Squeezing by FM @ 2 \omega_o

Parametric modulation of the frequency of a classical harmonic oscillator causes progressive squeezing of the two quadrature components of its motion.

Consider a mass \( m \) which moves in a potential

\[ U(t) = \frac{1}{2} k x^2 + \frac{\varepsilon}{2} k x^2 \sin 2 \omega_0 t \text{ with } \omega_o = \sqrt{k/m}. \]  

(1)

The corresponding equation of motion, which represents a harmonic oscillator potential which is modulated in frequency by a small (\( \varepsilon \ll 1 \)) amount at twice its natural frequency, is

\[ \ddot{x} + \omega_0^2 x = -\omega_0^2 \varepsilon \sin (2 \omega_0 t) x \]  

(2)

Since \( \varepsilon \) is small, substitute

\[ x(t) = B(t) \cos \omega_0 t + C(t) \sin \omega_0 t, \]  

(3)

the usual solution for an undamped harmonic oscillator except that \( B \) and \( C \) are presumed to be slowly varying functions of time. Therefore

\[ \ddot{x}(t) = -\omega_0^2 x(t) - \omega_0 \dot{B} \sin \omega_0 t + \omega_0 \dot{C} \cos \omega_0 t \]

and substitution in the equation of motion (Eq. 2) gives

\[ -\omega_0 \dot{B} \sin \omega_0 t + \omega_0 \dot{C} \cos \omega_0 t = \omega_0^2 \varepsilon [B \cos \omega_0 t + C \sin \omega_0 t] \sin 2 \omega_0 t. \]

using trig identities [eg. \( \sin x \cos y = \frac{1}{2} (\sin (x+y) + \sin (x-y)) \)] and averaging away the terms rapidly oscillating at 3 \( \omega_0 \) gives

\[ -\dot{B} \sin \omega_0 t + \dot{C} \cos \omega_0 t = \frac{-\varepsilon \omega_0}{2} [B \sin \omega_0 t + C \cos \omega_0 t]. \]

The coefficients of \( \sin \omega_0 t \) and \( \cos \omega_0 t \) must be separately equal, hence

\[ \dot{B} = +\varepsilon \omega_0/2 B, \quad \dot{C} = -\varepsilon \omega_0/2 C \]

and hence the coefficients \( B(t) \) and \( C(t) \) are

\[ B(t) = B_0 e^{+\varepsilon \omega_0 t/2} \quad C(t) = C_0 e^{-\varepsilon \omega_0 t/2} \]  

(4)

Thus squeezing parameter, \( r \), is \( r = \varepsilon \omega_0 t/2 \). Thus one quadrature component of the motion increases with time which the other one decreases. This represents a progressive squeezing of the motion -- after a long time almost any initial conditions will lead to motion predominately in the \( \cos \omega_0 t \) component. The total energy of the oscillator will then be further amplified by the parametric drive term.