Nonlinear Shallow Water Three-dimensional Solitary Waves Generated by High Speed Vessels

by

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Submitted to the Department of Ocean Engineering in partial fulfillment of the requirements for the degree of Master of Science in Naval Architecture and Marine Engineering at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Abstract

We study the nonlinear long waves generated by a disturbance moving at subcritical, critical and supercritical speeds in shallow water. In a shallow channel of finite width, there's no steady state disturbance and two dimensional upstream solitons are radiated periodically ahead of the disturbance when it travels in the vicinity of the critical speed $\sqrt{gH}$, where $g$ is the gravitational acceleration and $H$ is the water depth. It is an open question as to whether there exists a steady state and what the shallow water wave disturbance looks like when the domain is unbounded.

A modified generalized Boussinesq equation is derived to formulate the propagation of three dimensional nonlinear long waves in shallow water. An implicit finite difference scheme is developed to simulate the generation of upstream solitons and downstream Kelvin-like wake numerically. The calculation demonstrates that three dimensional solitary waves with significant amplitude are generated with a periodicity by a moving pressure distribution on the free surface. The crestlines of these solitons are almost perfect parabolas with decreasing curvature with respect to time. Behind the disturbance, a complicated, divergent Kelvin-like wave pattern is formed. The case of the free propagating three dimensional solitary waves is also investigated. Unlike the wave breaking phenomena in a narrow channel at $F_h \geq 1.2$, the three dimensional upstream solitons form several parabolic water humps ahead of the disturbance at supercritical speed when the domain is unbounded.

A thin ship approximation is used to investigate the nonlinear long waves generated by a high speed vessel in shallow water. The wave patterns of the upstream-running solitary waves are found to be similar with those generated by a moving pressure disturbance. The downstream wave system is dependent on the ship geometry and is composed of waves with smaller wave lengths compared with that generated by the applied pressure. The numerical results for the solitons generated by a ship in a shallow channel show satisfactory agreement with the experiment done by Ertekin, Webster & Wehausen(1984).
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Chapter 1

Introduction

In the water around northern Europe, Mediterranean, Japan and Hongkong, high speed vessels are used widely as a fast means of transportation. A large amplitude wake wash is generated by those fast ships and propagates shoreward, which has been becoming a raising concern for coastal communities. The large waves have a significant impact on the safety of people and crafts, the erosion on coasts and bottoms and the biological environment. In a fatal accident which happened in Harwich, a port on England’s east coast, in July 1999, one surviving victim reported that the wave looked like "the white cliffs of Dover" (Hamer, 1999). Research done in Europe shows that the soliton produced by the fast ferry should be responsible for the disaster.

In restricted waters, solitary waves can be generated ahead of the ship bow propagating upstreamward keeping their shape and velocity constant. This phenomenon was first discovered by Scott Russel in 1834 when he was watching a canal boat pulled by horses stopped suddenly. In ship hydrodynamics, it has been observed (Thews & Landweber, 1935, 1936) in a towing tank for long time that the test ship advancing steadily can radiate waves which move faster than the ship and a steady state of the wave resistance cannot be reached. The extensive experimental, theoretical and numerical investigations of this type of waves are pioneered by the systematic experiments by Huang et al (1982) for ship models moving at various transcritical speeds. Investigating the extensive experimental results for a Series 60 ship model, Ertekin,
Webster & Wehausen (1984) pointed out that the blockage coefficient \( A/Wh \) (\( A \) is the maximum cross-sectional area of the ship, \( W \) is the width of the channel) is the dominant parameter for the generation of solitons.

Wu & Wu (1982) presented numerical computations for the nonlinear long waves forced by a moving pressure patch in the vicinity of the critical speed \( U = \sqrt{gh} \) in a two dimensional tank, based on a generalized Boussinesq (gB) model (Wu, 1981) assuming a balance between the nonlinear effects and dispersion. It was shown that a solitary wave first emerges ahead of the disturbance, and finally propagates upstream freely. Starting from the break down of the linear solution, Akylas (1984) and Cole (1985) developed the nonlinear theory which accounts for the finite-amplitude effects and found that the generated waves are governed by a forced Korteweg-de Vries (fKdV) equation. Ertekin, Webster & Wehausen (1984) carried out the numerical calculation by using the Green-Naghdi fluid sheet equations. The forcing is taken as a pressure distribution on the free surface or the underwater topography and a similar phenomenon is reported: a succession of upstream-running solitons are generated periodically ahead of the disturbance, while a weakly nonlinear and dispersive wave train develops downstream of an elongated depressed water surface, trailing the disturbance. Wu (1987) present a preliminary study of the underlying basic mechanism of the phenomenon by analyzing the stability of the solutions of the fKdV equation. In a joint numerical and experimental study, Lee, Yate & Wu (1989) found that both the gB and fKdV models obtain qualitatively similar predictions of the phenomenon of the precursor solitons, showing a satisfactory agreement with experiments. Casciola & Landrini (1996) used an accurate boundary integral approach to simulate the flow and carried out a detailed comparison between the fully nonlinear model, gB and fKdV models. Zhang & Chwang (1999) investigated the influence of viscous effects on a two dimensional submerged body moving at transcritical speed by solving the Navier-Stokes equations with the complete set of nonslipping boundary conditions numerically.
In a restricted channel of shallow water, the two dimensional upstream solitons are generated by a disturbance with three dimensional geometry at a transcritical speed. Mei(1986) derived a one dimensional inhomogeneous KdV equation for the flow around thin bodies extending throughout the water depth. The corresponding result is two dimensional for both upstream and downstream waves if the channel width is small: $W \ll h^2/a$, $a$ being the typical wave amplitude. Ertekin, Webster and Wehausen(1986) used the restricted Green-Naghdi theory of fluid sheets to perform the three dimensional calculation of waves by an impulsively started pressure patch traveling at the transcritical speed. The two dimensional solitons propagate upstream periodically, whereas a three dimensional doubly corrugated set of waves is formed behind the disturbance. By analyzing the linear dispersive relation near the critical speed, Katsis and Akylas(1987) derived a forced nonlinear Kadomtsev-Petviashvili(KP) equation to describe the linear dispersive, nonlinear and transverse effects governing the nonlinear long waves excited by a moving pressure distribution. The sidewall is not essential for the radiation of upstream waves but for the transformation of curved waves to straight-crested solitons. Pedersen(1988) studied the wave patterns generated by a pressure field, source distribution and bottom topography in wide channels based on the Boussinesq equations. The formation of two dimensional solitons is related to the Mach reflection at the sidewall of the channel. Some other researchers solved the Laplace equation with the exact free surface condition numerically. Bai, Kim and Kim(1989) and Choi et al(1990) studied the nonlinear free surface flow produced by a three dimensional ship hull by means of the Finite Element method, respectively.

From the viewpoint of applications, the real ship geometry has to be considered for the demands of high speed vessel design. Using the matched asymptotic expansion method, Choi & Mei(1989) obtained the homogeneous KP equation with flux conditions on the symmetric plane under the assumption of a slender body. The normal condition of wave elevation is related to the second $x$ derivative of the longitudinal sectional area. Chen & Sharma(1995) extended the previous method and took the
local wave elevation and longitudinal disturbance velocity into account. The wave force, hydrodynamic lift force and trim moment are calculated for the fixed-hull and sinkage and trim are calculated for the freehull case.

It's natural to ask if there exists a steady state or what the unsteady state looks like for a disturbance moving at critical speed of long wave in horizontally unbounded domain, in which the blockage parameter tends to zero. Katsis & Akylas (1987) first dealt with this problem with the forced KP equation and a nonlinear curved wave emerges in front of the disturbance at the critical condition. They suggested that no nonlinear steady state could be reached in that case. Pedersen (1988) applied a radiation condition at the open seaward boundary and simulated the problem for a sufficiently long time. He suggested that there always exists a stationary state in unbounded sea and the wave pattern is extended some distance ahead of the disturbance when the depth Froude number is close to unity. This might be not true as the following studies presented opposite predictions. Lee & Grimshaw (1990) also employed the KP equation and reported various characteristics of upstream advancing waves in open sea. A similarity solution was presented under the assumption that the amplitude is constant along the isophasal line of the leading three dimensional soliton. The solitary wave amplitude diminishes in a manner proportional to \( O(t^{-\frac{3}{2}}) \) and the crestline, which is a parabola, decreases its curvature as it moves. Choi et al (1991) reported the numerical results of a pressure distribution traveling at the critical speed in an open domain and found that the crestline of the leading soliton fits well with a parabola when the upstream wave develops.

In the present study, the main concern is focused on the three dimensional upstream solitary waves generated by a moving disturbance at subcritical, critical and supercritical speeds in both restricted and unbounded shallow water. In chapter 2, a modified generalized Boussinesq(mgB) equation is derived in terms of the depth averaged velocity potential and water elevation. No specific limitation is imposed on the scale of transverse variation and time, hence allowing the modelling of unsteady
shallow water waves forced by either pressure patch on the free surface or a source distribution underwater. The numerical methods, based on both explicit and implicit finite-difference algorithms, are described in chapter 3. An open boundary condition is applied on both downstream and open seaward boundaries to make the wave propagate outward without reflection. The numerical results and discussion of the solitary waves generated by the pressure distribution are presented in chapter 4. The wave evolution forced by a disturbance moving at subcritical, critical and supercritical speeds is analyzed in detail and the properties of the crestline of the leading soliton are exploited. It is found that the crestline is nearly a perfect parabola with curvature diminishing as the soliton moves forward. The phase velocity of the leading soliton is proportional to its amplitude, which decays in the order of $O(t^{-\frac{1}{3}})$. The case of free three dimensional soliton is also studied. The free soliton decays at a rapid rate. In chapter 5, a thin ship approximation is used to simulate the upstream-running waves in channels and open sea. The geometry of the crestline of the leading soliton is the same as that of the pressure-forcing solitary wave. At supercritical speeds, the steady state could be reached with solitary waves moving at the same speed as that of the disturbance. Conclusions are presented in chapter 6.
Chapter 2

Mathematical Model

2.1 Governing Equations

2.1.1 Modified Generalized Boussinesq Equations

Let a pressure distribution advance at the constant speed $U$ acting on the surface of a layer of water with uniform depth $h$. The disturbance is stationary in a reference coordinate system moving with the pressure. $Oz$ points upward and the $(x, y)$ plane lies on the undisturbed free surface. In this reference frame, a steady current is moving in the positive $x$ direction with a speed of $U$. Under the assumption of incompressible, inviscid and irrotational flow, the water wave motion is described by the velocity potential $\Phi(x, y, z, t)$ and the free surface water elevation $\zeta(x, y, t)$.

In this reference frame, the velocity potential $\Phi(x, y, z, t)$ can be decomposed as

$$\Phi(x, y, z, t) = \phi(x, y, z, t) + Ux$$  \hspace{1cm} (2.1)

in which $\phi(x, y, z, t)$ is the disturbance velocity potential representing the flow motion induced by the external disturbances such as pressure distribution on the free surface, floating bodies or underwater topography.

The velocity potential $\phi(x, y, z, t)$ satisfies the Laplace equation

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$  \hspace{1cm} (2.2)
The dynamic and kinematic free surface conditions are

\[ \phi_t + g\zeta + U\phi_x + \frac{1}{2} \nabla \phi \cdot \nabla \phi + \frac{p}{\rho} = 0 \quad \text{at } z = \zeta \]  \hspace{1cm} (2.3)

\[ \zeta_t + U\zeta_x + \phi_x \zeta_x + \phi_y \zeta_y = \phi_z \quad \text{at } z = \zeta \]  \hspace{1cm} (2.4)

On the bottom the non-flux boundary condition is

\[ \phi_z = 0 \quad \text{at } z = -h \]  \hspace{1cm} (2.5)

where \( g \) is the gravitational acceleration, \( \rho \) is the fluid density and \( p \) is the forcing pressure on the free surface.

We choose the typical wave amplitude as \( a \), the characteristic wave number as \( k \), the characteristic horizontal velocity as \( \sqrt{gh} \) and vertical scale as \( h \). The above variables are thus normalized as follows:

\[ (x, y) = \left( \frac{x', y'}{k} \right), \quad \zeta = a \zeta', \quad z = h z', \]

\[ \phi = \frac{\phi'}{a} \sqrt{\frac{g}{h}}, \quad t = \frac{\nu}{k \sqrt{gh}}, \quad p = \rho g a p' \]

Three dominant parameters should be noted as

\[ \epsilon = \frac{a}{h}, \quad \mu = k h, \quad F_h = \frac{U}{\sqrt{gh}} \]

in which \( F_h \) is the so called depth Froude number. The Boussinesq approximation is adopted to assume that \( \epsilon \) is of the same order of \( \mu^2 \), which indicates the balance between the nonlinear and dispersive effects for nonlinear long waves.

In terms of these dimensionless variables, (2.2), (2.3), (2.4) and (2.5) become

\[ \mu^2 (\phi_{xx} + \phi_{yy}) + \phi_{zz} = 0 \]  \hspace{1cm} (2.6)

\[ \phi_t + \zeta + F_h \phi_x + \frac{\epsilon}{2} (\phi_x^2 + \phi_y^2 + \frac{\phi_z^2}{\mu^2}) + \frac{p}{\rho} = 0 \quad \text{at } z = \epsilon \zeta \]  \hspace{1cm} (2.7)
\[ \zeta_t + F \zeta_{xx} + \varepsilon (\phi_x \zeta_x + \phi_y \zeta_y) = \frac{\phi_z}{\mu^2} \quad \text{at } z = \varepsilon \zeta \quad (2.8) \]

\[ \phi_z = 0 \quad \text{at } z = -1 \quad (2.9) \]

Hereafter, we drop primes from (2.6) to (2.9).

The velocity potential \( \phi(x, y, z, t) \) is assumed analytic and we can expand it in power series with respect to the vertical coordinate about \( z = -1 \).

\[ \phi(x, y, z, t) = \sum_{n=0}^{\infty} (z + 1)^n \phi_n(x, y, t) \quad (2.10) \]

The derivatives with respect to \( z \) become

\[ \phi_z = \sum_{n=0}^{\infty} (n + 1)(z + 1)^n \phi_{n+1}(x, y, t) \quad (2.11) \]

\[ \phi_{zz} = \sum_{n=0}^{\infty} (n + 2)(n + 1)(z + 1)^n \phi_{n+2}(x, y, t) \quad (2.12) \]

Substituting (2.12) into the governing equation (2.6), we obtain

\[ \mu^2(\phi_{xx} + \phi_{yy}) + \phi_{zz} = \sum_{n=0}^{\infty} (z + 1)^n ((n + 1)(n + 2)\phi_{n+2} + \mu^2 \nabla^2 \phi_n) = 0 \quad (2.13) \]

where the operator \( \nabla \) denotes the gradient in the horizontal plane, i.e. \( \nabla = (\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j}) \). The value of \( z \) is arbitrary in the range of \((-1, \zeta)\) and the coefficients of the power of \( z + 1 \) must vanish to satisfy (2.13), thus we get

\[ (n + 1)(n + 2)\phi_{n+2} + \mu^2 \nabla^2 \phi_n = 0 \quad n = 0, 1, 2, \ldots \quad (2.14) \]

Combining (2.11) and the bottom normal boundary condition (2.9), it is obvious that

\[ \phi_1 = 0 \quad (2.15) \]

From the recursive relation (2.14), the velocity potential components with odd
subscripts all vanish,

$$\phi_1 = \phi_3 = \cdots = \phi_{2m+1} = \cdots = 0 \quad m = 0, 1, \cdots \quad (2.16)$$

In accordance to (2.14), the velocity components of even order can be expressed by the zero order term $\phi_0$ as follows,

$$\phi_{2m} = -\frac{\mu^2}{2m(2m-1)} \nabla^2 \phi_{2m-2}$$

$$= (-1)^n \frac{\mu^{2m}}{(2m)!} \frac{\nabla^2 \cdots \nabla^2 \phi_0}{m} \quad m = 1, 2, \cdots \quad (2.17)$$

Consequently the velocity potential can be expressed as

$$\phi(x, y, z, t) = \phi_0 - \frac{\mu^2}{2} (z + 1)^2 \nabla^2 \phi_0 + \frac{\mu^2}{24} (z + 1) \nabla^2 \nabla^2 \phi_0 + O(\mu^6)$$

with an error order $O(\mu^6)$.

Furthermore, we introduce a depth-averaged velocity potential, defined by

$$\overline{\phi}(x, y, t) = \frac{1}{1 + \epsilon \zeta} \int_{-1}^{\epsilon} \phi(x, y, z, t) dz$$

$$= \frac{1}{1 + \epsilon \zeta} \int_{-1}^{\epsilon} (\phi_0 - \frac{\mu^2}{2} (z + 1)^2 \nabla^2 \phi_0) dz + O(\mu^4)$$

$$= \phi_0 - \frac{\mu^2 (1 + \epsilon \zeta)^2}{6} \nabla^2 \phi_0 + O(\mu^4)$$

and thus

$$\phi_0(x, y, t) = \overline{\phi}(x, y, t) + \frac{\mu^2 H^2}{6} \nabla^2 \overline{\phi}(x, y, t) + O(\mu^4)$$

where $H = 1 + \epsilon \zeta$ is the free surface elevation measured from the bottom.

Substituting (2.20) into (2.18), the three dimensional velocity potential describing the water wave motion in shallow water is expressed by the velocity potential which is averaged along the vertical coordinate up to an error of $O(\mu^6)$, or
\[ \phi(x, y, z, t) = \bar{\phi} + \mu^2 \left( \frac{H^2}{6} - \frac{(z + 1)^2}{2} \right) \nabla^2 \bar{\phi} - \mu^4 \left( \frac{(z + 1)^2}{12} \nabla^2 (H^2 \nabla^2 \bar{\phi}) - \frac{(z + 1)^4}{24} \nabla^2 \nabla^2 \bar{\phi} \right) + O(\mu^6) \quad (2.21) \]

To account for the coupled nonlinear and dispersive effects for shallow water waves, the dimensionless variable \( \epsilon \), representing nonlinearity, and \( \mu \), representing dispersion, are assumed to be related as follows:

\[ O(\epsilon) = O(\mu^2) \]

The order of the time derivative and transverse derivative of \( \bar{\phi} \) and \( \zeta \) is crucial to the formulation of the problem. Many previous studies showed that the first order time derivative is of order \( O(\mu^2) \) and the first order transverse derivative of order \( O(\mu) \) (Katsis & Akylas, 1987) in a channel of finite width or \( O(\mu^{1/2}) \) (Mei, 1986) in a narrow channel. For the three dimensional nonlinear long waves generated by a moving disturbance, the evolution of upstream-running waves is transient and three dimensional. It is difficult to determine the characteristic transverse scale and the time scale explicitly due to the variation of the wave as it develops. Meanwhile, the upstream wave systems are supposed to exhibit different properties from those downstream wave trains, whose counterpart is the Kelvin wake in classical linear theory. To explore the physics of both the precursor waves and the downstream Kelvin-like waves, we assume that the primary time variable is a slow variable and the order of its derivative is set to be higher than unity and lower than \( O(\epsilon^2) \) to catch the transient wave motions. Any combination of a time derivative with orders of \( O(\epsilon) \) or \( O(\mu^2) \) is omitted to leading order, or

\[ O(\epsilon \frac{\partial}{\partial t}) \ll O(\epsilon) \quad O(\mu^2 \frac{\partial}{\partial t}) \ll O(\mu^2) \]

Following the above assumption, the substitution of (2.21) into the free surface
condition (2.7) leads to

\[
\bar{\phi}_t + F_h \bar{\phi}_x = -\zeta + \frac{\mu^2}{3} F_h \nabla^2 \bar{\phi}_x - \frac{\epsilon}{2} (\bar{\phi}_x^2 + \bar{\phi}_y^2) - p
\] (2.22)

in which terms of order higher than \( O(\epsilon, \mu^2) \) have been omitted.

Hence, (2.8) can be rewritten as

\[
\zeta_t + F_h \zeta_x = -\nabla \cdot ((1 + \epsilon \zeta) \nabla \bar{\phi}) + O(\mu^4)
\] (2.23)

The above equation (2.23) is valid for any arbitrary \( \epsilon \). (2.22) and (2.23) is the expression of the generalized Boussinesq(gB) equation (Wu, 1981) in the frame moving with a two dimensional pressure patch on free surface of shallow water of uniform depth.

The free surface elevation \( \zeta \) can be expressed in terms of \( \bar{\phi} \) explicitly according to (2.22)

\[
\zeta = -F_h \bar{\phi}_x - \bar{\phi}_t + \frac{\mu^2}{3} F_h \nabla^2 \bar{\phi}_x - \frac{\epsilon}{2} (\bar{\phi}_x^2 + \bar{\phi}_y^2) - p
\] (2.24)

Specific restrictions are applied to the scale of the slope of the pressure distribution to make further simplification of the equations. The pressure is assumed to vary with respect to the slow variable in both the longitudinal and transverse directions, respectively. Thus the order of the pressure derivatives are

\[
O\left(\frac{\partial p}{\partial x}\right) = O\left(\frac{\partial p}{\partial y}\right) < O(1)
\]

Rewrite the nonlinear term \( \nabla \cdot (\epsilon \zeta \nabla \bar{\phi}) \) in (2.23) by plugging in (2.24) for the free surface elevation \( \zeta \) and omit all terms higher than \( O(\epsilon, \mu^2) \), we obtain a new form of long-wave model in which all nonlinear terms are expressed by the derivatives of \( \bar{\phi} \) uniquely. Eventually, the so called modified generalized Boussinesq(mgB) equations are obtained combining with (2.22),

\[
\bar{\phi}_t + F_h \bar{\phi}_x = -\zeta + \frac{\mu^2}{3} F_h \nabla^2 \bar{\phi}_x - \frac{\epsilon}{2} (\bar{\phi}_x^2 + \bar{\phi}_y^2) - p
\] (2.25)
\[ \zeta_t + F_h \zeta_x = -\nabla^2 \Phi + \epsilon F_h (2\Phi_x \Phi_{xx} + \Phi_x \Phi_{yy} + \Phi_y \Phi_{xy}) \quad (2.26) \]

In terms of the physical variables, (2.25) and (2.26) take the form

\[ \Phi_t + U \Phi_x = -g \zeta + \frac{U h^2}{3} \nabla^2 \Phi_x - \frac{1}{2} (\Phi_x^2 + \Phi_y^2) - \frac{p}{\rho} \quad (2.27) \]

\[ \zeta_t + U \zeta_x = -h \nabla^2 \Phi + \frac{U}{g} (2\Phi_x \Phi_{xx} + \Phi_x \Phi_{yy} + \Phi_y \Phi_{xy}) \quad (2.28) \]

### 2.1.2 Kadomtsev-Petviashvili(KP) Equations

We assume that the disturbance is local and the forcing term vanishes in the far field. Combining (2.25) with (2.26), we obtain

\[ \Phi_{tt} + 2F_h \Phi_{xt} + (F_h^2 - 1) \Phi_{xx} - \Phi_{yy} = O(\epsilon, \mu^2) \quad (2.29) \]

The soliton evolution must involve dispersive and nonlinear effects at large time. \( \Phi_{xt} \) should play a dominant role superior to the second time derivative \( \Phi_{tt} \), which indicates that

\[ \frac{\partial}{\partial t} = O(\mu^2). \]

It is believable that there exists a balance between dispersion, nonlinearity and transverse evolution. To ensure that, we can derive from (2.29) that the second transverse derivative is

\[ \frac{\partial^2}{\partial y^2} = O(\mu^2) \]

Near the critical speed, the velocity of the disturbance is assumed of the form

\[ F_h = 1 + \lambda \mu^2 \quad (2.30) \]
Following the above scale analysis, we introduce two slow variables

\[ \tau = \mu^2 t \]
\[ \eta = \mu y \]

Differentiating (2.25) twice and (2.26) once, combining the resulting equations and omitting all terms higher than \( O(\epsilon, \mu^2) \), we can obtain the leading order governing equation as the forced Kadomtsev-Petviashvili (KP) equation

\[ \zeta_{\tau x} + \lambda \zeta_{xx} - \frac{\zeta_{xxxx}}{6} - \frac{\epsilon}{\mu^2} \frac{3}{4} (\zeta^2)_{xx} - \frac{1}{2} \zeta_{\eta \eta} = \frac{p_{xx}}{2\mu^2} \] (2.31)

Set

\[ \epsilon = \mu^2 \]

Assuming that the wave vanishes at infinity, we can integrate (2.31) once with respect to \( x \) and obtain

\[ \zeta_\tau + \lambda \zeta_x - \frac{\zeta_{xxx}}{6} - \frac{3}{4} (\zeta^2)_x - \frac{1}{2} \int_{-\infty}^{x} \zeta_{\eta \eta} dx = \frac{p_x}{2\epsilon} \] (2.32)

Katsis & Akylas (1987) and Lee & Grimshaw (1990) used the above K-P equation to investigate the wave generation in shallow water near resonance. In the shallow water Boussinesq equations described in the previous section, the contribution from the terms of order \( O(\mu^4, \epsilon^2, \epsilon \mu) \) is anticipated in the combination of the two equations about the velocity potential and free surface elevation. It is equivalent to the K-P equation to leading order and when time is large.
2.2 Boundary Conditions

2.2.1 Thin Ship Approximation

On the ship hull, the fluid particle cannot penetrate the solid body surface. Therefore, the normal component of the flow velocity is equal to the corresponding normal velocity of the rigid hull.

$$\frac{\partial \Phi(\vec{x}, t)}{\partial n} = \vec{V}_B(\vec{x}, t) \cdot \vec{n}(\vec{x}) \quad \vec{x} \in S_B$$  \hspace{1cm} (2.33)

in which $\Phi(x, y, z, t)$ is defined in (2.1) and $\vec{V}_B(\vec{x}, t)$ is the velocity of the point on the submerged body surface $S_B$. $\vec{n}(\vec{x})$ is the normal vector at $\vec{x}$, pointing outward of the fluid field.

$\vec{V}_B(\vec{x}, t)$ vanishes in a reference frame fixed on the ship, moving at the constant speed $U$. Also substituting (2.1) into (2.33), we obtain

$$\frac{\partial \Phi(\vec{x}, t)}{\partial n} = \frac{\partial \phi(\vec{x}, t)}{\partial n} + U n_z = 0 \quad \vec{x} \in S_B$$  \hspace{1cm} (2.34)

Assume that the geometry of ship hull is expressed in the form

$$y = Y(x; z) = B \cdot Y\left(\frac{x}{L}, \frac{z}{d}\right) = B \cdot Y(X, Z)$$  \hspace{1cm} (2.35)

where $B, L, d$ is the half beam-width, half ship length and the draft of ship. $X, Y, Z$ is the dimensionless variables normalized by $L, B, d$ respectively.

The normal vector on the ship surface is

$$\vec{n} = \frac{\frac{B}{L} Y_x \vec{i} - \vec{j} + \frac{B}{d} Y_z \vec{k}}{\left(1 + \left(\frac{B}{L} Y_x\right)^2 + \left(\frac{B}{d} Y_z\right)^2\right)^{\frac{1}{2}}}$$  \hspace{1cm} (2.36)

Following the same procedure of nondimensionization resulting from (2.6) to (2.9), the nondimensional version of the boundary condition on the ship hull is

$$\phi_y - \frac{1}{\mu} \frac{B}{d} Y_z \phi_z = \frac{1}{\epsilon L} Y_X (F_h + \epsilon \phi_x) \quad \text{at} \quad y = \frac{B}{h} Y(X, Z)$$  \hspace{1cm} (2.37)
From (2.18), it's easy to find that the vertical velocity component is a high order variable compared with the corresponding components in horizontal directions.

\[ O(\phi_z) = \mu O(\phi_x, \phi_y) \]  \hspace{1cm} (2.38)

A Taylor series expansion of (2.37) about the symmetric plane \( y = 0 \) leads to

\[
\phi_y(x, 0, z) + \frac{B}{h} \frac{F}{L} Y \phi_{yy}(x, 0, z) - \frac{1}{\mu} \frac{B}{d} Y Z \phi_z(x, 0, z) - \frac{B}{d} Y Y Z \phi_{yz}(x, 0, z)
\]

\[
= \frac{F}{L} \frac{B}{\epsilon} Y X + \frac{B}{L} Y X \phi_z(x, 0, z) + \frac{B}{h} Y Y X \phi_{xy}(x, 0, z) + \ldots \]  \hspace{1cm} (2.39)

where we keep two Taylor series terms in (2.37).

We assume that

\[
\frac{B}{L} = O(\epsilon), \quad \frac{B}{d} = O(\mu), \quad \frac{B}{h} = O(\mu)
\]

which implies that the geometry of the ship hull is slender and thin.

Therefore, the leading order term in (2.39) becomes

\[
\phi_y = \frac{F}{L} \frac{B}{\epsilon} Y X \quad \text{at} \quad y = 0 \]  \hspace{1cm} (2.40)

Thus the nonflux boundary condition on the ship hull is simplified into a Neumann condition on \( y = 0 \), which is convenient for theoretical analysis and numerical calculation.

Taking the average with respect to the vertical variable \( z \) from \(-1\) to the free surface \( \epsilon \zeta \) for (2.40), we obtain

\[
\bar{\phi}_y = \frac{F}{L} \frac{B}{\epsilon} \frac{1}{1 + \epsilon \zeta} \int_{-1}^{\epsilon \zeta} Y X d\zeta = \frac{F}{2\epsilon} \bar{S}_x + O(\epsilon) \]  \hspace{1cm} (2.41)

where \( \bar{S} \) is the nondimensional cross-sectional area of the ship

\[
\bar{S}_x = \frac{2B}{L} \int_{-1}^{0} Y X d\zeta \]  \hspace{1cm} (2.42)
and

\[ \tilde{S} = \frac{k}{h} S \]  

(2.43)

where \( S \) is the cross-sectional area of the real ship hull under the still waterline.

In physical variables, the Neumann boundary condition at \( y = 0 \) takes the form

\[ \frac{\tilde{\phi}_y}{2h} = \frac{U}{S_x} \]  

(2.44)

2.2.2 Open Boundary Condition

The present problem is treated as an initial boundary value problem. Waves propagate away from the disturbance and are subject to the radiation condition that waves vanish at infinity.

\[ \phi \to 0, \quad \zeta \to 0 \quad |R| \to \infty \]  

(2.45)

In numerical calculations, the computational field is truncated at some distance away from the disturbance in both the longitudinal and transverse directions. In general, waves can reflect from the truncated boundaries and contaminate the flow in the computational domain. Special concern should be taken to implement suitable open boundary conditions to make the waves pass through the boundaries with minimum reflection. The details of the numerical implementation of open boundary conditions will be discussed in the next chapter.

2.3 Two- and Three-dimensional Initial Boundary Value Problems

In summary, the problem of three dimensional nonlinear long waves generated by a pressure distribution on the free surface and a thin symmetric ship hull has been formulated in nondimensional variables as follows

\[ \tilde{\phi}_t + F_h \tilde{\phi}_x = -\zeta + \frac{\mu^2}{3} F_h \nabla^2 \tilde{\phi}_x - \frac{\epsilon}{2} (\tilde{\phi}_x^2 + \tilde{\phi}_y^2) - p \]  

(2.46)
\[ \zeta_t + F_h \zeta_x = -\nabla^2 \phi + \epsilon F_h (2\phi_x \phi_{xx} + \phi_x \phi_{yy} + \phi_y \phi_{xy}) \quad (2.47) \]
\[ \phi_y = \frac{F_h}{2\epsilon} \sigma_x \quad \text{at } y = 0 \quad (2.48) \]
\[ \phi \to 0 , \quad \zeta \to 0 , \quad |R| \to \infty \quad (2.49) \]

In physical variables, the above equations take the form
\[ \overline{\phi}_t + U \overline{\phi}_x = -g \zeta + \frac{U h^2}{3} \nabla^2 \overline{\phi}_x - \frac{1}{2}(\overline{\phi}_x^2 + \overline{\phi}_y^2) - \frac{p}{\rho} \quad (2.50) \]
\[ \zeta + U \zeta_x = -h \nabla^2 \phi + \frac{U}{g} (2\phi_x \phi_{xx} + \phi_x \phi_{yy} + \phi_y \phi_{xy}) \quad (2.51) \]
\[ \phi_y = \frac{U}{2h} S_x \quad \text{at } y = 0 \quad (2.52) \]
\[ \phi \to 0 , \quad \zeta \to 0 , \quad |R| \to \infty \quad (2.53) \]

Though the same notation is used for both nondimensional variables and physical variables, the difference between them should be noted.

Furthermore, we introduce another way of normalizing the physical variables, which will be used in the following numerical tests.

\[ (x, y, \zeta) = h (x^*, y^*, \zeta^*) , \quad S = h^2 S^* , \quad t = \sqrt{\frac{h}{g}} t^* \]
\[ \phi = h \sqrt{gh} \overline{\phi}^* , \quad P = \rho gh \overline{p}^* \quad (2.54) \]

Substituting into (2.50) to (2.53), we obtain
\[ \overline{\phi}^* \overline{t} + F_h \overline{\phi}^* x^* = -\zeta^* + \frac{F_h}{3} \nabla^2 \overline{\phi}^* x^* - \frac{1}{2} (\overline{\phi}^* x^* + \overline{\phi}^* y^*) - \overline{p}^* \quad (2.55) \]
\[ \zeta^* + F_h \zeta^* x^* = -\nabla^2 \overline{\phi}^* + F_h (2\overline{\phi}^* x^* \overline{\phi}^* x^* x^* + \overline{\phi}^* x^* \overline{\phi}^* y^* y^* + \overline{\phi}^* y^* \overline{\phi}^* y^* y^*) \quad (2.56) \]
\[ \overline{\phi}^* y^* = \frac{F_h}{2} S_x^* \quad \text{at } y^* = 0 \quad (2.57) \]
\[ \overline{\phi}^* \to 0 , \quad \zeta^* \to 0 , \quad |R^*| \to \infty \quad (2.58) \]

in which \( \nabla^2 = \frac{\partial^2}{\partial x^*} + \frac{\partial^2}{\partial y^*} \)

For two dimensional problems, the y-derivative vanishes and the corresponding
The above equations apply only for the nonlinear waves generated by a pressure distribution. For the waves induced by an advancing floating or submerging two dimensional object, nonflux boundary conditions should be satisfied on the exact body surface.
Chapter 3

Numerical Methods

In this chapter, both explicit and implicit finite difference schemes are described to implement the numerical investigation of the nonlinear long waves generated by the disturbance advancing at the transcritical speeds, in which the interaction between nonlinearity and dispersion is dominant. Several finite difference schemes were proposed by Taha & Ablowitz (1984) for the numerical solutions of the Korteweg-de Vries equation. Wu & Wu (1982) employed the modified Euler method to solve the two dimensional generalized Boussinesq equations. For the three dimensional case, Katsis & Akylas (1987) developed an explicit scheme for the governing equation which results from integrating the KP equation once with respect to $x$ from negative infinity to $x$. The same method was also applied by Choi & Mei (1989) to study a slender ship moving in restricted water. The drawback of the explicit scheme is that the time step $\Delta t$ is of the order of $\Delta x^3$, in which $\Delta x$ is the spatial step, which would result in rapid increase of computation time for the simulation with refined spatial meshes. Chen & Sharma (1995) developed a more efficient numerical technique using the fractional step algorithm with Crank-Nicolson-like schemes in each half step.

In this chapter, we design both explicit and implicit finite difference algorithms to solve the modified generalized Boussinesq equations formulated in the previous chapter.

$$\phi_t + F_h \phi_x = -\zeta + \frac{F_h}{3} \nabla^2 \phi_x - \frac{1}{2} (\phi_x^2 + \phi_y^2) - p(x, y) \quad (3.1)$$
\[
\begin{align*}
\zeta_t + F_h \zeta_x &= -\nabla^2 \phi + F_h (2\phi_x \phi_{xx} + \phi_x \phi_{yy} + \phi_y \phi_{xy}) \quad (3.2) \\
\phi_y &= \frac{F_h}{2} S_x \quad \text{at } y = 0 \quad (3.3) \\
\phi \to 0, \quad \zeta \to 0, \quad |R| \to \infty \quad (3.4)
\end{align*}
\]

We note that we will use the above dimensionless form of the mgB equations, normalized according to (2.54), from now on. The overbar and star superscript in the equations from (2.55) to (2.58) have been omitted for brevity.

### 3.1 Explicit Finite Difference Scheme

Define the notation
\[
\begin{align*}
\phi^n_{i,j} &= \phi(i\Delta x, j\Delta y, n\Delta t) \\
\zeta^n_{i,j} &= \zeta(i\Delta x, j\Delta y, n\Delta t)
\end{align*}
\]

Here \( \Delta x, \Delta y \) and \( \Delta t \) represent spatial and temporal increments. \( i, j \) are indices denoting the spatial grid points and \( n \) represents the time levels.

Variables at the next time step, \( \phi^{n+1}_{i,j} \) and \( \zeta^{n+1}_{i,j} \), can be obtained by a Taylor series expansion,
\[
\begin{align*}
\phi^{n+1}_{i,j} &= \phi^n_{i,j} + \Delta t (\phi_t)^n_{i,j} + \frac{1}{2} \Delta t^2 (\phi_{tt})^n_{i,j} + O(\Delta t^3) \quad (3.5) \\
\zeta^{n+1}_{i,j} &= \zeta^n_{i,j} + \Delta t (\zeta_t)^n_{i,j} + \frac{1}{2} \Delta t^2 (\zeta_{tt})^n_{i,j} + O(\Delta t^3) \quad (3.6)
\end{align*}
\]

The first time derivatives of \( \phi \) and \( \zeta \) can be obtained from the mgB equations (3.1) and (3.2) directly.
\[
\begin{align*}
\phi_t &= -\zeta - F_h \phi_x + \frac{F_h}{3} \nabla^2 \phi_x - \frac{1}{2} (\phi_x^2 + \phi_y^2) - p(x, y) \quad (3.7) \\
\zeta_t &= -F_h \zeta_x - \nabla^2 \phi + F_h (2\phi_x \phi_{xx} + \phi_x \phi_{yy} + \phi_y \phi_{xy}) \quad (3.8)
\end{align*}
\]

The second time derivatives of \( \phi \) and \( \zeta \) can be obtained by taking the derivative
of (3.7) and (3.8) once with respect to time.

\[
\phi_t = -\zeta_t - (F_h + \phi_x)\phi_{xt} - \phi_y\phi_{yt} + \frac{F_h}{3} \nabla^2 \phi_{xt} \tag{3.9}
\]

\[
\zeta_t = -F_h \zeta_{xt} - \nabla^2 \phi_t + F_h(2\phi_x\phi_{xt} + 2\phi_x \phi_{xxt} + \phi_y \phi_{xt} + \phi_x \phi_{ytyt} + \phi_{xy} \phi_{yt} + \phi_y \phi_{xyt}) \tag{3.10}
\]

The mixed temporal and spatial terms can be determined by taking spatial derivatives of (3.7) and (3.8)

\[
\zeta_{xt} = -F_h \zeta_{xx} - \nabla^2 \phi_x + F_h(2\phi_x^2 + 2\phi_x \phi_{xxx} + \phi_x \phi_{yy} + \phi_x \phi_{xy} + \phi_x^2 + \phi_y \phi_{xxy}) \tag{3.11}
\]

\[
\phi_{xt} = -\zeta_x - F_h \phi_{xx} + \frac{F_h}{3}(\phi_{xxxx} + \phi_{xxyy}) - \phi_x \phi_{xx} - \phi_y \phi_{xy} - p_x \tag{3.12}
\]

\[
\phi_{yt} = -\zeta_y - F_h \phi_{yy} + \frac{F_h}{3}(\phi_{xxxx} + \phi_{yxyy}) - \phi_x \phi_{xy} - \phi_y \phi_{yy} - p_y \tag{3.13}
\]

\[
\phi_{xxt} = -\zeta_{xx} - F_h \phi_{xxx} + \frac{F_h}{3}(\phi_{xxxxxx} + \phi_{xxxxyy}) - \phi_x^2 - \phi_x \phi_{xxx} - \phi_x \phi_{xy} - \phi_y \phi_{xxy} - p_{xx} \tag{3.14}
\]

\[
\phi_{yyt} = -\zeta_{yy} - F_h \phi_{xyy} + \frac{F_h}{3}(\phi_{xxxxxx} + \phi_{yyyy}) - \phi_x^2 - \phi_x \phi_{xyy} - \phi_y \phi_{xyy} - p_{yy} \tag{3.15}
\]

\[
\phi_{xyt} = -\zeta_{xy} - F_h \phi_{xyy} + \frac{F_h}{3}(\phi_{xxxxxx} + \phi_{xyyy}) - \phi_x \phi_{xy} - \phi_x \phi_{xyy} - \phi_y \phi_{xyy} - p_{xy} \tag{3.16}
\]

\[
\phi_{xxxt} = -\zeta_{xxx} - F_h \phi_{xxx} + \frac{F_h}{3}(\phi_{xxxxxx} + \phi_{xxxxyy}) - 3\phi_x \phi_{xxx} - \phi_x \phi_{xxx} - 3\phi_y \phi_{xyy} - \phi_y \phi_{xxx} - p_{xxx} \tag{3.17}
\]

\[
\phi_{xyyt} = -\zeta_{xyy} - F_h \phi_{xyy} + \frac{F_h}{3}(\phi_{xxxxxx} + \phi_{xyyy}) - \phi_x \phi_{xy} - \phi_x \phi_{xyy} - 2\phi_y \phi_{xyy} - \phi_y \phi_{yy} - p_{xy} \tag{3.18}
\]

All derivatives are discretized by centered differences as follows,

\[
f_{\xi}^{(1)} = \frac{f_{j+1} - f_{j-1}}{2\Delta \xi}
\]
\[ f^{(2)}_\xi = \frac{f_{j+1} - 2f_{j} + f_{j-1}}{\Delta \xi^2} \]
\[ f^{(3)}_\xi = \frac{f_{j+2} - 2f_{j+1} + 2f_{j-1} - f_{j-2}}{2\Delta \xi^3} \]
\[ f^{(4)}_\xi = \frac{f_{j+2} - 4f_{j+1} + 6f_{j} - 4f_{j-1} + f_{j-2}}{\Delta \xi^4} \]
\[ f^{(5)}_\xi = \frac{f_{j+3} - 4f_{j+2} + 5f_{j+1} - 5f_{j-1} + 4f_{j-2} - f_{j-3}}{2\Delta \xi^5} \]
\[ f^{(6)}_\xi = \frac{f_{j+3} - 6f_{j+2} + 15f_{j+1} - 20f_{j} + 15f_{j-1} - 6f_{j-2} + f_{j-3}}{\Delta \xi^6} \]

in which \( f \) denotes \( \phi \) or \( \zeta \) and \( \xi \) denotes the spatial variable \( x \) or \( y \).

The truncation error of the above explicit Lax-Wendroff scheme is \( O(\Delta x^2, \Delta t^2, \Delta t^3) \).

Assuming that the nonlinear terms don't essentially affect the stable properties of the finite difference equations of (3.7) and (3.8), we investigate the corresponding linear dispersive equations,

\[ \phi_t + F_h \phi_x = -\zeta + \frac{F_h}{3} \nabla^2 \phi_x \]  
(3.20)
\[ \zeta_t + F_h \zeta_x = -\nabla^2 \phi \]  
(3.21)

The stability condition of the linear problem can be analyzed by the amplification matrix method, which leads to the following stability condition

\[ \frac{\Delta t}{\Delta x^3} \leq \frac{3}{4F_h} \cdot \frac{1}{1 + (\frac{\Delta \xi}{\Delta y})^2} \]  
(3.22)

For the two dimensional case of the linear dispersive problem, we can set \( \Delta y \to \infty \) in (3.22) and the corresponding stability condition becomes

\[ \frac{\Delta t}{\Delta x^3} \leq \frac{3}{4F_h} \]  
(3.23)

### 3.2 Implicit Finite Difference Scheme

The unknown \( \phi \) and \( \zeta \) of the grid \((i\Delta x, j\Delta y)\) in the computational domain at \((n+1)\)th time level satisfy the governing equation (3.1) and (3.2). The discretized finite
difference equations can be written in the following form.

\[
(\phi_t)_{i,j}^{n+1} + F_h(\phi_x)_{i,j}^{n+1} - \frac{F_h}{3}(\phi_{xxx})_{i,j}^{n+1} = -\zeta_{i,j}^{n+1} + \frac{F_h}{3}(\phi_{yy})_{i,j}^{n+1} \\
\quad - \frac{1}{2}(\phi_x^2 + \phi_y^2)_{i,j}^{n+1} - p_{i,j}
\]  

(3.24)

\[
(\zeta_t)_{i,j}^{n+1} + F_h(\zeta_x)_{i,j}^{n+1} = -(\phi_{xx} + \phi_{yy})_{i,j}^{n+1} \\
\quad + F_h(2\phi_x\phi_{xx} + \phi_x\phi_{yy} + \phi_y\phi_{xy})_{i,j}^{n+1}
\]  

(3.25)

The three time-level scheme is used to approximate the time derivative

\[
(\phi_t)_{i,j}^{n+1} = \frac{3\phi_{i,j}^{n+1} - 4\phi_{i,j}^{n} + \phi_{i,j}^{n-1}}{2\Delta t}
\]  

(3.26)

\[
(\zeta_t)_{i,j}^{n+1} = \frac{3\zeta_{i,j}^{n+1} - 4\zeta_{i,j}^{n} + \zeta_{i,j}^{n-1}}{2\Delta t}
\]  

(3.27)

A centered scheme is applied to approximate all spatial derivatives at the inner nodes of computational mesh. The solution of the nonlinear equations (3.24) and (3.25) can be obtained iteratively. The initial value for the variables at the next time step are taken as the value at the n-th step,

\[
\phi_{i,j}^{n+1,0} = \phi_{i,j}^{n} \\
\zeta_{i,j}^{n+1,0} = \zeta_{i,j}^{n}
\]

in which the second superscript indicates the index of iteration. At each iterative step, the linear terms with the x derivative are approximated implicitly at the left side of equations and all other terms with nonlinearity and cross derivatives are put on the right side as the known for the next iteration. The iteration is done line by line on the nodes with same transverse superscript \( j \), from the symmetric plane \( (j = 0) \) to the truncated outer boundary \( (j = j_{max}) \). All variables take the latest value from the iteration. A sub-relaxation scheme is used to update the variables after each iteration.
At each iterative step \( k \),

\[
\begin{align*}
3\tilde{\zeta}_{i,j}^{n+1,k} - 4\zeta_{i,j}^n + \zeta_{i,j}^{n-1} &+ \frac{F_h}{2\Delta x} (\zeta_{i+1,j}^{n+1,k} - \zeta_{i-1,j}^{n+1,k}) \\
\frac{3\phi_{i,j}^{n+1,k} - 4\phi_{i,j}^n + \phi_{i,j}^{n-1}}{2\Delta t} + \frac{F_h}{2\Delta x} (\phi_{i+1,j}^{n+1,k} - \phi_{i-1,j}^{n+1,k}) &+ \frac{3\tilde{\phi}_{i+1,j}^{n+1,k} - 4\tilde{\phi}_{i,j}^n + \tilde{\phi}_{i,j}^{n-1}}{6\Delta t} + \frac{F_h}{2\Delta x} (\tilde{\phi}_{i+2,j}^{n+1,k} - \tilde{\phi}_{i-2,j}^{n+1,k}) \\
&= -\left(\phi_{xx} + \phi_{yy}\right)_{i,j}^{n+1,k'} + \frac{F_h}{2\Delta x} (2\phi_{xx} + \phi_{yy} + \phi_y\phi_{xy})_{i,j}^{n+1,k'}
\end{align*}
\]  

(3.28)

\[
\begin{align*}
\frac{3\tilde{\phi}_{i+1,j}^{n+1,k} - 4\tilde{\phi}_{i,j}^n + \tilde{\phi}_{i,j}^{n-1}}{6\Delta t} + \frac{F_h}{2\Delta x} (\phi_{xx} + \phi_{yy})_{i,j}^{n+1,k'} &- \frac{1}{2} (\phi_x^2 + \phi_y^2)_{i,j}^{n+1,k'} - p_{i,j}
\end{align*}
\]  

(3.29)

in which \( \tilde{\zeta}_{i,j}^{n+1,k} \) and \( \tilde{\phi}_{i,j}^{n+1,k} \) are the prediction values for the \( \zeta_{i,j} \) and \( \phi_{i,j} \) after the \( k \)-th iteration. The superscript \( k' \) is used in the terms of the right-hand side since both the values at \((k-1)\)th and \(k\)th iterations are used to evaluate the derivatives.

The \( x \) derivatives on the right sides of (3.28) and (3.29) are obtained by the \( k \)-th iterative value

\[
\begin{align*}
(f_x)_{i,j}^{n+1,k'} &= \frac{f_{i+1,j}^{n+1,k} - f_{i-1,j}^{n+1,k}}{2\Delta x} \\
(f_{xx})_{i,j}^{n+1,k'} &= \frac{f_{i+1,j}^{n+1,k} - 2f_{i,j}^{n+1,k} + f_{i-1,j}^{n+1,k}}{\Delta x^2}
\end{align*}
\]

in which \( f \) denotes the velocity potential \( \phi \).

The transverse derivatives are evaluated by

\[
\begin{align*}
(f_y)_{i,j}^{n+1,k'} &= \frac{f_{i,j+1}^{n+1,k} - f_{i,j-1}^{n+1,k}}{2\Delta y} \\
(f_{yy})_{i,j}^{n+1,k'} &= \frac{f_{i,j+1}^{n+1,k} - 2f_{i,j}^{n+1,k} + f_{i,j-1}^{n+1,k}}{\Delta y^2}
\end{align*}
\]

in which \( f \) denotes the velocity potential \( \phi \) or its first \( x \) derivative \( \phi_x \).

The \( k \)-th iterative values of \( \phi \) and \( \zeta \) are obtained by

\[
\begin{align*}
\phi_{i,j}^{n+1,k} &= \phi_{i,j}^n + \omega_\phi (\tilde{\phi}_{i,j}^{n+1,k} - \phi_{i,j}^n) \\
\zeta_{i,j}^{n+1,k} &= \zeta_{i,j}^n + \omega_\zeta (\tilde{\zeta}_{i,j}^{n+1,k} - \zeta_{i,j}^n)
\end{align*}
\]  

(3.30)

(3.31)
in which $\omega_\phi$ and $\omega_\zeta$ are the relaxative factors for $\phi$ and $\zeta$ respectively. The relaxative factor is related to the eigenvalue of the set of linear equations. A rigorous analysis of the eigenvalue of the discretized set of linear equations is almost impossible because of the complexity of the system. According to numerical experience, the under-relaxative factors $\omega_\phi$ and $\omega_\zeta$ take the values in the range of $(0.2, 1.0)$, which are the function of the spatial grid size and time step. Fine computational grids and a small time step ask for smaller relaxative factors than a numerical test with coarser grids. From the numerical tests, it is found that there exist critical relaxative factors, beyond which the iteration cannot converge and below which the number of iterations needed to reach convergence increases as the relaxative factors decrease. These optimal relaxative factors $\omega_{\phi, \text{opt}}$ and $\omega_{\zeta, \text{opt}}$ can be obtained by running the computer code for a few time steps and one can select the best values for the simulation of the free surface flow for long time.

The convergent criterion for the iterative solution of (3.24) and (3.25) at each time step is

$$\max(|\phi_{i,j}^{n+1,k} - \phi_{i,j}^{n+1,k-1}|) \leq \epsilon_\phi$$

$$\max(|\zeta_{i,j}^{n+1,k} - \zeta_{i,j}^{n+1,k-1}|) \leq \epsilon_\zeta \quad \text{for all } i, j$$

in which $\epsilon_\phi$ and $\epsilon_\zeta$ are the maximum permitted error in each time step for velocity potential and free surface elevation, respectively.

The implicit scheme is unconditionally stable and is second order for temporal and spatial accuracy, i.e. $O(\Delta t^2, \Delta x^2, \Delta y^2)$.

### 3.3 Open Boundary Condition

We can extend the computational domain wide enough upstream to ensure the validity of the no disturbance condition

$$\phi = 0 \quad \zeta = 0 \quad x \to -\infty$$  \hspace{1cm} (3.32)
The speed of wave propagation downstream is much higher than that of wave propagating upstream. Thus it isn’t suitable to extend the downstream domain far enough downstream to satisfy the no disturbance boundary condition. Suitable open boundary conditions are needed to act on the truncated boundaries and make the wave propagate outward with minimum reflection. It was shown that the reflective effect of side walls is the origin of the two dimensional straight-crest upstream solitons in the channels of finite depth (Katsis & Akylas, 1987, Pedersen, 1988). Therefore, the open boundary condition is crucial in order to capture the phenomenon of three dimensional upstream solitons. A Sommerfield type radiation condition is imposed on the downstream boundary and transverse truncated boundary.

\[
\begin{align*}
\frac{\partial f}{\partial t} + C_f \frac{\partial f}{\partial x} &= 0 \quad \text{at downstream boundary} \quad (3.33) \\
\frac{\partial f}{\partial t} + C_f \frac{\partial f}{\partial y} &= 0 \quad \text{at transverse boundary} \quad (3.34)
\end{align*}
\]

in which \( f \) denotes the velocity potential \( \phi \) and free surface elevation \( \zeta \) respectively. \( C_f \) is the local phase velocity for \( f \), which is a function of position and time and can be only determined numerically in general.

An Orlanski scheme (Orlanski, 1976) is applied to obtain the next step value, \( \phi^{n+1} \) and \( \zeta^{n+1} \) at boundary nodes. Assume that the normal vector of wave propagation is \( \vec{l} \) and thus the radiation condition is written as

\[
\frac{\partial f}{\partial t} + C_f \frac{\partial f}{\partial l} = 0 \quad (3.35)
\]

The above is discretized using the Leapfog scheme,

\[
\frac{f_i^n - f_i^{n-2}}{2\Delta t} = -C_f \frac{f_i^{n} + f_i^{n-2} - f_i^{n-1}}{\Delta l} \quad (3.36)
\]

The phase velocity of variable \( f \) can be determined by the inner node \((i - 1)\Delta l\) at the \( n \)-th time level

\[
C_f = \frac{-\frac{f_i^{n-1} - f_i^{n-2}}{\Delta l} \Delta l}{\frac{f_i^{n-1} + f_i^{n-2}}{2\Delta t} - f_i^{n-1} \Delta t} \quad (3.37)
\]
The value of $C_f$ is calculated numerically and we have to impose two additional constraints on it. The phase velocity is set to zero if the numerical result from (3.37) is negative. $C_f$ also cannot exceed the characteristic speed of the grid, i.e. $C^* = \frac{\Delta t}{\Delta t}$ and will be set to $C^*$ if it’s larger than the critical speed.

From (3.36) we can get the value for $f$ at the next time step $(n + 1)\Delta t$,

$$f_{n+1} = \frac{1 - C_f \Delta t}{1 + C_f \Delta t} f_n - 1 + \frac{2 C_f \Delta t}{1 + C_f \Delta t} f_{n-1}$$

(3.38)

The above algorithm can be implemented by substituting a normal vector $l$ with $x$ for the downstream boundary and $y$ for the transverse boundary. Both $\phi$ and $\zeta$ will substitute the general variable $f$ from (3.36) to (3.38).

### 3.4 Data Smoothing

With the progress of the calculation, error accumulation is unavoidable in the computational domain. The noise consists of waves with short wave length and its amplification will grow and eventually contaminate the computational domain. Two spatial filters are introduced to smooth the free surface profile (Longuet-Higgins & Cokelet, 1976).

**Five-point filter:**

$$\bar{f}_j = \frac{1}{16} (-f_{j-2} + 4f_{j-1} + 10f_j + 4f_{j+1} - f_{j+2})$$

(3.39)

**Seven-point filter:**

$$\bar{f}_j = \frac{1}{32} (-f_{j-3} + 9f_{j-1} + 16f_j + 9f_{j+1} - f_{j+3})$$

(3.40)

The spatial filtering is applied each 40-100 time steps to reduce its side smoothing effects.
Chapter 4

Nonlinear Long Waves Generated by a Moving Pressure Distribution

In this chapter, the mathematical model and numerical schemes described in the previous chapters are applied to simulate the nonlinear long waves induced by a pressure distribution traveling in the vicinity of the critical speed in shallow water.

In the first section, the explicit and implicit finite difference schemes are studied for the wave problem in a finite channel and the latter one is selected for its computational advantage. The convergence test is carried out in Section 2 and the dependence of the wave elevation on the grid size and time step is investigated. The two dimensional problem is described briefly in Section 3. Since most previous research was focused on the upstream solitons in narrow channel, we consider the case of the restricted channel first in Section 4. A series of numerical calculations are carried out to investigate the mechanism of the generation of two dimensional upstream solitons in channels with different width, in a range of $W = 20h$ to $W = 240h$. In the last section, the three dimensional solitary waves generated by a pressure distribution at sub-critical, critical and super-critical speed are simulated. The properties of the three dimensional solitons are discussed in detail. One interesting discovery is that the crestline of the leading soliton is nearly a perfect parabola. The free three dimensional soliton is also studied in Section 5.
4.1 Explicit and Implicit Finite Difference Schemes

To study the performance of the explicit and implicit finite difference schemes presented in chapter 3, we consider the case of nonlinear long waves generated by a sinusoidal pressure distribution at the critical speed \( F_h = 1.0 \) in a shallow channel. The geometrical parameters are

\[
W = 10.0, \quad B = 2.0, \quad \frac{L}{B} = 2.5
\]

in which \( W, L \) and \( B \) are the nondimensional channel width, length and width of the pressure distribution normalized by the water depth, respectively.

The pressure acts inside a rectangle symmetric about the x- and y-axes with the length \( L \) and width \( B \) and vanishes outside the rectangle. The pressure distribution is defined as

\[
p(x, y) = P \cos^2\left(\frac{\pi x}{L}\right) \cos^2\left(\frac{\pi y}{B}\right) \quad \frac{L}{2} < x < \frac{L}{2}, \quad \frac{B}{2} < y < \frac{B}{2}
\]

in which \( P \) is the peak value of the pressure distribution. Due to symmetry, computation is only carried out in the positive half fluid domain with \( y \geq 0 \). The computational mesh is orthogonal and the longitudinal and transverse grid sizes are \( \Delta x = 0.25 \) and \( \Delta y = 0.25 \), respectively. Both the explicit and implicit schemes are tested and the numerical results are shown in Figure 4-1. The time step is taken as \( \Delta t = 0.20 \) in the implicit scheme and \( \Delta t = 0.001, 0.002 \) and \( 0.004 \) respectively in the explicit scheme. As shown in Figure 4-1, two dimensional solitons are generated in front of the pressure distribution and propagate upstream. The results from the explicit method tend to converge to those from the implicit method when the time step decreases. The amplitude and phase velocity of the upstream-running solitons increase when the time step increases.

The implicit scheme is much more efficient than the explicit one. In the Lax-Wendroff type explicit scheme, over thirty temporal and spatial derivatives need to be calculated at each time step, however the implicit one only calculates six spatial
derivatives. In this numerical test, the ratio of the time step for the two schemes are 200 to achieve a convergent solution. In order to obtain satisfactory results, more than 200 times the computational effort can be saved with the application of the implicit finite difference method. The following numerical results are all computed with the implicit finite difference method described in session 3.2.
Figure 4-1: Wave profile due to a pressure with the explicit and implicit finite difference methods. \( F_h = 1.0, W = 20.0, \tilde{P} = 0.5, \Delta x = \Delta y = 0.25, t = 100. \) Top: \( 2y/W = 0.0; \) Middle: \( 2y/W = 0.5; \) Bottom: \( 2y/W = 1.0 \)
4.2 Convergence Test

Before implementing the time-consuming computation of the three dimensional solitary waves, we first perform systematic numerical tests to verify the convergence and accuracy of the present implicit method. For simplicity, we take the solitary wave in a shallow channel of finite width as the test problem. The geometrical parameters are chosen to be the same as those in section 4.1.

First we keep the grid size $\Delta x$ and $\Delta y$ fixed and let the time step change from $\Delta t = 0.05$ to $\Delta t = 0.10, 0.20$. In Figure 4-2, the wave cut along the plane $y = 0$ and $y = \frac{W}{4}$ converge very well. The wave amplitude of upstream soliton converges up to the fourth significant figure for these three time steps. We can conclude that the influence of the time step on the accuracy is small if we adopt a reasonably small time step which satisfies the stability condition.

Figure 4-3 shows the results for the two longitudinal wave cuts for different $\Delta x$, keeping $\Delta t = 0.20$ and $\Delta y = 0.25$ fixed. The longitudinal grid size varies from $\Delta x = 0.25$ to $\Delta x = 0.32, 0.40$. From the figure, we can find that the numerical dispersion effect is obvious and influences the numerical result significantly, especially in the downstream region at a large distance from the disturbance. The reason is that the dominant factor on the wave propagation far downstream is the dispersion, which is balanced by the nonlinearity near the disturbance and in the upstream region. The present research emphasizes the upstream nonlinear long waves and we should note that the assumptions, on which the mathematical model is based, are applicable mainly to the upstream waves. It’s still not clear if this model can also apply on the accurate prediction of the downstream wave system. Small grid size about $\Delta x = 0.25$ is used in the subsequent computations to anticipate little numerical dispersion ‘contamination’ on the downstream waves.

The convergence test with respect to the transverse grid size $\Delta y$, which is in the range of 0.25 to 0.50, is shown in Figure 4-4. The transverse grid size doesn’t play an important role on the prediction of the upstream soliton and introduces a smaller numerical dispersive effect compared with that of the longitudinal grid size.
\( \Delta x \). Though only nine nodes are used to interpolate the pressure distribution in the \( y \) direction, the numerical result shows that this grid size can predict the solitons with enough accuracy. For the three dimensional waves in unbounded sea, the width of the computational domain should be more than 200. The convergent property of \( \Delta y \) makes it possible to use an economic computational scheme with a large transverse grid.

Taking both the accuracy and efficiency into account, we use \( \Delta t = 0.10 - 0.20 \), \( \Delta x = 0.20 - 0.25 \) and \( \Delta z = 0.40 - 0.50 \) in all subsequent computations in both the restricted and unbounded domain.
Figure 4-2: Wave profile due to a pressure with different time step. $F_h = 1.0$, $W = 20.0$, $\bar{P} = 0.5$, $\Delta x = \Delta y = 0.25$, $t = 100$. Top: Perspective view; Middle: $2y/W = 0.0$; Bottom: $2y/W = 0.5$.
Figure 4-3: Wave profile due to a pressure with different longitudinal grid sizes. $F_h = 1.0$, $W = 20.0$, $P = 0.5$, $\Delta y = 0.25$, $\Delta t = 0.20$, $t = 100$. Top: Perspective view; Middle: $2y/W = 0.0$; Bottom: $2y/W = 0.5$
Figure 4-4: Wave profile due to a pressure with different transverse grid sizes. $F_h = 1.0$, $W = 20.0$, $P = 0.5$, $\Delta x = 0.25$, $\Delta t = 0.20$, $t = 100$. Top: Perspective view; Middle: $2y/W = 0.0$; Bottom: $2y/W = 0.5$
4.3 Two Dimensional Solitary Waves

In the two dimensional case, the governing equations are described by (2.59) and (2.60). The numerical computation is much easier to implement, however it can provide the basic understanding of such nonlinear long waves. We take the maximum pressure $\bar{P} = 0.05$, the grid size $\Delta x = 0.1$ and time step $\Delta t = 0.02$. Figure 4-5 shows the free surface elevation at $t = 100$ induced by a moving two dimensional pressure patch at the sub-critical($F_h = 0.90$), critical($F_h = 1.0$) and super-critical($F_h = 1.1$) speed. A succession of solitary waves are radiated ahead of the pressure disturbance periodically. A train of dispersive waves is formed downstream behind a depressed water surface which is elongated with respect to time.

At sub-critical speeds, $F_h = 0.9$, the upstream soliton series have a mean water level above the still free surface. The leading soliton is followed by a succession of solitons with decreasing wave amplitude. The wave amplitude of the downstream dispersive wave decreases as the distance from the disturbance increases. When the depth Froude number increases, the wave amplitude of solitons also increases and reaches a significant value about $0.6h$ in the super-critical case $F_h = 1.1$. However, the amplitude of the downstream wave train become smaller. From the figure we can observe that the wave amplitude of solitons tends to be uniform when $F_h$ exceeds unity. The period of emitting upstream solitons decreases as the speed increases. At $t = 100$, there have been six solitons for $F_h = 0.9$ and only four solitons have been formed at $F_h = 1.1$. The solitons have a larger wave slope at higher Froude numbers. At $F_h = 1.1$, the four upstream solitons are separated from each other and can be treated as free solitons.

As the figure shows, the expansion rate of the depressed water surface increases as the increment of the Froude number. The troughs of the upstream solitons are all non-negative and thus there exists an amplifying net mass in the upstream region. The mean level of the downstream waves is on the still free surface and there is no mass transfer downstream of the depressed water region. The upstream mass comes from the area with water surface depression. As the Froude number increases, the mean
depth of the depressed water area decreases and its length increases correspondingly.
Figure 4-5: Two dimensional solitons generated by a pressure distribution. $F_h = 1.0$, $L = 10.0$, $P = 0.05$, $\Delta x = 0.10$, $\Delta t = 0.02$, $t = 100$. (a): $F_h = 0.90$; (b): $F_h = 1.00$; (c): $F_h = 1.10$
4.4 Solitary Waves in Restricted Channels

The nonlinear long waves forced by an impulsively started three dimensional pressure distribution traveling at the critical speed in a shallow channel of different width are studied. The pressure is distributed inside a rectangle, whose center coincides with the origin, and is symmetric about the centerplane of the channel $y = 0$. Due to symmetry, the computational domain is chosen to be $x_{\text{upstream}} \leq x \leq x_{\text{downstream}}$, $0 \leq y \leq W/2$. $x_{\text{upstream}}$ is the position where we truncate the computational domain upstream. No radiation condition except the zero disturbance condition is imposed on the upstream boundary and the computation has to be stopped before the upstream-running wave reaches the boundary. An Orlanski type numerical radiation condition is used at $x_{\text{downstream}}$ to make the weakly nonlinear and dispersive waves propagate outward smoothly. Figures 4-6, 4-7 and 4-8 show the wave evolution of solitons in the channel with width (1) $W/2 = 10.0$; (2) $W/2 = 40.0$ and (3) $W/2 = 120.0$. For case (1) and (2) $x_{\text{upstream}}$ is $-50$ and for (3) is $-120$. $x_{\text{downstream}}$ is set to be 100 for all three tests. The pressure magnitude is $\bar{P} = 0.30$.

In the narrow channel of $W/2 = 10.0$, the calculation shown in Figure 4-6 illustrates the similar properties of upstream solitons as the numerical results reported by Ertekin et al(1986) and Katsis & Akylas(1987). Each soliton originally emerges as a three dimensional disturbance and adjusts itself after it hits the sidewall and straightens the crest to become a two dimensional spanwise one. The downstream wave system is much more complex and we can see that they are formed by the multiple interactions between the divergent wave following the disturbance and its reflection from the sidewall. The first oblique divergent wave originating from the trailing edge plays a role as the breakwater between upstream solitons and the downstream divergent wave system. An elongating depressed water surface is generated between this wave and the downstream corrugated wave systems.

Figure 4-7 illustrates the soliton generation in a wide channel. Initially a ring-like free surface elevation develops near the pressure distribution, which is washed downstream due to the incident uniform current. Because of nonlinearity and disper-
sion, the upstream half ring of water is higher than the undisturbed free surface and wrinkles emerge continuously behind the pressure patch with similar curvature as the upstream wave. The wave amplitude along the hump decreases with respect to the increase of the $y$ coordinate. When the wave is reflected by the side wall, the local wave height near the wall increases and the wave adjusts itself spanwise. For a two dimensional solitary wave, the phase velocity is related to its amplitude by,

$$C_{sm} = (1 + A_s)^{\frac{1}{2}} - 1 \approx \frac{A_s}{2}$$

in which $A_s$ is the dimensionless wave amplitude of the soliton and $C_{sm}$ is its phase velocity observed in the frame moving with the critical speed. Therefore, the phase velocity near the wall is larger than that of the wave away from the wall. In the transverse direction, the wave behaves as a diffusion which make the near-wall wave tend to lower itself and make the wave amplitude distribute uniformly along the crestline. With this self-adjusting mechanism, a straight-crest soliton of constant amplitude is eventually formed spanning the whole channel width. In Figure 4-7(d)-(f), it is obvious that the downstream wave system is generated by the interaction among divergent waves and their reflective waves from the sidewall. The crestline of the trailing wave near the symmetric plane is in the spanwise direction and a transverse wave train is formed in a wedge region behind the pressure disturbance.

The soliton generation in a very wide channel, $W/2 = 120.0$, is shown in Figure 4-8. The wave doesn’t feel the existence of the sidewall when $t < 300$. In Figure 4-8(a) a group of divergent waves are formed behind the pressure distribution with decreasing amplitude and a transverse wave train develops in the wedge region enclosed by the divergent waves. In Figure 4-8(b), the leading upstream solitary wave escapes from the disturbance and the second one is being formed. The interaction between the reflective wave of upstream soliton and downstream wave systems is clearly seen. In Figure 4-8(c)-(e), the process of the generation of the leading soliton is depicted and also the interaction between the upstream three dimensional solitons is shown. An interesting phenomenon is that the wave length of the trailing transverse wave
increases as the time passes by. An elongating depressed water surface is produced between the first and second crest of the transverse wave train. In Figure 4-8(f) a two dimensional soliton has been formed spanning the very wide channel. The wavelength of the upstream solitons is large and the leading soliton propagates freely from its successors.

With the increase of the channel width, the amplitude of the upstream-running solitons decreases and the wave length of solitons and the period of their generation increase.
Figure 4-6 (a) - (c)
Figure 4-6: Wave evolution in a shallow channel. $W/2 = 10.0$, $F_h = 1.0$, (a): $t = 5$; (b): $t = 10$; (c): $t = 25$; (d): $t = 50$; (e): $t = 125$; (f): $t = 250$
Figure 4-7 (a) - (c)
Figure 4-7: Wave evolution in a wide channel. $W/2 = 40.0$, $F_h = 1.0$, (a): $t = 20$; (b): $t = 40$; (c): $t = 80$; (d): $t = 200$; (e): $t = 400$; (f): $t = 600$
Figure 4-8 (c) - (d)
Figure 4-8: Wave evolution in a very wide channel. $W/2 = 120.0$, $F_h = 1.0$, (a): $t = 80$; (b): $t = 200$; (c): $t = 400$; (d): $t = 800$; (e): $t = 1200$; (f): $t = 1600$
4.5 Three Dimensional Solitary Waves in an Unbounded Domain

In this section, we present numerical results for the generation and evolution of forced and free three dimensional solitary waves in a horizontally unbounded domain. The pressure distribution takes the form in equation (4.1) and we choose the nondimensional size of the rectangle as follows:

\[ L = 10.0, \quad B = 4.0 \]

The pressure is assumed to be forced impulsively at the initial time when the velocity and free surface elevation are all zero. The computational domain is \( x_{\text{upstream}} = -60.0, x_{\text{downstream}} = 100.0 \) and the transverse boundary is truncated at \( y_{\text{span}} = 240.0 \). Oranski open boundary conditions are applied on the downstream boundary and the transverse boundary to make the outgoing wave propagate outside the computational domain without reflection.

4.5.1 Three Dimensional Solitary Waves at Critical Speed

Figure 4-9 presents perspective views of the wave patterns generated by a pressure distribution traveling at a critical speed with magnitude \( \bar{P} = 0.30 \) at different stages of the development of the waves. Compared with the waves in a very wide channel shown in Figure 4-8, there is no reflection from the side boundary and the crestlines of upstream waves are always curved with the curvature decreasing as it move forward. At \( t = 1200 \), four upstream solitary waves have been radiated from the disturbance and a new one is developing. The wave peak occurs ahead of the front edge of the pressure distribution and a nearly steady wave hump origins there and extends downstream obliquely, behaving like a barrier between the upstream solitary wave region and the downstream divergent wave region. The wave reaches its minimum beneath the pressure distribution. Both transverse and divergent wave systems develop behind the disturbance and the interaction between these two wave systems is clear.
The transverse wave packet has a mean downstream group velocity and a depressed water surface, analogous to that in two dimensional case, forms behind the trailing edge of the pressure patch.

Figure 4-10 shows the wave profile on the symmetric plane $y = 0$ induced by pressure distribution with different magnitudes $\bar{P} = 0.20, 0.30, 0.40, 0.50$. It is clear that a succession of solitary waves radiate upstream, the wave amplitude decreases when it moves forward. The mean level behind the pressure distribution is lower than the still free surface and connects to a transverse wave train propagating downstream. There exists significant interaction, including the transfer of mass, momentum and energy between neighboring solitons, which can be proved by the study of free solitons in the next section, which shows a different type of wave development without forcing. The magnitude of the pressure plays an important role for the wave amplitude, phase velocity and the period of soliton generation. With a higher forcing pressure, the wave amplitudes and velocities of both upstream and downstream waves increase.

For two dimensional case, the nondimensional phase velocity of the soliton is half of its amplitude from equation (4.2), which indicates that the displacement of the wave crest is

$$\Delta x_c = \frac{1}{2} A_c \Delta t$$  \hspace{1cm} (4.3)

in which $x_c$ is the position of the soliton crest and $A_c$ is its amplitude. We define the maximum value of the three dimensional solitary wave at the symmetric plane as its amplitude and shows the correlation between the wave amplitude and crest position in Figure 4-11. For different pressures, the relation between $x_c$ and $A_c t$ is linear and can be approximated by the following formula

$$x_c + \frac{2}{3} A_c t = C(\bar{P})$$  \hspace{1cm} (4.4)

in which $C(\bar{P})$ is the constant varying with the pressure distribution.

The amplitudes of the leading solitons at different times are drawn in Figure 4-12.
<table>
<thead>
<tr>
<th>$\bar{P}$</th>
<th>$A_1$</th>
<th>$t_0$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>0.5350</td>
<td>53.86</td>
<td>-0.3407</td>
</tr>
<tr>
<td>0.30</td>
<td>0.5981</td>
<td>43.07</td>
<td>-0.3303</td>
</tr>
<tr>
<td>0.40</td>
<td>0.6968</td>
<td>36.36</td>
<td>-0.3345</td>
</tr>
<tr>
<td>0.50</td>
<td>0.7696</td>
<td>30.87</td>
<td>-0.3346</td>
</tr>
</tbody>
</table>

Table 4.1: Approximation for the wave amplitude of the leading soliton generated by a pressure distribution, $F_h = 1.0$

and are approximated by the formula

$$A_c(t) = A_1(t - t_0)^{\gamma}$$  \hspace{1cm} (4.5)

in which $A_1$ and $t_0$ are constants related to the initial wave amplitude and the escaping time of the soliton from the pressure disturbance. $\gamma$ is the decay rate of the leading soliton. Table 4.1 shows the result of the approximation and shows that the wave amplitude of the leading soliton decays asymptotically in the order of $O(t^{-\frac{1}{2}})$ when $t$ is large enough.

Figure 4-13 shows the evolution of the crestline of the leading soliton generated by the pressure distribution of different magnitudes. The first crestline is recorded at $t = 200$, in which the soliton has detached from the forcing, and the time interval among the crestlines is $\Delta t = 80$. As shown, the curvature of the isophaseal crestline decreases as the soliton moves ahead. The crestlines are drawn in a log-log scale in Figure 4-14(a) to find its geometry property. Surprisingly, it predicts a straight line with almost the same slope at different time. The crestline of the leading soliton is approximated by the following formula

$$y = \alpha_1(t)(x - x_c)^{\beta(t)}$$  \hspace{1cm} (4.6)

The approximating indices $\beta(t)$ at different times are plotted in Figure 4-14(b). It is nearly 0.50 and varies very slowly with respect to time, independent on the forcing pressure magnitude and the wave amplitude. This shows that the crestline is nearly
a parabola and can be approximated by

\[ y^2 = \alpha(t)(x - x_c) \]  

(4.7)

in which \( \alpha(t) \) is the length of the *latus rectum* (the chord perpendicular to the symmetric axis and passes through the focus of a parabola). Its dependence on the time is shown in Figure 4-14(c) and a linear correlation is found. The slope of the \( \alpha(t) \) is about 1.40 and independent on the magnitude of the forcing pressure.

Based on the analysis shown above, we may conclude that, when the time is large enough, the crestline of the leading soliton forced in open sea takes the asymptotical form

\[ x + A_0 t^\frac{3}{2} = B_0 \frac{y^2}{t} \]  

(4.8)

in which \( A_0 \) and \( B_0 \) are constants determined by the initial value of the waves.

The wave amplitude along the crestline of the leading wave is shown in Figure 4-15 with the time interval \( \Delta t = 80 \). The wave is steeper at its wave front in the \( y \) direction, which is the consequence of the impulsive start of the forcing pressure. The evolution of the wave amplitude in the transverse direction is analogous to the one dimensional heat transfer in the infinite conduct with high temperature concentrated near \( y = 0 \) at initial time. We can see that the diffusion is the dominant factor in the transverse direction for the three dimensional solitary wave. The slope of the crestline decreases slowly when the soliton moves forward.

### 4.5.2 Three Dimensional Solitary Waves at subcritical and Supercritical Speeds

The three dimensional upstream solitary waves in the subcritical and supercritical cases are also studied. Figure 4-16 shows the perspective and contour plots of the wave patterns at \( F_h = 0.9, 1.0 \) and 1.1. The wave profiles on the symmetry plane are drawn in Figure 4-17. For \( F_h = 0.90 \) the time interval is 200 and for \( F_h = 1.10 \) is 400. According to linear theory (Wehausen & Laitone, 1960), the disturbance advancing
steadily in the unbounded sea generates steady waves in its wake. In deep water, there exist two wave systems, the transverse and divergent; both are prominent in a wedge region with half-angle 19.28°, which is independent on the Froude number. In shallow water, the transverse and divergent wave systems coexist when \( F_h \), the depth Froude number, is below unity. The half angle of the wedge rises rapidly toward 90° as \( F_h \) approaches unity. The transverse wave disappears and the half angle decreases from 90° when the disturbance moves beyond the critical speed. In the contour plots in Figure 4-16, we see that the wake behind the disturbance has similar properties to the wake predicted by linear theory, though it is weakly nonlinear and dispersive effects that take effect in shallow water. At \( F_h = 0.90 \), the pronounced waves are restricted in a wedge region with half angle about 42.0° with curved waves of small amplitude stretching outward. At the critical speed \( F_h = 1.0 \), the wavelength of the transverse wave increases and the divergent wave is toward to be normal to the x axis when \( y \) is large enough. At the supercritical speed \( F_h = 1.1 \), the transverse wave vanishes and a depressed water surface is formed in the wedge region.

In the transcritical region, three dimensional solitary waves are generated and propagate upstream. At \( F_h = 0.90 \), the leading soliton is followed by a succession of solitons with decreasing wave amplitude. The positive peak of the wave is reached at the trailing edge of the pressure patch and the downstream wake is much more significant than the upstream waves. The wave amplitude and phase velocity of the solitons increase as the Froude number increases, whereas the amplitude of the downstream waves decreases. As shown in Figure 4-17, there doesn’t exist a depressed water behind the disturbance at \( F_h = 0.9 \). However, the mean water level of the central line of the downstream region is below \( z = 0 \) at \( F_h = 1.1 \). These properties can also be found in the two dimensional case which has been discussed in section 4.3. A very interesting property which exists exclusively in three dimensions is that the upstream soliton is blocked at the super-critical case. The phase velocity of the leading soliton tends to that of the disturbance \( F_h \) as its amplitude decreases to a critical value, which is 0.23 seen in the Figure 4-17(b). The leading soliton shifts a little from \( t = 1600 \) to \( t = 2000 \) with almost no change of the amplitude. During this
time, the curvature of the leading soliton decreases slowly to form a more straight crestline.

4.5.3 Free Three Dimensional Solitary Waves at Critical Speed

Our study on the solitary waves forced by a pressure distribution shows that the upstream solitons interact with each other as they radiate from the disturbance and move ahead. In this section we investigate the free propagating three dimensional solitary waves in an unbounded domain. Instead of letting a steady pressure act on the free surface for all time \( t > 0 \), we remove the pressure distribution at some time, in this test we use \( t_{stop} = 200 \), when the first crest has been generated and the second has not escaped from the source. The free surface elevation and velocity at \( t = 200 \) become the initial value for the nonlinear wave evolution without forcing.

Figure 4-18 shows the wave profile at the symmetric plane without forcing by the finite-time action of the pressure of magnitude \( \bar{P} = 0.40 \) and 0.60. The large free surface elevation around the pressure distribution is smoothed rapidly. The leading soliton detaches from the second soliton and the mean water level between two successive waves becomes zero. The wave amplitude of the second soliton is smaller than the leading one. The interaction between the leading soliton and its successor is negligible and it can be considered as a free soliton in three dimensions.

The dependence of the crest displacement \( x_c \) and the product \( A_c t \) is shown in Figure 4-19. The slope of the \( A_c t \) as a function of \((-x_c)\) of the data points decrease from \( \frac{3}{2} \), which is illustrated in Figure 4-11, to nearly \( \frac{1}{3} \). The slope of the free soliton is independent of the initial values which are different for the wave pattern generated by a pressure with different magnitude. Therefore, the relation between \( A_c t \) and \( x_c \) for a free-running three dimensional soliton is

\[
x_c + 3A_c t = C(\bar{P})
\]

in which \( C(\bar{P}) \) is a constant determined by the initial condition.

The wave amplitude of the leading free soliton at different times are displayed in
Table 4.2: Approximation for the wave amplitude of the leading free soliton, $F_h = 1.0$

<table>
<thead>
<tr>
<th>$\bar{P}$</th>
<th>$A_1$</th>
<th>$t_0$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.30</td>
<td>7.79</td>
<td>85.06</td>
<td>-0.7854</td>
</tr>
<tr>
<td>0.40</td>
<td>9.65</td>
<td>62.17</td>
<td>-0.7863</td>
</tr>
<tr>
<td>0.50</td>
<td>11.16</td>
<td>57.91</td>
<td>-0.7861</td>
</tr>
<tr>
<td>0.60</td>
<td>12.36</td>
<td>38.20</td>
<td>-0.7830</td>
</tr>
</tbody>
</table>

Figure 4-20 for $\bar{P} = 0.30, 0.40, 0.50$ and $0.60$. We use the formula in the same form as equation (4.5) and the result is listed in Table 4.2. The decay rate of the free soliton amplitude with respect to time is about $-0.785$.

The decay rate of a free three dimensional soliton is larger than a forced one which has interaction with its successive soliton. In two dimensions, a free soliton can keep its shape and propagate permanently. By the conservation of energy, Lee & Grimshaw (1990) deduced that the amplitude of a free propagating soliton decays in a rate of $t^{-\frac{3}{2}}$ by assuming that the amplitude along the isophasal crestline is constant. In the present problem, the diffusion of the wave amplitude along the crestline is important and results in a more rapid decay.

Figure 4-21 shows the geometry properties of the crestline. The log-log scale plot in Figure 4-21(a) predicts a straight line and the dependence of the slope on time for different initial values is shown in (b). Same as the forcing soliton, We assume the crestline takes the form in equation (4.6). The index $\beta(t)$ is shown in Figure 4-21(b) and it varies slowly with respect to time and has the mean value approximately at 0.50, which illustrates that the crestline is nearly parabolic. The latus rectum of the parabola $\alpha(t)$ is drawn in Figure 4-21(c) and it’s a linear function of time with steeper slope 1.60, compared with the forced solitons. Thus the isophasal crestline of the free three dimensional solitary wave has the following asymptotic form

$$x + A_1 t^{0.22} = B_1 \frac{y^2}{t}$$

in which $A_1$ and $B_1$ are constants determined by the initial state of the waves.
Figure 4-9 (a) - (b)
Figure 4-9 (c) - (d)
Figure 4-9: Wave evolution generated by a pressure distribution in unbounded domain. $F_h = 1.0$, $P = 0.30$, (a). $t = 200$; (b). $t = 400$; (c). $t = 600$; (d). $t = 800$; (e). $t = 1000$; (f). $t = 1200$
Figure 4-10 (a)-(b)
Figure 4-10: Wave profile at $y = 0$ generated by a pressure distribution, $F_h = 1.0$,
(a). $\bar{P} = 0.20$; (b). $\bar{P} = 0.30$; (c). $\bar{P} = 0.40$; (d). $\bar{P} = 0.50$
Figure 4-11: $x_c - A_c t$ of the leading soliton generated by a pressure distribution, $F_h = 1.0$
Figure 4-12: Wave amplitude $A_c(t)$ of the leading soliton at different time, $F_h = 1.0$
Figure 4-13: Crestline of the leading soliton generated by a pressure distribution from $t = 200$ to $t = 1000$ with interval 80. $F_h = 1.0$, (a). $\bar{P} = 0.20$; (b). $\bar{P} = 0.30$; (c). $\bar{P} = 0.40$; (d). $\bar{P} = 0.50$
Figure 4-14: Geometry properties of the crestline of the leading soliton, $F_h = 1.0$,
Top: log-log scale plot of crestline of $P = 0.30$; Middle: index $\beta(t)$; Bottom: latus rectum $\alpha(t)$
Figure 4-15: Wave amplitude along the crestline of the leading soliton generated by a pressure distribution from \( t = 200 \) to \( t = 1000 \) with interval 80. \( F_h = 1.0 \), (a). \( \bar{P} = 0.20 \); (b). \( \bar{P} = 0.30 \); (c). \( \bar{P} = 0.40 \); (d). \( \bar{P} = 0.50 \)
Figure 4-16 (a)
Figure 4-16 (b)
Figure 4-16: Wave patterns and contours at transcritical speeds. $\bar{P} = 0.30$. (a). $F_h = 0.9$, $t = 400$; (b). $F_h = 1.0$, $t = 1200$; (c). $F_h = 1.1$, $t = 1200$
Figure 4-17: Wave profile at $y = 0$ generated by a pressure distribution, $\bar{P} = 0.30$, (a). $F_h = 0.9$; (b). $F_h = 1.1$
Figure 4-18: Wave profile at $y = 0$ of a free three dimensional soliton, $F_h = 1.0$, (a). $\bar{P} = 0.30$; (b). $\bar{P} = 0.60$
Figure 4-19: \( x_e - A_e t \) of the free leading three dimensional soliton, \( F_h = 1.0 \)
Figure 4-20: Wave amplitude $A_c(t)$ of the free leading three dimensional soliton, $F_h = 1.0$
Figure 4-21: Geometry properties of the crestline of the free leading soliton, $F_h = 1.0$, Top: log-log scale plot of crestline of $P = 0.30$; Middle: index $\beta(t)$; Bottom: latus rectum $\alpha(t)$
Chapter 5

Nonlinear Long Waves Generated by a High Speed Vessel

The three dimensional nonlinear long waves generated by a high speed ship advancing at subcritical, critical and supercritical speeds in both the restricted and open shallow water is studied. The scale of the upstream waves is much larger than that of the ship and it is reasonable to make the thin ship approximation. The ship is simplified into a line of sources at the symmetric plane $y = 0$ along which the source strength is related to the ship's geometry.

In the first section, the solitons generated by an advancing ship in a channel of shallow water is computed. It shows satisfactory agreement with the experiment (Ertekin et al., 1984) when the blockage coefficient, i.e. the ratio of the maximum cross-sectional area and the cross-sectional area of the channel, is below 0.08. At supercritical speed, the computation shows that there exist a critical blockage coefficient, below which no soliton will be radiated upstream. The upstream solitary waves in an unbounded domain is investigated in section 2. Similar wave patterns in the upstream region is found compared with the waves generated by a pressure distribution in chapter 4. The downstream wave system is strongly related to the slenderness of the ship. The crestline of the leading soliton is almost a perfect parabola. Some other properties of the leading upstream soliton are discussed in detail in that section.
5.1 Solitary Waves Generated by a Ship in a Channel

Ertekin et al (1984) found that the dominant parameter for the amplitude and period of generation of the upstream solitons generated by a ship advancing in the shallow water channel is the blockage coefficient, which is defined as

\[ C_{\text{block}} = \frac{A_m}{W h} \] (5.1)

in which \( A_m \) is the maximum area of the ship section. \( W \) and \( h \) are the width and still water depth of the channel, respectively. The geometry of the ship doesn’t play an important role for the solitons. For simplicity, we use a strut which extends to the sea bed and has the same width along the depth to simulate a high speed vessel with more complicate ship hull. The shape of the waterline is defined as

\[ b(x) = \frac{B}{L} \cos \left( \frac{\pi x}{L} \right) \] (5.2)

in which \( B \) and \( L \) are the beam width and length of the ship. An advantage of this choice is that there doesn’t exist any jump for the normal vector along the waterline near the bow and stern.

In our calculation, we choose

\[ \frac{L}{h} = 10.16, \]

which is corresponding to the case of \( L = 152.4 \text{cm} \) and \( h = 15.0 \text{cm} \) in the experiments of Ertekin et al (1984). The longitudinal range of the computational domain is \(-60 \leq x \leq 100\) and the grid size is \( \Delta x = 0.25 \).

First we keep the width of the channel fixed and vary the blockage coefficient by changing the width of the strut. The nondimensional half width of the channel is \( W/2 = 16.27 \), corresponding to the width \( W = 4.88 \text{m} \) and water depth \( h = 15 \text{cm} \) in the experiments. According to (2.57), the normal velocity at \( y = 0 \) is proportional
to the $x$ derivative of the cross-sectional area of the ship. Thus the variation of the blockage coefficient in a fixed-width channel can be also obtained by varying the draft of the ship. 40 grid points are used in the $y$ direction.

We also change the blockage coefficients by keeping the shape of the ship fixed and modifying the width of the channel, which is convenient in the numerical simulation rather than the realistic experiment. One test case of the experiments is selected to verify the validity of the present theory and numerical method. The ship width is $B = 23.4\, cm$ and the draft is $d = 10\, cm$ and thus the slenderness of the ship in the computation is obtained as follows

$$
\frac{B}{L} = \frac{23.4}{152.4} \cdot \frac{10.0}{15.0} = 0.1024
$$

The number of grid points in the transverse direction increases with the increase of the channel width. The average transverse grid size $\Delta y$ is 0.40.

Figure 5-1 illustrates the evolution of the nonlinear long waves generated by a strut of $\frac{B}{L} = 0.1024$ traveling at $F_h = 1.0$. The width of the channel is fixed to 16.94 and the blockage coefficient is $S = 0.06$. The upstream straight-crest solitons are radiated in front of the strut periodically and a complicated wave system is formed behind it. Both the pressure distribution on the free surface and the body in the water have the same effect upon the generation of upstream two dimensional solitons in the channel of finite width in shallow water.

Figure 5-2 shows the wave pattern at $t = 200$ in the case of $F_h = 1.0$ for different blockage coefficients $S = 0.02, 0.04, 0.08$ in a fixed-width channel. The wave amplitude and the period of generation of the upstream solitons increase when the blockage coefficients increase, which indicates a larger width or deeper draft of the ship. The transverse wave length of the downstream wave system decreases and shows a more complicated structure in the wake as the blockage coefficient increases. At $S = 0.08$, the spanwise distribution of the wave amplitude of the solitons has not become uniform at a distance of 4 - 5 wave lengths ahead of the strut. Several periods are taken to form the two dimensional straight-crest soliton propagating upstream. This
phenomena, either for the numerical calculation or the experiment, have not been reported before and they may be due to the thin ship approximation. The theory taking the real ship geometry into account is necessary in order to investigate this problem carefully.

The nonlinear waves generated by a strut moving at a critical speed in a channel of variable width are also studied. Figure 5-3 shows the wave patterns with the same blockage coefficients $S = 0.02, 0.04, 0.08$ as before, keeping the $B/L$ ratio fixed and varying the channel width. Compared with Figure 5-2, there have been more solitons in front of the ship, though the wave amplitude and period of the solitons are close in these two cases. At $S = 0.08$, the upstream solitons have constant amplitude along the crest. An interesting phenomenon is that the wave length in the transverse direction of the downstream wave system increases as the blockage coefficient increases, which is contrary to the results obtained by increasing the ship transverse section area in the fixed-width channel.

Figure 5-4 shows the normalized period of the generation of solitons $T_s$ and the wave amplitude of the leading upstream going soliton $A_s$ as a function of the blockage coefficient $S$ by keep $B/L$ fixed and $W$ variable and vice versa. The experimental results done by Ertekin et al and the computational results by Katsis & Akylas (1987) using a K-P equation are also displayed. The present numerical results compare well with the experiment and the K-P equation’s prediction of the wave amplitude and period of the solitons. The limit of the slendernessness of the thin ship approximation is about 0.08, beyond which the prediction of the wave amplitude is larger than the experimental data.

The radiation of the solitons by the strut at different transcritical speeds is investigated. The phase velocity of the leading soliton is computed as follows

$$C_s = F_h + C_{sm}$$  \(5.3\)

in which $C_s$ is the phase velocity in the frame fixed with respect to earth and $C_{sm}$ is that in the moving frame fixed on the strut. The results of the wave amplitude, period
of soliton generation and phase velocity for a strut with $\frac{B}{L} = 0.1024$ and $S = 0.06$ are listed in Table 5.1. The results are in satisfactory agreement with the experiment. The numerical prediction of the phase velocity of the leading soliton agrees well with the theoretical result of Rayleigh.

$$C_s = (1 + A_s)^{\frac{1}{2}}$$  \tag{5.4}$$

The dependence of the soliton generation on the Froude number $F_h$ is illustrated in Figure 5-5 where $F_h$ varies from 0.70 to 1.20. At all these speeds, the upstream two dimensional solitons are radiated. For the subcritical speed range, $F_h < 1.0$, the leading soliton is pronounced and followed by a succession of solitons with decreasing amplitude. When the strut moves faster than the critical speed, the upstream solitons have the same amplitude. The wave amplitude and period of generation of the solitons increase as the strut moves faster. The wave pattern of the downstream wave system are very similar at different $F_h$, though the amplitude is a function of strut speed. The ratio of $\frac{B}{L}$ and the width of the channel play an important role on the downstream wave patterns.

Ertekin et al(1986) found that there exists a critical Froude number for each moving pressure distribution, beyond which no solitons are generated in front of the disturbance. The analogous phenomenon does exist for a high speed ship advancing in the channel. From equation (5.4), the upstream soliton exists only when the wave

<table>
<thead>
<tr>
<th>$F_h$</th>
<th>$A_s(calc)$</th>
<th>$A_s(exp)$</th>
<th>$F_hT_z(calc)$</th>
<th>$F_hT_z(exp)$</th>
<th>$C_s$</th>
<th>$(1 + A_s)^{\frac{1}{2}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>0.072</td>
<td>0.068</td>
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<td>28.0</td>
<td>1.028</td>
<td>1.035</td>
</tr>
<tr>
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<td>0.146</td>
<td>0.139</td>
<td>40.7</td>
<td>38.3</td>
<td>1.054</td>
<td>1.071</td>
</tr>
<tr>
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<td>0.278</td>
<td>41.6</td>
<td>42.4</td>
<td>1.120</td>
<td>1.141</td>
</tr>
<tr>
<td>1.0</td>
<td>0.496</td>
<td>0.468</td>
<td>43.8</td>
<td>50.8</td>
<td>1.203</td>
<td>1.223</td>
</tr>
<tr>
<td>1.1</td>
<td>0.740</td>
<td>0.600</td>
<td>49.2</td>
<td>55.2</td>
<td>1.321</td>
<td>1.319</td>
</tr>
<tr>
<td>1.2</td>
<td>1.050</td>
<td>—</td>
<td>59.9</td>
<td>—</td>
<td>1.431</td>
<td>1.432</td>
</tr>
</tbody>
</table>

Table 5.1: Wave amplitude, period of soliton generation and phase velocity of soliton generated by a strut of $\frac{B}{L} = 0.1024$, $S=0.06$
amplitude satisfies
\[ A_s > F_h^2 - 1 \quad (5.5) \]

If the free surface elevation in front of the disturbance can be accumulated beyond the critical wave amplitude \( A_{s,\text{crit}} = F_h^2 - 1 \), a soliton will be generated and radiated upstream. In this case, we anticipate a critical blockage coefficient for the soliton radiation in the supercritical speed region. Figure 5-6 illustrates the wave evolution at \( F_h = 1.1 \) with blockage coefficient \( S = 0.03 \). The solitons are generated periodically in front of the strut. Figure 5-7 shows the wave pattern in the case of \( S = 0.0096 \). No soliton is generated for a long time up to \( t = 1000 \) and a steady state is reached. Figure 5-8 shows the wave amplitude of the leading soliton as a function of blockage coefficient \( S \) at \( F_h = 1.1 \). The numerical results compare well with the experiment when \( S \) is less than 0.04. The wave amplitude tends to be constant when the blockage coefficient increases continuously beyond 0.06. The computation predicts an increasing wave amplitude when \( S \) increases and deviates from the experiment result for larger blockage coefficients. The present theory is based on the assumption of small wave elevation, i.e. \( \epsilon = \frac{A}{h} \ll 1 \). The assumption is violated when the wave amplitude of the soliton is larger than 0.64, the corresponding blockage coefficient being 0.04. The critical blockage coefficient of about 0.01 is found below which there is no upstream-running soliton generated ahead of the ship periodically.
Figure 5-1: Wave evolution generated by a strut in a shallow channel. \( W/2 = 8.47, S = 0.06, F_h = 1.0 \), (a). \( t = 40 \); (b). \( t = 120 \); (c). \( t = 200 \)
Figure 5-2: Wave pattern generated by strut with variable cross section in a shallow channel of fixed width. \( W/2 = 16.27, \ F_h = 1.0, \) (a). \( S = 0.02; \) (b). \( S = 0.04; \) (c). \( S = 0.06 \)
Figure 5-3: Wave pattern generated by a strut in a shallow channel of variable width. 
$\frac{B}{L} = 0.1024$, $F_h = 1.0$, (a). $W/2 = 25.4$, $S = 0.02$; (b). $W/2 = 12.7$, $S = 0.04$; (c). $W/2 = 8.47$, $S = 0.06$
Figure 5-4: Period of soliton generation and wave amplitude of the leading soliton in a shallow channel, $F_h = 1.0$
Figure 5-5: Wave pattern generated by a strut travelling at transcritical speeds. \( \frac{b}{L} = 0.1024, S = 0.06 \), (a). \( F_h = 0.7, t = 120 \) (b). \( F_h = 0.8, t = 160 \) (c). \( F_h = 0.9, t = 200 \) (d). \( F_h = 1.0, t = 200 \) (e). \( F_h = 1.1, t = 200 \) (f). \( F_h = 1.2, t = 200 \)
Figure 5-6: Wave evolution generated by a strut moving at supercritical speed in a shallow channel. $S = 0.03, F_h = 1.1$, (a). $t = 40$; (b). $t = 100$; (c). $t = 200$
Figure 5-7: Wave evolution generated by a strut moving at supercritical speed in a shallow channel. $S = 0.0096, F_h = 1.1$, (a). $t = 100$; (b). $t = 600$; (c). $t = 1000$
Figure 5-8: Wave amplitude of the leading soliton in a shallow channel with different blockage coefficients, $F_h = 1.1$
5.2 Solitary Waves Generated by a Ship in an Unbounded Domain

In this section, the case of the zero blockage coefficient $S = 0.0$ is studied by investigating the three dimensional upstream solitary waves generated by a high speed vessel traveling in an unbounded domain. Following the same computational procedure as the previous section on the soliton generation in restricted channels, an extra open boundary condition is implemented on the transverse boundary to ensure the wave propagation outward without reflection from the side boundary.

To make it simple, we still consider a strut of length $L = 10.16$. Different beam widths are used to vary the slenderness of the strut. The computational domain is chosen as

$$-80 \leq x \leq 100; \quad 0 \leq y \leq 240$$

The grid size is taken as $\Delta x = 0.25$ and $\Delta y = 0.40$. The time step is $\Delta t = 0.15$.

Figure 5-9 shows the perspective views of the wave pattern at $t = 800$ generated by a strut of slenderness $\frac{B}{L} = 0.05, 0.10, 0.15, 0.20$ traveling at the critical speed $F_h = 1.0$. As shown, three dimensional solitary waves radiate in front of the strut periodically. The amplitude of the three dimensional soliton decays as it moves upstreamward. The wave amplitude and period of generation increase as the strut becomes less slender. Shorter waves are generated in the downstream area as $\frac{B}{L}$ increases, though the geometry of upstream solitons doesn’t change much. A transverse wave system is formed in a wedge-like region behind the strut. The performance of the Oranski open boundary condition becomes a little worse when the incident wave height increases as the strut becomes fat. Small reflective waves can be seen to interact with the upstream solitons and generate small fluctuation along the crests.

The free surface elevations on the symmetry plane $y = 0$ generated by struts with different beam-length ratio traveling at a critical speed are drawn in Figure 5-10. The peak value of free surface elevation occurs at about $x = -4.0$, a small distance behind
the bow, a phenomenon observed in steady ship bow waves. The maximum negative wave elevation occurs near \( x = 4.0 \), a little ahead of the stern. The mean water level at \( y = 0 \) in the downstream region is below the still sea level and the depressed region elongates with respect to time. As the slenderness increases, the downstream waves are pronounced and their wave heights increase. Compared with Figure 4-10, the downstream wave systems generated by a pressure distribution are more smooth than those by the strut. The ratio between the maximum negative wave elevation and the maximum positive wave elevation is smaller for the waves of strut than those of the pressure distribution.

The dependence of the phase velocity of the leading soliton on the symmetry plane \( y = 0 \) on the soliton amplitude is investigated by showing the \( A_c t \) as a function of the position of wave crest \( -x_c \) in Figure 5-11. Just as the solitons generated by a pressure distribution, a linear relation exists between \( -x_c \) and \( A_c t \) with a slope approximately about \( \frac{3}{2} \). The slope is independent on the slenderness of the strut. Combined with the understanding from the pressure induced wave, we may conclude that the relation

\[
V_c = -\frac{2}{3} A_c \quad (5.6)
\]

is applicable for the forced wave generated by pressure distribution, source or underwater topography traveling at critical speed.

The relation between the soliton amplitude \( A_c \) and time is approximated by the following formula

\[
A_c(t) = A_1(t - t_0)^{\frac{1}{2}} \quad (5.7)
\]

in which \( A_1 \) and \( t_0 \) are constants related to the initial wave amplitude and the time of the soliton generation. As shown in Figure 5-12, the data points match well with curves decaying at the order of nearly \( O((t - t_0)^{-\frac{3}{2}}) \). Table 5.2 lists the approximate results. This relation has also been found by the analysis of three dimensional solitary waves generated by pressure distribution on the free surface. When time is large, we have

\[
A_c(t) \sim O(t^{-\frac{3}{2}}) \quad (5.8)
\]
<table>
<thead>
<tr>
<th>$\frac{B}{L}$</th>
<th>$A_1$</th>
<th>$t_0$</th>
<th>$\gamma$</th>
</tr>
</thead>
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<td>41.10</td>
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</tr>
<tr>
<td>0.10</td>
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<td>-0.3323</td>
</tr>
<tr>
<td>0.15</td>
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<td>27.28</td>
<td>-0.3356</td>
</tr>
<tr>
<td>0.20</td>
<td>1.1265</td>
<td>21.48</td>
<td>-0.3373</td>
</tr>
</tbody>
</table>

Table 5.2: Approximation for the wave amplitude of the leading soliton generated by a strut, $F_h = 1.0$

The crestlines of the leading three dimensional soliton at different stages are shown in Figure 5-13 for the struts of $\frac{B}{L} = 0.05, 0.10, 0.15, 0.20$. It can be seen that the crestlines are almost parabolas with the curvature decreasing with respect to time. The following mathematical formula is used to approximate the geometry properties of the soliton crestlines.

$$y = \alpha_1(t)(x - x_c)^{\beta(t)}$$

in which $x_c$ is the position of the soliton crest at the central plane. Figure 5-14(a) shows the log-log plot of the crestline of the leading soliton. At each instant, it fits the straight line well. The slope of the log-log line, the index $\beta(t)$ in equation (5.9), is plotted in Figure 5-14(b). $\beta(t)$ varies slightly around the constant 0.50 and there seems no obvious evidence that it is dependent on the slenderness of the strut. If we set the index $\beta(t)$ as the constant 0.50, we define a new parameter

$$\alpha(t) = \alpha_1^2(t) = \frac{y^2}{x - x_c(t)}$$

in which $\alpha(t)$ is the length of the latus rectum of the parabolic curve. The numerical result is shown in Figure 5-14(c) and it is a linear function of time with slope about 1.40, which is independent of the geometry of the strut.

From the numerical simulation for the three dimensional solitary waves generated by either pressure distribution or a displacement ship traveling at the critical speed in shallow water, the crestline of the forced leading soliton in open sea takes the
asymptotical form

\[ x + A_0 t^3 = B_0 \frac{y^2}{t} \]  (5.11)

when time is large. The constants \( A_0 \) and \( B_0 \) are related to the strength of the forcing disturbance.

Figure 5-15 shows the amplitude distribution along the crestline of the leading soliton generated by struts of different slenderness. The higher gravitational energy near the symmetric plane tends to dissipate outward. The slope of the wave height along the crestline decreases as the time increases.

The perspective views and contours of the wave patterns generated by a strut of \( \frac{B}{L} = 0.15 \) at speed \( F_h = 0.9, 1.0, 1.1 \) are shown in Figure 5-16. At the subcritical speed, \( F_h = 0.9 \), the wave amplitudes of upstream solitary waves are lower than those of the downstream systems. The wave profile at the symmetric plane \( y = 0 \) is displayed in Figure 5-17(a). The leading soliton is followed by three solitary waves with decreasing wave amplitudes. Both transverse and divergent wave systems can be seen behind the strut. Outside the wedge-like area, in which short waves are dominant, divergent long waves are present with small wave height behind the strut. At the critical speed, \( F_h = 1.0 \), the half angle of the downstream wedge tends to 90°. When the strut moves beyond the critical speed, the half angle of the wedge decreases.

As shown in Figure 5-17(b), at supercritical speed \( F_h = 1.1 \), the leading soliton slows itself when its amplitude tends to 0.22, which is nearly \( \sqrt{F_h^2 - 1} \), the critical wave amplitude in a restricted channel for supercritical solitons. The leading soliton will move at the same speed as the strut eventually. The mean sea level of the downstream wave profile at \( y = 0 \) is below the still free surface. Both the upstream and downstream profile at the symmetric plane tend to be steady.
Figure 5-9 (a) - (b)
Figure 5-9: Wave pattern generated by a strut in open sea. $F_h = 1.0$, $t = 800$, (a). $B/L = 0.05$; (b). $B/L = 0.10$; (c). $B/L = 0.15$; (d). $B/L = 0.20$
Figure 5-10 (a)-(b)
Figure 5-10: Wave profile at $y = 0$ generated by a strut, $F_h = 1.0$, (a). $\frac{B}{L} = 0.05$; (b). $\frac{B}{L} = 0.10$; (c). $\frac{B}{L} = 0.15$; (d). $\frac{B}{L} = 0.20$
Figure 5-11: $x_c - A_c t$ of the leading soliton generated by a strut, $F_h = 1.0$
Figure 5-12: Evolution of the wave amplitude $A_c(t)$ of the leading soliton wave generated by a strut, $F_h = 1.0$
Figure 5-13: Crestline of the leading soliton generated by a strut from $t = 200$ to $t = 1000$ with interval 80. $F_h = 1.0$, (a). $\frac{B}{L} = 0.05$; (b). $\frac{B}{L} = 0.10$; (c). $\frac{B}{L} = 0.15$; (d). $\frac{B}{L} = 0.20$. 
Figure 5-14: Geometry properties of the crestline of the leading soliton generated by a strut, $F_h = 1.0$. Top: log-log scale plot; Middle: index $\beta(t)$; Bottom: latus rectum $\alpha(t)$
Figure 5-15: Wave amplitude along the crestline of the leading soliton generated by a strut from $t = 200$ to $t = 1000$ with interval 80. $F_h = 1.0$, (a) $\frac{B}{L} = 0.05$; (b) $\frac{B}{L} = 0.10$; (c) $\frac{B}{L} = 0.15$; (d) $\frac{B}{L} = 0.20$
Figure 5-16: Wave patterns and contours at transcritical speeds generated by a strut. $\frac{B}{L} = 0.10$. (a) $F_h = 0.9, t = 400$; (b) $F_h = 1.0, t = 800$; (c) $F_h = 1.1, t = 800$
Figure 5-17: Wave profile at $y = 0$ generated by a strut, $\frac{B}{L} = 0.10$, (a). $F_h = 0.9$; (b). $F_h = 1.1$
Chapter 6

Conclusions

A modified generalized Boussinesq equation is derived to formulate the nonlinear long waves generated by a disturbance, which could be a pressure distribution, ship or underwater topography, traveling at transcritical speeds in shallow water in a restricted channel or open sea. The balance between nonlinearity and dispersive effects for the long wave causes the generation of solitary waves propagating upstream ahead of the disturbance. An effective implicit finite difference method is applied to solve the nonlinear wave problem numerically. At each time step, an under-relaxation iterative procedure is used. A thin ship approximation is used to study the solitons generated by a high speed vessel in shallow water.

In restricted shallow waters, a succession of straight-crest solitons are emitted periodically and infinitely ahead of the disturbance advancing at the transcritical speed. In the subcritical case, the leading upstream-running wave is followed by its successors with decreasing wave amplitudes. The wave amplitude of the leading soliton and the period of the generation of soliton increase as the disturbance moves faster. The present numerical results verify that the blockage coefficient is the dominant parameter governing the soliton generation, which is found by Ertekin et al(1984) experimentally. At supercritical speeds, there exists a critical blockage coefficient beyond which no soliton can be produced upstream.
In the horizontally unbounded domain, three dimensional solitary waves are generated ahead of the disturbance of a pressure distribution or an advancing ship. The crestline of the leading soliton is approximately a parabola with its *latus rectum* proportional to time, making the crestline's curvature decreasing as it moves forward. The decay rate of the wave amplitude is found to be $O(t^{-\frac{1}{2}})$ for the forced leading soliton and $O(t^{-0.78})$ for the free propagating three dimensional solitary wave. The wave amplitude of upstream soliton tends to $F_h^2 - 1$ at supercritical case.
Bibliography


