Quantifying the effect of dispersion
in Continental Shelf sound propagation

by

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Submitted to the Department of Ocean Engineering
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Abstract

In a medium in which all waves have the same speed of propagation, a broadband signal of any form, will propagate without distortion. However, in a dispersive medium, such as a multi-modal ocean acoustic waveguide the situation is otherwise. All propagation is accompanied by a change in the temporal and spatial form of the signal. A quasi-statistical method is introduced, in which the temporal and the spatial moments of the signal can be derived. Those measures, which also have physical interpretations, give an insightful view of the signal behavior due to waveguide dispersion and can be related to some practical measurements, such as travel time, range error and time and frequency spread. The method is then applied to cases where a broadband LFM signal propagates in various ocean waveguide environments and is received at a single hydrophone.

Since a broadband pulse propagating in a dispersive medium is no longer time-invariant, and cannot be expressed as a replica of the source signal, some difficulties arise when a matched-filter approach is applied in a waveguide. Analytic expression for the degradation of match-filter performance due to dispersion is developed and applied in various oceanic waveguides.

Finally, the asymptotic properties of the maximum-likelihood estimator (MLE) are used to examine the statistical errors and biases that occur in the ocean acoustic inverse problem of localizing a broadband source signal in range and in depth in a shallow water waveguide with a single receiver. From the asymptotic properties, necessary conditions on sample size and signal-to-noise ratio (SNR) are determined for the localization MLE to become asymptotically unbiased and attain the Cramér-Rao Lower Bound.
"There are no classes in life for beginners; right away you are always asked to deal with what is most difficult"

— Rainer Maria Rilke

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To my beloved wife, Liz
in appreciation of her patience, understanding and help

and in loving memory of my son, Yarin. I miss you 🌙
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1. Introduction

1.1. Background

A broadband pulse generated in an ocean waveguide can spread in both time and space as it propagates to greater ranges. This effect is called dispersion [1], and it is caused by the variation of the phase velocity with frequency. While phase velocity is the speed at which a constant phase surface appears to move along in a given direction, group velocity is the speed at which energy is transported in a waveguide. In an ocean waveguide, the group velocity is also varies with frequency and more specifically, each normal mode of the received signal has associated with it a group velocity.

Previous heuristic methods suggested that the dispersion could be characterized by examining group velocity. One standard way of analyzing such time-frequency structure of a broadband pulse propagating is by the sonogram method [2]. A sonogram output exhibits the time and frequency dispersion relationship and can be interpreted as a display of modal group velocities versus ranges. Sonogram outputs are often so complicated that the mode structure cannot be identified. Also, the uncertainty in the group velocity measurement using this method is quite large, resulting in a poor frequency dispersion measurement especially in a shallow water environment. A method that improved the group velocity measurements using the stationary phase approximation [3], is based on the fine structure of the frequency dispersion. This was found to work in a deep water (i.e. upward-refracting Arctic environment with negligible bottom interaction) but is limited in shallow water, where significant bottom interaction leads to multi-modal propagation even at longer ranges. Many other methods exist that attempt to improve the estimation of modal group velocity [4].

Travel-time between a source and a receiver is the most commonly used measurement datum in ocean acoustic source localization [5], environmental imaging [1] and tomography [6]. The accuracy with which travel time and other parameters can be measured (or estimated) is fundamentally limited by (i) ambient acoustic noise, (ii)
interference between unresolved ray paths due to surface reflections and bottom interaction (iii) uncertainties in sound speed structure as a function of depth and range.

An acoustic signal propagating from a point source in a free-space retains the time dependence of its waveform with a time shift in the amount $r/c$, where $r$ is the range from the source to the measurement point and $c$ is the speed of sound. Information understood at the source then can be conveyed over distance $r$ without the necessity for translation at the receiver. Difficulties start to arise from the multi-modal and dispersive nature of waveguide propagation and scattering of sound in the ocean, where boundary interaction and interference causes the waveform received at some distance to differ from the source in its time dependence by more than a simple delay. There is no absolute method for determining the time it takes a signal to propagate a certain fixed distance in a dispersive medium. The received signal is then not simply a replica of the transmitted signal, but is the sum of unresolved signals propagating over many paths, all with different travel time. This gives the rise to a variety of multipath spreading effects, such as travel time spreading, frequency spreading and range spreading. This effect of dispersion can be also explained by the normal mode theory. In a dispersive medium, many modes are trapped within the waveguide even at long ranges. The different modes each propagate with different group velocity interfere with each other causing signal distortion that influences the performance of range estimation. Thus, the matched-filter approach which relies upon the time invariance of the signal, to within a constant shift to be determined, is no longer optimal when it is applied in a waveguide.

All these effects complicate the spatial structure of the field incident on or scattered from a submerged target or rough surface patch, and also distort the temporal dependence of the original source signal. These issues are not found in free space and are not significant in deep-water but must be accounted for in shallow water environment due to dispersion. In this class of problems, statistical analysis and spectral analysis are used in the study of the unresolved signals, especially if source localization is the goal. These techniques often require the non-linear inversion of acoustic field data measured at the receiver [7]. Furthermore, the measured data, which is also randomized by additive noise, yields estimates with biases and variance that are difficult to predict, and it has becomes popular to use estimation bounds on mean square errors, instead of attempting to compute
them directly. One particular popular estimation bound is the Cramér-Rao Lower bound (CRLB), which gives the minimum variance possible for an unbiased estimator. It is only close to the true mean square error when the signal-to-noise ratio is high enough.

The main objective of my thesis is to investigate and quantify the Dispersive effects a broadband pulse may encounter while propagating to longer range, and to examine the coupling of the shallow-water environment to range and depth localization performance. All the analysis is done for a passive sonar system, one-way propagation, using a single hydrophone receiver. I chose to concentrate on three specific, practical areas:

1) Quantitative measures of dispersion in time and frequency.
2) Matched-Filter degradation in a waveguide.
3) Range and depth performance of the Maximum-Likelihood Estimator (MLE) for one way transmission in a shallow water waveguide.

1.2. Thesis outline

The structure of my thesis is as follows; First, I chose to present and discuss the theoretical issues (Chapters 2 through 4), and then I demonstrate the main concepts using various illustrative examples (Chapter 5).

Chapter 2 introduces a general quasi-statistical mathematical method, which provides quantitative definitions for the temporal and spatial moments of a broadband signal; such as time-spread, time-delay, bandwidth-spread and bandwidth-delay. These quantities have a physical meaning, and can be used to characterize the effect of dispersion.

In Chapter 3, an analytical expression, which quantifies the degradation in the performance of a Matched-Filter approach is developed, and applied to the case where a received broadband signal is matched to the transmitted pulse in a waveguide.

In Chapter 4, higher order approximations to the first-order bias and second-order variance of the MLE are applied to the problem of localizing a broadband signal in range and depth in a shallow water waveguide. These expressions are then used, to determine a
necessary condition on the sample size for the MLE to become unbiased and attain the minimum variance as expressed by the Cramér-Rao Lower bound.

In Chapter 5, the concepts of the previous chapters are illustrated and analyzed for a broadband LFM signal propagating in various representative shallow-water environments.

Chapter 6 contains the summary and conclusion of the thesis.
2. Time-Frequency Characteristics

2.1. Fourier Synthesis of Frequency Domain Solutions

In many problems of practical importance in underwater acoustics (long-range propagation of pulses of "finite" bandwidth) spectral analysis is used to compute the acoustic field due to the transmission of a finite duration, time domain pulse or transient by a sound source.

Let \( q(t) \) be the time-dependence of the source signal. We can associate with this its frequency description by means of its Fourier transform \( Q(f) \), the complex frequency spectrum of \( q(t) \).

\[
q(t) = \int_{-\infty}^{\infty} Q(f)e^{-j2\pi ft} df \quad (2-1)
\]

\[
Q(f) = \int_{-\infty}^{\infty} q(t)e^{j2\pi ft} dt \quad (2-2)
\]

The inhomogeneous, time-dependent wave equation (assuming constant density within each layer) can be expressed by:

\[
\nabla^2 \Phi(r, t) - \frac{1}{c^2} \frac{\partial^2 \Phi(r, t)}{\partial t^2} = q(t)\delta(r - r_0) \quad (2-3)
\]

Assuming harmonic time-dependence for both field and source, we can obtain the frequency-domain wave equation, or the inhomogeneous Helmholtz equation:

\[
\left[ \nabla^2 + k^2(r) \right] \Phi(r, f) = Q(f)\delta(r - r_0) \quad (2-4)
\]

where \( k(r) \) is the medium wavenumber at radial frequency \( \omega \), \( k(r) = \omega/c(r) \).

The channel filter between the source at location \( r_0 \) and any other location in the channel at the receiver \( r \) is characterized in the time-domain by an impulse response function \( g(r|r_0|t) \), or in the frequency-domain by the transfer response function \( G(r|r_0|f) \).
As mentioned above, since most computational solutions of the time-dependent wave equation are implemented in the frequency-domain, it is conventional to use the channel transfer function, or the time-harmonic waveguide Green’s function $G(r|r_0|f)$ representation. Green’s function by definition [8] is the solution to the Helmholtz Equation, when the source signal is a point source, which is the case we wish to investigate in this thesis. The solution to the time-dependent wave equation (Eq.2-3) can be obtained via a Fourier transform of the frequency-domain solution as\(^1\),

$$
\Phi(r|r_0|t) = \int_{-\infty}^{\infty} \Phi(r|r_0|f)e^{-i2\pi f t} df
$$

$$
= \int_{-\infty}^{\infty} Q(f)G(r|r_0|f)e^{-i2\pi f t} df
$$

where $\Phi(r|r_0|t)$ is the complex analytic signal representation of the received field in time. We can see that the pulse solution is obtained by simply multiplying the time-harmonic solution by $Q(f)$ and then integrating over frequency $f$ from $-\infty$ to $\infty$. In other words, we “sum” the contributions from all the frequency components contained in the transmitted pulse. $\Phi(r|r_0|f)$ is the spatial-dependent part of the acoustic pressure obtained from the time-harmonic solution, and can be expressed as,

$$
\Phi(r|r_0|f) = \int_{-\infty}^{\infty} \Phi(r|r_0|t)e^{i2\pi f t} dt = Q(f)G(r|r_0|f)
$$

thus, $\Phi(r|r_0|t)$ and $\Phi(r|r_0|f)$ mutually determine one another.

**Example 2.1 – Free Space**

In free-space, for a point source at location $r_0$, and a receiver at range $r$, the time-harmonic Green’s function $G(r|r_0|f)$ in spherical coordinates is given by,

$$
G(r|r_0|f) = \frac{1}{4\pi} \frac{e^{i|r-r_0|}}{|r-r_0|}
$$

and the received field can than be written as,

\[
\Phi(r|r_0|t) = \int_{-\infty}^{\infty} Q(f) \frac{1}{4\pi} \frac{e^{i[r-r_0]}}{|r-r_0|} e^{-i2\pi f t} df = \frac{1}{4\pi \cdot |r-r_0|} \int_{-\infty}^{\infty} Q(f) e^{i2\pi(r-r_0)e^{-i f}} df 
\] (2-9)

Example 2.2 – Waveguide

Using the modal formulation far field approximation in cylindrical coordinates, the Green's function in the waveguide between a point at the origin (source location) \( r_0 \) and a field point at the receiver \( r \) can be expressed as a sum of normal modes [9],

\[
G(r|r_0|f) = \frac{ie^{-i\pi/4}}{\rho(z_0)\sqrt{8\pi}} \sum_m \Psi'_m(z_0) \Psi_m(z) \frac{e^{ik_m R}}{\sqrt{k_m R}} 
\] (2-10)

where \( R \) is the horizontal range from the source to the receiver, \( R = \sqrt{(x-x_0)^2 + (y-y_0)^2} \); \( k_m \) is the horizontal wavenumber for the \( m \)th mode; \( z_0, z \) are the depths of the source and the receiver, respectively; \( \rho(z_0) \) is the density at the source; and \( \Psi_m \) are the mode functions. The received signal can than be expressed using the source spectrum and the Green's function as,

\[
\Phi(r|r_0|t) = \frac{ie^{-i\pi/4}}{\rho(z_0)\sqrt{8\pi}} \int_{-\infty}^{\infty} Q(f) e^{-i2\pi f} df \sum_m \Psi'_m(z_0) \Psi_m(z) \frac{e^{ik_m R}}{\sqrt{k_m R}} 
\] (2-11)

2.2. Time-Frequency measures

2.2.1. General concept

The approach adopted here is to employ the quasi-statistical mathematical method of Gabor [10] to characterize the temporal and spatial moments of a signal. In mechanics, this kind of method is used to define the moment of inertia; in probability theory it expresses the second moment about the mean, or the variance. The following implementation is a method of analyzing signals in which time and frequency play symmetrical parts, and which contains "time analysis" and "frequency analysis" as special cases.
2.2.2. Definitions and Common Equalities

The definitions introduced below are adapted from the signal moment concept described in Gabor’s paper, where the normalized magnitude square of the received field $\Phi(r \mid r_0 \mid t)$, plays the role of a probability density or weighting function:

$$p(r \mid r_0 \mid t) = \frac{\Phi(r \mid r_0 \mid t)^2}{E(r \mid r_0 \mid t)}$$  \hspace{1cm} (2-12)

where,

$$E(r \mid r_0 \mid t) = \int_{-\infty}^{\infty} \Phi(r \mid r_0 \mid t)^2 \, dt$$  \hspace{1cm} (2-13)

The mean time of propagation can be written as,

$$\tau(r, r_0) = \frac{\int_{-\infty}^{\infty} t \Phi(r \mid r_0 \mid t)^2 \, dt}{\int_{-\infty}^{\infty} \Phi(r \mid r_0 \mid t)^2 \, dt}$$  \hspace{1cm} (2-14)

while the time delay for propagation between $r_0$ and $r$ can then be taken as,

$$\tau_d(r, r_0) = \frac{\int_{-\infty}^{\infty} t \Phi(r \mid r_0 \mid t)^2 \, dt}{\int_{-\infty}^{\infty} \Phi(r \mid r_0 \mid t)^2 \, dt} - \frac{\int_{-\infty}^{\infty} t \Phi(r_0 \mid r_0 \mid t)^2 \, dt}{\int_{-\infty}^{\infty} \Phi(r_0 \mid r_0 \mid t)^2 \, dt}$$  \hspace{1cm} (2-15)

The effective signal duration is measured by,

$$\Delta \tau(r, r_0) = \left[ \frac{\int_{-\infty}^{\infty} t^2 \Phi(r \mid r_0 \mid t)^2 \, dt}{\int_{-\infty}^{\infty} \Phi(r \mid r_0 \mid t)^2 \, dt} - \frac{\int_{-\infty}^{\infty} t \Phi(r \mid r_0 \mid t)^2 \, dt}{\int_{-\infty}^{\infty} \Phi(r \mid r_0 \mid t)^2 \, dt} \right]^{1/2}$$  \hspace{1cm} (2-16)

This simply represents the standard deviation of the signal from its mean arrival time. While the effective signal duration of the original signal in free-space is
simply $\Delta \tau (r_0, r_0)$, the time spread of the received signal due to waveguide dispersion is defined as

$$\Delta \tau _s (r, r_0) = \Delta \tau (r, r_0) - \Delta \tau (r_0, r_0) \quad (2-17)$$

Similarly, the mean frequency can be defined as,

$$\bar{\beta}(r, r_0) = \frac{\int \Phi(r|r_0,f)^2 \, df}{\int |\Phi(r|r_0,f)|^2 \, df} \quad (2-18)$$

the frequency delay (or frequency shift) as,

$$\beta_d = \bar{\beta}(r, r_0) - \bar{\beta}(r_0, r_0) \quad (2-19)$$

and the effective bandwidth as,

$$\Delta \beta(r, r_0) = \left[ \left( \frac{\int f^2 \Phi(r|r_0,f)^2 \, df}{\int |\Phi(r|r_0,f)|^2 \, df} \right)^2 \right]^{1/2} \quad (2-20)$$

In essence, $\Delta \beta(r, r_0)$ is the second moment of the normalized spectrum about its mean. The mean frequency and the effective bandwidth of the original signal in free-space are respectively $\bar{\beta}(r_0, r_0)$ and $\Delta \beta(r_0, r_0)$. Again, in the same manner as the time spread was defined earlier in Eq. (2-17), let us define the bandwidth spread of the received signal due to the waveguide dispersion as,

$$\Delta \beta _s (r, r_0) = \Delta \beta (r, r_0) - \Delta \beta (r_0, r_0) \quad (2-21)$$

Some useful elegant reciprocal relations between the time domain and the frequency domain take the form
\[
\int_{-\infty}^{\infty} t^n |\Phi(r|r_0|t)|^2 \, dt = \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \right)^n \frac{d^n \Phi(r|r_0|f)}{df^n} \Phi^*(r|r_0|f) \, df \tag{2-22}
\]

\[
\int_{-\infty}^{\infty} f^n |\Phi(r|r_0|f)|^2 \, df = \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \right)^n \frac{d^n \Phi(r|r_0|t)}{dt^n} \Phi^*(r|r_0|t) \, dt \tag{2-23}
\]

\[
\int_{-\infty}^{\infty} |\Phi(r|r_0|t)|^2 \, dt = \int_{-\infty}^{\infty} |\Phi(r|r_0|f)|^2 \, df \tag{2-24}
\]

where the last equation, is well known as the Parseval’s relation for analytic signals.

Some remarks:

- Often in the literature (for example, [11]), expressions for RMS time and RMS bandwidth are used. To obtain those expressions, we simply need to multiply the effective time duration and the effective bandwidth by a factor of \(2\pi\), so they can be written as follows,

\[
r_{ms} = \left[ \frac{(2\pi)^2 \int_{-\infty}^{\infty} t^2 |\Phi(t)|^2 \, dt}{\int_{-\infty}^{\infty} |\Phi(t)|^2 \, dt} \right]^{1/2} = 2\pi \cdot \Delta \tau \tag{2-25}
\]

\[
\beta_{ms} = \left[ \frac{(2\pi)^2 \int_{-\infty}^{\infty} f^2 |\Phi(f)|^2 \, df}{\int_{-\infty}^{\infty} |\Phi(f)|^2 \, df} \right]^{1/2} = 2\pi \cdot \Delta \beta \tag{2-26}
\]

- The time-frequency definitions mentioned in this section assumed that the weighting function is a complex signal. Since the Green’s function has in general a complex value, those definitions in their original form are suitable for our case. In order to be consistent with the complex representation, it’s necessary to express the source function as a complex signal too.

- The signal \(\Phi(r|r_0|t)\) itself is needed only in absolute value, which makes this method particularly robust.
Example 2.3

The time-harmonic Green's function \( G(\mathbf{r}|\mathbf{r}_0|f) \) in free-space is given by Eq. (2-8). Let us assume that the source spectrum \( Q(f) \) has a general form of,

\[
Q(f) = Q_0 e^{2\pi i f} \\
\text{then,} \quad \Phi(\mathbf{r}|\mathbf{r}_0|f) = \alpha \cdot e^{2\pi i f} e^{2\pi i f} = \frac{Q_0}{4\pi|\mathbf{r} - \mathbf{r}_0|} (2-28)
\]

\[
\Phi'(\mathbf{r}|\mathbf{r}_0|f) = \frac{d\Phi(\mathbf{r}|\mathbf{r}_0|f)}{df} = 2\pi \left[ \frac{f}{|\mathbf{r} - \mathbf{r}_0|} + \varphi'(f) \right] \Phi(\mathbf{r}|\mathbf{r}_0|f) (2-29)
\]

\[
|\Phi(\mathbf{r}|\mathbf{r}_0|f)|^2 = \Phi(\mathbf{r}|\mathbf{r}_0|f) \Phi'(\mathbf{r}|\mathbf{r}_0|f) = \alpha^2 (2-30)
\]

In free-space there is no dispersion throughout the medium, thus the time dependence of the waveform at the source is the same as the time dependence of the waveform in free-space \( \Rightarrow \Phi(\mathbf{r}_0|\mathbf{r}_0|t) = q(t) \).

Now, substituting Eqs. (2-28) & (2-29), into Eq. (2-14), we obtained the following expression for the time delay,

\[
\Rightarrow \tau_d(\mathbf{r}, \mathbf{r}_0) = \frac{\int_{-\infty}^{\infty} \alpha^2 \left[ \frac{f}{|\mathbf{r} - \mathbf{r}_0|} + \varphi'(f) \right] df}{\int_{-\infty}^{\infty} \alpha^2 df} = \frac{|\mathbf{r} - \mathbf{r}_0|}{c} (2-31)
\]

which is the expected linear relation. Since there is no dispersion, \( \Delta \tau_s(\mathbf{r}, \mathbf{r}_0) = 0 \)

Example 2.4

Let us assume that we have a Gaussian base-band source signal \( q(t) \), which propagates in free-space. Its waveform can be represented as,

\[
q(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} (2-32)
\]

where \( \sigma^2 \) is the variance of the Gaussian function \( q(t) \). As mentioned in the previous example, in a free-space medium, the received field at the source \( \Phi(\mathbf{r}_0|\mathbf{r}_0|t) \) is
equal to $q(t)$. Therefore, $|\Phi|^2$ is a square Gaussian weighting function. The effective signal duration is then $\Delta \tau(r_0, r_0) = \sigma_q / \sqrt{2} = \sigma / \sqrt{2}$.

The spectrum of this Gaussian source signal has the form,

$$Q(f) = e^{-\frac{1}{2} \sigma^2 (2\pi)^2}$$

thus, the variance of the spectrum is $\sigma_Q = 1/2\pi \sigma$, and the effective bandwidth is than $\Delta \beta(r_0, r_0) = \sigma_Q / \sqrt{2} = 1/2\sqrt{2} \pi \sigma$.

The Gaussian signal is not usually used in practical systems and experiments as a source function. However, due to the fact that all its parameters are well known and can be derived analytically both in time-domain and frequency-domain, the Gaussian signal is a mathematical convenient waveform to demonstrate various concepts. As so, I used in my thesis the Gaussian pulse only for validity checks of simulations, but a different, more practical signal (as will be presented later) to illustrate the main concepts.

2.3. Effective medium velocity

The mean horizontal propagation speed of a signal in a medium or the effective medium velocity, is defined here as

$$c_{\text{eff}}(r, r_0) = \frac{\sqrt{(x-x_0)^2 + (y-y_0)^2}}{\tau_d(r, r_0)} = \frac{R}{\tau_d(r, r_0)}$$

(2-34)

where $\tau_d(r, r_0)$ is the time delay of propagation between $r_0$ and $r$ as defined in Eq. (2-15). In normal mode theory, the general definition for the travel time, or time of arrival, of the $m$th trapped mode at the horizontal range $R$ from the source is given by,

$$\tau_{r,m}(f) = \frac{R}{c_{g,m}(f)}$$

(2-35)

where $c_{g,m}$ is the group velocity in the horizontal radial direction of the $m$th mode.
In a configuration where relatively few propagating modes exist, the medium sound speed is dominated by the lowest mode group velocity, and the effective medium velocity, in this case will approach $c_{g,1}$, especially at long ranges. However, in general, due to the waveguide dispersion, we expect that the effective medium velocity will change according to the "degree of dispersion" in the medium, as will be illustrated later.

2.4. The Uncertainty principle

In practice, it is neither feasible nor desirable to operate with infinite bandwidths and infinite time intervals. Time functions must be measured within limited frequency band, and frequency spectra within limited time intervals, in violation of the definition of those functions as a pair of Fourier transforms. It is to be expected that such measurements can be performed only approximately when the available intervals in time and frequency are finite. These difficulties introduced by the combination of the two incompatible concepts in time and frequency are expressed by the uncertainty principle. Using both definitions of effective duration and effective bandwidth, the uncertainty principle given by Gabor [10], can be written as,

$$\frac{\Delta \tau(r, r_0) \cdot \Delta \beta(r, r_0)}{(TB)\text{product}} \geq \frac{1}{4\pi}$$

so long as the integrals defining $\Delta \tau(r, r_0)$ and $\Delta \beta(r, r_0)$ exist. The product between the effective duration and the effective bandwidth is called $TB$ product, and has a lower limit. The waveform that achieves the lower limit of the time-bandwidth product is a constant-carrier pulse with a Gaussian envelope. In order to preserve this lower bound at all ranges, the medium should not alter a signal’s shape as it propagates. For all other signal envelopes the $TB$ product will be higher, as is demonstrated nicely in Ref. [12].
Example 3.4 – LFM Signal

The following example shows a more detailed form of the *uncertainty principle*. Suppose $\mu(t)$ is an arbitrary signal, which has the form $\mu(t) = |\mu(t)|e^{i\phi(t)}$. The *uncertainty principle* can be written as

$$
(2\pi\Delta \tau)^2 (2\pi\Delta \beta)^2 \geq \pi^2 + \alpha^2 \Rightarrow \Delta \tau \Delta \beta \geq \frac{1}{4\pi} \sqrt{1 + \left(\frac{\alpha}{\pi}\right)^2}
$$

(2-37)

where $\alpha$ is the correlation between $\varphi'(t)$ and time $t$, and define as,

$$
\alpha = 2\pi \int_{-\infty}^{\infty} t\varphi'(t)|\mu_n(t)|^2 dt
$$

(2-38)

and $\mu_n$ is the normalized signal, $\mu_n(t) = \mu(t)/\sqrt{\int_{-\infty}^{\infty} |\mu(t)|^2 dt}$.

The parameter $\alpha$ essentially measures the linear FM content of the signal and indicates how closely the actual frequency modulation of the signal $(1/2\pi)\varphi'(t)$ approaches linear FM. We can see that as $\alpha$ increases the lower bound for the $TB$ product will increase too. Thus, for LFM signals we can never obtained the lower limit of Eq. (2-36).

---

3. The Matched-Filter (*Mean-Square Criteria*)

3.1. General Concept

Since the received signal is always measured with independent and additive noise from other sources, the receiver must be optimized in some manner. The question of what constitutes an optimum receiver can seldom be answered exactly if such problems as prior information about the environment, consequences of errors in the interpretation of the measurement, and practical system implementation are taken into consideration. Among the entire class of receivers, the matched-filter receiver is a unique compromise between theoretical performance and ease of implementation [14].

For a problem of detecting a single target at unknown time delay in stationary white Gaussian noise, an engineering approach to receiver optimization would be to maximize the signal-to-noise ratio (SNR) at the receiver output. This approach leads to the matched-filter receiver. It has been shown that this type of receiver preserves all the information the received signal contains. Although it may be argued that preserving the information is not an adequate criterion for receiver optimality, practical aspects favor the matched-filter receiver.

To arrive at the matched-filter concept, we consider the problem of maximizing the SNR at the output of a passive linear filter [13]. Let us assume the filter has an impulse response $h(t)$, and the corresponding spectrum $H(f)$. The filter's input is composed of a signal $s_i(t)$ in addition to a white Gaussian noise (WGN), with constant power spectral density of $N_o/2 [W/Hz]^3$.

\[
s_i(t) + n_i(t) \xrightarrow{\text{Passive Linear Filter}} h(t) \leftrightarrow H(f) \xrightarrow{} s_o(t) + n_o(t)
\]

*Figure 3-1 - The Matched Filter Setup*

---

3 This is the "double-sided" power spectral density, covering positive and negative frequencies. The "single-sided" physical power density (positive frequencies only) is thus $N_o$. 

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If the noise passed through the filter, the noise power density at the filter output is 

\[ \frac{N_0}{2} |H(f)|^2 \]

and the average output noise power is therefore,

\[ \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df \]  

(3-1)

Further, if \( S_i(f) \) is the input signal spectrum, then \( S_i(f)H(f) \) is the output signal spectrum, and the output peak signal power is,

\[ |s_o(\tau)|^2 = \left| \int_{-\infty}^{\infty} S_i(f)H(f)e^{j2\pi f\tau} df \right|^2 \]  

(3-2)

where \( \tau \) represents the signal delay in the filter. The ratio of the output peak signal power (Eq.3-2) to the average noise power (Eq.3-1) is the ratio we wish to maximize, or

\[ \text{SNR}(\tau) = \frac{\int_{-\infty}^{\infty} S_i(f)H(f)e^{j2\pi f\tau} df}{N_0 \int_{-\infty}^{\infty} |H(f)|^2 df} \]  

(3-3)

To find the maximum SNR, we use the Schwartz inequality for complex functions:

\[ \left| \int_{a}^{b} u(x)v(x) dx \right|^2 \leq \int_{a}^{b} |u(x)|^2 dx \cdot \int_{a}^{b} |v(x)|^2 dx \]  

(3-4)

The equality holds only if \( u(x) = kv^*(x) \), where \( v^*(x) \) is the complex conjugate of \( v(x) \), and \( k \) is an arbitrary constant. Identifying \( u(x) \) with \( H(f) \) and \( v(x) \) with \( S_i(f)e^{j2\pi f\tau} \) and applying the Schwarz inequality on the SNR term in Eq. (3-3), we obtain [14]:

\[ \text{SNR}(\tau) \leq \text{SNR}_{\text{max}} = \frac{2E}{N_0} \]  

(3-5)

where,

\[ 2E = \int_{-\infty}^{\infty} |S_i(f)|^2 df \]  

(3-6)

The maximum SNR at the filter output is seen to depend only on the signal energy and the noise power density. Most significantly, it does not depend on the signal waveform.
The filter that maximized the SNR is,

\[ H(f) = ke^{-j2\pi f\tau_m}S^*(f) \quad (3-7) \]

reaching its maximum at \( \tau = \tau_m \), where \( \tau_m \) represents the signal delay in the filter. The impulse response of this filter is,

\[ h(t) = ks^*(t_m - t) \quad (3-8) \]

The impulse response of the filter is equal to the reversed time input waveform, except for an insignificant amplitude factor and a translation in time. As for the frequency domain, the matched filter frequency response is the complex conjugate of the signal spectrum modified by an arbitrary scale factor and phase.

### 3.2. Imperfect Matched-Filter

Let us return to Figure (3-1). Assume that \( s_i(t) \) is the received signal, and \( n(t) \) the noise input to the filter. Since the optimum filter \( h(t) \) is a linear time-invariant function, the filter output is given by,

\[ y(\tau) = \int_{s_i(t)} s_i(\tau - t)h(t)dt + \int_{n_i(t)} n_i(\tau - t)h(t)dt \quad (3-9) \]

where \( s_i(t) \) is the signal component, and \( n_i(t) \) is the noise part. The general expression of the SNR was defined in Eq. (3-3). The variance of the noise can be then written as follows,

\[ E[n^2(\tau)] = E\left(\left[\int n_i(\tau - t)h(t)dt\right]^2\right) = \int \int R_m(t_1, t_2)E[n_i(\tau - t_1)n_i(\tau - t_2)]dt_1dt_2 \quad (3-10) \]

where \( R_m(t_1, t_2) \) is the autocorrelation of the noise. If the noise is a stationary WGN with constant “double-sided” spectral density of \( N_0/2 \) [W/Hz], the autocorrelation function is simply,

\[ R_m(t_1, t_2) = \frac{N_0}{2} \delta(t_1 - t_2) \quad (3-11) \]

and substituting the above into Eq. (3-10), we obtain
\[ E\left[n^2_0(\tau)\right] = \frac{N_0}{2} \int h(t_1)h(t_2)\delta(t_1-t_2)dt_1dt_2 = \frac{N_0}{2} \int [h(t_1)]^2 dt_1 \]  

(3-12)

The SNR at the output of the filter is then,

\[ SNR(\tau) = \frac{s_\tau(\tau)}{E[n_0^2(\tau)]} = \left[ \int s_\tau(\tau-t)h(t)dt \right]^2 \]  \[ N_0/2 \int [h(t)]^2 dt \]  

(3-13)

and if \( \tau_m \) is the time that maximizes the value of the SNR then,

\[ SNR_{\text{max}} = SNR(\tau_m) = \frac{\left[ \int s_\tau(\tau_m-t)h(t)dt \right]^2}{N_0/2 \int [h(t)]^2 dt} \]  

(3-14)

**Example 3.1**

Let \( h(t) \) be an exact matched-filter replica of the transmitted signal \( s_\tau(t) \),

\[
h(t) = \begin{cases} 
  ks_\tau^*(\tau_m-t) & 0 \leq t \leq T \\
  0 & \text{elsewhere} 
\end{cases}
\]  

(3-15)

The filter is simply a time-shift version of the original signal multiply by a gain factor \( k \), where \( k \) is arbitrary and is henceforth taken as unity. Note that this filter expresses a non-dispersive propagation medium, since the shape of the signal remains the same.

Plug Eq. (3-15) into Eq. (3-14), we obtained the following result,

\[ SNR_{\text{max}} = \frac{\left[ \int s_\tau(t)dt \right]^2}{N_0/2 \int [h(t)]^2 dt} = \frac{2E}{N_0} \]  

(3-16)

where \( E \) is the energy of the signal. This result is known as the maximum gain of an ideal matched-filter as was also derived earlier - Eq. (3-5), and it’s the maximum performance of SNR that can be achieved.

\( \diamond \)

Now, let us rewrite Eq. (3-14) as follows [15],

\[ SNR_{\text{max}} = SNR(\tau_m) = \frac{\left[ \int s_\tau(t)dt \right]^2}{N_0/2} \cdot \frac{\left[ \int s_\tau(\tau_m-t)h(t)dt \right]^2}{\int [h(t)]^2 dt \int [s_\tau(t)]^2 dt} \]  

\[ \left[ R_{s_\tau}(\tau_m) \right]^2 \]  

(3-17)
The maximum SNR is then the optimum performance of an ideal matched-filter multiplied by the square magnitude of the correlation coefficient $R_{h,s}(t)$ between the filter $h(t)$ and the received signal $s_r(t)$, where the correlation coefficient is defined as,

$$R_{h,s}(\tau_m) = \frac{\int s_r(\tau_m - t)h(t)dt}{\left[\int [h(t)]^2 dt\int [s_r(t)]^2 dt\right]^{1/2}} \quad (3-18)$$

The highest correlation is achieved when the medium is non-dispersive and the filter has a form of Eq. (3-15). The correlation coefficient is exactly equal to one in this case, which is known as the ideal matched-filter. However, when the medium is dispersive, the form of the matched-filter replica in not exactly known and the optimum performance will not be achieved, thus $R_{h,s}(t)$ be less than unity.

### 3.3. Matching the source signal

Let us frame our problem in the manner depicted in the following figure,

\[ \text{Unknown filter } \quad g(t) \leftrightarrow G(f) \]

\[ \text{Optimum filter } \quad h(t) \leftrightarrow H(f) \]

\[ s_r(t) \quad n(t) \]

**Figure 3-2 - General model setup**

Let $s_r(t)$ be the transmitted signal. The unknown filter $g(t)$ may be some transmission medium, dispersive or not, the characteristics of which we wish to quantify. When the SNR is small, and it is assumed that nothing whatever is known about $g(t)$ except possibly its maximum duration, the optimum filter (for the *mean-square criteria*) turns out to be matched to the input signal if the noise is white [14]. Thus, choosing the matched-filter approach is a natural and a practical selection.
Suppose $s_2(t)$ is the received signal to which a filter is to be matched, where $n(t)$ is the input noise. Since, we want to derive a general analytical expression for the maximum SNR, we assume that the received signal has unspecified form.

The right part of the figure (after the dash-line) is the exact setup of the matched-filter as shown earlier in Figure 3-1, so we can identify $s_2(t)$ as $s_1(t)$. Let $h(t)$ be a matched-filter to the transmitted signal $s_1(t)$,

$$
h(t) = \begin{cases} 
ks_1^*(\tau_1 - t) & \text{if } 0 \leq t \leq T \\
0 & \text{elsewhere}
\end{cases}
$$

(3-19)

where $k$ is arbitrary and is henceforth taken as unity and $\tau_1$ is also an arbitrary time shift constant and will be taken as zero. Substituting Eq. (3-19) into Eq. (3-17), we obtained an expression for the maximum SNR,

$$
\text{SNR}(\tau_m) = \frac{\int |s_2(t)|^2 \, dt}{N_0/2} \left| R_{s_1,s_2}(\tau_m) \right|^2
$$

(3-20)

$$
R_{s_1,s_2}(\tau_m) = \frac{\int s_2(t+\tau_m) s_1^*(t) \, dt}{\left[ \int |s_1(t)|^2 \, dt \right] \left[ \int |s_2(t)|^2 \, dt \right]^{1/2}}
$$

(3-21)

The correlation coefficient $R_{s_1,s_2}$ is a quantitative expression that indicates the amount of degradation in the SNR due to the medium dispersion characteristics. When the medium is non-dispersive the coefficient is exactly unity and less than unity otherwise. However in a dispersive medium, modal interference varies with time, range and depth, which results in degradation in the ideal matched-filter performance, resulting in smaller value for the correlation coefficient.

### 3.4. Range Estimation

Let us take the time delay corresponding to the peak amplitude of a matched filter output – Eq. (3-20), as our estimated time-delay of propagation, denoted as $\tau_m$. A time-delay error can then be written as,
\[ \tau_e = \left| \tau_m - \tau_{\text{ref}} \right| \]  
(3-22)

and a range error as,

\[ \Delta R = \tau_e \cdot c_{\text{ref}} \]  
(3-23)

\( \tau_{\text{ref}} \) is the expected time of arrival; \( \tau_{\text{ref}} \equiv R/c_{\text{ref}} \). Where, \( c_{\text{ref}} \) is a reference sound speed based on a constant approximation and \( R \) is the true horizontal range between the source and the receiver.

The reference sound speed in a non-dispersive medium (i.e. free-space), where only few modes exist within the environment is usually chosen as the group speed of the lowest mode. However, in a waveguide where the medium is dispersive and the sound speed is not constant but depends on many parameters, such as frequency, environmental parameters and range. Usually, one selects a constant value for \( c_{\text{ref}} \) that gives a reasonable error resolution for the specific environment.

As will be demonstrated later in Chapter 5, a good selection that gives reasonable resolution for the expected medium velocity is the asymptotic value of the effective medium velocity \( c_{\text{eff}} \) as range becomes large.

The duration of the received signal is typically larger than the duration of the source signal due to dispersion. As the medium becomes more disperse the received pulse is more spread in time. Thus, in order to calculate the correlation coefficient using Eq. (3-18), the integration time should be selected carefully to include all the received signal information.

### 3.5. Minimum variance related to the signal moments

A Matched-Filter processor can be an optimum procedure for estimating the arrival time of a signal in a non-dispersive medium measured with WGN as discussed in the previous sections. The square root of the Cramér-Rao Lower Bound (CRLB) of an unbiased estimate after signal demodulator becomes,

\[ \sigma_r \geq \sqrt{\text{CRLB}} = \frac{1}{\beta_{\text{rms}} \sqrt{\text{SNR}_{\text{max}}}} \]  
(3-24)
where $\beta_{\text{rms}}$ is the RMS bandwidth as defined in Eq. (2-26), $\text{SNR}_{\text{max}} = 2E/N_0$ is the SNR obtained with a matched-filter, $\sigma_t$ is the standard deviation of any unbiased estimate of time delay $\tau$.

The inverse of the effective bandwidth is a natural measure of a signal's temporal coherence and so appears as a physical factor in Fisher's information measure on time-delay estimation (range resolution) for a deterministic signal in additive Gaussian noise [16].

**Example 3.2 - Rectangular Pulse**

Let $q(t)$ be a perfectly rectangular pulse of length $T$,

$$q(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases} \quad (3-25)$$

The computation of $(\Delta \beta)^2$ for this pulse results in $(\Delta \beta)^2 = \infty$, this implies that the minimum RMS range error is zero and that the measurement can be made with no error. However, in practice, pulses are not perfectly rectangular since a zero rise time or zero fall time requires infinite bandwidth. Finite bandwidths result in finite $(\Delta \beta)^2$.

So, in order to compute $(\Delta \beta)^2$ for a "practical" rectangular pulse, it will be assumed that the pulse spectrum is of the form, $Q(f) = \sin(\pi f T)/\pi f$ for $-B/2 \leq f \leq B/2$ just as with a perfectly rectangular pulse of width $T$, but that the bandwidth is limited so that $Q(f) = 0$ elsewhere. The time waveform of the output filter will be a pulse of width approximately $T$ for $B > \frac{1}{T}$. The integrals in the expression for the effective bandwidth extend, in this instance, to the interval $\pm B/2$ instead of $\pm \infty$. Using the expression for the effective bandwidth, Eq. (2-20), which for $BT \gg 1$ take the form of$^4$,

$$\frac{1}{(2\pi)^2} \frac{2B}{T} \propto \frac{B}{T} \quad (3-26)$$

Therefore, the minimum RMS error in time delay measurement is approximately,

$^4$ [11] - Sec.10
\[
\sigma_r \approx \left( \frac{2\pi^2 T}{B \cdot SNR_{\text{max}}} \right)^{1/2} \propto \frac{1}{\sqrt{B/T}} \frac{1}{\sqrt{SNR_{\text{max}}}} \quad (3-27)
\]

The pulse width \( T \) in the above expression is that of the perfectly rectangular pulse before it is band-limited. It is a good approximation to the width of the band-limited pulse when \( BT \) is large.

The medium does not behave as a perfectly rectangular filter of bandwidth \( B \). However, when the source signal is bandlimited with bandwidth \( B \) (LFM signal, for example), the received signal through the medium will have roughly a band of \( B \). When the medium is non-dispersive, the matched-filter approach can be applied and the minimum error will be roughly as Eq. (3-27). The minimum error in range estimation in a dispersive medium is given in section 5.5.

\[\diamond\]

### 3.6. Dispersive versus Non-Dispersive Media

Methods developed for estimating the time delay of a signal after propagation in free-space, such as taking the delay corresponding to the peak amplitude of a matched filter output, also rely upon the time invariance of the signal, to within the constant shift and phase. The matched filter approach is the most popular since, for a deterministic signal in additive Gaussian noise, it is optimal according to criteria set by information theory and signal-to-noise-ratio (SNR) maximization, and is asymptotically optimal in high SNR according to classical estimation theory. In high SNR, the resolution of a matched filter time delay estimate becomes inversely proportional to the RMS bandwidth of the signal, as has been demonstrated analytically in a number of ways for free-space [17].

Difficulties arise with the matched filter approach when it is applied in a waveguide, since, due to dispersion the signal is no longer time-invariant. When the signal remains deterministic, and can be accurately replicated at the receiver with a waveguide propagation model, the matched filter approach can be used by correlating the
measured data at a given range with the data modeled at that range, after appropriate normalization to scale out spreading loss.

Only when the time-spread of the received signal due to waveguide dispersion is negligible compared to the temporal coherence scale, or inverse RMS bandwidth of the received signal, can a time-delayed version of the original source time function be used as a matched filter replica. This is an especially important issue in designing active detection systems that exploit the temporal coherence of the received signal to extract it from ambient noise.

In Section 5.4, the main concepts presented in this section were examined using various types of shallow water waveguide environments.
4. Source localization using the MLE

4.1. Estimation of Signal Parameters

Let us assume that a signal at the receiver was detected. The receiver system is often required to provide estimates of specific parameters such as the range between the source and the receiver. As in the detection problem, the received waveform includes the desired target signal in combination with noise. We therefore choose for an estimate of a given parameter that value that maximizes the conditional probability density function for the parameter, given the received signal plus noise data.

In many practical problems the relationship between the data and the parameters to be estimated is nonlinear. This yields estimates with biases and mean-square errors that are difficult to quantify. It has become popular in recent years to compute limiting bounds, since those bounds are usually much easier to obtain. The most widely used limiting bound is the Cramér-Rao Lower Bound (CRLB), which describes the minimum possible variance of any unbiased estimator. For a given estimation problem, a minimum variance unbiased estimator, one that attains the CRLB, does not necessarily exists. However, if such an estimate exists, for sufficiently large data records or high SNR it is guaranteed to be the Maximum Likelihood Estimator (MLE) [16].

4.2. Maximum Likelihood Estimator (MLE)

The MLE estimators are highly practical and follow a well-defined systematic procedure for obtaining estimates from the data. Let us assume that $X_i$ is a vector of $n$ independent identically distributed experimental measurements; $X_i = x_i, ..., x_n$ and $P(X; \theta) = \prod_{i=1}^{n} P(X_i; \theta)$ is the conditional probability density function (PDF) of the data that depends on $m$ parameters, given $\theta = \theta_1, ..., \theta_m$ ($m \leq n$). When $X$ is known but $\theta$ is unknown, $P(X; \theta)$ is referred to as the likelihood function. The MLE for the vector
parameter $\theta$, is defined as the one that maximizes the likelihood function, which is equivalent to maximizing the logarithm of the likelihood function,

$$
\frac{d \ln(P(X;\theta))}{d\theta} \bigg|_{\theta=\hat{\theta}} = 0
$$

(4-1)

### 4.3. Asymptotic expressions for the bias and the covariance of the MLE

Following the theory and notation adopted in [19], the log-likelihood function $l(\theta)$, is defined as $l(\theta) = \ln(P(X;\theta))$. The first-order derivative as $l_r = \partial l(\theta) / \theta^r$, where $\theta^r$ is the $r^{th}$ component $\theta$. The higher log-likelihood derivatives are written in the following notation,

$$
l_{a_1 a_2 ... a_i} = \frac{\partial^2 l(X;\theta)}{\partial \theta_{a_1} \partial \theta_{a_2} ... \partial \theta_{a_i}}
$$

(4-2)

and the joint moments of the log-likelihood derivatives as,

$$
\nu_{a_1 a_2 ... a_i, b_1 b_2 ... b_s, z_1 z_2 ... z_s} = E[l_{a_1 a_2 ... a_i} l_{b_1 b_2 ... b_s} l_{z_1 z_2 ... z_s}]
$$

(4-3)

for example, $\nu_{s,\text{a}} = E[l_{\text{a}}, l_{\text{a}}]$. The expected Fisher information matrix is defined as $i_{rs} = E[l_{r} l_{s}]$, for arbitrary indices $r, s$. We use the lifted notation to denote inverse matrices and the Einstein summation convention is used, so that whenever an index appears as both superscript and subscript in a term, summation over that index is implied. For instance, $i^n = [i^{-1}]_{rs}$

The asymptotic expressions can be written as follows,

- The first-order bias of the MLE [18],

$$
B_r(\hat{\theta}^r; n) = E\{ (\hat{\theta} - \theta)^r \} = \frac{1}{2} \frac{i^{r a b c} \nu_{abc} + 2 \nu_{a b c}}{\sigma_r(n^{-1})}
$$

(4-4)

where the symbol $O_{\rho}(n^{-1})$ denotes the polynomial of exactly order $n^{-1}$, where $n$ is the sample size. A necessary condition for the MLE to become asymptotically
unbiased is for the first-order bias to become much smaller than the true value of the parameter $\theta^*$ [19] 

- The first-order covariance of the MLE,

$$V_1(\hat{\theta}, \hat{\theta}^a; n) = \lim_{n \to \infty} \frac{i^{aq}}{O_p(n^{-1})}$$

(4-5)

this term is known as the CRLB or the inverse of the Fisher information.

- The second-order covariance of the MLE is [19]

$$V_2(\hat{\theta}, \hat{\theta}^a; n) = 2i^{mb_{nc}}v_{ln}(i^{rs_{la}} + i^{as_{lr}})v_{s, b, c}(n^1) + \frac{1}{2}i^{cd_{ef}}(i^{rs_{lab}} + i^{as_{lrb}})v_{bce, d, f, s}(n^2)$$

$$+ i^{tu}(i^{rs_{lab}} + i^{rd_{loc}} + i^{ad_{lrb}} + i^{id_{lrc}})v_{s, t, u, b, c, d}(n^2)$$

$$+ \left( \frac{1}{4}i^{m_{ca}d_{e}}(i^{rs_{lab}} + i^{al_{lrb}}) + 2i^{mb_{nc}d_{e}}(i^{rd_{la}} + i^{al_{lrc}}) \right) v_{s, t, b, c, d}(n^2)$$

$$+ i^{cd_{ef}}(i^{rs_{la}} + i^{as_{lrb}})v_{s, b, c, d}(n^2) + \frac{1}{6}i^{mb_{nc}d_{e}}i^{m_{ca}d_{e}}v_{lmmo}(i^{rs_{la}} + i^{as_{lrb}})v_{s, b, c, d}(n^2)$$

$$+ 4i^{bm}(i^{rs_{lab}} + i^{as_{lrb}})v_{s, m, t, b, c, d}(n^1)$$

$$- \frac{1}{4}i^{rs_{lab}}i^{aw_{yz}}(v_{stu}v_{wyz} - v_{stu}v_{wyz}(n^1) - v_{stu}(n^1)v_{wyz}(n^1))$$

(4-6)

where the above expression is asymptotically a polynomial of order $n^2$ as $n$ becomes large. The notations such as $v_{bce, d, f, s}(n^2)$ indicates that in the joint moment $v_{bce, d, f, s}$ only polynomial terms of order $n^2$ are retained.

4.4. Minimum sample size needed to attain the CRLB

Observing the asymptotic expansion expressions for the variance, we can rewrite the total variance in a general form,

$$\text{var}(\hat{\theta}; n) = \frac{V_1(\hat{\theta}; n = 1)}{n} + \frac{V_2(\hat{\theta}; n = 1)}{n^2} + O_p(n^{-3})$$

(4-7)
where \( O_p(n^{-3}) \) is a polynomial in \( \frac{1}{n} \) containing higher powers than \( \left( \frac{1}{n} \right)^2 \). For large number of samples \( O_p(n^{-3}) \) approaches zero and we get,

\[
\text{var}(\hat{\theta};n) \approx \frac{V_1(\hat{\theta};1)}{n} + \frac{V_2(\hat{\theta};1)}{n^2}
\]

So, we can always find a sample size, where the second-order variance is smaller than the first-order variance by a wanted ratio \( \kappa \).

\[
\frac{V_2(\hat{\theta};n)}{V_1(\hat{\theta};n)} = \frac{V_2(\hat{\theta};1)}{n \cdot V_1(\hat{\theta};1)} < \kappa
\]

For convenience, a ratio of \( \frac{1}{10} \) between the second and first-order variance is used as a necessary condition for an MLE to asymptotically attain the CRLB. This leads to a necessary minimum sample size of approximately,

\[
n_{\text{neq}} \approx 10 \cdot \frac{V_2(\hat{\theta};1)}{V_1(\hat{\theta};1)} \tag{4-10}
\]

A minimum sample size \( n_{\text{neq}} \) is roughly necessary to be in the asymptotic region where the MLE is optimal.

### 4.5. Gaussian data measurements, deterministic signal

The received signal may be represented as shown in Fig. (3-1), by

\[
x(t) = s_1(t;\theta) + n(t) \quad 0 \leq t \leq T
\]

where \( x(t) \) is the sum of the noise \( n(t) \) and a deterministic signal \( s_1(t;\theta) \). It is assumed that a complete statistical description of the noise is known at the receiver. For most purposes, the noise may be assumed to be Gaussian. Gaussian noise is a
representative of many practical noise processes, and it also makes the mathematical analysis less complicated that do other distributions.

A number of observations are made of $x(t)$ either at discrete time intervals or else continuously for a finite time intervals. The discrete values are represented as,

$$x_j = (s_j)_j + n_j \quad j = 1, 2, \ldots, n$$

and

$$X_k = [x_1, x_2, \ldots, x_n]_k^T \quad (4-13)$$

Let us assume we have $n$ independent, identically distributed samples of the $N$-dimensional joint Gaussian vector $X_i$ stacked into a vector $X$, where $X = [X_1^T X_2^T X_3^T \ldots X_n^T]^T$. In addition, let us assume that the data depend on a set of $m$ unknown parameters $\theta$. The PDF for the multivariate real Gaussian case [16], where both the real-valued mean $\mu$, and the real-valued covariance matrix $C$, depend on the parameter vector $\theta$, is represented by,

$$p(X; \theta) = \frac{1}{(2\pi)^{nN/2}|C(\theta)|^{m/2}} \cdot \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} \left( X_i - \mu(\theta) \right)^T C^{-1}(\theta) \left( X_i - \mu(\theta) \right) \right\} \quad (4-14)$$

The general bias and variance expressions of Eqs. (4-4) – (4-6) are now applied to this specific case of data that obey the conditional Gaussian probability of Eq. (4-14). For the present study of underwater source localization, $x_i$ represents a time sample of the acoustic field received at a single hydrophone. The parameter vector $\theta$ includes the unknown parameters of interest, which in my case are either range $R$ from the source, depth of the source $z_0$ or both.

**Remark:**

The expression above is applicable for real random variables. For complex variables, which are usually used in ocean acoustics problems, see Ref. [19]-[20].

In the present study, I applied the asymptotic expressions to a specific case of practical interest, a deterministic signal in the presence if independent, additive noise. The noise has a *constant* power spectral density of $N_0$. 
\[ E[n(t;\theta)n(t+\tau;\theta)] = E[n(t)n(t+\tau)] = N_0 \delta(\tau) \] (4-15)

In this scenario, the covariance matrix \( C \) is independent of the parameter vector \( \theta \), while the mean \( \mu \) depends on \( \theta \). Using these assumptions and the PDF presented in Eq. (4-14), the log-likelihood joint moments required to evaluate the asymptotic expansions have been derived in Ref. [19] and are given in Appendix A. In this simplified form, only the knowledge of the first, second and third order derivatives of the log-likelihood function (or the received field) with respect to all combinations of the components of \( \theta \) are required.

4.6. Source Localization in a waveguide

As shown in Chapter 2, the received signal can be expressed using the source spectrum and the Green's function as,

\[ \Phi(t;\theta) = \int_{-\infty}^{\infty} Q(f)G(f;\theta)e^{-i2\pi ft}df \] (4-16)

where \( \Phi(t;\theta) \), represent the received signal \( s_i(t;\theta) \), and plays the role of the mean \( \mu \) of the PDF. Therefore, the derivatives of the received field (or equivalently, the derivatives of the time-harmonic Green's function) with respect to all combinations of the components of \( \theta \) are required. Typically, the Greens' function is generated using a propagation model (such as, KRAKEN [21], OASES [22]), and its derivatives are then evaluated numerically using finite difference equations – see Appendix B. For environments where the source is in a constant sound speed layer, such as Pekeris waveguide model for example, it is possible to determine analytically the derivatives with respect to the inversions parameters in the problem [20],[23].

4.6.1. One-parameter problem

In this case, \( \theta \) is a scalar unknown parameter; either the range from the source \((R)\) or the source-depth \((z_0)\). The time-harmonic Green's function in the waveguide
between the source and the receiver can be written as a sum of normal modes, using Eq. (2-10),

\[ G(f; z_0, R) = C \sum_{m} \psi_m(z_0) \psi_m(z) \frac{e^{ik_mR}}{\sqrt{k_mR}} \quad (4-17) \]

In order to calculate the bias and the variance using the asymptotic expansions, we need to calculate the first, second and third order derivative of the Green's function - Eq. (4-16) with respect to either \( R \) or \( z \). Again, this procedure is done numerically using finite difference equations (Appendix B). For the simple environment model of a Pekeris waveguide, the analytic expressions for the derivatives of the Green's function with respect to range or source-depth are derived in Appendix C.

4.6.2. Two-parameter problem

In the two-parameter problem, both range from the source \( (R) \) and source-depth \( (z_0) \) are unknowns. So, \( \theta \) is a vector of 2 parameters, \( \theta = [\theta_1, \theta_2] = [R, z_0] \). Now, in addition to the first, second and third derivatives of the pressure with respect to \( R \) and \( z_0 \) that were evaluated in the one-parameter problem, we also required to evaluate the appropriate mixed-derivatives \( \left( \frac{\partial^2 G}{\partial R \partial z_0}, \frac{\partial^3 G}{\partial R^2 \partial z_0}, \frac{\partial^4 G}{\partial R \partial z_0^2} \right) \).

4.6.3. Dispersive versus Non-Dispersive Media

In a Dispersive environment, where bottom interaction is significant and large numbers of propagating modes are trapped within the medium, even at longer ranges, the localization performance can be expected to improve. However, in a weakly dispersive medium, where the bottom interaction is limited and the field is dominated by only few modes, especially at large ranges, the acoustic field has less range-depth uniqueness and the localization performance will be degraded [24].
5. Illustrative Examples

5.1. Source function

I chose to illustrate the effect of dispersion on one specific source function\(^5\), a Linear Frequency Modulated (LFM) waveform. The waveform is modulated, since the phase of the signals is seldom known. This compression pulse is extensively used in radar and sonar applications, since its gain over an incoherent linear bandpass filter with the same bandwidth roughly equals the time-bandwidth product. Moreover, since its spectrum is very close to rectangular envelope it is easy to generate\(^6\). As for the specific set of signal parameters, I chose to follow a real experiment. The LFM signal was commonly used and transmitted in the recent Geological Clutter Acoustics experiment that is the cornerstone of a Major Research Program of the Office of Naval Research (ONR) [25]. The LFM signal that I used is identified as waveform "ID# WTLFM" in that experiment.

In general, the complex LFM source signal, is defined as [13],

\[
q(t) = \frac{A}{\sqrt{T_s}} e^{i k t} e^{-i 2 \pi f_s t}, \quad 0 \leq t \leq T_s
\]

(5-1)

where \(A\) is a scaling factor and \(T_s\) is the signal duration. The instantaneous frequency of the complex envelope is \(f_i = \frac{1}{2\pi} \frac{d\varphi(t)}{dt} = k t\), with \(k\) determines the slope of the FM. If the instantaneous frequency is swept over the bandwidth \(B\) during the signal duration \(T_s\), the absolute value of \(k\) is given by \(|k| = \frac{B}{T_s}\). The sign of \(k\) determines the direction of the sweep. The carrier frequency \(f_c\) is defined as, \(f_c = f_i + B/2\).

\(^5\) Another signal, the Gaussian Modulated waveform (GMW) was used for simulation debugging – more detailed about this waveform see Appendix E.

\(^6\) For further information about the LFM signals, see [13] - chapters 6-7.
In order to control frequency domain sidelobes, hence reduce wiggles in the spectrum, the LFM signal is often multiplied by a window function. I used the modified Tukey window [19] that was also used in the Geoclutter experiment [25] and has the form of,

\[
w(t) = \begin{cases} 
  p + (1 - p) \sin^2 \left( \frac{\pi}{2T_w} t \right) & 0 \leq t \leq T_w \\
  1 & T_w \leq t \leq T_s - T_w \\
  p + (1 - p) \sin^2 \left( \frac{\pi}{2T_w} \left( t - (T_s - 2T_w) \right) \right) & T_s - T_w \leq t \leq T_s 
\end{cases}
\]

where \( T_w \) is the window duration (\( T_w = 0.125 T_s \) in my case), and \( p \) is the pedestal for sources (\( p = 0.01 \) in my case). I used a signal with a carrier frequency of \( f_c = 415 \text{Hz} \), signal duration of 1sec, and pulse bandwidths between 10Hz and 300Hz.

One last remark: in my simulation, I shifted the source signal \( q(t) \) by \( t_d \) seconds for convenient computation reasons (\( t_d = 1 \text{sec} \) in my case). This allowed me to avoid dealing with negative time-axis.

The following graph, show the waveform and spectrum of the LFM signal with a bandwidth of 50Hz (denoted in the Geoclutter experiment as, WTLFM-2).
Source Signal - Waveform (1sec Pulse Duration)

Source Signal - Spectrum (50Hz Bandwidth)

Figure 5-1 - LFM Source signal Waveform & Spectrum
5.2. **Shallow-Water Environment**

The shallow water waveguide we consider here is a model that is bounded by a pressure release surface and a higher velocity fluid half-space. The boundary conditions are as follows, vanishing pressure at the sea surface, continuity of bottom particle displacement and continuity of bottom pressure. It is also assumed that surface and bottom are plane and parallel.

In all of the following illustrative examples, a water column of 100m depth is used to simulate a typical continental shelf environment. The sound speed structure of the water column varies from iso-velocity (Type A – Pekeris environmental model) to downward refracting layer (Type B model), with a constant density of 1g/cm³ and attenuation of $6.0 \times 10^{-5} \text{dB}/\lambda$.

I used four different ocean bottom half-space sediments (CASES 1-4) below the water column, in order to examine the effects of dispersion within the waveguide. All the sediments are modeled as fluids, which mean that they support only one type of sound wave, a compressional wave. This is often a good approximation since the rigidity of the sediment is usually considerably less than that of solid. The density, sound speed and attenuation of the bottoms are taken to be,

1) CASE1 - Silt Bottom – 1.4g/cm³, 1520m/sec, 0.3dB/λ.
2) CASE2 - Sand Bottom – 1.9g/cm³, 1700m/sec, 0.8dB/λ.
3) CASE3 - Gravel Bottom – 2.0g/cm³, 1800m/sec, 0.6dB/λ.
4) CASE4 - Moraine Bottom – 2.2g/cm³, 1900m/sec, 0.5dB/λ.

To complete the geometry of the problem, the source and the receiver are placed at a fixed depth, 20m and 50m, respectively.

---

7 Another case, a 3-layered environment (type B water column, CASE2 25m depth sediment layer, and CASE4 basement layer) that closely modeled one of the Geoclutter experiment [25] sites, was also examine. However, since it gave very similar results to type B/CASE2 waveguide, I didn’t include its result in my thesis.
Figures 5-2 through 5-3 illustrate the geometry of the two types of shallow water ocean waveguide used in this section.

**Atmosphere**

**Water Column**

- $c_w = 1500 m/s$
- $\rho_w = 1 g/cm^3$
- $\alpha_w = 6.0 \times 10^{-3} dB/\lambda$

**Sediment Half-Space**

<table>
<thead>
<tr>
<th>CASE 1</th>
<th>CASE 2</th>
<th>CASE 3</th>
<th>CASE 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_b = 1520 m/s$</td>
<td>$c_b = 1700 m/s$</td>
<td>$c_b = 1800 m/s$</td>
<td>$c_b = 1900 m/s$</td>
</tr>
<tr>
<td>$\rho_b = 1.4 g/cm^3$</td>
<td>$\rho_b = 1.9 g/cm^3$</td>
<td>$\rho_b = 2.0 g/cm^3$</td>
<td>$\rho_b = 2.2 g/cm^3$</td>
</tr>
<tr>
<td>$\alpha_b = 0.3 dB/\lambda$</td>
<td>$\alpha_b = 0.8 dB/\lambda$</td>
<td>$\alpha_b = 0.6 dB/\lambda$</td>
<td>$\alpha_b = 0.5 dB/\lambda$</td>
</tr>
</tbody>
</table>

Figure 5-2 – Type A Water Column – Pekeris Environmental Model
Atmosphere

\[ \rho_w = 1 \text{ g/cm}^3 \]
\[ \alpha_w = 6.0 \times 10^{-5} \text{ dB/\lambda} \]

Water Column

\[ c_w = 1470 \text{ m/s} \]
\[ \rho_w = 1 \text{ g/cm}^3 \]
\[ \alpha_w = 6.0 \times 10^{-5} \text{ dB/\lambda} \]

Sediment Half-Space

<table>
<thead>
<tr>
<th>CASE1</th>
<th>CASE2</th>
<th>CASE3</th>
<th>CASE4</th>
</tr>
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</tr>
<tr>
<td>( \rho_b ) = 1.4 g/cm(^3)</td>
<td>( \rho_b ) = 1.9 g/cm(^3)</td>
<td>( \rho_b ) = 2.0 g/cm(^3)</td>
<td>( \rho_b ) = 2.2 g/cm(^3)</td>
</tr>
<tr>
<td>( \alpha_b ) = 0.3 dB/\lambda</td>
<td>( \alpha_b ) = 0.8 dB/\lambda</td>
<td>( \alpha_b ) = 0.6 dB/\lambda</td>
<td>( \alpha_b ) = 0.5 dB/\lambda</td>
</tr>
</tbody>
</table>

Figure 5-3 – Type B Water Column
5.3. Time-Frequency Measures

In this section, I show how the time-frequency measures of a broadband LFM signal (defined in section 5.1) are affected when the signal is propagating in various shallow water environments. The horizontal range and source bandwidth dependencies of the different measures are plotted in Figures 5-4 through 5-17. The range-axis is up to 50 Km and the bandwidth of the source $B$ varies from 10 Hz to 300 Hz around a carrier frequency of 415 Hz. There are two types of figures (1) 2-D line graphs, where the quantities are plotted for a 50 Hz bandwidth source signal; (2) 3-D graphs, where the quantities are plotted versus horizontal range and source bandwidth. The time-frequency quantities are plotted in [dB] scale in order to increase the dynamic visual range, and the source bandwidth in logarithmic scale. The purpose of the 2-D graphs is mostly to give some quantitative values for the different measures, while the 3-D graphs show how the measures vary for narrower and wider source signals.

The transmitted source signal has a time-duration $T_v$ of 1 sec. The amplitude of the source function $q(t)$ was scaled such as the input SNR ($SNR_m$) is 100 dB. The input SNR was defined as follows,

$$SNR_m = 10 \log\left( \frac{E_q}{N_o} \right) [\text{dB}]$$

(5-3)

where $E_q = \int |q(t)|^2 dt$ is the energy of the complex source signal and $N_o$ is the noise power spectral density, and was taken as unity. The source and the receiver depths are fixed, 20 m and 50 m respectively. To evaluate the time-frequency measures, I set the integration time of the system $T$ to 10 seconds. For the following environmental implementation this value was found to work well, however, for more dispersive waveguides, one might need to increase this integration time value.

All the results were evaluated as follows: first, the time-harmonic waveguide Green's function was computed using KRAKEN software normal modes package [21], and was saved as a data output file. Then, using this Green's function, all the different measures were calculated and plotted using MATLAB.
In Figures 5-4 and 5-5, I plotted the time spread of the received signal, calculated using Eq. (2-16). The time delay of propagation between the source and the receiver \( \tau_d \) was computed using Eq. (2-15). This quantity is an increasing function of range, which in free space, can be written exactly as \( r/c \), where \( r \) is the range between the source and the receiver and \( c \) is the speed of sound. However, in a dispersive medium the relative time lag between the received signal and the transmitted signal is unknown, and it is not a constant value. So, I defined a new quantity, time delay error \( \tau_e \), as the difference between the time delay measure to an expected constant-valued time of arrival at the receiver \( \tau_{ref} = R/c_{ref} \),

\[
\tau_e = \left| \tau_d - \tau_{ref} \right| \quad \equiv \quad R \cdot \left| \frac{1}{c_{eff}} - \frac{1}{c_{ref}} \right| \quad \text{(5-4)}
\]

where \( c_{ref} \) is some expected constant-valued medium velocity at the receiver and \( R \) is the true horizontal range between the source and the receiver. Similarly, I defined the expected range error as,

\[
\Delta R = \tau_e \cdot c_{ref} \quad \text{(5-5)}
\]

In Figures 5-6 through 5-11, I plotted the time delay error and the range error for different values of \( c_{ref} \). In Figures 5-12 and 5-13, I plotted the effective medium velocity, calculated using Eq. (2-34).

The frequency measures, bandwidth spread and frequency delay were evaluated using Eq. (2-21) and (2-19), respectively, and are plotted in Figures 5-14 through 5-17. The effective bandwidth of a 50 Hz LFM source signal \( \Delta \beta(r_0,r_0) \) was calculated using Eq. (2-20) and is equal to roughly 12 Hz.

As an aid to interpret the results, I calculated the maximum number of trapped modes \( n_{max} \) within the waveguide in each different environment, where \( n_{max} \) was defined as follows [26], first, the total pressure field was calculated including all the possible modes \( m \). Then, the pressure field was calculated repeatedly including only \( i \) modes \( (1 \leq i \leq m) \), starting with \( i=1 \) and was compared to the total pressure field. At the point where the difference becomes less than 1dB, I set \( n_{max} = i - 1 \), which implies that all the higher modes above \( i + 1 \) are not contributing significantly to the total pressure field.
In Figures 5-20 through 5-23, I plotted the maximum number of trapped mode and the modal group velocities (for frequency of 415Hz) for all the different environments.

For time delay and range determination, using Eq. (5-4) and (5-5), I assumed that \( c_{\text{ref}} \) is the group velocity of the lowest mode, 1500 m/sec for Type A waveguide and 1470 m/sec for type B waveguide, as can be seen from Figures 5-20 through 5-23. For Type B waveguide, I also plotted the range error for other values of expected medium velocity: 1460 m/sec and 1450 m/sec in order to examine their effect on the range performance.

In Figure 5-18, I plotted the Time-Bandwidth product of the source and the received signal versus range for a 50 Hz bandwidth LFM source signal. The \( TB \) product was defined in section 2.4. The \( TB \) product for this LFM broadband signal is around 3, which is higher than the uncertainty principle bound \( (1/4\pi) \), as expected - see Example 3.4. Last, I plotted in Figure 5-19 the effective bandwidth versus the effective duration of the received signal for all the different environments.

Following are key features observed for the various measures, for the iso-velocity water column waveguide:

- The silt-bottom waveguide, with its low critical angle and high absorption, has the least dispersion. This leads to small time spread (less than 100 msec) and a small time delay error (an order of 200 msec), especially as the range increases. The mean horizontal sound speed fluctuates significantly at close ranges up to 30 Km, and especially at source bandwidth below 50 Hz. Beyond roughly 30 Km, the effective sound speed asymptotically approaches the free-space sound speed of 1500 m/s, due to the fact that only few modes (2-3 modes) are still trapped in the waveguide and dominates the field. The range error is small, less than roughly 300 m in all ranges and source bandwidths, but becomes significantly smaller as the range increases. A reduction in bandwidth-spread and frequency-delay with increasing range follows from the greater attenuation of higher frequencies in the sediment.

- In general, the sand-bottom waveguide has similar behavior as the silt-bottom waveguide with a slight increase in time spread and time delay error. The range error is still small, less than 300 m even at 50 Km. At large ranges, there are still 5 to 10
propagating modes within the waveguide, which cause a slight decrease in the mean horizontal sound speed at these ranges compared to the silt-bottom waveguide. There is also a small decrease in bandwidth-spread and frequency-delay compared to the silt-bottom case.

- On the other hand, both the gravel and the moraine-bottom waveguides, have much higher critical angles. This leads to an increase in time spread (an order of 150 msec and 300 msec, respectively) and similarly higher values for the time delay error. As a consequence, the range error is also increasing even at long ranges, and gets as high as 500 m in the moraine bottom waveguide at 50 Km. Many high modes are trapped inside the waveguide and constructively and destructively interfere with each other even at very large ranges. The mean horizontal sound speed is not approaching anymore 1500 m/s, but it’s asymptotically settles to 1-2% less, since the higher modes travel with smaller group velocity. Again, there is a continuing trend towards smaller values in the frequency measures compared to the silt and sand bottom waveguides.

- The Time-Bandwidth ($TB$) product of the silt and sand-bottom waveguides fluctuates around the source Time-Bandwidth product while for the more disperse mediums the $TB$ product increases as the signal propagates along the waveguide. In free-space, since the signal retains its time-frequency dependence shape, the $TB$ product should be conserved and equal to a constant value. However, when the signal is propagating in a dispersive environment both its time-frequency measures are spread as can be seen nicely in the time-frequency curves in Figure 5-19.

Following are key features observed for the various measures, for the downward refracting water column waveguide:

- Basically, the time frequency measures have similar behavior to those obtained for the iso-velocity water column over the same bottom types.

- However, since more modes are trapped within the waveguide for the same bottom types, the time spread and time delay error measures are slightly higher than the ones obtained for the iso-velocity case. For instance, the time spread for the gravel and
moraine-bottom waveguide, get as high as 200 msec and 500 msec at 50 Km respectively, compare to 150 msec and 300 msec for the iso-velocity waveguide.

- The group velocity of the first mode is now roughly 1470 m/s (as can be seen from Figures 5-18 through 5-21). Thus, the mean horizontal sound speed of the non-dispersive silt-bottom approaches this value for large ranges, and settles for 1-2% less for the highly disperse moraine-bottom case. Also, since the range error is a function of the expected medium velocity, choosing a smaller reference value like 1460 m/s (Figure 5-9 and 5-11), yields a smaller range error for the dispersive mediums, compared to 1470 m/s, but increase the range error for the silt bottom waveguide. Reducing $c_{nf}$ even more to 1450 m/s, yields the best range error performance for the moraine bottom waveguide, but worsens the error significantly for all the other cases. It leads to the conclusion that the optimal selection for a constant-valued expected medium velocity is environment dependent, and for the Type B waveguide is somewhere between 1450 and 1470 m/s.

- The Time-Bandwidth curves (Figure 5-19), are much more spread, mostly in time, than the ones received for the iso-velocity waveguide for the same bottom types.

Common observations for both the iso-velocity and the downward refracting water column waveguide:

- For a given source bandwidth, the time spread and time delay error decrease as a function of range, as fewer modes are trapped within the waveguide, due to absorption of energy in the bottom.

- The time measures are smaller for broader signal bandwidths, especially when the bandwidth is higher than 50 Hz. Those time measures seem to be much smoother for broader signals and fluctuate significantly for narrower signals even at large ranges. The frequency measures behave exactly the opposite, large values are achieved for source bandwidths greater than 50 Hz.

- At large ranges above 40 Km, the time-frequency measures seem to settle around some constant value, which characterize the dispersion properties of a given environment.
The magnitude of effective bandwidth can be written using Eq. (2-21) as,

$$\Delta \beta (r, r_0) = \Delta \beta (r_0, r) + |\Delta \beta_s|$$

(5-6)

where for a 50Hz LFM source signal, $\Delta \beta (r_0, r)$ is equal to roughly 12 Hz. From Figure 5-14 we can see that the magnitude of the bandwidth spread $\Delta \beta_s$ is $\approx 3$-4 Hz for the silt and sand bottoms, and $\approx 1$-2 Hz for the gravel and moraine bottoms. Thus, the inverse of the effective bandwidth is in the order of 0.06-0.08 seconds. Running the simulation for all the different environment, I explore a useful rule-of-thumb, when the time spread of the received signal due to the medium dispersion $\Delta \tau_s (r, r_0)$ is smaller than the inverse of the received signal effective bandwidth, the integration time $T$ of the system can be set to roughly $T_e$. It implies that the effects of dispersion are small and the medium can be considered as non-dispersive, as for the silt-bottom case.

In the following Tables 5-2 and 5-3, I summarized the main quantities obtained for the various waveguides, for a 50Hz source bandwidth, at ranges greater than 50 Km,

| Bottom type | $|\Delta \tau_s|$ | $|\Delta R|$ w/ $c_{ref} = 1500m/s$ | $c_{eff}$ |
|-------------|-----------------|-----------------|-----------|
| Silt        | $\leq 0.05$ sec | $\leq 200m$     | $\sim 1500m/s$ |
| Sand        | $\leq 0.05$ sec | $\leq 200m$     | $\sim 1500m/s$ |
| Gravel      | $\leq 0.15$ sec | $\leq 300m$     | $\sim 1490 + 1500m/s$ |
| Moraine     | $\leq 0.30$ sec | $\leq 500m$     | $\sim 1480 + 1490m/s$ |

Table 5-1 – Summary of results – Iso-Velocity water column

| Bottom type | $|\Delta \tau_s|$ | $|\Delta R|$ w/ $c_{ref} = 1470m/s$ | $c_{eff}$ |
|-------------|-----------------|-----------------|-----------|
| Silt        | $\leq 0.05$ sec | $\leq 300m$     | $\sim 1470m/s$ |
| Sand        | $\leq 0.075$ sec | $\leq 300m$     | $\sim 1460m/s$ |
| Gravel      | $\leq 0.20$ sec | $\leq 500m$     | $\sim 1450 + 1460m/s$ |
| Moraine     | $\leq 0.45$ sec | $\leq 800m$     | $\sim 1440 + 1450m/s$ |

Table 5-2 – Summary of results – Downward Refracting water column
Figure 5-4 – Time-Spread - 50Hz Signal Bandwidth
Top graph: Type A waveguide, Bottom graph: Type B waveguide
(Note: Time-axis of Moraine bottom/Type B waveguide is different than the other cases)
Figure 5-5 – Time-Spread - vs. Source bandwidth
Top graph: Type A waveguide, Bottom graph: Type B waveguide
Figure 5-6 – Time-Delay error - 50Hz Signal Bandwidth \( \tau_e = \tau_d - R/c_{ref} \)

Top graph: Type A waveguide w/ \( c_{ref} = 1500 \text{m/s} \), Bottom graph: Type B waveguide w/ \( c_{ref} = 1470 \text{m/s} \)

(Note: Time-axis of Moraine bottom/Type B waveguide is different than the other cases)
Figure 5-7 - Time-Delay error - vs. Source bandwidth

Top graph: Type A waveguide w/ \( c_{ref} = 1500 \text{m/s} \), Bottom graph: Type B waveguide w/ \( c_{ref} = 1470 \text{m/s} \)
Figure 5-8 – Range error - 50Hz Signal Bandwidth ($\Delta R = \tau_e \cdot c_{ref}$)

Top graph: Type A waveguide w/ $c_{ref} = 1500$ m/s, Bottom graph: Type B waveguide w/ $c_{ref} = 1470$ m/s

(Note: Range error scale of Moraine bottom /Type B waveguide is different than the other cases)
Figure 5-9 – Range error - 50Hz Signal Bandwidth
Top graph: Type B waveguide w/ $c_{ref} = 1460\text{m/s}$, Bottom graph: Type B waveguide w/ $c_{ref} = 1450\text{m/s}$
(Note: Range error scale of Moraine bottom is different than the other cases)
Figure 5-10 – Range error vs. Source bandwidth
Top graph: Type A waveguide w/ $c_{ref} = 1500$ m/s, Bottom graph: Type B waveguide w/ $c_{ref} = 1470$ m/s
Figure 5-11 – Range error - vs. Source bandwidth

Top graph: Type B waveguide w/ $c_{ref} = 1460$ m/s, Bottom graph: Type B waveguide w/ $c_{ref} = 1450$ m/s
Figure 5-12 – Effective Medium velocity - 50Hz Signal Bandwidth
Top graph: Type A waveguide, Bottom graph: Type B waveguide
Figure 5-13 – Effective Medium velocity vs. Source bandwidth
Top graph: Type A waveguide, Bottom graph: Type B waveguide
Figure 5-14 – Bandwidth-Spread - 50Hz Signal Bandwidth
Top graph: Type A waveguide, Bottom graph: Type B waveguide
Figure 5-15 – Bandwidth-Spread - vs. Source bandwidth
Top graph: Type A waveguide, Bottom graph: Type B waveguide
Figure 5-16 – Bandwidth-Delay error - 50Hz Signal Bandwidth

Top graph: Type A waveguide, Bottom graph: Type B waveguide
Figure 5.17 – Bandwidth-Delay error vs. Source bandwidth
Top graph: Type A waveguide, Bottom graph: Type B waveguide
Figure 5-18 – Time-Bandwidth Product - 50Hz Signal Bandwidth
Top graph: Type A waveguide, Bottom graph: Type B waveguide
Figure 5-19 – Time-Bandwidth Curve - 50Hz Signal Bandwidth
Top graph: Type A waveguide, Bottom graph: Type B waveguide
Figure 5-20 – Dispersion Curves: Silt Bottom
(Top graph: Type A waveguide, Bottom graph: Type B waveguide)
Figure 5-21 – Dispersion Curves: Sand Bottom
(Top graph: Type A waveguide, Bottom graph: Type B waveguide)
Figure 5-22 – Dispersion Curves: Gravel Bottom
(Top graph: Type A waveguide, Bottom graph: Type B waveguide)
Figure 5-23 – Dispersion Curves: Moraine Bottom
(Top graph: Type A waveguide, Bottom graph: Type B waveguide)
5.4. Matched-Filter Performance

In this section, I show how the various environments affect the performance of the matched-filter. The range and source bandwidth dependencies of the correlation coefficient are given in Figure 5-24. The correlation coefficient $R_{1,2}$ between the source signal and the received signal for each environment was evaluated using Eq. (3-21), and plotted in dB scale. The figure consists of two graphs: (1) Top graph – for the Iso-velocity water column waveguide, (2) Bottom graph – for the Downward refracting water column waveguide. Again, the transmitted source signal has a time-duration $\tau_s$ of 1 sec and an input SNR of 100 dB (The input SNR was defined in Eq.(5-3) ). The source and receiver depth are fixed, 20 m and 50 m respectively.

Following are key features observed for the correlation coefficient:

- The Iso-velocity water column silt-bottom case has the least dispersion. Only the first few modes contribute to the total pressure field, and the correlation coefficient can be well approximated as a constant, very close to unity. At all ranges and source bandwidths, the received signal is almost a perfect replica of the transmitted signal and the maximum SNR is almost the same as for the ideal matched-filter $2E/N_0$. At ranges greater than 25 Km the match field is the best.
- As the sound speed of the bottom sediment increases, the medium becomes more dispersive and the received signal is no longer a simple replica of the transmitted signal. The correlation coefficient then decreases significantly below unity, so that the maximum SNR is smaller for the same input signal, and the performance of the matched-filter is degraded.
- For all cases, the correlation coefficient decreases with increasing signal bandwidth. It implies that the performance of the matched-filter is worse for broader signals.
- I also observe that, for source bandwidths below 50 Hz the correlation coefficient fluctuates around a constant value.
- As expected, the correlation coefficient for the downward refracting water column is smaller than for the iso-velocity case, due to the fact that more modes are trapped
within the waveguide, which cause an increasing modal-interference. The correlation is significantly smaller for broader signals.

In Figures 5-25 through 5-26, I plotted the range error of the matched-filter estimate. The estimate range error was calculated using Equation (3-23) for various values of expected medium velocity $c_{\text{ref}}$. Following are key features observed:

- The range error for the iso-velocity water column silt-bottom case is insignificant, less than 25 m in all ranges and source bandwidths.
- For the other iso-velocity waveguide sediment bottoms, the range error is higher compared to the silt-bottom, but is still small, less than 250 m in all ranges for source bandwidths above 50 Hz. As the source signal bandwidth increases, the range error decreases.
- As expected, the range error for the downward refracting water column increases compared to the iso-velocity case, and strongly depends on the selection of the reference sound speed. For example, for the silt bottom choosing $c_{\text{ref}} = 1470$ m/s, seems to give the smallest error, however for all the other cases, $c_{\text{ref}} = 1460$ m/s seems to work even better, and choosing $c_{\text{ref}}$ as 1450 m/s, gives the worst performance. The optimal value of $c_{\text{ref}}$ that gives the smallest range error is probably between 1460-1470 m/s (finding the exact value is out of the scope of my thesis).
  Note, that those values are closely related to the asymptotic values (for large ranges) of the corresponding effective medium velocity (Figure 5-15).
- For all bottom types Type B waveguide, excluding the silt bottom case, I observe that the range error is not clearly affected by changing the signal bandwidth over the range shown and in general the range error looks similar for all these cases. This fact probably defines the optimum performance that can be achieved using the matched-filter technique.
Figure 5-24 – Imperfect Matched-Filter Correlation Coefficient
(Top graph: Type A waveguide, Bottom graph: Type B waveguide)
Figure 5-25 – Range Error - output of Matched-Filter ($\Delta R = R - \tau_m \cdot c_{ref}$)

(Top graph: Type A waveguide w/ $c_{ref} = 1500$ m/s, Bottom graph: Type B waveguide w/ $c_{ref} = 1470$ m/s)
Figure 5-26 – Range Error - output of Matched-Filter
(Top graph: Type B waveguide w/ $c_{ref} = 1460\text{m/s}$, Bottom graph: Type B waveguide w/ $c_{ref} = 1450\text{m/s}$)
5.5. Source Localization

The purpose of the following section is to analyze the source localization performance of the MLE for a deterministic, broadband, LFM signal embedded in zero-mean, additive Gaussian noise in a dispersive shallow water environment. Again, I used the same LFM signal that was defined in section 5.1, the source bandwidth $B$ varies from 10 Hz to 300 Hz around a carrier frequency of 415 Hz, and the time duration $T_r$ is 1 sec. All the following examples are done for the Pekeris waveguide model (Type A) and for various ocean sediment bottoms (CASES 1-4). I will illustrate the performance for both the one-parameter problem (section 4.6.1) and the two-parameter problem (section 4.6.2).

As in the previous sections, the source and the receiver depths are fixed, 20m and 50m respectively.

The evaluation of the bias, first-order (CRLB) and second-order variance using the asymptotic expressions defined in Eqs. (4-4) - (4-6), requires the knowledge of the first, second and third-order derivatives of the received pressure field $\Phi(r|\phi_0|t)$ with respect to range and source-depth. This is done either numerically (see Appendix B) or analytically for this special Pekeris waveguide model (see Appendix C). Numerically, it is done by solving finite difference equations for the derivatives, with the use of a well-chosen step-size increment $h$, one that gives a stable derivative value. For example, in the one-parameter range localization problem, I plotted in Fig. 5-27 the first and second-order variance for various values of step-size (between $10^{-4}$ to $10^3$ meters) for the silt bottom case. For this illustrative example, I use a LFM signal with an arbitrary input SNR of 100 dB ($SNR_{in}$ was defined in Eq. (5-3)). From the graph, we can see that any step-size increment between $10^{-1}$ and $10^{-3}$ meters, gives a stable value for the variance. The same result was achieved for all the other bottom cases. Thus, I set the step-size increment in my simulation to $10^{-2}$ meter. In general, it is best to test the highest frequency components in the finite difference, since it is most sensitive and may still have errors if the lower frequencies dominate the broadband result. But since the step size is much smaller than the shortest wavelength, the results are likely to be fine. For this Pekeris waveguide model, the first three derivatives of the pressure field were evaluated using the analytic expressions derived in Appendix C. The results from the two methods
were identical. As so, I chose to use the analytic derivation to evaluate the asymptotic expressions and to analyze the source localization performance of the MLE.

In all figures where the magnitude of the first-order bias, the first-order and the second-order variance are plotted, the source level has been adjusted, so that the SNR at a receiver located at 5 Km is 0 dB. The SNR is defined as follows,

\[
SNR = \frac{E_s}{E_n}
\]

(5-7)

where \(E_s\) is the energy of the received field calculated using Eq. (2-24) and \(E_n\) is the noise power density \(N_0\). Note that the same equation for the SNR was used in the matched-filter section, Chapter 3, Eq. (3-5).

### 5.5.1. One-parameter problem

The unknown parameter in this problem is either the range from the source \(R\) or the source-depth \(z_0\). In Figures 5-28 through 5-31, the magnitude of the first-order bias ('Bias'), the first-order variance ('\(\text{Var}_1\)') and the second-order variance ('\(\text{Var}_2\)') are plotted versus range for a 50Hz LFM broadband signal. All the values are for a single realization (\(n=1\)) of the full signal, consists of \(N\) measurements over a time-window of \(T=10\) seconds. Each figure consists of the source localization performance for both range and source-depth. Note that the square root of the first-order and the second-order variance are being plotted in the figures. In Figures 5-32 through 5-35, I plotted the necessary sample size to attain the CRLB, using Eq. (4-10), as a function of SNR. In Figures 5-36 and 5-37, I plotted the ratio between the second-order and the first-order variance, as well as \(1/SNR\) for various source bandwidths (modulated about a carrier frequency of 415 Hz) for the silt and sand bottom cases in order to examine the dependence on the source bandwidth.

The following are the key results,

- The magnitude of the first-order bias, the first-order variance and the second-order variance are shown to increase as a function of source range.
The second-order variance grows much more rapidly than the first-order, exceeding the CRLB by almost an order of magnitude at 50 Km for all medium for the given SNR values.

The range biases are negligible for all case, less than 0.01 m for the silt bottom and less than 0.001 m for all other bottoms. Thus, the MLE is effectively unbiased for the given SNR values.

The depth biases are much larger than the range biases, and especially high for the silt bottom case reaching a maxima of 400 m at around 47 Km. As the medium becomes more dispersive, the range biases decrease significantly, reaching a maxima of 20 m for sand bottom, 2 m for gravel bottom and less than 1 m for moraine bottom, for the given SNR values.

The first-order and second-order variances, both show smaller values for both range and source-depth variances as the medium becomes more dispersive. However, the decrease in the source-depth variance is much more noticeable than the decrease in the range variance.

Clearly, the range and source-depth biases and variances are smaller as the medium becomes more dispersive, due to the fact that more propagating modes exists in the more dispersive environments, which improves the MLE performance. Localization performance improves as signal dispersion increases, as expected [24]. The degradation in the range and source-depth estimation performance is especially notable for the silt bottom case.

For all cases, the first-order variance equalized with the second-order variance near a SNR of roughly 0dB, for both range and depth localization. A similar result was obtained in Ref.[20] for the case of monopole source signal received at a vertical array. The time-window here for the broadband signal plays the same role as spatial aperture in Ref.[20].

The number of necessary samples to attain the CRLB increases with range and 1/SNR. For SNR values smaller than roughly -10 dB, the CRLB underestimates the true parameter variance (of either range or source-depth) at all ranges, and many data samples (in the order of tens and thousands) are required to make the second-to-first order variance negligible, so the MLE asymptotically can attain the CRB. Even for
higher SNR values between -10 dB and 20 dB there are ranges were many samples are needed to attain the CRLB.

- As the SNR increases, the CRLB becomes a good approximation to the true variance, and the ratio between the second-to-first order variance becomes small. From Figures 5-32 through 5-35, it seems that for SNR's above 30dB, only one sample will be necessary to attain the CRLB for all ranges up to 50 Km, both for range and source-depth localization.

- From Eq. (4-10), we can see that the second-to-first order variance ratio is a function of 1/SNR (or 1/n) up to a factor \( \alpha = V_2(\hat{\theta};1)/V_1(\hat{\theta};1) \). For the range localization, the second-to-first order variance ratio behaves exactly the same as 1/SNR, independent of the source bandwidth and the environment, as seen in Figures 5-36 and 5-37, which implies that the factor \( \alpha \) is a constant. The same result was received in Ref. [19] for the time-delay estimation problem. However, it is hard to identify from these figures the same pattern for the second-to-first order variance ratio for the source-depth localization, the factor \( \alpha \) is probably a range and depth dependent. Also, it seems that the second-to-first order variance ratio (which relates to the necessary sample sizes to attain the CRLB) is more sensitive for narrower signals (less than 50 Hz).

5.5.2. Two-parameter problem

The two unknown parameters in this problem are both the range from the source \( R \) and the source-depth \( z_0 \). Introducing another unknown parameter to the estimation problem will increase the general uncertainty [23] and is expected to cause some degradation in source localization. Similar figures to the one-parameter problem are plotted again for this two-parameter case. In Figures 5-38 through 5-41, the magnitude of the first-order bias ('Bias'), the first-order variance ('Var_1') and the second-order variance ('Var_2') are plotted versus range for a 50Hz LFM broadband signal, where the sample size is taken as unity \((n=1)\). In Figures 5-42 through 5-45, I plotted the necessary sample size to attain the CRLB for different values of source levels and in Figures 5-46
and 5-47, the ratio between the second-order and the first-order variance, as well as $1/SNR$ have been plotted.

Following are key features observed,

- The range localization is very sensitive to the addition of another unknown parameter. The range biases are much higher in this situation compared to the one-parameter problem. For instance, the biases reach a maxima of 8 m, 0.04 m, 0.007 m and 0.002 m, for the silt, sand, gravel and moraine bottoms, respectively, which is much higher than the values obtained in the one-parameter problem. The variances are also slightly higher than the one-parameter problem, but almost unnoticeable compared to the biases.

- The source-depth biases and variances remain almost the same as the one-parameter problem, and seem to be unaffected from the uncertainty in range.

- For SNR values smaller than roughly 0 dB (compared to -10 dB for the one-parameter problem), many data samples (in the order of tens and thousands) are required to make the second-to-first order variance negligible compared to the CRLB.

- Again, we can see that the second-to-first order variance ratio is more sensitive for narrower signals (less than 50 Hz).

In the following Table 5-3 and 5-4, I summarized the main quantities obtained for the first-order bias, CRLB and the second-order variance for both range and source-depth estimate for the various waveguides, for both the one-parameter and the two-parameter source localization. The quantities in the tables are the maximum values obtained from the different figures.
<table>
<thead>
<tr>
<th>Quantity</th>
<th>Silt bottom</th>
<th>Sand bottom</th>
<th>Gravel bottom</th>
<th>Moraine bottom</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\text{Bias}(R)</td>
<td>$</td>
<td>0.01 m</td>
<td>0.001 m</td>
</tr>
<tr>
<td>$\sqrt{\text{Var}_1(R)}$</td>
<td>5 m</td>
<td>4 m</td>
<td>4 m</td>
<td>4 m</td>
</tr>
<tr>
<td>$\sqrt{\text{Var}_2(R)}$</td>
<td>40 m</td>
<td>30 m</td>
<td>30 m</td>
<td>20 m</td>
</tr>
<tr>
<td>$</td>
<td>\text{Bias}(z_o)</td>
<td>$</td>
<td>400 m</td>
<td>20 m</td>
</tr>
<tr>
<td>$\sqrt{\text{Var}_1(z_o)}$</td>
<td>200 m</td>
<td>70 m</td>
<td>40 m</td>
<td>30 m</td>
</tr>
<tr>
<td>$\sqrt{\text{Var}_2(z_o)}$</td>
<td>2000 m</td>
<td>1000 m</td>
<td>600 m</td>
<td>300 m</td>
</tr>
</tbody>
</table>

Table 5-3 – Summary of results – One-parameter problem

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Silt bottom</th>
<th>Sand bottom</th>
<th>Gravel bottom</th>
<th>Moraine bottom</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\text{Bias}(R)</td>
<td>$</td>
<td>8 m</td>
<td>0.4 m</td>
</tr>
<tr>
<td>$\sqrt{\text{Var}_1(R)}$</td>
<td>6 m</td>
<td>4 m</td>
<td>4 m</td>
<td>4 m</td>
</tr>
<tr>
<td>$\sqrt{\text{Var}_2(R)}$</td>
<td>100 m</td>
<td>40 m</td>
<td>40 m</td>
<td>40 m</td>
</tr>
<tr>
<td>$</td>
<td>\text{Bias}(z_o)</td>
<td>$</td>
<td>700 m</td>
<td>20 m</td>
</tr>
<tr>
<td>$\sqrt{\text{Var}_1(z_o)}$</td>
<td>200 m</td>
<td>70 m</td>
<td>40 m</td>
<td>30 m</td>
</tr>
<tr>
<td>$\sqrt{\text{Var}_2(z_o)}$</td>
<td>2000 m</td>
<td>1000 m</td>
<td>600 m</td>
<td>300 m</td>
</tr>
</tbody>
</table>

Table 5-4 – Summary of results – Two-parameter problem
Figure 5-27 – Finding suitable step-size increment

Type A waveguide, for source Bandwidth of 50Hz
Figure 5-28 – 1-parameter localization (Silt) - 50Hz Signal Bandwidth
Figure 5-29 – 1-parameter localization (Sand) - 50Hz Signal Bandwidth
Figure 5-30 – 1-parameter localization (Gravel) - 50Hz Signal Bandwidth
Figure 5-31 – 1-parameter localization (Moraine) - 50Hz Signal Bandwidth
Range Localization: Pekeris WG, BW=50Hz, SILT Bottom

Depth Localization: Pekeris WG, BW=50Hz, SILT Bottom

Figure 5-32 – 1-parameter / Necessary Samples (Silt) - 50Hz Signal Bandwidth
Figure 5-33 – 1-parameter / Necessary Samples (Sand) - 50Hz Signal Bandwidth
Range Localization: Pekeris WG, BW=50Hz, GRAVEL Bottom

Depth Localization: Pekeris WG, BW=50Hz, GRAVEL Bottom

Figure 5-34 – 1-parameter / Necessary Samples (Gravel) - 50Hz Signal Bandwidth
Figure 5-35 – 1-parameter / Necessary Samples (Moraine) - 50Hz Signal Bandwidth
Figure 5-36 – 1-parameter / Variance ratio (Silt bottom) - vs. Signal Bandwidth
Figure 5-37 – 1-parameter / Variance ratio (Sand bottom) - vs. Signal Bandwidth
Figure 5-38 – 2-parameter localization (Silt) - 50Hz Signal Bandwidth
Figure 5-39 – 2-parameter localization (Sand) - 50Hz Signal Bandwidth
Figure 5-40 – 2-parameter localization (Gravel) - 50Hz Signal Bandwidth
Figure 5-41 – 2-parameter localization (Moraine) - 50Hz Signal Bandwidth
Figure 5-42 – 2-parameter / Necessary Samples (Silt) - 50Hz Signal Bandwidth
Figure 5-43 – 2-parameter / Necessary Samples (Sand) - 50Hz Signal Bandwidth
Figure 5-44 – 2-parameter / Necessary Samples (Gravel) - 50Hz Signal Bandwidth
Figure 5-45 – 2-parameter / Necessary Samples (Moraine) - 50Hz Signal Bandwidth
Figure 5-46 – 2-parameter / Variance ratio (Silt bottom) - vs. Signal Bandwidth
Figure 5-47 – 2-parameter / Variance ratio (Sand bottom) - vs. Signal Bandwidth
6. Summary and Conclusions

In this thesis I examine the effects of dispersion in ocean acoustic propagation in continental shelf environments. I concentrate on three particular areas of practical interest: parameters that succinctly describe the temporal and spatial form of the signal, the degradation of the matched-filter performance and the localization performance of the MLE for one-way propagation scenarios.

I used a quasi-statistical method to derive the moments of the signal, which give quantitative characteristics of the signal behavior in a dispersive medium. Dispersion causes an increase in time-spread and a shift in the mean arrival time of the pulse. The magnitude of dispersion-induced time-spread is highly dependent upon the given ocean waveguide. For fast bottoms, it is apparently small (typically less than 0.1 sec) when the sediment layer has sound speed roughly of ten percent greater than water, as it does for silt or sand bottoms. As the sediment-layer sound speed increases beyond this value as it does for gravel or harder bottoms, the dispersion-induced time-spread becomes significant (as high as 0.4 seconds) even after tens of kilometers of propagation. The amount of time-frequency dispersion is a function of range. In a dispersive medium, many modes are trapped within the waveguide even at large ranges, each mode with different group velocity that varies with frequency. The interference between the modes causes a change in the spatial and temporal structure of the signal as it propagates along the medium. When the time-spread of the received signal is smaller than its effective bandwidth, the medium can be treated as a time invariant filter, and the matched-filter approach can be applied to determine range by time delay estimation using an effective sound speed.

The higher order expressions for the bias and variances of the Maximum Likelihood Estimator have been applied to the problem of localizing an acoustic source both in range and depth, in an ocean waveguide environment. I show that these inversion
parameters can very easily have large variances that do not attain the CRLB. The results suggest that as the signal-to-noise ratio at the receiver falls below 0 dB, the MLE exhibits significant biases and variances that can exceed the CRLB by orders of magnitude. In this case, high sample sizes are required to obtain an unbiased, minimum variance estimates. The more unknown parameters introduced simultaneously to the problem, the higher the biases and variances of the estimates will be, which cause degradation in the source localization performance. In a dispersive medium, the fact that more propagating modes exists, improves the MLE performance significantly compared to a non-dispersive medium.
7. References

15. Y. Lai, N.C. Makris, “Detection and localization of non-vocal humpback whales using vocal whales as sources of opportunity”, to be submitted to JASA.


Appendix A – Joint moments: Deterministic signal in Independent additive Gaussian noise

For this case, the covariance matrix is independent of the parameter to be estimated, i.e. $\partial C/\partial \theta' = 0$ for all $i$, $\mu$ is the mean of the random data. The joint moments required to evaluate the first-order bias and the first and second-order variance are [19],

\begin{align*}
i_{ab} &= n \mu_a^T C^{-1} \mu_b \quad (A-1) \\
v_{abc}(n^1) &= -\frac{n}{2} \sum_{a,b,c} \mu_{ab}^T C^{-1} \mu_c \quad (A-2) \\
v_{a,b,c}(n^1) &= 0 \quad (A-3) \\
v_{ab,c}(n^1) &= n \mu_{ab}^T C^{-1} \mu_c \quad (A-4) \\
v_{abcd}(n^1) &= -\frac{1}{8} \sum_{a,b,c,d} \mu_{ab}^T C^{-1} \mu_{cd} - \frac{1}{6} \sum_{a,b,c,d} \mu_{abc}^T C^{-1} \mu_d \quad (A-5) \\
v_{a,b,c,d}(n^2) &= \frac{n^2}{8} \sum_{a,b,c,d} \mu_a^T C^{-1} \mu_b \mu_c^T C^{-1} \mu_d \quad (A-6) \\
v_{a,b,c,d}(n^1) &= 0 \quad (A-7) \\
v_{a,b,c,de}(n^2) &= \frac{n^2}{2} \sum_{a,b,c} \mu_a^T C^{-1} \mu_b \mu_c^T C^{-1} \mu_{de} \quad (A-8) \\
v_{a,b,cd,ef}(n^2) &= \sum_{(a,b) \times (cd,ef)} \mu_a^T C^{-1} \mu_{cd} \mu_b^T C^{-1} \mu_{ef} + n^2 \mu_{cd}^T C^{-1} \mu_{ef} \mu_a^T C^{-1} \mu_b \quad (A-9) \\
v_{a,b,cd,ef}(n^2) &= \frac{n^2}{2} \sum_{a,b,c} \mu_a^T C^{-1} \mu_b \mu_c^T C^{-1} \mu_{def} \quad (A-10)
\end{align*}

where the notation $\sum_{(a,b) \times (cd,ef)}$ indicates a sum over all possible permutations of $a$ and $b$ combined with permutations of $cd$ and $ef$ orderings, leading to a total of four terms.
Appendix B - Numerical derivative calculations

For the case where the data is given by a deterministic signal, embedded in zero-mean, additive white Gaussian noise, the evaluation of Equations (4-4) through (4-6) requires the knowledge of the first, second and third order derivatives of the log-likelihood function with respect to all combinations of the components of $\theta$. In our case, the derivatives of the Green's function with respect to any perturbation in the environment are required. Numerical differentiation can lead to instability and inaccuracy, so the preferred method is to derive analytic expressions for the derivatives. However, in ocean acoustic, the cases were the pressure field can be written in a close form, where the derivatives can be written explicitly are limited to few simple environmental ocean acoustic models, for example, the zeroth order waveguide model or the Pekeris waveguide model [8]. In general, since the Green's function is a complicated function, analytic expressions for the derivatives are difficult to obtain, and therefore are often computed numerically using finite difference techniques.

B.1 Finite Difference Equations

Let $f$ be a general function, in which the derivatives with respect to parameter $x$ need to be evaluated. The finite-difference equations can be written as,

$$f'(x) = \frac{f(x+h) - f(x)}{h} \quad (B-1)$$

$$f''(x) = \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} \quad (B-2)$$

$$f'''(x) = \frac{f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)}{h^3} \quad (B-3)$$

where $h$ is the step-size.

B.2 Determination of a sufficient step-size

A necessary part of accurately executing the finite difference equations for function derivatives is to select a well-chosen step-size, one that gives a stable derivative
value. If the step-size is too small, computer quantization error will be an issue (i.e. the step-size is smaller or on the order of the smallest quantity the computer can resolve, thus calculated perturbed function values will either not change from the unperturbed function or change in a spurious manner due to sudden jumps in the perceived perturbation). On the other hand, when the step-size is too large, some features of the function may be missed and inaccurate derivative will be achieved.

Clearly, the step-size that will lead to a stable derivative value is somewhere in between. For complicated functions that are non-linearly related to the parameter, which one would like to take derivatives, such as the received pressure field in our case, it can be difficult to predict ahead the sufficient step-size, and it must be checked carefully using simulation.

Notice that for a problem, which includes a vector of unknown parameters, it might be impossible to find one specific step-size that will be suitable for all calculation of the derivatives and the mixed-derivatives, and a different step-size for every function may be necessary.
Appendix C - Analytic Derivatives for Pekeris Waveguide

C.1 Modal Formulation

Assume that the receiver is far away from the source; the pressure field can be modeled in general as the sum of a complete set of normal modes,

\[
p(r; z, z_0) = \frac{ie^{-i\pi/4}}{\rho(z_0)\sqrt{8\pi}} \sum_m \Psi_m^*(z_0) \Psi_m(z) \frac{e^{ik_mR}}{\sqrt{k_mR}}
\]

where, \( R \) is the horizontal range from the source to the receiver; \( \rho(z_0) \) is the density at the source; \( k_m \) is the horizontal wavenumber of the \( m \)th mode; \( z_0, z \) are the depths of the source and the receiver, respectively; and \( \Psi_m \) are the mode functions.

Pekeris waveguide is a canonical example of a particular interest in ocean acoustics because it embodies many of the fundamental features of acoustic propagation in shallow water. The waveguide consists of a homogenous fluid layer bounded above by pressure-release surface and below by a higher velocity, homogeneous fluid half-space. For the Pekeris waveguide the modal functions within the waveguide can be written explicitly as [8],

\[
\begin{align*}
\Psi_m(z) &= A_m \sin(k_mz) \\
\Psi_m(z_0) &= A_m \sin(k_mz_0)
\end{align*}
\]

where, \( A_m \) is the modal amplitude,

\[
A_m = \sqrt{2} \left[ \frac{1}{\rho} \left( h - \frac{\sin(2k_mh)}{2k_m} \right) + \frac{1}{\rho_1} \frac{\sin^2(k_mh)}{k_{1m}} \right]^{-1/2}
\]

Let \( k = \frac{\omega}{c} \) and \( k_1 = \frac{\omega}{c_1} \) be the medium wavenumbers for the water column and bottom half-space, respectively; \( \rho, \rho_1 \) and \( c, c_1 \) are the corresponding densities and sound speeds;
than, $k_{zm}, k_{1zm}$ are the vertical wavenumbers for the water column and bottom half-space which satisfy, $k^2 = k_{rm}^2 + k_{zm}^2$ and $k_1^2 = k_{rm}^2 + k_{1zm}^2$. Finally, $h$ is the waveguide depth;

C.2 pressure derivatives

For 2-parameter estimation problem, range and source-depth, Eq. (C-1) can be written as follows,

$$p(R, z_0) = C \sum_m A_m^2 \frac{\sin(k_{zm}z) \cdot \sin(k_{zm}z_0) \cdot e^{ik_{rm}R}}{f_m(z_0) \sqrt{k_{rm}R}} = C \sum_m N_m$$

(C-4)

The first, second and third derivatives of the pressure field with respect to those two quantities and all the mixed-derivatives can be worked out using the chain rule, and the result is as follows,

$$p_r = \frac{\partial p}{\partial R} = C \sum_m \left( jk_{rm} - \frac{1}{2R} \right) \cdot N_m$$

(C-5)

$$p_{rr} = \frac{\partial^2 p}{\partial R^2} = C \sum_m \left[ \left( jk_{rm} - \frac{1}{2R} \right)^2 + \frac{1}{2R^2} \right] \cdot N_m$$

(C-6)

$$p_{rrr} = \frac{\partial^3 p}{\partial R^3} = C \sum_m \left[ \left( jk_{rm} - \frac{1}{2R} \right)^3 + \frac{3}{2R^2} \left( jk_{rm} - \frac{1}{2R} \right) - \frac{1}{R^3} \right] \cdot N_m$$

(C-7)

$$p_d = \frac{\partial p}{\partial z_0} = C \sum_m k_{zm} \cdot \cot(k_{zm}z_0) \cdot N_m$$

(C-8)

$$p_{dd} = \frac{\partial^2 p}{\partial z_0^2} = C \sum_m \left( -k_{zm}^2 \right) \cdot N_m$$

(C-9)

$$p_{ddd} = \frac{\partial^3 p}{\partial z_0^3} = C \sum_m \left( -k_{zm}^3 \right) \cdot \cot(k_{zm}z_0) \cdot N_m$$

(C-10)

$$p_{rd} = \frac{\partial^2 p}{\partial R \partial z_0} = C \sum_m \left( jk_{rm} - \frac{1}{2R} \right) \cdot k_{zm} \cdot \cot(k_{zm}z_0) \cdot N_m$$

(C-11)

$$p_{rdd} = \frac{\partial^3 p}{\partial R^2 \partial z_0} = C \sum_m \left[ \left( jk_{rm} - \frac{1}{2R} \right)^2 + \frac{1}{2R^2} \right] \cdot k_{zm} \cdot \cot(k_{zm}z_0) \cdot N_m$$

(C-12)

$$p_{rdd} = \frac{\partial^3 p}{\partial R^2 \partial z_0^2} = C \sum_m \left( jk_{rm} - \frac{1}{2R} \right) \cdot \left( -k_{zm}^2 \right) \cdot N_m$$

(C-13)
Appendix D — Gaussian Modulated source signal

The Gaussian modulated waveform (GMW) has the same Gaussian envelope that was introduced in Example 2.4, and defined as,

\[ q(t) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(t-t_d)^2}{2\sigma^2}} e^{-2\pi i \epsilon (t-t_d)} \]  \hspace{1cm} (D-1)

The Gaussian signal was selected since it’s a good signal for fast validity checks. All its parameters are well known and can be derived analytically, it has a Gaussian waveform spectrum, and for free-space propagation the Uncertainty Principle reached its lower bound. The analytical expression for this waveform spectrum as the form of,

\[ Q(f) = e^{-\frac{1}{2} \sigma^2 [2\pi (f-f_c)]^2} \]  \hspace{1cm} (D-2)

In my simulation I used the following parameters, a carrier frequency of \( f_c = 200\,Hz \) and variance of \( \sigma = 0.1\,\text{sec} \). Again, I shifted the signal by \( t_d \) seconds for convenient computation reasons (\( t_d = 1\,\text{sec} \) in my case).

Using the results from Example 2.4, the effective signal duration and the effective bandwidth are as follows,

\[ \Rightarrow \Delta \tau(r_0, r_0) = \frac{\sigma}{\sqrt{2}} \approx 0.0707\,\text{sec}. \]

\[ \Delta \beta(r_0, r_0) = \frac{1}{2\sqrt{2\pi} \sigma} \approx 1.125\,\text{Hz} \]

We can see that the time-bandwidth product (via the uncertainty principle) is conserved for this case, reaching its lower bound, as expected; \( \Delta \tau \cdot \Delta \beta \approx 0.07957(\approx \frac{1}{4\pi}) \)

The validity of this result was checked using the time-harmonic Green’s function \( G(r|r_0|f) \) for free-space, as defined in Eq.(2-8). When this signal is propagating in a non-dispersive medium, like silt bottom, those quantities versus range are almost conserved. However, for more dispersive medium those quantities start to increase, as expected.