WITT SPACES: A GEOMETRIC CYCLE THEORY FOR KO-HOMOLOGY

AT ODD PRIMES

by

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ABSTRACT

We solve a problem posed by Dennis Sullivan: to construct a class of piecewise-linear cycles with signature which represent the connective version of KO homology at odd primes, $ko_* \otimes \mathbb{Z}[\frac{1}{p}]$. The construction is one of the first applications of the recent intersection homology theory of Goresky and MacPherson. It demonstrates the power of the theory in the study of the geometry of spaces with singularities. We also present what appears to be the first direct adaptation of classical surgery (spherical modification) to the setting of singular spaces.

Specifically, we consider a class of stratified p.l. pseudomanifolds which we call Witt spaces, whose rational intersection homology groups satisfy a Poincare duality theorem. To each Witt space $X$, we associate a p.l. invariant $w(X)$ taking values in $W(\mathbb{Q})$, the Witt ring of the rationals. This invariant generalizes the signature of manifolds, and it possesses many of the same beautiful properties: cobordism invariance, additivity, and a product formula. Sullivan has shown how to use such an invariant to define a canonical orientation in $ko_* \otimes \mathbb{Z}[\frac{1}{p}]$. The orientations induce a natural transformation $\mu$ of homology theories, from Witt space bordism to $ko_* \otimes \mathbb{Z}[\frac{1}{p}]$. To complete the construction, we use rational surgery to prove that the cobordism class of a Witt space $X$ is determined by $w(X)$, and we obtain an explicit description of the cobordism groups of Witt spaces. The only nontrivial groups occur in dimensions $4k$, and for $k > 0$, they are precisely $W(\mathbb{Q})$. The structure of $W(\mathbb{Q})$ is known, and we conclude that $\mu$ is an equivalence at odd primes.

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Introduction

In this thesis, we construct a geometric cycle theory with signature, representing connected KO-homology at odd primes, $\text{ko}_* \otimes \mathbb{Z}[\frac{1}{2}]$. The construction is one of the first applications of the intersection homology theory of Goresky and MacPherson. We also present what appears to be the first direct adaptation of classical surgery (spherical modification) to the setting of "manifolds with singularity."

To put the result in perspective, we give some of the historical background to the problem before describing the construction. Dennis Sullivan [22] discovered the Conner-Floyd type theorem [9] relating smooth oriented bordism, the signature, and connected K0-theory at odd primes:

Theorem (Sullivan): For compact p.l. pairs, there is a canonical isomorphism (equivalence) of homology theories:

$$
\Omega^*_{S^0}(X,A) \otimes \mathbb{Z}[\frac{1}{2}] \to \text{ko}_*(X,A) \otimes \mathbb{Z}[\frac{1}{2}],
$$

where the theories are regarded as $\mathbb{Z}/4\mathbb{Z}$-graded.
He found (at least) two interesting applications:

(1) A geometric description of connected $K_0$ homology at odd primes, $K_{0*} \otimes \mathbb{Z}[rac{1}{2}]$, as a bordism theory with join-like singularities obtained by geometrically "killing" generators of smooth bordism. [See [6], [8], [23]]. There is a commutative diagram:

\[
\begin{array}{ccc}
\Omega^S_0(X,A) \otimes \mathbb{Z}[rac{1}{2}] & \xrightarrow{i} & J_*(X,A) \\
\downarrow{p} & & \downarrow{\psi} \\
\Omega^S_0(X,A) \otimes S_0(\text{pt}) & & \mathbb{Z}[rac{1}{2}] \cong K_{0*}(X,A) \otimes \mathbb{Z}[rac{1}{2}]
\end{array}
\]

in which $J_*$ is the bordism theory with singularities, $i$ is the natural inclusion (gotten by regarding a smooth manifold as a cycle in the bordism theory), $p$ is the natural projection, and $\psi$ gives an equivalence of theories [8].

(2) A construction of a canonical $K_{0*} \otimes \mathbb{Z}[rac{1}{2}]$ orientation for a p.l. block bundle over a finite complex, and, in particular, a canonical $K_{0*} \otimes \mathbb{Z}[rac{1}{2}]$ orientation for p.l. manifolds via Alexander duality [22]. The techniques used for this construction generalize to give a more general result. If the collection of cycles of a geometric homology
theory is closed under the operations of taking cartesian product and transversal intersections with p.l. manifolds, and if it possesses a "nice" signature invariant extending the classical signature, then the cycles have canonical $K_0\otimes \mathbb{Z}[(1/2)]$ orientations. Note that among the properties of a "nice" signature are cobordism invariance, additivity, and a product formula with respect to closed p.l. manifolds. For further details see Chapter VI and the Appendix.

A natural synthesis of the themes of these two applications would be achieved by solving the following

**Problem:** Construct a geometric cycle theory for $K_0\otimes \mathbb{Z}[(1/2)]$ built from a class of p.l. cycles $\mathcal{F}$ with "nice" signature invariant, such that the equivalence (at odd primes)

$$
\mu : \mathcal{F} \to K_0 \otimes \mathbb{Z}[(1/2)]
$$

is induced by the orientations described in (2) above.

Goresky and MacPherson were also interested in finding a class of spaces with cobordism invariant signature for the purpose of extending the Hirzebruch L-class to the setting of "manifolds with singularity." Their investigation of cycle intersection phenomena in stratified spaces resulted in their beautiful intersection homology theory [11]. They proved that for a stratified p.l. pseudomanifold $X$ of
dimension $4k$ with only even codimension strata, the intersection homology group $H^m_{2k}(X,A)$ is self-dual, and they proceeded to define a cobordism invariant signature and $L$-class for the collection of spaces with only even codimension strata.

This paper begins where [11] left off, and makes use of intersection homology theory to solve the problem stated above. Techniques for computing the cobordism groups of stratified spaces with even codimension strata are not currently available, so we enlarge the class of spaces to permit certain odd codimension strata. Specifically, we impose the following local intersection homological link condition: If $L^{2\ell}$ is the link of an odd codimension stratum, then $H^m_{A}(L;\mathbb{Q}) = 0$.

There are three observations which confirm that this definition is reasonable.

Observation 1: Atiyah in [4] studied examples of smooth fibrations

$$M^{2\ell} \longrightarrow X$$

$$\downarrow$$

$$N^{2k}$$

with sign $(X) \neq 0$, where sign $(X)$ denotes the classical signature of $X$. Using a theorem of Putz, we may regard
the fibre bundle as being p.l., and the mapping cylinder of the fibration provides a stratified p.l. cobordism of X to zero. The cobordism has odd codimension stratum N with link M. If the extended signature is to be cobordism invariant, we need a general condition on M which ensures that \( \text{sign}(X) = 0 \). Atiyah's analysis provides one:
\[ H_\ell(M;\mathbb{Q}) = 0. \]
For a manifold M, intersection homology coincides with ordinary homology, so we get the same link condition as above.

**Observation 2:** We want the extended signature to be additive. An appealing geometric proof of additivity follows immediately from cobordism invariance if the "pinch cobordism" is a permissible cobordism (see Figure 1.). The pinch point v is the only odd codimension singularity introduced. Its link \( L^4 \) is a suspension, with suspension points identified. A geometric calculation, carried out in Chapter IV, shows that \( \overline{H}_2(L;\mathbb{Z}) = 0 \); a fortiori, \( \overline{H}_2(L;\mathbb{Q}) = 0 \).

**Observation 3:** An intersection homology spectral sequence argument proves that \( \overline{H}_k(X^{2\ell};\mathbb{Q}) \) is self-dual if the link condition is satisfied in X. This result is proved in Chapter III.

Using Observation 3, we associate to each permissible
stratified p.l. pseudomanifold $X^{4k}$ (that is, to each $X^{4k}$ satisfying the link condition), the inner product space $H_{2k}^m(X;\mathbb{Q})$. The inner product space is determined up to permissible cobordism by its equivalence class in $W(\mathbb{Q})$, the Witt group of the rationals. We call this equivalence class the Witt class of $X$, and denote it by $w(X)$. If $\dim (X) \not\equiv 0 \pmod{4}$, we set $w(X) = 0$. This p.l. invariant incorporates a "nice" signature in the sense used above.

Applying Sullivan's result, we obtain canonical orientations in $ko_* \otimes \mathbb{Z}[\frac{1}{2}]$ for permissible cycles. It remains to compute the cobordism groups. By applying a technique of rational surgery, adapted from classical spherical modification to the setting of stratified spaces and intersection homology, we show that the Witt class $w(X)$ determines the cobordism class of $X$. We also construct geometrical representatives in positive dimensions $q \equiv 0 \pmod{4}$ for each class in $W(\mathbb{Q})$.

Therefore, we give the collection of permissible spaces the name Witt spaces, and denote the bordism theory based on them by $\Omega_*^{Witt}$. These results can be rephrased as:

**Theorem:** There are isomorphisms

...
\[ \Omega_0^{\text{Witt}}(\text{pt}) \cong \mathbb{Z} \]
\[ \Omega_q^{\text{Witt}}(\text{pt}) \cong W(\mathbb{Q}), \quad q > 0 \text{ and } q \equiv 0 \pmod{4} \]
\[ \Omega_q^{\text{Witt}}(\text{pt}) \cong 0, \quad q \not\equiv 0 \pmod{4} \]

determined by the Witt class invariant.

The \( ko_* \otimes \mathbb{Z}[\frac{1}{2}] \) orientations induce a natural transformation \( \mu \) of homology theories:

\[ \mu^{\text{Witt}} : \Omega_* \rightarrow ko_* \otimes \mathbb{Z}[\frac{1}{2}] \]

which reduces to the signature homomorphism on coefficient groups when \( q \equiv 0 \pmod{4} \):

\[ \text{sign} : \Omega_q^{\text{Witt}}(\text{pt}) \rightarrow \mathbb{Z} \subset \mathbb{Z}[\frac{1}{2}] \cong ko_q(\text{pt}) \otimes \mathbb{Z}[\frac{1}{2}] \]

The group structure of \( W(\mathbb{Q}) \) is classical [13]. There is a canonical isomorphism

\[ W(\mathbb{Q}) \cong \mathbb{Z} \oplus T, \]

where \( T \) is a direct sum of cyclic 2-groups of order 2 or 4. The composition gotten by projection onto the infinite cyclic summand
\[ \Omega_q^{Witt}(pt) \cong W(\mathbb{Q}) \to \mathbb{Z}, \]

coincides with the signature homomorphism above.

Tensoring with \( \mathbb{Z}[\frac{1}{2}] \), we conclude:

**Theorem:** The natural transformation

\[
\mu_{\text{Witt}} \otimes \mathbb{Z}[\frac{1}{2}] : \Omega_\ast^{\text{Witt}} \otimes \mathbb{Z}[\frac{1}{2}] \to \text{k}_{\ast} \otimes \mathbb{Z}[\frac{1}{2}]
\]

is an equivalence of homology theories, reducing to the signature homomorphism (tensored with \( \mathbb{Z}[\frac{1}{2}] \)) on coefficient groups.

Finally, we note here that John Morgan has also extended the signature to a class of stratified spaces whose cobordism groups he has computed. These spaces give a cycle theory for the homology theory associated to \( G/PL \). He uses different techniques from those in this thesis, and thoroughly studies the 2-torsion in his theory.

We gratefully acknowledge some helpful remarks regarding 2-torsion which he provided early on during the preparation of this thesis.
CHAPTER I: Geometric Preliminaries

In this chapter we present definitions and results from intersection homology theory and the theory of bordism with singularities. All topological spaces are piecewise linear unless otherwise stated.

Section 1: Stratified p.l. pseudomanifolds.

The contents of sections 1 - 6 are taken for the most part from [11].

**Definition 1.1.:** A (closed) n-dimensional (p.l.) pseudomanifold $X$ is a compact space containing a closed subspace $\Sigma_X$ satisfying:

1. $\dim (\Sigma_X) \leq n - 2$
2. $X - \Sigma_X$ is an n-dimensional oriented (p.l.) manifold
3. $\mathcal{C}(X - \Sigma_X) = X$.

**Definition 1.2.** A stratification of a pseudomanifold $X$ is a filtration by closed subspaces

$$X = X_n \supset X_{n-1} \supset X_{n-2} = \Sigma_X \supset X_{n-3} \supset \ldots \supset X_1 \supset X_0$$

such that for each point $p \in X_i - X_{i-1} = \gamma_i$, there is a
filtered compact p.l. space:

\[ V = V_{n} \supset V_{n-1} \supset \ldots \supset V_{i} \text{ is a point} \]

and a mapping \( V \times B^{i} \rightarrow X \) which, for each \( j \), takes \( V_{j} \times B^{i} \) (p.l.) homeomorphically to a neighborhood of \( p \) in \( X_{j} \). Here, \( B^{i} \) is the p.l. \( i \)-ball and \( p \) corresponds to \( V_{i} \times \) (interior point of \( B^{i} \)).

If \( X_{i} \) is not empty, it is a manifold of dimension \( i \), called the \( i \)-dimensional stratum of the stratification.

We also consider relative objects:

**Definition 1.3.** An \( n \)-dimensional pseudomanifold with boundary is a compact pair \( (X,A) \) satisfying the properties:

1. \( A \) is an \( (n-1) \)-dimensional pseudomanifold with singular set \( \Sigma_{A} \).
2. There is a closed subspace \( \Sigma_{X} \) with \( \dim \Sigma_{X} \leq n - 2 \) such that \( X - (\Sigma_{X} \cup A) \) is an oriented \( n \)-manifold which is dense in \( X \).
3. \( A \) is collared in \( X \); i.e., there is a closed neighborhood \( N \) of \( A \) in \( X \) and an orientation preserving (p.l.) homeomorphism \( \theta : A \times [0,1] \rightarrow N \) such that

\[ \theta(\Sigma_{A} \times [0,1]) = \Sigma_{X} \cap N. \]
Here is the relative definition of stratification.

Definition 1.4.: A stratification of an n-dimensional pseudomanifold with boundary \((X, A)\) is a filtration by closed subspaces:

\[
X = X_n \supset X_{n-1} = X_{n-2} = \ldots \supset X_1 \supset X_0
\]

such that the filtration of \( A \) given by \( A_{j-1} = X_j \cap A \) stratifies \( A \), the \( X_j - A_{j-1} \) stratify \( X - A \) (in the obvious sense), and the filtrations respect the collaring of \( A \) in \( X \). By the last condition, we mean

\[
\theta(A_{j-1} \times [0,1]) = X_j \cap N.
\]

Section 2: Piecewise-linear chains and perversity

Assume throughout that \( X \) is an n-dimensional pseudomanifold with fixed stratification.

Definition 2.1.: \( C_\ast(X) \), the complex of p.l. geometric chains is the direct limit under refinement of the simplicial complexes \( C^T_\ast(X) \), with \( T \) ranging over all triangulations of \( X \) compatible with the p.l. structure.

For \( C \in C^T_1(X) \), the support of \( C \), denoted \( |C| \), is the union of the closures of the i-simplices \( \sigma \) whose coefficient in \( C \) is not zero. As a p.l. subspace of \( X \),
|C| is invariant under refinement. Therefore, a chain D ∈ C∗(X) has a well defined support, |D|.

**Definition 2.2.** A perversity is a sequence of integers \( \overline{p} = (p_2, \ldots, p_n) \) such that \( p_2 = 0 \) and \( p_{c+1} = p_c \) or \( p_{c} + 1 \). We also write \( \overline{p}(c) \) for \( p_c \), on occasion.

If \( i \) is an integer and \( \overline{p} \) a perversity, a subspace \( Y \subset X \) is called \( (\overline{p},i) \) allowable if \( \dim(Y) \leq i \) and 
\[
\dim(Y \cap X_{n-c}) \leq i - c + p_c
\]
for all \( c \geq 2 \).

**Definition 2.3.** \( C_i^p(X) \) is the subgroup of \( C_i(X) \) consisting of those chains \( C \) such that

1) |C| is \( (\overline{p},i) \) allowable
2) |∃C| is \( (\overline{p},i-1) \) allowable.

**Remark 2.4.** We will frequently use an equivalent form of the condition in 2.2. Namely, \( Y \) is \( (\overline{p},i) \) allowable if \( \dim(Y) \leq i \) and 
\[
\dim(Y \cap X_{n-c}) \leq i - c + p_c
\]
for all \( c \geq 2 \).

**Proof of equivalence:** Since \( X_{n-c} \subset X_{n-c'} \), we have 
\[
\dim(Y \cap X_{n-c'}) \leq \dim(Y \cap X_{n-c}),
\]
so the second condition implies the first. On the other
hand,
\[
\dim (Y \cap X_{n-c}) = \max_{j=0, \ldots, n-c} \{\dim (Y \cap X_{n-c-j})\}.
\]

For each \( j \), \( p_{c+j} \leq p_c + j \), so the second condition implies,
\[
\dim (Y \cap X_{n-c}) \leq i - (c+j) + p_{c+j} \leq i + c + p_c.
\]

Therefore,
\[
\dim (Y \cap X_{n-c}) \leq i - c + p_c.
\]

The \( i \)th intersection homology group of perversity \( \bar{p} \), \( H_i^{\bar{p}}(X) \), is the \( i \)th homology group of the chain complex \( C_{\bar{p}}(X) \).

There are homomorphisms for each perversity \( \bar{p} \):
\[
\alpha^{-}_{\bar{p}} : H^*(X) \rightarrow H_{n-\ast}^{\bar{p}}(X)
\]
\[
\omega^{-}_{\bar{p}} : H_{\ast}^{\bar{p}}(X) \rightarrow H_{\ast}(X)
\]

which factor the classical cap product homomorphisms:
\[
\cap [X] : H^*(X) \rightarrow H_{n-\ast}(X).
\]

These are roughly defined as follows. The map \( \omega^{-}_{\bar{p}} \) is
induced by chain level inclusion. For the map $\alpha_p^\pm$, assume $T$ is a triangulation. Define $\alpha_p^-$ on the cochain level by sending the elementary cochain $\delta^i$, where $\sigma^i$ is an $i$-simplex in $T$, to the $p,l$ chain supported on the dual to $\sigma^i$ in $X$, corresponding to the appropriate fundamental class in $H_{n-i}(D(\sigma), \partial D(\sigma))$.

Section 3: Transversality and Intersection products.

**Definition 3.1.** Suppose $p + q = r$ and $i + j - n = \ell$. A chain $C \in C_i^p(X)$ and a chain $D \in C_j^q(X)$ are said to be dimensionally transverse (written $C \pitchfork D$) if $|C| \cap |D|$ is $(r, \ell)$ allowable.

In [11], the following lemma is proved:

**Lemma 3.2.** Let $C, D$ be as in 3.1. If $C \pitchfork D, \partial C \pitchfork D, C \pitchfork \partial D$, then the intersection chain $C \cap D \in C_{\ell}^r$ is defined and

$$\partial (C \cap D) = \partial C \cap D + (-1)^{n-i} C \cap \partial D.$$
Suppose \( \partial A \cong B \). Then there is a p.f. isotopy \( K : X \times [0,1] \to X \) such that

1. \( K(x,0) = x \), for all \( x \in X \)
2. \( K(x,t) = x \) for all \( x \in \partial A \), all \( t \in [0,1] \)
3. \( K(x,t) \in X_i - X_{i-1} \) whenever \( x \in X_i - X_{i-1} \)
4. \( K(|A|,1) = K_1(|A|) \) supports \( K_1(A) \), which is dimensionally transverse to \( B \). Here \( K_1 \) denotes the induced map on \( C_*(X) \).

**Corollary 3.4.** Suppose \( C \in C_i^p(X) \), \( D \in C_j^q(X) \), and \( \bar{p} + \bar{q} = \bar{r} \). Suppose \( \partial C \cong D \). Then there exists \( C' \in C_i^p(X) \) and \( E \in C_{i+1}^p(X) \) such that \( \partial E = C' - C \), \( C' \cong D \), and \( \partial C' = C \).

Now we come to the result which gives intersection homology theory its name.

**Theorem 3.5.:** (1) Suppose \( \bar{p} + \bar{q} \leq \bar{r} \). There is a unique intersection pairing:

\[
\cap : H_i^p(X) \times H_j^q(X) \to H_{i+j-n}^r(X)
\]

defined by:

\[
[C] \cap [D] = [C \cap D]
\]
for every dimensionally transverse pair of cycles $C \in \tilde{C}^p_1(X)$ and $D \in \tilde{C}^q_j(X)$.

(2) If $\xi \in H^p_1(X)$, $\eta \in H^q_j(X)$, and $\omega \in H^s_k(X)$, and if $\bar{p} + \bar{q} + \bar{s} \leq \bar{t} = (0, 1, 2, \ldots, n-2)$, then

$$\xi \cap \eta = (-1)^{(n-i)(n-j)} \eta \cap \xi$$

$$\xi \cap (\eta \cap \omega) = (\xi \cap \eta) \cap \omega$$

(3) The intersection pairings are compatible with the cup and cap product. That is, if $A, B \in H^*(X)$, $C \in H^*_k(X)$ and $\bar{p} + \bar{q} = \bar{r}$, then

$$\omega_{\bar{p}} A = A \cap [X] \quad \text{(cap product)}$$

$$\alpha_{\bar{p}} (A \cup B) = \alpha_{\bar{p}} (A) \cap \alpha_{\bar{q}} (B)$$

$$A \cap \omega_{\bar{r}} (C) = \omega_{\bar{r}} (\alpha_{\bar{q}} (A) \cap C)$$

$$\langle A, \omega_{\bar{r}} (C) \rangle = \varepsilon (\omega_{\bar{r}} (\alpha_{\bar{q}} (A) \cap C)).$$

Here $\langle , \rangle$ is the Kronecker pairing $H^* \times H_* \rightarrow \mathbb{Z}$ and $\varepsilon : H_* \rightarrow \mathbb{Z}$ is the usual augmentation.

Section 4: The basic sets $\tilde{Q}^p_i$ and the intersection pairing theorem.

For any compatible triangulation $T$ of $X$ and perversity $\bar{p}$, Goresky and MacPherson define the basic
subsets \( \{ Q^P_i \} \). For \( 0 \leq i \leq n \), \( Q^P_i \) is a full \( i \)-dimensional subcomplex of \( T' \), the first barycentric subdivision of \( T \), and \( Q^P_i \subseteq Q^P_{i+1} \) for all \( i \) (modulo \( n \)). The relation of these sets to intersection homology is given by the lemmas below.

**Lemma 4.1.**: There is an isomorphism

\[
\psi : \text{Im}(H_i(Q^P_i) \rightarrow H_i(Q^P_{i+1})) \xrightarrow{\cong} H_i^P(X)
\]

induced by inclusion of cycles.

**Lemma 4.2.**: If \( i > 1 \) and \( \bar{p} + \bar{q} = \bar{t} = (0,1,2,\ldots,n-2) \), then there are canonical simplex preserving deformation retractions:

\[
X - (Q^P_{n-i+1} \cap |T_{n-2}|) \rightarrow Q^P_i
\]

\[
X - Q^P_{n-i+1} \rightarrow Q^P_i \cap |T_{n-2}|
\]

where \( T_{n-2} \) is the \( n-2 \) skeleton of \( T \).

**Lemma 4.3.**: If \( i + j = n \), and \( \bar{p} + \bar{q} = \bar{t} \),

\[
H^j_1(X) \cong \text{Im}(H^j_1(Q^P_{j+1}) \rightarrow H^j_1(Q^P_j)),
\]
canonically.

Lemma 4.1 implies that the groups \( H^p_i(X) \) are finitely generated and independent of the stratification of \( X \), since any two stratifications have a common subordinate triangulation.

Lemmas 4.2 and 4.3, combined with some elementary algebra, yield the **Intersection Pairing Theorem**.

**Theorem 4.4.** Let \( \varepsilon : H_0^\bar{\ell}(X) \to \mathbb{Z} \) be the usual augmentation, where \( \bar{\ell} = (0,1,2,\ldots,n-2) \). If \( i + j = n \) and \( \bar{p} + \bar{q} = \bar{\ell} \), then the augmented intersection pairing:

\[
H^p_i(X) \times H^q_j(X) \to H_0^\bar{\ell}(X) \xrightarrow{\varepsilon} \mathbb{Z}
\]

is nondegenerate if these groups are tensored with the rationals \( \mathbb{Q} \).

**Section 5:** The basic sets \( R^p_i \) and complexes \( W^p_k \).

Whereas the basic sets \( Q^p_i \) are perfectly suited to proving the intersection pairing theorem, there are different basic sets which are better adapted to other applications. The main drawback of the sets \( Q^p_i \) is that if \( \bar{p} \leq \bar{q} \), then it is generally not true that \( Q^p_i \subset Q^q_i \).

The basis sets \( R^p_i \) behave well under inclusion relations, on the other hand; they play an important role in Chapter III.
They are defined as follows.

Let $T$ be a triangulation of $X$ with first barycentric subdivision $T'$. Consider the stratification of $X$ given by the skeleta of $T$,

$$X = |T_n| \supset \Sigma_{x} = |T_{n-2}| \supset |T_{n-3}| \supset \cdots \supset |T_0|.$$  

Define $R^p_i$ to be the subcomplex of $T'$ consisting of all simplexes which are $(p,i)$ allowable with respect to this stratification.

Let $W^p_i$ be the subgroup of $C^T_i(R^p_i)$ consisting of those simplicial $i$-chains with boundary supported on $R^p_{i-1}$.

Two fundamental properties of the complexes $W^p_*$ are:

Property 5.1.: The inclusion

$$i : W^p_i \hookrightarrow C^p_i(X)$$

induces an isomorphism on homology, so

$$\text{Im}(H_i(R^p_1) \rightarrow H_i(R^p_{i+1})) \cong H_i^p(X).$$

Property 5.2.: If $p \leq q$, there is an inclusion of chain complexes:

$$W^p_* \hookrightarrow W^q_*.$$
Section 6: Intersection homology with coefficients and the $\mathbb{Q}/\mathbb{Z}$ intersection pairing.

Let $G$ be an abelian group. Define $C^p_i(X;G)$ to be the subgroup of chains $C \in C_i(X) \otimes G$ such that $|C|$ is $(p,i)$ allowable and $|\partial C|$ is $(p,i-1)$ allowable. (Note: This complex may differ from $C^p_*(X) \otimes G$).

One can check that:

$$H^p_i(X;G) \cong \text{Im}(H_i(Q^p_i;G) \to H_i(Q^p_{i+1};G))$$

and

$$H^p_i(X;G) \cong \text{Im}(H_i(R^p_i;G) \to H_i(R^p_{i+1};G))$$

Suppose there is a pairing $G \otimes_{\mathbb{Z}} G' \to G''$, and suppose $p + q = r$ and $i + j - n = \lambda$. If $C \in C^p_i(X;G)$ and $D \in C^q_j(X,G')$, with $C \pitchfork D$, $\partial C \pitchfork D$, and $C \pitchfork \partial D$, then the arguments in [11] can be modified to yield an intersection chain $C \cap D \in C^r_\lambda(X;G'')$, and the Lefschetz boundary formula holds. By applying the transversality lemma, we get intersection products with coefficients.

Theorem 6.1: If $\tilde{p} + \tilde{q} \leq \tilde{r}$, there is an intersection pairing:

$$\cap : H^p_i(X;G) \times H^q_j(X;G') \to H^r_{i+j-n}(X;G'')$$
such that \([C] \cap [D] = [C \cap D]\) for every dimensionally transverse pair of cycles \(C \in C^p_i(X;G)\) and \(D \in C^q_j(X;G')\).

In particular, let \(G = \mathbb{Q}/2\mathbb{Z},\ G' = \mathbb{Z},\) and \(G'' = \mathbb{Q}/2\mathbb{Z}\). Consider the pairing induced by the abelian group structure of \(\mathbb{Q}/2\mathbb{Z}\):

\[
\mathbb{Q}/2\mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Q}/2\mathbb{Z}.
\]

The group \(\mathbb{Q}/2\mathbb{Z}\) is divisible, so the universal coefficient theorem for cohomology implies, for a finite complex,

\[
H^i(A;\mathbb{Q}/2\mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}}(H_i(A),\mathbb{Q}/2\mathbb{Z})
\]

with the isomorphism induced by chain level evaluation. Some elementary algebra, and an argument very much like that used to prove Theorem 4.4 yields the \(\mathbb{Q}/2\mathbb{Z}\) intersection pairing theorem:

**Theorem 6.2.** Let \(i + j = n\) and let \(\bar{p} + \bar{q} = \bar{t}\). There is a nonsingular augmented intersection pairing:

\[
H^\bar{p}_i(X;\mathbb{Q}/2\mathbb{Z}) \times H^\bar{q}_j(X;\mathbb{Z}) \rightarrow H^\bar{t}_0(X;\mathbb{Q}/2\mathbb{Z}) \rightarrow \mathbb{Q}/2\mathbb{Z}.
\]
Section 7: Bordism with singularities.

In [23], Sullivan introduced the concept of a bordism theory based upon a class of cycles and homologies with specified singularities, or "bordism with singularities." Let $\mathcal{F}^n$, $n \in \mathbb{Z}$, be a sequence of classes of compact p.l. pairs. Akin [2] gave a list of axioms which ensure that $\{\mathcal{F}^n\}$ is rich enough to support a homology theory $\Omega_*(T,T_0)$, defined for compact p.l. pairs $(T,T_0)$, and built upon cobordisms of continuous maps $f : (X,X_0) \to (T,T_0)$, where $(X,X_0) \in \mathcal{F}^n$, some $n$. The list represents the bare necessities, and produces $\mathbb{Z}/2\mathbb{Z}$-bordism theories (that is, unoriented). We present slightly stronger axioms providing criteria for $\{\mathcal{F}^n\}$ to support a $\mathbb{Z}$-bordism theory (that is, oriented). In chapter VI, we give an interesting example of such a theory.

First, recall the following standard definitions:

**Definition 7.1.** Let $(X,X_0)$ be a compact p.l. pair. $X_0 \subseteq X$ is **collared** if there is a closed neighborhood $N$, $X_0 \subseteq N \subseteq X$, and a p.l. homeomorphism $h : X_0 \times I \to N$ such that $h(X_0 \times 0) = X_0 \subseteq N$. (Note definition 1.3 (3) is a special case).

**Definition 7.2.** A locally compact polyhedron $X$ is totally $n$-dimensional if $X = \text{Cl}(I^n(X) - I^{n-1}(X))$. Here
$I^k(X)$ is the intrinsic $k$-skeleton of $X$, as in [1].

Remark 7.3.: Akin [2] shows that $X$ is totally $n$-dimensional iff some (and hence any) triangulation $K$ of $X$ satisfies the following condition: Every simplex of $X$ is a (not necessarily proper) face of an $n$-simplex of $K$.

Now, let $\{F^n\}$ be a sequence of compact p.l. pairs, $n \geq 0$. Write $X \in F^n$ for $(X, \phi) \in F^n$. Assume $\phi \in F^n$, all $n$. Suppose $\{F^n\}$ satisfies the following axioms.

Axiom 1: If $(X, X_0) \approx (Y, Y_0)$ and $(X, X_0) \in F^n$, then $(Y, Y_0) \in F^n$.

Axiom 2: If $(X, X_0) \in F^n$, then $X$ is totally $n$-dimensional and $X_0 \subset X$ is collared.

Axiom 3: If $(X, X_0) \in F^n$ and $(X, X_0) \neq (\cdot, \cdot, \phi)$, then $H_n(X, X_0; \mathbb{Z}) \cong \bigoplus_{i=1}^J \mathbb{Z}$, some finite $J$. An orientation for $(X, X_0)$, denoted $[X, X_0]$ is a sum of generators of the infinite cyclic summands of $H_n(X, X_0; \mathbb{Z})$.

Axiom 4: If $(X, X_0) \in F^n$, then $X_0 \in F^{n-1}$. If $[X, X_0]$ is an orientation for $(X, X_0)$, then $\partial_* [X, X_0] \in H_{n-1}(X_0; \mathbb{Z})$ is an orientation for $X_0$, called the orientation induced by $[X, X_0]$. Here $\partial_* : H_n(X, X_0; \mathbb{Z}) \rightarrow H_{n-1}(X_0; \mathbb{Z})$ is the
connecting homomorphism in ordinary homology.

**Axiom 5**: If \((X, X_0) \in \mathcal{F}^n\), then \((X \times I, X_0 \times I) \in \mathcal{F}^{n+1}\).

**Axiom 6** (Cutting): Let \((X, X_0) \in \mathcal{F}^n\), \(A \subset X\) be a closed subpolyhedron, \((V, V_0)\) a regular neighborhood pair for \(A\) in \((X, X_0)\). If \(Y = \text{cl}(X-V)\) and \(Y_0 = \text{cl}(X_0 - V_0)\), then \((Y, \text{Bd } V \cup Y_0) \in \mathcal{F}^n\), \((V, \text{Bd } V \cup V_0) \in \mathcal{F}^n\) and \((\text{Bd } V, \text{Bd } V \cap Y_0) \in \mathcal{F}^{n-1}\).

**Axiom 7** (Pasting). Let \(X \supset X_0 \supset X_1\), \(Y \supset Y_0 \supset Y_1\), \(Z \supset Z_0 \supset \emptyset\) be p.l. triples with \(X\) and \(Y\) disjoint.
Suppose \((X, X_0) \in \mathcal{F}^n\), \((Y, Y_0) \in \mathcal{F}^n\), and \((Z, Z_0) \in \mathcal{F}^{n-1}\).
Let \([X, X_0]\), \([Y, Y_0]\), and \([Z, Z_0]\) be orientations. Let \(i_X : Z \rightarrow X_0\) and \(i_Y : Z \rightarrow Y_0\) be p.l. embeddings which are orientation preserving and reversing, respectively, with respect to the restrictions of the induced orientations on \(X_0\) and \(Y_0\).

Furthermore, suppose \(X_0 = i_X(Z) \cup X_1\),
\[Y_0 = i_Y(Z) \cup Y_1,\]
\[i_X^{-1}(X_1) = i_Y^{-1}(Y_1) = Z_0\]
and \((X_1, i_X(Z_0)), (Y_1, i_Y(Z_0)) \in \mathcal{F}^{n-1}\).

Then, \((W, W_0) = (X \cup_Z Y, X_1 \cup_Z Y_1) \in \mathcal{F}^n\) and there is an orientation \([W, W_0]\) such that the natural embeddings \(X \leftrightarrow W\) and \(Y \leftrightarrow W\) are orientation preserving.

Now we define the notions of oriented cobordism and
Definition 7.4.: Let \((X,X_0)\) and \((Y,Y_0)\) be in \(\mathcal{F}^n\), with orientations \([X,X_0]\) and \([Y,Y_0]\), respectively.

An oriented cobordism between \((X,X_0)\) and \((Y,Y_0)\) is an oriented pair \((W,W_0)\) in \(\mathcal{F}^{n+1}\), with orientation \([W,W_0]\) satisfying

1) There are p.l. embeddings \(i_X : X \rightarrow W_0\) and \(i_Y : Y \rightarrow W_0\) which are orientation preserving and reversing respectively with respect to the induced orientation on \(W_0\).

2) \(i_X(X) \cap i_Y(Y) = \emptyset\)

3) There is a compact subspace \(W_1 \subset W_0\) such that
   a) \(i_X(X) \cap W_1 = i_X(X_0)\), \(i_Y(Y) \cap W_1 = i_Y(Y_0)\),
   and \(W_0 = i_X(X) \cup W_1 \cup i_Y(Y)\)
   b) \((W_1, i_X(X_0) \cup i_Y(Y_0)) \in \mathcal{F}^n\)

An oriented map of \((X,X_0)\), with its orientation \([X,X_0]\), is a continuous map \(f : (X,X_0) \rightarrow (T,T_0)\), where \((T,T_0)\) is a compact p.l. pair with \(T_0 \subset T\) closed. We denote this \(\{f,[X,X_0]\}\), and refer to \(\{f,[X,X_0]\}\) as a singular \(\mathcal{F}\)-cycle.

Given oriented maps \(f : (X,X_0) \rightarrow (T,T_0)\) and \(g : (Y,Y_0) \rightarrow (T,T_0)\), an oriented cobordism between \(f\) and

...
g is an oriented cobordism \((W,W_0)\) between \((X,X_0)\) and \((Y,Y_0)\), together with a continuous map \(F : (W,W_0) \rightarrow (T,T_0)\), such that \(F \circ i_X = f\) and \(F \circ i_Y = g\).

Cobordism of oriented maps into \((T,T_0)\) is an equivalence relation, as is easily checked. Denote by \(\Omega_n(T,T_0)\) the set of equivalence classes of oriented maps \(\{f,[X,X_0]\}\) into \((T,T_0)\). Under disjoint union, this has an abelian group structure: the zero element is \(\phi : (\phi,\phi) \rightarrow (T,T_0)\), and \(\{f,[X,X_0]\} + \{f,-[X,X_0]\} = 0\), using the product cobordism \(\{f \circ \pi_{X \times 1, X \times I} \cup X_0 \times I\}\).

Define the connecting homomorphism:

\[ \partial_n : \Omega_n(T,T_0) \rightarrow \Omega_{n-1}(T_0) \]

by

\[ \partial_n\{f,[X,X_0]\} = \{f|_{X_0}, \partial*[X,Y_0]\}, \]

where \(\partial*[X,X_0]\) is the induced orientation on \(X_0\), as before. Finally, if \(F : (S,S_0) \rightarrow (T,T_0)\) is continuous, define \(F_* : \Omega_n(S,S_0) \rightarrow \Omega_n(T,T_0)\) by

\[ F_*\{f,[X,X_0]\} = \{F \circ f,[X,X_0]\}. \]

Let \(\Omega_*\) denote the functor from compact p.l. pairs to graded abelian groups, along with the connecting morphisms.
Theorem 7.5.: $\Omega_*$ is a homology theory.

Proof: The proof proceeds as in [2]; simply keep track of orientations.

Such a homology theory is called an oriented bordism theory with singularities.

Definition 7.6.: Let $\mathcal{S} = \{\mathcal{S}^n\}$ be a sequence of classes of compact p.l. spaces, for $n \geq 0$. We say $\mathcal{S}$ is a class of singularities if:

1. $\phi \in \mathcal{S}^n$, all $n$
2. If $L \in \mathcal{S}^n$, then $L$ is total $n$-dimensional
3. If $L \in \mathcal{S}^n$ and $L \neq \phi$, then $n(L; \mathbb{Z}) = \bigoplus \mathbb{Z}$, some $J \geq 1$. An orientation for $L$ is denoted $[L]$, as before.
4. For $n \geq 0$, $L \in \mathcal{S}^n$ iff $S^0 \ast L \in \mathcal{S}^{n+1}$, where $S^0$ is the 0-sphere. This is closure under suspension and desuspension.

Let $\mathcal{S}^n = \{(X, X_0) | (X, X_0) \text{ is a totally n-dimensional compact pair, orientable, with } X_0 \subset X \text{ collared, such that for } x \in X \setminus X_0, \mathcal{L}(x; X) \in \mathcal{S}\}$. Here $\mathcal{L}(x; X)$ denotes the link.
of \( x \) in \( X \). Then \( J_{\mathcal{P}} = \{ J_{\mathcal{P}n} \} \) satisfies axioms 1 - 7, as is readily verified [2]. The associated homology theory \( \Omega_{\mathcal{P}}^* \) is called the oriented bordism theory based on the class \( \mathcal{P} \) of singularities.

Section 8: Bordism theory with \( \mathbb{Z}/k\mathbb{Z} \) coefficients.

Suppose we have an oriented bordism theory \( \Omega_{\mathcal{F}}^* \). Coefficients in an abelian group \( G \) can be introduced geometrically into the theory, as described in [8], yielding the theory \( \Omega_{\mathcal{F}}^* ( ; G ) \). The case \( G = \mathbb{Z}/k\mathbb{Z} \) admits a particularly simple and convenient description first realized by Morgan and Sullivan in [18] in the setting of \( \Omega_{\mathcal{F}}^* ( ; \mathbb{Z}/k\mathbb{Z} ) \).

In general, a \( \mathbb{Z}/k\mathbb{Z} \)-cycle of dimension \( n \) is obtained from an oriented pair \( ( X, X_0 ) \in \mathcal{F}^n \), equipped with an oriented embedding \( \bigsqcup_{i=1}^{k} Y_0 \hookrightarrow X_0 \) of \( k \) copies of \( Y_0 \) onto \( X_0 \), where \( Y_0 \in \mathcal{F}^{n-1} \), by identifying the copies of \( Y_0 \) in an orientation preserving manner. A \( \mathbb{Z}/k\mathbb{Z} \)-cycle with boundary is defined similarly, by embedding \( k \) copies of \( ( Y_0, Y_1 ) \) into \( X_0 \), and identifying the copies of \( ( Y_0, Y_1 ) \). Here \( ( Y_0, Y_1 ) \in \mathcal{F}^{n-1} \). The boundary consists of \( k \) \( X_0 - \bigsqcup_{i=1}^{k} \{ \text{interior of image of } Y_0 \} \), with identifications along the images of \( Y_1 \).
The homology theory \( \Omega_*(-;\mathbb{Z}/k\mathbb{Z}) \) is the bordism theory built from maps of \( \mathbb{Z}/k\mathbb{Z} \)-cycles, modulo maps of \( \mathbb{Z}/k\mathbb{Z} \)-cycles with boundary.

We need the following properties of bordism with \( \mathbb{Z}/k\mathbb{Z} \) coefficients, both easily verified. These facts play a role in the discussion in the Appendix.

**Property 8.1.** The homomorphism \( i: \mathbb{Z}/k\mathbb{Z} \to \mathbb{Z}/k^2\mathbb{Z} \), which is multiplication by \( k \), induces a natural transformation of theories:

\[
\Omega_*(-;\mathbb{Z}/k\mathbb{Z}) \to \Omega_*(-;\mathbb{Z}/k^2\mathbb{Z})
\]

This is Sullivan's "replication" homomorphism.

**Property 8.2.** The projection \( \mathbb{Z}/k \cdot k \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z} \), which is reduction mod \( k \), induces a natural transformation

\[
\Omega_*(-;\mathbb{Z}/k \cdot k \mathbb{Z}) \to \Omega_*(-;\mathbb{Z}/k\mathbb{Z})
\]
Chapter II: Algebraic Preliminaries.

This chapter contains a description of the Witt theory of classification of inner product spaces.

Section 1: Witt rings.

We employ the terminology of [13]. Another treatment is found in [3].

Let $R$ be a commutative ring with unit. Let $X$ be a finitely generated projective module over $R$.

**Definition 1.1.** An inner product $\beta$ on $X$ is a bilinear form $\beta : X \times X \rightarrow R$ which is non-degenerate, i.e., the two homomorphisms from $X$ to $\text{Hom}_R(X,R)$ given by:

$$x_0 \mapsto \beta(x_0, \cdot)$$

and

$$y_0 \mapsto \beta(\cdot, y_0)$$

are bijective.

We call $(X, \beta)$ an inner product space; there is the obvious notion of isomorphism of inner product spaces over $R$.

The inner product is symmetric or skew-symmetric according as
\[ \beta(x,y) = \beta(y,x) \]

or \( \beta(x,y) = -\beta(y,x) \).

In this thesis, we will be concerned primarily with the cases \( R = \mathbb{Q}, \ R = \mathbb{Z}/p\mathbb{Z} \) for \( p \) prime, and \( R = \mathbb{Z} \). Then \( X \) is necessarily a free \( R \)-module (a vector space or free abelian group). Let \( e_1, \ldots, e_n \) be a basis for \( X \). We can associate to \((X, \beta)\) the matrix:

\[ B = (\beta_{ij}) \]

where \( \beta_{ij} = \beta(e_i, e_j) \).

**Lemma 1.2.**

a) \( B \) is invertible

b) If \((X, \beta)\) has a matrix \( B \) and \((X, \beta')\) has matrix \( B' \) (with respect to \((e_1, \ldots, e_n)\)), then \((X, \beta) = (X, \beta')\) if and only if \( B' = ABA^t \), for some invertible matrix \( A \).

**Proof:** [13, p. 3].

We sometimes write \((X, \beta) = \langle B \rangle\).

Given inner product spaces \((X_1, \beta_1), \ldots, (X_n, \beta_n)\), we define the **orthogonal sum** \((X_1 \oplus \cdots \oplus X_n, \beta)\) to be \(X_1 \oplus \cdots \oplus X_n\) with the form:
Clearly \((X_1 \oplus \ldots \oplus X_n, \beta)\) is an inner product space. We may also construct the tensor product \((X_1 \otimes \ldots \otimes X_n, \beta')\) which is \(X_1 \otimes \ldots \otimes X_n\) with the unique form \(\beta'\) satisfying

\[
\beta'(x_1 \otimes \ldots \otimes x_n, y_1 \otimes \ldots \otimes y_n) = \sum_{i=1}^{n} \beta_i(x_i, y_i).
\]

Again, \((X_1 \otimes \ldots \otimes X_n, \beta')\) is an inner product space [13, p. 10].

**Definition 1.3.** Let \((X, \beta)\) be a symmetric inner product space over \(R\). \(X\) is **split** if there exists a submodule \(N \subset X\) such that \(N\) is a direct summand of \(X\), and \(N = N^\perp\), its orthogonal complement. \((N^\perp = \{x \in X | \beta(x,n) = 0 \text{ for all } n \in N}\)).

**Remark 1.4.** A split inner product space \(X\) is called "metabolic" in [3].

In the cases of interest to us, we have the following convenient description of split spaces.

**Lemma 1.5.** Let \(R\) be a ring such that every inner product space \((X, \beta)\) over \(R\) is free. Then \((X, \beta)\) is split iff it has a basis with respect to which the matrix of \(\beta\) has
the form: \( \begin{pmatrix} 0 & I_n \\ I_n & A \end{pmatrix} \), where \( I_n \) is the identity matrix of rank \( n \). If 2 is a unit, then with respect to some basis, the matrix of \( \beta \) is \( \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \).

**Proof:** [13, p. 13].

In the case of skew-symmetric inner product spaces, we have an analogue of the concept of being split.

**Definition 1.6.** \((X, \beta)\) is **symplectic** if \( \beta(x, x) = 0 \), all \( x \in X \).

Parallel to Lemma 1.5 is:

**Lemma 1.7.** If \( R \) is either a Dedekind domain (e.g., \( \mathbb{Z} \)) or a local ring (e.g., a field), then every symplectic inner product space over \( R \) is free, and possesses a **symplectic basis**; that is, a basis such that the matrix of \( \beta \) is

\[
\begin{pmatrix}
0 & I_n \\
-I_n & 0
\end{pmatrix}
\]

**Proof:** [13, p. 7].

Finally, we distill the collection of symmetric inner product spaces over \( R \) into the Witt ring \( W(R) \) by means
of an equivalence relation.

Definition 1.8.: Two symmetric inner product spaces $X, X'$ belong to the same Witt class, denoted $X \sim X'$, if there exist split spaces $S$ and $S'$ such that $X \oplus S$ is isomorphic to $X' \oplus S'$. (For convenience, we omit explicit mention of the forms $\beta, \beta'$).

By abuse of terminology, we say that a split space is equivalent to the form $\langle 0 \rangle$.

This is a useful relation, it satisfies the following properties.

Lemma 1.9.: If $X \sim X'$ and $Y \sim Y'$, then:

1) $X \oplus Y \sim X' \oplus Y'$
2) $X \oplus Y \sim X' \oplus Y'$
3) $(X, \beta) \oplus (X, -\beta) \sim \langle 0 \rangle$
4) $\langle 1 \rangle \oplus X \sim X$

Proof: [13, p. 14].

Consequently, the collection of Witt classes, denoted $W(R)$ has attractive structure, suitable for study in terms of standard algebraic concepts.

Theorem 1.10.: $W(R)$ is a commutative ring with unit, using orthogonal sum as addition operation, and tensor
product as multiplication operation.

Proof: Immediate from 1.9.
note that if we fix a prime $p \in \mathbb{Z}$, then any non-zero rational number $q$ can be written uniquely as:

$$q = p^n \cdot u,$$

where $n \in \mathbb{Z}$ and $u$ is a unit in $\mathbb{Z}_{(p)}$, the subring of rationals which, when written in reduced form, have denominators prime to $p$. (That is, $\mathbb{Z}_{(p)} = (\mathbb{Z}-(p))^{-1}\mathbb{Z}$).

Lemma 2.2.: Fix a prime $p \in \mathbb{Z}$. There is a unique additive homomorphism

$$\vartheta_p : W(\mathcal{Q}) \rightarrow W(\mathbb{Z}/p\mathbb{Z})$$

which maps each generator $\langle p^n u \rangle$ to $\langle \bar{u} \rangle$ or $0$, according as $n \equiv 1 \pmod{2}$ or $n \equiv 0 \pmod{2}$. (Here $\bar{u}$ is the residue class of $u$ in $\mathbb{Z}_{(p)}/(p) \cdot \mathbb{Z}_{(p)} = \mathbb{Z}/p\mathbb{Z}$).

**Proof:** [13, p. 85].

**Remark 2.3.:** This homomorphism is the second residue class form homomorphism.

We can collect the homomorphisms $\vartheta_p$ for all primes $p$ and thereby obtain an additive homomorphism
Note that we also have a homomorphism

\[ i : W(\mathbb{Z}) \to W(\mathbb{Q}) \]

gotten by regarding an integral inner product space as a rational one.

The next, remarkable theorem is proved in [13, p. 88-90]. First recall a standard definition.

**Definition 2.4.** The **signature** of a rational inner product space is the integer \( d^+ - d^- \) where \( d^+ \) (resp. \( d^- \)) is the number of positive (resp. negative) entries on the diagonal of a diagonalized matrix for the form on the space.

**Theorem 2.5.** 1) The sequence

\[ 0 \to W(\mathbb{Z}) \xrightarrow{i} W(\mathbb{Q}) \xrightarrow{\vartheta} \bigoplus_{p \text{ prime}} W(\mathbb{Z}/p\mathbb{Z}) \to 0 \]

is split exact.

2) \( W(\mathbb{Z}) \cong \mathbb{Z} \) canonically, and the isomorphism is given by the signature homomorphism,

\[ \text{sign}_{\mathbb{Z}} : W(\mathbb{Z}) \to \mathbb{Z}, \]
which takes \((X, \beta) \in W(\mathbb{Z})\) to its signature (that is, the signature of \(i(X, \beta) \in W(\mathbb{Q})\)).

3) The exact sequence is split by the homomorphism
\[\text{sign}^{-1} \circ \text{sign}_Q : W(\mathbb{Q}) \to W(\mathbb{Z}),\]
where \(\text{sign}_Q : W(\mathbb{Q}) \to \mathbb{Z}\) is the signature homomorphism.

Therefore there is a canonical isomorphism:

\[W(\mathbb{Q}) \cong W(\mathbb{Z}) \oplus \bigoplus_{p} W(\mathbb{Z}/p\mathbb{Z})\]

\[\cong \mathbb{Z} \oplus \bigoplus_{p} W(\mathbb{Z}/p\mathbb{Z}).\]

It remains to compute \(W(\mathbb{Z}/p\mathbb{Z})\) for all primes \(p\).

**Lemma 2.6.** The additive structure of \(W(\mathbb{Z}/p\mathbb{Z})\) is:

1) \(W(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\), generated by \(\langle 1 \rangle\).

2) \(W(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}\), generated by \(\langle 1 \rangle\) for \(p \equiv 3 \pmod{4}\)

3) \(W(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\), with generators \(\langle 1 \rangle\) and \(\langle d \rangle\) where \(d\) is a non-square in \(\mathbb{Z}/p\mathbb{Z}\), for \(p \equiv 1 \pmod{4}\).

**Proof:** [19, p. 47].

**Remark 2.7.** [19, p. 47] actually shows that the ring structure is described by:
1) \( \langle 1 \rangle \otimes \langle 1 \rangle = \langle 1 \rangle \) for \( p = 2 \)

2) \( \langle 1 \rangle \otimes \langle 1 \rangle = \langle 1 \rangle \) for \( p \equiv 3 \pmod{4} \)

3) \( \langle 1 \rangle \otimes \langle 1 \rangle = \langle 1 \rangle \)
\[ \langle d \rangle \otimes \langle d \rangle = \langle 1 \rangle \) for \( p \equiv 1 \pmod{4} \).

We isolate the following obvious corollary to Theorem 2.5 and Lemma 2.6, as it has important ramifications in Chapter VI.

**Corollary 2.8.**: \( W(\mathbb{Q}) \cong \mathbb{Z} \oplus T \) (canonically), where \( T \) is an infinite direct sum of cyclic 2-groups of order 2 or 4.

It is also useful to state the following fact about \( W(\mathbb{Q}) \).

**Proposition 2.9.**: If \( (X, \beta) \) is a symmetric rational inner product space such that its equivalence class in \( W(\mathbb{Q}) \) is the zero element, then \( (X, \beta) \) is split.

**Proof**: This follows from the definition of \( W(\mathbb{Q}) \) and Witt's theorem [13, p. 8].
Chapter III: Witt spaces and generalized Poincaré duality.

Section 1: Introduction.

In section 2, we define the class of Witt spaces. Roughly, a Witt space is a stratified p.l. pseudomanifold satisfying the following intersection homological link condition:

If $L^{2\ell}$ is the link of an odd codimension stratum at a point $x$ in $X$, then

$$H^m_{\ell}(L; \mathbb{Q}) = 0,$$

where $m = (0, 0, 1, 1, 2, 2, \ldots)$; that is $m(c) = \lfloor \frac{c-2}{2} \rfloor$.

In section 3, for an arbitrary triangulated p.l. pseudomanifold $X$, we define an increasing filtration on $W^m_* (X; \mathbb{Q})$ corresponding to the increasing sequence of perversities:

$$m = (0, 0, 1, 1, 2, 2, \ldots, \lfloor \frac{\dim X-2}{2} \rfloor)$$

$$\vdots$$

$$(0, 0, 1, 1, 2, 3, \ldots, \lfloor \frac{\dim X-1}{2} \rfloor)$$

$$(0, 0, 1, 2, 2, 3, \ldots, \lfloor \frac{\dim X-1}{2} \rfloor)$$

$$n = (0, 1, 1, 2, 2, 3, \ldots, \lfloor \frac{\dim X-1}{2} \rfloor)$$
Recall $\tilde{n}(c) = \left\lfloor \frac{c-1}{2} \right\rfloor$.

Notice that the perversity index of each odd co-dimension stratum gets incremented by 1, in order of decreasing codimension.

Associated to the filtration is an $E^1$-spectral sequence, converging to $\tilde{W}_{\ast}(X;\mathbb{Q})$. We prove that the spectral sequence collapses for a Witt space. That is, $E^\infty_{s} = 0$ for $s > 0$. The generalized Poincare duality theorem for closed Witt spaces $X$, stating that $\tilde{H}_{\ast}(X;\mathbb{Q})$ is self-dual under augmented intersection product, then follows from the intersection pairing theorem (I.4.4).

Finally in section 4, we introduce the Witt class $w(X)$ of a Witt space $X$, a p.l. invariant taking its values in $W(\mathbb{Q})$, the Witt group of the rationals. If $\dim X = 4k$, $k > 0$, $w(X)$ is the equivalence class of the rational inner product space $\tilde{H}_{2k}(X;\mathbb{Q})$ in $W(\mathbb{Q})$. If $\dim X = 0$, $w(X)$ is merely the rank of $H_0(X;\mathbb{Q})$ times $\langle 1 \rangle$ in $W(\mathbb{Q})$. Otherwise, we set $w(X) = 0$.

For discussion and applications of a special case of the Witt class, see [3], [10].

Section 2: Witt spaces.

Let $X^q$ be a q-dimensional p.l. pseudomanifold with
(possibly empty) collared boundary $\partial X$. Let $x \in X - \partial X$. The link of $x$, $\mathcal{L}(x,X)$, is unique up to p.l. homeomorphism [20]. Let $d(x)$ be the intrinsic dimension of $X$ at $x$. Then there is a p.l. homeomorphism: $\mathcal{L}(x,X) \cong S^{d(x)-1} \ast L(x)$. The space $L(x)$ is a pseudomanifold of dimension $\ell(x) = q - d(x) - 1$, unique up to p.l. homeomorphism [1, p. 420 and 424].

**Definition 2.1.** Let $X^q$ be a $q$-dimensional p.l. pseudomanifold as above. We say $X$ is a Witt space if $\tilde{H}_{\ell(x)/2}(L(x); \mathbb{Q}) = 0$, for all $x \in X - \partial X$ such that $\ell(x) \equiv 0 \pmod{2}$.

**Remark 2.2.** The definition places intersection homological restrictions on the links of odd codimension intrinsic strata only. The discussion of the spectral sequence in section 3 will provide justification.

For applications, we must translate the Witt condition in (2.1) into a statement about arbitrary stratifications of $X$. To this end, we prove a few lemmas which culminate in Proposition 2.6. Recall that $\bar{m}$ refers to the perversity with $\bar{m}(c) = \lfloor \frac{c-2}{2} \rfloor$. 
Lemma 2.3.: Let $L$ be a closed p.l. pseudomanifold of dimension $2\ell$. Suppose $L = S^0 \ast K$ for some p.l. pseudomanifold $K$ of dimension $2\ell - 1$. Then $H^m_\ell(L; \mathbb{Z}) = 0$.

Proof: The 0-sphere $S^0$ is the discrete set $\{a, b\}$. Let $T$ be a triangulation of $K$, and $\mathcal{J}$ the associated stratification: $\mathcal{J} = (T_{2\ell-1} \supset \Sigma_T = T_{2\ell-3} \supset \ldots \supset T_0)$. Let $S$ be the triangulation of $L$ induced by suspension, and let $\mathcal{M}$ be the stratification on $L$ similarly induced. That is:

$$\mathcal{M} = (S^0 \ast T_{2\ell-1} \supset S^0 \ast \Sigma_T = S^0 \ast T_{2\ell-3} \supset \ldots \supset S^0 \ast T_0 \supset S^0)$$

Note that $\mathcal{M}$ is not the stratification associated to $S$.

Consider a cycle $z \in C^m_\ell(L)$, where perversity is defined with respect to $\mathcal{M}$. Then, $|z| \cap S^0 = \emptyset$. Choose a triangulation $S_1$ of $L$ subordinate to $S$ such that $z$ is a sum of $\ell$-simplices with coefficients. Each vertex $v$ of $|z|$ lies in the interior of a unique simplex $\sigma$ of $L$. If $\text{Int}(\sigma) \subset K$, define $f(\sigma) = \hat{\sigma}$, the barycenter of $\sigma$. If $v \in \text{Int}(a \ast \tau)$ or $v \in \text{Int}(b \ast \tau)$, where $\text{Int}(\tau) \subset K$, then define $f(\sigma) = \hat{T}$. Extend the map $f$ linearly to simplices of $|z|$. By [20, p. 17], $f$ determines a p.l. geometric chain which we denote $f(z)$ in $C^m_\ell(L)$, supported in $|K|$. One checks easily that $f(z)$ is a cycle and, in fact,
\[|f(z)| \text{ is } (\bar{m}, \bar{\ell})\text{-allowable. We claim that } f(z) \text{ represents}
\]
the same homology class as } z \text{ in } H_{\bar{\ell}}^{\bar{m}}(L). \text{ The triangulation of } |z| \text{ can be used to construct a triangulation of}
\[|z| \times [0,1] \text{ having as its vertices the set}
\{(v,0),(v,1) | v \in |z|\} \text{ and subordinate to the product cell}
structure on } |z| \times [0,1]. \text{ See, for instance, [11].}
Define } H : |z| \times [0,1] \rightarrow L \text{ by } H(v,0) = v \text{ and}
H(v,1) = f(v), \text{ and extend linearly over simplices of}
\[|z| \times [0,1]. \text{ Appealing again to [20, p. 17], we obtain}
a \text{ p.l. chain } H(z \times [0,1]) \text{ in } C_{\bar{\ell}+1}(L), \text{ with}
\partial H(z \times [0,1]) = f(z) - z. \text{ Moreover, it is easy to see}
that } |H(z \times [0,1])| \text{ is } (\bar{m}, \bar{\ell}+1)\text{-allowable. This proves}
the claim. \]

Let } W \text{ be the obvious p.l. chain in } C_{\bar{\ell}+1}(L) \text{ with}
\[|W| = a \ast |f(z)| \text{ and } \partial W = f(z). \text{ Then } W \in C_{\bar{\ell}+1}^{m}(L),
\text{ implying that the homology class of } f(z) \text{ in } H_{\bar{\ell}}^{m}(L) \text{ is}
trivial. Therefore } z \text{ represents } 0 \text{ in } H_{\bar{\ell}}^{m}(L). \text{ Since } z
\text{ was arbitrary, we conclude } H_{\bar{\ell}}^{m}(L) = 0. \]

**Corollary 2.4.:** Let } L \text{ be a closed p.l. pseudomanifold of}
dimension } 2\ell. \text{ Suppose } L = S^j \ast K \text{ for some p.l. pseudo-}
manifold } K, \text{ where } 0 \leq j \leq 2\ell. \text{ Then, } H_{\bar{\ell}}^{m}(L) = 0.

**Proof:** Rewrite } L \text{ as } S^0 \ast (S^{j-1} \ast K), \text{ and apply 2.3.}
Corollary 2.5.: Let $L$ be as in 2.4. Then $H^m_L(L;\mathbb{Q}) = 0$.

Proof: The result follows from 2.4 and the isomorphism of groups: $H^m_L(L;\mathbb{Q}) \cong H^m_L(L) \otimes \mathbb{Q}$, induced by the definition:

$$C^m_*(L;\mathbb{Q}) = C^m_*(L) \otimes \mathbb{Q}.$$ 

Proposition 2.6.: Let $X^p$ be a stratified p.l. pseudo-manifold, with (possibly empty) boundary $\partial X$ and stratification: $\mathcal{J} = (X_n \supset X_{n-1} = X_{n-2} \supset \ldots \supset X_0)$. Let $L(x_i, x)$ be the link of $x_i = X_i - \bar{X}_{i-1}$ at $x$, where $x \in (X - \partial X) \cap \chi_i$, and $\chi_i \neq \emptyset$. Then, $X$ is a Witt space if and only if

$$H^m_L(L(x_i, x);\mathbb{Q}) = 0$$

for $i = q - (2\ell + 1)$, whenever $\ell \geq 1$.

Proof: Suppose $X$ is Witt. There are p.l. homeomorphisms:

$$\mathcal{L}(x, X) = S^d(x) \ast L(x)$$

$$\mathcal{L}(x, X) = S^i \ast L(x_i; x)$$

Let $j = d(x) - i$. Then $j \geq 0$, and

$$L(x_i; x) = S^{j-1} \ast L(x)$$

by [1, p. 423].
Case 1: If $j - 1 = -1$, then $\mathcal{L}(\chi_1; x) \simeq L(x)$, so the result follows from the fact $X$ is a Witt space.

Case 2: If $j - 1 \geq 0$, the result follows from 2.5.

The converse follows from the fact that any p.l. stratification of $X$ is subordinate to the intrinsic stratification.

Section 3: The intersection homology spectral sequence and duality.

Assume $X$ is a $q$-dimensional p.l. pseudomanifold with triangulation $T$, and associated stratification $\mathcal{J}$. Let $r$ be the largest integer satisfying $2r+1 \leq q$ (assume $q \geq 2$). Assume rational coefficients unless otherwise specified.

Definition 3.1.: Let $\tilde{p}_k$ be the perversity defined by:

$$\tilde{p}_k(c) = \begin{cases} \lceil \frac{c-2}{2} \rceil & \text{for } c \leq k \\ \lceil \frac{c-1}{2} \rceil & \text{for } c > k \end{cases}$$

where $1 \leq k \leq q$ and $2 \leq c \leq q$. 
Since $\bar{m}(c) = \bar{n}(c)$ for $c$ even, we may assume $k$ odd. Note that $\bar{p}_{2r+1} = \bar{m}$ and $\bar{p}_1 = \bar{n}$.

There is a filtration $F$ on the chain complex $\bar{W}_n(X)$ [see 1.5]:

$$F_s\bar{W}_n(X) = \begin{cases} 0 & \text{for } s < 0 \\ \bar{p}_{(2r+1)-2s}(X) & \text{for } 0 \leq s \leq r \\ \bar{W}_n(X) & \text{for } s > r \end{cases}$$

The filtration is induced by the inclusions of complexes:

$$0 \subset \bar{W}_n = \bar{W}_{2r+1} \subset \bar{W}_{2r-1} \subset \ldots \subset \bar{W}_{*} \subset \bar{W}_{*} = \bar{W}_n.$$

Following [21, p. 469], there is a convergent $E^1$-spectral sequence with

$$E^1_{s,t} = H_{s+t}(F_s\bar{W}_n(X)/F_{s-1}\bar{W}_n(X))$$

and $E^\infty$ isomorphic to the bigraded module $G(\bar{W}_n(X))$, the associated graded to the filtration

$$F_s(H_*(X)) = \text{Im}(H_{*(2r+1)-2s}(X) \rightarrow H_*(X)).$$

**Theorem 3.2.:** Assume $X$ is a Witt space. Then

1. $E^\infty_{0,t} = \text{Im}(H^m_t(X) \rightarrow H^t(X))$
2. $E^\infty_{s,t} = 0$ for $s > 0$.

**Proof:** Part (1) is immediate, since $F_{-1}(H^t(X)) = 0$.  

Part (2) follows from the equality
\[
\text{Im}(H_{2s+3}(X) \rightarrow H_{2s+1}(X)) = H_{2s+1}(X), \text{ for } 0 \leq s \leq r - 1.
\]

We now prove (3). Fix \( s \) with \( 0 \leq s \leq r - 1 \). Note that
\[
\bar{p}_{2s+3}(X) = \bar{p}_{2s+1}(X) \text{ for } j \leq s + 1 \text{ and } j \geq q - s,
\]
so assume \( s + 1 < j < q - s \).

Let \( \xi \in \bar{p}_{2s+1}(X) \) be represented by the cycle \( z \in \bar{p}_{2s+1}(X) \). We construct a chain \( w \in \bar{p}_{2s+1}(X) \) such that
\[
\partial w = z - y,
\]
where \( y \in \bar{p}_{2s+3}(X) \). This will prove (3) and therefore prove the theorem.

The chain \( w \) is constructed locally to reduce the dimension of \( |z| \cap \chi_{q-(2s+3)} \), where \( \chi_{q-(2s+3)} \) is the \( q-(2s+3) \) stratum of \( \mathcal{J} \). To begin with, we have:
\[
\dim(|z| \cap \chi_{q-(2s+3)}) \leq j - (2s+3) + \bar{p}_{2s+1}(2s+3)
\]
\[= j - s - 2.\]
If, in fact, the following stronger inequality holds:

\[(4) \quad \dim(|z| \cap \chi_{q-(2s+3)}) \leq j - (2s+3) + \bar{p}_{2s+3}(2s+3) = j - s - 3,\]

then \(z \in W_{j}^{2s+3}(X)\). This follows from the fact that \(\bar{p}_{2s+1}(c) = \bar{p}_{2s+3}(c)\) if \(c \neq 2s+3\).

In this case, we may set \(y = z\) and \(w = 0\), and we're done.

Suppose (4) does not hold. The open stratum \(\chi_{q-(2s+3)}\) is the disjoint union of interiors of simplexes \(\sigma\) in \(T\), where \(\dim \sigma = q - (2s+3)\). Let \(\bar{C}\) denote the non-empty subset of these simplexes for which \(\dim(|z| \cap \Int(\sigma)) = j - s - 2\). Let \(\sigma \in \bar{C}\), and consider the set \(\{\tau_{i}\}_{i \in I}\) (where \(I\) is a finite index set) of \(j - s - 2\) simplexes in the first barycentric subdivision of \(\sigma\) satisfying

\[|z| \cap \Int(\sigma) \supset \Int(\tau_{i}).\]

Orient each \(\tau_{i}\). The subchain of \(z\) consisting of \(j\)-simplices in \(T'\) whose closures intersect \(\Int(\sigma)\) in \(\Int(\tau_{i})\) is:
\[ z_i = \tau_i \ast v_i \]

where \( v_i \in C_{s+1}(\text{lk}(\sigma,T')) \), and \( \text{lk}(\sigma,T') \) denotes the boundary of the classical dual to \( \sigma \) in \( T', \) dual(\( \sigma,T' \)).

Lemma 3.3.: \( \partial v_i = 0 \), for all \( i \).

Proof: Fix \( i \in I \). Decompose \( z \) as \( z = z_i + (z-z_i) \).

Since \( z \) is a cycle, we have:

\[ (5) \quad \partial z_i = -\partial (z-z_i). \]

Now, \( \partial z_i = \partial (\tau_i \ast v_i) \)

\[ = \partial \tau_i \ast v_i + (-1)^{j-1} \tau_i \ast \partial v_i. \]

Therefore, \( |\partial z_i| \cap \text{Int}(\sigma) \) contains \( \text{Int}(\tau_i) \) if and only if \( \partial v_i \neq 0 \). By definition of \( z_i \), however,

\[ |\partial (z-z_i)| \cap \text{Int}(\sigma) \]

does not contain \( \text{Int}(\tau_i) \). It follows, by taking supports in (5), that \( \partial v_i = 0 \).

The canonical simplicial isomorphism of \( \text{lk}(\sigma,T') \) and \( \text{lk}(\sigma,T)' \) maps \( v_i \) to a cycle \( \bar{v}_i \in C_{s+1}(\text{lk}(\sigma,T)) \). We have the complex \( \bar{W}_\ast(\text{lk}(\sigma,T)) \) associated to the restriction of
Lemma 3.4.: \( \bar{v}_i \in \mathcal{W}_{s+1}(\text{lk}(\sigma,T)) \), for all \( i \).

Proof: Clearly, \( \bar{v}_i \) is a sum of \((s+1)\)-simplices in the first barycentric subdivision of \( \text{lk}(\sigma,T) \). Since \( \partial \bar{v}_i = 0 \), it remains to show only that \( |\bar{v}_i| \) is \((\bar{m},s+1)\)-allowable, for each \( i \).

Fix \( i \in I \). The cycle \( v_i \) can be written uniquely as

\[
(6) \quad v_i = \sum_{\sigma < \mu_0, \ldots, \mu_{s+1}} a(\mu_0, \ldots, \mu_{s+1}) \langle \hat{\mu}_0, \ldots, \hat{\mu}_{s+1} \rangle.
\]

Here \( a(\mu_0, \ldots, \mu_{s+1}) \) is the rational coefficient of the simplex in \( T' \) spanned by \( \hat{\mu}_0, \ldots, \hat{\mu}_{s+1} \). For \( \mu > \sigma \), define \( d(\mu) \) to be the maximum integer \( k \) such that \( \hat{\mu} = \hat{\mu}_k \) in a simplex \( \langle \hat{\mu}_0, \ldots, \hat{\mu}_{s+1} \rangle \) with non-zero coefficient in (6).

Set \( d(\mu) = -1 \) if \( \hat{\mu} \) does not appear. Then

\[
(7) \quad \dim(|z_i| \cap \text{Int}(\mu)) = j - s - 2 + d(\mu) + 1.
\]

On the other hand, since \( z \in C_{\bar{m}}^{2s+1}(X) \), we get the following inequality, valid when codimension of \( \mu \) in \( X \) is greater than or equal to 2 (\( \text{codim}(\mu,X) \geq 2 \)):
Now, \( p_{2s+1} (\text{codim}(\mu, X)) = \tilde{m} (\text{codim}(\mu, X)) \), since \( \mu > \sigma \), implying \( \text{codim}(\mu, X) \leq 2s+1 \). Combining (7) and (8), we get:

\[
(9) \quad d(\mu) \leq (s+1) - \text{codim}(\mu, X) + \tilde{m} (\text{codim}(\mu, X)).
\]

Each \( \mu > \sigma \) has a unique expression as:

\[
\mu = \sigma \ast \nu,
\]

where \( \nu \in \text{lk}(\sigma, T) \). The cycle \( \tilde{v}_i \) then has the expression:

\[
(10) \quad \tilde{v}_i = \sum_{\sigma < \mu_0 < \ldots < \mu_{s+1}} a(\mu_0, \ldots, \mu_{s+1}) \langle \hat{\nu}_0, \ldots, \hat{\nu}_{s+1} \rangle.
\]

If \( \mu = \sigma \ast \nu \), then \( d(\mu) \) is the maximum integer \( k \) such that \( \hat{\nu} = \hat{\nu}_k \) in a simplex \( \langle \hat{\nu}_0, \ldots, \hat{\nu}_{s+1} \rangle \) with non-zero coefficient in (10). Therefore,

\[
(11) \quad d(\mu) = \dim (|\tilde{v}_i| \cap \text{Int}(\nu)).
\]

Observe that

\[
(12) \quad \text{codim}(\nu, \text{lk}(\sigma, T)) = \text{codim}(\mu, X).
\]
Then (9), (11), and (12) yield the inequality:

\[
\dim(|\Delta_i| \cap \text{Int}(v)) \leq (s+1) - \text{codim}(v, \ell k(\sigma,T)) + m(\text{codim}(v, \ell k(\sigma,T)))
\]

when \( \text{codim}(v, \ell k(\sigma,T)) \geq 2 \).

This inequality implies that \( |\Delta_i| \) is \((m,s+1)\)-allowable.

By hypothesis, \( X \) is Witt. Proposition 2.6 implies that \( H_{s+1}^m(\ell k(\sigma,T)) = 0 \). Therefore, there is a chain \( \bar{x}_i \in W_{s+2}^m(\ell k(\sigma,T)) \) with \( \partial \bar{x}_i = \Delta_i \). Let \( x_i \) be the corresponding chain in \( \ell k(\sigma,T') \). We define:

\[
w_i = (-1)^{j-s} \tau_i \ast x_i
\]

and

\[
w_0 = \sum_{i \in I} w_i
\]

Note that:

\[
\partial w_i = (-1)^{j-s} \partial \tau_i \ast x_i + \tau_i \ast \partial x_i.
\]

**Lemma 3.5.** \( w_0 \in W_{j+1}^{2s+1}(X) \).

**Proof:** It suffices to prove \( w_i \in W_{j+1}^{2s+1}(X) \), for all \( i \).
The argument proceeds as in 3.4. First we show that $|w_i|$ is $(\tilde{p}_{2s+1}, j+1)$-allowable. Let $\mu \in \text{star}(\sigma, T)$, with $\text{codim}(\mu, X) \geq 2$.

**Case 1:** Suppose $\mu > \sigma$. A counting argument analogous to that in 3.4 proves:

$$\dim(|w_i| \cap \text{Int}(\mu)) \leq (j+1) - \text{codim}(\mu, X) + \tilde{m}(\text{codim}(\mu, X)).$$

Since $\text{codim}(\mu, X) \leq 2s+1$, $\tilde{m}(\text{codim}(\mu, X)) = \tilde{p}_{2s+1}(\text{codim}(\mu, X))$ so

$$\dim(|w_i| \cap \text{Int}(\mu)) \leq j+1 - \text{codim}(\mu, X) + \tilde{p}_{2s+1}(\text{codim}(\mu, X)).$$

**Case 2:** Suppose $\mu \leq \sigma$. Then

$$\dim(|w_i| \cap \text{Int}(\mu)) = \dim(|\tau_i \ast x_i| \cap \text{Int}(\mu))$$

$$= \dim(|\tau_i \ast v_i| \cap \text{Int}(\mu))$$

$$\leq \dim(|z| \cap \text{Int}(\mu)).$$

Since $|z|$ is $(\tilde{p}_{2s+1}, j)$-allowable,

$$\dim(|w_i| \cap \text{Int}(\mu)) \leq j - \text{codim}(\mu, X) + \tilde{p}_{2s+1}(\text{codim}(\mu, X)).$$
Case 3: Suppose $\mu \not\in \text{star}(\sigma, T)$. Then $|w| \cap \text{Int}(\mu) = \emptyset$.

These three cases prove the assertion that $|w|^{\prime}$ is $(p_{2s+1}, j+1)$-allowable.

Next, we must verify that $|w^{\prime}|$ is $(p_{2s+1}, j)$-allowable. Note that $|w^{\prime}| = |w^{\prime} \cap x| \cup |w^{\prime} \cap v|$. Let $\mu$ be as above. We treat three cases again.

Case 1: Suppose $\mu > \sigma$. Then

\begin{align*}
\text{(13)} \quad \dim(|w^{\prime} \cap x| \cap \text{Int}(\mu)) &= \dim(|w^{\prime} \cap v| \cap \text{Int}(\mu)) - 1 \\
&= \dim(|w^{\prime} \cap \text{Int}(\mu)) - 1 \\
&\leq j - (\text{codim}(\mu, X) + p_{2s+1}(\text{codim}(\mu, X))).
\end{align*}

Also,

\begin{align*}
\text{(14)} \quad \dim(|w^{\prime} \cap v| \cap \text{Int}(\mu)) &= \dim(|w^{\prime} \cap v| \cap \text{Int}(\mu)) \\
&\leq \dim(|w^{\prime} \cap \text{Int}(\mu)) \\
&\leq j - \text{codim}(\mu, X) + p_{2s+1}(\text{codim}(\mu, X)).
\end{align*}

Together, (13) and (14) imply that

$$
\dim(|w^{\prime} \cap \text{Int}(\mu)) \leq j - \text{codim}(\mu, X) + p_{2s+1}(\text{codim}(\mu, X)).
$$
Case 2: Suppose \( \mu \leq \sigma \). Then

\[
\dim(|\partial w_i| \cap \text{Int}(\mu)) \leq \dim(|\tau_i| \cap \text{Int}(\mu)) \\
\leq \dim(|z| \cap \text{Int}(\mu)) \\
\leq j - \text{codim}(\mu, X) + p_{2s+1}(\text{codim}(\mu, X))
\]

Case 3: Suppose \( \mu \notin \text{star}(\sigma, T) \). Then \( |\partial w_i| \cap \text{Int}(\mu) = \emptyset \).

These three cases prove that \( |\partial w_i| \) is \((p_{2s+1}, j)\)-allowable, completing the proof of the lemma.

Let \( y_\sigma = z - \partial w \). Then \( y_\sigma \) is a cycle and \( y_\sigma \in \bar{\omega}_j(\Sigma) \). Moreover, it is clear from the construction of \( w_\sigma \) that

\[
\dim(|y_\sigma| \cap \text{Int}(\sigma)) \leq j - s - 3.
\]

Now repeat the argument for each of the simplices remaining in \( \mathcal{C} \). Denote the \( j+1 \)-chains obtained at all of the steps by \( \{ w_\sigma \}_{\sigma \in \mathcal{C}} \). If we set

\[ w = \sum_{\sigma \in \mathcal{C}} w_\sigma, \]

and
we have:

\[ \dim(|y| \cap \text{Int}(\sigma)) \leq j - s - 3 \]

for all \( \sigma \in T_{q-(2s+3)} \). Therefore, (4) holds and

\[ y \in W_{j}^{(2s+3)}(X). \]

The theorem is proved.

Remark 3.6.: The proof of 3.2 shows that if the Witt condition on links is satisfied only for odd codimension strata of codimension less than or equal to \( 2k+1 \), then the spectral sequence collapses partially:

\[ E_{s,t}^{\infty} = 0 \quad \text{for} \quad s > (r-k). \]

We now use 3.2 to prove a proposition which, when combined with the intersection pairing theorem, yields Theorem 3.8, generalized Poincaré duality for Witt spaces. We include the rational coefficients for emphasis.
Proposition 3.7.: Let $X^q$ be a Witt space, $q \geq 0$. The homomorphism of graded groups,

$$i_* : H^m_*(X; \mathbb{Q}) \to H^n_*(X; \mathbb{Q})$$

induced by inclusion of chain complexes, is an isomorphism.

Proof: For $q = 0$ or $1$, the result claimed is obvious. For $q \geq 2$, Theorem 3.2 implies:

(15) \[ \text{Im}(H^*_m(X; \mathbb{Q}) \to H^*_n(X; \mathbb{Q})) = H^*_n(X; \mathbb{Q}) \]

Therefore:

(16) \[ \dim_{\mathbb{Q}} H^m_j(X; \mathbb{Q}) \geq \dim_{\mathbb{Q}} H^n_j(X; \mathbb{Q}) \text{ for } 0 \leq j \leq q. \]

The intersection pairing theorem (I.4.4) implies:

(17) \[ \dim_{\mathbb{Q}} H^m_j(X; \mathbb{Q}) = \dim_{\mathbb{Q}} H^n_{q-j}(X; \mathbb{Q}) \text{ for } 0 \leq j \leq q. \]

Combined, (16) and (17) yield:

(18) \[ \dim_{\mathbb{Q}} H^m_j(X; \mathbb{Q}) = \dim_{\mathbb{Q}} H^n_j(X; \mathbb{Q}) \text{ for } 0 \leq j \leq q. \]

The proposition follows from (15) and (18).
Finally, we get generalized Poincaré duality for Witt spaces.

**Theorem 3.8.** Let $X^q$ be a Witt space. There is a nondegenerate rational pairing:

$$
\tilde{H}^m_i(X; \mathbb{Q}) \times \tilde{H}^m_j(X; \mathbb{Q}) \rightarrow \mathbb{Q}
$$

for $i + j = q; \ i, j \geq 0$.

The pairing is given by augmented intersection product.

**Proof:** Theorem 1.4.4 provides a nondegenerate rational pairing

$$
\tilde{H}^m_i(X; \mathbb{Q}) \times \tilde{H}^n_j(X; \mathbb{Q}) \rightarrow \mathbb{Q} \quad \text{for} \quad i + j = q; \ i, j \geq 0.
$$

Proposition 3.7 gives an isomorphism (induced by inclusion on the chain level):

$$
i_j : \tilde{H}^m_j(X; \mathbb{Q}) \xrightarrow{\cong} \tilde{H}^n_j(X; \mathbb{Q}) \quad \text{for} \quad 0 \leq j \leq q.
$$

Intersection product of cycles is compatible with this homomorphism by [11]. The theorem follows.
Section 4: The Witt class $w(X)$.

Let $X$ be a Witt space of dimension $4k$, $k > 0$.

Theorem 3.8 gives a nondegenerate symmetric bilinear form on the vector space $H_{2k}^m(X;\mathbb{Q})$. That is, $H_{2k}^m(X;\mathbb{Q})$ is a symmetric inner product space.

Definition 4.1: (1) Let $X^q$ be a Witt space of dimension $q$, $q \geq 0$.

If $q = 4k$, $k > 0$, the Witt class of $X$, denoted $w(X)$, is the equivalence class of the inner product space $H_{2k}^m(X;\mathbb{Q})$ in $W(\mathbb{Q})$, the Witt group of the rationals.

If $q = 0$, $w(X) = \text{rank}(H_0(X;\mathbb{Q})),\langle 1 \rangle$ in $W(\mathbb{Q})$.

If $q \neq 4k$, set $w(X) = 0$ in $W(\mathbb{Q})$.

(2) If $(X,\mathfrak{a}X)$ is a Witt space of dimension $q \equiv 0 \pmod{4}$, set $w(X) = w(\hat{x})$, where $\hat{x} = X \cup \text{cone}(\mathfrak{a}X)$.

Remark 4.2.: Clearly $w(X)$ is a p.l. invariant.

Let $\text{sign}_Q : W(\mathbb{Q}) \to \mathbb{Z}$ be the signature homomorphism of II.2.5.

Definition 4.3.: The signature of $X$, $\text{sign}(X)$, is the integer $\text{sign}_Q(w(X))$. 
Remark 4.4.: This signature extends that for pseudo-manifolds which can be stratified with strata of only even codimension and, therefore, the classical signature for p.l. manifolds [11].
Chapter IV: Properties of the Witt class $w(X)$.

Section 1: Introduction.

In this chapter, we study properties of the Witt class, $w(X)$. For the most part, these are analogues of properties of the signature of manifolds.

1. Cobordism invariance:
   \[
   X^{4k} = \exists w^{4k+1}, \text{ then } w(X) = 0.
   \]

2. Additivity:
   \[
   Y^{4k-1} = \exists X^{4k} \text{ and } -Z^{4k-1} = \exists Y^{4k},
   \]
   then $w(X \cup_Z Y) = w(X) + w(Y)$.

3. Multiplicativity with respect to signature of closed manifolds:
   \[
   \text{If } X \text{ is a Witt space, and } M \text{ is a closed manifold},
   \]
   \[
   \text{(*) } w(M \times X) = \text{sign}(M) \cdot w(X).
   \]

These properties are crucial in the application of Sullivan's $ko_* \otimes \mathbb{Z}[\frac{1}{2}]$ orientation argument to Witt bordism theory. See Chapter VI and the Appendix.

In section 2, we give a proof of Witt cobordism invariance of the Witt class. The proof is modeled on the classical proof of cobordism invariance of the signature for
manifolds [12].

In section 3, we give a geometric proof of additivity. By means of the "pinch cobordism," we deduce additivity easily from cobordism invariance. Incidentally this result implies the Novikov additivity theorem for the signature of manifolds. [cf. 5, p. 588]

Section 4 contains a Kunneth formula for intersection homology of a product $M \times X$, where $M$ is a closed manifold and $X$ a closed Witt space. Specifically, we prove that chain level cross-product induces an isomorphism:

$$H_*(M; \mathbb{Q}) \otimes H_*(X; \mathbb{Q}) \xrightarrow{\times} H_*(M \times X; \mathbb{Q}),$$

for all perversities $\bar{p}$. From this, we deduce the product formula (*), which generalizes a similar formula stated in [10] for the case where $X$ is the quotient space of a prime order diffeomorphism on a smooth manifold.

**Section 2: Cobordism invariance.**

**Theorem 2.1.:** Let $(Y, \partial Y)$ be a $(4k+1)$-dimensional Witt space with boundary. Then $w(\partial Y) = 0$.

**Proof:** The argument resembles that in [11], which in turn follows [12].
Let \( \hat{Y} = Y \cup \text{cone}(\partial Y) \) be the space obtained from \( Y \) by adjoining the cone on the boundary. We will construct a commutative diagram of rational vector spaces (assume rational coefficients):

\[
\begin{array}{cccc}
H^{2k+1}_{2k+1}(\hat{Y}) & \overset{i}{\longrightarrow} & H_{2k}^m(\partial Y) & \longrightarrow \ H_{2k}^m(\hat{Y}) \\
\downarrow \alpha & & \downarrow \beta & \downarrow \gamma \\
H^n_{2k+1}(\hat{Y}) & \longrightarrow & H^n_{2k}(\partial Y) & \longrightarrow \ H^n_{2k}(\hat{Y})
\end{array}
\]

Here \( \bar{m} + (c) = \bar{m}(c) \) for \( 2 \leq c \leq 4k \), \( \bar{m} + (4k+1) = \bar{n}(4k+1) = 2k \), and \( \bar{n}-(c) = \bar{n}(c) \) for \( 2 \leq c \leq 4k \), \( \bar{n}-(4k+1) = \bar{m}(4k+1) = 2k - 1 \).

In the diagram, the row sequences are exact and, in fact, dual to each other (over \( \mathbb{Q} \)). The vertical map \( \alpha \) is surjective, and \( \beta \) is the isomorphism induced by inclusion, as in III.3.7. A standard argument in linear algebra [12] implies that \( i(H^{m+}_{2k}(\hat{Y})) \) is self-annihilating in \( H_{2k}^m(\partial Y) \), under the pairing of III.3.3, and

\[
\dim_{\mathbb{Q}} i(H^{m+}_{2k+1}(\hat{Y})) = \frac{1}{2} \dim_{\mathbb{Q}} H_{2k}^m(\partial Y). \]

It follows that \( w(\partial Y) = 0 \).

Here are the details. The top exact row fits into the long exact sequences on homology corresponding to the inclusion of chain complexes:

1. \( \overline{C}^m_*(\hat{Y}) \subseteq \overline{C}^{m+}_*(\hat{Y}) \).

Similarly, the bottom row is a segment of the long
exact sequence gotten from:

\[
(2) \quad C^*_{\ast} (\hat{Y}) \rightarrow C^*_{\ast} (\hat{Y})
\]

Allowability is defined with respect to the stratifications on \( \partial Y \) and \( \hat{Y} \) induced by that of \( Y \).

Let \( \{v_i\}_{i \in I} \) be the set of point singularities in the stratification of \( Y \), and let \( \{L_i\}_{i \in I} \) be the corresponding set of 4k-dimensional links (here \( I \) is a finite index set).

By reasoning just as in [11], we find that the quotient complex arising from (1) has only two non-zero terms:

\[
C^m_{2k+1}(\hat{Y})/C^m_{2k+1}(\hat{Y}) \cong \bigoplus_{i \in I} Z^m_{2k}(L_i) \oplus Z^m_{2k}(\partial Y)
\]

and

\[
C^m_{2k+2}(\hat{Y})/C^m_{2k+2}(\hat{Y}) \cong \bigoplus_{i \in I} B^m_{2k}(L_i) \oplus B^m_{2k}(\partial Y).
\]

The differential on the quotient complex identifies with the inclusion homomorphism. Therefore

\[
H_{2k+1}(C^*_{\ast} (\hat{Y})/C^*_{\ast} (\hat{Y})) \cong \bigoplus_{i \in I} H^m_{2k}(L_i) \oplus H^m_{2k}(\partial Y).
\]

The Witt condition on \( Y \) ensures that

\[
H^m_{2k}(L_i) = 0, \text{ all } i \in I,
\]

so

\[
H_{2k+1}(C^*_{\ast} (\hat{Y})/C^*_{\ast} (\hat{Y})) \cong H^m_{2k}(\partial Y).
\]
This proves the existence of the top row of the diagram. An entirely analogous argument yields the bottom row.

The duality of the row sequences follows from the intersection pairing theorem (I.4.4). Note that $(\bar{m}^+)+(\bar{n}^-)=\bar{t}$. It remains to show that $\alpha$ is surjective. For this, refer to Remark III.3.6, noting that the Witt condition on links is satisfied for all odd codimension strata except $v$, the cone point.

Section 3: Additivity.

The cobordism invariance of the Witt class $w(X)$ suggests a simple geometric proof of the additivity of $w(X)$. "Additivity" refers to the property described in the following proposition.

Proposition 3.1.: Let $Y$ be an oriented $4k$-dimensional Witt space, and $Z, X_1, X_2$ subspaces of $Y$ such that:

1. $Y = X_1 \cup X_2$
2. $X_1 \cap X_2 = Z$, and $Z$ is bicollared in $Y$
3. $(X_1, Z)$ and $(X_2, -Z)$ are oriented $4k$-dimensional Witt spaces with boundary, so $[Y] = [X_1] + [X_2]$.

Then, $w(Y) = w(X_1) + w(X_2)$. 
Proof: Let $K_1$, $K_2$ be triangulations of $X_1$, $X_2$ respectively. Denote by $K_1^+$, $K_2^+$ the complexes with isomorphic simplicial collars added on the outside [20]. The induced triangulation of $Z$ (on the outside edge of the collars) will be denoted $L$. Then $|K| = |K_1^+ U_L K_2^+|$ is p.l. homeomorphic to $Y$, as in [20, p. 24]. The "pinched space" is the polyhedron underlying the complex $J$ defined by:

$$J = [K_1 U \text{cone}(L_1)] U_v [K_2 U \text{cone}(L_2)]$$

where $L_1$, $L_2$ are the triangulations of $Z$ in $X_1$, $X_2$, and $v$ is the common cone vertex.

Now, define a continuous map: $p: |K| \rightarrow |J|$ by collapsing $|L|$ to $v$ in $|J|$. Orient $|J|$ so that the induced map $p_*$ on homology carries the orientation of $|K|$ to that of $|J|$. By constructing a triangulation explicitly, we see that the mapping cylinder $C_p$ is triangulable. See Figure 2. Attach a collar $|J| \times I$ to $C_p$, forming the space $P$. It is not difficult to see that $P$ is a pseudomanifold with collared boundary. It can be oriented so that $\partial P = |K| - |J|$. We call $P$ the pinch cobordism.

To check that $P$ is a Witt space, it is enough to check that $H^m_{2k}(\partial k(v,P); \mathbb{Q}) = 0$. But $\partial k(v,P)$ is p.l. homeomorphic to $Z \times [-1,1] U \text{cone}(\partial(Z \times [-1,1]))$. This is
homeomorphic to the suspension of $\mathbb{Z}$, with suspension points identified. Therefore, $S^0 \# \mathbb{Z}$ has the same normalization as $\mathcal{Lk}(v,P)$, implying

$$H^m_{2k}(\mathcal{Lk}(v,P); \mathbb{Q}) \cong H^m_{2k}(S^0 \# \mathbb{Z}; \mathbb{Q}) \cong 0,$$

with the latter isomorphism from III.2.5.

By cobordism invariance of the Witt class, we have

$$w(Y) = w(|J|) = w(\hat{X}_1) + w(\hat{X}_2)$$

$$= w(X_1) + w(X_2) \quad \text{(by definition III.4.1)}$$

Section 4: Multiplicativity with respect to signature of manifolds.

Suppose $M$ is a p.l. manifold, and $X$ is a Witt space, both without boundary. Then $M \times X$ is a Witt space without boundary. In this section, we prove the product formula

$$w(M \times X) = \text{sign}(M) \cdot w(X)$$

where $\text{sign}(M) \cdot w(X)$ is $\text{sign}(M)$ times the element $w(X)$ in the abelian group $W(\mathbb{Q})$. 
The proof proceeds as follows. For closed manifold $M$ and arbitrary pseudomanifold $X$, we construct an isomorphism

$$i^*: H_*(M; \mathbb{Q}) \otimes H_\bar{p}^*(X; \mathbb{Q}) \to H_\bar{p}^*(M \times X; \mathbb{Q}),$$

for every perversity $\bar{p}$. This is done in Theorem 3.1.

Assume $X$ is a Witt space, $\dim(M \times X) = 4k$, and specialize to the case $\bar{p} = \bar{m}$. We have an isomorphism

$$\bigoplus_{i+j=2k} H_i(M; \mathbb{Q}) \otimes H_j^m(X; \mathbb{Q}) \to H_{2k}^m(M \times X; \mathbb{Q})$$

which can be regarded as an isomorphism of rational inner product spaces. The inner product on the right hand side is that given by Theorem III.3.8, while that on the left side is uniquely determined by:

$$\langle a \otimes b, c \otimes d \rangle = \begin{cases} (-1)^{\dim b \dim c} \langle a, c \rangle_M \langle b, d \rangle_X & \text{if } \dim a + \dim c = \dim M \quad \dim b + \dim d = \dim X \\ 0 & \text{otherwise} \end{cases}$$

where $\langle , \rangle_M$ and $\langle , \rangle_X$ are the intersection pairings on $H_*(M; \mathbb{Q})$ and $H_*^m(X; \mathbb{Q})$. It follows at once that $w(M \times X) = 0$ if $\dim M \equiv 1 \pmod{2}$ or $\dim M \equiv 2 \pmod{4}$ and
w(M\times X) = \text{sign}(M) \cdot w(X) \text{ if } \dim M \equiv 0 \pmod{4}.

Now we formulate and prove Theorem 3.1. First, define a chain homomorphism

\[ i^P : C_\ast(M) \otimes C_\ast^P(X) \rightarrow C_\ast^P(M \times X) \]

as follows. Given \( c_i \otimes d_j \in C_\ast(M) \otimes C_\ast^P(X) \), let \( \tilde{c}_i \) (resp. \( \tilde{d}_j \)) be the homology class in \( H_\ast(|c_i|,|\partial c_i|) \) (resp. \( H_j(|d_j|,|\partial d_j|) \)) corresponding to \( c_i \) (resp. \( d_j \)).

Denote by \( c_i \times d_j \) the chain in \( C_{i+j}(|c_i| \times |d_j|) \subset C_{i+j}(M \times X) \) corresponding to the ordinary exterior cross product \( \tilde{c}_i \times \tilde{d}_j \) in \( H_{i+j}(|c_i| \times |d_j|, |\partial c_i| \times |d_j| \cup |c_i| \times |\partial d_j|) \). With respect to the product stratification on \( M \times X \), \( c_i \times d_j \) lies in \( C_{i+j}^P(M \times X) \). Set \( \tilde{i}^P(c_i \otimes d_j) = c_i \times \tilde{d}_j \). Clearly \( \tilde{i}^P \) is a chain map. Let \( \tilde{i}_\ast^P \) be the homomorphism induced on rational homology:

\[ \tilde{i}_\ast^P : H_\ast(C_\ast(M; \mathbb{Q}) \otimes C_\ast^P(X; \mathbb{Q})) \rightarrow H_\ast^P(M \times X; \mathbb{Q}). \]

**Theorem 3.1:** \( \tilde{i}_\ast^P \) is an isomorphism.

**Proof:** The proof breaks down into two main lemmas (injectivity and surjectivity) and two technical lemmas.

We assume rational coefficients throughout.

**Lemma 3.2:** \( \tilde{i}_\ast^P \) is injective.
By the classical Kunneth theorem, there is a canonical isomorphism

\[ H_*(C_*(M;\mathbb{Q}) \otimes C_*(X;\mathbb{Q})) \cong H_*(M;\mathbb{Q}) \otimes H_*(X;\mathbb{Q}). \]

Let \( \bar{q} \) be the perversity such that \( \bar{p} + \bar{q} = \bar{t} \), and consider the homomorphism:

\[ i_q^* : H_*(M) \otimes H_*^q(X) \to H_*^q(M \times X). \]

The dual homomorphism is:

\[ (i_q^*)^* : (H_*^q(M \times X))^* \to (H_*(M) \otimes H_*^q(X))^* \]

There are canonical isomorphisms

(2) \( (H_*^{\bar{q}}(M \times X))^* \cong H_*^\bar{p}(M \times X) \)

(3) \( (H_*(M) \otimes H_*^{\bar{q}}(X))^* \cong (H_*(M))^* \otimes (H_*^{\bar{q}}(X))^* \)

\[ \cong H_*(M) \otimes H_*^\bar{p}(X) \]

by III.3.8.

So, we regard \( (i_q^*)^* \) as a homomorphism

\[ (i_q^*)^* : H^\bar{p}(M \times X) \to H_*(M) \otimes H_*^\bar{p}(X). \]
Injectivity of \( \tilde{i}^* \) follows from the following claim.

**Claim:** The composition \((i^q)^* \circ i^p\) is the identity homomorphism.

To verify the claim, observe that

\[
\langle i^p \langle \sum \alpha_i \otimes \beta_j \rangle, i^q \langle \sum \gamma_k \otimes \delta_l \rangle \rangle_{M \times X} = \langle \sum \alpha_i \otimes \beta_j, \sum \gamma_k \otimes \delta_l \rangle_\Theta,
\]

where \( \langle \cdot, \cdot \rangle_\Theta \) is the pairing of \( H_*(M) \otimes H^\tilde{p}(X) \) with \( H_*(M) \otimes H^q(X) \), and that, by III.3.8, intersection numbers determine the isomorphisms (2) and (3) above.

**Lemma 3.3.** \( \tilde{i}^* \) is surjective.

The cases of dimension 0 and \( m+x \) are obvious. Assume \( \alpha \in H_l(M \times X) \), with \( 0 < l < m+x \). Given a representative cycle \( z \) for \( \alpha \), we produce a cycle \( z' \) homologous to \( z \) and such that \( z' = \sum c_i \times d_j \). In the course of the construction, we use some technical lemmas whose proofs are deferred to the end of the section (3.4 and 3.5).

Let \( S \) be a triangulation of \( M \), with \( i \)-dimensional skeleton \( S_i \), and \( (m-j) \)-dimensional coskeleton \( S^{(m-j)} \).

Let \( T \) be a triangulation of \( X \) with \( i \)-dimensional skeleton \( T_i \), and basic sets \( \{ Q^p_i \} \) and \( \{ Q^q_i \} \) for perversities \( \tilde{p} \) and \( \tilde{q} \). (We will refer to a chain as allowable with respect to perversity \( \tilde{p} \) if its support is so.)
The cycle \( z \) is \((\bar{p},\lambda)\)-allowable with respect to the (product) intrinsic stratification on \( M \times X \) since any other stratification refines it. Assume initially that \( \lambda \leq m \). We treat the case \( \lambda > m \) later.

The first step is to apply stratified general position [16] to shift \( z \) so as to be disjoint from

\[
S^{(m-\lambda+1)} \times \{Q_x \cap T_x^{\lambda-2}\}.
\]

Denote the shifted chain by \( z_1 \). By lemma 3.4, there is a strong deformation retraction:

\[
r : M \times X - S^{(m-\lambda+1)} \times \{Q_x \cap T_x^{\lambda-2}\} \rightarrow M \times Q_1^p \cup S_{\lambda-2} \times X
\]

which preserves products of simplices in \( S^1 \) and \( T^1 \) and, therefore, intrinsic skeleta of \( M \times X \). Moreover, it also follows from 3.4 that \( |z_1| \) can be triangulated so that the map of vertices \( v \mapsto r(v) \) has a linear extension to simplices of \( |z_1| \), yielding a cycle \( z_2 = r_*(z_1) \) with support in \( M \times Q_1^p \cup S_{\lambda-2} \times X \). Also, the definition of \( r \), and the fact that \((M \times Q_1^p) \cap (S_{\lambda-2} \times X)\) has dimension \( \lambda-1 \), imply that \( z_2 \) is \((\bar{p},\lambda)\)-allowable.

An argument entirely analogous to that in the proof of III.2.3 proves that \( z_2 \) is homologous to \( z_1 \).

Decompose \( z_2 \) into the sum of two chains:
where $|u| \subset M \times \bar{Q}_{1}^{p}$ and $|v| \subset S_{x-2} \times X$. Note that $|\partial u| = |\partial v| \subset S_{x-2} \times \bar{Q}_{1}^{p}$.

Since $M \times \bar{Q}_{1}^{p}$ lies in the nonsingular part of $M \times X$, and $\bar{Q}_{1}^{p}$ is a regular cell complex with 0-skeleton $\bar{Q}_{0}^{p}$, we can find a chain $w$ homologous to $u$ with $\partial w = \partial u$ and $w = y_{0} + y_{1}$, where $|y_{0}| \subset S_{x} \times \bar{Q}_{0}^{p}$, $|\partial y_{0}| \subset S_{x-1} \times \bar{Q}_{0}^{p}$ and $|y_{1}| \subset S_{x-1} \times \bar{Q}_{1}^{p}$, $|\partial y_{1}| \subset S_{x-1} \times \bar{Q}_{0}^{p} \cup S_{x-2} \times \bar{Q}_{1}^{p}$. By lemma 3.5, $y_{0} = i^{P}( \sum_{i=1}^{J} c^{i}_{x} \times d^{i}_{0} )$ and $y_{1} = i^{P}( \sum_{i=1}^{K} c^{i}_{x-1} \theta d^{i}_{1} )$

for chains $c^{i}_{x} \in C_{x}(M)$, $d^{i}_{0} \in C_{0}^{p}(X)$ and $c^{i}_{x-1} \in C_{x-1}(M)$, $d^{i}_{1} \in C_{1}(X)$.

So we now have a cycle $z_{3} = w + v$, with $w \in \text{Image}(i^{P})$.

Since $|v|$ is $(\bar{p}, \ell)$-allowable and $|\partial v|$ is disjoint from $s^{(m-\ell+2)} \times [Q_{x-1}^{q} \cap T_{x-2}^{q}]$, using stratified general position we shift $v$ so as to be disjoint from this subspace, keeping $\partial v$ fixed. The shifts can be chosen to keep the chain in $S_{x-2} \times X$, since they take place in $\bigcup \text{Int}(\sigma_{x-2}) \times X$ which is the interior of a stratified pseudo manifold of dimension $x + \ell - 2$ and
involve closed subsets \(|v|\) of dimension \(l\), and

\[
(S_{l-2} \times X) \cap (S^{(m-l+2)} \times [Q_{x-1} \cap T_{x-2}]) = \bigcup_{q_{l-2} \in S_{l-2}} [q_{l-2} \cap T_{x-2}]
\]

of dimension \(x-3\). Denote the shifted chain by \(v_1\).

Now, use the deformation retraction, given by 3.4,

\[
r : S_{l-2} \times X \to S^{(m-l+2)} \times [Q_{x-1} \cap T_{x-2}]
\]

\[
S_{l-2} \times X \cup S_{l-2} \times Q_2^p
\]

to get a \((p,\ell)\)-allowable chain \(r_*(v_1)\) homologous to \(v\) with support in \(S_{l-3} \times X \cup S_{l-2} \times Q_2^p\). Decompose \(r_*(v_1)\) as \(y_2 + v'\), where \(y_2\) is the subchain with support in \(S_{l-2} \times Q_2^p\). Then \(y_2 = i^p(L \prod_{i=1}^l c_{l-2}^i \times d_2)\) by 3.5, where \(c_{l-2}^i \in C_{l-2}^i(M)\) and \(d_2^i \in C_{l-2}^i(X)\). Now, proceed in the same fashion, shifting \(v'\) to \(v_1'\) and deforming \(v_1'\) to \(r_*(v_1')\).

The chain \(r_*(v_1')\) will have a decomposition

\[
r_*(v_1') = y_3 + v'',\]

where \(y_3 \in \text{Image}(i^p)\), and so on. Continue until the process stops. We will then have constructed a cycle \(y\), homologous to \(z\), with

\[
y = \sum_{j+k=l} i^p(L \prod_{i} c_{j}^i \otimes d_{k}^i)
\]

where \(c_{j}^i \in C_{j}^i(M)\) and \(d_{k}^i \in C_{k}^i(X)\).
This completes the proof for the case $k < m$.

If $k > m$, the procedure is very similar, with changes in indices being the primary modifications. Specifically, first shift $z$ so as to be disjoint from

$$S^{(0)} \times [Q_{x-(k-m)+1} \cap T_{x-2}],$$

yielding a chain $z$.

Use the deformation retraction:

$$r: M \times X - S^{(0)} \times [Q_{x-(k-m)+1} \cap T_{x-2}] \rightarrow M \times Q_{k-m}^p \cup S_{m-1} \times X$$

to get a chain $r_*(z_1)$ homologous to $z$, with support in $M \times Q_{k-m}^p \cup S_{m-1} \times X$, and which is $(p,k)$-allowable.

Decompose $r_*(z_1)$ into a sum of two chains

$$r_*(z_1) = y_1 + v,$$

where $|y_1| \subset M \times Q_{k-m}^p$ and $|v| \subset S_{m-1} \times X$. By 3.5, $y_1$ lies in $\text{Image}(i_p)$.

Now, repeat the shift, deform, decompose procedure on $v$, using the deformation retraction:

$$r: S_{m-1} \times X - S^{(1)} \times [Q_{x-(k-m)} \cap T_{x-2}] \rightarrow S_{m-2} \times X \cup S_{m-1} \times Q_{k-m+1}^p.$$
And so on, as in the case where \( l \leq m \).

The first technical lemma provides the deformation retractions used in the proof of 3.3. We thank Javier Bracho for his assistance on this argument.

**Lemma 3.4.** Given integers \( i, j \) with \( 1 \leq i \leq x \) and \( 0 \leq j \leq m \), there is an embedding:

\[
M \times X \hookrightarrow S^{(j)} \times [Q_i^d \cap T_{x-2}] \ast [S_{m-j-1} \times X \cup M \times Q_{x-i+1}^P]
\]

which is the identity on

\[
S^{(j)} \times [Q_i^d \cap T_{x-2}]
\]

and

\[
S_{m-j-1} \times X \cup M \times Q_{x-i+1}^P
\]

and which takes \( M \times X \) to a union of join lines.

**Proof:** It suffices to define compatible embeddings on products of simplices in \( S' \) and \( T' \):

\[
\Delta^P \times \Delta^q \hookrightarrow (\Delta^a \times \Delta^c) \ast (\Delta^b \times \Delta^q \cup \Delta^P \times \Delta^d)
\]

where \( \Delta^P \in S' \), \( \Delta^a = \Delta^P \cap S^{(j)} \), \( \Delta^b = \Delta^P \cap S_{m-j-1} \).
and \( \Delta^q \subset T' \), \( \Delta^p = \Delta^q \cap [Q_i \cap T_{x-2}] \), \( \Delta^d = \Delta^q \cap Q_{x-i+1} \).

Let \( \{v^a_i\}, \{v^b_i\}, \{v^c_i\}, \{v^d_i\} \) be vertices of the simplices in these decompositions: \( \Delta^p = \Delta^a * \Delta^b \)
\( \Delta^q = \Delta^c * \Delta^d \).

Any points \( x \in \Delta^P \) and \( y \in \Delta^Q \) have unique expressions in terms of barycentric coordinates:

\[
x = \sum a_i v^a_i + \sum b_i v^b_i, \quad \text{with} \quad \sum a_i + \sum b_i = 1, \quad a_i, b_i \geq 0
\]

and \( y = \sum c_i v^c_i + \sum d_i v^d_i, \quad \text{with} \quad \sum c_i + \sum d_i = 1, \quad c_i, d_i \geq 0. \)

Define \( t : \Delta^p \rightarrow I \) by \( t(x) = \sum b_i \)
and \( s : \Delta^q \rightarrow I \) by \( s(y) = \sum d_i \).

Both are p.l. maps.

Consider the closed convex subspaces of \( \Delta^p \times \Delta^q \):

\[
X = \{(x,y) \mid t(x) \leq s(y)\}
\]
\[
Y = \{(x,y) \mid t(x) \geq s(y)\}
\]
\[
Z = \{(x,y) \mid t(x) = s(y)\}.
\]

See Figure 3.

We define the embedding separately on these subspaces. It is easy to check compatibility on intersections.

Define \( h_z : Z \rightarrow \Delta^a \times \Delta^c \star \Delta^b \times \Delta^d \subset \Delta^c \star \Delta^a \times (\Delta^b \times \Delta^d) \cup (\Delta^a \times \Delta^d) \)

by
Define \( h_X : \Delta^a \times \Delta^c \star \Delta^p \times \Delta^d \rightarrow \Delta^a \times \Delta^c \star \Delta^p \times \Delta^d \) by

\[
(x, y) \mapsto \begin{cases} 
\left( \frac{1}{1-t}, \frac{1}{1-s} \right) (1-t) + \left( \frac{1}{1-t}, \frac{1}{1-s} \right) (1-s) \\
\left( \frac{1}{1-t}, \frac{1}{1-s} \right) (1-t) + \left( \frac{1}{1-t}, \frac{1}{1-s} \right) (1-s) \\
(x, y) & \text{if } t = t(x) \neq 0,1 \\
(x, y) & \text{if } t = t(x) = 0,1.
\end{cases}
\]

Define \( h_Y : \Delta^a \times \Delta^c \star \Delta^p \times \Delta^d \rightarrow \Delta^a \times \Delta^c \star \Delta^p \times \Delta^d \) similarly:

Continuity is checked using barycentric coordinates, and the definitions agree on overlaps. The image is easily seen to be a union of join lines. Since \( \Delta^p \times \Delta^q \) is compact Hausdorff and the map is injective, we have an embedding as
Thus, the strong deformation retractions along join lines in the join induce the strong deformation retractions in $\Delta^p \times \Delta^q$ which we used in 3.3. Specifically, we obtain the deformation retraction

$$r : \Delta^p \times \Delta^q \to \Delta^b \times \Delta^q \cup \Delta^p \times \Delta^d,$$

which preserves the convex sets $X$, $Y$, and $Z$, and carries them to the convex sets $\Delta^p \times \Delta^d$, $\Delta^b \times \Delta^q$, and $\Delta^b \times \Delta^d$ respectively.

The next lemma was used to conclude that the chains $y_i$ in a product space $S_{-i} \times Q_{-i}$ are in the image of $i_{-i}$.

**Lemma 3.5.** Let $(X,X_0)$ and $(Y,Y_0)$ be finite complexes with $\dim X = x$, $\dim X_0 = x-1$ and $\dim Y = y$, $\dim Y_0 = y-1$. Given a chain $c \in Z_{x+y}(X \times Y, X \times Y_0 \cup X_0 \times Y)$ there are chains $a_j \in Z_x(X,X_0)$, $b_j \in Z_y(Y,Y_0)$, for $j = 1, \ldots, J$, with $J$ finite, such that

$$c = \sum_{j=1}^{J} a_j \times b_j.$$ 

**Proof:** By the Kunneth formula [21, p. 235] for top-dimensional chains we have the isomorphism:
\[ H_x(X, X_0) \otimes H_y(Y, Y_0) = Z_x(X, X_0) \otimes Z_y(Y, Y_0) \]
\[ \cong Z_{x+y}(X \times Y, X \times Y_0 \cup X_0 \times Y) \]
\[ = H_{x+y}(X \times Y, X \times Y_0 \cup X_0 \times Y). \]

Since \( c \in Z_{x+y}(X \times Y, X \times Y_0 \cup X_0 \times Y) \), we can find an element of the tensor product \( \sum a^j \otimes b^j \) which maps to \( c \) under the isomorphism. Therefore, \( c = \sum a^j \times b^j \).

This completes the proof of Theorem 3.1.
Chapter V: Rational surgery on Witt spaces.

Section 1: Description of program.

In this chapter, we develop a technique for performing rational surgery on Witt spaces. We use it to prove that the Witt class $w(X)$ determines the cobordism class of $X$. In particular, if $X$ is a closed Witt space of dimension $4k$, and $w(X) = 0$, we find a "surgery basis" $\{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$ for $\tilde{H}_{2k}^m(X; \mathbb{Q})$ with respect to which the inner product has matrix $\text{diag}[(1, 0), \ldots, (1, 0)]$. By collapsing a regular neighborhood of an irreducible representative for $\alpha_1$ (see Section 2), we obtain a Witt space $X_1$ with

(1) $w(X_1) = 0$

and (2) $\text{rank } \tilde{H}_{2k}^m(X_1; \mathbb{Q}) = \text{rank } \tilde{H}_{2k}^m(X; \mathbb{Q}) - 2$.

Roughly speaking, the homology classes $\{\alpha_1, \beta_1\}$ are killed. Moreover, the mapping cylinder of the collapse (with a collar added) gives a Witt cobordism between $X$ and $X_1$. We say that $X_1$ is obtained from $X$ by an elementary surgery, and call the cobordism the trace of the surgery.

Iterating this procedure $n$ times, we construct a Witt space $X_n$ with $\tilde{H}_{2k}^m(X_n; \mathbb{Q}) = 0$ and a Witt cobordism $Y$ between $X$ and $X_n$. Coning off the copy of $X_n$ in $\partial Y$ produces a Witt cobordism of $X$ to zero.
A similar procedure works when the dimension of $X$ is $4k+2$. In that case, we find a symplectic basis
$$\{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$$
with respect to which the skew-symmetric form on $H_{2k+1}^m(X; \mathbb{Q})$ has matrix
$$\text{diag}([0, 1, \ldots, 0, 1]).$$
Then proceed as above.

The chapter is organized as follows. In section 2 we define and prove existence of irreducible representative cycles. In section 3, we study the regular neighborhood of an irreducible cycle and show that the "mapping cylinder" of a collapse as described above is a Witt cobordism.

Finally, in section 4, we construct a surgery exact diagram very much in the classical vein [17] to compute the effect of surgery on intersection homology, verifying (1) and (2) above.

Section 2: Irreducible p.l. geometric cycles.

Definition 2.1.: A representative cycle $z$ for $\alpha \in H^j_{\cdot}(X; G)$, $G = \mathbb{Z}$ or $\mathbb{Q}$, is irreducible if

1. $H_j(|z|; \mathbb{Z}) = \mathbb{Z}$
2. the generator of $H_j(|z|; \mathbb{Z})$ has coefficient $\pm 1$ on every $j$-simplex of $|z|$, in some triangulation of $|z|$.

Remark 2.2.: Condition (2) implies that the generator of
\[ H_j(|z|; \mathbb{Z}) \] has coefficient \( \pm 1 \) on every \( j \)-simplex of \( |z| \), in any triangulation of \( |z| \).

Let \( X^{2k} \) be a p.l. pseudomanifold of dimension \( 2k \) with irreducible fundamental class \([X]\). By abuse of language, we say \( X \) is irreducible. This section is devoted to a proof of the following lemma.

**Lemma 2.3.** Let \( X^{2k} \) be an irreducible p.l. pseudomanifold. Given \( \alpha \in H^\mathbb{P}_j(X; \mathbb{Q}) \), there exists a representative cycle \( z \) which is irreducible, if \( 0 \leq j < 2k-1 \).

**Proof:** The proof proceeds in 2 steps. Step 1 constructs, by a general position argument, a representative cycle \( v \) such that \( |v| \) supports a simplicial cycle with coefficient \( \pm 1 \) on every \( j \)-simplex. Step 2 uses the technique of piping [20, p. 67] to produce a cycle \( z \) which, additionally, satisfies the condition \( H_j(|z|, \mathbb{Z}) = \mathbb{Z} \). If \( J \) is a \( j \)-cycle, \( \Sigma_J \) denotes the complement of the intrinsic open \( j \)-stratum.

**Step 1:** In some triangulation \( T \) of \( X \), choose a representative simplicial cycle for \( \alpha \). By reorienting \( j \)-simplices if necessary, we may assume all non-zero coefficients of \( j \)-simplices are positive. Clear denominators so as to obtain an integral cycle \( y = \sum_{\sigma \in T_\sigma} n(\sigma) \sigma \) such that
all distinct non-zero coefficients share no common factor.

If \( n(\sigma) = 1 \) whenever \( n(\sigma) \neq 0 \), then set \( v = y \) and Step 1 will be completed.

If not, let \( E_1, \ldots, E_n \) be the connected components of the intrinsic open \( j \)-stratum of \( Y = |y| \). To each component we can associate a unique coefficient \( n_i \), the coefficient of simplices in that component. Let \( \tilde{W} \) be the disjoint union of \( n_i \) copies of \( \tilde{E}_i \), \( i = 1, \ldots, n \), and \( f: \tilde{W} \to Y \) the obvious p.l. map. Let \( W \) be the quotient space of \( \tilde{W} \) obtained by identifying \( x \) to \( x' \) if \( x, x' \in f^{-1}(\Sigma_y) \) and \( f(x) = f(x') \). Then \( \cdot W \) supports a simplicial \( j \)-cycle \( w \) having coefficient \( +1 \) on all \( j \)-simplices, and such that \( f_*(w) = y \), where \( f_* \) denotes the induced map on chains.

Note that \( f(W - \Sigma_W) \subseteq X - \Sigma_X \). Perform a finite sequence of shifts of the map \( f \), bringing \( f \) into general position with respect to \( \Sigma_W \) [1, p. 418], [20, p. 51]. We move \( f(\tilde{\tau}) \) only if \( \text{Int}(\tau) \subseteq W - \Sigma_W \); that is, only if \( f(\text{Int}(\tau)) \subseteq X - \Sigma_X \), the nonsingular part of \( X \). Let \( g: W \to X \) be the resulting map. The linear trace of the shifts yields a p.l. homotopy \( H: \tilde{W} \times [0,1] \to X \) with \( H_0 = f \) and \( H_1 = g \). Then \( g_*(w) \) is a p.l. geometric cycle which, when triangulated, has coefficient \( +1 \) on each \( j \)-simplex. Its support is also \((p,j)\)-allowable.

Give \( W \times [0,1] \) the orientation such that \( \Theta(w \times [0,1]) = w \times 1 - w \times 0 \). Then \( H_*(W \times [0,1]) = g_*(w) - f_*(w) = g_*(w) - y \). Since \( H(x,t) = H(x,0) = f(x) \)
when \( x \in \Sigma_{W} \), for all \( t \), and \( H(x,t) \subseteq X - \Sigma_{X} \) when \( x \in W - \Sigma_{W'} \), for all \( t \), it follows that \( |H_{\ast}(w \times [0,1])| \) is \((p,j+1)\)-allowable. We conclude that \( g_{\ast}(w) \) represents the same homology class as \( y \) in \( H^{p}_{j}(X;\mathbb{Q}) \). Let \( v \) be the multiple of \( g_{\ast}(w) \) such that \( [v] = \alpha \in H^{p}_{j}(X;\mathbb{Q}) \). Then \( v \) is the desired representative, and Step 1 is complete.

**Step 2:** Let \( \{F_{i}\}_{i=1}^{k} \) be the connected components of the intrinsic open \( j \)-stratum of \( |v| \). We now construct an orientation respecting pipe \( P_{i} \) from \( F_{i} \) to \( F_{i+1} \), for \( 1 \leq i \leq k-1 \), with distinct pipes disjoint. [20, p. 67]. Note that since \( |v| \) is a subcomplex of some triangulation of \( X \), the pair \((X-\Sigma_{X}, |v| \cap (X-\Sigma_{X}))\) is locally unknotted at all points in the interiors of \( j \)-dimensional simplices of \( |v| \). Therefore the piping construction can always be carried out.

Let \( Z \) be the p.l. subspace of \( X \) we obtain after the piping is done. Then \( Z \) supports a simplicial cycle which has coefficient 1 on every simplex in \( Z \). Since \( Z - \Sigma_{Z} \) is connected, this cycle generates \( H_{j}(Z;\mathbb{Z}) \). Clearly, its support is \((p,j)\)-allowable and, by "filling in the pipes," we see that it represents the same homology class as \( g_{\ast}(w) \) in \( H^{p}_{j}(X;\mathbb{Q}) \). Let \( z \) be the multiple of this cycle such that \( [z] = [v] \in H^{p}_{j}(X;\mathbb{Q}) \). Then \( z \) satisfies the conditions (1) and (2) of definition 2.1.
Remark 2.4.: If $X^2$ is an irreducible p.l. pseudo-manifold of dimension 2, it is easy to see that any class $\alpha \in \tilde{H}_1^p(X;\mathbb{Q})$ is represented by a cycle whose support is a p.l. embedded $S^1$. 
Section 3: Regular neighborhoods.

Let $X^{2k}$ be an irreducible pseudomanifold. Let $z$ be an irreducible representative for $a \in \tilde{H}_k^m(X;\mathbb{Q})$, with $U$ a (closed) regular neighborhood of $|z|$ in $X$. Then $U$ is a pseudomanifold with collared boundary $\partial U$, and stratification induced from that of $X$ [1, p. 437]. Let $\hat{U} = U \cup \text{cone}(\partial U)$ be $U$ with the cone on $\partial U$ adjoined. The stratification of $U$ induces a stratification on $\hat{U}$: corresponding to a stratum $X$ in $U$ is the stratum $X \cup \text{cone}(X \cap \partial U) - \{v_0\}$, where $v_0$ is the cone point. Since $|z|$ is $(\bar{m},k)$-allowable, the cone point $v_0$ constitutes the entire 0-stratum of the stratification of $\hat{U}$. The main result of this section is:

**Proposition 3.1.** If $X$ is a Witt space and $\langle \alpha, \delta \rangle = 0$, then:

$$\tilde{H}_k^m(\hat{U};\mathbb{Q}) = 0.$$ 

To prove this, we prove a series of preliminary lemmas.

Let $i : C_\ast^{\bar{m}}(\hat{U};\mathbb{Z}) \rightarrow C_\ast^m(\hat{U};\mathbb{Z})$ be the inclusion of chain complexes, where $\bar{m}$ is the perversity: $\bar{m} - (c) = \bar{m}(c)$ for $2 \leq c \leq 2k-1$ and $\bar{m} - (2k) = \bar{m}(2k) - 1 = k - 2$. An argument identical to that given in the proof of IV.2.1 proves that the quotient complex $C_\ast^m(\hat{U};\mathbb{Z})/C_\ast^{\bar{m}}(\hat{U};\mathbb{Z})$ has only two non-trivial terms, and there are canonical iso-
morphisms of these terms:

\[ \frac{\tilde{C}_{k+2}}{\tilde{C}_{k+2}} \cong \frac{\tilde{B}_k}{\partial} \]

\[ \frac{\tilde{C}_{k+1}}{\tilde{C}_{k+1}} = \tilde{Z}_k(\partial). \]

The boundary homomorphism on the left is identified with inclusion on the right. We thereby obtain:

**Lemma 3.2.** There is an exact sequence of integral intersection homology groups (integer coefficients assumed):

\[ H_{k+1}(U) \rightarrow H_k(\partial U) \rightarrow H_k^{-}(\hat{U}) \rightarrow H_k^+(\hat{U}) \rightarrow 0 \]

We made no use of the irreducibility of \( z \) in the proof of 3.2, but it is crucial in the next lemma.

**Lemma 3.3.** Let \( u \) be a generator of \( H_k(|z|; \mathbb{Z}) = \mathbb{Z}^{|z|} \). Then \( \tilde{H}_k^{-}(\hat{U}; \mathbb{Z}) = \mathbb{Z} \) and the homology class of \( u \) is a generator.

**Remark 3.4.** (1) Since \( |u| \) is \((\tilde{m}, k)\)-allowable, it is also \((\tilde{m}, k)\)-allowable.

(2) Since \([z] \in H_k^m(X; \mathbb{Q})\) is non-trivial, \([u] \in H_k^+ (\hat{U}; \mathbb{Z})\) is non-trivial. If \( u = \partial w \), for some
w ∈ \( \tilde{C}_{k+1}(\hat{U}; \mathbb{Z}) \), then \(|w| \cap v_0 = \phi\), so by pseudoradial projection, we may assume \(|w| \subset U\). Since \(w ∈ \tilde{C}_{k+1}(\hat{U}; \mathbb{Z})\) also, this would imply \([u]\) is trivial in \(H^\pi_k(X; \mathbb{Z})\) a contradiction. Therefore \([u] ∈ \tilde{H}^\pi_k(\hat{U}; \mathbb{Z})\) is non-trivial.

**Proof of 3.3.** Let \(N\) denote a collar of \(\partial J\) in \(U\).
There is a p.l. homeomorphism of \(N \cup \text{cone}(\partial U)\) with \(\text{cone}(\partial U) = v_0 \ast \partial U\). Let \(U_0 = \text{Cl}(U-N)\). Suppose \(ξ ∈ \tilde{H}^\pi_k(\hat{U}; \mathbb{Z})\), with representative cycle \(y ∈ \tilde{C}_k(\hat{U}; \mathbb{Z})\). Then \(|y| \cap v_0 = \phi\). By pseudoradial projection, we can assume \(|y| \subset U_0\).

Give \(U\) the structure of a derived neighborhood \([20, \text{p. 33}]\). Then every \(σ ∈ U\) satisfies:

\[|σ| = |σ| \cap |z| \ast |σ| \cap \partial U\]

where \(A \ast \phi = \phi \ast A = A\), by the usual convention.

Triangulate the cycle \(y\) so that each simplex lies in a unique simplex with interior in \(U\). If a vertex \(v ∈ |y|\) lies in \(\text{Int}(σ)\), then \(|σ| \cap |z| \neq \phi\), and we let \(τ\) denote the simplex in \(|z|\) with \(|τ| = |σ| \cap |z|\).

Define \(f(v) = \hat{τ}\). Extend this map on vertices of \(|y|\) linearly over simplices, obtaining a p.l. map \(f : |y| \to |z| \subset \hat{U}\). Then \(f\) determines a geometric cycle \(f_*(y)\) in \(C_k(\hat{U}; \mathbb{Z})\). Since the support of \(f_*(y)\) lies in
\[ |z|, \ f_\ast(y) \in C_k^{m-}(\hat{U};\mathbb{Z}). \] Proceed as in III.2.3. to define a p.l. map \( H : |y| \times I \to \hat{U} \) and geometric chain \( H_\ast(y \times I) \) satisfying \( \Delta H_\ast(y \times I) = f_\ast(y) - y. \) The definition of \( f \) implies that \( H_\ast(y \times I) \in C_k^{m-}(\hat{U};\mathbb{Z}). \) Therefore \( f_\ast(y) \) represents \( \xi. \) But, \( f_\ast(y) \) is a \( k \)-cycle supported in \( |z|, \) and \( H_k(|z|;\mathbb{Z}) = \mathbb{Z} \) by hypothesis. Therefore \( f_\ast(y) = j \cdot u \) for some \( j \in \mathbb{Z}, \) and so \( \xi = j \cdot [u]. \)

By the intersection pairing theorem (I.4.4), there is a nondegenerate pairing induced by intersection product:

\[
\langle \cdot , \cdot \rangle : H_k^{m-}(\hat{U};\mathbb{Q}) \times H_k^{n+}(\hat{U};\mathbb{Q}) \to \mathbb{Q}.
\]

Here, \( \bar{n}+ \) is the perversity satisfying \( (\bar{m}-) + (\bar{n}+) = \bar{t}. \)

That is, \( \bar{n}+(c) = \bar{n}(c) \) for \( 2 \leq c \leq 2k-1 \) and

\[
\bar{n}+(2k) = \bar{n}(2k)+1 = k
\]

Choose \( \beta \in H_k^{\bar{n}+}(\hat{U};\mathbb{Q}) \) a generator such that

\[
\langle a, \beta \rangle = 1.
\]

**Lemma 3.5.** The following diagram commutes:
All maps are induced by inclusion of chain complexes.

The map $g$ is given by:

$$g(\xi) = \langle \xi, \alpha \rangle \cdot \beta.$$ 

**Proof:** Commutativity is clear. Since $\tilde{C}^m_k(\hat{U}; \mathbb{Q}) = \tilde{C}^m_k(U; \mathbb{Q})$, the map $j$ is injective. Similarly, $\tilde{C}^m_k(U; \mathbb{Q}) = \tilde{C}^m_k(U; \mathbb{Q})$ implies that $h$ is surjective. Finally, the description of $g$ is immediate from the definition of $\beta$.

We may now prove Proposition 3.1.

**Proof of 3.1.** Refer to the diagram of 3.5. Under the hypothesis of the proposition, $g$ is the zero map. (Recall that $\alpha$ generates $\tilde{H}^m_k(U; \mathbb{Q})$). Injectivity of $j$ and commutativity imply that $i \circ h$ is the zero map also. $\hat{U}$ inherits a Witt space structure from $X$, so III.3.7 implies that $i$ is an isomorphism, so $h$ must in fact be the zero map. The proposition now follows from surjectivity of $h$. 

\[ \]
One corollary will be useful when we analyze the surgery exact diagram in Section 4. Refer to 3.2 for notation.

**Corollary 3.6.** Assume rational coefficients. Under the condition of 3.1,

\[ H^m_k(\partial U) \cong \{\gamma\} \oplus \text{Image}(q_*) \text{, where } \gamma \in H^m_k(\partial U) \text{ with } \partial_*(\gamma) = \alpha. \]

**Section 4:** The surgery exact diagram.

Let \( X \) be an irreducible Witt space of dimension \( 2k \). Assume there is a decomposition of \( H^m_k(X; \mathbb{Q}) \):

\[ H^m_k(X; \mathbb{Q}) = V_0 \oplus V_1 \]

where \( V_0 = (\gamma_1, \ldots, \gamma_\ell) \), \( V_1 = (\alpha_1, \beta_1) \) and with respect to this basis the intersection pairing is given by:

\[ \langle \alpha_1, \gamma_i \rangle = \langle \beta_1, \gamma_i \rangle = 0, \quad i = 1, \ldots, \ell \]

\[ \langle \alpha_1, \alpha_1 \rangle = \langle \beta_1, \beta_1 \rangle = 0 \]

\[ \langle \beta_1, \alpha_1 \rangle = \pm 1. \]
Let $U$ be a regular neighborhood of an irreducible representative cycle $z$ for $\alpha_1$. We may assume that $\partial U$ is bicollared in $X$ [1, p. 437]. Form the space

$$X_1 = (X - \text{Int}(U)) \cup \text{cone}(\partial U).$$

The passage from $X$ to $X_1$ is called an **elementary surgery**. Note $X_1$ is homeomorphic to $\tilde{X}$, the space obtained from $X$ by collapsing $U$ to a point.

The following sequence of propositions, whose proofs will be deferred briefly, partially describe the effect of the elementary surgery.

**Proposition 4.1.**: $X$ is an irreducible Witt space.

**Proposition 4.2.**: Let $Y$ be the mapping cylinder of the collapse $X \to \tilde{X}$, with a collar $\tilde{X} \times I$ attached. Then $Y$ can be given the structure of a Witt cobordism between $X$ and $X_1$. (We call $Y$ the trace of the elementary surgery on $X$).

**Proposition 4.3.**: There is an isomorphism

$$H_k^m(X_1; \mathbb{Q}) \cong V_0$$

which preserves the intersection form.
Repeated application of these propositions yields the result alluded to in the introduction.

**Theorem 4.4.** Let $X$ be an irreducible Witt space of dimension $2k$, $k \geq 1$. If $w(X) = 0$, then $X$ is Witt cobordant to zero.

**Proof:** Suppose $\dim X \equiv 0 \pmod{4}$. Then $w(X) = 0$ implies that the inner product space $\tilde{H}_k(X;\mathbb{Q})$ is split, by II.2.9. Choose a basis $\{a_1, \beta_1, a_2, \beta_2, \ldots, a_n, \beta_n\}$ with respect to which the matrix of the form is $\text{diag}[(0 1), \ldots, (0 1)]$.

Let $V_0 = (a_2, \beta_2, \ldots, a_n, \beta_n)$ and $V_1 = (a_1, \beta_1)$, and apply 4.1 - 4.3. We obtain $X_1$, Witt cobordant to $X$ with $\tilde{H}_k(X_1;\mathbb{Q}) = (a_2', \beta_2', \ldots, a_n', \beta_n')$, isomorphic to $V_0$ as rational inner product space. Now decompose $\tilde{H}_k(X_1;\mathbb{Q})$, setting $V'_0 = (a_3', \beta_3', \ldots, a_n', \beta_n')$ and $V'_1 = (a_2', \beta_2')$, and repeat the procedure. Continue until we obtain $X_n$, cobordant to $X$, with $\tilde{H}_k(X_n;\mathbb{Q}) = 0$. The cobordism to $X$ is obtained by pasting the traces of the elementary surgeries along their boundaries. Now attach $\text{cone}(X_n)$ to the cobordism.

If $\dim X \equiv 2 \pmod{4}$, then the intersection form on $\tilde{H}_k(X;\mathbb{Q})$ is skew-symmetric. Choose a symplectic basis $\{a_1, \beta_1, \ldots, a_n, \beta_n\}$ with respect to which the matrix of the form is $\text{diag}[(0 1), \ldots, (0 1)]$. The same reasoning as above shows that a sequence of $n$ surgeries will yield a
cobordism of $X$ to zero.

We now supply the proofs of 4.1 - 4.3.

**Proof of 4.1.** The fact that $X$ is a Witt space and $\partial U$ is bicollared in $X$ immediately implies that $X_1$ is a Witt space. If $k > 1$, general position proves that the nonsingular part of $X - \text{Int}(U)$ is path connected. Therefore $X_1$ is irreducible.

If $k = 1$, then $X$ has at most point singularities. Therefore $z$ is an embedded $S^1$ and $U$ is homeomorphic to $S^1 \times [-1,1]$. The assumption that $[z] \in H^1_1(X; \mathbb{Q})$ is nontrivial implies $X - \text{Int}(U)$ is connected, so $X_1$ is irreducible.

**Proof of 4.2.** The trace of the elementary surgery is homeomorphic to the following polyhedron, defined analogously to the trace of a spherical modification [17]. To $(X - \text{Int}(U)) \times [0,1]$, attach $U$ along $\partial U \times 0$ and cone $(\partial U)$ along $\partial U \times 1$. There is an embedding of $\hat{U}$ in the resulting space as $\partial U \times I$ with $U$ and cone$(\partial U)$ attached as above. Now attach cone$(\hat{U}) = v^* \hat{U}$ along $\hat{U}$ via this embedding, yielding the trace of the surgery $Y$. $Y$ is clearly a pseudomanifold with collared boundary, and it can be oriented so that $\partial Y = X_1 - X$. To prove that $Y$ is a Witt space, it suffices to check that $\text{lk}(\sigma, Y)$ satisfies
\[ H^k_k(\mu k(v,Y); \mathbb{Q}) = 0. \]

But \( \mu k(v,Y) \approx \hat{U} \), so Proposition 3.1 completes the proof.

**Proof of 4.3.:** The proof is broken into a number of steps. Steps 1 - 4 provide the ingredients for the surgery exact diagram, which constitutes step 5. Step 6 completes the proof using the diagram.

**Step 1:** Transverse chains.

Let \( U \) denote (by abuse of notation) the fundamental cycle of \( X \) restricted to the regular neighborhood \( U \), and \( \partial U \) its boundary chain. Let \( \text{TC}^m_i(X) \) be the subcomplex of \( C^*_i(X) \) consisting of chains \( c \) for which \( c \) and \( \partial c \) are dimensionally transverse to \( \partial U \). To each element \( c \in \text{TC}^m_i(X) \), we can associate the intersection chain \( c \cap U \), and, by the Lefschetz boundary formula:

\[ \partial (c \cap U) = \partial c \cap U + (-1)^{2k-i} c \cap \partial U. \]

This defines a homomorphism

\[ t : \text{TC}^m_i(X) \rightarrow C^m_i(\hat{U}) \quad \text{for each } i \]

by \( t(c) = c \cap U. \)
(We use the inclusion $U \subset \hat{U}$).

Define $\overline{C}_i(U, \partial U)$ to be the image of $t$ in $\overline{C}_i(\hat{U})$. Give the graded group $\overline{C}_*(U, \partial U)$ the structure of a chain complex, defining

$$\delta : \overline{C}_i(U, \partial U) \to \overline{C}_{i-1}(U, \partial U)$$

by $\delta(t(c)) = \partial c \cap U = t(\partial c)$.

Note that $\delta$ is well-defined: if $t(c) = 0$, then

$$\partial(t(c)) = \partial(c \cap U) = \partial c \cap U + (-1)^{2k-i} c \cap \partial U = 0.$$ 

Dimensional transversality implies that

$$\dim(|\partial c \cap U| \cap |c \cap \partial U|) \leq i-2$$

so

$$\partial c \cap U = 0$$

and

$$c \cap \partial U = 0.$$ 

In particular, $\delta(t(c)) = \partial c \cap U = 0$.

It is easy to check that $\delta \circ \delta = 0$. We conclude that $t$ is a chain homomorphism. Let $R^m_*$ denote the kernel of $t$, with the induced chain complex structure. There is a short exact sequence of complexes:
and an associated long exact sequence in homology:

\[ H_k(\tilde{m}(U, \partial U)) \rightarrow H_k(\tilde{m}) \rightarrow H_k(TC_\ast(X)) \rightarrow H_k(C_\ast(U, \partial U)) \rightarrow \]

In steps 2 - 4 we analyze the homology groups occurring in this sequence.

**Step 2:** \( H_\ast(TC_\ast(X)) \cong H_\ast(X) \).

The inclusion of chain complexes

\[ i : \tilde{m}(X) \rightarrow C_\ast(X) \]

induces an isomorphism on homology. In other words,

\[ H_\ast(TC_\ast(X)) \cong H_\ast(X) \].

**Proof of surjectivity:** Apply the stratified general position lemma (I.3.3) (or its corollary I.3.4) to a cycle \( z \in C_i(X) \).

**Proof of injectivity:** Apply the relative part of I.3.4 to a chain \( w \in C_{i+1}(X) \), where \( \partial w = z \) and \( z \in TC_i(X) \).

**Step 3:** The complex \( \tilde{m}^+ (U) \).

Let \( \tilde{m}^+ \) denote the sequence with
\( m^+(c) = \bar{m}(c) \quad \text{for} \quad 2 \leq c \leq 2k-1 \)

\( m^+(2k) = \bar{m}(2k)+1 = k \)

Although \( \bar{m}^+ \) is not a perversity, we may define \((\bar{m}^+, j)\)-allowability of the support of a geometric chain with respect to a stratification. Consider the complex \( C_*^{\bar{m}^+}(\hat{U}) \) consisting of chains \( c \) of dimension \( j \) such that \( |c| \) is \((\bar{m}^+, j)\)-allowable and \( |\partial c| \) is \((\bar{m}^+, j-1)\)-allowable with respect to the stratification on \( \hat{U} \). Recall that the cone point is the 0-stratum.

The inclusion \( C_*^{\bar{m}}(\hat{U}) \hookrightarrow C_*^{\bar{m}^+}(\hat{U}) \) induces a short exact sequence of complexes. We can analyze the homology of the quotient as in Lemma 3.1 and obtain an isomorphism:

\[
H^m_{k+1}(\hat{U}) \cong H^{\bar{m}^+}_{k+1}(\hat{U})
\]

and an exact sequence

\[
0 \rightarrow H^m_k(\hat{U}) \rightarrow H^m_k(\hat{U}) \rightarrow H^m_{k-1}(\partial U) \rightarrow H^m_{k-1}(\hat{U}) \rightarrow \ldots
\]

We get another long exact sequence from the short exact sequence induced by the inclusion \( C_*^{\bar{n}^+}(\hat{U}) \rightarrow C_*^{\bar{n}^+}(\hat{U}) \), where \( \bar{n}^+ \) is the perversity:
\[ n^+(c) = n(c) \quad \text{for} \quad 2 \leq c \leq 2k - 1 \]

\[ n^+(2k) = n(2k + 1) = k. \]

There is a commutative ladder:

\[
\begin{array}{cccccc}
0 & \rightarrow & H^m_k(\hat{U}) & \rightarrow & H^{m+}_k(\hat{U}) & \rightarrow & H^m_{k-1}(\partial U) & \rightarrow & H^m_{k-1}(\hat{U}) & \rightarrow & \\
& \downarrow & & \downarrow & \phi & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^n_k(\hat{U}) & \rightarrow & H^n_k(\hat{U}) & \rightarrow & H^n_{k-1}(\partial U) & \rightarrow & H^n_{k-1}(\hat{U}) & \rightarrow & 
\end{array}
\]

All vertical maps are induced by chain complex inclusions. Since \( X \) is Witt, \( \hat{U} \) and \( \partial U \) are also Witt. Proposition III.3.7 and the 5-lemma imply that \( \phi \circ \partial \) is an isomorphism.

We now identify \( H^m_j(U, \partial U) \) with \( H^m_j(\hat{U}) \) in dimensions \( j \geq k \). Let \( N \) be a simplicial collar of \( \partial U \) in \( U \). There is a p.l. homeomorphism \( \text{cone}(\partial U) \cup N \cong \text{cone}(\partial U) \) preserving cone lines in the cone on the left. In \( \hat{U} \), we can triangulate any chain so that its intersection with \( \text{cone}(\partial U) \cup N \) will be subordinate to the simplicial structure on \( \text{cone}(\partial U) \cup N \).

Let \( p_* \) denote the chain map induced by pseudoradial projection along cone lines from the cone vertex \( v \) to the boundary of \( \text{Cl}(U-N) \), \( p_* : C_*(\hat{U}) \rightarrow C_*(\hat{U}) \). Note that \( p_*(v) = v \), and \( p_* \circ p_* = p_* \).

Define homomorphisms:
The discussion of $t$ in Step 1 shows that $\psi_j$ is well-defined.

For $j \geq k + 1$,

$$\psi_{j-1}(\delta t(c)) = \psi_{j-1}(t(\partial c)) = p_*(t(\partial c)) + (-1)^j v^*(c \cap \partial U) = p_*(\partial(t(c)) + (-1)^{j+1} v^*(c \cap \partial U)) = \partial p_*(t(c)) + (-1)^{j+1} v^*(c \cap \partial U) = \partial \psi_j(t(c)).$$

Therefore $\psi_j$ induces a homomorphism

$$(\psi_j)_* : \overline{H}_j^m(U, \partial U) \rightarrow \overline{H}_j^{m+}(\hat{U})$$

on homology for $j \geq k$.

In fact, $(\psi_j)_*$ is an isomorphism for $j \geq k$.

Proof of surjectivity: Let $z \in \overline{Z}_j^{m+}(\hat{U})$. The cycle $p_*(z)$
represents the same class as $z$ in $\tilde{H}_j^m(\hat{U})$, by an argument just like that used in previous pseudoradial projection arguments. The chain $p_*(z) \cap U$ lies in $C_j^m(U, \partial U)$ and satisfies: $\psi_j(p_*(z) \cap U) = p_*(z)$.

Proof of injectivity: Let $z \in Z_j^m(U, \partial U)$ and suppose $\psi_j(z) = \omega w$, where $w \in C_{j+1}^m(\hat{U})$. Then $p_*(w) \cap U$ lies in $C_{j+1}^m(U, \partial U)$ and

$$\delta(p_*(w) \cap U) = \partial p_*(w) \cap U = p_*(\partial w) \cap U = p_*(\psi_j(z) \cap U) = \psi_j(z) \cap U, \text{ since } p* \circ p* = p_*.$$ 

Since $\psi_j(z) = p_*(z + (-1)^{j+1} v_*(z \cap \partial U))$, the chains $\psi_j(z)$ and $z + (-1)^{j+1} v_*(z \cap \partial U)$ are homologous in $C_j^m(\hat{U})$, via a chain we denote by $y$. By the relative stratified general position lemma, we can assume $y \perp \partial U$. Then $y \cap U \in C_{j+1}^m(U, \partial U)$ and $\delta(y \cap U) = (\psi_j(z) \cap U) - z$. Therefore $\psi_j(z) \cap U$ represents the same class as $z$ in $H_j^m(U, \partial U)$. We conclude that $(\psi_j)_*$ is injective.

Step 4: The complex $D_j^m(\hat{X} - U)$.

Let $\hat{X} - U = (X - \text{Int}(U)) \cup \text{cone}(\partial U)$. Let $D_j^m(\hat{X} - U)$ be the subgroup of $C_j^m(\hat{X} - U)$ consisting of those chains such that:
\[ \dim(|c_j| \cap v) \leq j - 2k + (k-2) \]
\[ \dim(|\partial c_j| \cap v) \leq (j-1) - 2k + (k-2), \]

where \( v \) is the cone vertex of \( X^\wedge U \). Allowability of chain supports is with respect to the stratification on \( X^\wedge U \) induced by that on \( X-\text{Int}(U) \). Thus the cone point is contained in, but may not be all of, the 0-stratum.

Define homomorphisms for all \( j \):

\[ \phi_j : K_j \rightarrow D^m_j(X^\wedge U) \]

(by means of the inclusion \( X - \text{Int}(U) \subset X^\wedge U \)) by:
\[ \phi_j(c) = c \cap (X-\text{Int}(U)), \]
where by abuse of notation, \( X-\text{Int}(U) \) represents the restriction of the fundamental cycle of \( X \) to \( X-\text{Int}(U) \). Then \( \phi_j \) is injective, because

1. \( c = c \cap (X-\text{Int}(U)) + c \cap U \) for \( c \in \text{TC}_j^m(X) \)

and

2. \( t(c) = c \cap U = 0 \) for \( c \in K_j \).

Clearly the \( \phi_j \) are chain maps and we regard them as an inclusion of chain complexes.

We will show that the induced maps on homology \((\phi_j)_* \) are isomorphisms for \( j \leq k \).

**Proof of surjectivity:** Let \( N \) be a collar of \( \partial U \) in
X - \text{Int}(U)$, and $p_\ast$ be the chain map on $C_\ast(X^\wedge U)$ induced by pseudoradial projection along cone lines to the inner boundary of $N$. If $z \in D^\overline{m}_j(X^\wedge U)$ for $j \leq k$, then $|z| \wedge v = \phi$. If $z$ is a cycle, $p_\ast(z)$ represents the same homology class in $H_j(D^\overline{m}_\ast(X^\wedge U))$. But $p_\ast(z) \in K_j$, since its support lies in $X-U$.

**Proof of injectivity:** Let $a \in H_j(K_\ast)$. If the cycle $z$ represents $a$, then $p_\ast(z)$ represents $a$ also. Assume $p_\ast(z) = \partial w$, where $w \in D^\overline{m}_{j+1}(X^\wedge U)$. If $j \leq k$, then $|w| \wedge v = \phi$, and $p_\ast(w) \in K_{j+1}$. Moreover,

$$\partial p_\ast(w) = p_\ast(\partial w) = p_\ast p_\ast(z) = p_\ast(z)$$

This proves the claim about $(\phi_j)_\ast$, $j \leq k$.

Finally, analysis of the quotient complex $C^\overline{m}_\ast(X^\wedge U)/D^\overline{m}_\ast(X^\wedge U)$ yields an exact sequence:

$$H_k(\partial U) \to H_k(D^\overline{m}_\ast(X^\wedge U)) \to H_k^m(X^\wedge U) \to 0.$$

**Step 5:** The surgery exact diagram.

Recall that $X$ is a Witt space of dimension $2k$. There is a commutative diagram of rational homology groups, with exact row and column (assume rational coefficients):
The homomorphism $q_*$ appears in Lemma 3.2. The isomorphism on the left was proved in Step 3.

The horizontal sequence is a segment of the exact sequence in Step 1. We have used the isomorphisms proved in Steps 2, 3, and 4 to relabel some of the groups.

The vertical exact sequence is a segment of the long exact homology sequence obtained from the inclusion of complexes $D_*^m(X-U) \xrightarrow{\partial_*} C_*^m(X-U)$, as discussed in Step 4.

The commutativity of the upper left triangle follows from the chain-level definitions of the connecting homomorphism of the horizontal and vertical sequences and the map $q_*$.

The homomorphism $g$ is defined to make the lower right triangle commute. Here $\phi$ is the rational isomorphism of Step 3, and $t_*$ is the homomorphism induced by the chain map $t$ of Step 1. In fact, $g$ is given by: $g(\gamma) = \langle \gamma, \alpha_1 \rangle \cdot \beta$, ...
where $\beta$ generates $\tilde{H}^{n+}_k(\hat{U})$ as in 3.5, and $\langle \cdot , \cdot \rangle$ is the inner product on $\tilde{H}^m_k(X)$.

**Step 6: Conclusion.**

Assume rational coefficients. Recall the basis $\{\alpha_1, \beta_1, \gamma_1, \ldots, \gamma_\ell\}$ of $\tilde{H}^m_k(X)$ and the decomposition $H^m_k(X) \cong V_0 \oplus V_1$. By remarks in Step 5,

\[ t_*(\gamma_i) = 0, \quad i = 1, \ldots, \ell \]

and

\[ t_*(\alpha_1) = 0. \]

Similarly,

\[ \phi(t_*(\beta_1)) = \beta. \]

By exactness,

\[ H_k(D^-(X^\sim U)) = V'_0 \oplus (\alpha'_1) \oplus \text{Image}(\partial_*) \]

as a rational vector space, and $V'_0 \oplus (\alpha'_1)$ is isomorphic to $V_0 \oplus (\alpha_1)$ by an isomorphism compatible with intersection product. Now, corollary 3.6 and commutativity of the upper left triangle imply:

\[ \text{Image}(j) = (\alpha'_1) \oplus \text{Image}(\partial_*) \]
Finally, since the vertical sequence is exact, we conclude that:

\[ \check{H}^m_k(X^\wedge U) \cong V'_0 \cong V_0. \]

The isomorphism is compatible with intersection product.

This completes the proof of 4.3.
Chapter VI: Witt space bordism theory.

Section 1: Introduction.

In this chapter, we show that Witt spaces provide a geometric cycle theory for connected KO homology at odd primes. Specifically, in section 2, the bordism theory based on the class of Witt spaces, $\Omega^{\text{Witt}}_*$, is introduced and using results from Chapter V and a simple construction given in section 3 of this chapter, it follows that the coefficient groups are:

\[
\begin{align*}
\Omega_0^{\text{Witt}}(\text{pt}) &\cong \mathbb{Z} \\
\Omega_q^{\text{Witt}}(\text{pt}) &\cong 0 \quad \text{if } q > 0 \text{ and } q \not\equiv 0 \pmod{4} \\
\Omega_q^{\text{Witt}}(\text{pt}) &\cong \mathbb{W}(\mathbb{Q}) \quad \text{if } q > 0 \text{ and } q \equiv 0 \pmod{4}
\end{align*}
\]

The isomorphisms are induced by the Witt class.

Finally, in section 4, the technique of D. Sullivan [24] is used to construct canonical orientations in $\text{ko}_* \otimes \mathbb{Z}[\frac{1}{2}]$ for Witt spaces, and the induced natural transformation of theories:

\[\mu^{\text{Witt}} : \Omega_*^{\text{Witt}} \rightarrow \text{ko}_* \otimes \mathbb{Z}[\frac{1}{2}],\]

is shown to be an equivalence at odd primes.
Section 2: The bordism theory $\Omega^*_\text{Witt}$.

Let $\mathcal{Z}$ be the class of oriented pseudomanifolds $L$ satisfying:

1. $L$ is a Witt space
2. $H^m(L;\mathbb{Q}) = 0$ if $\dim L = 2\ell$.

It is not difficult to check that $\mathcal{Z}$ is a class of (oriented)singularities (use III.2.3). The associated bordism sequence $\{\mathcal{Z}^n\}$ is the class of all oriented Witt spaces. Let $\Omega^*_\text{Witt}$ denote the bordism theory based on the class of singularities $\mathcal{Z}$.

**Proposition 2.1.** The coefficient groups $\Omega^*_\text{Witt}(pt)$ are:

\[
\begin{align*}
\Omega^0\text{Witt}(pt) &= \mathbb{Z} \\
\Omega_q\text{Witt}(pt) &= 0, \text{ if } q \nmid 0 \mod 4 \\
\Omega_q\text{Witt}(pt) &\cong W(q), \text{ if } q > 0 \text{ and } q \equiv 0 \mod 4.
\end{align*}
\]

The isomorphisms in dimension $q > 0$ are induced by the Witt class.

**Proof:** The case $q = 0$ is evident. When $q \equiv 1 \mod 2$, observe that any odd dimensional Witt space bounds the cone on itself.
The case $q \equiv 2 \pmod{4}$ follows from cobordism invariance (IV.2.1) and surgery (V.4.4). In applying V.4.4, we can use the fact that any Witt space $X^{2k}$ which is not irreducible is Witt cobordant to an irreducible Witt space $X'$. Simply take connected sum of components of the $2k$-stratum of $X$, say $E_1, \ldots, E_m$, in order, so that the $2k$-stratum of $X'$ is $E_1 \# E_2 \# \ldots \# E_n$.

Finally, in the case $q \equiv 0 \pmod{4}$, IV.2.1 and V.4.4 show that the homomorphism

$$w : \Omega^{Witt}_{4k}(pt) \rightarrow W(Q)$$

induced by the Witt class is injective. In section 2, we construct explicit generators for $\Omega^{Witt}_{4k}(pt)$ and prove that $w$ is also surjective.

Section 3: Generators for $\Omega^{Witt}_{*}(pt)$.

We use the method of plumbing disk bundles over spheres, due to Milnor [7] to construct representative Witt spaces for each bordism class in $\Omega^{Witt}_q(pt)$, when $q > 0$ and $q \equiv 0 \pmod{4}$. The representatives can be stratified in such a way that the singularity subset $\Sigma$ is a single point. In particular, they have stratifications with only even dimensional strata. This yields information on $\Omega^e_*$.
Theorem 3.1. [7]: Let $B$ be an $n \times n$ matrix with integer entries, symmetric, and with even entries on the diagonal. Then, for $k \geq 1$, there is an manifold with boundary $(M, \partial M)$ of dimension $4k$ such that:

1. $M$ is $(2k-1)$-connected, $\partial M$ is $(2k-2)$-connected, and $H_{2k}(M)$ is free abelian;
2. The matrix of intersections $H_{2k}(M) \otimes H_{2k}(M) \rightarrow \mathbb{Z}$ is given by $B$.

Given an element $[V, \beta] \in W(\mathbb{Q})$, it is possible to find a basis $\{v_i\}$ for $V$ with respect to which the matrix of the inner product, $B$, is an integral symmetric matrix with even entries on the diagonal. Theorem 2.1 produces a manifold with boundary $(M, \partial M)$ such that $H_{2k}(M; \mathbb{Q})$ represents the Witt class $[V, \beta]$ in $W(\mathbb{Q})$. In the long exact sequence of rational homology groups (rational coefficients assumed):

$$
\cdots \rightarrow H_{2k}(\partial M) \rightarrow H_{2k}(M) \rightarrow H_{2k}(M, \partial M) \rightarrow H_{2k-1}(\partial M) \rightarrow \cdots
$$

the transformation $i_*$ has matrix $B$ with respect to the bases $\{v_i\}$ and $\{v_i^*\}$, where $\{v_i^*\}$ is dual to $\{v_i\}$ under the intersection pairing of $H_{2k}(M)$ with $H_{2k}(M, \partial M)$.
\[ H_{2k}(M) \times H_{2k}(M, \partial M) \to \mathbb{Q}. \]

Therefore the Novikov group of \((M, \partial M)\) and the isomorphic group \(\hat{H}_{2k}^{\mathbb{M}}(\hat{M}; \mathbb{Q})\), where \(\hat{M} = M \cup \text{cone}(\partial M)\), represent the class of \((V, \beta)\). That is:

\[ w(\hat{M}) = [V, \beta] \in W(\mathbb{Q}). \]

This proves the result used in section 2:

**Proposition 2.2.** The homomorphisms

\[ w : \Omega_{4k}^{\text{Witt}}(\text{pt}) \to W(\mathbb{Q}), \quad k > 0, \]

which are induced by the Witt class, are surjective.

There is also the corollary:

**Corollary 3.3.** Every cobordism class in \(\Omega_{q}^{\text{Witt}}\), for \(q > 0\) and \(q \equiv 0 \pmod{4}\), has a representative \(\hat{M} = M \cup \text{cone}(\partial M)\), where \((M, \partial M)\) is a \(q\)-dimensional manifold with boundary, obtained by plumbing disk bundles over spheres.

In particular, since \(\hat{M}\) has a natural stratification with only even dimensional strata, we obtain the following concrete result about \(\Omega_{*}^{\text{ev}}\).
Corollary 3.4. The homomorphism of graded cobordism groups

\[ \Omega^\text{ev}_*(pt) \rightarrow \Omega^\text{Witt}_*(pt) \]

is surjective.

Remark 3.5.: Simple examples show that the homomorphism in 3.4 is not injective.

Section 4: Witt spaces: a geometric cycle theory for \( k_{0*} \oplus \mathbb{Z}[^{\frac{1}{2}}] \).

Suppose that \( \Omega_* \) is a bordism theory based on a class \( \mathcal{F} \) of p.l. spaces satisfying the following properties:

1. \( \mathcal{F} \) is closed under the operations of taking cartesian product with a p.l. manifold, and intersecting transversely with a closed p.l. manifold in Euclidean space.
2. \( \mathcal{F} \) has a signature invariant defined on the cycle level which is cobordism invariant:

There is a homomorphism

\[ \text{sign} : \Omega_*(pt) \rightarrow \mathbb{Z} \]

which extends the identification

\[ \Omega_*(pt) \rightarrow \mathbb{Z}. \]
(3) The signature can be extended to relative cycles (X, ∂X), so that it is additive:

If X, Y are relative cycles such that Z = ∂X, -Z = ∂Y, then \( \text{sign}(X \cup Z, Y) = \text{sign}(X, Z) + \text{sign}(Y, Z) \).

(4) The signature is multiplicative with respect to closed manifolds:

If X is a cycle in \( \mathcal{F} \), and M is a closed manifold, then:

\[
\text{sign}(M \times X) = \text{sign}(M) \cdot \text{sign}(X),
\]

where \( \text{sign}(M) \) denotes the classical signature of M.

Then, Dennis Sullivan [24] has defined a natural transformation of homology theories:

\[
\mu^\mathcal{F} : \Omega^\mathcal{F} \to ko_* \otimes \mathbb{Z}[\frac{1}{2}].
\]

Recall that \( ko_* \otimes \mathbb{Z}[\frac{1}{2}] \) is \( \mathbb{Z}/4\mathbb{Z} \)-periodic. The coefficient groups are:

\[
ko_q(pt) \otimes \mathbb{Z}[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}] \quad \text{if} \quad q \equiv 0 \pmod{4}
\]
\[
\cong 0 \quad \text{otherwise}.
\]

Sullivan shows that for \( [X] \in \Omega^\mathcal{F}(pt) \), where \( q \equiv 0 \pmod{4} \),
\[ \mu_{\text{pt}}([X]) = \text{sign}(X) \in \mathbb{Z}[\frac{1}{2}] \cong Ko_{q}(pt) \otimes \mathbb{Z}[\frac{1}{2}]. \]

In chapters III and IV, we demonstrated that properties (1) through (4) hold for the class of Witt spaces, so we receive a natural transformation:

\[ \mu_{\text{Witt}} : W_{\text{Witt}} \rightarrow ko_{\bullet} \otimes \mathbb{Z}[\frac{1}{2}]. \]

In Chapter V and Chapter VI, sections 1 and 3, we computed \( W_{\text{Witt}}(pt) \). Recall there are canonical isomorphisms:

\[ W_{\text{Witt}}(pt) \cong \mathbb{Z} \]
\[ W_{q}(pt) \cong W(\mathbb{Q}) \text{ if } q \equiv 0 \pmod{4} \text{ and } q > 0 \]

and \[ W_{q}(pt) \cong 0 \text{ otherwise.} \]

Recall from (II.2.8) the canonical isomorphism

\[ W(\mathbb{Q}) \xrightarrow{\cong} \mathbb{Z} \oplus T, \]

where \( T \) is a direct sum of 2-groups of order 2 and 4. The projection onto the \( \mathbb{Z} \) summand:

\[ W_{q}(pt) \xrightarrow{\cong} W(\mathbb{Q}) \rightarrow \mathbb{Z}, \]
when \( q > 0, \ q \equiv 0 \pmod{4} \), is the signature homomorphism. Therefore, the natural transformation

\[
\mu^\text{Witt} \otimes \mathbb{Z}[\frac{1}{2}] : \Omega^\text{Witt} \otimes \mathbb{Z}[\frac{1}{2}] \rightarrow KO_\ast \otimes \mathbb{Z}[\frac{1}{2}]
\]

induces an isomorphism on coefficient groups. We have proved:

**Theorem 4.1:** The natural transformation (5) is an equivalence of homology theories.

For details of Sullivan's construction, see the Appendix.
Appendix: Sullivan's construction of $\text{ko}_\ast \otimes \mathbb{Z}[\frac{1}{2}]$ orientations.

In chapter VI, we defined a natural transformation

$$\mu_{\text{Witt}} : \Omega_{\ast} \rightarrow \text{ko}_\ast \otimes \mathbb{Z}[\frac{1}{2}]$$

by assigning to every Witt space $(X^q, \partial X)$ a canonical orientation class in $\text{ko}_q(X, \partial X) \otimes \mathbb{Z}[\frac{1}{2}]$. What is needed to carry out this assignment is the alchemy of Dennis Sullivan: Given a geometric homology theory $\Omega_\ast$ based on a class of cycles $\mathcal{F}$, with signature invariant, and satisfying the properties listed in section VI.4, Sullivan translates the signature data into canonical $\text{ko}_\ast \otimes \mathbb{Z}[\frac{1}{2}]$ orientations for $\mathcal{F}$-cycles. Specifically, he constructs a natural transformation of homology theories: $\mu : \Omega_\ast \rightarrow \text{ko}_\ast \otimes \mathbb{Z}[\frac{1}{2}]$, such that the induced homomorphism on coefficient groups is the signature homomorphism.

The construction is found in [22] and [24]. Since [22] may be unavailable to the reader, and [24] will almost certainly be so, we sketch the arguments here.

The first ingredient is the following algebraic result, which Sullivan calls Pontrjagin-duality for KO-theory.

**Theorem:** [Sullivan] Let $X$ be a finite complex. The group $\text{KO}_i(X) \otimes \mathbb{Z}[\frac{1}{2}]$ is naturally isomorphic to the group
of commutative diagrams:

\[
\begin{array}{ccc}
KO_i(X; \mathbb{Q}) & \xrightarrow{\sigma} & \mathbb{Q} \\
\downarrow i_* & & \downarrow i \\
KO_i(X; \mathbb{Q}/\mathbb{Z}[\frac{1}{2}]) & \xrightarrow{\tau} & \mathbb{Q}/\mathbb{Z}[\frac{1}{2}],
\end{array}
\]

where \( i_* \) is the map on homology induced by the homomorphism \( i \) of coefficient groups, and \( \sigma \) and \( \tau \) are to be specified. (The group structure on the collection of diagrams is the obvious one).

**Sketch of proof:** Sullivan constructs an exact sequence:

\[
(2) \quad 0 \to KO^i(X) \otimes \mathbb{Z}[\frac{1}{2}] \to KO^i(X) \otimes KO^i(X; \mathbb{Q}) \to KO^i(X) \otimes \mathbb{Q} \to 0
\]

from the arithmetic square:

\[
\begin{array}{ccc}
\mathbb{Z}[\frac{1}{2}] & \to & \mathbb{Q} \\
\downarrow & & \downarrow \\
\mathbb{Z} [\frac{1}{2}] & \to & \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{Q}.
\end{array}
\]

Here \( \mathbb{Z}[\frac{1}{2}] \) = \( \lim_{n \text{ odd}} \mathbb{Z}[\frac{1}{2}]/n\mathbb{Z}[\frac{1}{2}] \), where the inverse limit is taken over the direct system of odd positive integers,
partially ordered by the relation:

\[ m \leq n \]

if \( m/n \).

The corresponding homomorphism:

\[ \mathbb{Z}[\frac{1}{2}]/n\mathbb{Z}[\frac{1}{2}] \rightarrow \mathbb{Z}[\frac{1}{2}]/m\mathbb{Z}[\frac{1}{2}] \]

is reduction modulo \( m \).

Similarly, \( KO^i(X) = \lim_{\text{n odd}} KO^i(X; \mathbb{Z}[\frac{1}{2}]/n\mathbb{Z}[\frac{1}{2}]) \).

The inclusion:

\[ \mathbb{Z}[\frac{1}{2}]/n\mathbb{Z}[\frac{1}{2}] \hookrightarrow \mathbb{Q}/\mathbb{Z}[\frac{1}{2}] \]

induces a natural map on homology:

\[ KO^i(X; \mathbb{Z}[\frac{1}{2}]/n\mathbb{Z}[\frac{1}{2}]) \rightarrow KO^i(X; \mathbb{Q}/\mathbb{Z}[\frac{1}{2}]) \]

By composition with the homomorphism \( \tau \) in the diagram, we get a collection of homomorphisms:

\[ \tau_n : KO^i(X; \mathbb{Z}[\frac{1}{2}]/n\mathbb{Z}[\frac{1}{2}]) \rightarrow \mathbb{Z}[\frac{1}{2}]/n\mathbb{Z}[\frac{1}{2}] \subseteq \mathbb{Q}/\mathbb{Z}[\frac{1}{2}] \]
Now, there is a canonical isomorphism of cohomology theories induced by evaluation (cap product) for \( n \) odd:

\[
KO^i(X; \mathbb{Z} [\frac{1}{n}] / n\mathbb{Z} [\frac{1}{n}]) \xrightarrow{\cong} \text{Hom}(KO_i(X; \mathbb{Z} [\frac{1}{n}] / n\mathbb{Z} [\frac{1}{n}]), \mathbb{Z} [\frac{1}{n}] / n\mathbb{Z} [\frac{1}{n}])
\]

This isomorphism is discussed in [22]. We can therefore regard each \( \tau_n \) as an element of \( KO^i(X; \mathbb{Z} [\frac{1}{n}] / n\mathbb{Z} [\frac{1}{n}] ) \) and thereby obtain an element \( \hat{\tau} = \lim \tau_n \) of \( KO^i(X) \).

Using the isomorphism obtained from the universal coefficient theorem for KO-theory, namely:

\[
KO^i(X; \mathbb{Q}) \xrightarrow{\cong} \text{Hom}(KO_i(X; \mathbb{Q}), \mathbb{Q}),
\]

we identify \( \sigma \) in the diagram with a unique element of \( KO^i(X; \mathbb{Q}) \). The commutativity of the diagram implies that \( \hat{\tau} \otimes \sigma \in KO^i(X) \otimes KO^i(X; \mathbb{Q}) \) is sent to \( 0 \) in \( KO^i(X) \otimes \mathbb{Q} \) in the short exact sequence (2). Therefore, we can associate a unique element \( v \in KO^i(X) \otimes \mathbb{Z} [\frac{1}{n}] \) with the diagram (1).

The next result, also Sullivan's, enables us to translate the above isomorphism into more geometric terms. It is a Conner-Floyd type theorem (cf. [9]) relating smooth cobordism to KO-theory at odd primes.
Theorem [Sullivan]: There is a canonical isomorphism of \( \mathbb{Z}/4\mathbb{Z} \) periodic theories:

\[
\Omega^\text{SO}_*(X,A) \otimes_{\Omega^*_\text{pt}} \mathbb{Z}[\frac{1}{2}] \to KO^*_*(X,A) \otimes \mathbb{Z}[\frac{1}{2}],
\]

where \((X,A)\) is a compact p.l. pair and where \(\mathbb{Z}[\frac{1}{2}]\) has the \(\Omega^*_\text{pt}\)-module structure induced by the signature.

Sketch of proof: Sullivan constructs an element \(\Lambda_{4k}\) of \(KO^0(\text{MSO}(4k)) \otimes \mathbb{Z}[\frac{1}{2}]\) with the property

\[
\phi^{-1}(e \cdot \text{ch}(\Lambda_{4k} \otimes 1)) = L^{-1} \epsilon H^*(B\text{SO}(4k); \mathbb{Q}),
\]

where \(L\) is the universal L-class. Moreover, if \(f: \text{MSO}(4q) \wedge \text{MSO}(4r) \to \text{MSO}(4(q+r))\) is the natural map, then

\[
f^*\Lambda_{4(q+r)} = \Lambda_{4q} \times \Lambda_{4r},
\]

"\(\times\)" denotes the external product in KO-cohomology.

Proceeding as in [9], one uses \(\Lambda\) to define a natural transformation of cohomology theories

\[
\Omega^*_\text{SO}(X,A) \xrightarrow{\Lambda^*} KO^*_*(X,A) \otimes \mathbb{Z}[\frac{1}{2}],
\]
or, by Alexander duality, a natural transformation of homology theories:

$$\Omega^*_\mathrm{SO}(X,A) \xrightarrow{\Delta^*} \mathrm{KO}^*_\mathrm{SO}(X,A) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$$

such that

$$\Omega^*_{4k}(pt) \xrightarrow{\Delta^*} \mathrm{KO}^*_{4k}(pt) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \cong \mathbb{Z}\left[\frac{1}{2}\right]$$

is the signature homomorphism:

$$[M] \mapsto \text{sign}(M).$$

(That is,

$$\Omega^*_{4k}(pt) \cong \Omega^*_{\mathrm{SO}}(pt) \xrightarrow{\Delta^*} \mathrm{KO}^*_{4k}(pt) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \xrightarrow{\text{ph}} \mathbb{Z}\left[\frac{1}{2}\right] \subset \mathcal{Q},$$

is the signature homomorphism).

By the multiplicativity of the elements $\Delta^*_{4k}$, one deduces that, in fact, there is a natural transformation of $\mathbb{Z}/4\mathbb{Z}$-periodic theories:

$$\Omega^*_\mathrm{SO}(X,A) \otimes \Omega^*_{\mathrm{SO}}(pt) \mathbb{Z}\left[\frac{1}{2}\right] \xrightarrow{} \mathrm{KO}^*_\mathrm{SO}(X,A) \otimes \mathbb{Z}\left[\frac{1}{2}\right].$$

Since this transformation induces an isomorphism on
coefficient groups, we conclude that it is an equivalence of theories.

From the two theorems above, we see that elements of $\text{KO}^i(X) \otimes \mathbb{Z}[\frac{1}{2}]$ are commutative diagrams:

$$
\begin{array}{ccc}
\Omega_{i+4}^\ast(X) \otimes \mathbb{Q} & \xrightarrow{\sigma} & \mathbb{Q} \\
\downarrow i_* & & \downarrow i \\
\Omega_{i+4}^\ast(X; \mathbb{Q}/\mathbb{Z}[\frac{1}{2}]) & \xrightarrow{T} & \mathbb{Q}/\mathbb{Z}[\frac{1}{2}]
\end{array}
$$

where $\sigma, T$ are homomorphisms satisfying the "periodicity relations":

$$
\tau((V \xrightarrow{f} X) \times (M \twoheadrightarrow \text{pt})) = \text{sign}(M) \cdot T(V \xrightarrow{f} X)
$$

and

$$
\sigma((V \xrightarrow{f} X) \times (M \twoheadrightarrow \text{pt})) = \text{sign}(M) \cdot \sigma(V \xrightarrow{f} X).
$$

The elements of $\text{KO}^i(X,A) \otimes \mathbb{Z}[\frac{1}{2}]$ (that is, in the relative case) are analogously described.

Now, suppose that $X$ is a cycle in the geometric homology theory $\Omega_*^\mathbb{Q}$ mentioned earlier. Let $h : X^\mathbb{Q} \rightarrow \mathbb{R}^N$ be a p.l. embedding into a large Euclidean space such that $X$ has codimension $4k$, and let $U$ be a regular neighborhood of $X$. A canonical orientation $\mu_X$ in $\text{ko}_n(X) \otimes \mathbb{Z}[\frac{1}{2}]$ corresponds by Alexander duality to a canonical element of
ko^{4k}(U, dU) \otimes \mathbb{Z}[\frac{1}{k}]

That is, it corresponds to a commutative "diagram":

\[
\begin{array}{ccc}
\Omega_4^*(U, dU) \otimes \mathbb{Q} & \overset{\sigma}{\longrightarrow} & \mathbb{Q} \\
\downarrow & & \downarrow \\
\Omega_4^*(U, dU; \mathbb{Q}/\mathbb{Z}[\frac{1}{k}]) & \overset{\tau}{\longrightarrow} & \mathbb{Q}/\mathbb{Z}[\frac{1}{k}]
\end{array}
\]

(3)

which satisfies the periodicity relation. Sullivan described how to obtain such a diagram from the signature invariant (in $\mathcal{F}$) and transversality.

Given $[(M, \partial M), f]$ in $\Omega_{i+4*}(U, dU)$, the relative block transversality theorem ([8], [15]) implies that we can assume $f^{-1}(X) \subset M$ is a cycle with the same local structure as $X$, with dimension $4k$, say. That is, $f^{-1}(X)$ is a cycle in $\mathcal{F}$, and has a signature $\text{sign}(f^{-1}(X)) \in \mathbb{Z}$. By relative transversality again, and by cobordism invariance of the signature, this procedure associates to $[(M, \partial M), f]$ a well-defined integer $\sigma_X([(M, \partial M), f])$. Define $\sigma$ in (3) to be $\sigma_X \otimes \mathbb{Q}$. For $\tau$, it suffices to define a compatible collection of homomorphisms: $\tau_n : \Omega(U, dU; \mathbb{Z}/n\mathbb{Z}) \rightarrow \mathbb{Z}/n\mathbb{Z}$ for $n$ odd, which we can then tensor with $\mathbb{Z}[\frac{1}{k}]$.

To do this, recall that $\Omega_*^{\cdot}(\cdot; \mathbb{Z}/n\mathbb{Z})$ is, geometrically, bordism of $\mathbb{Z}/n\mathbb{Z}$-manifolds. Using transversality as above, assign to a singular $\mathbb{Z}/n\mathbb{Z}$-cycle $[\tilde{M}, g]$ a $\mathbb{Z}/n\mathbb{Z}$-cycle
$g^{-1}(X)$ in the bordism theory with $\mathbb{Z}/n\mathbb{Z}$-coefficients associated to $\mathfrak{F}$ (see I.8).

If $Y$ is the relative cycle from which $g^{-1}(X)$ is obtained by identifications on the boundary, we define:

$$\text{sign}(g^{-1}(X)) = \text{sign}(Y),$$

as in the case of $\mathbb{Z}/n\mathbb{Z}$-manifolds. Additivity of the signature invariant of $\mathfrak{F}$ guarantees that the correspondence:

$$\tau_n : \Omega_*(U, U; \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$$

$$[\bar{M}, g] \mapsto \text{sign}(g^{-1}(X))$$

is a well-defined homomorphism. Let $\tau = \lim_{\text{odd } n} \tau_n \otimes \mathbb{Z}[\frac{1}{2}].$

By multiplicativity of the signature with respect to manifolds, it is clear that $\sigma$, $\tau$ satisfy the periodicity relations. Finally, the diagram (3) corresponding to $\sigma$, $\tau$ commutes. The definition of orientations for relative cycles proceeds similarly.

Note that the orientation $\mu_X$ satisfies:

$$c_*(\mu_X) = \text{sign}(X) \in ko_n(pt) \otimes \mathbb{Z}[\frac{1}{2}]$$

where $c : X \to pt$ is the collapse map.

Specializing now to the bordism theory associated to
the class of Witt spaces, we have the theorem:

**Theorem:** Every Witt space \((X, \mathcal{E}X)\) of dimension \(q \geq 0\) has a canonical orientation class \(\mu_X \in \text{ko}_q(X, \mathcal{E}X) \otimes \mathbb{Z}[\frac{1}{2}]\). If \(\mathcal{E}X = \emptyset\) then the homomorphism

\[
c_* : \text{ko}_q(X) \otimes \mathbb{Z}[\frac{1}{2}] \rightarrow \text{ko}_q(\text{pt}) \otimes \mathbb{Z}[\frac{1}{2}]
\]

carries \(\mu_X\) to \(\text{sign}(X)\).
References.

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Figure 2.
Figure 3.
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BIOGRAPHICAL NOTE

The author was born September 12, 1953 in Berkeley, California. He grew up in Framingham, Massachusetts during its quaint pre-"Golden Mile" era, and attended Framingham North High School.

In September 1971, he began studies at M.I.T., where he was elected to Phi Beta Kappa in 1974. He graduated with a B.S. in Mathematics in June 1975, and stayed on at M.I.T. for his graduate work in topology. He held a four year teaching assistantship in the Department of Mathematics.

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