \quad (4.5.4) \]

The two quantities \(bg_3\) and \(bmn + b\ell + z\) are exactly equal when \(g_3 = mn + \frac{b\ell + z}{b}.\) Recall, too, from Lemma 4.27 that \((\ell; \frac{n}{m}, \frac{n_2}{m_2})\) is strongly triangulating and \(\left\lfloor \frac{n_2}{m_2} \right\rfloor \geq \left\lfloor \frac{m}{b^2} \right\rfloor,\) so we have the following trichotomy:

Case (a): We have \(\left\lfloor \frac{m}{b^2} \right\rfloor - 1 < \frac{n_2}{m_2} < \frac{b\ell + z}{b}.\)
We have already mentioned the case $\frac{n_A}{m_2} = \frac{bl+z}{b}$, so assume now that $\frac{n_A}{m_2} < \frac{bl+z}{b}$. Recall from Lemma 4.22, part (i), that $D > \frac{cm}{b}$, so we have $D \geq \frac{bg}{cm}$ regardless of how $\frac{n_A}{m_2}$ compares to $\frac{m}{b}$.

If $\frac{n_A}{m_2} < \frac{m}{b}$, then $D > \frac{cm}{b} > \frac{bg}{cm}$, and if $\frac{n_A}{m_2} > \frac{m}{b}$, then we use (4.5.4). Thus, $D > \frac{cm}{b}$, so $S_j(D, \frac{n_A}{m_2}) \subseteq S_j(D, \frac{m}{b})$ for all $j \in \mathbb{N}$.

Then it is easy to see that Lemma 4.19 applies: for the lower bound we have

$$m + n + \left\lfloor \frac{m}{b^2} \right\rfloor - 1 = (b^2 + c^2 + 1)\ell + a^2 + ac + 1 + \left\lfloor \frac{a^2}{b^2} \right\rfloor - 1$$

$$= (a + c)(\ell + a) + \left\lfloor \frac{a^2}{b^2} \right\rfloor$$

$$\geq (1 + c) \cdot 1 + 1 = c + 2,$$

so

$$1 \cdot mn + \left\lfloor \frac{m}{b^2} \right\rfloor - 1 \geq mn - (m + n) + c + 2 = \max(\mathbb{Z}\setminus W_2) + c + 2.$$

Case (b): We have $\frac{bl+z}{b} < \frac{n_A}{m_2} < \frac{(bl+z)^2}{m}$ and $g_3 + n < c(bl+z)D$. (The condition on $D$ is meant to be an additional assumption; we are not claiming it as an immediate consequence of the bounds on $\frac{n_A}{m_2}$.) In this case, according to (4.5.4), we know

$$bm + bl + z \leq cmD < bg_3.$$

Then Lemma 4.30 applies.

Case (c): We have $g_3 + n > c(bl+z)D$. Then Lemma 4.32 applies.

This completes the proof. \(\Box\)

### 4.5.2 Cases where $\frac{n_1}{m_i} \neq \frac{\ell F_{k+1}^2 + F_{k+2}^2}{\ell F_{k-1}^2 + F_{k-3}^2}$

In stark contrast to the case of $(m, n) = \ell(b^2, c^2) + (a^2, b^2 + 2)$ discussed in the previous subsection, most of the remaining cases do not admit many extensions to triangulating sequences of length 2.

**Lemma 4.34.** Let $2 \leq a < b < c < d$ denote consecutive odd-indexed Fibonacci numbers. There are no weakly triangulating sequences $(D; \frac{n_1}{m}, \frac{n_2}{m_2})$ with $m_2 \geq 2$ for the following $\frac{n_A}{m}$:

1. $(m, n) = (a, c)$ with $a \geq 5$. 
2. $(m, n) = \left(\frac{a+b}{3}, \frac{c+d}{3}\right)$ with $a \geq 34$. 
3. $(m, n) = \left(\frac{a+2b}{3}, \frac{c+2d}{3}\right)$ with $a \geq 89$. 
4. $\frac{n}{m}$ is a fair upper approximation to $\frac{F_5}{3}$. 
5. $(m, n) = (m, 9m - 1)$ with $m \geq 2$.

For the following $\frac{n_1}{m}$, there are only finitely many $\frac{n_2}{m_2}$ completing a weakly triangulating sequence $(D; \frac{n_1}{m}, \frac{n_2}{m_2})$:

1’. For $(m, n) = (a, c) = (2, 13)$, there are only two possibilities $(m_2, n_2)$ making making $(D; \frac{n_1}{m}, \frac{n_2}{m_2})$ a weakly triangulating sequence, namely $(m_2, n_2) = (2, 3)$ with $\frac{52}{9} \leq m_2 D < \frac{94}{9}$, and $(m_2, n_2) = (3, 4)$ and $\frac{203}{13} \leq m_2 D < \frac{172}{11}$.
2'. For \((a, b, c, d) = (5, 13, 34, 89), (m, n) = \left(\frac{a + b}{3}, \frac{c + d}{3}\right) = (6, 41),\) there is only one possibility
\((m_2, n_2)\) making \((D; \frac{n_2}{m_2})\) a weakly triangulating sequence, namely
\((m_2, n_2) = (2, 1),\) with \(568 \leq m_2D < \frac{722}{23}.\)

3'. For \((a, b, c, d) = (13, 34, 89, 23)\) and \((m, n) = \left(\frac{a + 2b}{3}, \frac{c + 2d}{3}\right) = (27, 185),\) there is only one pos-
sibility \((m_2, n_2)\) making \((D; \frac{n_2}{m_2})\) a weakly triangulating sequence, namely
\((m_2, n_2) = (5, 1),\) with \(122 \leq m_2D < 332.\)

\[\text{Proof.}\] The finitely many cases of \((m, n)\) listed above that do admit weakly triangulating sequences
\((D; \frac{n}{m}, \frac{n_2}{m_2})\) can all be checked by a finite sequence of operations on a computer, so we will not discuss
them any further. Let us concern ourselves with \((1)-(5).\) Suppose for the sake of contradiction that
we had some counterexample \((D; \frac{n}{m}, \frac{n_2}{m_2})\) in any of those cases. Write \(n_0 = \frac{n_2}{m_2}\)
and \(g_3 = mn + n_0.\)

We will apply the \(j\)th triangulating conditions for several \(j,\) omitting the routine verifications that
\(j\) is small enough that these triangulating conditions are required of \((D; \frac{n}{m}, \frac{n_2}{m_2}).\)

Case 1: \((m, n) = (a, c)\) with \(a \geq 5,\) and (since \((D; \frac{n}{m}, \frac{n_2}{m_2})\) must satisfy the \(b\)th triangulating
condition) by Proposition 4.22, part (i), and the results in subsection 4.4.3, we have
\[
\frac{ac}{b} \leq D < \frac{ac - b}{b - 1}.
\]

Since \(b \geq 13,\) note that
\[
\frac{ac + b}{b + 1} < \frac{ac}{b} \leq D < \frac{ac - b}{b - 1} < \frac{ac + 5b + 1}{b + 1}. \tag{1}
\]

For \(j = b + 1,\) we have
\[
\max\{g_3 + 2a, ac + b\} > (b + 1)D.
\]

Indeed, if this were not the case, then \(S_j(D; \frac{n}{m}, n_0)\) would contain at least
\[
3 + ac + b + 1 - \frac{(a - 1)(c - 1)}{2} = \frac{6 + 2ac + 2b + 2 - ac + a + c - 1}{2} = \frac{ac + 5b + 7}{2} = \frac{b^2 + 5b + 8}{2} > \left(\frac{j + 2}{2}\right)
\]
elements, contradicting the \(j\)th triangulating condition. (Similar reasons can be given for the
remaining such inequalities in this proof, and we will not belabor them by writing out the calculations
in each case.)

Putting together the inequalities so far, we see that it must be that
\[
g_3 + 2a > (b + 1)D.
\]

For \(j = b + 5,\) we have
\[
\min\{g_3 + 6a + c, ac + 5b + 1\} \leq (b + 5)D,
\]
and by the above it must be that
\[
g_3 + 6a + c \leq (b + 5)D.
\]
Subtraction yields

\[ 4a + c = (g_3 + 6a + c) - (g_3 + 2a) < (b + 5)D - (b + 1)D = 4D < 4 \cdot \frac{ac - b}{b - 1}. \]

So

\[ 0 > (4a + c)(b - 1) - 4(ac - b) = z(b - 1) - 4 \geq 2 \cdot 12 - 4, \]

which is false.

Case 2: \((m, n) = (\frac{a+b}{3}, \frac{c+d}{3})\), with \(a \geq 34\), and

\[ \frac{m + n}{3} + \frac{1}{3b} = \frac{cm}{b} \leq D < \frac{m + n}{3} + \frac{2}{b + c - \eta}, \]

where \(\eta\) is either 4 or 5, and \(\eta \equiv a \equiv b + c \mod 3\). Note that \(m + n = b + c\). For \(j = 3m = a + b\), we have

\[ \min\{g_3 + m^2, m^2 + mn + 1\} \leq 3mD, \]

and for \(j = 2b - 3\), we have

\[ \max\{g_3 + (3c - 2n - 1)m + (m - a - 1)n, (2c - n - 1)m + (m - 1)n\} > (2b - 3)D. \]

Let’s rewrite these two inequalities in terms of \(\xi\) instead of \(g_3\):

\[ \min\{\xi, 1\} \leq 3mD - m(m + n) < \frac{6m}{b + c - \eta} = \frac{2(a + b)}{b + c - \eta} < 1, \]

\[ \max\{\xi, 1\} > (2b - 3)D - (2c - 1)m + n + 1, \]

where we have incorporated the bounds on \(D\) that we computed in subsection 4.4.3. It follows that \(\xi < 1\) and, consequently, \(\max\{\xi, 1\} = 1\), and

\[ D < \frac{(2c - 1)m - n}{2b - 3} = \frac{m + n}{3} + \frac{2}{3(2b - 3)}. \]

We can plug this back into our first inequality involving \(\xi\):

\[ \xi = \min\{\xi, 1\} \leq 3mD - m(m + n) < \frac{2m}{2b - 3} < \frac{1}{2}, \]

as one can easily check the last bit under the assumption \(a \geq 34\). In particular, the denominator \(m_2\) of \(\xi\) is at least 3, so we may apply the triangulating conditions for slightly higher \(j\).

For \(j = c\), for example, we have

\[ \min\{2g_3 + (b - 2m)n, bn + 1\} \leq cD, \]

and for \(j = a + 3b - 3\), we have

\[ \max\{2g_3 + (2b - 1)m + (b - 2m - 1)n, g_3 + (b + 3c - 2n - 1)m + (m - 1)n\} > (a + 3b - 3)D. \]

Let’s both of these inequalities in terms of \(\xi\), and use the upper bound for \(D\) that we derived earlier:

\[ \min\{2\xi, 1\} \leq cD - bn < c \left( \frac{m + n}{3} + \frac{2}{3(2b - 3)} \right) - bn = \frac{2c - 2b + 3}{3(2b - 3)} < 1, \]

\[ \max\{2\xi - 1, \xi\} > (a + 3b - 3)D - (b + 3c - 1)m + n. \]
So in fact
\[ \xi = \max\{2\xi - 1, \xi\} > (a + 3b - 3)D - (b + 3c - 1)m + n, \]
so we have
\[ (a + 3b - 3)D - (b + 3c - 1)m + n < \xi \leq \frac{cD - bn}{2}, \]
from which we can calculate
\[ D < \frac{m + n}{3} + \frac{1}{3a + 3b - 6} < \frac{m + n}{3} + \frac{1}{3b}, \]
which contradicts the bound in subsection 4.4.3.

Case 3: \((m, n) = (a + 2b, c + 2d)\) with \(a \geq 89\), and
\[ \frac{b + 2c}{3} + \frac{1}{3b} = \frac{cm}{b} \leq D < \frac{b + 2c}{3} + \frac{3}{b + 2c - \eta}, \]
where \(\eta\) is either 4 or 5, and \(\eta \equiv c \equiv b + 2c \mod 3\). (Note that \(am - cm = 2\) and \(dm - bn = 1\))
For \(j = c\), we have
\[ \min\{g_3 + (b - m)n, dm\} \leq cD, \]
and for \(j = 3(b - 1)\), we have
\[ \max\{g_3 + 3cm - mn - m - n - 1, 3cm - m - n\} > 3(b - 1)D, \]
and for \(j = b + c - 3\), we have
\[ \max\{g_3 + (c - 1)m + (b - m - 1)n, (c - 1)m + (b - 1)n\} > (b + c - 3)D. \]

Let's rewrite all this in terms of \(\xi\), and use the upper bound for \(D\) we found in subsection 4.4.3:
\[ \min\{\xi, 1\} \leq cD - bn < c \left(\frac{b + 2c}{3} + \frac{3}{b + 2c - \eta}\right) - bn < c \cdot \frac{bn + 1}{c} - bn = 1, \]
\[ \max\{\xi, 1\} > 3(b - 1)D - 3cm + m + n + 1, \]
\[ \xi = \max\{\xi, 0\} > (b + c - 3)D - (c - 1)m - (b - 1)n. \]
so
\[ \xi \leq cD - bn < 1, \]
and so
\[ 1 > 3(b - 1)D - 3cm + m + n + 1, \]
which means
\[ D < \frac{3cm - m - n - 1}{3(b - 1)} = \frac{m + n}{3} + \frac{1}{3(b - 1)}, \]
and in particular
\[ \xi \leq c \cdot \left(\frac{(m + n)}{3} + \frac{1}{3(b - 1)}\right) - bn = \frac{c}{3(b - 1)} - \frac{2}{3} = \frac{b - a + 2}{3(b - 1)} < \frac{1}{4}, \]
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as one can easily check the last bit under the assumption $a \geq 89$. At the same time, if we recall that \( \frac{cm}{b} \leq D \), then

\[
\xi > (b + c - 3) \cdot \frac{cm}{b} - (c - 1)m - (b - 1)n \\
= \frac{(b - 1)(dm - bn) - m}{b} \\
= \frac{b - m - 1}{b} = \frac{b - a - 1}{3b} > \frac{1}{5}
\]

under our assumption that $a \geq 89$. In particular, the denominator $m_2$ of $\xi$ is at least 9, so we may apply the triangulating conditions for slightly higher $j$.

For $j = 4c$, we have

\[
\min\{5g_3 + (4b - 5m)n, \ 4bn + 1\} \leq 4cD,
\]

which is to say

\[
\min\{5\xi, \ 1\} \leq 4(cD - bn) < 1,
\]

thanks to the upper bound on $D$ that we derived above. But then

\[
(b + c - 3)D - (c - 1)m - (b - 1)n < \xi < \frac{4(cD - bn)}{5},
\]

from which we can calculate

\[
D < \frac{cm}{b} + \frac{m + 5 - b}{b(5b + c - 15)} < \frac{cm}{b},
\]

which is a contradiction.

Case 4: \( \frac{m}{n} \) is a fair upper approximation to \( \frac{64}{12} \). When \( 2 \leq m \leq 7 \), a computer program can verify (by brute force) that the triangulating conditions for $j \leq 5m$ are contradictory, so there cannot be any $D$ and $\frac{m}{n}$ for which \( (D; \frac{m}{n}, \frac{m}{n} \) is triangulating. So assume \( (m, n) = (9\ell + 8, \ 64\ell + 57) \), with

\[
\frac{8m}{3} \leq D < \frac{8m}{3} + \frac{1}{12(6\ell + 5)}.
\]

For $j = 24\ell + 40$, note

\[
mn + 7m + 6n = \min\{g_3 + 7m + 6n, \ mn + 7m + 6n\} \leq (24\ell + 40)D,
\]

so

\[
\frac{8m}{3} + \frac{1}{12(6\ell + 5)} = \frac{mn + 7m + 6n}{24\ell + 40} \leq D < \frac{8m}{3} + \frac{1}{12(6\ell + 5)},
\]

which is a direct contradiction.

Case 5: \( (m, n) = (m, 9m - 1) \), with

\[
3m - \frac{1}{3} \leq D < 3m - \frac{m - 1}{3m - 2}.
\]

For $j = 3m + 2$, we see

\[
\frac{m - 1 + 1}{(3m - 2) + 4} = \frac{mn + 6m}{3m + 2} \leq D < \frac{m - 1}{3m - 2},
\]

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Lemma 4.35. If \((D; \frac{c^2}{b^2}, \frac{m_2}{m_2})\) is weakly triangulating with \(b, c, m_2 \geq 2\), then \(n_2 = 1\) and
\[
bc + \frac{1}{bcm_2} = \frac{g_3}{bc} \leq D < bc + \frac{1}{bcm_2 - 1}.
\]

Proof. First of all, \((\frac{D}{b^2}; \frac{c^2}{b^2})\) must be triangulating, so the results in subsection 4.4.3 tell us that
\[
bc \leq D < bc + \frac{1}{bc - 1}.
\]  
(4.5.5)

In particular, \((D; \frac{c^2}{b^2}, \frac{m_2}{m_2})\) is required to satisfy the jth triangulating condition for all \(j < bcm_2\).

Our strategy is to exploit the fact that \((\frac{c}{b}, \frac{c^2}{b^2})\) is strongly triangulating. This is equivalent to saying, “\((bc; \frac{c^2}{b^2}; 0)\) is strongly triangulating,” although I suppose we have not defined triangulating sequences involving zeros, so perhaps this should be taken as merely an intuitive guide to what follows.

First let us prove that \(n_2 = 1\). Suppose for the same of contradiction that \(n_2 \geq 2\). Let \(p = \left\lceil \frac{m_2}{n_2} \right\rceil\), so \(p\) is the smallest positive integer such that \(\frac{1}{p} < \frac{m_2}{n_2}\). Note that \(m > p\). For \(j = pbc\), a comparison to the strongly triangulating sequence \((\frac{c}{b}, \frac{c^2}{b^2})\) shows that
\[
 pb^2c^2 + 1 = \min\{pg_3, pb^2c^2 + 1\} \leq pbcD < \max\{pg_3, pb^2c^2 + 1\} = pg_3.
\]

Let us consider two cases, depending on \(p\):

- Case 1: \(p = 1\). For \(j = bc + 1\), we see that
\[
\min\{g_3 + 2b^2, b^2c^2 + bc + 1\} \leq (bc + 1)D < \max\{g_3 + 2b^2, b^2c^2 + bc + 1\},
\]
but we already know from above that
\[
(bc + 1)D \geq bcD + D \geq (b^2c^2 + \frac{1}{p}) + bc,
\]
so it must be that
\[
b^2c^2 + bc + 1 \leq (bc + 1)D < g_3 + 2b^2,
\]
and
\[
\frac{n_2}{m_2} > b(c - 2b) + 1 + \frac{1}{bc} > 3.
\]

For \(j = 2(bc - 1)\), we have
\[
\min\{g_3 + (bc - 2)bc - 1, (2bc - 2)bc + 2\} \leq 2(bc - 1)D < \max\{g_3 + (bc - 2)bc - 1, (2bc - 2)bc + 2\},
\]
but since we know \(\frac{n_2}{m_2} > 3\), it must be the case that
\[
(2bc - 2)bc + 2 \leq 2(bc - 1)D < g_3 + (bc - 2)bc - 1,
\]
and in particular
\[
D \geq \frac{(2bc - 2)bc + 2}{2(bc - 1)} = bc + \frac{1}{bc - 1},
\]
which directly contradicts the upper bound in (4.5.5).
• Case 2: $p \geq 2$. For $j = pbc + 1$, a comparison to the strongly triangulating sequence $(\frac{c}{p}; \frac{2}{bc})$ shows that

$$(pbc + 1)D < \max\{pg_3 + 2b^2, (p - 1)g_3 + b^2c^2 + bc, pb^2c^2 + bc + 1\}.$$  

By the minimality of $p$, we know that $\frac{p}{m} \leq \frac{1}{p-1}$ and $g_3 < b^2c^2 + \frac{1}{p-1}$, so, as $2b^2 + \frac{1}{p-1} \leq bc$, we have

$$pg_3 + 2b^2 < (p - 1)g_3 + bc < pb^2c^2 + bc + 1.$$  

So it must be that

$$(pbc + 1)D < pb^2c^2 + bc + 1,$$  

but this blatantly contradicts the fact that

$$pbcD \geq pb^2c^2 + 1$$  

and

$$D \geq bc.$$  

Thus, we have shown that $n_2 = 1$, so $\frac{m+n}{m_2}$ is a fair upper approximation to $0$. A comparison to the strongly triangulating sequence $(\frac{c}{p}; \frac{2}{bc})$ shows that

$$b^2c^2 + \frac{1}{m_2} = g_3 \leq bcD$$  

and

$$(bcm_2 - 1)D < (bcm_2 - 1)bc + 1,$$  

as claimed. \hfill \Box 

Lemma 4.36. Let $(m, n) = (\frac{E_{3k}}{3}, \frac{E_{k+4}}{3})$ and $0 \leq j < m + n$. Then

$$\#S_j\left(\frac{m+n}{3m} \cdot \frac{n}{m}\right) - \#S_j(\varphi^2; \varphi^4) = \begin{cases} \left\lceil \frac{j}{3m} \right\rceil & \text{if } 3 \mid j, \\ 0 & \text{otherwise}. \end{cases}$$

Remark. Note that $m, n$ are always relatively prime integers, but $\frac{m+n}{3} = \frac{E_{k+2}}{3}$ is never an integer.

Proof. Let us prove this with induction by threes. For $j < 3$, this can be checked directly. Suppose that $j \geq 3$, and the claim holds for $j - 3$. Note that, if $x, y > 0$, then

$$(x, y) \in S_j(\frac{m+n}{3m}; \frac{n}{m}) \iff (x - 1, y - 1) \in S_{j-3}(\frac{m+n}{3m}; \frac{n}{m}),$$

$$(x, y) \in S_j(\varphi^2; \varphi^4) \iff (x - 1, y - 1) \in S_{j-3}(\varphi^2; \varphi^4),$$

so we are done by induction if we can show that, for $x, y \in \mathbb{N}$,

$$(0, y) \in S_j(\frac{m+n}{3m}; \frac{n}{m}) \iff (0, y) \in S_{j-3}(\varphi^2; \varphi^4),$$

$$(x, 0) \in S_j(\frac{m+n}{3m}; \frac{n}{m}) \iff (x, 0) \in S_{j-3}(\varphi^2; \varphi^4) \cup \left\{ \left(\frac{(m+n)j}{3m}, 0\right) \right\}.$$
For the first equivalence, we have to show there is no integer \( y > 0 \) such that
\[
\phi^2 \leq \frac{j}{y} < \frac{3n}{m+n} = \frac{F_{4k+4}}{F_{4k+2}},
\]
and indeed this is true, because \( \frac{F_{4k+4}}{F_{4k+2}} \) is a fair upper approximation to \( \phi^2 \), and we are assuming \( j < m + n \leq 3n \).

For the second equivalence, we have to find all integers \( x \geq 0 \) such that
\[
\phi^2 < \frac{x}{j} \leq \frac{m+n}{3m} = \frac{F_{4k+2}}{F_{4k}}.
\]
Note that \( \frac{F_{4k+2}}{F_{4k}} \) and \( \frac{F_{4k+4}}{F_{4k+2}} \) are two consecutive fair upper approximations to \( \phi^2 \), and \( j < m + n = F_{4k+2} \), so in fact
\[
\phi^2 < \frac{x}{j} \leq \frac{m+n}{3m} \iff \frac{x}{j} = \frac{m+n}{3m},
\]
as claimed. \( \square \)

**Lemma 4.37.** Let \( (m,n) = (\frac{F_{4k}}{3}, \frac{F_{4k+4}}{3}) \). Then the sequence \( (\frac{m+n}{3} ; \frac{n}{m} , \frac{1+\varepsilon}{9} ) \) is a strongly pseudo-triangulating sequence.

**Remark.** Note that \( (m+n)^2 = 9mn+1 \), so indeed \( (\frac{m+n}{3} ; \frac{n}{m} , \frac{1+\varepsilon}{9} )^2 = mn + \frac{1}{9} \), which is of course a necessary condition for \( (\frac{m+n}{3} ; \frac{n}{m} , \frac{1+\varepsilon}{9} ) \) to be a strongly pseudo-triangulating sequence.

**Proof.** First let us prove that \( (\frac{m+n}{3} ; \frac{n}{m} , \frac{1+\varepsilon}{9} ) \) satisfies the \( j \)th triangulating condition for every \( j \) with \( 0 \leq j < m + n \). For such \( j \), in light of Lemma 4.36, it suffices to show that
\[
#S_j \left( \frac{m+n}{3m} ; \frac{n}{m} \right) - #S_j \left( \frac{m+n}{3} ; \frac{n}{m} , \frac{1+\varepsilon}{9} \right) = \begin{cases} \left\lfloor \frac{j}{3m} \right\rfloor & \text{if } 3 \mid j, \\ 0 & \text{otherwise}. \end{cases}
\]
Let
\[
T := \{ (X,Y,Z) \in \mathbb{N}^3 \mid X < n \}.
\]
Every member of \( S_j \left( \frac{m+n}{3m} ; \frac{n}{m} \right) \) can be expressed as \( (X+nZ,Y) \) for a unique \( (X,Y,Z) \in T \), just as every member of \( S_j \left( \frac{m+n}{3} ; \frac{n}{m} , \frac{1+\varepsilon}{9} \right) \) can be expressed as \( (mX+nY,Z) \) for a unique \( (X,Y,Z) \in T \). Now for \( (X,Y,Z) \in T \), it is clear that
\[
(mX+nY,Z) \in S_j \left( \frac{m+n}{3} ; \frac{n}{m} , \frac{1+\varepsilon}{9} \right) \iff \begin{aligned}
mX+nY \leq &\left( mn + \frac{1+\varepsilon}{9} \right) Z \\
\leq &\frac{(m+n)j}{3} \\
\implies &m(X+nZ)+nY \leq \frac{(m+n)j}{3} \\
\implies & (X+nZ,Y) \in S_j \left( \frac{m+n}{3m} ; \frac{n}{m} \right).
\end{aligned}
\]
Thus, if we let
\[
U_j := \{ (X,Y,Z) \in T \mid (X+nZ,Y) \in S_j \left( \frac{m+n}{3m} ; \frac{n}{m} \right) \text{ and } (mX+nY,Z) \notin S_j \left( \frac{m+n}{3} ; \frac{n}{m} , \frac{1+\varepsilon}{9} \right) \},
\]
then we must prove that
\[
#U_j = \begin{cases} \left\lfloor \frac{j}{3m} \right\rfloor & \text{if } 3 \mid j, \\ 0 & \text{otherwise}. \end{cases}
\]
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Indeed, for \((X, Y, Z) \in T\), we have \((X, Y, Z) \in U_j\) if and only if

\[
m(X + nZ) + nY < \frac{(m + n)j}{3} < mX + nY + \left( mn + \frac{1 + \varepsilon}{9} \right) Z,
\]

which, if we remember that \(\frac{(m+n)j}{3} < \frac{(m+n)^2}{3} = 3(mn + \frac{1}{9})\), is equivalent to the condition

\[
Z \in \{1, 2\} \text{ and } m(X + nZ) + nY = \frac{(m + n)j}{3}.
\]

Obviously, this can happen only when 3 \(\mid j\). For any fixed \(Z \in \{1, 2\}\), there exists at most one pair \((X, Y) \in \mathbb{N}^2\) with \(X < n\) and

\[
mX + nY = \frac{(m + n)j}{3} - mnZ,
\]

and there is such a pair \((X, Y)\) precisely when the right-hand side is a non-negative linear combination of \(m\) and \(n\). The minimum \(j\) for which this occurs is \(j = 3mZ\), since

\[
\frac{(m + n) \cdot 3mZ}{3} - mnZ = m^2Z
\]

but

\[
\frac{(m + n)(3mZ - 3)}{3} - mnZ = m(mZ - 1) - n = m(mZ - n - 1) + (m - 1)n,
\]

and both \(-1\) and \(mZ - n - 1 \leq 2m - n - 1\) are strictly negative. Thus \#\(U_j\) includes one element of the form \((X, Y, 1)\) when \(3 \mid j \geq 3m\), and one element of the form \((X, Y, 2)\) when \(3 \mid j \geq 6m\), exactly in accordance with our claims.

So indeed we have proven that \((m+n, \frac{n}{m}, \frac{1+\varepsilon}{9})\) satisfies the \(j\)th triangulating condition for every \(j\) with \(0 \leq j < m + n\). To prove the same for higher values of \(j\), let us induct in increments of \(m + n\). So suppose that \(j \geq m + n\), and \((m+n, \frac{n}{m}, \frac{1+\varepsilon}{9})\) satisfies the \((j - m - n)\)th triangulating condition. Let us count the number of elements \((W, Z) \in S_j(m+n, \frac{n}{m}, \frac{1+\varepsilon}{9})\):

- If \(Z = 0\), then \(W \leq \frac{(m+n)j}{3} \frac{n}{m}\). There are \(\left\lfloor \frac{(m+n)j}{3} \frac{n}{m} \right\rfloor + 1\) non-negative integers \(W\) with that property, of which \(\left(\frac{m-1}{2}(n-1)\right)\) fail to be non-negative linear combinations of \(m\) and \(n\). So there are

  \[
  \left\lfloor \frac{(m+n)j}{3} \right\rfloor + 1 - \left( \frac{m-1}{2}(n-1) \right)
  \]

  elements \((W, Z) \in S_j(m+n, \frac{n}{m}, \frac{1+\varepsilon}{9})\) with \(Z = 0\).

- If \(Z \in \{1, 2\}\), then \(W < \frac{(m+n)j}{3} - (mn + \frac{1}{9})Z\). There are

  \[
  \left\lfloor \frac{(m+n)j}{3} - mn - \frac{1}{9} \right\rfloor - \left( \frac{m-1}{2}(n-1) \right) = \left\lfloor \frac{(m+n)j}{3} \right\rfloor - mn - \left( \frac{m-1}{2}(n-1) \right)
  \]

  elements \((W, Z) \in S_j(m+n, \frac{n}{m}, \frac{1+\varepsilon}{9})\) with \(Z = 1\), and

  \[
  \left\lfloor \frac{(m+n)j}{3} - 2mn - \frac{2}{9} \right\rfloor - \left( \frac{m-1}{2}(n-1) \right) = \left\lfloor \frac{(m+n)j}{3} \right\rfloor - 2mn - \left( \frac{m-1}{2}(n-1) \right)
  \]

  elements \((W, Z) \in S_j(m+n, \frac{n}{m}, \frac{1+\varepsilon}{9})\) with \(Z = 2\).
• If \( Z \geq 3 \), then \((W, Z - 3) \in S_{j-3}(\frac{m+n}{3}; \frac{n}{m}, \frac{1+\varepsilon}{9})\) and

\[
(W, Z) \neq \left( \frac{(j - (m + n))(m + n)}{3}, 3 \right).
\]

By the induction hypothesis, there are

\[
\begin{cases} 
1 & \text{if } j \equiv m + n \pmod{3} \\
0 & \text{otherwise}
\end{cases}
\]

elements \((W, Z) \in S_{j}(\frac{m+n}{3}; \frac{n}{m}, \frac{1+\varepsilon}{9})\) with \( Z \geq 3 \).

So it remains to prove that the four numbers above add up to \( \binom{j+2}{2} \), which is to say

\[
\binom{j+2}{2} = \binom{(m+n)j}{3} + 2 \binom{(m+n)j}{3} + 1 - 3mn - \frac{3(m-1)(n-1)}{2} + \binom{j-m-n+2}{2}
\]

\[
- \begin{cases} 
1 & \text{if } j \equiv m + n \pmod{3} \\
0 & \text{otherwise}
\end{cases}
\]

Note that

\[
3mn + \frac{3(m-1)(n-1)}{2} = \frac{9mn - 3(m+n) + 3}{2} = \frac{(m+n)^2 - 3(m+n) + 2}{2} = \frac{m+n-1}{2},
\]

and

\[
\frac{(m+n)j}{3} + 2 \frac{(m+n)j}{3} + 1 - 3mn - \frac{3(m-1)(n-1)}{2} + \binom{j-m-n+2}{2} = (m+n)j,
\]

and it remains to prove that

\[
(m+n)j = \left\lfloor \frac{(m+n)j}{3} \right\rfloor + 2 \left\lfloor \frac{(m+n)j}{3} \right\rfloor - \begin{cases} 
1 & \text{if } j \equiv m + n \pmod{3} \\
0 & \text{otherwise}
\end{cases}
\]

Well, recall that \( \frac{m+n}{3} \) is never an integer. If \( j \equiv 0 \pmod{3} \), this if obviously true. If \( j \equiv m + n \pmod{3} \), then

\[
\left\lfloor \frac{(m+n)j}{3} \right\rfloor = \frac{(m+n)j - 1}{3}, \quad 2 \left\lfloor \frac{(m+n)j}{3} \right\rfloor = 2 \cdot \frac{(m+n)j + 2}{3},
\]

so this is true again. Finally, if \( j \equiv -(m+n) \pmod{3} \), then

\[
\left\lfloor \frac{(m+n)j}{3} \right\rfloor = \frac{(m+n)j - 2}{3}, \quad 2 \left\lfloor \frac{(m+n)j}{3} \right\rfloor = 2 \cdot \frac{(m+n)j + 1}{3},
\]

so yet again it is true. Thus, we conclude by induction that \( \left( \frac{m+n}{3}; \frac{n}{m}, \frac{1+\varepsilon}{9} \right) \) is strongly pseudo-triangulating.

\[\square\]

**Lemma 4.38.** If \((m, n) = (\frac{F_k}{3}, \frac{F_{k+1}}{3})\), and \((D; \frac{n}{m}, \frac{n^2}{m^2})\) is weakly triangulating, then \( \frac{n^2}{m^2} \) must be a fair upper approximation to \( \frac{1}{9} \).
Proof. By Proposition 4.19, we know this to be true if

\[ 0 < \frac{n_2}{m_2} < \frac{1}{6}, \]

and \( \frac{1}{6} \) itself is a fair upper approximation to \( \frac{1}{6} \). So let us suppose that \( \frac{n_2}{m_2} > \frac{1}{6} \), and \( \frac{n_2}{m_2} \) is not a fair upper approximation to \( \frac{1}{6} \). Let \( p \in \{1, 2, 3, 4, 5, 6\} \) be the least positive integer such that \( \frac{1}{p} < \frac{n_2}{m_2} \). Note that if \( p > 1 \), then

\[ \frac{1}{p} < \frac{n_2}{m_2} < \frac{1}{p-1}, \]

so that \( m_2 \geq 2p - 1 \).

Our proof proceeds by applying the \( j \)th triangulating condition for certain values of \( j \), listed below. For each of these values of \( j \), we will endeavor to show

\[ S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) = S_j \left( \frac{m+n}{3}; \frac{n}{m}, \frac{1}{9} + \epsilon \right), \]

or something close enough to that for our purposes. Then we will derive several inequalities involving \( g_3 \) and \( D \), culminating in a contradiction one way or another. Let us begin by noting that \( \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) \) is required to satisfy the \( j \)th triangulating condition whenever \( j < \frac{m_2(m+n)}{3} \), and in particular in each of the following six cases:

1. When \( j = n - 2m \), as long as \( m_2 \geq 2 \), since \( n - 2m < \frac{2(m+n)}{3} \).

2. When \( j = \frac{2(m+n) + \eta - 9}{3} \), as long as \( m_2 \geq 2 \), if for this proof \( \eta \) denotes the unique number such that \( \eta \in \{4, 5\} \) and \( \eta \equiv m + n \pmod{3} \), just as in subsection 4.4.3.

3. When \( j = m + n \), provided that \( m_2 \geq 4 \);

4. When \( j = 3(m + i) < \frac{(p+1)(m+n)}{3} \), provided that \( p \geq 2 \);

5. When \( j = 2(m + n) \), provided that \( m_2 \geq 7 \);

6. When \( j = 3i + m + n < \frac{(p+1)(m+n)}{3} \), provided that \( p \geq 3 \).

Let us consider each of these six cases in turn, and derive some consequences from each.

Step 1. Let \( j = n - 2m = F_{4k+1} \), and note that

\[ j \cdot (m + n) - 3m \cdot (2n - m) = F_{4k+1} \cdot F_{4k+2} - F_{4k} \cdot F_{4k+3} = 1, \]

so

\[ m(2n - m) < m(n - m) + \left( mn + \frac{1}{9} \right) < m(2n - m) + \frac{1}{3} = \frac{j(m+n)}{3}. \]

Therefore,

\[ (m(2n - m), 0), (m(n - m), 1) \in S_j \left( \frac{m+n}{3}; \frac{n}{m}, \frac{1}{9} + \epsilon \right), \]

and a comparison of \( S_j \left( \frac{m+n}{3}; \frac{n}{m}, \frac{1}{9} + \epsilon \right) \) with \( S_j (D; \frac{n}{m}, \frac{n_2}{m_2}) \) shows that

\[ \min \{m(2n - m) + 1, m(n - m) + g_3\} \leq jD < \max \{m(2n - m) + 1, m(n - m) + g_3\}. \]
Now, from the bounds in subsection 4.4.3, we know that
\[ \frac{m+n}{3} \leq D < \frac{m+n}{3} + \frac{1}{m+n-\eta}. \]

It is not hard to see that \( \frac{1}{m+n-\eta} \leq \frac{2}{3j} \), so
\[ D < \frac{m+n}{3} + \frac{2}{3j} = \frac{j(m+n) + 2}{3j} = \frac{3m(2n-m) + 3}{3j} = \frac{m(2n-m) + 1}{j}, \]
so we conclude that
\[ m(n-m) + g_3 \leq jD < m(2n-m) + 1, \]
or in other words
\[ \frac{n_2}{m_2} = g_3 - mn \leq (n - 2m)D - m(2n-m) = \frac{1}{3} + (n - 2m) \left( D - \frac{m+n}{3} \right). \quad (4.5.6) \]

In particular,
\[ \frac{n_2}{m_2} < \frac{1}{3} + j \cdot \frac{2}{3j} = 1, \quad (4.5.7) \]
and \( p \geq 2 \).

Step 2. Let \( j := \frac{2(m+n)+(\eta-9)}{3} \), and note that \( \frac{5n+14m+n-9}{9}, \frac{5n-31m+n-9}{9} \in \mathbb{N}, \) and
\[ \frac{5m+5n+\eta-9}{9} m + \frac{5n-31m+\eta-9}{9} n = \frac{j + m + n}{3} m + \frac{j + n - 11m}{3} n = \frac{j(m+n) + 1}{3} - mn, \]
so that
\[ \frac{j(m+n) + 1}{3} - mn \in W_2, \]
while
\[ \frac{j(m+n) + 1}{3} < \left( \frac{j(m+n) + 1}{3} - mn \right) + \left( mn + \frac{1}{9} \right) < \frac{j(m+n)}{3}. \]

Therefore,
\[ \left( \frac{j(m+n) + 1}{3}, 0 \right), \quad \left( \frac{j(m+n) + 1}{3} - mn, 1 \right) \notin S_j \left( \frac{m+n}{3}; \frac{n}{m}, \frac{1}{9} + \varepsilon \right), \]
and a comparison of \( S_j \left( \frac{m+n}{3}; \frac{n}{m}, \frac{1}{9} + \varepsilon \right) \) with \( S_j(D; \frac{n}{m}, \frac{p_2}{m_2}) \) shows that
\[ \min \left\{ \frac{j(m+n) + 1}{3}, \frac{j(m+n) + 1}{3} - mn + g_3 - 1 \right\} \leq jD \]
\[ < \max \left\{ \frac{j(m+n) + 1}{3}, \frac{j(m+n) + 1}{3} - mn + g_3 - 1 \right\}. \]

From (4.5.7) we know that \( g_3 < mn + 1, \) so
\[ \frac{j(m+n) + 1}{3} - mn + g_3 - 1 \leq jD < \frac{j(m+n) + 1}{3}. \]

Thus,
\[ D < \frac{m+n}{3} + \frac{1}{3j} = \frac{m+n}{3} + \frac{1}{2(m+n) + \eta - 9}. \quad (4.5.8) \]
Putting this together with (4.5.6), we have, after a straightforward calculation,
\[
\frac{n_2}{m_2} \leq \frac{1}{3} + (n - 2m) \left( D - \frac{m + n}{3} \right) < \frac{1}{3} + \frac{n - 2m}{2(m + n) + \eta - 9} < \frac{2}{3}.
\]
In particular, \( \frac{n_2}{m_2} \neq \frac{2}{3} \), so \( m_2 \geq 4 \).

Step 3. Let \( j = m + n \); since we showed in Step 2 that \( m_2 \geq 4 \), we know that \( (D; \frac{n}{m}, \frac{n_2}{m_2}) \) must satisfy the \( j \)th triangulating condition. Note that
\[
3mn < mn + 2 \left( mn + \frac{1}{9} \right) < \frac{j(m + n)}{3},
\]
so
\[
(3mn, 0), (mn, 2) \in S_j \left( \frac{m + n}{3}; \frac{n}{m}, \frac{1}{9} + \varepsilon \right).
\]
A comparison of \( S_j \left( \frac{m+n}{3}; \frac{n}{m}, \frac{1}{9} + \varepsilon \right) \) with \( S_j(D; \frac{n}{m}, \frac{n_2}{m_2}) \) shows that
\[
\min\{3mn + 1, mn + 2g_3\} \leq jD < \max\{3mn + 1, mn + 2g_3\}.
\]
But with a bit of arithmetic, we know from Step 2 that
\[
D < \frac{m + n}{3} + \frac{1}{2(m + n) + \eta - 9} < \frac{m + n}{3} + \frac{2}{3(m + n)} = \frac{3mn + 1}{m + n},
\]
so it must be that
\[
\frac{mn + 2g_3}{(m + n)D} < 3mn + 1.
\]
(4.5.9)
In particular, \( \frac{n_2}{m_2} < \frac{1}{2} \), so \( p \geq 3 \) and \( m_2 \geq 2p - 1 = 5 \).

Step 4. Let \( i \) be any positive integer such that \( 3(i + m) < (\frac{p+1)(m+n)}{3}) \); such \( i \) exists because \( p \geq 3 \). Let \( j = 3(i + m) \). Note that
\[
\frac{m + n}{3} \leq \frac{(m + n)i + m^2 + mn + \frac{1}{9}}{3i + 3m},
\]
so
\[
((m + n)i + m^2, 1) \notin S_j \left( \frac{m + n}{3}; \frac{n}{m}, \frac{1}{9} + \varepsilon \right).
\]
We claim that
\[
((m + n)i + m^2, 1) \notin S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right),
\]
as well. If this were not the case, then by the \( j \)th triangulating condition, the sets \( S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) \) and \( S_j \left( \frac{m+n}{3}; \frac{n}{m}, \frac{1}{9} + \varepsilon \right) \) have the same size, so there would need to exist some
\[
(x, y) \in S_j \left( \frac{m + n}{3}; \frac{n}{m}, \frac{1}{9} + \varepsilon \right) \setminus S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right).
\]
Since \( \frac{n_2}{m_2} > \frac{1}{9} \), we must have \( x < (m + n)i + m^2, y > 1 \), and in fact
\[
\frac{mn + \frac{1}{9}}{y - 1} < \frac{(m + n)i + m^2 - x}{y - 1} < \frac{mn + n_2}{m_2}.
\]
By assumption, \( \frac{1}{p} \) is the simplest fraction strictly between \( \frac{1}{9} \) and \( \frac{m_2}{m_2} \), so it must be the case that \( y - 1 \geq p \). In particular,

\[
(0, p + 1) \in S_j \left( \frac{m + n}{3}; \frac{n}{m}, \frac{1}{9} + \varepsilon \right),
\]

which contradicts the assumption that \( j < \frac{(p+1)(m+n)}{3} \). So indeed we must have \( ((m+n)i+m^2, 1) \notin S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2} \right) \), which is to say

\[
(m+n)i+m^2+g_3 > 3(m + i)D.
\]

If we combine this with (4.5.9), we find that

\[
\frac{mn + 2g_3}{m + n} \leq D < \frac{(m+n)i+m^2+g_3}{3(i+m)},
\]

so, with a bit of arithmetic,

\[
\frac{6n_2/m_2 - 1}{3(m+n)} \leq D - \frac{m + n}{3} < \frac{n_2/m_2}{3(i+m)},
\]

and in particular

\[
\frac{n_2}{m_2} < \frac{i + m}{6(i + m) - (m + n)}.
\]

We already showed in Step 3 that \( p \geq 3 \). If \( p = 3 \), then we may set \( i + m = \frac{4(m+n)-\eta}{g} \), so that

\[
\frac{n_2}{m_2} < \frac{i + m}{6(i + m) - (m + n)} = \frac{4(m+n)-\eta}{6[4(m+n)-\eta]-9(m+n)} = \frac{4(m+n)-\eta}{15(m+n) - 6\eta} < \frac{1}{3} = \frac{1}{p},
\]

a contradiction. If \( p = 4 \), then we may set \( i + m = \frac{5(m+n)+\eta-9}{g} \), so that

\[
\frac{n_2}{m_2} < \frac{i + m}{6(i + m) - (m + n)} = \frac{5(m+n)+\eta-9}{6[5(m+n)+\eta-9]-9(m+n)} = \frac{5(m+n)+\eta-9}{21(m+n) + 6(\eta-9)} < \frac{1}{4} = \frac{1}{p},
\]

a contradiction.

So we conclude that \( p > 5 \). Thus, we may set \( i + m = \frac{6(m+n)+3(\eta-6)}{g} = \frac{2(m+n)+\eta-6}{3} \), so

\[
D - \frac{m + n}{3} < \frac{n_2/m_2}{3(i+m)} = \frac{1}{2(m+n)+\eta-6} \cdot \frac{n_2}{m_2}.
\]

For future reference, let us note in particular that

\[
D - \frac{m + n}{3} < \frac{1}{4[2(m+n)+\eta-6]} < \frac{1}{6(m+n)}.
\]

Step 5. Let \( j = 2(m+n) \). We know from Step 4 that \( p \geq 5 \), so in particular \( m_2 \geq 2p - 1 \geq 9 \geq 7 \), and \( (D; \frac{n}{m}, \frac{n_2}{m_2}) \) satisfies the \( j \)th triangulating condition. Now note that \( (0, 6) \) just barely fails to belong to \( S_j \left( \frac{m+n}{3}; \frac{n}{m}, \frac{1}{9} + \varepsilon \right) \), and for each integer \( y \) such that \( 0 \leq y \leq 5 \), we have \( (x, y) \in S_j \left( \frac{m+n}{3}; \frac{n}{m}, \frac{1}{9} + \varepsilon \right) \) if and only if \( x \in W_2 \) and

\[
x + \left( mn + \frac{1}{9} \right) y \leq \frac{2(m+n)^2}{3} = \frac{2(9mn + 1)}{3} = 6mn + \frac{2}{3}.
\]
with strict inequality if \( y > 0 \). But in all cases at hand, \( \frac{y}{9} < \frac{2}{9} \leq \frac{5}{9} \), so really the inequality is equivalent to
\[
x + mny \leq 6mn,
\]
with no strictness requirement even when \( y > 0 \). This means that a comparison between \( S_j\left(\frac{m+n}{3} ; \frac{n}{m}, \frac{1}{3} + \epsilon\right) \) and \( S_j(D; \frac{n}{m}, \frac{n_2}{m_2}) \) shows that
\[
\min\{mn + 5g_3, 6mn + 1\} \leq jD < \max\{mn + 5g_3, 6mn + 1\}.
\]
In Step 4, we proved that
\[
D - \frac{m+n}{3} < \frac{1}{6(m+n)},
\]
so
\[
jD < \frac{2(m+n)^2}{3} + \frac{2(m+n)}{6(m+n)} = 6mn + \frac{2}{3} + \frac{1}{3} = 6mn + 1.
\]
It follows that
\[
mn + 5g_3 \leq jD < 6mn + 1,
\]
In particular, \( \frac{n_2}{m_2} < \frac{1}{3} \), so \( p = 6 \). Let us rewrite the above result as
\[
D - \frac{m+n}{3} \geq \frac{mn + 5g_3}{j} - \frac{m+n}{3} = \frac{15n_2/m_2 - 2}{6(m+n)}. \tag{4.5.10}
\]

Step 6. Let \( i \) be any positive integer such that \( i < \frac{(p-2)(m+n)}{9} \); such \( i \) exist because \( p = 6 \). Let \( j = 3i + m + n \), and note that \( j < \frac{(p+1)(m+n)}{3} < \frac{m_2(m+n)}{3} \), so the \( j \)th triangulating condition must hold. We claim that \( (3i + m + n)D < (m + n)i + 3g_3 \). Indeed, if this were false, then we would have
\[
(i(m+n), 3) \in S_j\left(D; \frac{n_2}{m_2}\right) \setminus S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right).
\]
Since \( \frac{n_2}{m_2} > \frac{1}{3} \), for every \((x, y) \in S_j\left(\frac{m+n}{3} ; \frac{n}{m}, \frac{1}{9} + \epsilon\right) \setminus S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right)\), we must have \( x < i(m+n) \) and \( 3 < y < 7 \), but then
\[
(2i(m+n) - x, 6 - y) \in S_j\left(D; \frac{n_2}{m_2}\right) \setminus S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right).
\]
The upshot is that \( \#S_j\left(D; \frac{n}{m}, \frac{n_2}{m_2}\right) > \#S_j\left(\frac{m+n}{3} ; \frac{n}{m}, \frac{1}{9} + \epsilon\right) \), so the \( j \)th triangulating condition cannot possibly hold for \( D; \frac{n}{m}, \frac{n_2}{m_2} \), contrary to our assumption. We conclude that in fact
\[
jD < (m + n)i + 3g_3
\]
which is to say
\[
j \left( D - \frac{m+n}{3} \right) = (3i + m + n) \left( D - \frac{m+n}{3} \right) < 3g_3(mn) - \frac{1}{3} = \frac{3n_2}{m_2} - \frac{1}{3}.
\]
Let us combine this with (4.5.10):
\[
\frac{15n_2/m_2 - 2}{6(m+n)} < D - \frac{m+n}{3} < \frac{9n_2/m_2 - 1}{3j},
\]
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from which it follows that
\[
\frac{n_2}{m_2} < \frac{2i}{15i - (m + n)},
\]
provided that the denominator is positive. Since \( p = 6 \), we may in fact set \( i = \frac{4(m + n) - \eta}{9} \), so that
\[
\frac{n_2}{m_2} < \frac{2[4(m + n) - \eta]}{15[4(m + n) - \eta] - 9(m + n)} = \frac{8(m + n) - 2\eta}{51(m + n) - 9\eta} < \frac{1}{6},
\]
a contradiction. \( \Box \)

**Lemma 4.39.** If \((m, n) = (\frac{F_k}{3}, \frac{F_{k+4}}{3})\), and \((D; \frac{n}{m}, \frac{n_2}{m_2})\) is weakly triangulating, then
\[
D \geq \frac{m + n}{3},
\]
and one of the following must hold:

- \( \frac{n_2}{m_2} = \frac{1}{2} \) and \( m_2 D < \frac{(m+n)m_2}{3} + \frac{2}{2(m+n)+\eta-9} \),
- \( \frac{n_2}{m_2} = \frac{1}{3} \) and \( m_2 D < \frac{(m+n)m_2}{3} + \frac{1}{m+n-1} \),
- \( \frac{n_2}{m_2} = \frac{1}{4} \) and \( m_2 D < \frac{(m+n)m_2}{3} + \frac{3}{4(m+n)-\eta} \),
- \( \frac{n_2}{m_2} = \frac{1}{5} \) and \( m_2 D < \frac{(m+n)m_2}{3} + \frac{3}{5(m+n)+\eta-9} \),
- \( \frac{n_2}{m_2} = \frac{1}{6} \) and \( m_2 D < \frac{(m+n)m_2}{3} + \frac{1}{2(m+n)-1} \),
- \( \frac{n_2}{m_2} = \frac{1}{7} \) and \( m_2 D < \frac{(m+n)m_2}{3} + \frac{2}{7(m+n)-\eta} \),
- \( \frac{n_2}{m_2} = \frac{n_2}{m_2-1} \) and \( m_2 D < \frac{(m+n)m_2}{3} + \frac{1}{m_2(m+n)+\eta-9} \).

**Proof.** Note \((m + n)^2 = 9mn + 1\) and \( g_3 = mn + \frac{n_2}{m_2} \). Let \( j \) be the integer closest to \( \frac{m_2(m+n)-4}{3} \). By Lemma 4.38, \( \frac{n_2}{m_2} \) must appear on the above list, and the lower bound on \( D \) comes from the lower bound in subsection 4.4.3, so it just remains to derive the upper bounds on \( D \).

- If \( \frac{n_2}{m_2} = \frac{1}{2} \), then \( j = \frac{2(m+n)+\eta-9}{3} \in \mathbb{N} \), and the \( j \)th triangulating condition implies that
  \[
  jD < \frac{(m + n)j + 1}{3} = 2mn + \frac{(\eta - 9)(m + n) + 5}{9} \in W_2,
  \]
  so
  \[
  D < \frac{m + n}{3} + \frac{1}{3j},
  \]
as desired.

- If \( \frac{n_2}{m_2} = \frac{1}{3} \), then \( j = m + n - 1 \). If \( \eta = 4 \), then the \( j \)th triangulating condition implies that
  \[
  jD < \frac{(m + n)j}{3} + \frac{n_2}{m_2} = g_3 + 2mn - \frac{m + n - 1}{3} \in g_3 + W_2,
  \]
  whereas if \( \eta = 5 \), then the \( j \)th triangulating condition implies that
  \[
  jD < \frac{(m + n)j + 1}{3} = 3mn - \frac{m + n - 2}{3} \in W_2.
  \]
So in either case,
\[ D < \frac{m+n}{3} + \frac{1}{3j}, \]
as desired.

- If \( \frac{n_2}{m_2} = \frac{1}{4} \), then \( j = \frac{4(m+n)-\eta}{3} \in \mathbb{N} \), and the \( j \)th triangulating condition implies that
  \[ jD < \frac{(m+n)j}{3} + \frac{n_2}{m_2} = g_3 + 3mn - \frac{\eta(m+n) - 4}{9} \in g_3 + W_2, \]
  so
  \[ D < \frac{m+n}{3} + \frac{1}{4j}, \]
as desired.

- If \( \frac{n_2}{m_2} = \frac{1}{5} \), then \( j = \frac{5(m+n)+\eta-9}{3} \in \mathbb{N} \) and
  \[ jD < \frac{(m+n)j}{3} + \frac{n_2}{m_2} = g_3 + 4mn + \frac{(\eta-9)(m+n) + 5}{9} \in g_3 + W_2, \]
  so
  \[ D < \frac{m+n}{3} + \frac{1}{5j}, \]
as desired.

- If \( \frac{n_2}{m_2} = \frac{1}{6} \), then \( j = 2(m+n) - 1 \). If \( \eta = 4 \), then the \( j \)th triangulating condition implies that
  \[ jD < \frac{(m+n)j}{3} + 3\left(\frac{n_2}{m_2} - \frac{1}{9}\right) = 3g_3 + 3mn - \frac{m+n-1}{3} \in 3g_3 + W_2, \]
  whereas if \( \eta = 5 \), then the \( j \)th triangulating condition implies that
  \[ jD < \frac{(m+n)j}{3} + \frac{n_2}{m_2} = g_3 + 5mn - \frac{m+n-2}{3} \in g_3 + W_2. \]
  So in either case,
  \[ D < \frac{m+n}{3} + \frac{1}{6j}, \]
as desired.

- If \( \frac{n_2}{m_2} = \frac{1}{7} \), then \( j = \frac{7(m+n)-\eta}{3} \) and
  \[ jD < \frac{(m+n)j}{3} + 3\left(\frac{n_2}{m_2} - \frac{1}{9}\right) = 3g_3 + 4mn - \frac{\eta(m+n) - 4}{9} \in g_3 + W_2, \]
  so
  \[ D < \frac{m+n}{3} + \frac{2}{21j} \]
as desired.

- If \( \frac{n_2}{m_2} = \frac{n_2}{9n_2-1} \), then \( j = \frac{n_2(m+n)+\eta-9}{3} \) and the \( j \)th triangulating condition implies that
  \[ jD < \frac{(m+n)j}{3} + 3\left(\frac{n_2}{m_2} - \frac{1}{9}\right) = 3g_3 + (m_2-3)mn + n_2 + \frac{(\eta-9)(m+n) - 4}{9} \in g_3 + W_2, \]
  so
  \[ D < \frac{m+n}{3} + \frac{1}{3m_2j}, \]
as desired.

So in all cases we have proved the Lemma.

\( \square \)
4.6 Assorted results on triangulating sequences of length 3

In principle, we could keep going indefinitely, and try to classify triangulating sequences of all lengths. Perhaps there is a general pattern waiting to be discovered. In the meantime, let us prove some partial results on length 3 triangulating sequences.

**Lemma 4.40.** There are no weakly triangulating sequences \((D; \frac{n_1}{m_1}, \frac{n_2}{m_2}, \frac{n_3}{m_3})\) with \(m_1, m_2, m_3 \geq 2\) in the following cases:

- \(\frac{n_1}{m_1} = \frac{c^2}{b^2}\), where \(b\) and \(c\) are consecutive odd-indexed Fibonacci numbers, with \(2 \leq b < c\).
- \(\frac{n_1}{m_1} = \frac{F_{4k+4/3}}{F_{4k+5/3}}\), where \(k \geq 2\).
- \(\frac{n_1}{m_1} = \frac{3}{2}\) and \(\frac{n_2}{m_2} > \frac{46}{9}\).

**Proof.** Suppose for the sake of contradiction that \((D; \frac{n_1}{m_1}, \frac{n_2}{m_2}, \frac{n_3}{m_3})\) really is triangulating.

Part 1: \(\frac{n_1}{m_1} = \frac{c^2}{b^2}\). Then by Lemma 4.35, we have \(n_2 = 1\) and

\[
\frac{bcn_2}{bc} < D < \frac{m_2}{bcn_2 - 1}.
\]

Thus, \((D; \frac{n_1}{m_1}, \frac{n_2}{m_2}, \frac{n_3}{m_3})\) must satisfy the \(j\)th triangulating condition for \(j \leq bcn_2m_3\), and, in particular, for \(j \leq bcn_2 + 5\).

Note that the first 20 elements of the semigroup generated by \(b^2\) and \(c^2\) are, in ascending order,

\[
0, b^2, \frac{2b^2}{3}, \frac{3b^2}{4}, \frac{4b^2}{5}, \frac{5b^2}{6}, \frac{6b^2}{7}, \frac{7b^2}{8}, b^2 + c^2, \frac{2b^2 + c^2}{9}, \frac{3b^2 + c^2}{10}, \frac{4b^2 + c^2}{11}, \frac{5b^2 + c^2}{12}, \frac{6b^2 + c^2}{138}.
\]

Using this list, and paying particular attention to the positions of the boxed expressions above, we may apply the \(j\)th triangulating conditions for \(j \in \{bcn_2 + 1, bcn_2 + 5\}\). For \(j = bcn_2 + 1\), we find that

\[
\min\{(g_3 + bc)m_2, g_4 + 2b^2m_2\} \leq (bcn_2 + 1)D < \max\{(g_3 + bc)m_2, g_4 + 2b^2m_2\}.
\]

But as

\[
D \geq bcn_2 + \frac{1}{bc} \geq \frac{(g_3 + bc)m_2}{bcn_2 + 1},
\]

the only possibility is

\[(g_3 + bc)m_2 \leq (bcn_2 + 1)D < g_4 + 2b^2m_2.
\]

For \(j = bcn_2 + 5\), we find that

\[
\min\{(g_3 + 5bc)m_2 + 1, g_4 + (6b^2 + c^2)m_2\} \leq (bcn_2 + 5)D < \max\{(g_3 + 5bc)m_2 + 1, g_4 + (6b^2 + c^2)m_2\}.
\]

But as

\[
D < bcn_2 + \frac{m_2}{bcn_2 - 1} < \frac{(g_3 + 5bc)m_2 + 1}{bcn_2 + 5},
\]

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the only possibility is that
\[ g_4 + (6b^2 + c^2)m_2 \leq (bcm_2 + 5)D < (g_3 + 5bc)m_2 + 1. \]
Thus,
\[ (g_3 + bc - 2b^2)m_2 < g_4 < (g_3 + 5bc - 6b^2 - c^2)m_2 + 1, \]
and in particular
\[ bc - 2b^2 \leq 5bc - 6b^2 - c^2, \]
or
\[ (c - 2b)^2 \leq 0, \]
which is patently absurd, since \( c > 2b \).

Part 2: \( \frac{n_2}{m_1} = \frac{n}{m} = \frac{F_{n+4}}{F_{n+3}} \). Then by Lemma 4.38 we know that \( \frac{n_2}{m_2} \) is a fair upper approximation to \( \frac{1}{n} \), so either \( n_2 = 1 \) and \( m_2 \leq 7 \), or else \( m_2 = 9n_2 - 1 \). Note that \( g_3 = mn \cdot m_2 + n_2 \). Let us consider the various cases:

- If \( \frac{n_2}{m_2} = \frac{1}{3} \), then let \( j = m + n \). The \( j \)th triangulating condition implies that, no matter how small \( \frac{n_2}{m_3} \) is (as long as it is positive), we must have
  \[ mn \cdot m_2 + 2g_3 \leq (m + n)D, \]
or else the set \( S_j \left( D; \frac{n}{m}, \frac{n_2}{m_2}, \frac{n_3}{m_3} \right) \) will simply be too small. Thus,
  \[ D \geq \frac{mn \cdot m_2 + 2g_3}{m + n} = \frac{(m + n)m_2}{3} + \frac{4}{3(m + n)}. \]

Lemma 4.39 says that
\[ D < \frac{(m + n)m_2}{3} + \frac{2}{2(m + n) + \eta - 9}, \]
so
\[ \frac{4}{3(m + n)} < \frac{2}{2(m + n) + \eta - 9}, \]
which is false.

- The case of \( \frac{n_2}{m_2} = \frac{1}{4} \) will be treated together with the case of \( \frac{n_2}{m_2} = \frac{1}{6} \); see below.

- If \( \frac{n_2}{m_2} = \frac{1}{4} \), then let \( j = 2n - m \). The \( j \)th triangulating condition implies that
  \[ m(n - m) \cdot m_2 + 4g_3 \leq (2n - m)D, \]
which is to say
  \[ D \geq \frac{m(n - m) \cdot m_2 + 4g_3}{2n - m} = \frac{(m + n)m_2}{3} + \frac{4}{3(2n - m)}. \]

Lemma 4.39 says that
\[ D < \frac{(m + n)m_2}{3} + \frac{3}{4(m + n) - \eta}, \]
so
\[ \frac{4}{3(2n - m)} < \frac{3}{4(m + n) - \eta}, \]
which is false.
• If $\frac{m_n}{m_2} = \frac{1}{k}$, then let $j = 2(m + n)$. The $j$th triangulating condition implies that

$$mn \cdot m_2 + 5g_3 \leq 2(m + n)D,$$

which is to say

$$D \geq \frac{mn \cdot m_2 + 5g_3}{2(m + n)} = \frac{(m + n)m_2}{3} + \frac{5}{6(m + n)}.$$

Lemma 4.39 says that

$$D < \frac{(m + n)m_2}{3} + \frac{3}{5(m + n) + \eta - 9},$$

so

$$\frac{5}{6(m + n)} < \frac{3}{5(m + n) + \eta - 9},$$

which is false.

• If $\frac{m_n}{m_2} = \frac{1}{k}$ with $k \in \{1, 2\}$, then for $j = k(m + n)$ the $j$th triangulating condition implies that

$$1 + k^2(m + n)^2 = 9k^2mn + 3k - 1 = mn \cdot m_2 + (3k - 1)g_3 \leq k(m + n)D.$$

Note that $mn \cdot m_2 + (3k - 1)g_3$ is large enough that every integer exceeding it belongs to $W_3$, and that the first 20 elements of the semigroup generated by $m$ and $n$ are, in ascending order,

0, m,  
$\underline{2m}$, 3m, 4m,  
5m, 6m, n, 7m,  
$m + n$, 8m, 2m + n, 9m, 3m + n,  
10m, 4m + n, 11m, 5m + n, 12m, $\underline{6m + n}$.

Using this list, and paying particular attention to the positions of the boxed expressions above, we may apply the $j$th triangulating conditions for $j \in \{k(m + n) + 1, k(m + n) + 5\}$. For $j = k(m + n) + 1$, the $j$th triangulating condition says that

$$\min \{g_4 + 2mm_2, 1 + k(m + n)(k(m + n) + 1)\} \leq (k(m + n) + 1)D$$

$$< \max \{g_4 + 2mm_2, 1 + k(m + n)(k(m + n) + 1)\}.$$

But recall and compute that

$$D \geq \frac{1 + k^2(m + n)^2}{k(m + n)} = \frac{1}{k(m + n)} + \frac{1}{k(m + n)} > k(m + n) + \frac{1}{k(m + n) + 1},$$

so it must be that

$$1 + k(m + n)(k(m + n) + 1) \leq (k(m + n) + 1)D < g_4 + 2mm_2.$$

For $j = k(m + n) + 5$, the $j$th triangulating condition says that

$$\min \{g_4 + (6m + n)m_2, 2 + k(m + n)(k(m + n) + 5)\}$$

$$\leq (k(m + n) + 5)D < \max \{g_4 + (6m + n)m_2, 2 + k(m + n)(k(m + n) + 5)\}.$$
But recall and compute that
\[ D < k(m + n) + \frac{1}{k(m + n) - 1} < k(m + n) + \frac{2}{k(m + n) + \xi}, \]
so it must be that
\[ g_4 + (6m + n)m_2 \leq (k(m + n) + 5)D < 2 + k(m + n)(k(m + n) + 5). \]
Thus, putting this together with our results for \( j = k(m + n) + 1 \), we see that
\[ 1 + k(m + n)(k(m + n) + 1) - 2mm_2 < g_4 < 2 + k(m + n)(k(m + n) + 5) - (6m + n)m_2. \]
In particular, this implies that
\[ 3k(4m + n) = (4m + n)m_2 < k(m + n)(5 - 1) + (2 - 1) = 4k(m + n) + 1, \]
and
\[ 3(4m + n) \leq 4(m + n), \]
which is easily seen to be false.

- If \( \frac{n}{m_2} = \frac{1}{5} \), then let \( j = 3(m + n) \). The \( j \)th triangulating condition says that
  \[ \min\{(8mn + 1)m_2 + g_3, \ mn \cdot m_2 + g_3 + g_4\} \leq 3(m + n)D \]
  \[ < \max\{(8mn + 1)m_2 + g_3, \ mn \cdot m_2 + g_3 + g_4\}, \]
but
\[ g_4 \geq g_3m_2 = (7mn + 1)m_2, \]
so in fact we must have
\[ (8mn + 1)m_2 + g_3 \leq 3(m + n)D, \]
which is to say
\[ D \geq \frac{(8mn + 1)m_2 + g_3}{3(m + n)} = \frac{(m + n)m_2}{3} + \frac{1}{3(m + n)}. \]
Lemma 4.39 says that
\[ D < \frac{(m + n)m_2}{3} + \frac{2}{7(m + n) - \eta}, \]
so we must have
\[ \frac{1}{3(m + n)} < \frac{2}{7(m + n) - \eta}, \]
which is false.

- If \( \frac{n_2}{m_2} = \frac{n_2}{m_2 - 1} \), then let \( j = (3n_2 + 2)(m + n) \). The \( j \)th triangulating condition says that
  \[ (m + n)^2m_2n_2 + 6g_3 \leq (3n_2 + 2)(m + n)D, \]
which is to say
\[ D \geq \frac{(m + n)m_2}{3} + \frac{2}{3(3n_2 + 2)(m + n)}; \]
Lemma 4.39 says that
\[ D < \frac{(m+n)m_2}{3} + \frac{1}{m_2(m+n)+\eta-9}, \]
so we must have
\[ \frac{2}{3(3n_2+2)(m+n)} < \frac{1}{m_2(m+n)+\eta-9}, \]
which is false.

Part 3: \( \frac{n_1}{m_1} = \frac{3}{2} \) and \( \frac{n_2}{m_2} > \frac{46}{9} \). Then by Corollary 4.29, \( \frac{n_2}{m_2} \) is a fair upper approximation to \( \frac{46}{9} \). The cases where \( n_2 = 1 \) and \( m_2 \leq 7 \) can be checked by computer, so let us pass to the case where
\[ \frac{n_2}{m_2} = \frac{46k-5}{9k-1} \]
for some \( k \geq 1 \). Let \( j = 30k + 20 \). The \( j \)th triangulating condition implies that, no matter how small \( \frac{n_2}{m_2} \) is (as long as it is positive), we must have
\[ 100k \cdot m_2 + 6g_3 \leq (30k + 20)D, \]
or else the set \( S_j(D; \frac{n_1}{m_1}, \frac{n_2}{m_2}, \frac{n_3}{m_3}) \) will simply be too small. Combining this with the result in Proposition 4.33, we see that
\[ \frac{10m_2}{3} + \frac{2}{3(30k + 20)} = \frac{100k(9k - 1) + 6g_3}{30k + 20} \leq D < \frac{10m_2}{3} + \frac{1}{3(30k - 5)}, \]
which implies that
\[ \frac{2}{30k + 20} < \frac{1}{30k - 5}, \]
which is evidently false even for \( k = 1 \).

**Corollary 4.41.** If \( (D; \frac{n_1}{m_1}, \frac{n_2}{m_2}, \frac{n_3}{m_3}) \) is weakly triangulating with \( m_1, m_2, m_3 \geq 2 \), then there exist consecutive odd-indexed Fibonacci numbers \( b, c \) and \( \ell \in \mathbb{Z} \) such that \( (m_1, n_1) = \ell(b^2, c^2) + (a^2, b^2 + 2) \).

**Proof.** In light of the Lemma, it remains to eliminate a finitely many cases, enumerated in Lemma 4.34. But this is easily verified by computer.

**Lemma 4.42.** The sequence \( \left( \frac{9m'+1}{3}; \frac{2}{3}, \frac{3m'+1}{m}, \frac{9m'+1}{9} + \varepsilon \right) \) is strongly pseudo-triangulating for all \( m' \geq 1 \).

**Proof.** Let
\[ T_i := \{(X, Y, Z) \in \mathbb{N}^3 \mid X < 9m' + 1 \text{ and } X + 9Y + (9m' + 1)Z \leq i \}. \]
Note by Lemma 4.21 and Proposition 4.16 that \( (3; \frac{2}{3}, \frac{3}{1}) \) is a strongly triangulating sequence. Therefore, for each \( j \in \mathbb{N} \), we have
\[
\binom{j+2}{2} = \#S_j\left( 3; \frac{2}{3}, \frac{3}{1} \right)
= \#\{(X, Y) \in \mathbb{N}^2 \mid X \neq 1 \text{ and } X + 9Y \leq 3j \}
= \#\{(X, Y, Z) \in \mathbb{N}^2 \mid X < 9m' + 1 \text{ and } (9m' + 1)Z + X \neq 1 \text{ and } (9m' + 1)Z + X + 9Y \leq 3j \}
= \#\{(X, Y, Z) \in T_{3j} \mid X \neq 1 \text{ or } Z \geq 1 \}
= \#T_{3j} - \#\{(X, Y, Z) \in T_{3j} \mid X = 1 \text{ and } Z = 0 \}
= \#T_{3j} - \left[ \frac{3j}{9} \right].
\]
Meanwhile, the semigroup \( W' \) generated by \( 2m', 3m' \), and \( 9m' + 1 \) is contained in the semigroup generated by \( m' \) and \( 9m' + 1 \); more precisely, each element of \( W' \) has a unique representation as \( m'X + (9m' + 1)Y \), where \( (X, Y) \in \mathbb{N}^2 \) and the following two conditions both hold: (i) \( X < 9m' + 1 \), and (ii) \( X \neq 1 \) or \( Y \geq m' \). Thus, \( S_j \left( \frac{9m' + 1}{3}, \frac{2m' + 1}{3}, \frac{9m' + 1}{9} + \varepsilon \right) \) is in bijection with the set of triples \( (X, Y, Z) \in \mathbb{N}^3 \) with \( (X, Y) \) satisfying the conditions (i) and (ii), such that

\[
m'X + (9m' + 1)Y + \left( m'(9m' + 1) + \frac{9m' + 1}{9} + \varepsilon \right) Z \leq \frac{9m' + 1}{3} j,
\]
or in other words

\[
\frac{9m'X}{9m' + 1} + 9Y + (9m' + 1 + \varepsilon)Z \leq 3j, \tag{4.6.2}
\]
for all sufficiently small \( \varepsilon > 0 \). But as \( X < 9m' + 1 \), the inequality 4.6.2 is easily seen to be equivalent to the conjunction of the following two conditions: (iii) \( X + 9Y + (9m' + 1)Z \leq 3j \), and (iv) if \( X + 9Y + (9m' + 1)Z = 3j \), then \( X \geq 1 \) or \( Z = 0 \). The upshot is that

\[
\#S_j \left( \frac{9m' + 1}{3}, \frac{2m' + 1}{3}, \frac{9m' + 1}{9} + \varepsilon \right) = #T_{3j} - \#\{(X, Y, Z) \in T_{3j} \mid X = 1 \text{ and } Y < m'\} - \#\{(X, Y, Z) \in T_{3j} \mid X = 0 \text{ and } Z \geq 1 \text{ and } X + 9Y + (9m' + 1)Z = 3j\} = #T_{3j} - \#\{(Y, Z) \in \mathbb{N}^2 \mid Y < m' \text{ and } 1 + 9Y + (9m' + 1)Z \leq 3j\} - \#\{(Y, Z) \in \mathbb{N}^2 \mid Z \geq 1 \text{ and } 9Y + (9m' + 1)Z = 3j\}. \tag{4.6.3}
\]

Now we claim that, for \( (Y, Z) \in \mathbb{N}^2 \) with \( Y < m' \), we have

\[
1 + 9Y + (9m' + 1)Z \leq 3j \iff Y + m'Z < \frac{3m'j}{9m' + 1}. \tag{4.6.5}
\]

Indeed, on the one hand, if \( Y + m'Z < \frac{3m'j}{9m' + 1} \), then in particular

\[
Z < \frac{3j}{9m' + 1},
\]
so

\[
9(Y + m'Z) < \frac{3 \cdot 9m'j}{9m' + 1} = 3j - \frac{3j}{9m' + 1} < 3j - Z,
\]
and \( 1 + 9Y + (9m' + 1)Z \leq 3j \), as desired. On the other hand, if \( Y + m'Z \geq \frac{3m'j}{9m' + 1} \), then in particular

\[
m'(Z + 1) = m' + m'Z > \frac{3m'j}{9m' + 1},
\]
so

\[
Z > \frac{3j}{9m' + 1} - 1.
\]
Thus, \( Z \geq \left\lfloor \frac{3j}{9m' + 1} \right\rfloor \), so

\[
1 + 9Y + (9m' + 1)Z \geq 1 + 9Y + 9m'Z + Z + \left\lfloor \frac{3j}{9m' + 1} \right\rfloor > \frac{3 \cdot 9m'j}{9m' + 1} + \frac{3j}{9m' + 1} = 3j,
\]
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as needed. We conclude that (4.6.5) holds, so
\[
\# \{(Y, Z) \in \mathbb{N}^2 \mid Y < m' \text{ and } 1 + 9Y + (9m' + 1)Z \leq 3j\} = \\
\# \left\{ Y + m'Z \in \mathbb{N} \mid Y + m'Z < \frac{3m'j}{9m' + 1} \right\} = \left\lceil \frac{3m'j}{9m' + 1} \right\rceil. \quad (4.6.6)
\]

Meanwhile,
\[
\# \{(Y, Z) \in T_{3j} \mid Z \geq 1 \text{ and } 9Y + (9m' + 1)Z = 3j\} = \# \left\{ Z \in \mathbb{N}_{>0} \mid Z \equiv 3j \pmod{9} \text{ and } Z \leq \frac{3j}{9m' + 1} \right\}. \quad (4.6.7)
\]

Now
\[
3j + 9 - 9 \left\lfloor \frac{j}{3} \right\rfloor
\]
is the least strictly positive integer congruent to 3j modulo 9, so any given Z counted in the right-hand side of equation (4.6.7) may be written as
\[
Z = 9Z' + 3j + 9 - 9 \left\lfloor \frac{j}{3} \right\rfloor
\]
for a unique Z' \in \mathbb{N}. Thus,
\[
\# \left\{ Z \in \mathbb{N}_{>0} \mid Z \equiv 3j \pmod{9} \text{ and } Z \leq \frac{3j}{9m' + 1} \right\} = \# \left\{ Z' \in \mathbb{N} \mid 9Z' + 9 + 3j - 9 \left\lfloor \frac{j}{3} \right\rfloor \leq \frac{3j}{9m' + 1} \right\} = \\
\# \left\{ Z' \in \mathbb{N} \mid Z' \leq \left\lceil \frac{j}{3} \right\rceil + \frac{j}{3(9m' + 1)} - \frac{j}{3} - 1 \right\} = \left\lceil \frac{j}{3} \right\rceil + \left\lceil \frac{j}{3(9m' + 1)} - \frac{j}{3} \right\rceil = \\
\left\lceil \frac{j}{3} \right\rceil + \left\lceil \frac{-3m'j}{9m' + 1} \right\rceil = \left\lceil \frac{j}{3} \right\rceil - \left\lceil \frac{3m'j}{9m' + 1} \right\rceil. \quad (4.6.8)
\]

Putting equations (4.6.4), (4.6.6), (4.6.7), (4.6.8), and (4.6.1) together, we have
\[
\# S_j \left( \frac{9m' + 1}{3}, \frac{2}{3}, \frac{3m' + 1}{m'}, \frac{9m' + 1}{9} + \varepsilon \right) = \# T_{3j} - \left\lceil \frac{3m'j}{9m' + 1} \right\rceil - \left( \left\lceil \frac{j}{3} \right\rceil - \left\lceil \frac{3m'j}{9m' + 1} \right\rceil \right) = \\
\# T_{3j} - \left\lceil \frac{j}{3} \right\rceil = \# S_j \left( \frac{3}{3}, \frac{2}{3}, \frac{3}{1} \right) = \left( \frac{j + 2}{2} \right),
\]
as desired. \qed
Bibliography


