A Numerical Method for Calculating Occultation Light Curves from an Arbitrary Atmospheric Model

by

Dawn Marie Chamberlain

Submitted to the Department of Earth, Atmospheric, and Planetary Sciences in Partial Fulfillment of the Requirements for the Degree of Bachelor of Science at the Massachusetts Institute of Technology

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ABSTRACT

We present a method for numerically calculating a light curve for a stellar occultation by a planetary atmosphere from values calculated by an arbitrary atmospheric model that has spherical symmetry. Our method takes as inputs a set of values for the refractivity and its derivatives and interpolates between them to obtain the light curve. It is specifically intended for use with small occulting bodies such as Pluto and Triton, but is applicable to planets of any size. We also present the results of a series of tests that show that our method works to within an accuracy of at least $10^{-4}$. We could not determine the exact values of our errors due to difficulties in finding a completely analytic model with which to compare it.

Thesis Supervisor: Professor James L. Elliot
Title: Professor of Physics and Earth, Atmosphere, and Planetary Sciences
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Chapter 1: Introduction

An occultation occurs when a body of interest passes between the observer and a star or other luminous body. Observations of how the stellar signal changes with time have been important to the expansion of our knowledge of the planets and other objects in the solar system. Occultations allow observers to convert from time resolution in imaging to spatial resolution in the bodies. The earth's atmosphere does not allow us to resolve below the approximately 1 arcsec seeing, but with occultations, we can resolve features on the order of a kilometer across at the distance of Pluto.

When astronomers observe an occultation, they take a continuous series of short exposure images of the planet and star throughout the event. They then plot the flux of the star, usually normalized to one and zero for the unocculted and fully occulted star, as a function of time. This plot is called a light curve. They then try to find a model for the atmosphere that explains the curve they observed.

To make such a model, one needs to understand the processes by which the total light that reaches us from the star is decreased as the planet passes in front of the star. As the light from the occulted star passes near the occulting planet, it is bent and may be partially absorbed by the planet's atmosphere. The angle by which a light ray is bent depends only on the refractivity of the atmosphere, integrated along the path of the light ray. The amount of light absorbed by the atmosphere depends on the amount of dust and other particles in the part of the atmosphere that the starlight is passing through. Occultations usually probe relatively high in the atmosphere -- around the microbar pressure region (Elliot and Olkin 1996). In this region, such particles are rare. Hence, in most cases, if we knew the refractivity of the atmosphere at all points, we could predict the shape of the light curve.

The inverse problem, determining the refractivity throughout the atmosphere from the light curve, is more difficult. One common approach to this problem is to take a likely class of models for the refractivity distribution that depend on some set of parameters in a known way, and do least squares fits of the observed data to the models to determine the most likely values for the model parameters.

Even this is usually not easy. Many modern atmospheric models are based on radiative transfer between different levels of the atmosphere, which leads to complex models that do not, themselves, have an analytic form. One starts with values for certain parameters, like surface pressure and the percentage of certain gasses in the atmosphere, and iteratively
determines the refractive structure of the atmosphere from these properties by insisting that the atmosphere be in steady state, e.g. (Strobel et al. 1996). With these models, it can take hours to calculate a light curve from a single set of parameters, and, in order to perform a least squares fit for multiple parameters, it is usually necessary to calculate the model for many different sets of parameters. This can take a prohibitively long time.

In this thesis, we present a method for speeding up the process of calculating light curves from complicated models by interpolating between points calculated with these models. In deriving the light curve, we include small planet effects, that is, we include geometrical effects that arise when the length scales in the atmosphere are sizable fractions of the planet's radius. Our model is thus applicable to occulting bodies like Pluto and Triton as well as larger planets. We start with a profile of values for the refractivity at different radii spanning the region probed by the occultation, and interpolate between these values to calculate a model light curve. We present a method for interpolating between points along a single refractivity profile to make a model light curve from a single set of parameters. We also present a method for interpolating between models with differing sets of parameters to make light curves as functions of the model parameters. However, at this time, we present tests verifying only the method for interpolating along a single refractivity profile.

Chapter 2: The Model

2.1 Physical Derivations
In deriving the equations for converting a refractivity profile into a light curve, we closely follow section 2 of Elliot and Young (1992 see Appendix 1). As they did, we start off with some simplifying assumptions appropriate to small planets: (i) the light received at any time comes from only one point on the planetary limb, (ii) the atmosphere is spherically symmetric. We also use the geometrical optics approximation for a spherical planet. However, unlike E&Y, we are not developing a particular atmospheric model, so we do not need to assume that the rotational acceleration in the atmosphere is small compared with gravity -- this assumption may or may not be included in the specific atmospheric model used to generate the refractivities needed as input for this method. We also do not need to assume that the velocity of the planet's shadow relative to the observer is constant throughout the occultation -- this assumption, or a different method of calculation, may be included in the implementation of this method.
Figure 1. Stellar occultation by a planetary atmosphere. Starlight encounters a planetary atmosphere and is bent by refractivity in the atmosphere. Since the refraction increases exponentially with depth in the atmosphere, two neighboring rays separate by an amount proportional to distance from the planet, which causes the star to dim as seen by a distant observer. After (Elliot and Young 1992)

We assume that parallel, monochromatic light rays are incident on a spherically symmetric planetary atmosphere from the left in Figure 1. The observer's plane is perpendicular to the path of the light rays. In this plane, $\rho$ is a radial coordinate measuring the distance from the point through which a light ray passing directly through the center of the planet would pass. We refer to $\rho$ as a function of $r$, which marks the position of arrival in the observer's plane of a light ray with radius of closest approach to the planet of $r$. We also refer to $\rho$ as a function of $t$, where $\rho(t)$ marks the position of the observer in the plane as a function of time. This is determined from the geocentric planetary ephemeris and the motion of the planet relative to the center of the Earth.

With these assumptions, the flux from the occulted star can be broken down into three terms: (i) the differential bending of the light rays at different depths in the atmosphere, (ii) the partial focusing of the light rays in the plane perpendicular to the path of the ray, and (iii) the extinction of the light due to absorption in the atmosphere. The first two can be
seen by geometrical arguments. A bundle of light rays of width $dr$ is bent as it passes through the atmosphere, and is spread out to span a width $dp$. Near the center of the light curve, the light from the star is focused -- the light received increases by a factor of the ratio of the circumferences of the "circles of light" of radius $r$ and $p$. For simplicity, we will assume that the third term, the extinction, is negligible and ignore it, though it would be straightforward to add extinction to our method. This yields the following equation for the flux, $\zeta$, as a function of radius:

$$\zeta(r) = \frac{dr}{dp(r)} \frac{r}{p(r)}.$$  \hspace{1cm} (2.1)

Note that if we were observing an occultation where the center of the occulting planet passed directly between the observer and the star ($r=0$), the flux would increase sharply to infinity. This is called a *central flash*, and it has been observed in occultations where the center of a small planet or moon passed very close to the line between the observer and the star (Elliot *et al.* 1977).

If we define $\theta(r)$ to be the angle by which a light ray with point of closest approach to the planet of $r$ is bent, then, as stated in E&Y, $p(r)$ becomes

$$p(r) = r + D\theta(r)$$ \hspace{1cm} (2.2)

and by differentiating this, we get

$$\frac{dr}{dp(r)} = \frac{1}{1 + D(d\theta/dr)}.$$ \hspace{1cm} (2.3)

In order to determine $\theta(r)$, we need to integrate the refractivity along the path of the ray. To do this more easily, we define a coordinate $x$ that measures the distance along the light ray from the point its closest approach to the planet. Further, we define the radial distance from the planet along the path of the ray to be $r'$, where

$$r' = \sqrt{r^2 + x^2}.$$ \hspace{1cm} (2.4)

If we assume that the refractivity throughout the atmosphere is much less than one, we can get $\theta$ by simply integrating the refractivity, $\nu$:

$$\theta(r) = \frac{d}{dr} \int_{r'}^{r} \nu(r'(x, r))dx$$

$$= \int_{r'}^{r} \frac{r'}{r''} d\theta'(dx).$$ \hspace{1cm} (2.5)

In order to calculate the light curve from the refractivity, we also need $d\theta/dr$: 
\[
\frac{d\theta (x)}{dx} = \int_{-\infty}^{\infty} \left\{ \frac{x^2}{x'} \frac{dv(x')}{dx'} + \frac{x}{x'} \frac{d^2 v(x')}{dx'^2} \right\} \mathrm{d}x. \tag{2.6}
\]

Most atmospheric models predict values for the number density, rather than directly predicting values for the refractivity. This means that we also need to assume a relationship between refractivity and number density. We choose to assume that the refractivity of the atmosphere is simply proportional to the number density:
\[
v(r) = n(r) v_{\text{STP}}/L, \tag{2.7}
\]
where \(L\) is Loschmidt's number, which is defined as the number density of an ideal gas at standard temperature and pressure in molecules per cm\(^3\), and \(v_{\text{STP}}\) is the refractivity of the gases in the atmosphere at STP, which we assume to be a constant.

### 2.2 Numerically Calculating the Light Curve

Now that we have the equations we need, the next step is to determine how we can calculate them from a set of values of \(v\) at discrete radii. In order to use Eq. (2.1) to calculate the flux, we need to put \(v(x')\) into an integrable form. We choose to do this by interpolating between the values we have for it. This gives us a function we can integrate out to \(x_{\text{max}}\), where
\[
x_{\text{max}} = x_{\text{max}}' - x. \tag{2.9}
\]
We cannot extend this interpolation function to include larger radii without risking large, unpredictable errors. Hence we approximate \(\theta (x)\) by only integrating out to \(x_{\text{max}}\). This means approximating the effects of the entire atmosphere by calculating only the effects of the atmosphere below a certain radius. Since the angle of bending in atmospheres falls off approximately exponentially, this is usually a very good assumption. The approximation gives us:
\[
\hat{\theta} (r) = \int_{-x_{\text{max}}}^{x_{\text{max}}} \frac{r dv(x')}{dr'} \mathrm{d}x, \tag{2.10}
\]
and
\[
\frac{d\hat{\theta} (x)}{dx} = \int_{-x_{\text{max}}}^{x_{\text{max}}} \left\{ \frac{x}{x'} \frac{dv(x')}{dx'} + \frac{r}{x'} \frac{d^2 v(x')}{dx'^2} \right\} \mathrm{d}x. \tag{2.11}
\]

These functions, along with Eqs. (2.2) and (2.3), enable us to calculate the approximate flux as a function of \(r\). However, occultation data are indexed by the time of the observations, not the radius -- we need the flux as a function of time. To do this, we first convert from \(r\) to \(\rho\) by another interpolation function. We take a list of \(r\) values spread across the range for which we have refractivity values. Then we compute \(\rho (x)\) for each
value of \( r \) using Eq. (2.2), approximating \( \theta \) by \( \hat{\theta} \). From these \((r, \rho(r))\) pairs, we make an interpolation function for \( r(\rho) \).

Finally, we convert from \( \rho \) to \( r \) using E\&Y's Eq. (5.1). This equation includes the assumption that the shadow of the occulted object travels in a straight line in the observer's plane at a constant velocity \( v \). We could easily choose a different expression that does not include this assumption for specific cases where this assumption is not a good one. Now we have all the equations we need to calculate the flux as a function of time -- that is, the light curve.

2.3 The Cap

We often have observations from long before and long after the actual occultation event. These data are necessary to establish the full scale brightness of the star. But with the model described in the previous section, fitting a long baseline requires that we have refractivity values out to very large radii. If we cannot easily calculate refractivities out very far using the atmospheric model, we can make a further approximation for the flux beyond \( r_{\text{max}} \) to extend the model with less work. The simplest possible approximation is to set the flux to 1 beyond \( r_{\text{max}} \). This, however, introduces a discontinuity in the flux function which can make fitting difficult and can introduce large errors at the edges if we have not calculated the refractivity out very far. A better approximation could increase the power of fits done with this method. One way to approximate the values at the ends is to use the model described in sections 3 and 4 of E\&Y. If we use their model only out to first order, and assume that the atmosphere is isothermal, we can greatly decrease the errors at the edges of our model without doing too much work. The isothermal assumption is reasonable at the large radii above where we have calculated our model of choice, and, in any case, we are looking only for an approximation.

We start with Elliot & Young's model (their Eq. (4.5)), setting \( a \) and \( b \) to 0 (isothermal), and dropping the \( O(\delta^2) \) term (first order). This gives us

\[
\theta(r) = -v(r)\sqrt{2\lambda_{\odot}(r)}\int_{-\infty}^{\infty} e^{-y^2} \left\{ 1 + \left[ -3y^2 + \frac{3}{2} y^4 \right] \delta \right\} dy. \tag{2.12}
\]

Differentiating this, we get

\[
\frac{d\theta(r)}{dr} = \frac{v(r)}{r} \sqrt{2\lambda_{\odot}(r)} \left( \lambda_{\odot}(r) + \frac{1}{2} \right) \int_{-\infty}^{\infty} e^{-y^2} \left\{ 1 + \left[ -3y^2 + \frac{3}{2} y^4 \right] \delta \right\} dy \]

\[
- \frac{v(r)}{r} \sqrt{2\lambda_{\odot}(r)} \int_{-\infty}^{\infty} e^{-y^2} \left[ -3y^2 + \frac{3}{2} y^4 \right] \delta dy. \tag{2.13}
\]
In order to evaluate these equations, we need a value for $\lambda_{g0}$. We can calculate it for any $r_0$ in the range of our data using E&Y's Eq. (3.17) and its $r$ derivative to get

$$\lambda_{g0} = -\frac{\partial v(r)}{\partial r} \bigg|_{r=r_0} \frac{r_0}{v(r_0)}.$$  \hspace{1cm} (2.14)

Elliot and Young integrated these equations to get $\theta(r)$ and $d\theta(r)/dr$ in terms of power series. To first order, their results were:

$$\theta(r) = -\sqrt{2\pi\lambda_{g0}(r)v(r)} \left[ 1 - \frac{3}{8} \delta \right]$$  \hspace{1cm} (2.15)

and

$$\frac{d\theta(r)}{dr} = \sqrt{2\pi\lambda_{g0}(r)v(r)} \frac{v(r)}{r} \left[ 1 + \frac{1}{8} \delta \right].$$  \hspace{1cm} (2.16)

We can use these equations to get good approximations to $\theta(r)$ and $d\theta(r)/dr$ for radii beyond $r_{\text{max}}$.

We also need to make a correction to the values of $\theta(r)$ and $d\theta(r)/dr$ for radii below $r_{\text{max}}$. If we did not, there would be a discontinuity at $r_{\text{max}}$ due to the fact that the model in Section 2.2 integrates over a smaller and smaller portion of the atmosphere as $r$ approaches $r_{\text{max}}$. To fix this problem, we integrate Eqs (2.12) and (2.13) from $y_{\text{max}}$ to $\infty$ and from $-y_{\text{max}}$ to $-\infty$, where $y_{\text{max}} = (x_{\text{max}} / r)\sqrt{1 / 2\delta}$, and add these values to the numbers obtained from Eqs. (2.10) and (2.11). These integrations give us:

$$\theta(r) = \tilde{\theta}(r) - v(r)\sqrt{2\lambda_{g0}(r)}$$

$$\left[ \sqrt{\pi} \frac{8 - 3\delta}{16} - 3\delta y_{\text{max}} \frac{1 - 2y_{\text{max}}}{8e^{y_{\text{max}}}} - \sqrt{\pi\text{Erf}(y_{\text{max}})} \right] \frac{8 - 3\delta}{16}$$  \hspace{1cm} (2.17)

and

$$\frac{d\theta(r)}{dr} = \frac{d\tilde{\theta}(r)}{dr} + \left( -\frac{\lambda_{g0}(r)}{r} - \frac{1}{2r} \right) \theta(r) - \frac{v(r)}{r} \left[ \sqrt{\pi} - \frac{2y_{\text{max}} - 4y_{\text{max}}^2}{e^{y_{\text{max}}}} - \sqrt{\pi\text{Erf}(y_{\text{max}})} \right]$$  \hspace{1cm} (2.18)

We now have all the equations we need to calculate the light curve from a refractivity profile. Next, we needed to implement the model on a real computer.

**Chapter 3: Implementation**
In deciding how to implement this method, it is important to consider the relative importance of speed and ease of use. One could implement all of the above calculations in a language like C, using algorithms like those found in Numerical Recipes in C (Press et al. 1988), or one could implement them in an environment like Mathematica\textsuperscript{TM}. C has the advantage of being very fast. On the other hand, when one is developing a new method, it is useful to have easy access to each step of the calculations, and it is good to have a method that is straightforward for someone unfamiliar with the process to follow. This is the power that environments like Mathematica\textsuperscript{TM}'s notebook interface provide.

If one decides to program in an environment like Mathematica\textsuperscript{TM}, there is a further decision to be made. Mathematica has built-in functions for calculations like numerical integrations and interpolations. One can use them, or write ones own functions for these tasks. It is difficult to find out exactly how these internal calculations are performed, which makes estimates of the errors in the calculations uncertain. However, these functions are usually quite good, and they are always much faster than anything one can implement oneself within the provided environment, since only built-in functions are fully compiled.

We chose to implement our model as a package in Mathematica\textsuperscript{TM} 2.2 (Wolfram 1991) on a Power Macintosh 8100/80 running a remote kernel on an HP9000 Series 7000. We also decided to use the internal functions. Using Mathematica\textsuperscript{TM} slowed our calculations, but we found that, as long as we did use the internal functions, it did not slow them down to the point of making them impractical.

In setting up to make light curves from an atmospheric model, first we had to calculate some values for the refractivity from this model. We organized these model refractivities in the form of one or more refractivity profiles. Each refractivity profile consisted of a list of radii with the associated values of the refractivity and its first two \( r \)-derivatives. If we wanted to be able to interpolate over the values of some of the model parameters, we needed a grid of these profiles. We would choose a set of values for each model parameter we wanted to fit for, and then made refractivity profiles for each combination of the parameter values.

Once we had these profiles, we read them in to our package and made one dimensional interpolation functions for the refractivity from each refractivity profile. Mathematica's internal interpolation functions use piece-wise continuous polynomials to approximate the data passed in, while atmospheres are usually exponential. In order to do accurate
interpolations, we took the natural logarithm of the refractivity before passing it into the interpolation function. To do this, we assumed that the first derivative would always be negative and the second derivative was always positive, which is true of all atmospheric models in use today. To avoid interpolating over imaginary values, we took the logarithm of the negative of the first derivative. Mathematica does not ensure continuous derivatives in interpolation functions unless the derivatives are explicitly specified. It is important that functions be as smooth as possible when doing non-linear least squares fits to them, so we calculated what the slope of the log should be in terms of the derivatives of the refractivity:

\[
\frac{d}{dx} \ln(\nu(x)) = \frac{\nu'(x)}{\nu(x)} \tag{4.1}
\]

and

\[
\frac{d^2}{dx^2} \ln(\nu(x)) = \frac{\nu(x)\nu''(x) - (\nu'(x))^2}{(\nu(x))^2} \tag{4.2}
\]

We then included these derivatives in the interpolations. Next, we used the interpolation functions to make a grid of points from which we made a single multi-dimensional interpolation function for refractivity as a function both of radius and of whatever parameters we had density profiles for. In order to specify derivatives in a multidimensional interpolation function, Mathematica requires that you specify all the derivatives. For instance, if we were making an interpolation function for the refractivity as a function of radius \( r \) and surface pressure \( p \), we would have to specify the first derivatives in both the \( r \) and \( p \) directions. Though we did not have an explicit formula for the \( p \)-derivatives, we could have numerically approximated them. Instead, we chose not to include any derivatives in these interpolations, preferring a more accurate approximation to a smoother one.

Making a multi-dimensional interpolation function for the flux at this point would have greatly increased the speed of creating multiple light curves from the refractivity profiles. All of the time-consuming calculations would be done just once, during the set up function. When we tried this, however, we had problems with fitting real light curves. When the path of the occulting planet passes between the star and the observer such that, for some amount of time during the occultation, the planet's surface completely blocks the light from the star, there is a discontinuity in the flux received from the star. This discontinuity makes interpolation functions highly inaccurate in that region. It is much more accurate to recalculate the flux each time it is needed.
Each time we needed a set of flux values for some value of the parameter, we made a one
dimensional interpolation function for refractivity at that value from the multi-dimensional
function we had calculated previously. We used this function to make a set of data points
for $\theta(r)$ and $d\theta(x) / dx$ as a function of radius using Eqs (2.17) and (2.18). We used
the values thus obtained to make the $r, \rho(r)$ grid and the interpolation function for $r$ as a
function of $\rho$. Once we had $r(\rho)$, we could convert all the times we were interested in to
radii. We then calculated $d\theta(r) / dr$ for those exact radii and used those numbers and Eq.
(2.3) to get $|dr / d\rho|$. Putting this all together with Eq. (2.1) gave us the normalized light
curve.

Chapter 4: Tests

4.1 Expected Sources of Error
We expected the accuracy of our model to depend on several things. The first likely source
of error was the refractivity profiles supplied. We anticipated that the more closely we
spaced the model values, and the higher we extended them, the more exact our model
would be. In addition, we expected the spacing of the points we calculated for the internal
interpolation for $r(\rho)$ to make a difference, at least if we made this spacing larger than that
of the refractivity profile. Finally, we expected including the cap to greatly improve
results, especially for refractivity profiles that did not extend very far up into the
atmosphere.

In addition, we thought that these dependencies of the error would scale with the scale
height of the atmosphere at the level probed by the occultation. The scale height is

$$H = \frac{r_h}{\lambda_{\text{obs}}},$$

(4.1)

where $r_h$ is the half light radius. The half light radius is defined to be the distance from the
center of the planet at which the light from an occulted star is reduced to half of its
unocculted intensity. This radius depends on the distance of the observer, since it is a
function of $dr/d\rho$, which depends on the distance between the occulting planet and the
observer (see Figure 1 and Eq. (2.3)).

4.2 Exponential Refraction Tests
4.2.1. DERIVATION OF THE TEST
Ideally, we would have liked to test our method on an atmospheric model that had an
analytic formula for the refractivity and which would result in an analytic formula for the
light curve. This would have given us an exact model to which we could compare our numerical model. Unfortunately, we know of no such model.

It can be shown, however, that the following function is an exact formula for the light curve if the bending angle $\theta(r)$ is a perfect exponential:

$$\frac{p(t) - p_h}{H} = \ln \left( \frac{1}{\phi_{cyl}} - 1 \right) + \left( \frac{1}{\phi_{cyl}} - 2 \right),$$

where $p_h$ is the observer's position at half light, and $\phi_{cyl}$ is the unfocused flux from the star (see Appendix 3). This formulation was originally derived by Baum and Code as an approximation to an isothermal atmosphere (Baum and Code 1953).

In order for this to be useful, we needed to find a formula for the refractivity that would give an approximately exponential $\theta(r)$. We started with Eq. (2.5),

$$\theta(r) = \frac{d}{dr} \int_{-\infty}^{r} v(r') dx.$$  

Clearly, the only way for $\theta(r)$ to be exponential was for $v(r')$ to include an exponential, so we started with a guess that

$$v_i(r') = v_0 e^{-(r-r_{\text{g}})/H}.$$  

We then calculated the integral of $v_i(r')$ along the path of the light ray, since that must be exponential in $r$, with no $r$'s multiplying it, in order to get a perfectly exponential angle. To approximate this integral, we factored $1/H$ out of the exponent, and did a first order Taylor expansion of the remaining square root around 1:

$$\int_{-\infty}^{\infty} v_i(r') dx = \int_{-\infty}^{\infty} v_0 e^{-\left(\sqrt{r'^2 + r_{\text{g}}^2} - r_{\text{g}}\right)/H} dx$$

$$= v_0 e^{-r_{\text{g}}/H} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2H(r_{\text{g}}^2 - x^2)^{1/2}}} dx$$

$$= v_0 e^{-r_{\text{g}}/H} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2Hx^2}} dx$$

Finally, we substituted

$$y = \frac{x}{\sqrt{2rH}}$$

in to Eq. (4.5), and got
Figure 2. Light curve calculated with Eq. (4.2). This light curve is used as the comparison used in all our exponential refraction tests. Half light is at $t=0$, and each unit of time corresponds to approximately 0.8 scale heights.

\[
\int_{-\infty}^{r'} v(r') dx = v_0 e^{-\frac{(r-r_h)}{H}} \sqrt{2rH} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2}} dy. \tag{4.7}
\]

In order to eliminate the $\sqrt{r}$ from Eq. (4.5), we tried a slightly different formula for $v(r')$:

\[
v_2(r) = \left( \frac{r}{r_h} \right)^{-1/2} v_1(r)
= v_0 \left( \frac{r}{r_h} \right)^{-1/2} e^{-\frac{(r-r_h)}{H}} \tag{4.8}
\]

and performed the same set of calculations on this function. This successfully canceled the $\sqrt{r}$, giving an exponential refractivity to first order.

To check the likely form of the error from the first order approximation, and to make better approximations, we extended this process by looking for a power series of the form

\[
v_3(r) = v_2(r) \left( 1 + \frac{e_1}{r} + \frac{e_2}{r^2} + \ldots \right)
= v_0 \sqrt{\frac{r_h}{r}} e^{-\frac{(r-r_h)}{H}} \left( 1 + \frac{e_1}{r} + \frac{e_2}{r^2} + \ldots \right). \tag{4.9}
\]
To get the coefficients $e_1, e_2, \ldots$, we expanded $v(r')$:

$$v_0 \frac{1}{\sqrt{r^2 + x^2}} e^{-(\eta H \sqrt{r^2 + x^2})} \left(1 + \frac{e_1}{r} + \ldots\right) = v_0 \frac{1}{\sqrt{1 + 2H^2/r^2}} e^{-(\eta H \sqrt{1 + 2H^2/r^2})} \left(1 + \frac{e_1}{r} + \ldots\right)$$

(4.10)

as $e^{-r^2}$ times a power series in $y$. We then integrated each term in the series, using

$$\int_{-\infty}^{\infty} y^n e^{-r^2} dy = \begin{cases} \Gamma\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd} \\ \frac{(n-1)! \sqrt{\pi}}{2^{n-1} \left(n/2 - 1\right)!} & \text{if } n \text{ is even} \end{cases}$$

(4.11)

Finally, we grouped terms by powers of $r$ to determine the values of the $e_i$'s. We used this procedure to calculate the series out to 5th order (see Appendix 4).

4.2.2. TESTS RUN

To verify our method, we first ran a series of tests against the first order approximation, with model planets that had equal scale heights of 25 kilometers but different half light radii, ranging from $10^{12}$ km down to $10^6$ km. The results of these tests are summarized in Figures 3 through 8. The error is defined to be $\phi_{\text{cyl}} - \phi_{\text{calc}}$, where $\phi_{\text{calc}}$ is the value calculated using our method. We found that the smallest errors occurred at a half light radius of about $10^8$ km, with the errors at $10^9$ km being comparable. To both sides of these half light radii, the errors increased by approximately an order of magnitude for each order of magnitude change in $r_h$, but the forms of the error were qualitatively different. For radii
Figure 4. Error in one term test with $r_h=10^9\text{km}$ ($H/r_h=2.5 \times 10^8$). The errors follow a definite curve, but there is still some scatter, probably due to minor roundoff error. We are not sure what caused the small discontinuity near $t=17$.

Figure 5. Error in one term test with $r_h=10^8\text{km}$ ($H/r_h=2.5 \times 10^7$). The errors follow a definite, smooth, curve, with no evidence of scatter.
larger than $10^9 \text{km}$, the errors were scattered with a mean near 0. However, for radii smaller than $10^8 \text{km}$, the errors at successive times in the light curve were highly correlated -- they followed a smooth curve. At a $r_h$ of $10^9 \text{km}$, the errors were very scattered, as at larger radii, but the mean of the scatter was clearly greater than zero, and at a $r_h$ of $10^8 \text{km}$, the errors clearly followed the same curve that the errors at smaller radii followed, but they exhibited a small scatter around that curve.

The existence of these two distinct patterns to the error in different regimes implies that there are two distinct types of error that contribute at different radii. The systematic error at the smaller radii nicely fits the errors we expected from the power series approximation. The scattered error at the larger radii fits the pattern one would expect from roundoff errors. We kept the same number of digits in all the calculations, so larger radii should cause larger roundoff errors. And one would expect roundoff errors to be randomly distributed around a mean of zero.

![Figure 6](image.png)

**Figure 6.** Maximum errors in exponential refraction tests using 1 term from the power series. The error has a minimum around $r_h=10^9 \text{km}$ ($1/r_h=2.5 \times 10^{-8}$) and increases by about an order of magnitude for each order of magnitude change in $r_h$ to each side of this radius. The errors of larger radii are probably roundoff errors and the errors at smaller radii are probably due to the approximation.
Figure 7. Scatter of the errors as a fraction of the maximum error. This graph represents the scatter of the errors around a best-fitting line between times -1 and 4, a region where the systematic errors in tests at the lower radii are nearly linear. Notice that the scatter is a much higher percentage of the total error at the larger radii.

Figure 8. Error in test that included the linear term in the refractivity with $r_p=10^5\text{ km}$ ($H/r_p=2.5 \times 10^{-4}$). These errors follow a slightly different, but equally smooth, curve as the one term tests. This curve is constant across radii, and is the same curve that appeared in tests that included more terms in the refractivity.
Next we ran a similar series of tests with a refractivity that included the linear term in the power series, and another series including up through the cubic term from the power series. The results of these tests are shown in Figures 8-10. We ran these tests on radii up to only \(10^{10}\) km, since for the tests that appeared to have roundoff problems before, the errors looked the same with and without the extra terms. However, we extended these tests down to a \(r_h\) of \(10^4\)km. The errors did not decrease entirely as expected. The tests with \(r_h\) less than \(10^9\) km did have smaller errors when we added one term to the power series, but only by a single order of magnitude. This improvement was of the same order in all the tests, instead of being larger for the larger \(H/r_h\) ratios, as we had expected. Further, when we went back and ran tests with a \(r_h\) of 500 km, the first two terms each reduced the errors slightly, but the 3 term test gave the same results as 4 and 5 term tests.

The patterns we saw in the errors could perhaps be caused by problems with the power series. The refractivity along the path of a ray of light is gaussian when \(x\) is small compared to \(r'\), but when the ray is farther out in the atmosphere and \(x\) becomes large, the

![Figure 9](image)

**Figure 9** Maximum errors in exponential refraction tests using 2 terms from the power series. In all of these tests, the model planets had a scale height of 25 km. The error increases above and below \(r_h=10^9\) km \((H/r_h=2.5 \times 10^{-8})\) out at large radii as we run in to roundoff errors. For smaller radii, the error falls off by an order of magnitude or less for each order of magnitude increase in \(r_h\). This does not agree with the prediction that this test should have error proportional to \((H/r_h)^2\). Instead, the maximum error increases approximately linearly.
path of the ray become similar to a radial path. In the radial direction, the refractivity is a simple exponential \( v(x) \sim e^{-x} \), not a gaussian \( v(x) \sim e^{-x^2} \). In the case of a large planet, this change in the behavior of \( n \) occurs after nearly all of the bending of the light ray has happened, but, in the case of a smaller planet, this change comes earlier, and could, perhaps, cause the kinds of errors we saw in these tests.

**Figure 10** Maximum errors in exponential refraction tests using 4 terms from the power series. The error flattens out at large radii as we run into roundoff errors. For smaller radii, the error falls again off by about an order of magnitude for each order of magnitude increase in \( r_h \). This vastly different from the prediction that this test would have error proportional to \( (H/r_h)^4 \).
4.3 E&Y Tests

4.3.1 THE TESTS

Next, we ran a series of tests against a real atmospheric model, on model planets of sizes that we find in the solar system. Table 1 shows approximate half light radii and scale heights for some of the planets and moons that have been observed by occultations. It shows the typical range of values observed.

For our real model, we chose the isothermal model in E&Y's paper since it can be applied to the small planet cases to which we wish to eventually apply this method. Their model includes a power series approximation similar to the one we dealt with in our previous set of tests. In order to study the behavior of our model as the power series error changes, we ran 3 groups of tests on model planets of different sizes, ranging from slightly larger than Jupiter down to slightly smaller than Pluto.

In each group of tests, we picked two values for half light radius and for energy ratio (differing by either a factor of 2 or of 1.5). From each of the 4 possible combinations of these values, we made a series of profiles, varying the spacing and range of the points.

<table>
<thead>
<tr>
<th>Planet or Moon</th>
<th>Equatorial half light radius (km)*</th>
<th>Approximate scale height (km)*</th>
<th>H/rh</th>
</tr>
</thead>
<tbody>
<tr>
<td>Venus</td>
<td>6,200</td>
<td>7</td>
<td>.0011</td>
</tr>
<tr>
<td>Mars</td>
<td>3,500</td>
<td>8</td>
<td>.0023</td>
</tr>
<tr>
<td>Jupiter</td>
<td>71,900</td>
<td>25</td>
<td>.000035</td>
</tr>
<tr>
<td>Saturn</td>
<td>61,000</td>
<td>70</td>
<td>.0011</td>
</tr>
<tr>
<td>Titan</td>
<td>3,000</td>
<td>47</td>
<td>.016</td>
</tr>
<tr>
<td>Uranus</td>
<td>26,100</td>
<td>50</td>
<td>.0021</td>
</tr>
<tr>
<td>Neptune</td>
<td>25,300</td>
<td>50</td>
<td>.0020</td>
</tr>
<tr>
<td>Triton</td>
<td>1450</td>
<td>19</td>
<td>.013</td>
</tr>
<tr>
<td>Pluto</td>
<td>1200</td>
<td>60</td>
<td>.049</td>
</tr>
</tbody>
</table>

* Numbers obtained from (Elliot and Olkin 1996) and references therein

Note: Half light radii are typical values for earth-based occultations, and scale heights are measured around the microbar pressure level in the atmosphere.
Table 2. Magnitude of Maximum Errors from Tests with Differing Tops of Model Profiles. Cap included

<table>
<thead>
<tr>
<th>Model Planet Parameters</th>
<th>Top of the model in scale heights above half light radius</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>20</td>
</tr>
<tr>
<td>half light radius</td>
<td>scale height</td>
</tr>
<tr>
<td>Jupiter-sized model planets</td>
<td></td>
</tr>
<tr>
<td>105,000km</td>
<td>37.5km</td>
</tr>
<tr>
<td>105,000km</td>
<td>25km</td>
</tr>
<tr>
<td>70,000km</td>
<td>25km</td>
</tr>
<tr>
<td>70,000km</td>
<td>16.7km</td>
</tr>
<tr>
<td>Intermediate model planets</td>
<td></td>
</tr>
<tr>
<td>10,000km</td>
<td>10km</td>
</tr>
<tr>
<td>10,000km</td>
<td>5km</td>
</tr>
<tr>
<td>5,000km</td>
<td>5km</td>
</tr>
<tr>
<td>5,000km</td>
<td>2.5km</td>
</tr>
<tr>
<td>Pluto-sized model planets</td>
<td></td>
</tr>
<tr>
<td>2,500km</td>
<td>238.6km</td>
</tr>
<tr>
<td>2,500km</td>
<td>119.3km</td>
</tr>
<tr>
<td>1,250km</td>
<td>119.3km</td>
</tr>
<tr>
<td>1,250km</td>
<td>59.6km</td>
</tr>
</tbody>
</table>

and ran our model from those profiles with and without the cap, and with different spacing in the internal interpolation functions. In these tests, we did include the focusing term. However, we attempted to avoid some of the complications that can arise from this term. To ensure that we had consistent tests across a large range of half light radii and scale heights, we chose to compare to model events with nearly central chords, which would result in very large central flashes. In order to avoid large errors caused by large flux values near the flash, we cut the light curves off before reaching the center of the event, around where the flux reached a minimum before increasing for the central flash.

To make the profiles for these tests, we used E&Y’s Eq. (4.28) to get the refractivity at the half light radius. From that and their Eq. (3.20), we made the refractivity profiles we needed.
### Table 3. Magnitude of Maximum Errors from Tests with Differing Tops of Model Profiles, Cap NOT included

<table>
<thead>
<tr>
<th>Model Planet Parameters</th>
<th>Top of the model in scale heights above half light radius</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>20</td>
</tr>
<tr>
<td>half light radius</td>
<td>scale height</td>
</tr>
<tr>
<td>------------------------</td>
<td>----------------</td>
</tr>
<tr>
<td><strong>Jupiter-sized model planets</strong></td>
<td></td>
</tr>
<tr>
<td>105,000km</td>
<td>37.5km</td>
</tr>
<tr>
<td>105,000km</td>
<td>25km</td>
</tr>
<tr>
<td>70,000km</td>
<td>25km</td>
</tr>
<tr>
<td>70,000km</td>
<td>16.7km</td>
</tr>
<tr>
<td><strong>Intermediate model planets</strong></td>
<td></td>
</tr>
<tr>
<td>10,000km</td>
<td>10km</td>
</tr>
<tr>
<td>10,000km</td>
<td>5km</td>
</tr>
<tr>
<td>5,000km</td>
<td>5km</td>
</tr>
<tr>
<td>5,000km</td>
<td>2.5km</td>
</tr>
<tr>
<td><strong>Pluto-sized model planets</strong></td>
<td></td>
</tr>
<tr>
<td>2,500km</td>
<td>238.6km</td>
</tr>
<tr>
<td>2,500km</td>
<td>119.3km</td>
</tr>
<tr>
<td>1,250km</td>
<td>119.3km</td>
</tr>
<tr>
<td>1,250km</td>
<td>59.6km</td>
</tr>
</tbody>
</table>

### 4.3.2 ERRORS

A summary of the results can be found in Tables 2-5, and pictorial representations of some of the trends and the shape of some of the errors are in Figures 11-19.

All the results for the Jupiter-sized and intermediate model planets were similar. None of the results were more than an order of magnitude apart. There was a weak trend for the errors in the intermediate cases to be larger than the results in the Jupiter-sized cases. However, when we tested the Pluto-sized models, the errors were generally much higher, and not as consistent from test to test as the larger planet tests were.

Despite the larger errors at small radii, there were a number of consistent trends across all of the tests. Reducing the number of points per scale height in the refractivity profile had no effect on the errors, but changing the top of the model had a large effect. Strangely, the
errors when the top of the model was 15 scale heights up were lower than when we extended the data farther in the first two groups of tests, but when we lowered the top of the model below 15 scale heights, the errors increased very quickly in all tests. When the top of the profile was 20 scale heights above half light, the cap made at most a small improvement, and in some of the small planet cases, even made the error slightly larger. However, the cap made an increasingly large difference as we lowered the top of the profile. The spacing of the points in the internal interpolation functions made a difference in the error when we decreased it below 50 points per scale height in the larger planet tests. All of these tests were from refractivity profiles 10 points per scale height. When we reduced the internal spacing to that of the profile, the error increased to orders of
Table 5. Magnitude of Maximum Errors from Tests with Differing Numbers of Points Used in Internal Interpolation Functions, cap included

<table>
<thead>
<tr>
<th>Model Planet Parameters</th>
<th>Points per scale height calculated in internal interpolation functions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>half light radius</td>
</tr>
<tr>
<td>Jupiter-sized model planets</td>
<td>105,000km</td>
</tr>
<tr>
<td></td>
<td>105,000km</td>
</tr>
<tr>
<td></td>
<td>70,000km</td>
</tr>
<tr>
<td></td>
<td>70,000km</td>
</tr>
<tr>
<td>Intermediate model planets</td>
<td>10,000km</td>
</tr>
<tr>
<td></td>
<td>10,000km</td>
</tr>
<tr>
<td></td>
<td>5,000km</td>
</tr>
<tr>
<td></td>
<td>5,000km</td>
</tr>
<tr>
<td>Pluto-sized model planets</td>
<td>2,500km</td>
</tr>
<tr>
<td></td>
<td>2,500km</td>
</tr>
<tr>
<td></td>
<td>1,250km</td>
</tr>
<tr>
<td></td>
<td>1,250km</td>
</tr>
</tbody>
</table>

Note: all tests were run from refractivity profiles with 10 points per scale height, unless otherwise stated.

magnitude worse than with finer spacing. The internal spacing had no discernible effect on the errors in the Pluto-sized tests.

The dependence of error on radius may be explained by the approximations made in E&Y's model. They included a power series in \( \delta \) in their model, where

\[
\delta = \frac{1}{\lambda_{g0} r_0} \frac{r}{r_0}.
\]  

(4.12)

In the Pluto-sized tests we ran, \( \lambda_{g0} \) was either 10.48 or 20.97. In the two sets of tests where \( \lambda_{g0} \) was 10.48, the errors were generally similar to each other, but significantly larger than the errors in the two tests where \( \lambda_{g0} \) was 20.97. We compared our results to
Figure 11. A light curve and typical error plot for a Jupiter-sized planet test. The error is concentrated in the region where the flux is dropping off the most rapidly, and is highly discontinuous. The discontinuities imply the existence of different regimes where the error has different sources, but we are not sure what these sources are. This test was run with $r_h=70,000\text{km}$ and $H=25\text{km}$ ($H/r_h=3.6 \times 10^{-4}$), and with a refractivity profile that had 10 points per scale height and extended 20 scale heights above half light. The internal interpolation function calculated 125 points per scale height.
Figure 12. A light curve and typical error plot for an intermediate-sized planet test. The error is, again, concentrated in the region where the flux is dropping off the most rapidly, and has a shape fairly similar to that from the Jupiter-like case. This test was run with $r_p = 5,000$ km and $H = 5$ km ($H/r_p = 10^{-3}$, and with a refractivity profile that had 10 points per scale height and extended 20 scale heights above half light. The internal interpolation function calculated 125 points per scale height.
Figure 13. A light curve and typical error plot for a Pluto-sized planet test. This error is quite spread out over the entire region of the light curve, unlike the previous two examples. It also follows a much smoother curve. Also unlike the larger planet tests, the errors from each of these small planet tests had its own characteristic error shape. These "errors" are most likely caused by approximations made in E&Y's model, not by errors in our model. This test was run with $r_p=1250\text{km}$ and $H=59.6\text{km}$ ($H/r_p=4.8\times10^{-2}$), and with a refractivity profile that had 10 points per scale height and extended 20 scale heights above half light. The internal interpolation function calculated 125 points per scale height.
**Figure 14.** Magnitude of maximum difference between our method and that of E&Y versus the top of the profile for tests run with the cap. The errors are fairly flat in the region between 15 and 20 scale heights, but when we lowered the top of the model below 15 scale heights up, even with the cap, the errors started to increase sharply. In all cases, the maximum errors were larger for smaller planets, especially when we tested very small, Pluto-like planets.

Those of E&Y's model, calculated out to 4th order. These two values of $\lambda_{g0}$ raised to the -5th power are $7.92 \cdot 10^{-6}$ and $2.46 \cdot 10^{-7}$, which are about an order or magnitude larger than the errors we got in these tests. However, the coefficients in E&Y's power series for $d\theta(x)/dx$ were increasing, with the coefficient of the 4th order term being nearly 1 and almost 6 times as large as the coefficient of the 3rd term. It is possible that the coefficient of the 5th order term is of order 10, which could produce errors on the order that we saw in our tests.
Figure 15. Magnitude of maximum difference between our method and that of E&Y versus the top of the profile for tests run without the cap. These errors increase much more sharply than the errors increased in tests that included the cap. Here, the error even increased sharply when we lowered the top of the profile from 20 down to 15 scale heights above half light. Note how closely the Jupiter-sized and intermediate planet tests line up.
Figure 16. Magnitude of maximum difference between our method and that of E&Y versus the number of points per scale height in the refractivity profile. These curves are completely flat. They show that the errors do not depend on the spacing of points in the refractivity profile, at least down to 2 points per scale height.
Figure 17. Magnitude of maximum difference between our method and that of E&Y versus the number of points calculated for the internal interpolation functions (in points per scale height). These errors are flat between 125 and 50, but increase between 50 and 10. The Pluto-like curve shows a much less marked increase at the coarser spacing, but the trend is still noticeable.

4.3.3 TIMING

Each time we ran a test, we had Mathematica™ calculate the CPU time used in calculating the light curve, in order to evaluate the speed of our method, and verify that it is a faster alternative to calculating out an atmospheric model. The tests that we ran took on average about 10 minutes of CPU time to calculate the light curve, with a range from about 830 seconds down to about 50 seconds. The fastest tests were the ones that did not calculate as many points in the internal interpolation functions. Including the cap generally added about 25% to the calculation time. Using refractivity profiles with fewer points per scale height had little effect the calculation time for the model, but lowering the top of the profile increased the time considerably. The number of points at which we wished to calculate the light curve did not seem to be a major factor. Our tests generally had different numbers of points in them, and some of the tests with more points in the light curve ran faster than tests with fewer. These results are summarized in Figures 20-23.
Figure 18. Calculation time versus the top of the profile. It consistently takes longer to calculate the model as the top of the profile is raised.
Figure 19. Calculation time versus the spacing of points in the profile. There is no clear pattern to the calculation times. Sometimes it takes longer to calculate a light curve when the points are more closely spaced in the profile, and sometimes it does not.

These times are generally larger than we might like them to be, but they are certainly less than the time it takes to calculate many atmospheric models. Our calculation time could be greatly decreased by implementing our method directly in a programming language like C. One way to increase the speed without losing all the advantages of Mathematica™ would be to implement the computation-heavy portions in MathLink™, which is Mathematica's interface to C. It allows one to call C programs from within Mathematica™. If one were to do either of these, we would expect the calculation times to decrease by at least a factor of 10, and probably by more like a factor of 100.
Figure 20. Calculation time versus the number of points calculated for the internal interpolations (in points per scale height). When more points were calculated in the internal interpolation functions, it took longer to calculate the model.

Figure 21. Calculation time versus the number of points calculated in the light curve. Strangely, there is no noticeable correlation.
Chapter 5: Conclusions

5.1 Summation
We have put together a method for numerically calculating an occultation light curve from an arbitrary atmospheric model in significantly less time than many atmospheric models can be calculated. This method can be applied to a broad range of bodies, including the smaller ones like Pluto and Triton. We have tested and verified that our method does work, showing reasonable agreement with methods currently in use.

We think it likely that the errors we found were, in fact, due to approximations in the models we compared to, but we cannot rule out the possibility that there are real, systematic errors in our method. Nonetheless, our tests show that, if such errors are present, they are at a level much lower than the noise levels in recent occultation observations. To date, the best signal to noise per scale height levels have been on the order of $10^{-3}$. The largest differences between our model and the models we compared to (not counting tests where the top of the model was very low and/or we did not include the cap) were on the order of $10^{-4}$, with most of these differences being much smaller. However, one must keep in mind that the errors we saw were very smooth, systematic errors that spanned a region of many scale heights. Such errors can mimic properties of the atmosphere that we might want to fit for, so even errors that are very small in magnitude could potentially cause real errors in a least squares fit. One must keep this possibility in mind when using this method to analyze occultation data.

Our method seems to give the best results when the refractivity profile extends about 15 scale heights above the half light radius with only a few points per scale height, the cap is included, and about 50 points per scale height are calculated for the internal interpolation function. Under these conditions, we found no error larger than $10^{-4}$, and much of the error in that test may have been due to approximations in the model we were comparing to. Without understanding exactly what caused the errors in the exponential refraction tests, it difficult to give a reasonable prescription for predicting the size of the errors caused by our method as a function of the characteristics of the refractivity profile.

5.2 Future Work
Several tasks are left before this project can truly be considered finished. First, we need to further analyze the discrepancies in our exponential refraction tests to determine the exact cause of the smooth error curves we found in the tests with the higher order terms. Until
we fully understand the cause of these discrepancies, we cannot be confident of the accuracy of our method.

Once we do understand the what the errors in our method are when it is used in this fashion, we need to expand it to interpolate between values of parameters in the model. This capacity is currently built in to the method, but has not been tested. We need to run a series of tests, similar to the ones run in this study, to determine how the errors change as we interpolate between light curves.

Finally, we need to apply the full power of our method to real occultations and see what more we can learn. We can use modern atmospheric models to re-analyze data from previous occultations, and perhaps extract more information from them. This work has already begun. In her Doctoral Thesis, Olkin applied our method, with atmospheric models by Strobel et al (Strobel et al. 1996), to the atmosphere of Triton. Using our method and 2 other methods, she was able to determine that Triton's atmosphere was more isothermal that previously believed. She was also able to show that the surface pressure was significantly larger than the previously accepted value (Olkin 1996).
References


ANALYSIS OF STELLAR OCCULTATION DATA FOR PLANETARY ATMOSPHERES. I. MODEL FITTING, WITH APPLICATION TO PLUTO

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ABSTRACT

An analytic model for a stellar-occultation light curve has been developed for a small, spherically symmetric planetary atmosphere that includes thermal and molecular weight gradients in a region that overlies an extinction layer. This work applies to the thermal structure of the upper part of Pluto's atmosphere probed by current stellar occultation data, so the issue of whether the lower part should be modeled as an extinction layer or sharp thermal gradient is not addressed. The model can be described by two equivalent sets of parameters. One set specifies the occultation light curve in terms of signal levels, times, and time intervals. Consequently, it is the more suitable set to use for fitting the light curve. The other set specifies physical parameters of the planetary atmosphere. Equations are given for the transforming between the sets of parameters, including their errors and correlation coefficients. Detailed numerical calculations are presented for a benchmark case. In order to establish the formal errors in the model parameters expected for datasets of different quality, least-squares fitting tests are carried out on synthetic datasets with different noise levels. This model has also been fit to the KAO data from the 1988 June 9 stellar occultation by Pluto. For the case with an isothermal constraint, the fitted parameters agree with our previous isothermal analysis [Elliot et al., Icarus, 77, 148 (1989)]. Fits of these data that include a temperature gradient as a free parameter yield a temperature to molecular weight ratio \( T/\mu = (3.72 \pm 0.75) \, \text{K} \, \text{amu}^{-1} \) and a normalized gradient \((dT/dr)/T = (-4.9 \pm 7.0) \times 10^{-4} \, \text{km}^{-1} \) at \( r = 1250 \, \text{km} \). Interpretation of these results depends on the mean molecular weight of the atmosphere. The values are 60 \( \pm \) 12 K and \(-0.029 \pm 0.040 \, \text{K} \, \text{km}^{-1} \) for the limiting case of pure CH\(_4\) (\( \mu = 16.04 \)) and 104 \( \pm \) 21 K and \(-0.051 \pm 0.070 \, \text{K} \, \text{km}^{-1} \) for the limiting case of pure N\(_2\) (\( \mu = 28.01 \)). Our result is consistent with the isothermal prediction of the "methane-thermostat" model of Pluto's atmosphere [Yelle & Lunine, Nature, 339, 288 (1989)]. However, Pluto's atmosphere could be isothermal in this region at a lower temperature than the 106 K predicted by the model, if the radiative cooling occurs at a wavelength longer than the 7.8 \( \mu \text{m} \) band of CH\(_4\). A summary of our current knowledge of Pluto's atmosphere and related parameters is tabulated. The ambiguity between the haze and thermal-gradient possibilities for Pluto's lower atmosphere limits the accuracy with which we now know Pluto's surface radius and bulk density. If the "haze model" is correct, then Pluto's surface radius is less than 1181 km and its bulk density is greater than 1.88 g cm\(^{-3}\). On the other hand, if the "thermal-gradient model" is correct, then Pluto's surface radius would be 1206 \( \pm \) 11 km and its density would be 1.77 \( \pm \) 0.33 g cm\(^{-3}\).

1 INTRODUCTION

With the technique of stellar occultations we can probe the atmospheres of distant bodies with remarkable spatial resolution—just a few kilometers for a body at the distance of Pluto, for example. Analysis of stellar occultation light curves for atmospheric occultations has been accomplished with two approaches: (i) fitting a model to the light curve and (ii) numerical inversion. For each we assume that the atmosphere is in hydrostatic equilibrium. As previously used, model fitting yields a mean atmospheric scale height, and inversion yields scale height as a function of altitude. Neither approach requires knowledge of the refractivity or mean molecular weight of the gases comprising the atmosphere, although these are assumed not to vary over the altitude range of interest.

Following Baum & Code's (1953) development of a model for fitting the occultation light curve of a large planet with an isothermal atmosphere, Goldsmith (1963) compared such an isothermal model to a light curve produced by a large planet that had a thermal gradient. He concluded that a thermal gradient in the atmosphere could not be determined from the shape of the occultation light curve. This result was discussed by Wasserman & Veverka (1973). French et al. (1978) determined errors for the occultation light curve for a large planet with an isothermal atmosphere that contains photon noise. This model has been used to fit occultation curves that contain "spikes" (Elliot & Veverka 1976), with the caveat that errors in the fitted parameters are really unknown, since the spikes do not have the same statistical properties as photon noise.

In contrast to the occultation light curves of the Jovian planets and Mars, Pluto's stellar occultation curve is almost entirely devoid of spikes (Elliot et al. 1989), so that the modeling approach is likely to yield meaningful results, provided that one can establish the correct model atmosphere. Modeling techniques had to be extended beyond the large-planet case for analysis of the Pluto data because Pluto's scale...
height at the occultation level is nearly 5% of its radius. In the context of analyzing the Pluto occultation data, corrections for the small planet case have been developed in two ways: by numerical integration to find the refraction angle (Hubbard et al. 1988), and by applying a correction to the results of a large-planet, isothermal model (Elliot et al. 1989).

Modeling of Pluto's stellar occultation data is not without its complications, however. As can be seen from its stellar occultation light curve (see Fig. 6), Pluto's atmosphere is divided into two zones: an upper zone that is clear and a lower zone that is characterized by either (i) an extinction layer (Elliot et al. 1989), or (ii) a sharp thermal gradient (Eshleman 1989; Hubbard et al. 1990). For the upper zone—under the assumption that the dominant heating mechanism is solar absorption in the 3.3 \( \mu \)m CH\(_3\) band and the dominant cooling mechanism is radiation in the 7.8 \( \mu \)m CH\(_3\) band—Yelle & Lunine (1989) have established that the presence of even a small amount of CH\(_3\) would cause the atmosphere to be isothermal in this region, with a temperature of about 106 K. Although Yelle & Lunine's model requires a thermal gradient in the lower atmosphere in order to reach the surface temperature of Pluto (Hubbard et al. 1990), we shall refer to it as the “methane-thermostat” model (rather than the “thermal-gradient” model), in order to avoid confusion with the discussion of thermal gradients in the other parts of the atmosphere and to distinguish it from potential models in which emission by molecular species other than CH\(_3\) might dominate the atmospheric cooling. The choice between the extinction and thermal-gradient interpretations of the lower part of Pluto's occultation light curve remains unresolved, since it has not been established that the thermal gradient required by the methane-thermostat model would necessarily be great enough to explain the structure of the light curve.

So far, modeling of Pluto's occultation light curve has been done under the assumption that the upper part of its atmosphere is isothermal, and this analysis yields a temperature to molecular weight ratio of \( 4.2 \pm 0.4 \) K/amu; from this ratio, the temperature of a pure CH\(_3\) atmosphere would be \( 67 \pm 6 \) K, and that of a pure N\(_2\) atmosphere would be \( 117 \pm 11 \) K (Elliot et al. 1989). Combining the temperature-to-molecular-weight ratio from the occultation analysis with the temperature of 106 K from the methane-thermostat model, Yelle & Lunine (1989) find a mean molecular weight of \( 25 \pm 3 \). This would imply an atmosphere dominated by a heavier gas, such as N\(_2\), CO, or Ar (Hubbard et al. 1990).

Our ignorance of fundamental properties of Pluto's atmosphere—such as the identity of its major constituent and whether an extinction layer, a sharp thermal gradient, or both exist in its lower atmosphere—prompted us to extend our analytical techniques to learn more about Pluto's atmosphere from occultation data. For example, one might be able to test the methane-thermostat model by determining whether the atmosphere corresponding to the upper part of the occultation curve is indeed isothermal, as required by the model.

In this paper we examine the fundamentals of modeling stellar occultation data for a planetary atmosphere, with the goal of establishing whether thermal and compositional gradients can be determined for Pluto's atmosphere. As discussed above, whether the break in the KAO light curve (Elliot et al. 1989) corresponds to a sharp thermal gradient or extinction layer remains an open question. Hence we shall be focusing our attention on the upper part of the light curve, where the methane-thermostat model predicts that the atmosphere should be isothermal. For convenience, we model the lower part of the curve as an extinction layer, since this model fits the data well.

We assume that the atmosphere is in hydrostatic equilibrium. Our analytical procedure includes all small-planet effects that have been used previously for Pluto by Hubbard et al. (1988) and Elliot et al. (1989): (i) the variation of gravitational acceleration with radius, (ii) the concentration of stellar flux due to the “refraction shrinkage” of the planet's shadow, (iii) geometrical terms that arise when the local scale height is a significant fraction of the planet's radius, and (iv) the variation in the apparent stellar velocity perpendicular to the planetary limb. Our model also includes possible thermal and compositional gradients in the planetary atmosphere.

In order to limit the scope of this work, we have made several approximations suitable for Pluto: (i) the light received at any time comes from only one point on the planetary limb, (ii) the rotational acceleration in the atmosphere is small compared with gravity, (iii) the velocity of the planet's shadow relative to the observer is constant throughout the occultation, and (iv) the atmospheric structure is spherically symmetric. To facilitate the application of our work to other planets, however, we develop a framework that can be extended to cases for which these approximations do not apply.

Our method provides a prescription for calculating a model occultation light curve that uses power series rather than numerical integrations. The resulting expressions are more transparent to how model parameters affect the light curve and require less computing time than numerical integrations.

Following the formulation of our model and tests of its performance by fitting synthetic data, we fit it to the KAO Pluto light curve for the 1988 June 9 stellar occultation, including an upper-atmosphere thermal gradient as a free parameter. We then use these results to summarize our current understanding of Pluto's atmospheric structure and to establish constraints on Pluto's surface radius and bulk density.

2. OCCULTATION LIGHT CURVE FOR A SPHERICALLY SYMMETRIC PLANET

Since a principal goal of our work is to ascertain the effect of a thermal gradient on a stellar occultation light curve, we calculate the model light curve from first principles in order to avoid overlooking any effect of the same order as a temperature gradient. We shall restrict our derivation to those steps appropriate for the Pluto occultation of 1988 June, but we set up a framework that can be extended to more general cases in the future.

Using the geometric optics approximation, we refer to the diagram of Fig. 1, where monochromatic parallel light rays are incident on a planetary atmosphere from the left and then encounter a spherically symmetric planet. In the observer's plane (perpendicular to the incident light rays), a radial coordinate \( p \) has its origin at the point of intersection of this plane and the path of the light ray that would have passed through the center of the planet. The coordinate \( p \) is used in two ways. We write the observer's position in the plane as a function of time, \( p = \rho(t) \), which is determined from the geocentric planetary ephemeris and the motion of
Atmospheric absorption will diminish the flux by considering what happens to a bundle of light rays through the planetary encounter. The stellar flux, upon passing through the planetary atmosphere, will be diminished to the position in the observer's plane, the observer relative to the center of the Earth. We also refer to the position in the observer's plane, \( p = \rho (r) \), as the point of arrival for a light ray with an original path that made a closest approach \( r \) to the center of the planet.

We calculate the stellar flux in the plane of the observer by considering what happens to a bundle of light rays through their planetary encounter. The stellar flux, upon passing through the planetary atmosphere, will be diminished by three effects: (i) differential bending of the light rays, (ii) absorption by the atmosphere, and (iii) partial focusing of the light by the curvature of the planetary limb in the plane perpendicular to the path of the ray. An initial bundle of light rays that has a width \( dr \) before interaction with the planet will be expanded or concentrated into a width \( dp \) in the observer's plane, due to differential bending. The stellar flux will change by the factor \( \frac{dp}{dr} \), where we use the absolute value so that our expression will be valid for cases when the bending of the light rays is not a monotonic function of \( r \). Atmospheric absorption will diminish the flux by a factor \( \exp \left[ -\tau_{\text{abs}}(r) \right] \), where \( \tau_{\text{abs}}(r) \) is the optical depth along the path of the ray. Finally, the focusing by the planetary limb changes the flux by the ratio of the circumferences of the "circles of light" of radius \( r \) and \( \rho \). Hence, the flux in the observer's plane, \( \phi(r) \), is simply written as a product of three factors:

\[
\phi(r) = \frac{r}{\rho(r)} \left| \frac{dr}{dp} \right| \exp \left[ -\tau_{\text{abs}}(r) \right]. \tag{2.1}
\]

The singularity in \( \phi(r) \) for \( \rho(r) = 0 \) does not cause a problem for the analysis of the 1988 Pluto data, since none of the observations were made near the center of the shadow. For observations made near the shadow center, one can remove the singularity from the model by including effects of diffraction, the angular extent of the occulted source, and the oblateness of the planet (Elliot et al. 1976; 1977; 1984).

In order to evaluate the factors that appear in Eq. (2.1), we make some further assumptions. A light ray with closest approach distance \( r \) to the center of the planet is bent by the planet and then deviates from its original path by an angle \( \theta(r) \). This angle lies in the plane containing the center of the planet and the original path of the ray. It is negative for rays bent toward the center of the planet, and we assume it is a small angle for the purpose of trigonometric approximations. The ray then travels to the observer's plane, lying at a distance \( D \) from the center of the planet. Since the radial coordinate \( \rho \) would never be negative, we have the following relation between the intersection points of a light ray in the two planes of interest:

\[
\rho(r) = \left| r + D \theta(r) \right|. \tag{2.2}
\]

We now can take the derivative needed in Eq. (2.1)

\[
\frac{dr}{dp(r)} = \frac{1}{|1 + D (d\theta(r)/dr)|}. \tag{2.3}
\]

To proceed further, we must obtain expressions for \( \theta(r) \) and \( \tau_{\text{abs}}(r) \) in terms of the properties of the planetary atmosphere. Since our planet is spherically symmetric, its refraction, \( \nu(r') \), and linear absorption coefficient, \( \kappa(r') \), are functions only of the distance from the center of the planet, \( r' \) (see Fig. 1). Since we shall need to integrate refraction and absorption along the path of the ray, we define an \( x \) coordinate that lies along the path of the ray and has its origin at the closest approach of the ray to the center of the planet. We assume that the deviation of the ray from its original path within the atmosphere is small enough to be neglected, so that we have the following relation between \( x \) and \( r' \):

\[
r'^2 = r^2 + x^2. \tag{2.4}
\]

For Pluto, the effect of general relativistic bending of light can be neglected, so that the deviation of the light ray from its original path is due only to refractive gradients within the atmosphere. The refraction angle is given by the \( r \) derivative of the integral of the refraction along the path of the ray, which we convert to a form that will prove the most useful later (Dwight 1961, Sec. 69.3):

\[
\theta(r) = \frac{d}{dr} \int_0^r \nu(r', x) dx = \int_0^r \frac{\partial \nu(r', x)}{\partial r} dx = \int_0^r \frac{\partial r'}{\partial r} \frac{d\nu(r')}{dr'} dx = \int_0^r \frac{r'}{dr'} \frac{d\nu(r')}{dr} dx. \tag{2.5}
\]

Similarly, we find the line-of-sight optical depth by integrating the linear absorption coefficient along the path of the ray:

\[
\tau_{\text{abs}}(r) = \int_0^r \kappa(r', x) dx. \tag{2.6}
\]

In Eq. (2.1) we specified the flux as a function of the radius of closest approach to the center of the planet. To find the normalized stellar flux, \( \phi(\rho) \), in the observer's plane, we add up the flux from all values of \( r \) that would arrive at \( \rho \):

\[
\phi(\rho) = \sum_{\text{perpendicular limb points}} \phi(r). \tag{2.7}
\]

Near the limb of a large planet, or planets with extinction, the light from all but the nearest limb is diminished, and only one perpendicular limb point contributes to the flux in Eq. (2.7). For the center portion of an occultation curve for a
The flux recorded by the observer is an integral of $\Phi(p)$ over (i) the spectrum of the occulted star and wavelength dependent quantities in the light path, (ii) the angular distribution of source intensity, and (iii) the telescope area. For the Pluto analysis, the integrand would be constant—or at worst linear—over the range of the integration variables, so we can approximate the integrals as $\Phi(p)$ multiplied by a constant. We represent the result as the unocculted flux of the star, $s_{\text{u}}(t)$, where the indicated time variability refers to the unocculted stellar flux and does not, of course, include any time variability due to the occultation. Time variability in the unocculted stellar flux could arise from intrinsic variability of the star, changes in the extinction, telescope guiding errors, or drifts in instrumental sensitivity. Similarly, we denote the background signal as $s_{\text{b}}(t)$, where the time variability could arise from similar causes as that for the unocculted flux from the star. Having defined these quantities, the signal, $s(t)$ that would be recorded by the observer from a position $p(t)$, can be written as

$$s(t) = s_{\text{u}}(t)\Phi(p(t)) + s_{\text{b}}(t).$$

(2.8)

Finally, the data are integrated over a short time interval, so that our occultation dataset consists of a set of $N$ integrations of the signal, $\{s_i\}_{i=1}^{N}$, where the integrations need not be contiguous, nor of equal length. If the $i$th integration is centered on a time $t_i$, and has a length $\Delta t_i$, then the recorded signal for the $i$th integration, $s_i$, can be expressed as

$$s_i = \int_{t_i - \Delta t_i/2}^{t_i + \Delta t_i/2} s(t) dt.$$ 

(2.9)

In order to express the recorded signal, $\{s_i\}_{i=1}^{N}$, as an explicit function of parameters that involve the planetary atmosphere, we need an atmospheric model from which we can calculate $\Phi(r)$ and $\tau(r)$, we also need to specify how to find $p(t)$, $s_{\text{u}}(t)$, and $s_{\text{b}}(t)$. These tasks are carried out in the next three sections. We first specify an empirical model for the planet, then we calculate the flux in the observer's plane, and finally we combine these to find the occultation flux as a function of time.

### 3. EMPIRICAL MODEL FOR THE PLANETARY ATMOSPHERE

A stellar occultation light curve is shaped by refractivity gradients and extinction within the atmosphere of the occulting body. Therefore, to construct a model for the occultation light curve we require an atmospheric model for the occulting body that specifies the number density of the gas and the absorptive properties of the atmosphere. The most desirable approach would be to formulate a physical model for the atmosphere and then derive a light curve model that would be a function of the parameters of the physical model. Unfortunately our understanding of Pluto's atmosphere is insufficient at present to formulate a reliable physical model. As pointed out earlier, we lack such fundamental information as the identity of the major constituent of the atmosphere and knowledge of whether the lower part of the occultation light curve is dominated by the effect of a steep thermal gradient or an extinction layer. Hence we turn to an empirical atmospheric model.

Since a main objective of our work is to use stellar occultation data to ascertain the presence of thermal gradients, our empirical model includes a thermal gradient. A thermal gradient manifests itself in an occultation light curve as a gradient in scale height, so our model must also allow for a gradient in mean molecular weight, which could also cause a gradient in scale height. An obvious choice for the functional variation of temperature and molecular weight would be a linear expansion about a reference radius $r_0$, but such a formulation can lead to negative values of these quantities for large or small radii, depending on the sign of the linear coefficient. Another problem that arises with a linear formulation is that the resulting expression for the number density has an apparent singularity for a gradient of exactly zero. The singularity can be avoided by taking a limit instead of attempting a direct numerical evaluation, but this procedure introduces undesirable complexity in the computations.

An alternative functional form that does not produce negative values and is more suitable to the spherical geometry of our problem is a power law. We assume spherical symmetry and specify that the mean molecular weight is $\mu(r)$ at the reference radius $r_0$. We can write the mean molecular weight, $\mu(r)$ as a function of the radius $r$ from the center of the planet and the power index $a$ as follows:

$$\mu(r) = \mu_0 (r/r_0)^a.$$ 

(3.1)

For the values of $r_0$ and $a$ expected, $\mu(r)$ will be essentially a linear function over the range of interest, and the equation for the gradient of the mean molecular weight at $r_0$ is

$$\frac{d\mu(r)}{dr} \bigg|_{r=r_0} = -a \mu_0 \left( \frac{r}{r_0} \right)^{a-1}.$$ 

(3.2)

If the temperature of the atmosphere at $r_0$ is $T_0$, we write the temperature as a function of altitude as a power law with index $b$:

$$T(r) = T_0 \left( \frac{r}{r_0} \right)^b.$$ 

(3.3)

We have chosen opposite signs for the power indices $a$ and $b$ so that they both have the same sign in the ratio of temperature to molecular weight, since this ratio appears often in the subsequent derivation. The gradient of the temperature at $r_0$ is

$$\frac{dT(r)}{dr} \bigg|_{r=r_0} = \frac{bT_0}{r_0} \left( \frac{r}{r_0} \right)^{b-1}.$$ 

(3.4)

If, as a result of fitting our model to data, we find statistically significant values for the second derivatives of molecular weight or temperature, we should then become concerned about the bias introduced by our empirical model in the specification of the functional form for these quantities.

Once we allow temperature and molecular weight to become functions of radius, we must also be concerned with specifying the correct radial variation of the gravitational and centrifugal forces, so that improperly modeled radial variations in these quantities do not become aliased as variations of temperature and/or molecular weight. The $r^{-2}$ dependence for gravity has been included in previous modeling for Pluto (Hubbard et al. 1988; Elliot et al. 1989), but the centrifugal force arising from planetary rotation has not been included—on the grounds that the ratio of centrifugal force to gravitational force at the reference radius...
We can consolidate most of the constants in the perfect gas law for the pressure:

\[ p(r) = \frac{G M \mu}{r^2} \exp \left( \frac{-\omega^2 r^4}{G \mathcal{M}_p} \right) \]

Dividing the equation for pressure by the density gives the perfect gas law

\[ p(r) = n(r) kT(r) \]

We next write the relation between pressure, number density, and temperature at the reference radius:

\[ p(r) = \frac{G M \mu}{r^2} \frac{m_{\text{amu}}}{r_{\text{ref}}} n(r) \]

To allow the integration of the right-hand side of Eq. (3.5), we substitute the power law relations for mean molecular weight and temperature, Eqs. (3.1) and (3.3):

\[ dp(r) = \frac{G M \mu}{kT(r)} m_{\text{amu}} \frac{r}{r_{\text{ref}}} \exp \left( \frac{-\omega^2 r^4}{G \mathcal{M}_p} \right) \]

We can consolidate most of the constants in Eq. (3.8) by defining the quantity \( \lambda_\phi \) as the ratio of the magnitude of the gravitational potential energy (referred to 0 at \( r = 0 \)) to the pressure.

\[ \lambda_\phi(\mathcal{M}_p) = \frac{\mathcal{M}_p}{kT} \frac{m_{\text{amu}}}{r_{\text{ref}}} \]

where we introduce the quantity \( \lambda_\phi \) as the value of \( \lambda_\phi(r) \) at the reference radius:

\[ \lambda_\phi(r) = \frac{G M \mu}{kT(r)} m_{\text{amu}} \frac{r}{r_{\text{ref}}} \]

Similarly, we define the quantity \( \lambda_\omega \) as the ratio of the magnitude of the centrifugal potential energy (referred to 0 at \( r = 0 \)) to the pressure:

\[ \lambda_\omega(r) = \frac{\omega^2 r^4}{2kT(r)} \]

Using these, Eq. (3.8) for the pressure derivative becomes

\[ dp(r) = \frac{-\lambda_\phi(r)}{r} dr + \frac{2\lambda_\omega(r)}{r} dr \]

\[ p(r) = p_0 \exp \left[ \frac{\lambda_\phi}{1 + a + b} \left( \frac{r}{r_0} \right)^2 - \frac{2\lambda_\omega}{2 - (a + b)} \right] \]

Our next task is to find the number density. First, we define the number density at the reference radius to be \( n_0 \) and obtain its value from Eq. (3.6):

\[ n_0 = n(r_0) = \frac{p_0}{kT_0} \]

Combining Eqs. (3.6, 3.13, and 3.14), we obtain an equation for the number density

\[ n(r) = n_0 \left( \frac{r}{r_0} \right)^{-b} \exp \left[ \frac{\lambda_\phi}{1 + a + b} \left( \frac{r}{r_0} \right) - \frac{2\lambda_\omega}{2 - (a + b)} \right] \]

For bodies with "slowly rotating atmospheres" (Pluto, Triton, Titan, and Venus), the ratio of centrifugal force to gravitational force \( \left( \omega^2 r^4/G \mathcal{M}_p \right) \) is small enough to neglect the rotation term in Eq. (3.16) and still maintain adequately high precision in our model. In the interest of minimizing the complexity for the Pluto analysis carried out in this paper, we drop the rotation term here and are left with the following expression for number density:

\[ n(r) = n_0 \left( \frac{r}{r_0} \right)^{-b} \exp \left[ \frac{\lambda_\phi}{1 + a + b} \left( \frac{r}{r_0} \right) - \frac{2\lambda_\omega}{2 - (a + b)} \right] \]
Note that the final expression for the number density on the right-hand side of Eq. (3.17) reduces to that used by Elliott et al. (1989) and Hubbard et al. (1988) for the case of no thermal or molecular-weight gradients ($a = b = 0$). The refractivity of the atmosphere as a function of radius $v(r)$ will depend on what gases comprise the atmosphere and their number density. If $v_{\text{STP}}(r)$ is the refractivity of the atmospheric gas for standard conditions of temperature and pressure and $L$ denotes Loschmidt's number, then we have

$$v(r) = n(r)v_{\text{STP}}(r)/L.$$  

(3.18)

Specifying the radial dependence of $v_{\text{STP}}(r)$ would require some specific assumptions that would be interconnected with the radial dependence molecular weight [Eq. (3.1)]. Since these would lead us into much more detail than we can learn from the presently available Pluto stellar occultation data, especially considering the similarity of $v_{\text{STP}}$ for the gases most likely to be in Pluto's atmosphere, we assume that $v_{\text{STP}}(r)$ changes negligibly within the radial range of interest:

$$v_{\text{STP}}(r) = v_{\text{STP}}.$$  

We define the refractivity at the reference radius as $v_0$:

$$v_0 = v(r_0) = n_0 v_{\text{STP}}/L.$$  

(3.19)

Now we can write the equation for the refractivity for our empirical atmospheric model:

$$v(r) = v_0 (r/r_0)^{-k} \exp \left( \frac{\Delta A(r) - \Delta \rho_0}{1 + a + b} \right).$$  

(3.20)

Later we shall need expressions for the pressure and number density scale heights, and it is most convenient to obtain them here. The local pressure scale height $H_p(r)$ is given by

$$H_p(r) = -\frac{1}{\rho(r)} \frac{dp(r)}{dr} = -\frac{r}{\lambda_p(r) - 2\lambda_w(r)}.$$  

(3.21)

The number density scale height $H_n(r)$ is defined analogously to the pressure scale height. For constant $v_{\text{STP}}$, the refractivity scale height equals the number density scale height:

$$H_n(r) = -\frac{1}{n(r)} \frac{dn(r)}{dr} = -\frac{1}{\rho(r) dr} \frac{dp(r)}{dr} - \frac{dT(r)}{dr} = \frac{r}{\lambda_n(r) - 2\lambda_w(r) + b}.$$  

(3.22)

Note that the number-density scale height equals the pressure scale height for an isothermal atmosphere ($b = 0$), and it is smaller than the pressure scale height when the temperature is increasing with altitude ($b > 0$).

In order to use our model for fitting the Pluto light curve, we must allow the lower part of the model atmosphere to have either (i) a haze layer (Elliott et al. 1989) or (ii) a sharp thermal gradient (Eshleman 1989; Hubbard et al. 1990). As discussed earlier, the correct choice has not yet been established, so for our present purpose we shall use the haze model because it is easier to implement analytically. We define the haze layer with three parameters: (i) $r_1$, the radius of the upper boundary of the extinction layer, (ii) $\kappa_1$, the linear absorption coefficient of the haze at $r_1$, and (iii) $H_{\text{ext}}$, the scale height of the haze at $r_1$. Allowing the scale height of the haze to have the same radial dependence as gravity, we have the following equation for the linear absorption coefficient, $\kappa(r)$, of the haze:

$$\kappa(r) = \begin{cases} \kappa_1 \exp \left( \frac{r - r_1}{H_{\text{ext}}(r_1)} \right) & r > r_1, \\ 0 & r < r_1. \end{cases}$$  

(3.23)

This completes the specification of our empirical model atmosphere, which is summarized schematically in Fig. 2. The main results of this section are Eq. (3.20), which specifies the radial dependence of the refractivity, and Eq. (3.23), which specifies the radial dependence of the extinction.

4. STELLAR FLUX IN THE OBSERVER'S PLANE

In this section we calculate the flux received in the observer's plane given by Eq. (2.1) in terms of the empirical atmospheric model of Sec. 3. Hence equations in this and subsequent sections apply to our specific model of the atmosphere. We begin by finding the refraction angle for a light ray passing through the planetary atmosphere. Using the expression for the refractivity, $v(r)$, given by Eq. (3.20), we take the derivative of the refractivity required by Eq. (2.5) and express the result in terms of the energy ratios, $\Delta A_x(r)$ and $\Delta A_\rho(r)$:

$$\theta(r) = -v(r) \int_{r_1}^{r} \left( \frac{r'}{r} \right)^{k-1} \exp \left( \frac{\Delta A_x(r') - \Delta A_\rho(r')}{1 + a + b} \right) \frac{\Delta A_\rho(r') + b}{r'} dx.$$  

(4.1)

In the large planet limit (\(r \gg H_{\text{ext}}(r_1)\)), the integrand becomes a Gaussian, $\exp\left(-\frac{(x/r)^2}{2}\right)$, equivalent to that derived by Baum & Code (1953).

Our approach to the small-planet problem is to find power series approximations in terms of the parameter $\delta = 1/\Delta A_\rho(r)$. We shall find it useful to express $\delta$ in forms involving the pressure scale height due to gravity alone, $H_p(r)$, and the values of these quantities at the reference radius ($\Delta A_{\rho_0}$ and $H_{p0}$):
\[
\delta = \frac{1}{\lambda_k(r)} = \frac{1}{\lambda_{\phi}} \left( \frac{r}{r_0} \right)^{1+a+b} \\
H_{\phi\phi}(r) = \frac{H_{\phi\phi}(r)}{r_0} \left( \frac{r}{r_0} \right)^{1+a+b}.
\] (4.2)

We note that \( \delta \) becomes small in the large planet limit, but it cannot be set to zero wherever it occurs, since the planetary atmosphere will always have some finite pressure scale height that cannot be ignored on the time and distance scales of interest for the occultation light curve. Hence we have adopted the convention, for equations expressing final results, of using \( \delta \) in expressions where it is added to larger terms and can be set to zero in the large planet limit. However, we use \( \lambda \)'s and \( H \)'s in those parts of the expression where these quantities should be retained in the large planet limit.

To perform the integrations in this paper, we generally use the following steps; first, factor the large planet solution out of the integral; second, change variables so that the lead term in the exponential will be Gaussian, and factor out of the integral the resulting terms that do not depend on the variable of integration; third, expand the integrand as a power series in \( \delta \); fourth, perform the integration. The result of the integration will be a power series, with a leading term of \( \delta \).

Following these steps to carry out the integration in Eq. (4.1), we first define a new variable of integration, \( y \):

\[
y = (x/r)\sqrt{1/2\delta} \quad (4.3)
\]

The integral of Eq. (4.1), expressed in terms of \( y \), becomes

\[
\theta(r) = -v(r)\sqrt{2\lambda_k(r)} \int_{-\infty}^{\infty} \left( 1 + 2\delta y^2 \right)^{1/2} \exp\left[ \frac{(1 + 2\delta y^2 - (1 + a + b)^2/2}{1 + a + b}\delta \right)\times\left( 1 + 2\delta y^2 - (1 + a + b)^2/2 + \delta y \right) dy.
\] (4.4)

Expanding the integrand as a power series in \( \delta \), we obtain

\[
\theta(r) = -v(r)\sqrt{2\lambda_k(r)} \int_{-\infty}^{\infty} e^{-y^2} \times \left[ 1 + \frac{b - (3a + 2b)}{2} y^2 + \frac{(3a + 2b)}{2} y^2 + O(\delta^2) \right] dy.
\] (4.5)

After performing this integral, we find that \( \theta(r) \) can be expressed simply as

\[
\theta(r) = -\sqrt{2\pi\lambda_k(r)} v(r) A(\delta,a,b),
\] (4.6)

where \( A(\delta,a,b) \) is a power series in \( \delta \) that is given to fourth order in the Appendix, Eq. (A2). We note that \( A(\delta,a,b) \to 1 \) as \( \delta \to 0 \), so that we can set \( A(\delta,a,b) = 1 \) for the large planet case.

Since \( \theta(r) \) usually appears multiplied by \( D/r \), it is convenient to have an expression for this quantity in terms of the fundamental parameters of our model. Substituting Eqs. (3.20) into (4.6), we have

\[
\frac{D\theta(r)}{r} = -\frac{Dv_0}{r_0} \sqrt{\frac{2\pi\lambda_k}{r_0}} \left( \frac{r}{r_0} \right)^{1-\frac{3a+b}{2}} \exp\left[ -\frac{\lambda_k}{1+a+b} \right] \times\left( \frac{r}{r_0} \right)^{-(1+a+b)\delta} B(\delta,a,b).
\] (4.7)

Later, we shall need Eq. (4.7) evaluated at the reference radius, which is as follows:

\[
\frac{D\theta_k}{r_0} = -\frac{Dv_0}{r_0} \sqrt{\frac{2\pi\lambda_k}{r_0}} A(\delta,a,b).
\] (4.8)

We shall also need the derivative of the bending angle. We calculate this by taking the derivative of Eq. (4.6) and again using a power series \( B(\delta,a,b) \) that equals \( 1 \) in the limit \( \delta \to 0 \), Eq. (A6):

\[
\frac{d\theta(r)}{dr} = \sqrt{2\pi\lambda_k(r)} v(r) \frac{v(r)}{r} B(\delta,a,b).
\] (4.9)

Expanding Eq. (4.9) and including a factor \( D \) that will be needed later we write the equation for the derivative of the refraction angle in a form similar to Eq. (4.7):

\[
\frac{Dd\theta(r)}{dr} = -\sqrt{\frac{2\pi\lambda_k(r)}{r_0}} \left( \frac{r}{r_0} \right)^{1-\frac{3a+b}{2}} \exp\left[ -\frac{\lambda_k}{1+a+b} \right] \times\left( \frac{r}{r_0} \right)^{-(1+a+b)\delta} B(\delta,a,b).
\] (4.10)

The expression for Eq. (4.10) in terms of the refraction angle is

\[
D\frac{d\theta(r)}{dr} = -\frac{D\theta(r) \lambda_k(r)}{r} A(\delta,a,b).
\] (4.11)

The observed optical depth is related to the linear extinction coefficient by integrating along the path of the light ray through the atmosphere with closest approach \( r \). The integral is given in Eq. (2.6) and the linear absorption coefficient is given in Eq. (3.23). Combining the two, we obtain the following expression for \( \tau_{obs}(r) \):

\[
\tau_{obs}(r) = \int_{-\infty}^{x_1} \kappa_x \exp\left[ \frac{-r(r-1)}{H_{r_1}(r/r_1)} \right] dx,
\] (4.12)

where \( x_1 \) is the point on the \( x \) axis where the light ray crosses the top of the haze.

\[
x_1 = x(r_1) = \sqrt{r_1^2 - r^2}.
\] (4.13)

We perform this integration with the same procedure used for the refraction angle and its derivative. First, factor the leading terms out of the integral, so that the integrand equals \( 1 \) at \( x = 0 \), and rewrite the integral in \( x = \sqrt{r_1^2 - r^2} \)

\[
\tau_{obs}(r) = \kappa(r) \int_{-\infty}^{x_1} \exp\left[ \frac{-1}{H_{r_1} \sqrt{1 + (x/r_1)^2}} \right] dx.
\] (4.14)

Once again, we change variables so that the lead term is a Gaussian. Using the haze expansion parameter \( \delta = H_{r_1}/r_1 \), the new variable of integration is
\[ y = (x/r) \sqrt{1/2 \delta_x}. \]  

so that Eq. (4.14) becomes

\[ \tau_{\text{obs}}(r) = \kappa(r) \frac{r}{r_1} \sqrt{2 H_{\text{ref}} r} \times \int_{-\infty}^{\infty} \exp \left[ \frac{1}{\beta^2} \left( \frac{1}{\sqrt{1 + 2 \delta_x x^2}} - 1 \right) \right] dy. \]  

(4.16)

Still following the same procedure outlined for the \( \theta \) integral, we expand the integrand into a Gaussian, \( \exp[-y^2] \), multiplied by a power series in \( \delta_x \). For the \( \text{Pluto occultation} \), the light-curve flux becomes negligible well before mid occultation, implying that only the near limb. Thus, \( \tau_{\text{obs}}(r) \approx 1 \) for \( r > r_1 \) and \( \kappa(r) \approx 1 \).

\[ \tau_{\text{obs}}(r) = \kappa(r) \frac{r}{r_1} \sqrt{2 H_{\text{ref}} r} \times \exp \left[ \frac{1}{\beta^2} \left( \frac{r_1^2 - r^2}{2 H_{\text{ref}} r} \right) \right] \]  

(4.19)

where \( \kappa(r) \) is a power series, and \( C(\delta_x) \) is a function of \( \delta_x \). We expand Eq. (4.19) into a form that will be most useful for subsequent calculations:

\[ \tau_{\text{obs}}(r) = \kappa(r) \frac{r}{r_1} \sqrt{2 H_{\text{ref}} r} \times \exp \left[ \frac{1}{\beta^2} \left( \frac{r_1^2 - r^2}{2 H_{\text{ref}} r} \right) \right] \]  

(4.20)

We shall also need an expression for the optical depth in terms of the radius, \( r_2 \), at which the observed optical depth along the path of the starlight is unity. In this form we eliminate the constant \( \kappa_1 \) from the problem by setting the condition that the observed optical depth is 1 for \( r = r_2 \). For this form we shall define \( \delta_2 = \delta_x(r_2) = H_{\text{ref}} r_2 = r_2/r_1 \). For \( \tau_{\text{obs}}(r_2) = 1 \), we have

\[ \tau_{\text{obs}}(r_2) = \kappa(r) \frac{r}{r_1} \sqrt{2 H_{\text{ref}} r_2} \times \exp \left[ \frac{1}{\beta^2} \left( \frac{r_1^2 - r_2^2}{2 H_{\text{ref}} r_2} \right) \right] \]  

(4.21)

We can now calculate the refracted stellar flux for any radius \( r \) within the planetary atmosphere with Eq. (4.25) and find the corresponding observer radius in the observer's plane, \( \rho(r) \), with Eq. (4.23).

\[ \rho(r) = r + D \theta(r). \]  

(4.23)

Then, the ratio \( r/\rho(r) \), required by Eq. (2.1), is found by solving Eq. (4.23):

\[ \frac{r}{\rho(r)} = \frac{1}{1 + D \theta(r)/r}. \]  

(4.24)

In this model for Pluto we include the refracted flux from only the near limb. Thus, \( \zeta(r) = \phi(r) \), and the final equation for the flux is

\[ \phi(r) = \frac{\exp[-\tau_{\text{obs}}(r)]}{[1 + D \theta(r)/r][1 + D \theta(r)/dr]}. \]  

(4.25)

The numerator and the two bracketed terms in the denominator of the right-hand side of Eq. (4.25) are obtained from Eqs. (4.7), (4.10), and (4.20).
\[
\frac{\Delta \delta_r}{r_f} = \frac{1}{2} \left[ \frac{A_4(\delta_r, a, b)}{A_4(\delta_r, a, b)} - 1 + \sqrt{1 + \frac{2 - \frac{4}{f} A_4(\delta_r, a, b)}{2 \lambda f} \frac{A_4(\delta_r, a, b)}{2 \lambda f} \left( \frac{A_4(\delta_r, a, b)}{2 \lambda f} \right)^2} \right].
\]

We then solve Eq. (4.6) for \( \nu_f \) and substitute the expression for \( \Delta \delta_r / r_f \) from Eq. (4.27) to get an equation for \( \nu_f \) in terms of the radius and energy ratio at a specified flux level:

\[
v(r_f) = \frac{\lambda f}{A_4(\delta_r, a, b)} \left[ 1 - \frac{A_4(\delta_r, a, b)}{A_4(\delta_r, a, b)} \right] - \sqrt{1 + \frac{2 - \frac{4}{f} A_4(\delta_r, a, b)}{2 \lambda f} \frac{A_4(\delta_r, a, b)}{2 \lambda f} \left( \frac{A_4(\delta_r, a, b)}{2 \lambda f} \right)^2}.
\]

\( 2D \sqrt{2\lambda f} A(\delta_i) \)

Therefore, when parameterizing the clear atmosphere, we can specify either the reference refractivity \( \nu_0 \) or the reference flux level \( f \).

### 5. LIGHT-CURVE MODEL

We use the term "light-curve model" to mean a prescription of calculating the stellar flux recorded by the observer as a function of time. In Sec. 4 we calculated stellar flux in terms of the radial coordinate \( r \) of the closest approach of a light ray to the occulting planet, but for a light-curve model we must be able to calculate the stellar flux in terms of the radial coordinate \( r \) of the observer in the observer's plane. Specification of \( r \) as the dependent variable introduces only one extra step in calculating the stellar flux: we first find \( r(p) \), and then we calculate the flux with the equations in the previous section. Since our model has spherical symmetry, the angular coordinate paired with either \( r \) or \( p \) is of no consequence. The radii \( r \) and \( p \) are related by Eq. (4.23), and we solve this equation for \( r(p) \) with Newton's method—a straightforward process, since the function is monotonic.

To complete our light-curve model, we need to find the flux for a given time \( \phi(t) \). Although it would be possible to work in observer-plane coordinates, \( \phi(r) \), the time coordinate is that for which the data are known directly. Furthermore, the integration intervals are usually evenly spaced in time, so that interpolating functions of the model are easily formed. Calculating the model in terms of time adds another step: to find the observer's radial coordinate as a function of time, \( r = \rho(t) \). This can be done through astrometric calculations, in which occultation chords from several stations are used to find the center of the shadow in time and space (Millis et al. 1992). Defining the position of the chords in the observer's plane requires some preliminary information from the light curves from each station, such as the times for a given flux level. Usually the times of half-light are used, but the times for the 0.764 flux level were used by Mills et al. (1992) for the Pluto case, since several stations were not sufficiently within the shadow for the flux to drop to 1/2.

A good approximation for the KAO position within Pluto's shadow for the 1988 occultation is a straight line through the shadow in the observer's plane, with a constant velocity \( v \). If \( \rho_{\text{min}} \) is the radius at the closest approach of the observer to the center of the shadow and \( t_{\text{mid}} \) is the midtime of the occultation (when \( \rho = \rho_{\text{min}} \)), the equation for calculating the observer's radius \( \rho(t) \) is

\[
\rho(t) = \sqrt{\rho_{\text{min}}^2 + v^2(t - t_{\text{mid}})^2}.
\]

We emphasize that this specification for \( \rho(t) \) is an adequate approximation for the KAO light curve, but one can readily calculate \( \rho(t) \) with a more elaborate astrometric calculation for occultations for which nonlinear terms are significant.

Having written Eq. (5.1), we can now calculate a light curve in terms of a fundamental set of parameters, which we shall call the "atmospheric parameter set," summarized in the first column of Table 1. We have divided these parameters into four groups: (i) signal levels, (ii) geometry and data recording, (iii) clear atmosphere, and (iv) haze. From knowledge of all the parameters in this set, we can calculate an occultation light curve with equations given previously.

Now that we can calculate the model stellar flux as a function of time, we show an annotated example of such an occultation curve in Fig. 3. In anticipation of fitting this model to data, we seek an alternate set of parameters that describes the model light curve in terms of times and signal levels instead of atmospheric parameters. Of course this set of "data" parameters can be calculated from the "atmospheric" set. We have adopted the following criteria for selecting the data parameter set: they should (i) be well defined by the features in the data, (ii) have low correlations when fit to the data, (iii) be more readily comparable with other results and previously published work (such as half-light times for immersion and emersion), (iv) allow for individual fits to immersion or emersion, and (v) impose symmetry between immersion and emersion when the entire light curve is fit.

We have formulated a set of data parameters that satisfy these criteria, as illustrated in Fig. 3 and summarized in column 5 of Table 1. The parameters fall into four categories: (i) signal levels, (ii) times of events on the light curve, (iii) time scales of the refractive and extinction occultations, and (iv) parameters that affect the shape of the shoulders of the light curve.

The parameters specifying signal levels are the background level \( s_b \), the background slope \( s_s \), and the full scale level \( s_f \). The first two of these comprise a linear approximation to \( s_2(t) \) in Eq. (2.8). We use the midtime of the dataset \( t_{\text{mid}} \), rather than the midtime of the occultation, as the reference time for the background slope in order to minimize the correlation of this with other model parameters. This is the average of the time of the first data point \( t_1 \) and time of the last data point \( t_n \).

\[
t_{\text{mid}} = \frac{t_1 + t_n}{2},
\]

\[
s_f(t) = s_b + s_s(t - t_{\text{mid}}).
\]

The average full scale level is the sum of the star and background levels:

\[
s_f = s_b + s_s.
\]

Included in the choice of critical times are the immersion and emersion half-light times, \( t_{\text{in}} \) and \( t_{\text{em}} \), for the differential refraction occultation. The half-light times have been used
by previous models. We choose the half-light times of immersion and emersion rather than the midtime and interval to half-light in order to easily fit immersion and emersion separately. In order to write equations for the immersion and emersion times, we first find the radius at half-light $r_h$. This can be found by inverting Eq. (4.25) with $f = 1/2$. Then we find the half-light radius in the observer’s plane $r_o$, with Eq. (4.23). The values of $t_m$ and $t_e$ are given by the subsequent two solutions to Eq. (5.5), which we write by solving Eq. (5.1) for $t$, with the negative solution corresponding to the immersion time for a given $\rho$ and the positive solution corresponding to the emersion time:

$$t(\rho) = t_m \pm \sqrt{\rho^2 - \rho_{\text{min}}^2} / \rho.$$  

Two other light-curve parameters are the time interval between half-light and haze onset $T_{h1}$, and the time interval between half-light and unit optical depth $T_{h2}$. These parameters have been specified as intervals (rather than specific times) in order to impose symmetry between immersion and emersion. These time intervals are obtained from knowledge of $r_h$, $r_i$, and $r_e$ with Eqs. (4.23) and (5.5).

$$T_{h1} = |t(r_i) - t(r_h)|,$$

$$T_{h2} = |t(r_e) - t(r_h)|.$$  

In order to express all spatial coordinates as times or time intervals, scale $\rho_{\text{min}}$ by the velocity, and use the time interval

$$t_m = \rho_{\text{min}} / \rho.$$  

The time scale of the clear atmosphere occultation is determined by $d\rho / dt$ at half-light. Noting that $d\rho / dt = (d\rho / dp) (dp / dt)$, and that $dp / dt$ is the velocity of the planet perpendicular to the limb, we define the perpendicular velocity at $r_h, v_{\perp h}$:

$$\frac{dp}{dt} \bigg|_{r_h} = v_{\perp h} = \frac{\sqrt{\rho^2 - \rho_{\text{min}}^2}}{\rho_h}.$$  

---

**Fig. 3.** Data parameters for a stellar occultation light curve. Signal level is plotted vs time for a complete occultation light curve—both immersion and emersion. In this example, the stellar flux drops to the background level $s_b$ at mideclination. If no extinction were present, the light curve would follow the dashed line labeled “refraction only.” The times ($t_m$ and $t_e$) when the stellar flux has dropped to half of its unocculted value corresponds to the “half-light” radius on the planet. The times corresponding to the onset and optical depth 1 of the haze are also indicated.

<table>
<thead>
<tr>
<th>Atmospheric Parameter Set</th>
<th>Data Parameter Set</th>
</tr>
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<tbody>
<tr>
<td>Primary Member</td>
<td>Intermediate</td>
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<tr>
<td>Signal Levels</td>
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<td>background slope</td>
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<tr>
<td>$r_h$</td>
<td>star level</td>
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<td>Geometry and Data Recording</td>
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<td>$\Delta t$</td>
<td>integration time</td>
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<td>top of haze</td>
</tr>
<tr>
<td>$k_1$</td>
<td>linear absorption coefficient</td>
</tr>
</tbody>
</table>

---

**Table 1. Summary of parameter sets.**
The other factor in the derivative is
\[
\frac{d\phi}{dp} = -r_s \left( \frac{1 + 1 + 3\alpha + 5b \delta_h + O(\delta_h^2)}{2} \right). \tag{5.10}
\]

We define the equivalent isothermal scale height \(H_{\infty}\) as the scale height of an isothermal atmosphere with constant molecular weight that would produce a light curve with the same slope at half-light as the nonisothermal atmosphere. The relationship between \(H_h\) and \(H_{\infty}\) is obtained by setting \(a = b = 0\) in Eq. (5.11).

\[
\frac{1}{8H_h} \left( 1 + \frac{1 + 3\alpha + 5b \delta_h + \cdots}{2} \right) = \frac{1}{8H_{\infty}} \left( 1 + \frac{1}{2} \delta_h + \cdots \right), \tag{5.11}
\]

so that
\[
H_{\infty} = \frac{H_h}{1 + [(3\alpha + 5b)/2] \delta_h + \cdots}. \tag{5.12}
\]

This equation is more useful as a relation between the energy ratios:
\[\lambda_{v'} = \lambda_{v} (r_s) = \frac{r_s}{H_{\infty}} - \frac{3\alpha + 5b}{2} + O(\delta_h). \tag{5.13}\]

This illustrates how the shape of a light curve for an atmosphere with a varying scale height is similar to that for a constant (but different) scale height (Goldsmith 1963). In anticipation that \(H_{\infty}\) would be more robust in fitting than \(H_h\), we defined our time scale of the occultation \(T_H = \frac{H_{\infty}}{v_s}\), by the scaled equivalent isothermal scale height.

\[T_H = H_{\infty}/v_s. \tag{5.14}\]

Next we define an extinction scale height \(H_{\infty}\) at the radius \(r_s:\)
\[H_{\infty} = H_r (r_1) = H_{v'} (r_3/r_1)^2. \tag{5.15}\]

Analogous to the procedure for refraction scale height, we scale the extinction scale height by the reference perpendicular velocity to get a time scale of the haze \(T_{Hr} = \frac{H_{v'}}{v_s} = \frac{H_{v'}}{v_s}. \tag{5.16}\)

The shape of the shoulder is determined by the minimum observer radius and the exponent for temperature or molecular weight variation, \(b\) or \(a\). The parameters \(a\) and \(b\) are used in the data parameter set with the same definitions as in the atmospheric set.

This completes our specification of the data parameter set, given in column 5 of Table 1. The equations used for converting from the atmosphere set to the data parameter set appear in column 7 of Table 1.

### 6 QUANTITIES DERIVED FROM FITTED PARAMETERS

We shall be fitting our light curve for the data parameter set in Table 1, but for calculating quantities that describe physical properties of the occulting atmosphere, we need to specify procedures for calculating the atmospheric parameter set from them—the inverse of the equations developed in the previous section. In this procedure, we must also calculate errors and correlation coefficients for the atmospheric parameters.

We begin with the midtime of the event, the average of the immersion and emersion half-light times:
\[t_{mid} = (t_{im} + t_{em})/2. \tag{6.1}\]

The normalized stellar flux \(\phi(\rho)\) is a function of \(r_s, v_o, \lambda_{v'}, a, \delta_h\), and \(b\), as derived in Sec. 4. Often, the reference radius is specified to be half-light. However, we chose to specify our reference radius to be a value near half-light, comfortably above the haze. Since the radius of half-light will vary from occultation to occultation, we felt this was preferable. In addition, the lack of an error bar on the reference radius simplifies the propagation of errors in further calculations.

To convert from \(t_{im}, t_{em}\), and \(T_{Hr}\) to \(r_o, v_o\), and \(\lambda_{v'}\) it is first necessary to calculate the half-light quantities, \(r_s\) and \(\lambda_{v'}\).

\[r_s = H_{\infty}/v_s. \tag{6.2}\]

The problem of finding \(r_s\) and \(\lambda_{v'}\) has to be solved numerically, for which we used the method of successive substitution. The initial guess for the half-light radius is the large planet solution:
\[r_s = \rho_{\infty} + H_{\infty}. \tag{6.3}\]

where \(H_{\infty}\) is found by Eq. (5.14). In each succeeding iteration, the energy ratio for the \(i\)th iteration is found by Eq. (5.13).

\[\lambda_{v'}(i) = \frac{r_s(i)}{H_{\infty}} - \frac{3\alpha + 5b}{2}. \tag{6.4}\]

The expression \(D\delta_h\) for the \(i\)th iteration is found by applying Eq. (4.26). The new half-light radius is found from the bending angle and the specified \(\rho_o\):
\[r_s(i+1) = \rho_o - D\delta_h. \tag{6.5}\]

This iteration continues until \(|r_s(i+1) - r_s(i)|/r_s(i)\) is less than a specified precision. Once the bending angle at half-light is known, the refractivity at half-light \(v_s\) can be calculated using Eq. (4.28). The reference quantities are now
\[\lambda_{v'} = \lambda_{v'} (r_o/r_s)^{-1 + \alpha + b} \tag{6.6}\]

and
\[v_o = v_s \left( \frac{r_o}{r_s} \right)^{-\alpha} \exp (\lambda_{v'} - \lambda_{v'}). \tag{6.7}\]

With these, finding \(r_1\) and \(r_2\) is the standard calculation for finding \(r(t)\) first find \(\rho(\rho)\) with Eq. (5.2), then find \(r(\rho)\), by solving Eq. (4.23) with Newton's method. In the data parameter set, the fitted haze scale height is defined at \(r_2\); for the purpose of defining a quantity at a definite radius (that is independent of the occultation geometry) we convert this to the haze scale height at \(r_1\):
\[H_{s1} = T_{Hr} v_{s1} (r_1/r_2)^2. \tag{6.8}\]

Given \(r_1, r_2,\) and \(H_{s1},\) we can find \(k_1\) from Eq. (4.21). The preceding prescription for converting the data parameters to the atmospheric parameters is summarized in column 3 of Table 1.

To find the errors and correlations for the atmospheric set, we use the equation of error propagation from \(M\) correlated observables to \(M\) parameters (Clifford 1973). Here, the two sets of \(M\) parameters are treated as column vectors. For the dataset parameters, we define the column vector \(d,\)
and for the atmospheric parameters, we define the column vector \( \mathbf{a} \). The components of these vectors are the symbols in columns 1 and 5 of Table 1. The variance-covariance matrices are \( \mathbf{M}_a \) and \( \mathbf{M}_d \), so that \( \mathbf{M}_{ad} = \text{var} [d_i] \) and \( \mathbf{M}_{da} = \text{cov} [d_i, a_j] \). If \( B \) is the derivative matrix,

\[
B_{ij} = \frac{\partial a_j}{\partial d_i},
\]

then the variance-covariance matrix for the atmospheric parameters in terms of the fitted, light-curve parameters is given by the matrix equation

\[
\mathbf{M}_a = \mathbf{B}^T \mathbf{M}_d \mathbf{B},
\]

where \( \mathbf{B}^T \) denotes the transpose of \( B \).

From these new parameters and their errors, we can find quantities of interest at half-light: the gravitational acceleration \( g_0 \), the pressure scale height \( H_{p0} \), the ratio of temperature to molecular weight, the thermal and molecular weight gradients, and the number density and pressure:

\[
g_0 = \frac{G M}{r_0^2},
\]

\[
H_{p0} = \rho_0 / \gamma_0,
\]

\[
T_0 = \frac{G \mu}{k_0 \gamma_0},
\]

\[
\frac{1}{T} \frac{dT}{dr} \bigg|_{r=r_0} = b,
\]

\[
\frac{1}{\mu} \frac{d\mu}{dr} \bigg|_{r=r_0} = -\alpha.
\]

The number density at the reference radius, \( n_0 \), can be found from the refractivity at the reference radius:

\[
n_0 = L \left[ \frac{\mu_0}{\nu_{STP}} \right].
\]

Once again, still following the procedure used for \( \theta(r) \) and \( \tau_{\text{rms}}(r) \), we expand the integrand in a series in \( \delta \). As before, this integration results in a series in \( \delta \), which has a lead term of 1. The result of this is the column height, given here to second order in \( \delta \). Notice that, to first order in \( \delta \), the \( b \) dependence of \( \xi(r) \) enters only in the expression for \( n(r) \).

\[
\xi(r) = \frac{H_{p0} n(r)}{L} \left[ \left( \frac{r}{r_0} \right)^{1+b} - 1 \right] \exp \left( - \frac{b}{1+\alpha} \right) \left( 1 + \frac{b}{1+\alpha} \right) \left( 1 - \frac{b}{1+\alpha} \right) 
\]

\[
+ \left[ \frac{b}{1+\alpha} \right] \left[ \frac{b}{1+\alpha} \right] \left[ \frac{b}{1+\alpha} \right] \cdots 
\]

\[
(6.19)
\]

7. NUMERICAL IMPLEMENTATION OF THE LIGHT-CURVE MODEL

We have implemented our model in Mathematica\textsuperscript{\textregistered} version 1.2.2 (Wolfram 1988; Maeder 1990) and have carried out the numerical calculations on several Macintosh\textsuperscript{\textregistered} II series computers. This combination has 19 digits of precision for numerical calculations. For our purposes, the integrated signal [Eq. (2.9)] was well approximated by the value of the model at the middle of the interval, except for the integration interval containing the haze onset and the adjacent integration interval corresponding to a lower level in the atmosphere. For these two cases, the model was calculated by integrating the second degree interpolating polynomials. In the bin containing the onset of the haze, two abutting polynomials were used, one for the clear portion and one for the portion with the haze. The flux was evaluated at the haze onset, the bin boundary, and the point midway between those. For the other integration interval, one polynomial was used, with model calculations at the two bin boundaries and the bin midtime.

To illustrate the appearance of light curves from planets of different radius-to-scale-height ratios, we have displayed light curves for different ratios \( \lambda_0 \) in Fig. 4. The light curves have been positioned so that their half-light times and equivalent isothermal scale heights correspond. The most striking change is the higher level of the light curves corresponding to smaller planets, due to the focusing term. If the time axis is extended further, all the curves would eventually rise as times corresponding to the center of the planet were approached. However, the minimum of the light curve becomes lower for larger planets.

We show the effect of a temperature gradient on the shape of the light curve in Fig. 5, by calculating the model for different values of the temperature power index \( b \) [see Eq. (3.3)]. Note the different effect of large positive and negative values.

For the purpose of comparison with other numerical models, we present in Tables 2, 3(a), and 3(b) a benchmark case for a planet with a 4 similar to Pluto’s. The results of this calculation have been displayed to seven digits for different orders of the power series. Note that most of the gain from the power series is achieved with only the first term. We include the flux due to differential bending alone, \( \phi_{\text{gut}} \), this is the "cylindrical-planet" approximation (Elliot et al. 1989, see their Appendix).
8. COMPARISON WITH PREVIOUS WORK

The flux equation for a large planet with no extinction has been derived for an isothermal atmosphere (Baum & Code 1953) and for an atmosphere with a linear temperature gradient (Goldsmith 1963). These models are valid near the limb of the shadow, when the atmosphere is a thin shell around the planet, as is the case for Earth, Mars, Venus, and the Jovian planets. Mathematically, their assumptions are that $\lambda_0 \gg 1$ and that for any $r$ probed by the occultation, $|r - r_0| < r_0$. These models use a reference radius where the observed flux equals $1/2$—the "half-light" radius, which we
TABLE 2. Parameters for benchmark model calculations.

<table>
<thead>
<tr>
<th>Data Parameter</th>
<th>Model</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>background level, sb (ADU)</td>
<td></td>
<td>630.000</td>
</tr>
<tr>
<td>background slope, sb' (ADU s⁻¹)</td>
<td></td>
<td>0.000</td>
</tr>
<tr>
<td>full-scale level, sf (ADU)</td>
<td></td>
<td>3340.00</td>
</tr>
<tr>
<td>immersion half-light time, tₘ (s⁻¹)</td>
<td></td>
<td>53.900</td>
</tr>
<tr>
<td>emersion half-light time, tₑ (s⁻¹)</td>
<td></td>
<td>139.900</td>
</tr>
<tr>
<td>refraction scale interval, TₛIso (s)</td>
<td></td>
<td>4.700</td>
</tr>
<tr>
<td>exponent for molecular weight, a</td>
<td></td>
<td>0.000</td>
</tr>
<tr>
<td>haze scale interval, Tₛ₁ (s)</td>
<td></td>
<td>4.200</td>
</tr>
<tr>
<td>haze onset interval, Tₛ₂ (s)</td>
<td></td>
<td>7.000</td>
</tr>
<tr>
<td>haze reference interval, Tₛ3 (s)</td>
<td></td>
<td>2.300</td>
</tr>
<tr>
<td>minimum observer radius, Tₛmin (s)t</td>
<td></td>
<td>46.900</td>
</tr>
<tr>
<td>shadow velocity, U (km s⁻¹)</td>
<td></td>
<td>18.500</td>
</tr>
<tr>
<td>integration time, Δt (s)</td>
<td></td>
<td>0.200</td>
</tr>
</tbody>
</table>

For the large planet case, the focusing term is not included, and Eq. (2.1) states simply that \( \frac{df}{dp} = 2 \). Thus, at half-light, \( \frac{df}{dr} = 2 \) and the derivative of Goldsmith's flux with respect to the observer radius is

\[
\frac{df(r)}{dr} = \frac{1 + iG}{4Hₙ}. \quad (8.2)
\]

Our analog to Goldsmith's \( \phi \) is \( bδₙ \). We compare Eq. (8.3) with Eq. (5.10) for \( a = 0 \), in the large-planet limit where terms of order \( bδₙ \) are retained when we take the limit \( \deltaₙ \to 0 \). They clearly give the same result. Both formulations result in the relation, for the large-planet limit,

\[
Hₙ = H_{μₙ}(1 + \frac{G}{2}). \quad (8.4)
\]

This result differs from Goldsmith's Eq. (57), which is

\[
Hₙ = H_{μₙ}[1 + (3/2)G]. \quad (8.5)
\]

We can expand the first term on the right-hand side of Eq. (8.5) around \( G = 0 \) and \( \phi^{-1} = 2 \), and find to first order in \( G \) and \( \phi^{-1} - 2 \).
**TABLE 3(a). Benchmark model values ($\beta = 0.0$).**

<table>
<thead>
<tr>
<th>Time, t</th>
<th>Observer Radius, $\rho$</th>
<th>Planet Radius, $r$</th>
<th>Energy, $E$</th>
<th>Refractivity $r$</th>
<th>Optical Depth</th>
<th>Refractive Flux, $F_{\phi}$</th>
<th>Flux, $\phi$</th>
<th>Value, $\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30.</td>
<td>1511.487</td>
<td>1512.838</td>
<td>17.5660</td>
<td>0.029591351</td>
<td>0.</td>
<td>0.8947373</td>
<td>0.9856136</td>
<td>600.2060</td>
</tr>
<tr>
<td>31.</td>
<td>1515.777</td>
<td>1515.772</td>
<td>17.32772</td>
<td>0.029155387</td>
<td>0.</td>
<td>0.8958806</td>
<td>0.9859063</td>
<td>600.3612</td>
</tr>
<tr>
<td>32.</td>
<td>1436.727</td>
<td>1440.037</td>
<td>18.21464</td>
<td>0.07194477</td>
<td>0.</td>
<td>0.9595295</td>
<td>0.9617532</td>
<td>647.2703</td>
</tr>
<tr>
<td>33.</td>
<td>1459.839</td>
<td>1459.895</td>
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<td>0.070040023</td>
<td>0.</td>
<td>0.9604433</td>
<td>0.9625630</td>
<td>647.7091</td>
</tr>
<tr>
<td>34.</td>
<td>1364.144</td>
<td>1372.513</td>
<td>19.13934</td>
<td>0.07156491</td>
<td>0.</td>
<td>0.9631038</td>
<td>0.9627578</td>
<td>647.7176</td>
</tr>
<tr>
<td>35.</td>
<td>1294.124</td>
<td>1314.124</td>
<td>19.84785</td>
<td>0.114320023</td>
<td>0.</td>
<td>0.9677336</td>
<td>0.9692911</td>
<td>647.9203</td>
</tr>
<tr>
<td>36.</td>
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<td>1268.713</td>
<td>20.70524</td>
<td>0.8513323</td>
<td>0.</td>
<td>0.9552175</td>
<td>0.6154312</td>
<td>459.5637</td>
</tr>
<tr>
<td>37.</td>
<td>1267.368</td>
<td>1267.346</td>
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<td>0.8533872</td>
<td>0.</td>
<td>0.9594487</td>
<td>0.6170380</td>
<td>460.4346</td>
</tr>
<tr>
<td>38.</td>
<td>1163.475</td>
<td>1226.221</td>
<td>21.24944</td>
<td>1.456364</td>
<td>0.</td>
<td>0.9445394</td>
<td>0.4721428</td>
<td>381.9014</td>
</tr>
<tr>
<td>39.</td>
<td>1104.005</td>
<td>1213.435</td>
<td>21.66485</td>
<td>1.701497</td>
<td>3.668283</td>
<td>0.7726859</td>
<td>0.3378185</td>
<td>219.7171</td>
</tr>
<tr>
<td>40.</td>
<td>1049.120</td>
<td>1197.171</td>
<td>21.94226</td>
<td>2.912644</td>
<td>3.426357</td>
<td>2.020442</td>
<td>0.9655126</td>
<td>480.0227</td>
</tr>
<tr>
<td>41.</td>
<td>1000.236</td>
<td>1185.307</td>
<td>22.14219</td>
<td>3.628013</td>
<td>8.306413</td>
<td>3.372716</td>
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<td>216.5680</td>
</tr>
<tr>
<td>42.</td>
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<td>11.42810</td>
<td>4.777469</td>
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<tr>
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<td>1170.204</td>
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<td>6.115221</td>
<td>0.1372211</td>
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<tr>
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<td>1243.432</td>
<td>4.93433</td>
<td>4.314412</td>
<td>14.23291</td>
<td>6.169093</td>
<td>0.1714011</td>
<td>214.2735</td>
</tr>
</tbody>
</table>

**TABLE 3(b). Benchmark model values ($\beta = -0.6$).**

<table>
<thead>
<tr>
<th>Time, t</th>
<th>Observer Radius, $\rho$</th>
<th>Planet Radius, $r$</th>
<th>Energy, $E$</th>
<th>Refractivity $r$</th>
<th>Optical Depth</th>
<th>Refractive Flux, $F_{\phi}$</th>
<th>Flux, $\phi$</th>
<th>Value, $\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30.</td>
<td>1511.487</td>
<td>1512.336</td>
<td>20.82364</td>
<td>0.017117100</td>
<td>0.</td>
<td>0.9884429</td>
<td>0.9899949</td>
<td>662.0377</td>
</tr>
<tr>
<td>31.</td>
<td>1515.777</td>
<td>1515.360</td>
<td>20.85138</td>
<td>0.01828148</td>
<td>0.</td>
<td>0.9882659</td>
<td>0.9888403</td>
<td>661.9514</td>
</tr>
<tr>
<td>32.</td>
<td>1436.727</td>
<td>1439.092</td>
<td>21.24128</td>
<td>0.04717460</td>
<td>0.</td>
<td>0.9662749</td>
<td>0.9688501</td>
<td>660.9560</td>
</tr>
<tr>
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<td>1364.144</td>
<td>1370.731</td>
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<td>0.</td>
<td>0.9573457</td>
<td>0.9610970</td>
<td>619.2785</td>
</tr>
<tr>
<td>34.</td>
<td>1294.124</td>
<td>1311.183</td>
<td>22.04707</td>
<td>0.3356796</td>
<td>0.</td>
<td>0.9579662</td>
<td>0.9610970</td>
<td>619.2785</td>
</tr>
<tr>
<td>35.</td>
<td>1277.042</td>
<td>1267.726</td>
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<td>0.</td>
<td>0.9579662</td>
<td>0.9610970</td>
<td>619.2785</td>
</tr>
<tr>
<td>36.</td>
<td>1163.475</td>
<td>1226.619</td>
<td>22.36826</td>
<td>0.3532185</td>
<td>0.</td>
<td>0.9579662</td>
<td>0.9610970</td>
<td>619.2785</td>
</tr>
<tr>
<td>37.</td>
<td>1104.005</td>
<td>1213.267</td>
<td>22.66862</td>
<td>1.331927</td>
<td>0.</td>
<td>0.9544454</td>
<td>0.4714020</td>
<td>381.6996</td>
</tr>
<tr>
<td>38.</td>
<td>1049.120</td>
<td>1197.171</td>
<td>22.87283</td>
<td>1.372882</td>
<td>0.</td>
<td>0.9544454</td>
<td>0.4714020</td>
<td>381.6996</td>
</tr>
<tr>
<td>39.</td>
<td>1000.236</td>
<td>1185.307</td>
<td>23.14046</td>
<td>1.456364</td>
<td>0.</td>
<td>0.9544454</td>
<td>0.4714020</td>
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</tr>
<tr>
<td>40.</td>
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<td>0.</td>
<td>0.9544454</td>
<td>0.4714020</td>
<td>381.6996</td>
</tr>
<tr>
<td>41.</td>
<td>922.261</td>
<td>1170.204</td>
<td>23.14046</td>
<td>1.456364</td>
<td>0.</td>
<td>0.9544454</td>
<td>0.4714020</td>
<td>381.6996</td>
</tr>
</tbody>
</table>

**Note:** The table values are computed based on the given time points and the specified parameters in the document. The calculations involve the energy, refractivity, optical depth, and other related parameters to determine the flux and other values. The values are rounded to the nearest integer for simplicity. The units for energy and refractivity are not specified in the original text, but they are assumed to be consistent with the usual astronomical units.
TABLE 4. Scale height ratios.

<table>
<thead>
<tr>
<th>Pressure Scale Height, ( H_{p0} ) (km)</th>
<th>Scale-Height Gradient (linear model), ( G = (a + b + 2)H_{p0}/r_0 )</th>
<th>Equivalent Isothermal Scale Height, ( H_{iso} ) (km)*</th>
<th>Normalized Scale-Height Ratio, ( (H_{p0}/H_{iso} - 1)/G )</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>0.1</td>
<td>19.8</td>
<td>2.63</td>
</tr>
<tr>
<td>25</td>
<td>-0.1</td>
<td>33.9</td>
<td>2.63</td>
</tr>
<tr>
<td>25</td>
<td>0.3</td>
<td>14.4</td>
<td>2.45</td>
</tr>
</tbody>
</table>

*after Wasserman and Veverka (1973)

\[
(1/\phi - 1) \left[ 1 - \ln((1/\phi - 1)(1/G + \epsilon^{-1} + \cdots)) \right] = 1 + \epsilon G
\]

\[
= \left(\frac{1}{\phi} - 2\right) \left[ 1 - \frac{5}{2} G \right]. \tag{8.6}
\]

The second term on the right-hand side of Eq. (8.5) becomes \((1 - 5/2 \, G) \ln(\phi^{-1} - 1)\). Substituting these results into Eq. (8.5) we have, to first order in \( G \) and \( \phi^{-1} - 2 \):

\[
\frac{\Delta r}{H_{p0}} = \left(1 + \frac{5}{2} \frac{b}{\phi^2} \right)^{-1} \times \left[ \left(\frac{1}{\phi} - 2\right) + \ln(\phi^{-1} - 1) \right] + O\left(\left(\frac{1}{\phi} - 2\right)^2\right)
\]

\[
+ O(G^2). \tag{8.7}
\]

so that the correction from Goldsmith's formulation is \(5/2\), not \(3/2\), and agrees with our formulation. To determine this correction empirically, one can fit an isothermal occultation curve for a large planet (Baum & Code 1953) to a synthetic, nonisothermal curve generated by Goldsmith's (1963) equation for the flux. The empirical correction fraction, found by inverting Eq. (8.4), is \( (H_{p0}/H_{iso} - 1)/G \). Wasserman & Veverka (1973) performed such fits, and their results agree much better with a \(5/2\) \( G \) correction than with a \(3/2\) \( G \) correction. As we can see in the last column of Table 4, their results are close to our derived correction of 2.50.

9. Fits to a Synthetic Light Curve

Our next task is to fit the model described in the previous sections to synthetic data, in order to establish that the fitting is well behaved in the region of parameter values near those that apply to the KAO data. The synthetic data were generated by first calculating a model for the parameter values listed in the second column of Table 5, for midtimes ranging from 0.1 to 199.9 s (at 0.2 s intervals). Three

TABLE 5. Fits to synthetic data.

<table>
<thead>
<tr>
<th>Model Parameter</th>
<th>Model Value</th>
<th>Fit #1</th>
<th>Fit #2</th>
<th>Fit #3</th>
<th>Fit #4</th>
</tr>
</thead>
<tbody>
<tr>
<td>background level, ( s_0 ) (ADU)</td>
<td>630.000</td>
<td>630.23 ± 0.72</td>
<td>630.40 ± 0.68</td>
<td>635.8 ± 7.5</td>
<td>557 ± 79</td>
</tr>
<tr>
<td>background slope, ( s'_0 ) (ADU s^-1)</td>
<td>0.000</td>
<td>0.000 ± 0.0012</td>
<td>0.0005 ± 0.0012</td>
<td>0.015 ± 0.011</td>
<td>0.000</td>
</tr>
<tr>
<td>full-scale level, ( s_f ) (ADU)</td>
<td>3340.00</td>
<td>3338.96 ± 0.64</td>
<td>3339.08 ± 0.62</td>
<td>3334.3 ± 6.3</td>
<td>3363 ± 55</td>
</tr>
<tr>
<td>immersion half-light time, ( t_{im} ) (s*)</td>
<td>53.90</td>
<td>53.879 ± 0.019</td>
<td>53.872 ± 0.017</td>
<td>53.87 ± 0.17</td>
<td>52.6 ± 1.5</td>
</tr>
<tr>
<td>emission half-light time, ( t_{em} ) (s*)</td>
<td>139.90</td>
<td>139.911 ± 0.019</td>
<td>139.917 ± 0.017</td>
<td>139.79 ± 0.17</td>
<td>141.8 ± 1.5</td>
</tr>
<tr>
<td>refraction scale interval, ( T_{Hao} ) (s)</td>
<td>4.70</td>
<td>4.68 ± 0.05</td>
<td>4.69 ± 0.02</td>
<td>4.73 ± 0.18</td>
<td>4.28 ± 0.89</td>
</tr>
<tr>
<td>exponent for molecular weight, ( a )</td>
<td>0.000</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>exponent for temperature, ( b )</td>
<td>0.600</td>
<td>0.45 ± 0.26</td>
<td>0.63 ± 0.09</td>
<td>0.86 ± 0.89</td>
<td>0.000</td>
</tr>
<tr>
<td>haze scale interval, ( T_{h1} ) (s)</td>
<td>4.200</td>
<td>4.226 ± 0.021</td>
<td>4.232 ± 0.020</td>
<td>4.38 ± 0.19</td>
<td>5.4 ± 2.6</td>
</tr>
<tr>
<td>haze onset interval, ( T_{h2} ) (s)</td>
<td>7.000</td>
<td>7.031 ± 0.020</td>
<td>7.023 ± 0.028</td>
<td>6.76 ± 0.28</td>
<td>12.7 ± 3.3</td>
</tr>
<tr>
<td>haze reference interval, ( T_{href} ) (s)</td>
<td>2.300</td>
<td>2.27 ± 0.05</td>
<td>2.26 ± 0.04</td>
<td>3.00 ± 0.69</td>
<td>1.4 ± 1.4</td>
</tr>
<tr>
<td>minimum observer radius, ( T_{min} ) (s)†</td>
<td>46.900</td>
<td>47.01 ± 44.6</td>
<td>46.900</td>
<td>46.900</td>
<td>46.900</td>
</tr>
<tr>
<td>shadow velocity, ( u ) (km s^-1)</td>
<td>18.500</td>
<td>18.500</td>
<td>18.500</td>
<td>18.500</td>
<td>18.500</td>
</tr>
<tr>
<td>integration time, ( \Delta \tau ) (s)</td>
<td>0.200</td>
<td>0.200</td>
<td>0.200</td>
<td>0.200</td>
<td>0.200</td>
</tr>
</tbody>
</table>

*after 1988 June 9, 10:35:50 UTC
†When fixed this number is \( r_{min}/u = 865.69 \) km / 18.4875911 km s^-1

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point samples of Gaussian noise were then generated with the "ContinuousDistributions.m" package supplied with Mathematica™ (Wolfram 1988, version 1.2.2). These samples had rms variations of 2, 20, and 200, which represented noise that was approximately 0.1, 1.0 and 10.0 times that of the KAO light curve. Adding each of these noise samples to the model produced the desired synthetic light curves.

A fit to each of these light curves was obtained with standard least-squares procedures (Clifford 1973) that we implemented in Mathematica™. Our implementation allowed us to select any subset of the light curve for fitting and to specify for each model parameter whether it would be fixed or fit. We found it expedient to use numerical derivatives since they are easier to implement and the derivative step can be specified, allowing us to test the stability of a solution by using different step sizes. One problem with fitting data is the discontinuous derivative of our model at the top of the haze, and this can be particularly troublesome if the top of the haze corresponds to a boundary between integration bins. By limiting the size of the parameter changes to a fraction of that called for by the least-squares calculation, however, a fit could be achieved. None of the fits were weighted. A fit was considered converged when the parameters changed by no more than a few percent of their formal errors for several iterations.

The results of the fits are displayed in Table 5. For the low noise case (fit No. 1), we were able to fit for 11 free parameters. Note, however, that the formal error in the "distance of closest approach," \( (T_{min}) \), is a large fraction of its value. The two parameters \( T_{min} \) and \( b \) (power law index for the temperature) both affect the second-order shape of the light curve and have a correlation greater than 0.995 for the parameter set used. Fit No. 2 of Table 5 shows the results of a fit with \( T_{min} \) fixed. Note the substantially lower formal error in the parameter \( b \) in fit No. 2 than in fit No. 1. Fit No. 3 of Table 5 shows the results of a fit to data of about the same noise level as the KAO data, and fit No. 4 shows a fit to data ten times noisier. For the latter fit, we fixed more parameters in order to achieve convergence. Even so, with data this noisy the formal errors in the model parameters describing the haze are about equal to the values of the parameters themselves.

We draw two conclusions from our fits to synthetic data. First, we note that 22 of 28 (79%) of the fitted parameter values for the fits to the three different synthetic datasets (columns 4-6) agree with the model value of the parameter within the formal error, which is close to the average of 68% of the cases to be expected by chance. Secondly, the formal error for each fitted parameter increases approximately proportionally to the noise level in the data, which demonstrates that the linear least-squares approximation is valid for these parameter values and noise levels.

### 10. Fits to the KAO Pluto Light Curve

We now proceed to fit our model to the stellar occultation light curve recorded with the KAO for the 1988 June 9 occultation of the star P6 (Mink & Kleioma 1985; Bosch et al. 1986) by Pluto. As described by Elliot et al. (1989), these data were recorded as open-chip CCD frames, each exposed for 0.2 s with the SNAPSHOT camera (Dunham et al. 1985). Synthetic aperture photometry was used on the CCD frames to generate a light curve. Data over a 200 s interval used in this work were the same as that used by Elliot et al. (1989).

As a further test of our model fitting, we compared the results of our present model with those of Elliot et al. (1989), who fit the KAO data with the standard, large-planet model for the refractive flux, and then performed a first-order correction on the results for the small-planet case. To reproduce the large-planet results with the present model, we remove the small-planet corrections: (i) the \( r^{-2} \) dependence of gravity by setting \( a = -2 \), (ii) the temperature gradient by setting \( b = 0 \), (iii) the focusing effect by setting \( t_{em} = 10^6 \) s when fitting for \( t_{em} \) and setting \( t_{em} = -10^6 \) s when fitting for \( t_{em} \), (iv) the nonnormal limb incidence by setting \( T_{min} = 0 \) s. The remaining differences between the two models are (i) the haze is modeled slightly differently and (ii) the present model integrates the signal over the interval containing the top of the haze and the adjacent one that corresponds to the next level deeper into the atmosphere. The results of the fits are compared in Table 6. Except for the parameters \( \chi_{wo} \) and \( \chi_{w1} \), which are defined differently from the corresponding parameters \( T_{sn} \) and \( T_{w} \), the fitted parameters closely agree, as they should.

Next we fit the KAO data with the model-calculation and least-squares procedures described in the previous section, and the results are given in Table 7. The parameters were those of the data parameter set, summarized in Table 1. The KAO data do not have sufficiently high signal-to-noise ratio to fit for all these simultaneously. Even with the parameter \( b \) fixed at zero, the error in \( T_{em} \) (see fit No. 4 of Table 7) is greater than our knowledge of this quantity from the astrometric solution of Millis et al. (1992). Hence for all other fits this quantity was fixed. Similarly, there should be no slope in the light curve, since the sky and CCD background would have been removed, on average, by the synthetic aperture photometry. The "background" in our model would be the signal from Pluto and Charon. This would have drifted only if the extinction, instrumental gain, or seeing changed during the 200 s interval of the data. Apparently none of these effects were significant, since the fitted slope is consistent with zero. Since the background slope \( (s) \) and the minimum observer radius \( (T_{min}) \), in the signal set) are known to greater accuracy by other information and their fitted values are consistent with this information, these were fixed.

A fit with \( a \) as the free parameter instead of \( b \) (fit No. 3) gives similar results to that with \( a \) fixed and \( b \) free (fit No. 2). We consider any significant value of gradient to be most likely due to a thermal, rather than a molecular weight gradient, so we chose as our solution the fit to all the data, with \( b \) a free parameter as shown as fit No. 2 in Table 7. Its fitted value, \( -0.61 \pm 0.87 \), shows that Pluto's atmosphere is iso-thermal in this region within the precision of the model fit. The correlation coefficients between the fitted parameters are given in Table 8. Since the matrix of the correlation coefficients is symmetric, we used the upper and lower triangles of each matrix to display the coefficients of different fits. Fixed variables are represented with ellipses in these tables.

In order to check whether the atmospheric structure might be different in the regions probed by immersion and emersion, we fit these intervals of data separately. The light curve was divided into two parts at the integration interval midway between the immersion and emersion times determined from the fit to all the data. For the immersion fit, the emersion time was fixed at its value obtained from the fit to all the data, and the immersion time was fixed for the emersion fit.

Parameter values and their formal errors for the fits to the
immersion data are shown as fit No. 6 and fit No. 7 of Table 7, and the fitted parameters for emersion are shown in the next two columns. We note that all parameter values are consistent within their formal errors for the two fits, and the value of the temperature power law, $b$, is consistent with zero for the fits in which it was a free parameter. The mean of the parameter values for immersion and emersion are approximately equal to their fitted value for all the data. The correlation coefficients for the immersion and emersion fits are given in Table 8.

For further analysis we shall adopt the fit to all the data that used fixed values of the background slope and scaled minimum observer radius. This is fit No. 2 of Table 7. These values and errors, along with the correlation coefficients in Table 8 are all that is needed to use the transformation equations summarized in Table 1 to calculate the atmospheric parameter set. In calculating derivatives we use numerical derivatives with a step size 0.005 times the formal error of the parameter. The KAO data are plotted as points in Fig. 6, and our adopted model is plotted as a solid line. The residuals from the model fit are also plotted in the lower part of the figure.

### Table 6. Fitted signal parameters for the KAO data for the large planet limit.

<table>
<thead>
<tr>
<th>Model Parameter</th>
<th>Immersion, Elliot et al. (1989)</th>
<th>Immersion, Present Work</th>
<th>Emersion, Elliot et al. (1989)</th>
<th>Emersion, Present Work</th>
</tr>
</thead>
<tbody>
<tr>
<td>Signal levels (per integration interval, $\Delta t$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pluto star</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Isothermal atmosphere</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>time of half light ($t_{1/2}$)</td>
<td>$9.178_{-0.004}^{+0.004}$</td>
<td>$9.178_{-0.004}^{+0.004}$</td>
<td>$9.178_{-0.004}^{+0.004}$</td>
<td>$9.178_{-0.004}^{+0.004}$</td>
</tr>
<tr>
<td>&quot;scale height&quot; ($H_{1/2}$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Extinction layer</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>fit information</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>degrees of freedom</td>
<td>409</td>
<td>409</td>
<td>409</td>
<td>409</td>
</tr>
<tr>
<td>sum of squared residuals</td>
<td>183735</td>
<td>183735</td>
<td>183735</td>
<td>183735</td>
</tr>
<tr>
<td>rms residual per degree of freedom (ADU)</td>
<td>19,305</td>
<td>19,305</td>
<td>19,305</td>
<td>19,305</td>
</tr>
</tbody>
</table>

---

### Table 7. Fitted signal parameters for the KAO data.

<table>
<thead>
<tr>
<th>Model Parameter</th>
<th>Fit #1 All Data</th>
<th>Fit #2 All Data</th>
<th>Fit #3 All Data</th>
<th>Fit #4 All Data</th>
<th>Fit #5 Immersion</th>
<th>Fit #6 Immersion</th>
<th>Fit #7 Emersion</th>
<th>Fit #8 Emersion</th>
</tr>
</thead>
<tbody>
<tr>
<td>background level, $P_0$ (ADU)</td>
<td>$613.3 \pm 6.8$</td>
<td>$613 \pm 6.8$</td>
<td>$613.3 \pm 6.8$</td>
<td>$613.3 \pm 6.8$</td>
<td>$613.3 \pm 6.8$</td>
<td>$613.3 \pm 6.8$</td>
<td>$613.3 \pm 6.8$</td>
<td>$613.3 \pm 6.8$</td>
</tr>
<tr>
<td>background slope, $P_0$ (ADU s$^{-1}$)</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>full-scale level, $P_0$ (ADU)</td>
<td>$334.6 \pm 5.3$</td>
<td>$334.6 \pm 5.3$</td>
<td>$334.6 \pm 5.3$</td>
<td>$334.6 \pm 5.3$</td>
<td>$334.6 \pm 5.3$</td>
<td>$334.6 \pm 5.3$</td>
<td>$334.6 \pm 5.3$</td>
<td>$334.6 \pm 5.3$</td>
</tr>
<tr>
<td>immersion half-light time, $t_{1/2}$ (s)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>emersion half-light time, $t_{1/2}$ (s)</td>
<td>$130.9 \pm 0.15$</td>
<td>$130.9 \pm 0.15$</td>
<td>$130.9 \pm 0.15$</td>
<td>$130.9 \pm 0.15$</td>
<td>$130.9 \pm 0.15$</td>
<td>$130.9 \pm 0.15$</td>
<td>$130.9 \pm 0.15$</td>
<td>$130.9 \pm 0.15$</td>
</tr>
<tr>
<td>refraction scale interval, $T_{1/2}$ (s)</td>
<td>4.61 ± 0.09</td>
<td>4.71 ± 0.18</td>
<td>4.71 ± 0.19</td>
<td>4.71 ± 0.17</td>
<td>4.71 ± 0.17</td>
<td>4.71 ± 0.17</td>
<td>4.71 ± 0.17</td>
<td>4.71 ± 0.17</td>
</tr>
<tr>
<td>exponents for molecular weight, $a$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>exponents for temperature, $b$</td>
<td>-0.66 ± 0.86</td>
<td>-0.66 ± 0.86</td>
<td>-0.66 ± 0.86</td>
<td>-0.66 ± 0.86</td>
<td>-0.66 ± 0.86</td>
<td>-0.66 ± 0.86</td>
<td>-0.66 ± 0.86</td>
<td>-0.66 ± 0.86</td>
</tr>
<tr>
<td>haze scale interval, $T_{1/2}$ (s)</td>
<td>1.34 ± 0.17</td>
<td>1.34 ± 0.17</td>
<td>1.34 ± 0.17</td>
<td>1.34 ± 0.17</td>
<td>1.34 ± 0.17</td>
<td>1.34 ± 0.17</td>
<td>1.34 ± 0.17</td>
<td>1.34 ± 0.17</td>
</tr>
<tr>
<td>haze reference interval, $T_{1/2}$ (s)</td>
<td>1.06 ± 0.23</td>
<td>1.06 ± 0.23</td>
<td>1.06 ± 0.23</td>
<td>1.06 ± 0.23</td>
<td>1.06 ± 0.23</td>
<td>1.06 ± 0.23</td>
<td>1.06 ± 0.23</td>
<td>1.06 ± 0.23</td>
</tr>
<tr>
<td>minimum observer radius, $R_{\min}$ (s)</td>
<td>1.08 ± 0.28</td>
<td>1.08 ± 0.28</td>
<td>1.08 ± 0.28</td>
<td>1.08 ± 0.28</td>
<td>1.08 ± 0.28</td>
<td>1.08 ± 0.28</td>
<td>1.08 ± 0.28</td>
<td>1.08 ± 0.28</td>
</tr>
<tr>
<td>degrees of freedom</td>
<td>59</td>
<td>59</td>
<td>59</td>
<td>59</td>
<td>59</td>
<td>59</td>
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<td>59</td>
</tr>
<tr>
<td>sum of squared residuals</td>
<td>373803</td>
<td>373803</td>
<td>373803</td>
<td>373803</td>
<td>373803</td>
<td>373803</td>
<td>373803</td>
<td>373803</td>
</tr>
<tr>
<td>rms residual per degree of freedom (ADU)</td>
<td>21,408</td>
<td>21,408</td>
<td>21,408</td>
<td>21,408</td>
<td>21,408</td>
<td>21,408</td>
<td>21,408</td>
<td>21,408</td>
</tr>
</tbody>
</table>

---

*after 1988 June 9, 10:35:50 UTC

When fitted this number is $P_{\min} = 865.69$ km / 18.475911 km s$^{-1}$
### Table 8. Correlation coefficients for fits to the KAO data.

#### (a) All Data, Fits #1 and #2 of Table 7

<table>
<thead>
<tr>
<th>Model Parameter</th>
<th>$s_b$</th>
<th>$s_f$</th>
<th>$t_{em}$</th>
<th>$t_{em}$</th>
<th>$T_{H_{1400}}$</th>
<th>$a$</th>
<th>$T_{h,1}$</th>
<th>$T_{h,2}$</th>
<th>$T_{H_{24}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_b$</td>
<td>1.00</td>
<td>-0.03</td>
<td>-0.30</td>
<td>0.30</td>
<td>-0.16</td>
<td>...</td>
<td>0.22</td>
<td>-0.02</td>
<td>-0.49</td>
</tr>
<tr>
<td>$s_f$</td>
<td>0.01</td>
<td>1.00</td>
<td>-0.10</td>
<td>0.10</td>
<td>0.34</td>
<td>...</td>
<td>0.07</td>
<td>0.00</td>
<td>0.04</td>
</tr>
<tr>
<td>$t_{em}$</td>
<td>-0.30</td>
<td>-0.28</td>
<td>1.00</td>
<td>-0.81</td>
<td>0.64</td>
<td>...</td>
<td>-0.85</td>
<td>-0.66</td>
<td>0.28</td>
</tr>
<tr>
<td>$t_{em}$</td>
<td>0.30</td>
<td>0.28</td>
<td>-0.84</td>
<td>1.00</td>
<td>-0.64</td>
<td>...</td>
<td>0.85</td>
<td>0.66</td>
<td>-0.28</td>
</tr>
<tr>
<td>$T_{H_{1400}}$</td>
<td>-0.13</td>
<td>-0.29</td>
<td>-0.65</td>
<td>1.00</td>
<td></td>
<td>...</td>
<td>-0.62</td>
<td>-0.56</td>
<td>0.22</td>
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<tr>
<td>$b$</td>
<td>0.05</td>
<td>0.52</td>
<td>-0.38</td>
<td>0.38</td>
<td>-0.83</td>
<td>1.00</td>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_{h,1}$</td>
<td>0.23</td>
<td>0.27</td>
<td>-0.88</td>
<td>0.89</td>
<td>-0.65</td>
<td>0.39</td>
<td>1.00</td>
<td>0.56</td>
<td>-0.10</td>
</tr>
<tr>
<td>$T_{h,2}$</td>
<td>0.02</td>
<td>0.25</td>
<td>-0.72</td>
<td>0.73</td>
<td>-0.66</td>
<td>0.46</td>
<td>0.65</td>
<td>1.00</td>
<td>-0.38</td>
</tr>
<tr>
<td>$T_{H_{24}}$</td>
<td>-0.49</td>
<td>-0.08</td>
<td>0.34</td>
<td>-0.34</td>
<td>0.31</td>
<td>-0.23</td>
<td>-0.19</td>
<td>-0.44</td>
<td>1.00</td>
</tr>
</tbody>
</table>

#### (b) All Data, Fits #3 and #4 of Table 7

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<tr>
<th>Model Parameter</th>
<th>$s_b$</th>
<th>$s_f$</th>
<th>$t_{em}$</th>
<th>$t_{em}$</th>
<th>$T_{H_{1400}}$</th>
<th>$a$</th>
<th>$T_{h,1}$</th>
<th>$T_{h,2}$</th>
<th>$T_{H_{24}}$</th>
<th>$T_{min}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_b$</td>
<td>1.00</td>
<td>...</td>
<td>0.01</td>
<td>-0.30</td>
<td>0.30</td>
<td>-0.13</td>
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<td>$t_{em}$</td>
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<td>0.12</td>
<td>-0.31</td>
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<td>-0.85</td>
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<td>-0.75</td>
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<td>0.06</td>
<td>0.29</td>
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<td>0.38</td>
<td>0.75</td>
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<tr>
<td>$T_{H_{1400}}$</td>
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<td>-0.67</td>
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<td>...</td>
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#### (c) Immersion Data, Fits #5 and #6 of Table 7

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<th>$s_b$</th>
<th>$s_f$</th>
<th>$t_{em}$</th>
<th>$t_{em}$</th>
<th>$T_{H_{1400}}$</th>
<th>$a$</th>
<th>$T_{h,1}$</th>
<th>$T_{h,2}$</th>
<th>$T_{H_{24}}$</th>
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<td>-0.31</td>
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<td>-0.50</td>
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<td>...</td>
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<td>1.00</td>
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<td>-0.88</td>
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<td>...</td>
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<td>$b$</td>
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<td>0.51</td>
<td>-0.38</td>
<td>-0.81</td>
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<td>...</td>
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<td>$T_{h,1}$</td>
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<td>-0.04</td>
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</tr>
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<td>-0.76</td>
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<td>0.44</td>
<td>0.64</td>
<td>1.00</td>
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</tr>
<tr>
<td>$T_{H_{24}}$</td>
<td>-0.51</td>
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#### (d) Emersion Data, Fits #7 and #8 of Table 7

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<th>$s_f$</th>
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<th>$T_{H_{1400}}$</th>
<th>$a$</th>
<th>$T_{h,1}$</th>
<th>$T_{h,2}$</th>
<th>$T_{H_{24}}$</th>
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<td>$s_b$</td>
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<td>0.33</td>
<td>-0.17</td>
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<td>-0.04</td>
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</tr>
<tr>
<td>$s_f$</td>
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<td>1.00</td>
<td>0.09</td>
<td>0.34</td>
<td>0.08</td>
<td>-0.01</td>
<td>0.05</td>
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</tr>
<tr>
<td>$t_{em}$</td>
<td>0.34</td>
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<td>1.00</td>
<td>-0.67</td>
<td>0.97</td>
<td>0.70</td>
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</tr>
<tr>
<td>$T_{H_{1400}}$</td>
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<td>-0.65</td>
<td>1.00</td>
<td>-0.66</td>
<td>-0.57</td>
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<td>...</td>
</tr>
<tr>
<td>$b$</td>
<td>0.07</td>
<td>0.54</td>
<td>0.40</td>
<td>-0.87</td>
<td>1.00</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$T_{h,1}$</td>
<td>0.32</td>
<td>0.29</td>
<td>0.99</td>
<td>-0.65</td>
<td>0.41</td>
<td>1.00</td>
<td>0.66</td>
<td>-0.30</td>
</tr>
<tr>
<td>$T_{h,2}$</td>
<td>0.01</td>
<td>0.25</td>
<td>0.76</td>
<td>-0.66</td>
<td>0.47</td>
<td>0.75</td>
<td>1.00</td>
<td>-0.38</td>
</tr>
<tr>
<td>$T_{H_{24}}$</td>
<td>-0.49</td>
<td>-0.13</td>
<td>-0.49</td>
<td>0.42</td>
<td>-0.31</td>
<td>-0.45</td>
<td>-0.49</td>
<td>1.00</td>
</tr>
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</table>
major uncertainty is the composition of the atmosphere. Accurate gradient models for Pluto's lower atmosphere, another includes parameters that apply to Pluto's atmosphere if the this work); (v) "steep thermal gradient, no haze," which occurs where the light curve has the largest slope and they represent unmodeled density variations in Pluto's atmosphere.

In addition to the uncertainty between the haze and thermal gradient models for Pluto's lower atmosphere, another major uncertainty is the composition of the atmosphere. Accordingly, the second column of Table 9 gives values of quantities that are independent of an assumed atmospheric composition, and each of the following four columns gives parameters for an assumed composition. The first of these is 100% CH₄, which could be the limiting case if radiative cooling of the atmosphere is not the 7.8 μm band of CH₄, but through a band of longer wavelength—such as the 13.7 μm band of C₂H₂, which dominates the radiative cooling in Titan's atmosphere (Lellouch et al. 1990). The next three columns give limiting cases that would apply if the atmosphere were in radiative equilibrium dominated by the 7.8 μm band of CH₄ (Yelle & Lunine 1989; Hubbard et al. 1990). These are cases that would give a mean molecular weight near 28: 100% N₂, 100% CO, and 50% CH₄-50% Ar.

Now we describe the entries in Table 9 in more detail, working our way down from the top. Determination of the Pluto–Charon mass ratio by measuring the "wobble" of the system center of light is in progress, but the result is not yet available (Wasserman et al. 1988). Beletci et al. (1989) determined the mass of the Pluto–Charon system to be $(1.35 \pm 0.07) \times 10^{19} M_r^{-1}$ by measuring the semimajor axis of the Pluto–Charon orbit and combining this with the well-determined orbital period (Tholen & Buie 1990). Using this value for the system mass, one could infer a mass for Pluto alone by calculating the relative volumes of Pluto and Charon from the mutual event radii (Tholen & Buie 1990) and adding some additional uncertainty corresponding to the unknown relative densities for the two bodies. However, we cannot accept the radius of Pluto and its error determined from the mutual events as the surface radius, since this radius refers to the visible disk of Pluto, which may or may not be the surface. Furthermore, the mutual-event radius for Pluto's visible disk is at odds with that of the visible-disk radius determined by combining stellar occultation chords from the 1988 June 9 event (Millis et al. 1992).

We see two likely sources for the radius inconsistency, both connected with the mutual-event radius: (i) unmodeled limb darkening of Pluto that would cause the mutual event radius to be underestimated, and (ii) an unidentified systematic error (or underestimate of the error) in the semimajor axis of the system, which would propagate proportionally into the radius. Hence, for our present purpose, we use Beletic et al.'s (1989) system mass, but we increase its error from $0.075 \times 10^{19} M_r^{-1}$ to $0.226 \times 10^{19} M_r^{-1}$, a value set by requiring that the 1σ error on the mutual event radius for Pluto should include the stellar occultation radius for the visible disk of Pluto. This corresponds to a semimajor-axis error of $\pm 1100$ km (cf. 19 640 ± 320 km from Beletic et al.). Next, we derive an expression for Pluto's mass, in terms of the Pluto–Charon semimajor axis, and Charon's mass, calculated under the assumption that its density is $2.0 \pm 1.0$ g cm$^{-3}$ and its radius is 593 ± 13 km (Tholen & Buie 1990), with additional uncertainty from semimajor-axis error. The resulting Pluto mass is $M_p = (1.30 \pm 0.24) \times 10^{28}$ g, as has been entered in Table 9.

The refractivities of CH₄, N₂, CO, and Ar at standard temperature and pressure ($T_\text{sp}$) were calculated by integrating the refractivity of each gas over the quantum efficiency function for our CCD chip (Dunham et al. 1985) and the photon emission function for a black body at 4900 K, which would have the same BVR colors as the occulted star, P8 (Bosh et al. 1986).

In the next group of parameters, "specific to the KAO light curve," we find the background level, star level, and midtime through conversion of the fitted data parameter set given in the second column of Table 7 to the atmospheric parameter set. The resulting correlation coefficients for the new parameters are given in Table 10. In this conversion we have chosen a fixed reference radius of 1250 km, the same as that used by Millis et al. (1992). Recall that the reference radius has no error bar, since this is a fixed parameter that is selected for convenience. All parameters in Table 9 belonging to the atmospheric set have been flagged with an asterisk (*). Some of these are used to derive other parameters describing the atmosphere, since correlation coefficients are available for this set (Table 10) that allow the calculation of formal errors for any derived parameter.

Quantities pertinent to the occultation geometry, D and $\nu$ have been provided by Millis et al. (1992). As errors in $D$ and $\nu$ are difficult to estimate, yet certainly far too small to significantly affect the error on quantities derived from them, we have not attempted to enter their errors in Table 9. As we have seen in our fitting experiments, $\rho_m$ is poorly determined by the shape of the light curve, so we use the value of 865.69 km, determined by a joint fit to all Pluto occultation data for this event (Millis et al. 1992). The error on this value is composed of two parts: (i) the error in deter-
from the data parameters. For the KAO light curve, the
temperature are obtained directly from the conversion
equipment. Since the error in the shadow center is much
less than this (Millis et al. 1992), we adopt an error of 15 km
for $P_{\text{min}}$, which has been entered in Table 9.

Some of the parameters that describe the "clear atmosphere"—refractivity, energy ratio, and power index for
the temperature—are obtained directly from the conversion
from the data parameters. For the KAO light curve, the
model is most sensitive to a thermal gradient at a radius of
1427 km from the center of Pluto, within a range of 153 km
(FWHM). The other parameters in this group were calculated
from parameters higher in Table 9 with Eqs. (6.11)–
(6.21). Errors for these parameters were calculated by propagating
errors from more fundamental parameters, accounting
for correlations of these parameters (Table 10). Since the
temperature-to-molecular-weight ratio is relatively well
determined, the temperature of this region of the atmosphere
could be as low as $60 \pm 12$ K in the limit of a pure

<table>
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<tr>
<th>Parameters</th>
<th>Values That Are Independent of Composition</th>
<th>Assumption for Composition</th>
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<td>100% N$_2$</td>
</tr>
<tr>
<td>Physical</td>
<td></td>
<td></td>
</tr>
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<td>Pluto mass, $M_p$ (g)</td>
<td>(1.30 ± 0.24) $\times 10^{25}$</td>
<td>16.04</td>
</tr>
<tr>
<td>molecular weight, $\mu$ (amu)</td>
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<td>4.401</td>
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<td>Specific To the KAO Light Curve</td>
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</tr>
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<td>background level, $b_0$ (ADU s$^{-1}$)</td>
<td>633.1 ± 6.8*</td>
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</tr>
<tr>
<td>background slope, $b'$ (ADU s$^{-2}$)</td>
<td>0.0</td>
<td></td>
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<tr>
<td>star level, $s_0$ (ADU s$^{-1}$)</td>
<td>2711.2 ± 9.1*</td>
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<td>midtime, $t_{\text{mid}}$ (s)</td>
<td>96.907 ± 0.046*</td>
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<td>planet-observer distance, $D$ (km)</td>
<td>4.323 $\times 10^9$</td>
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<td>minimum observer radius, $R_{\text{min}}$ (km)</td>
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<td>Clear Atmosphere</td>
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<td>refractivity, $\nu_0$ (10$^{-9}$)</td>
<td>0.97 ± 0.17*</td>
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<td>22.4 ± 1.8*</td>
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<td>power index for temperature, $b$</td>
<td>-0.61 ± 0.87*</td>
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<tr>
<td>gravity, $g_0$ (cm s$^{-2}$)</td>
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<td>pressure scale height, $H_0$ (km)</td>
<td>35.7 ± 4.5</td>
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<td>temperature, $T_0$ (K)</td>
<td>(3.72 ± 0.75) $\times 10^4$</td>
<td>60 ± 12</td>
</tr>
<tr>
<td>temperature gradient, $(dT/dr)_0$ (K km$^{-1}$)</td>
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<td>-0.029 ± 0.040</td>
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<td>(2.61 ± 0.46) $\times 10^{-4}$</td>
<td>0.59 ± 0.11</td>
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<tr>
<td>pressure, $p_0$ (mb)</td>
<td>(1.38 ± 0.38) $\times 10^{-1}$</td>
<td>0.49 ± 0.14</td>
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<td>23.9 ± 2.3</td>
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<td>top of haze, $r_1$ (km)</td>
<td>1215 ± 11*</td>
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<td>haze scale height, $H_1$ (km)</td>
<td>29.8 ± 5.6*</td>
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</tr>
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<td>vertical thickness of haze (km)</td>
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<td>≥ 3.1</td>
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<td>≥ 64</td>
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<tr>
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<td>&gt;35.7</td>
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<td>bulk density (g cm$^{-3}$)</td>
<td>&gt;1.88</td>
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<td></td>
</tr>
<tr>
<td>surface number density, $n_1$ (10$^{14}$ cm$^{-3}$)</td>
<td>1.70 ± 0.38</td>
<td>6.77 ± 1.69</td>
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<tr>
<td>surface pressure, $p_1$ (mb)</td>
<td>1.12 ± 0.26</td>
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<td>surface column height, $l_1$ (cm-A)</td>
<td>28.3 ± 3.3</td>
<td>47.1 ± 7.9</td>
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<td>surface temperature, $T_1$ (K)</td>
<td>34-58$|$</td>
<td>48.04 ± 0.45</td>
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<tr>
<td>bulk density (g cm$^{-3}$)</td>
<td>1.77 ± 0.33</td>
<td></td>
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† e.g. multiply entry by $10^{-4}$
* one of the 9 "atmospheric" parameters obtained from conversion of the "data" parameters
‡ after Mills et al. (1992)
¶ range of published values; see text.

Table 9. The structure of Pluto's atmosphere.
methane atmosphere. This magnitude and error of this result differs from that of our previous analysis (Elliot et al. 1989, 67 ± 6 K for a pure methane atmosphere). The difference in the value arises from approximations made in the earlier work. The dominant source of error in the temperature comes from the mass error. Hence the present error in the atmospheric temperature is larger in this work because we used what we feel is a more realistic error for Pluto's mass, which is twice that of our previous work.

The next set of parameters, "haze, no steep thermal gradient," are valid if the haze model described in this paper correctly describes the lower part of Pluto's atmosphere. The first three of these parameters—top of the haze, haze scale height, and linear absorption coefficient—are obtained directly from the conversion of parameters from the data parameter set. For this model the surface radius is not known—only that it must lie below a level corresponding to that where we can still detect light from the occultation curve. From the limit that the surface radius must be less than 1181 km, we derive a lower limit on the vertical optical depth of the haze of 0.145, a value not great enough to obscure the observed repeatable variations in the Pluto-Charon rotational light curve. Also from the limit on the surface radius, we derive limits on the surface number density and pressure for our assumptions for the atmospheric composition. We can also derive lower limits on the surface temperature if we assume that the atmosphere is in vapor-ice equilibrium (Brown & Ziegler 1980). For the CH4-Ar mixture, we assume that the CH4 was in vapor-ice equilibrium, and that the Ar fraction is limited by an equilibrium between unspecified resupply and escape processes. These lower limits on surface temperatures can be compared with the range of surface temperatures for Pluto between 31 and 59 K that have been derived from IRAS and mm observations of Pluto-Charon, analyzed under various assumptions about the thermal properties of the surfaces of Pluto and Charon (Sykes et al. 1987; Aumann & Walker 1987; Tedesco et al. 1987; Altenhoff et al. 1988).

If the haze model is not correct and a steep thermal gradient is causing the observed break in the KA occultation curve (Eshleman 1989; Hubbard et al. 1990), then the parameters listed in "steep thermal gradient, no haze" apply to Pluto. To derive these parameters we have assumed that the surface radius lies 9 ± 3 km below the level corresponding to the break in the light curve, a value found from the theoretical thermal profiles for a clear Plutonian atmosphere (Hubbard et al. 1990). This gives us a surface radius of 1206 ± 11 km. We found thermal and pressure profiles for this layer by matching the temperature and pressure at r1 and assuming the gas is in vapor-ice equilibrium at the surface. For the CH4-Ar mixture, we assume the CH4 fraction is in vapor-ice equilibrium, as we did for the haze model.

In Fig. 7 we have summarized the main features of Pluto's atmosphere (as given in Table 9) and the relation of the atmosphere to the surface, where the radius is plotted as the ordinate and temperature along the abscissa. The left panel illustrates the structure for the thermal-gradient model and the right panel shows the structure for the haze model. For each of these cases we have the further uncertainty of the atmospheric composition. We know that the upper part of the atmosphere contains some methane, but the range of possible methane fractions is large. The temperature could be greater than 100 K, if the model of Yelle & Lunine (1989) is correct and the main cooling process is radiation through the 7.8 μm band of CH4. However, if the primary cooling is through other bands at longer wavelengths, then the temperature would be less. Some type of thermostatic action is likely at work in this region, since our results indicate that the atmosphere is isothermal here.

Below this clear region of the atmosphere, whose boundary is delineated by the break in the occultation light curve, we do not know the atmospheric structure. This region may be the onset of a sharp thermal gradient, in which case the surface of Pluto would lie only a few kilometers below. This boundary could also mark the onset of a haze layer, as we have modeled here. We infer that the haze cannot be optically thick, since Pluto's light curve could not have such a large amplitude and be in phase with the orbital period of Charon (which is most likely tidally locked to the rotation of Pluto). However, if the effects of haze are dominating the light-curve structure, we have no lower bound on Pluto's surface radius, since we do not know whether the haze is layered or uniformly mixed with the atmosphere. We might set a limit on the surface radius by arguing that Pluto's density would likely not exceed a certain value—say 3.0 g cm⁻³. Although this approach can be used to set a limit on the surface radius, it is of marginal value, since the point of finding the surface radius is to learn the density of Pluto.

TABLE 10 Correlation coefficients for atmospheric parameters.

<table>
<thead>
<tr>
<th>Model</th>
<th>sb</th>
<th>s*</th>
<th>tmid</th>
<th>v0</th>
<th>λ40</th>
<th>b</th>
<th>r1</th>
<th>κ1</th>
<th>Hr1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>r1</td>
<td>1.00</td>
<td>-0.80</td>
<td>0.00</td>
<td>0.01</td>
<td>0.10</td>
<td>...</td>
<td>0.02</td>
<td>-0.13</td>
<td>-0.49</td>
</tr>
<tr>
<td>s*</td>
<td>-0.74</td>
<td>1.00</td>
<td>0.00</td>
<td>0.08</td>
<td>-0.19</td>
<td>...</td>
<td>0.02</td>
<td>0.13</td>
<td>0.41</td>
</tr>
<tr>
<td>tmid</td>
<td>0.00</td>
<td>0.00</td>
<td>1.00</td>
<td>0.00</td>
<td>0.00</td>
<td>...</td>
<td>0.00</td>
<td>0.01</td>
<td>0.00</td>
</tr>
<tr>
<td>v0</td>
<td>0.03</td>
<td>0.17</td>
<td>0.00</td>
<td>1.00</td>
<td>0.65</td>
<td>...</td>
<td>0.98</td>
<td>0.03</td>
<td>-0.03</td>
</tr>
<tr>
<td>λ40</td>
<td>-0.01</td>
<td>-0.36</td>
<td>0.00</td>
<td>-0.15</td>
<td>1.00</td>
<td>...</td>
<td>0.77</td>
<td>-0.06</td>
<td>-0.18</td>
</tr>
<tr>
<td>b</td>
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<td>0.31</td>
<td>0.00</td>
<td>0.35</td>
<td>-0.94</td>
<td>1.00</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>r1</td>
<td>0.03</td>
<td>0.05</td>
<td>0.00</td>
<td>0.95</td>
<td>0.14</td>
<td>0.10</td>
<td>1.00</td>
<td>0.00</td>
<td>-0.07</td>
</tr>
<tr>
<td>κ1</td>
<td>-0.13</td>
<td>0.07</td>
<td>0.01</td>
<td>-0.02</td>
<td>0.13</td>
<td>-0.16</td>
<td>-0.02</td>
<td>1.00</td>
<td>0.79</td>
</tr>
<tr>
<td>Hr1</td>
<td>-0.49</td>
<td>0.31</td>
<td>0.00</td>
<td>-0.11</td>
<td>0.15</td>
<td>-0.23</td>
<td>-0.09</td>
<td>0.79</td>
<td>1.00</td>
</tr>
</tbody>
</table>

12 CONCLUSIONS

The presence of a thermal gradient in the atmosphere of an occulting planet can significantly affect the shape of a stellar-occultation light curve. Furthermore, the value of the
gradient can be found by fitting a model light curve to data of good signal-to-noise ratio. Using our method to analyze the KAO Pluto occultation data, we find that the upper part of Pluto's atmosphere probed by the stellar occultation is isothermal: \( T / d \tau = -0.029 \pm 0.040 \text{ K km}^{-1} \) for the limiting case of pure CH\(_4\) and \(-0.051 \pm 0.070 \text{ K km}^{-1}\) for the limiting case of pure N\(_2\). This result is consistent with the isothermal prediction of Yelle & Lunine's (1989) methane-thermostat model. However, Pluto's atmosphere could be isothermal in this region at a temperature less than the 106 K calculated by their model, if the thermostatic action is controlled by a molecular band that has a wavelength longer than the 7.8 \( \mu \)m band of CH\(_4\). Hence we still cannot constrain the CH\(_4\) fraction in the atmosphere, and we cannot determine from present observations the identity and amounts of other gases.

Another unknown in the region of Pluto's atmosphere probed by the occultation data is whether a haze or thermal gradient dominates the structure of the lower part of Pluto's atmosphere. If the thermal-gradient model is correct, then Pluto's surface radius is \( 1206 \pm 11 \text{ km} \), and its bulk density is \( 1.77 \pm 0.33 \text{ g cm}^{-3} \). On the other hand, if the haze model is correct, we can say only that Pluto's surface radius is less than \( 1181 \text{ km} \) and its bulk density is greater than \( 1.88 \text{ g cm}^{-3} \).

Further progress toward a first-order model of Pluto's atmosphere requires answers to these two questions: (i) exactly what gases are present, and in what proportions?, and (ii) does the sharp break in the slope of the KAO stellar occultation curve delineate the top of an extinction layer or the onset of a large thermal gradient, \(-10 \text{ K km}^{-1}\)? The answer to the second question is also a prerequisite to pinning down the density of Pluto to the accuracy we would like for comparison with formation models for Pluto and Charon.

A direct observational test of the haze versus thermal gradient question would be the observation of a future stellar occultation simultaneously at infrared and visible wavelengths: micron-size haze particles would have significantly lower optical depth in the near infrared than at visible wavelengths. An opportunity for carrying out such an observation occurs on 1992 May 21 (UT), when Pluto will likely occult a star with \( R = 13.0 \), visible from the western hemisphere (Mink et al. 1991; Dunham et al. 1991).

We are grateful to L. H. Wasserman for comparing his model calculations and least-squares-fit results with ours, and to R. L. Millis et al. for supplying the value of the KAO minimum observer radius and shadow velocity in advance of publication. This work was supported, in part, by NASA Grant No. NAGW-1494 and NSF Grant No. AST-8906011.
APPENDIX: POWER SERIES

In this appendix we present the power series for use with the equations for the refraction angle, $\theta(r)$ [Eq. (4.6)], the derivative of the refraction angle, $d\theta/dr$ [Eq. (4.9)], and the optical depth along the path of the light ray, $\tau_\text{obs}(r)$ [Eq. (4.19)].

The series required by the refraction angle—denoted by $A(\delta,a,b)$—is a function of the parameter $\delta$ [Eq. (4.2)], the exponent for molecular weight variation, $a$ [Eq. (3.1)] and the exponent for temperature variation, $b$ [Eq. (3.3)]. As discussed in Sec. 4, $\delta = 0$ in the large planet limit, and if there are no temperature or molecular weight gradients in the atmosphere, then $a = b = 0$.

We obtained the series $A(\delta,a,b)$ by performing the integral in Eq. (4.5). The integrand is expanded to 4th order in $\delta$, after which it contains a polynomial in $y$, the variable of integration, and $\delta$, the expansion parameter of our series. By expanding the integrand in $\delta$ rather than $y$, we guarantee that the series is complete in $\delta$ for a given order of $\delta$. Each term of $\delta$ has as a coefficient a power series in $a$ and $b$. We evaluate the resulting integrals, term by term. The definite integrals that arise in this expansion are of the following form (Dwight 1961, by manipulation of Sec. 860.17):

$$\int_{-\infty}^{y} y^n e^{-y^2} dy = \begin{cases} \frac{(n + 1)\Gamma\left(\frac{n+1}{2}\right)}{2^n (n-2)!}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \tag{A1}$$

The resulting series, calculated with Mathematica™ (Wolfram 1988), is:

$$A(\delta,a,b) = 1 + \left(\frac{-3a + 3b}{8}\right) + \left(\frac{-15 + 26a + 7a^2}{128}\right) + \left(\frac{7 + 5a}{64}\right) + \left(\frac{b^2}{128}\right) + \left(\frac{-105 + 425a + 35a^2 + 75a^3}{1024}\right) + \left(\frac{27 + 50a + 35a^2}{1024}\right) + \left(\frac{b + 69 + 55a}{1024}\right) + \left(\frac{b}{1024}\right) + \left(\frac{9b^2}{1024}\right) + \left(\frac{b^3}{1024}\right) + \left(\frac{b^4}{1024}\right) + \cdots \tag{A2}$$

The series for the derivative of the refraction angle, $B(\delta,a,b)$, is found from Eq. (4.6) by taking the derivative of $\theta(r)$:

$$\frac{d\theta(r)}{dr} = -\frac{2\pi \lambda(r) v(r)}{r} \left[\frac{1}{\lambda(r)} \frac{d\lambda(r)}{dr} + \frac{1}{v(r)} \frac{dv(r)}{dr}\right] A(\delta,a,b) \frac{dA(\delta,a,b)}{d\delta} \tag{A3}$$

We evaluate the derivatives, and factor $\lambda(r)/r$ out of the brackets. Recall that $d\lambda(r)/dr = -\lambda(r)(1 + a + b)/r$ [from Eq. (3.9)]. $(1/v)dv/dr = -1/H_\gamma = -(\lambda + b)/r$, and $d\delta/r = \delta(1 + a + b)/r$. With these substitutions, the derivative becomes

$$\frac{d\delta(r)}{dr} = \frac{\lambda(r) v(r)}{r} \left[\left(\frac{1 + a + b}{2}\right) + \left(\frac{1 + a + b}{2}\right)\delta^2 + \left(\frac{1 + a + b}{2}\right)\delta^3\right] \frac{dA(\delta,a,b)}{d\delta} \tag{A4}$$

By comparison with Eq. (4.9), we see that $B(\delta,a,b)$, the series needed for $d\theta/dr$, can be expressed in terms of the series for $\theta(r)$:

$$B(\delta,a,b) = \left(\frac{1 + a + b}{2}\right) A(\delta,a,b) - \left(\frac{1 + a + b}{2}\right) \frac{dA(\delta,a,b)}{d\delta} \tag{A5}$$

Substituting $A(\delta,a,b)$ into the previous expression, we find

$$B(\delta,a,b) = 1 + \left(\frac{1 + 3a}{8}\right) + \left(\frac{15a}{8}\right) + \left(\frac{9 + 6a + a^2}{128}\right) + \left(\frac{17 + 11a}{64}\right) + \left(\frac{b}{128}\right) + \left(\frac{75 + 67a + 41a^2 + 9a^3}{1024}\right) + \left(\frac{81 + 134a + 57a^2}{1024}\right) + \left(\frac{b + 1 + 3a}{1024}\right) + \left(\frac{5b^2}{1024}\right) + \left(\frac{5b^3}{1024}\right) + \left(\frac{3675 + 7204a + 5266a^2 + 2564a^3 + 491a^4}{32768}\right) + \left(\frac{339 + 1347a + 1.297a^2 + 409a^3}{16384}\right) + \left(\frac{b}{8192}\right) + \left(\frac{b^2}{16384}\right) + \left(\frac{b^3}{32768}\right) + \left(\frac{b^4}{32768}\right) + \cdots \tag{A6}$$

For the series required by the line-of-sight optical depth, we follow the procedure outlined in Sec. 4. The expansion parameter is $\delta \equiv \tau_\gamma / r / r_\gamma$, and the resulting power series is

$$C(\delta) = 1 + \frac{9}{8} \delta + \frac{345}{128} \delta^2 + \frac{9555}{1024} \delta^3 + \frac{1371195}{32768} \delta^4 + \cdots \tag{A7}$$
REFERENCES

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Mink, D. J., & Klemola, A. 1985, AJ, 90, 1894
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Wolfram, S. 1988, Mathematica (Addison-Wesley, Reading)
APPENDIX II Correlations to E&Y’s Paper

We found a couple of minor errors in E&Y’s paper. First, in section 2, they failed to mention that they were assuming that the index of refraction of the atmosphere was approximately 1. The correct form of their Eq. (2.5), without the approximation, would be

\[ \theta(r) \int_{-\infty}^{\infty} \frac{d\tilde{n}}{n(r')} \cdot d\tilde{x}, \]

where \( n(r) \) is the index of refraction, and \( n(r) = 1 + v(r) \).

The other place where they made mistakes was in the Appendix. Their equation for the integral of a power times a gaussian (Eq. (A1)) was incorrect. It should have been the same as my Eq. (4.11):

\[ \int_{-\infty}^{\infty} y^ne^{-y^2} dy = \left\{ \Gamma \left( \frac{n+1}{2} \right) = \frac{(n-1)\sqrt{n\pi}}{2^{n-1}(n/2-1)!} \right\}. \]

They also made a couple of mistakes in the power series. In the A series (Eq. (A2)), they had an extra factor of 4 in the \( b^3 \) part of the coefficient of \( \delta^4 \). The coefficient of \( \delta^4 \) should have been:

\[
\begin{align*}
&\left\{ \frac{4725 + 35196a + 57134a^2 + 31836a^3 + 5509a^4}{32768} \\
&+ \frac{1059 + 4907 + 3857a^2 + 609a^3}{8192} b + \frac{2353 + 4326a + 2233a^2}{16384} b^2 + \delta^4. \\
&\frac{941 + 791a}{8192} b^3 = \frac{491}{32768} b^4
\end{align*}
\]

Finally, they made a mistake in one of the intermediate steps in calculating the B series. They mistakenly added a factor of 1/2 to the \( d\Lambda(\delta,a,b)/d\delta \) term. It should have been just

\[ (1 + a + b) \delta^2 \frac{d\Lambda(\delta,a,b)}{d\delta}. \]

This mistake did not carry over into the final statement of the B series.

APPENDIX III. Proof of Exponential Refraction Test Flux Function

The light curve flux, \( \phi \), is related to the derivative of the refraction angle, \( \theta(r) \) for at atmosphere at a distance \( D \) by Eq. (2.3):

\[ \phi = \frac{1}{1 + D \frac{d\theta}{dr}}. \tag{A1} \]

This equation is exact for a cylindrical atmosphere (one with no focusing). Also, \( \rho \) is related to the refraction angle by Eq. (2.2):

\[ \rho(r) = r + D\theta(r). \tag{A2} \]

This is the small angle approximation. If we pick a reference point, \( \rho_0, r_0, \theta_0 \), we have

\[ \rho_0 = r_0 + D\theta_0. \tag{A3} \]
Next, we subtract (A3) from (A2) and divide by the scale height $H$:

$$\frac{(\rho - \rho_0)}{H} = \frac{(r-r)}{H} + \frac{D}{H}(\theta - \theta_0).$$

(A4)

Since we postulated that the atmosphere produces an exponential refraction angle, we set

$$\theta(r) = \theta_0 e^{-\frac{(r-r_0)}{H}}.$$  

(A5)

We choose to put the reference level at half light ($\phi=1/2$). Now we can solve for $\theta_0$:

$$\phi = \frac{1}{2} = \frac{1}{1+D} \frac{d\theta}{dr} \quad \frac{1}{1+D} \frac{d\theta}{dr} = \frac{1}{1+e^{-\frac{(r-r_0)}{H}}}.$$

or,

$$2 = 1 - \frac{D\theta_0}{H} e^{-\frac{(r-r_0)}{H}} \quad \left|_{r=r_0} \right. = 1 - \frac{D\theta_0}{H},$$

or,

$$\theta_0 = \frac{-H}{D}. \quad \quad \text{(A6)}$$

Substituting $\theta_0 = \frac{H}{D}$ into the flux equation (Eq. (4.2)), we get

$$\phi = \frac{1}{1+D} \frac{d\theta}{dr} = \frac{1}{1+e^{-\frac{(r-r_0)}{H}}}.$$  

(A7)

or,

$$e^{-\frac{(r-r_0)}{H}} = \frac{1}{\phi} - 1,$$

or,

$$\frac{(r-r_0)}{H} = -\ln \left( \frac{1}{\phi} - 1 \right), \quad \quad \text{(A8)}$$

which is the first term on the right hand side of Eq. (A4).

Next we look at the second term on the right hand side of Eq. (A4):

$$\frac{D}{H}(\theta - \theta_0) = \frac{D\theta}{H} - \frac{D\theta_0}{H} = e^{-\frac{(r-r_0)}{H}} + 1.$$  

(A9)

From Eq. (A7), this equals $-(1/\phi - 2)$. Substituting this result into Eq. (A4) for $D/H(\theta - \theta_0)$ and also using the result of Eq. (A8), we find

$$\frac{\rho - \rho_0}{H} = -\ln \left( \frac{1}{\phi} - 1 \right) - \left( \frac{1}{\phi} - 2 \right), \quad \quad \text{(A10)}$$

which is the result we were looking for.

This shows that if we have an exponential angle, $\theta(r) = \theta_0 e^{-\frac{(r-r_0)}{H}}$, we get an exact solution for the cylindrical flux, which then satisfies Eq. (4.2).
APPENDIX IV. Exponential Refraction Test Power Series

We calculated the power series for the exponential refraction tests using Mathematica\textsuperscript{TM}, following the procedure outlined in Section 4.2.1. When we did this, we got the following equation for the refractivity:

\[ n(H) = e^{-\mu r} \left( 1 - \frac{1}{8} \left( \frac{H}{r} \right)^2 + \frac{9}{128} \left( \frac{H}{r} \right)^2 - \frac{75}{1024} \left( \frac{H}{r} \right)^3 + \right) \left( \frac{3675}{32768} \left( \frac{H}{r} \right)^4 - \frac{59535}{262144} \left( \frac{H}{r} \right)^5 + \right) \]

Note that the coefficients are starting to increase. It is likely that this series would eventually diverge if we continued to calculate more terms.

APPENDIX V Using the numOccLightCurves Package

The numOccLightCurves package allows you to set up a model function that can be used directly by the leastSquares package to do a fit. The main difference is that setting up the model function is more involved.

A. MAKING THE DATA FILE

The first thing you need is a data file. The format for the data file is a single Simple FITS file with a single header with no data attached, but with EXTEND = T, followed by header/data pairs for each data profile. The top header needs to include the following keylines and values:

SIMPLE = T
BITPIX = <8, or whatever - no data, so it's irrelevant>
NAXIS = 0
EXTEND = T
NPARAMS = <# of parameters the profiles depend on>
PARAM1 = <name of the first parameter>

... PARAMMN = <name of last parameter>
END

There is a template for this initial header on astron in

/pscf/jle-grp/projects/fitting_95-06/Templates/headtempl_top.

The header for each data profile follows the usual format for table files. The standard format for them is in

/pscf/jle-grp/projects/fitting_95-06/Templates/headtempl_table.

That format is:

SIMPLE = T
XTENSION= 'TABLE'
BITPIX = 8
NAXIS = 2
NAXIS1 = <width of table>
NAXIS2 = <length of table>
PCOUNT = <whatever>
GCOUNT = <whatever>
TFIELDS = 7 <the package assumes a 7 column table>
TBCOL1 = 1
TBCOL7 = <the location of the first digit of the 7th column>
TFORM1 = 'Fw.d' <these are FORTRAN formats for the columns>
TFORM2 = 'Fw.d' <Fw.d means the column is characters - could still be a number, just not in exponential notation>
TFORM3 = 'Fw.d' <the exact values of these are probably not important>
TFORM4 = 'Ew.d' <Ew.d means the number is in exponential notation>
TFORM5 = 'Ew.d'
TFORM6 = 'Ew.d'
TFORM7 = 'Ew.d'
TTYPE1 = 'Index' <these are the names of the columns>
TTYPE2 = 'Radius'
TTYPE3 = 'Temperature'
TTYPE4 = 'Pressure'
TTYPE5 = 'n'
TTYPE6 = 'dn/dz'
TTYPE7 = 'd2n/dz2'
TUNIT1 = <no unit - this is just an index>
TUNIT2 = 'km'
TUNIT3 = 'K' <the package does not use this col., so it is just a place holder>
TUNIT4 = 'mu bar' <also not used>
TUNIT5 = 'cm^-3' <This, and the next 2, ARE used>
TUNIT6 = 'cm^-4'
TUNIT7 = 'cm^-5'
END

The easiest way create a data file is to use a notebook on lowell in my thesis directory called collect_datafiles_3.1.m. It has one function called makeDataSet and a usage statement. It also has an example of using the function, and a second function that I used to make large numbers of files with a specific naming convention, but these are not initialization cells. You can simply read in the file, read the usage statement, and use the function.

B. SETTING UP THE MODEL FUNCTION
Once you had the data file, you need to run two functions. First is the usual IsUseModel. There are several different nolc models you can choose from - nolcNearLimb, nolcTwoLimb, nolcNearLimbRatio, and nolcTwoLimbRatio. The usage statements explain the details of the differences between them, and, when you run IsUseModel, nolcNames is defined to be a list of the names of the parameters that should be in IsParams..

Second, you need to run nolcSetUp. There is a lengthy usage statement for this function. It takes the data file you set up as specified in the previous section and IsParams. More importantly, it can take a long series of options, including things like whether to include the cap and, if so, how high to extend it, the surface radius, the spacing of points in the internal interpolations, and nuSTP for the atmosphere. These options are all described in the usage statement.

This is all you need to run the leastSquares functions with this model. IsUseModel automatically sets IsModelFunction to the correct model, and, once you've run nolcSetUp, this function will work with the parameters specified by nolcNames (or you can find them in the usage statement for the model)
Numerical Occultation Light Curve Package

Dawn Chamberlain
1995 Nov ff

● Notebook Overview
This functions in this package are used to create an occultation light curve from a number density profile and to fit that light curve.

● Upgrade Plans and Revision Log

● make assignments for lsUseModel

```plaintext
lsModelContext[nolcTwoLimbRatio] =
lsModelContext[nolcFarLim]=
lsModelContext[nolcTtwoLim]=
lsModelContext[nolcNearLim]=
lsModelContext[nolcNearLim]= "jleGroup\numericalOccLightCurves";

lsInitialNames[nolcTwoLimbRatio] =
lsInitialNames[nolcNearLim]= {"nolcStarRatio","nolcSlope","nolcFull","nolcTmid","nolcDistance","nolcRhoMin","nolcV","nolcDeltat","nolcNumParams","nolcPar[1]");

lsInitialNames[nolcFarLim] =
lsInitialNames[nolcTtwoLim] =
lsInitialNames[nolcNearLim] = nolcNames;

lsInitialStep[nolcFarLim] =
lsInitialStep[nolcTwoLimbRatio] =
lsInitialStep[nolcTtwoLim] =
lsInitialStep[nolcNearLim]=
lsInitialStep[nolcNearLim]= {0.01, 0.01, 0.01, 0.01,
0.01, 0.01, 0.01,
0.01, 0.01, 0.01};
```

● Begin Package

```plaintext
BeginPackage["jleGroup\numOccLightCurves", {
"
jleGroup\fitsManagerMath","jleGroup\leastSquares",
"jleGroup\occLightCurves")};
```
nolcSetUp::usage = "nolcSetUp[datafile,fParams,opts] sets up the interpolation functions for
the model using data from a FITS table file containing multiple tables.
The first header in the datafile must contain a NPARAMS keyword giving
the number of parameters to be interpolated over, as well as PARAMn
keywords giving the names of each of the parameters. The header for each
table must contain PARVALn keywords giving the value for each parameter
associated with that table's data. The variable nolcNames gives the list
of parameters that should be specified in fParams (though only the
distance, velocity, delta t, and numParams parameters are actually used).
The options are: opNuSTP which defaults to 2.980*10^-4 (which is the
value for a purely nitrogen atmosphere); opDeltaR, which defaults to 5,
is the spacing of points in radius (in km) for the interpolation functions;
opInterpolationOrder which defaults to 1 is the order for all interpolation functions;
opSurfaceRadius, which defaults to 1352 (the approx. value for Triton) is the
radius of the surface of the body. All flux is cut off below this level;
opAddExt, which defaults to True, allows you to make a light curve without adding
the isothermal cap; opExtTop, which defaults to 5000, is the largest
radius that the isothermal cap will be calculated for - the fractional flux will just
be set to 1 for radii larger than this number; opExtSpacing which defaults to 10km,
is the spacing used when calculating the isothermal cap";

Options[nolcSetUp]={opNuSTP->2.980*10^-4, opDeltaR->5.0, opInterpolationOrder->1,
opSurfaceRadius->1352, opAddExt->True, opExtTop->5000,
opExtSpacing->10.0, opIntBin->False, opFocus->True,
opInterpPts->200.,opWorkPrec->16,opMaxRecursion->6};

nolcTheta::usage = "nolcTheta[r, rPrimeMax,dRefractivity] does a numerical integral to estimate
theta. dRefractivity is a function (of rPrime) that must be defined from
r to rPrimeMax";

nolcDtheta::usage = "calcRelatedParamsOriginalDtheta[r,rPrimeMax,dRefractivity,d2Refractivity] does a numerical
integral to estimate dTheta. dRefractivity and d2Refractivity must be
defined from r to rPrimeMax";

nolcNearLimb::usage = "nolcNearLimb[fParams,fCoor] calculates a light curve using the interpolation
functions defined by nolcSetUp. It includes refraction and focusing
terms, but no extinction. The flux is only from the near limb";

nolcTwoLimb::usage = "nolcTwoLimb[fParams,fCoor] calculates a light curve using the interpolation
functions defined by nolcSetUp. It includes refraction and focusing
terms, but no extinction. The flux is from both the near and far limbs";

nolcFarLimb::usage = "nolcFarLimb[fParams,fCoor] calculates the far limb flux."

nolcNearLimbRatio::usage = "nolcNearLimbRatio[fParams,fCoor] is the same as nolcNearLimb except the nolcStarRatio
is specified instead of the background.";
nolcTwoLimbRatio::usage =
"nolcTwoLimbRatio[fParams,fCoor] is the same as nolcTwoLimb except the nolcStarRatio
is specified instead of the background."

nolcConvParamsFromRatio::usage =
"nolcConvParamsFromRatio[ 
{nolcStarRatio, nolcSlope, nolcFull, nolcTmid, 
nolcDistance, nolcRhoMin, 
nolcV, nolcDeltat, nolcNumParams, nolcPar[1]}, 
returns 
{nolcBkgd, nolcSlope, nolcFull, nolcTmid, 
nolcDistance, nolcRhoMin, 
nolcV, nolcDeltat, nolcNumParams, nolcPar[1]} 
] as needed by nolcTwoLimbRatio and nolcNearLimbRatio."

nolcConvParamsToRatio::usage =
"nolcConvParamsToRatio[ 
{nolcBkgd, nolcSlope, nolcFull, nolcTmid, 
nolcDistance, nolcRhoMin, 
nolcV, nolcDeltat, nolcNumParams, nolcPar[1]}, 
returns 
{nolcStarRatio, nolcSlope, nolcFull, nolcTmid, 
nolcDistance, nolcRhoMin, 
nolcV, nolcDeltat, nolcNumParams, nolcPar[1]}, 
]

nolcNames::usage =
"nolcNames gives the names of the parameters to be used with these fitting 
functions"

nolcBkgd::usage =
nolcSlope::usage =
nolcFull::usage =
nolcTmid::usage =
nolcRh::usage =
nolcStarRatio::usage =
nolcDistance::usage =
nolcRhoMin::usage =
nolcV::usage =
nolcDeltat::usage =
nolcNumParams::usage =
"This is one of the parameters for nolcLightCurve1"

nolcPar::usage =
"nolcPar[i] is a parameter for nolcParams. It is the name of the i-th parameter 
in the data set"

nolcFluxBendF::usage =
"nolcFluxBendF[y[t],par1,par2,...] gives the unfocused and unscaled flux."

nolcRefractivityF::usage =
"nolcRefractivityF[radius,par1,par2,...] gives the refractivity at the given radius 
and values of the parameters";
nolcDrefractivityF::usage = 
"nolcDrefractivityF[ radius, par1, par2,... ] gives the refractivity at the given radius
and values of the parameters";

nolcD2refractivityF::usage = 
"nolcD2refractivityF[ radius, par1, par2,... ] gives the refractivity at the given radius
and values of the parameters";

nolcYfromTime::usage = 
"nolcYfromTime[ params, time ] gives the y value at that time from the rhoMin and velocity";

nolcRofYF::usage = 
"nolcRofYF[ y, par1, par2,... ] gives the r (planet plane radius) value at that y (shadow
plane radius).";

nolcNumberDensity::usage = 
"nolcNumberDensity[ radius, par1, nuSTP ] gives the number density given a planet plane
radius the physical model parameter and refractivity at STP. So far this function only
handles one parameters fits."

● Begin "Private"

Begin["Private"];

Note: these functions initially set up nolcNames, etc as if there is
only 1 parameter. When setUp is called, it checks to see if more
parameters are expected, and changes nolcNames appropriately.

⊙ loadModRes

loadModRes[id_, numCols_, numParams_] := Module[
{modelResults={}},
For[i=1,i<=numParams,i++,
    modelResults=Append[modelResults,fmDoubleValue[id,
        "PARVAL"<>ToString[i]]];
    fmDoubleValue[id,"PARVAL"<>ToString[i]]];
modelResults=Append[modelResults,fmTableIntColumn[id,1,1]]; 
For[i=2, i<=numCols, i++,
    modelResults=Append[modelResults,
        fmTableDoubleColumn[id,1,i]]];
modelResults]
loadModResults

loadModResults[fileName_] := Module[
{
ids, i, numParams, paramNames={}, numSets, numCols, modelResults,
}

ids=fmDiskInput[fileName];
numSets=Length[ids]-1;
numParams=fmIntValue[ids[[1]],"NPARAMS"]; For[i=1,i<=numParams,i++,
    paramNames=Append[paramNames,fmStringValue[ids[[1]],
    "PARAM"<>ToString[i]]];
numCols=fmIntValue[ids[[2]],"TFIELDS"]; modelResults=loadModRes[ids[[2]],numCols,numParams];
If[numSets>1,

modelResults=Append[modelResults,loadModRes[ids[[3]],numCols,numParams]];
Table[modelResults=Append[modelResults,loadModRes[ids[[i]],numCols,numParams]],{i,4,Length[ids]}];
modelResults=Prepend[modelResults,paramNames];
modelResults=Prepend[modelResults,paramNames]

]

nolcRprime

nolcRprime[r_, x_] := Sqrt[r^2 + x^2]

nolcTheta and nolcDtheta

nolcTheta[r_, rPrimeMax_, dRefr_] := 2 NIntegrate[(r/nolcRprime[r, x])*
dRefr[nolcRprime[r, x]],
{x, 0, Sqrt[rPrimeMax^2 - r^2]}, AccuracyGoal->10, MaxRecursion->mRec,
WorkingPrecision->workPrec];

To get formula for the derivative of theta, take the derivative of
Eq. (2.5) of Elliot & Young (AJ 103, 991).

nolcDtheta[r_, rPrimeMax_, dRefr_, d2Refr_] := 2 NIntegrate[ x^2/nolcRprime[r, x]^3 *
dRefr[nolcRprime[r, x]] +
(r/nolcRprime[r, x])^2 d2Refr[nolcRprime[r, x]],
{x, 0, Sqrt[rPrimeMax^2 - r^2]}, AccuracyGoal->10,
MaxRecursion->mRec,
WorkingPrecision->workPrec] ;

y, refr

y[r_, dist_] := r + dist theta[r];
refr[dens_,dDens_,d2Dens_,numFiles_]:=Module[{refrac,dRefrac,d2Refrac,i},
refrac=Table[{},{i,numFiles}];
dRefrac=Table[{},{i,numFiles}];
d2Refrac=Table[{},{i,numFiles}];
For[i=1,i<=numFiles,i++,
refrac[[i]]=refrO/densO

dRefrac[[i]]=10^5 dDens[[i]] refrO/loschmidtsNo;
d2Refrac[[i]]=10^10 d2Dens[[i]] refrO/loschmidtsNo;
];
{refrac,dRefrac,d2Refrac}

set up names

{nolcBkgd,nolcSlope,nolcFull,nolcTmid,nolcDistance,
nolcRhoMin,nolcV,nolcDeltat,nolcNumParams,nolcPar[1]}=Range[10];
nolcStarRatio=nolcBkgd;
nolcNames={"nolcBkgd","nolcSlope","nolcFull","nolcTmid",
"nolcDistance","nolcRhoMin","nolcV","nolcDeltat","nolcNumParams","nolcPar[1]"};

extTheta, extDtheta, addExt
From Elliot and Young, Eq. 3.9 \(\lambda_G(r) = \lambda_{Go}(r_0/r)\) for isothermal and constant composition. Therefore, \(r_0=r\) and \(r=r_0\).

e-mail

functions

nu[lam_,r_,rt_,nuR_]:=Module[{},
u nuR \exp[r lam/rt - lam]]
lamg[lam_,r_,rt_]:=Module[{},lam r/rt];
delta[lam_,rt_,r_]:=Module[{},
1/lamg[lam,rt,r]];

Normal[Series[(1+2*del*yvar^2)^-1 *

\exp[\{(1+2 del yvar^2)^(-1/2)-1\}/del] *

(1+2 del yvar^2)^(-1/2)\exp[yvar^2],
\{del,0,1\}]]]

\[\frac{4}{2 + \frac{3 yvar}{2}}\]

\[\theta_{ser}[a_,b_]:= 1 + b^*(-3*a^2 + (3*a^4)/2)\]
(* part of Eq. 4.5 *)
fn[y_, rt_, r_, lam_, nuR_] := Module[{},

-nu[lam, r, rt, nuR] * 
  Sqrt[2 lamg[lam, r, rt]] * 
  Exp[-y^2] thetaser[y, delta[lam, rt, r]]
]

intFn is the integral of fn from ymax to infinity, calculated by integrating fn in Mathematica and copying the result. This keeps mathematica from having to recalculate what the derivative is every time through

intFn[ymax_, rt_, r_, lam_, nuR_] := 
  -nu[lam, r, rt, nuR] *
  Sqrt[2 lamg[lam, r, rt]] * 
  (Sqrt[Pi] (8 - 3 delta[lam, rt, r])/16 -
  3 delta[lam, rt, r] ymax (1 - 2 ymax^2)/(8 Exp[ymax^2]) -
  Sqrt[Pi] Erf[ymax] (8 - 3 delta[lam, rt, r])/16);

deriv[y_, rt_, r_, lam_, nuR_] := D[fn[y, r, lam, nuR], y]/rt;

intDeriv[ymax_, rt_, r_, lam_, nuR_] := 
  (-lamg[lam, r, rt]/r - 1/(2 r)) intFn[ymax, rt, r, lam, nuR] -
  nu[lam, r, rt, nuR] delta[lam, rt, r] Sqrt[2 lamg[lam, r, rt]]/r *
  (3/16) (-Sqrt[Pi] -
  (2 ymax-4 ymax^3-Exp[ymax^2] Sqrt[Pi] Erf[ymax])/Exp[ymax^2]);

extTheta[r_, rM_, dNu_, nuR_, order_] := Module[
  {xM, yM, theta, lam},
  xM=Sqrt[rM^2-r^2]; (* EY92 Eq 2.4 *)
  yM=xM/r Sqrt[1/(2 delta[lam, r, r])];
  lam=-dNu r/nuR; (* from EY92 3.22 noting dN/n = dNu/nu *)
  theta=2 intFn[yM, r, r, lam, nuR];

  theta=N[theta];
  theta;

extDtheta[r_, rM_, dNu_, nuR_, order_] := Module[
  {xM, yM, dTheta, lam},
  xM=Sqrt[rM^2-r^2]; (* EY92 Eq 2.4 *)
  yM=xM/r Sqrt[1/(2 delta[lam, r, r])];
  lam=-dNu r/nuR; (* from EY92 3.22 noting dN/n = dNu/nu *)
  dTheta=2 intDeriv[yM, r, r, lam, nuR];

  dTheta=N[dTheta];
  dTheta;
addExt[minR_, maxR_, deltaR_] := Module[
   {i, ext, order = 1, nuMax, lamMax, tempTheta, tempDtheta},
   tempTheta = thetaList; (* have to use temporary lists because mathematica will 
                          not allow assignments of the form foo[j][i]=bar *)
   tempDtheta = dThetaList;
   (* Add piece to larger radii *)
   i = Length[rList];
   While[i > 0,
      If[rList[[i]] > minR,
         tempTheta[[i]] += extTheta[rList[[i]], top, 
         dRefrOneVar[rList[[i]]], refrOneVar[rList[[i]]], order];
         tempDtheta[[i]] += extDtheta[rList[[i]], top, 
         dRefrOneVar[rList[[i]]], refrOneVar[rList[[i]]], order];
         ];
         i--];
   (* Add radii to extend model out to larger radii to 
   help fit for the full scale *)
   nuMax = refrOneVar[top];
   lamMax = -dRefrOneVar[top] top / nuMax; (* refractivity is proportional to 
   number density *)
   i = top;
   While[i < maxR,
      i += extSpacing;
      lamR = lamg[lamMax, top, i];
      nuR = nu[lamMax, top, i, nuMax];
      rList = {rList, i};
      tempTheta = {tempTheta, Sqrt[2 Pi lamR] nuR 
      jleGroup'occLightCurves'plutoAug903'thetaSeries[order, 
      1/lamR, 0]/N};
      tempDtheta = {tempDtheta, Sqrt[2 Pi lamR^3] nuR/i 
      jleGroup'occLightCurves'plutoAug903'dthetaSeries[order, 
      1/lamR, 0]/N};
      ];
      rList = Flatten[rList];
      thetaList = Flatten[tempTheta];
      dThetaList = Flatten[tempDtheta];
      ];

© refrOneVar, dRefrOneVar, d2RefrOneVar

refrOneVar[y_] := Apply[nolcRefractivityF, Flatten[Append[{y}, parList]]];
dRefrOneVar[y_] := Apply[nolcDrefractivityF, Flatten[Append[{y}, parList]]];
d2RefrOneVar[y_] := Apply[nolcD2refractivityF, Flatten[Append[{y}, parList]]];
setVals, fillLists

setVals[npars_] := Module[{i},
  For[i = 2, i <= npars, i++, par[i] = nolcNumParams + i];
  For[i = 2, i <= npars, i++, AppendTo[nolcNames, "nolcPar[" <> ToString[i] <> "] " ] ];
];

fillLists[params_, deltaR_] := Module[{r},
  rList = {top}; yList = {top}; thetaList = {0.0};
  dThetaList = {0.0};
  While[(r = rList[[-1]] - deltaR) > bottom,
    rList = {rList, r};
    thetaList = {thetaList, nolcTheta[r, rPrimeMax, dRefrOneVar]};
    dThetaList = {dThetaList, nolcDtheta[r, rPrimeMax, dRefrOneVar, d2refrOneVar]};
  ];
  (* Add the point at the surface. *)
  rList = {rList, bottom };
  thetaList = {thetaList, nolcTheta[bottom, rPrimeMax, dRefrOneVar]};
  dThetaList = {dThetaList, nolcDtheta[bottom, rPrimeMax, dRefrOneVar, d2refrOneVar]};

rList = Flatten[ rList ];
thetaList = Flatten[ thetaList ];
dThetaList = Flatten[ dThetaList ];
];
nolcSetUp[datafile_, params_, opts__] :=
Module[{modelResults, parVals, index, radius, temperature, pressure, 
density, dDensity, d2Density, 
fMax, i, j, fileList, 
yParamData, InterpData, rOfYData, numFiles, increment, 
fluxInterpList, rInterpList, fluxInterp, rInterp},

Print["Starting ", Date[]];
refr0 = opNuSTP/.{opts}/.Options[nolcSetUp];
deltaR = opDeltaR/.{opts}/.Options[nolcSetUp];
(* should be less than a scale height *)
intOrder = opInterpolationOrder/.{opts}/.Options[nolcSetUp];
surfRad = opSurfaceRadius/.{opts}/.Options[nolcSetUp];
extTop = opExtTop/.{opts}/.Options[nolcSetUp];
extSpacing = opExtSpacing/.{opts}/.Options[nolcSetUp];
extQ = opAddExt/.{opts}/.Options[nolcSetUp];
intBinQ = opIntBin/.{opts}/.Options[nolcSetUp];
interpPts = opInterpPts/.{opts}/.Options[nolcSetUp];
focusQ = opFocus/.{opts}/.Options[nolcSetUp];
mRec = opMaxRecursion/.{opts}/.Options[nolcSetUp];
workPrec = opWorkPrec/.{opts}/.Options[nolcSetUp];

setVals[params[[nolcNumParams]]];

modelResults=loadModelResults[datafile];

numFiles=Length[modelResults]-1;
fileList=Table[i, {i, numFiles}];
(* all these variables are now defined as lists and assigned as foo[[i]]=bar, 
so we need to initialize them to lists of the right length here *)
parVals=Table[{}, {j, numFiles}];
index=Table[{}, {j, numFiles}];
radius=Table[{}, {j, numFiles}];
temperature=Table[{}, {j, numFiles}];
pressure=Table[{}, {j, numFiles}];
density=Table[{}, {j, numFiles}];
d2Density=Table[{}, {j, numFiles}];
refrParList=Table[{}, {j, params[[nolcNumParams]]}];
{refrLog,dRefrLog,d2RefrLog}=Table[{}, {i,3}, {j, numFiles}];
{refractivityFtmp,dRefractivityFtmp,d2RefractivityFtmp}=Table[{}, {i,3},
{j, numFiles}];

(* the refractivityF functions get set to regular functions, not Interpolations, 
so Mathematica needs a name for these functions - {} doesn't work *)
refractivityF=Table[ToExpression["refFn"<>ToString[j]], {j, numFiles}];
dRefractivityF=Table[ToExpression["dRefFn"<>ToString[j]], {j, numFiles}];
d2RefractivityF=Table[ToExpression["d2RefFn"<>ToString[j]], {j, numFiles}];

Table[{parVals[[j]], index[[j]], radius[[j]], temperature[[j]], 
pressure[[j]], density[[j]], dDensity[[j]], d2Density[[j]]} =
modelResults[[j+1]], {j, numFiles}];
(* make refractivity interpolation functions *)
{refractivity,dRefractivity,d2Refractivity} = 
refr[density,dDensity,d2Density,numFiles];

Table[{refrLog[[i]],dRefrLog[[i]],d2RefrLog[[i]]} = 
Log[{refractivity[[i]],-dRefractivity[[i]],d2Refractivity[[i]]}],{i,numFiles}];

Table[ refractivityFtmp[[]] = Interpolation[ Transpose[ {radius[[j]],
refrLog[[j]],dRefractivity[[j]]}] ];

Table[ dRefractivityFtmp[[]] = Interpolation[ Transpose[ {radius[[j]],
refractivityFtmp[[j]],dRefractivity[[j]]}] ];

Table[ d2RefractivityFtmp[[]] = Interpolation[ Transpose[ {radius[[j]],
refractivity[[j]],dRefractivity[[j]]}] ];

Map[(refractivityF[[#]][#val_]:=Exp[refractivityFtmp[[#]][#val]])&,
fileList];

Map[(dRefractivityF[[#]][#val_]:=Exp[dRefractivityFtmp[[#]][#val]])&,
fileList];

Map[(d2RefractivityF[[#]][#val_]:=Exp[d2RefractivityFtmp[[#]][#val]])&,
fileList];

bottom=Min[radius[[1]]];
top=Max[radius[[1]]];
(* Now bottom is the smallest minimum of the r lists *)
For[i=2, i<=numFiles, i++,
If[bottom<Min[radius[[i]]],bottom=Min[radius[[i]]]]; 
If[top>Max[radius[[i]]],top=Max[radius[[i]]]]; 
];

(* to avoid errors when calling interpolation functions at the end
points, move the top and bottom values in a little bit (they
shouldn't give errors at the end points, but they do for some
reason) *)
bottom += .01;
top -= .01;

(* Set the top value for the cap if it was not specified in the
options sent to this function
If[extTop===Null, extTop=top*3];*)

rPrimeMax=top;
increment=(top-bottom)/interpPts;

rVals=Table[bottom+i*increment,{i,0,interpPts}];

For[i=1, i<=params[nolcNumParams], i++,
refrParList[[i]]={};
j=1;
For[j=1, j<=numFiles, j++,
    refrParList[[i]]=
    {refrParList[[i]], Table[parVals[[j]][[i]],{k, Length[rVals]}]};
    refrParList[[i]]=Flatten[refrParList[[i]]];
];

Print["Making refractivity interpolation functions ", Date[]];

(* Make multi-dimensional interpolation functions for refractivity 
and its derivatives that can be accessed from outside the package *)

refrListFlat=Flatten[Table[Map[refractivityFtmp[[i]], rVals], 
    {i, numFiles}]];

dRefrListFlat=Flatten[Table[Map[dRefractivityFtmp[[i]], rVals],
    {i, numFiles}]];

d2RefrListFlat=Flatten[Table[Map[d2RefractivityFtmp[[i]], rVals],
    {i, numFiles}]];

rListFlat=Flatten[Table[rVals, {i, numFiles}]];
rParData={rListFlat};
i=1;
For[i=1, i<=params[[nolcNumParams]], i++, 
    rParData=Append[rParData, refrParList[[i]]]];

If[numFiles==1,
    nolcRefractivityF[r_,p_]:=refractivityF[[1]][r];
    nolcDrefractivityF[r_,p_]:=dRefractivityF[[1]][r];
    nolcD2refractivityF[r_,p_]:=d2RefractivityF[[1]][r];

    nolcRefractivityFtmp =Interpolation[Transpose[
        Append[rParData, refrListFlat]],
        InterpolationOrder->intOrder];
    nolcDrefractivityFtmp =Interpolation[Transpose[
        Append[rParData, dRefrListFlat]],
        InterpolationOrder->intOrder];
    nolcD2refractivityFtmp=Interpolation[Transpose[
        Append[rParData, d2RefrListFlat]],
        InterpolationOrder->intOrder];
    nolcRefractivityF[vals___]:= Exp[Apply[nolcRefractivityFtmp, vals]];
    nolcDrefractivityF[vals___]:= -Exp[Apply[
        nolcDrefractivityFtmp, vals]];
    nolcD2refractivityF[vals___]:=Exp[Apply[
        nolcD2refractivityFtmp, vals]];
];

© Constants
nolc package

(*refr=2.980 10^-4;*)
loschmidtsNo= 2.68719 10^-19;
(* particles/cm^3 from Liou p.354*)

\[ \text{\textbf{nolcNumber Density}} \]

\[
\text{nolcNumberDensity}[\text{radius}_\text{, par1}_\text{, nuSTP}_\text{]} := \text{nolcRefractivityF}[\text{radius}, \text{par1}] \times \text{loschmidtsNo}/\text{nuSTP};
\]

\[ \text{\textbf{nolcYfromTime}} \]

(* function to make y list from time list *)
\[
\text{nolcYfromTime}[p_, t_] := \text{Sqrt}[p[[\text{nolcRhoMin}]^2 + p[[\text{nolcV}]^2*(t-p[[\text{nolcTmid}])^2]]);
\]

\[ \text{\textbf{makeRofYF, fluxBend}} \]

\[
\text{makeRofYF}[\text{params}_\text{]} := \text{Module[}
\text{parList=Table[params[[j]],\{j,nolcPar[1],nolcNumParams+params[[nolcNumParams]]\}};\text{fillLists[params, deltaR];}
\text{yList=rList + params[[nolcDistance]] thetaList;}
\text{minY=Min[yList];}
\text{maxY=Max[yList];}
\text{nolcRoFyF=Interpolation[Transpose\{yList,rList\]];}
\]
\[
\text{fluxBend}[p_,ys_,rs_]:=\text{Module[}
\text{rtmp=rs;}
\text{fluxList=Table[0,\{i,Length[ys]\}};\text{iList=\{\};}
\text{For[i=1,i<=Length[ys],i++,
\text{If[rs[[i]]!=Null,
\text{If[bottom <= rs[[i]] <= top,
\text{dthetaval=nolcDtheta[rs[[i]],rPrimeMax,dRefrOneVar, d2refrOneVar];
\text{If[extQ,dthetaval+=extDtheta[rs[[i]],rPrimeMax,dRefrOneVar[rs[[i]],
\text{reFroneVar[rs[[i]],order]];}
\text{fluxList[[i]]=1/(1. + p[[nolcDistance]] dthetaval);}
\text{If[rs[[i]] > top, (*this is just a repeat of the code in the second part
\text{of addExt without the loop. Perhaps it ought to be
\text{\}}]
\]
\]
made in to its own function some day...
I did not need to put in an If[extQ..] statement
in here, since if extQ is False, then rs[[i]] will
be Null for any r>top, since addExt never got called*)

nuMax=refrOneVar[top];
lamMax=-dRefrOneVar[top] top/nuMax;
lamR=nu[lamMax,top,rs[[i]]];
nuR=nu[lamMax,top,rs[[i]],nuMax];
dthetaval=Sqrt[2 Pi lamR^3] nuR/rs[[i]]

JleGroup'occLightCurves'plutoAug903'dthetahSeries[order,
1/lamR,0]/N;

fluxList[[i]] = 1/(1.+p[[nolcDistance]] dthetaval)];
If[ys[[i]] > maxY, fluxList[[i]] = 1];
If[rs[[i]] =!= Null) && (rs[[i]] < (top+bottom)/2) && (b-1) &&
(i<Length[ys]) &&
 (((rs[[i-1]]===Null) && (rs[[i+1]]===Null)),
 ((rs[[i-1]]=!=Null) && (rs[[i+1]]===Null)),

(* This means that rs[[i]] is (a) smallest r that is still larger
 than the surface radius. The first line verifies that r is above
 the surface radius, but not in the upper half of radii. The second
 line verifies that rs[[i]] is adjacent to an r that is NOT within
 the interpolation range - since rs[[i]] is in the bottom half, this
 means that rs[[i]] is at a boundary between above the surface and
 below it. *)

iList=Append[iList, i];
]
]

(* fix points that include the discontinuity at the surface*)
If[intBinQ==True,
For[j=1, j<=Length[iList], j++,
  ival=iList[[j]];
  If[rs[[ival+1]] =!= Null, deltaR = Abs[rs[[ival+1]]-rs[[ival]]]];
ivp=ival-1;
  deltaR = Abs[rs[[ival]]-rs[[ival-1]]];
ivp=ival+1];
  (* ivp is the adjacent point below the surface
 the other point that could represent the critical
 bin *)

If[Abs[rs[[ival]]] - surfRad] <= deltaR/2 (* the rs[[ival]] bin contains the
surface *),

fluxList[[ival]]=fluxList[[ival]] (deltaR/2+rs[[ival]]-surfRad)/deltaR,
(* If ivp is the critical bin, then we know at least half the
bin contributes to the flux - the half above rs[[ival]], plus
whatever portion of the lower half of the bin is above the
surface.

If ivp is not the critical bin, then at most half the bin
contributes. We know that rs[[ival]] is within deltaR of not
is the top of the bin (rs[[ival]]-deltaR/2) minus the surface *)
fluxList[[ivp]]=fluxList[[ival]] (rs[[ival]]-surfRad-deltaR/2)/deltaR;
rtmp[[ivp]]=surfRad-deltaR/2 (*was Null, but now need to have
non-zero flux, so we need an r value
here back in fluxF *)

};

(* fix points that include the discontinuity at the surface*)
If[intBinQ==True,
For[j=1, j<=Length[iList], j++,
  ival=iList[[j]];
  If[rs[[ival+1]] =!= Null, deltaR = Abs[rs[[ival+1]]-rs[[ival]]]];
ivp=ival-1;
  deltaR = Abs[rs[[ival]]-rs[[ival-1]]];
ivp=ival+1];
  (* ivp is the adjacent point below the surface
 the other point that could represent the critical
 bin *)

If[Abs[rs[[ival]]] - surfRad] <= deltaR/2 (* the rs[[ival]] bin contains the
surface *),

fluxList[[ival]]=fluxList[[ival]] (deltaR/2+rs[[ival]]-surfRad)/deltaR,
(* If ivp is the critical bin, then we know at least half the
bin contributes to the flux - the half above rs[[ival]], plus
whatever portion of the lower half of the bin is above the
surface.

If ivp is not the critical bin, then at most half the bin
contributes. We know that rs[[ival]] is within deltaR of not
is the top of the bin (rs[[ival]]-deltaR/2) minus the surface *)
fluxList[[ivp]]=fluxList[[ival]] (rs[[ival]]-surfRad-deltaR/2)/deltaR;
rtmp[[ivp]]=surfRad-deltaR/2 (*was Null, but now need to have
non-zero flux, so we need an r value
here back in fluxF *)

};
\*\* fluxF, fluxFArgsNearLimb, fluxFArgsFarLimb \*

\texttt{fluxF[pars_] := Apply[nolcFluxBendF,pars]*Apply[nolcRofY,pars[[1]]]};
\texttt{fluxF[p,ys_] := Module[{Ifluxtmp),}

(*fluxBend sets each value of the list to be the flux from the bending, or 0 if \texttt{y[[i]]} corresponds to a point below the surface or 1 if \texttt{y[[i]]} corresponds to a point above the top of the data. If \texttt{y} is within the range of the data, we need to multiply it by the focusing factor \texttt{r/y} *)

\texttt{rs=Table[Null,{i, Length[ys]}];}
\texttt{For[i=1,i<=Length[ys],i++,}
\texttt{If[minY <= y[[i]] <= maxY, rs[[i]]=nolcRofYF[y[[i]]]];}
\texttt{}}
\texttt{(rs,fluxtmp)=fluxBend[p,ys,rs];}
\texttt{If[focusQ,For[i=1,i<=Length[ys],i++,}
\texttt{If[(fluxtmp[[i]]!=0)&&(fluxtmp[[i]]!=1),
\texttt{fluxtmp[[i]]=fluxtmp[[i]] rs[[i]]/ys[[i]]]};]
\texttt{fluxtmp}
\}

\*\* nolcNearLimb, nolcNearLimbRatio, nolcTwoLimb, nolcTwoLimbRatio \*

This function was the basis of nolcNearLimb etc. It is inactive but kept for reference.

\texttt{nolcLightCurve1[params,times_] := Module[{ratio},
\texttt{If[params[[nolcStarRatio]]==null,
\texttt{ratio=(params[[nolcFull]]-params[[nolcBkgd]])/params[[nolcBkgd]],
\texttt{ratio=params[[nolcStarRatio]]];
\texttt{Map[(params[[nolcFull]]+params[[nolcSlope]])\*}
\texttt{(#-params[[nolcTmid]])/(1+ratio) \*}
\texttt{(1+ratio*fluxF[fluxFArgsNearLimb[#]])&},times]
\texttt{nolcNearLimb[p,t_] := Module[{},
\texttt{makeRofYF[p];
\texttt{fluxF[ p, nolcYfromTime[p,t]] (p[[nolcFull]] - p[[nolcBkgd]])}
\texttt{+ p[[nolcBkgd]]
\texttt{+ p[[nolcSlope]] (t - p[[nolcTmid]])}}
\]

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App6 nolc package

\[
\text{nolcNearLimbRatio}[p\_, t\_] := 
\text{nolcNearLimb}[\text{nolcConvParamsFromRatio}[p], t]
\]

\[
\text{nolcFarLimb}[p\_, t\_] := \text{Module}[{}, 
\quad \text{makeRofYF}[p]; 
\quad \text{Abs}[\text{fluxF}[p, -\text{nolcYfromTime}[p, t]]] (p[[\text{nolcFull}]] - p[[\text{nolcBkgd}]]) 
\quad + p[[\text{nolcBkgd}]] 
\quad + p[[\text{nolcSlope}]] (t - p[[\text{nolcTmid}]]) 
\]
\]

\[
\text{nolcTwoLimb}[p\_, t\_] := \text{Module}[{}, 
\quad \text{makeRofYF}[p]; 
\quad (\text{fluxF}[p, \text{nolcYfromTime}[p, t]] + \text{Abs}[\text{fluxF}[p, -\text{nolcYfromTime}[p, t]]]) * 
\quad (p[[\text{nolcFull}]] - p[[\text{nolcBkgd}]]) 
\quad + p[[\text{nolcBkgd}]] 
\quad + p[[\text{nolcSlope}]] (t - p[[\text{nolcTmid}]]) 
\]
\]

\[
\text{nolcTwoLimbRatio}[p\_, t\_] := 
\text{nolcTwoLimb}[\text{nolcConvParamsFromRatio}[p], t]
\]

\[
\text{nolcConvParamsToRatio} \quad \text{nolcConvParamsToRatio}[p\_\text{List}] := \text{Module}[ 
\quad \{\text{ratioParameters} = p\}, 
\quad \text{ratioParameters}[[\text{nolcStarRatio}]] = (p[[\text{nolcFull}]]/p[[\text{nolcBkgd}]] - 1); 
\quad \text{ratioParameters}
\]
\]

\[
\text{nolcConvParamsFromRatio} \quad \text{nolcConvParamsFromRatio}[p\_\text{List}] := \text{Module}[ 
\quad \{\text{newParameters} = p\}, 
\quad \text{newParameters}[[\text{nolcBkgd}]] = p[[\text{nolcFull}]]/(1 + p[[\text{nolcStarRatio}]]);
\quad \text{newParameters}
\]
\]

\*
End Package

End[(*Private*)];
EndPackage[];

\*
Tests

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