Design of Non-serial, Non-parallel Flexural Transmissions as Applied to a Micro-machined MEMS Tuning Fork Gyroscope

by

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ABSTRACT

The purpose of this work is to develop and implement design rules for flexures that emphasize directionality. This work is important for flexure designs that cannot be broken down into equivalent series or parallel components. The impact of this work is illustrated in the implementation of a high-performance micro-electromechanical system (MEMS) tuning fork gyroscope (TFG) that may be used for inertial navigation, automobile rollover detection, video games, and smartphones. These design rules build upon Freedom Actuation and Constraint Topologies (FACT), pseudo-rigid body modeling (PRBM), and constraint based-design (CBD) to include directionality. Flexural transmissions may be used to couple the motion of a plurality of stages to have different types (translations, rotations, and screws), different transmission ratios, and different directions on different axes. These design rules are implemented to create a MEMS TFG that exhibits coupled mass motions and decoupled mode shapes. A MEMS gyroscope was designed and modeled and a meso-scale prototype was made to verify the models and test sensitivity to fabrication errors. The design has 49% separation between desired and undesired modes. The meso-scale prototype and finite element analysis (FEA) suggest that the TFG design developed from these rules exhibits a 4x reduction in sensitivity to quadrature error.
ACKNOWLEDGEMENTS

My family deserves much appreciation. They have always loved and supported me, pushed me to do my best, and showed me the value of an education. I literally and figuratively would not be here without them.

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The purpose of this work is to create flexure design rules for coupled motions and decoupled mode shapes. This work is important because it builds on previous design techniques to design flexures that cannot be broken down into serial/parallel components. These flexures allow coupled motions that impact micro-electromechanical system (MEMS) gyroscopes, which may be used for the inertial guidance of submarines and unmanned aerial vehicles (UAVs), or for consumer applications including cell phones and video games.

Flexure systems are used as bearings with low stiffness in the degrees of freedom and high stiffness in their degrees of constraint. The motion results from elastic deformation rather than using sliding or rolling interfaces [1], [2]. Flexures are ubiquitous in MEMS devices and precision machine design because they have no friction, wear, or backlash which allows repeatable motion down to the angstrom level [1], [3]. Flexures also typically have no assembly or maintenance and potentially infinite life, which is why they are ideal for MEMS sensors and actuators. Other flexure applications include energy harvesting devices [4], medical devices [5], micro/nano-manipulation [6], and consumer products [7]. There has been much work in the design and synthesis of flexures and other compliant mechanisms [8]; however, it is challenging to synthesize mechanisms that contain multiple stages moving in different directions. This work will demonstrate how to use Table 1.1 to design compliant mechanisms with an emphasis on directionality.
Table 1.1: Flexural Transmissions

<table>
<thead>
<tr>
<th>Input</th>
<th>Intermediate Stage</th>
<th>Output Type</th>
<th>Transmission Ratio (TR)</th>
<th>Axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>None</td>
<td>T</td>
<td>TR &gt; 0</td>
<td>Same</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>R</td>
<td>TR &gt; 0</td>
<td>Same</td>
</tr>
<tr>
<td>R</td>
<td>R</td>
<td>S</td>
<td>TR = 0</td>
<td>Off</td>
</tr>
<tr>
<td>S</td>
<td>Rigid</td>
<td>None</td>
<td>TR &lt; 0</td>
<td>Off</td>
</tr>
</tbody>
</table>

The design rules presented herein are used to design a MEMS tuning fork gyroscope (TFG) with coupled motions and decoupled mode shapes, shown in Figure 1.1.

Figure 1.1: A modally decoupled TFG. Material in blue is fixed. Flexures (yellow) control the “drive” mode, where proof masses $M_1$ and $M_2$ move co-axially in opposite directions. Flexures (red) control the “sense” mode, where the proof masses move in opposite directions on different axes. Material in orange moves in both the drive and sense modes. Vectors are not drawn to scale. a) Top view illustrates the desired gyroscope motions, b) Isometric view illustrates how the device functions as a MEMS TFG.
1.1 Flexure Design Synthesis Review

The purpose of this section is to review the current flexure synthesis approaches to understand their intents, advantages and drawbacks. There are several approaches for flexure design, including pseudo-rigid body modeling [7], constraint-based design [9], computer-based topological synthesis [10] and Freedom, Actuation, and Constraint Topologies (FACT) [11]. One limitation of these techniques is that they are typically used to design the motion of independent stages. Coupling the motion of several stages is a present topic of research in the field of flexure design [12]. Another challenging flexure synthesis problem involves flexures that cannot be described by equivalent springs that are neither in series nor parallel configurations [13]–[15]. The latter problem is described in the methods for combining flexure systems shown in Table 1.2.

Figure 1.2 shows a flexure topology that cannot be reduced to an equivalent spring via series and parallel flexures. Standard flexure design tools have difficulty in the design synthesis of this class of flexures. This structure was designed using flexures as transmission elements. The design rules outlined in Chapter 2 also explain how to combine flexures to create decoupled mode shapes once the mode shapes have been designed independently.

![Figure 1.2: An example of a hybrid flexure system.](image)
Table 1.2: Combining Flexure elements.

<table>
<thead>
<tr>
<th>Type</th>
<th>Stiffness Modeling</th>
<th>DOF modeling</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parallel</td>
<td>Sum of each stiffness</td>
<td>Intersection of DOFs</td>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td></td>
<td>$k_{eq} = \sum_i k_i$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Series</td>
<td>Reciprocal of sum of compliances</td>
<td>Union of DOFs</td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td></td>
<td>$k_{eq} = \left(\sum_i \frac{1}{k_i}\right)^{-1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Parallel-Series Hybrid</td>
<td>Recursively find each equivalent series or parallel stiffness</td>
<td>Recursively find each equivalent series or parallel DOF</td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
<tr>
<td></td>
<td>$k_{eq} = \frac{\partial^2 U}{\partial x^2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>None of the above</td>
<td>Flexural Transmissions</td>
<td></td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
</tbody>
</table>
1.1.1 Pseudo-Rigid Body Model (PRBM)

The pseudo-rigid body model (PRBM) transforms flexures into rigid body components with equivalent force-deflection characteristics [7]. The PRBM has been used to model flexures as linkages, shown in Figure 1.3 [7], [16]. This technique allows designers to apply the knowledge of linkage design to create equivalent compliant mechanisms. The kinematics are modeled by rigid links with appropriately placed pin joints; the force-deflection relationships are modeled with lumped parameter springs (typically torsional springs, but linear springs have been used to model strain stiffening [7], [17]). This technique is useful for modeling a compliant mechanism once the concept has been generated (the PRBM is used in Chapters 2 and 3), but flexure synthesis using this technique requires pattern recognition and previous experience with linkage design.

![Figure 1.3: Pseudo-Rigid Body Model example. a) A beam subjected to a load that causes a displacement, b) A pseudo-rigid body model of the same beam [18], © Elsevier. Reproduced with permission. All rights reserved. (License #3384551434956).](image)

1.1.2 Constraint-Based Design (CBD)

Constraint-based design uses rules of thumb and flexural building blocks to create compliant mechanisms. An example of a CBD building block is flexural beams with intersecting lines of action form an instant center about which a stage may rotate. The concepts generated from CBD are therefore limited by the building blocks that possess simple motions, i.e. motions that only involve pure translations and rotations. This technique is not useful for generating
compliant screw mechanisms, i.e. compliant mechanisms where the translations and rotations are coupled. The matrix of all arrangements of orthogonal constraints is shown in Figure 1.4 [9].

![Constraint-based matrix of orthogonal constraint arrangements](image)

Figure 1.4: Constraint-based matrix of orthogonal constraint arrangements [9].

1.1.3 Topological Synthesis

Topological synthesis uses a computer to iteratively synthesize the topology of a flexure by satisfying displacement (of input and output) requirements and force requirements. This technique has been used to generate designs of grippers (e.g. Figure 1.5) and other compliant mechanisms [8] that require directionality. There are several limitations of topological synthesis [8]:

- The “optimal” solutions are generally mesh-dependent and not unique.
• The solution depends on the volume (of available material) constraint, and on the starting point.
• It often generates mechanisms with lumped compliance. Lumped compliance mechanisms exhibit localized deformations around a stress concentration, which is often not ideal in practice.
• The optimal solution depends on the choice of output stiffness.
• The solution depends on the load magnitude when including geometric nonlinearity is taken into account.
• The designer may have no idea why the flexure is the shape that it is, or how to design a compliant mechanism without a computer.

![Figure 1.5: Topological synthesis example.](image)

Figure 1.5: Topological synthesis example. [12] © Elsevier. Reproduced with permission. All rights reserved. (License # 3380581322285).

1.1.4 Freedom, Actuation, and Constraint Topology (FACT)

FACT uses a comprehensive library of shapes that represent the mathematics of screw theory to enable flexure synthesis through the visual representation of all regions where compliant constraints allow the desired degrees of freedom [11], [12], [19]–[23]. The shapes in Figure 1.6 contain the information to design all possible configurations of a flexure that satisfy the degrees of freedom and constraint. FACT uses complementary geometric shapes, freedom spaces and constraint spaces, to illustrate: (i) the necessary constraints for the desired degree(s) of freedom (DOF), and (ii) the DOFs for a structure with given constraints [8]. FACT and screw theory are investigated further in Chapter 2 and FACT is used in Chapters 2 and 3 for flexure synthesis.
1.1.5 Flexural Transmissions

This research builds on FACT, constraint-based design, and pseudo-rigid body modeling to form a technique that allows the synthesis of compliant mechanisms that may have multiple stages moving in various directions. As shown in Table 1.1, the inputs and outputs may have different types, with different transmission ratios, moving on different axes, with a plurality of intermediate stages. Figure 1.2 shows an example of such a flexure: there is an input stage that moves to the left, an output stage that moves to the right and two intermediate stages that move up and down. That is (in terms of Table 1.1), the stages in Figure 1.2 have the same type (translation “T”), a transmission ratio of -1 (“TR < 0,” the negative sign indicates that the output stage moves in the opposite direction of the input stage), and they move along the same axis. This technique, described in detail in Chapter 2, illustrates how to build on previous work while overcoming their drawbacks.
1.2 Application: MEMS Gyroscopes

MEMS gyroscopes demonstrate the need of this research because they require coupled motions and decoupled mode shapes. A gyroscope is a sensor that measures rotation rate. Gyroscopes are used in inertial navigation (described in section 1.2.2) to guide submarines, unmanned aerial vehicles (UAVs) and other devices without the use of GPS. There are many types of gyroscopes, including spinning mass gyroscopes, ring laser gyroscopes, and MEMS tuning fork gyroscopes (TFGs). TFGs offer several advantages to other types of gyroscopes: (i) the concept scales well for miniaturization, (ii) they have no rotating parts that require bearings (iii) they are frictionless, (iv) the parts have no wear [24]. Smaller (i.e. MEMS) devices are desirable because they are often much cheaper and enable their use in more applications including micro-satellites, micro-robotics and implantable devices to cure vestibular disorders [24]. The problem is that these MEMS devices exhibit lower performance and higher sensitivity to fabrication errors. The reason these devices exhibit lower performance and are so sensitive to fabrication errors is because they are under-constrained. Several companies have worked for decades and spent millions of dollars on TFG designs that are under-constrained [25]. The synthesis of design that eliminates these problems is demonstrated herein.

Tuning fork gyroscopes operate on the principle of the Coriolis acceleration. If a mass moves with velocity $v$, and the reference frame rotates by an amount $\Omega$, then the mass will experience a Coriolis acceleration $a_C$ according to Equation (1.1). An example of a TFG is shown in Figure 1.7.

$$a_C = 2\Omega \times \vec{v} \tag{1.1}$$

The tuning fork gyroscope moves each mass with a velocity $v$ over a finite range by vibrating each mass back and forth with electrostatic comb drives. This mode shape is called the “drive” mode. The masses are mounted on springs, therefore the Coriolis force causes a relative displacement. The mode shape where one mass moves in the +Y direction and another mass moves in the –Y direction (as shown in Figure 1.7) is called the “sense” mode. The displacement is measured by relating it to the resulting capacitance change. Thus, the capacitance change gives a displacement which yields the force exerted by the springs, which gives the Coriolis acceleration, which (when divided by the velocity) gives the orthogonal angular rate [24], [25].
TFGs typically have (at least) two masses that are driven to move in coaxially in opposite directions to mechanically filter out any disturbance accelerations. That is, if the masses were driven to move in the same direction (or if there were only one mass), then the resulting Coriolis force would not be differentiable from a disturbance force (e.g. the gyroscope falling under the influence of gravity). Thus, the flexure design rules illustrated herein impact MEMS TFGs because directionality is a critical functional requirement.

1.2.1 Fundamental Issues

There are two inherent issues with state-of-the-art TFGs ([24], [26]–[65]) that preclude practical use: (i) TFGs have undesired modes in close proximity to useful modes with high coupling between desired and undesired modes, and (ii) TFGs have been designed as “springs” rather than bearing elements. These TFGs are under-constrained, and therefore sensitive to disturbance accelerations and undesired rotations (e.g. rotations about X and Z in Figure 1.7).
Useful information from these sensors may only be obtained if they are manufactured to tight and expensive tolerances. Structural improvements have been shown to decrease sensitivity to undesirable modes [24], [29], [32], [34], [63]. Chapter 3 demonstrates how to use flexural transmission design rules that are built on precision design process/principles [9], [12], [66] to yield a design without these inherent problems.

A TFG is driven at or near resonance to minimize energy requirements. Thus, mode ordering is critical to performance. Mode shapes where the proof masses move in the same direction must be >10% from the operating resonance frequency, or they will distort the sensor’s signal beyond utility [25]. TFG designs that exhibit coupled motions and decoupled mode shapes are shown in Figure 1.1 and Figure 3.4. The motion for the drive and sense mode shapes are kinematically decoupled. The drive and sense modes may be tuned independently by changing the flexures responsible for their specific kinematics. These designs have multiple advantages: (i) a ~5x increased separation of desired and undesired modes (ii) decoupled mode shapes, and (iii) a 4x decrease in sensitivity to fabrication errors. This TFG may retain a 2½D geometry and may therefore be fabricated via micromilling/electrodischarge machining for most metals or deep reactive ion etching for silicon.

1.2.2 Inertial Navigation

Inertial navigation uses sensors to calculate the position, orientation and velocity of an object relative to a known starting point (also known as dead reckoning). “Dead” (deduced) reckoning calculates the current position relative to a known starting point by using accelerometers to acquire the acceleration, which is then integrated twice with respect to time. Gyroscopes determine the angular orientation of the acceleration vectors with respect to an inertial frame. Gyroscopes measure a rotation rate, which is integrated with respect to time to give the angular orientation.

1.2.3 Gyroscope Performance Specifications

Errors in the measured rotation rate lead to orientation errors that increase over time. Thus, the accuracy of these inertial sensors must be as high as possible to mitigate the position and orientation errors. There are some cases, however when the accuracy of these sensors does not need to be high (e.g. rollover detection of automobiles), so there are different grades of
gyroscopes: inertial grade (highest accuracy), tactical grade (medium accuracy), and rate grade (low accuracy). Several gyroscope performance parameters are listed in Table 1.3.

<table>
<thead>
<tr>
<th>Grade</th>
<th>Angle Random Walk</th>
<th>Bias Drift</th>
<th>Scale Factor Accuracy</th>
<th>Full Scale Range</th>
<th>Max shock in 1 msec</th>
<th>Bandwidth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inertial</td>
<td>&lt; 0.001</td>
<td>&lt; 0.01</td>
<td>&lt; 0.001</td>
<td>&gt; 400</td>
<td>10^3</td>
<td>~100</td>
</tr>
<tr>
<td>Tactical</td>
<td>0.05-0.5</td>
<td>0.01-10</td>
<td>0.01-0.1</td>
<td>&gt; 500</td>
<td>10^3 - 10^4</td>
<td>~100</td>
</tr>
<tr>
<td>Rate</td>
<td>&gt; 0.5</td>
<td>&gt; 10</td>
<td>0.1-1</td>
<td>50-1000</td>
<td>10^3</td>
<td>&gt;70</td>
</tr>
</tbody>
</table>

IEEE has a standard protocol for testing MEMS TFGs that defines the parameters listed in Table 1.3 and how to measure them [68], which gives the following definitions. The angle random walk is the angular error that results from the angular rate white noise. Bias is defined as the gyro output (averaged over a specified time with specified operating conditions) that has no correlation with an input rotation. Bias instability is the variation in bias characterized by a 1/f power spectral density. The scale factor is the ratio of a change in output to a change in the input. One scale factor parameter is the linearity error: the residual error from a linear fit. Another scale factor specification is nonlinearity, which is the systematic deviation from the straight line that defines the input-output relationship. The operating range is the range of positive and negative angular rates that may be detected without saturation. Shock resistance is the maximum shock that the operating or non-operating device may endure without failure, and conform to all performance requirements after the shock. The bandwidth is the frequency range of the rotation rate that the gyroscope may detect. The linear and angular vibration sensitivity is the ratio of the change in output due to linear and angular vibration about a sensor axis to the amplitude of the vibration causing it [24], [68].

The design parameters that result from these functional requirements are sensitivity to fabrication errors, stiffness, modal separation, damping ratio, and range of motion. The importance of the modal separation is critical as it affects the scale factor accuracy and sensitivity to linear and angular vibration sensitivity. The stiffness affects the nonlinearity and
modal separation. The sensitivity to fabrication errors affects the stiffness. The damping ratio and the range of motion affect the impulse response and therefore the maximum shock the gyroscope can withstand. The bias drift and angle random walk are mostly attributed to temperature changes and other environmental factors [69]. The design of the flexure architecture is preferably symmetric to mitigate errors induced by thermal asymmetry.

### 1.2.4 Other Gyroscopes

There are many different types of gyroscopes that may be used for inertial guidance, perhaps the two most important for comparison to MEMS gyroscopes are spinning mass gyroscopes and ring laser gyroscopes, shown in Figure 1.8.

![Figure 1.8: Other gyroscopes, including a) spinning mass gyroscope (Creative Commons CC-BY-SA-2.5) [70], and b) ring laser gyroscope [71].](image)

Spinning mass gyroscopes use gimbaled structures to hold a spinning mass. As the rotation rate of the frame increases, the angle of the gimbaled structure changes with respect to the frame. They may also be suspended in a fluid to maintain a uniform temperature distribution. Spinning mass gyroscopes are not conducive to a small scale because they rely on bearing elements. Reducing the size of these gyroscopes would require hard bearing elements (e.g. sapphires for ball bearings) that are manufactured to extremely tight tolerances. The
disadvantages of the spinning mass gyroscopes (namely bearing friction and wear) make them not conducive to inertial guidance.

Ring laser gyroscopes operate on the Sagnac effect. A HeNe laser produces two counter-rotating light beams that exhibit an interference shift in the wave pattern when the system rotates. Ring laser gyroscopes eliminate mechanical limitations such as friction and vibration and shock sensitivity. Ring laser gyroscopes are valued for their precision, but are also several times more expensive than MEMS gyroscopes [24].

### 1.3 Impact of MEMS gyroscopes

The impact of MEMS gyroscopes is that they are smaller and cheaper than other gyroscopes, which allows them to be implemented as a sensor in numerous applications. MEMS gyroscopes are typically ~1mm². This allows them to be integrated in applications that would not otherwise be possible, for example cell phones and video games. This size reduction relative to other gyroscopes allows for more room to put additional equipment on devices such as submarines and UAVs. MEMS gyroscopes are much cheaper than higher quality counterparts such as ring laser gyroscopes. A high quality, low cost MEMS gyroscope would allow them to be ubiquitous sensors in portable (e.g. wearable) technology. There are many applications of MEMS gyroscopes, as shown in Figure 1.9 [24].
Figure 1.9: Applications of MEMS gyroscopes. a) Submarines may use gyroscopes for inertial guidance to navigate without GPS [72], b) Unmanned aerial vehicles (UAVs) may use gyroscopes for inertial navigation [73], c) Oil wells use gyroscopes to measure the angle and rate of their extractors [74], d) Automobiles use gyroscopes for rollover detection [75], e) Cell phones use gyroscopes for rolling detection and may be used for navigation without GPS [76] (Creative Commons CC-BY-SA-3.0), f) Video games like the Nintendo Wii [77] use gyroscopes to measure the user input. Images a), b), c), d) and f) are public domain.
1.4 Summary and Thesis Preview

The purpose of Chapter 1 is to introduce the new flexure design rules, review current flexure design rules and apply them to a MEMS gyroscope. Chapter 2 provides the new design rules and several examples illustrating their use. Chapter 2 will also describe the limitations of this theory (e.g. small motions, etc.) The design rules developed in Chapter 2 are implemented in the case study of MEMS TFGs in Chapter 3. Chapter 3 describes the technical details underlying the mechanism design and sensitivity analysis of the MEMS TFG. Chapter 4 shows the results of experimental validation of the sensitivity analysis. Data from a meso-scale prototype compare theory and behavior as well as demonstrate decreased sensitivity to fabrication errors relative to current state-of-the-art TFGs. Chapter 5 provides a summary of the thesis and outline the future work. The future work is to include more experimentation and move beyond the limitations presented in Chapter 2. Appendix A provides a more extensive look into the analysis of MEMS TFGs.
This chapter presents the development of the design theory of flexural transmissions. The limitations of the design theory presented are the same as with FACT [12]: no buckling, small motions, etc., with the exception that this includes directionality. This design approach is intended for designing flexure-based mechanisms that have not been deformed over a finite range. The flexure topology and force transmission mechanism are first designed assuming that it is ideal (i.e. infinite stiffness along its degrees of constraint, and zero stiffness along its degrees of freedom), then the elasto-mechanics are considered. Section 2.1 covers the definitions that are necessary for the design theory. Section 2.2 presents the design approach and illustrates it through several examples. Section 2.3 compares several designs for transmission elements.

2.1 Definitions

The definitions of a stage, input, output, transmission ratio, axis, and transmission element are necessary for understanding this work. A stage is a rigid body that has a desired motion path. An input is the motion that is supplied to the stage by an actuator described by a twist vector (i.e. a translation, rotation, or a screw). The output is the motion of a stage that results from the input motion, also described by a twist vector. The output stage may or not be the same as the input stage. There may or may not be an intermediate stage that provides the desired output. There may be a plurality of intermediate stages. There may be multiple inputs and multiple outputs (as in [12]). Flexural transmissions connect an output motion to a given input motion. A transmission ratio is the ratio of output motion to input motion. These motions may be of different types, so the transmission ratio is defined as the magnitude of the output twist vector divided by the magnitude of the input twist vector. The output motion may be in the
opposite direction of the input motion, which is accounted for by a plus sign or a minus sign in the transmission ratio.

Table 1.1 lists the possibilities for an input, intermediate stage, an output type, transmission ratio, and whether the input and output are along the same axis. A block diagram and an ordered set give the abstract representation of a flexure design. The complete ordered set is given in the same order as listed in Table 1.1: {\{input type(s)\}, \{intermediate type(s)\}, \{output type(s)\}, \{transmission ratio (TR)\}, and \{axis (or axes)\}}. The braces encompass the ordered set. The brackets represent subsets that are used in the case of multiple inputs, outputs or intermediate types. If there are no brackets, then there it is assumed that there is only one element in that subset. If the number of arguments in the set is incomplete, then the missing entries are allowed to be the any of the possibilities in Table 1.1. The “T, R, and S” in Table 1.1 represent translation, rotation and screw motions respectively. Screw theory proves that these are all possible types of motion for a 1 degree of freedom system [11]. These motions are shown in Table 2.1, along with the FACT representation, flexural examples, and the block diagrams that will be used throughout this work.

The blocks represent the input, output, or intermediate stage. The double-headed arrows in the block diagrams in Table 2.1 illustrate the bi-directionality of the flexures. Block diagrams may also have one direction, indicated by a single-headed arrow. The arrows for intermediate stages are drawn with dashed lines. The most general description would have the block diagrams as cubes and the block diagrams on faces corresponding to the motions in different directions, as shown in Figure 2.1. The general three-dimensional block diagrams will be covered in future work; two-dimensional block diagrams are sufficient for the scope of this research.

**Figure 2.1:** 3D block diagram example: a) Block Diagram, b) Flexural implementation [22] © Elsevier. Reproduced with permission. All rights reserved. (License # 3384560423923).
2.2 **Flexural Transmission Design Approach**

The types of inputs, outputs, intermediate stages, transmission ratios, etc. are listed in Table 1.1, reproduced here for convenience.

### Table 1.1: Flexural Transmissions

<table>
<thead>
<tr>
<th>Input</th>
<th>Intermediate Stage</th>
<th>Output Type</th>
<th>Transmission Ratio (TR)</th>
<th>Axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>None</td>
<td>T</td>
<td>TR &gt; 0</td>
<td>Same</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>R</td>
<td>TR &gt; 0</td>
<td>Same</td>
</tr>
<tr>
<td>R</td>
<td>R</td>
<td>S</td>
<td>TR = 0</td>
<td></td>
</tr>
<tr>
<td>S</td>
<td>Rigid</td>
<td>None</td>
<td>TR &lt; 0</td>
<td>Off</td>
</tr>
</tbody>
</table>
2.2.1 Motion Examples

The designs that may be constructed from this table will be illustrated in several examples. There are some basic building blocks that will be used to construct higher order concepts that will include multiple intermediate stages. Four distinct examples, \{T, none, T, TR > 0, off\}, \{T, none, R, TR < 0, off\}, \{R, none, R, TR < 0, off\} and \{T, S, R, TR < 0, same\}, are shown in Figure 2.2.

![Diagram showing basic building block examples](image)

Figure 2.2: Basic building block examples a) \{T, none, T\}, b) \{T, none, R\}, c) \{R, none, R\}, d) \{T, S, R\} [12] © Elsevier. Reproduced with permission. All rights reserved. (License # 3380581322285)

Rigid intermediate stages are typically used to extend the length of the stage, or approximate another motion as shown in Figure 2.3. Using intermediate stages allows the construction of parallel flexure systems that mimic the kinematics of serial flexure systems [78].
Figure 2.3: Rigid intermediate stage example a) Undeformed flexure, both ends and the central post are ground, b) Translation input turns into an approximate translation output in one axis, c) Translation turns into an approximate translation output along a different axis.

Transmission ratios of zero correspond to degenerate cases. In general, if the flexural transmission transmits a wrench that lies in the constraint space of the output, then the output does not move (i.e. TR=0). For example, \{T, S, T, TR<0\} cannot exist because the screw adds rotation that would be constrained by the output stage. Thus, if the input and output are the same and the intermediate stage is a screw, then there need to be intermediate stages of the opposite type surrounding the screw (i.e. either \{T, R, S, R, T\} or \{R, T, S, T, R\}). Another example is \{T, rigid, T, TR<0, same\}. The rigid intermediate stage transmits the motion from the input stage directly to the output stage, so the output stage cannot move along the same axis in an opposite direction. The output type “none” corresponds only to a transmission ratio of zero, which corresponds to inputs and outputs that are disconnected, or connected in a degenerate way, as shown in Figure 2.4.

Figure 2.4: Degenerate flexural transmissions (TR=0) examples: a) Degenerate case because the flexural transmission is only along the degree of constraint of the output. b) Degenerate case because the input and output are disconnected.
2.2.2 Combining Transmissions

This subsection shows how to combine several flexural transmissions into a single train with decoupled mode shapes. Decoupling is defined here as a ratio of the off-diagonal terms of the transfer function matrix to the on-diagonal terms. The general procedure is to design the bi-directional flexure stage for the input and output as disconnected pieces with motion in their respective axes, choose an intermediate stage (if necessary, or choose multiple intermediate stages) to connect them such that the desired transmission ratio is achieved. The flexure system that gives the necessary motion for each stage may be designed using any technique (i.e. they do not necessarily need to be designed from FACT). This procedure is done recursively on subsets of the inputs, outputs, and intermediate stages for multiple inputs and outputs.

For one degree of freedom (DOF), this technique may be illustrated with an example of “flexural gears,” i.e. a flexure with an input rotation and output rotation that exhibits conjugate action. First the requirements of the flexural gears must be stated in terms of the ordered set: \{R, R, TR<0, off\}. The block diagrams may be drawn, the stages may be designed independently, and the intermediate stage may be chosen with the ordered set. This procedure is visually outlined in Figure 2.5. The designer may now choose one of the following designs, or develop more that involve more intermediate stages.

A more complicated example involves multiple degrees of freedom. This example builds on the flexural gears, \{R, R, TR<0,off\}, by including the added requirement that they also translate in opposite directions off-axis, \{T, T, TR<0, off\}, to make a concatenated ordered set of \{[R, T], [R, T], [TR<0, TR<0], [off, off]\}. The procedure for multi-degree of freedom (MDOF) systems is to break the overall system down into subsystems and add stages in series with bi-directional flexures that do not interfere with the degrees of freedom of the other subsystem.
Figure 2.5: The output rotates in the opposite directions of the input, creating “flexural gears.” This example includes a) the block diagram of desired motion, b) the bi-directional block diagram (left) for independent rotating stages (right), c) the block diagram of \{R, none, R, TR<0, off\} (left), flexural embodiment (middle), 1st mode shape (right, the colors represent resultant displacement), d) the block diagram of \{R, T, R, TR<0, off\} (left), flexural embodiment (middle), 1st mode shape (right), e) the block diagram of \{R, R, R, TR<0, off\} (left), flexural embodiment (middle), 1st mode shape (right), and f) the block diagram of \{R, S, T, S, R, TR<0, off\} (left), flexural embodiment (middle), 1st mode shape (right).
Figure 2.6: MDOF Example {{R, T}, [R, T], [TR<0, TR<0], [off, off]}, including a) the MDOF block diagram, b) the expanded subsystem diagram, c) the flexural embodiments of sub-subsystems, d) the serial synthesis of the sub-subsystems, e) the synthesis of subsystems, and f) the first two mode shapes (the colors represent resultant displacement).
2.3 Flexural Transmission Elements

The goal of the flexural transmission element is to transmit the load from the input to the output without altering the mass or stiffness characteristics, i.e. to approximate a rigid, massless link with pin joints. Sections 2.1 and 2.2 neglect the elasto-mechanics and dynamics of the flexural transmission element, but the transmission element may introduce non-idealities such as buckling and dynamic contributions. Four example designs are shown in Figure 2.7.

![Figure 2.7: Four possible linkage designs. a) A straight beam, b) An elliptical notched flexure, c) A bar with small rotary joints in an “cross” configuration, d) A bar with rotary joints in a “diamond” configuration.](image)

There are several variables that are important for a flexural transmission element. The axial stiffness ($k_a$) needs to be as high as possible to mitigate the transmission loss. The volume ($V$) needs to be minimal (ideally massless) to avoid introducing undesired resonant modes near the desired modes. The buckling load ($P_{cr}$) must also be considered, as the load is not effectively transmitted if the flexure buckles. The out-of-plane stiffness ($k_{op}$) is also important as it affects the constraint quality [79]. The flexural transmission elements shown in Figure 2.7 are linkages with effective torsional spring stiffness at each joint ($k_t$). The stiffness torsional springs need to be minimized to attenuate the additional stiffness that the flexural transmission element introduces. The linkage length ($L_0$) should be as long as possible to minimize the angular change that results in energy storage in the torsion springs ($k_t$).
These variables of interest are combined into independent dimensionless variables via the Buckingham Π Theorem, starting with Equation (2.1). The $D$ matrix contains the dimensions of interest, namely Newtons and meters. $M$ is the dimensional matrix. The rows of $M$ correspond to the dimensions and the columns correspond to the variables of interest. For example, $k_a$ is a stiffness with units of N/m, so the corresponding column is $[1 \ -1]^T$; $V$ has units of m³, so $[0 \ 3]^T$ is its corresponding column, etc.

$$
\begin{bmatrix}
  k_a & k_{op} & L_0 & k_{1t} & V & P_{cr}
\end{bmatrix}^T = \begin{bmatrix} N & m \ \end{bmatrix} \begin{bmatrix}
  1 & 1 & 0 & 1 & 0 & 1 \\
  -1 & -1 & 1 & 1 & 3 & 0
\end{bmatrix}_D
$$

The independent dimensionless variables are found from the nullspace of the matrix $M$, given by Equation (2.2).

$$
N(M) = c_1 \begin{bmatrix}
  -1 \\
  1 \\
  0 \\
  0
\end{bmatrix} + c_2 \begin{bmatrix}
  -1 \\
  0 \\
  -2 \\
  1
\end{bmatrix} + c_3 \begin{bmatrix}
  0 \\
  0 \\
  -3 \\
  1
\end{bmatrix} + c_4 \begin{bmatrix}
  -1 \\
  0 \\
  0 \\
  1
\end{bmatrix}
$$

The independent dimensionless variables are shown in Table 2.2.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Name</th>
<th>Desire</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{k_{op}}{k_a}$</td>
<td>$C$</td>
<td>$\rightarrow 1$</td>
<td>Constraint factor</td>
</tr>
<tr>
<td>$\frac{k_t}{k_a L_0^2}$</td>
<td>$\kappa$</td>
<td>$\downarrow$</td>
<td>Bending stiffness over axial stiffness</td>
</tr>
<tr>
<td>$\frac{V}{L_0}$</td>
<td>$\psi$</td>
<td>$\downarrow$</td>
<td>Volume fraction of length</td>
</tr>
<tr>
<td>$\frac{P_{cr}}{k_a L_0}$</td>
<td>$\chi$</td>
<td>$\rightarrow 1$</td>
<td>Buckling Load Factor</td>
</tr>
</tbody>
</table>
A combination of dimensionless parameters that captures the desired behavior of all of the variables is a dimensionless parameter called $\lambda$, given in Equation (2.3).

$$\lambda = \frac{\chi}{\kappa^3 C \psi} = \frac{k_\alpha k_{opp} L_0^8 P_c}{k_t^3 V}$$ (2.3)

The torsional stiffness over axial stiffness ($\kappa$) had to be cubed to allow the axial stiffness term to move to the numerator. This allows $\lambda$ to capture all of the desired behaviors: the axial stiffness, out-of-plane stiffness, linkage length, and buckling load need to be maximized, while the torsional stiffness and the volume need to be minimized. This linkage parameter $\lambda$ should be maximized, which indicates the flexural transmission design that is the best at approximating a rigid massless link from a given set of options. The following subsections compare the designs presented in Figure 2.7 in terms of $\lambda$. The out-of-plane thickness and the overall length are constant across all of the transmission elements in this comparison.

### 2.3.1 Straight beam

The straight beam is a simple case that is used throughout the preceding sections. The nomenclature for the straight beam is illustrated in Figure 2.8.

![Figure 2.8: Straight Beam Nomenclature.](image)

The design parameters that may be tuned are the width ($t$), the length ($L$) and the out-of-plane thickness ($b$). The out-of-plane thickness and the overall length are constant across all of the transmission elements in this comparison. The only free parameter for the straight beam is the width. The linkage length $L_0$ for a beam is simply the beam length times $\gamma = 0.85$, shown in Equation (2.4) [7].
\[ L_0 = \gamma L \]  
\[ (2.4) \]

The torsional spring constant of a beam is given in Equation (2.5) [7].

\[ k_t = 2\gamma K_\theta \frac{Ebt^3}{12L} \]  
\[ (2.5) \]

The axial stiffness is given by Equation (2.6).

\[ k_a = \frac{Ebt}{L} \]  
\[ (2.6) \]

The out-of-plane stiffness is given by Equation (2.7).

\[ k_{op} = \frac{Etb^3}{L^3} \]  
\[ (2.7) \]

The volume of the linkage is given by Equation (2.8).

\[ V = Lbt \]  
\[ (2.8) \]

The critical buckling load is given by Equation (2.9), where \( K \) is a constant that depends on the end conditions according to Table 2.3. The end conditions considered are fixed-guided.

**Table 2.3: Effective length coefficients for an Euler column buckling with various end conditions** [80]

<table>
<thead>
<tr>
<th>End condition</th>
<th>( K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pinned-Pinned</td>
<td>1</td>
</tr>
<tr>
<td>Fixed-Fixed</td>
<td>0.5</td>
</tr>
<tr>
<td>Fixed-Guided</td>
<td>1</td>
</tr>
<tr>
<td>Fixed-Free</td>
<td>2</td>
</tr>
<tr>
<td>Fixed-Pinned</td>
<td>0.707</td>
</tr>
</tbody>
</table>
This all leads to a single equation for $\lambda$ as a function of width, given in Equation (2.10).

$$\lambda = \frac{18\gamma\pi^2 b}{K^2 K_\theta^3 t} \left( \frac{L_0}{t} \right)^4$$  \hspace{1cm} (2.10)

Equation (2.10) suggests that to maximize $\lambda$ while keeping the thickness and the length constant, the width must be minimized. The minimum width is governed by manufacturing constraints and the buckling load.

![Figure 2.9: Straight Beam $\lambda$ as a function of the beam width $t$, plotted on a log-log scale the slope of -5 indicates that $\lambda \sim 1/t^5$.](image)

### 2.3.2 Elliptical Notched Flexure

A notched flexure and its associated nomenclature are shown in Figure 2.10. The stiffness characteristics are given in [2]. The ratio of the major and minor axes is given by Equation (2.11).

$$\epsilon = \frac{a_x}{a_y}$$ \hspace{1cm} (2.11)
The ratios of the thickness of the notch to the major and minor diameters are given by Equations (2.12) and (2.13).

\[
\beta_x = \frac{t}{2a_x} \tag{2.12}
\]

\[
\beta_y = \frac{t}{2a_y} \tag{2.13}
\]

The relation of the radius of the notch to the major and minor axes is given in Equation (2.14).

\[
R = \frac{a_y + \epsilon a_x}{2} \tag{2.14}
\]

The sweep angle is given by Equation (2.15).
\[ \theta = 2 \sin^{-1} \left( \frac{a_x}{R} \right) \]  

The area of the circular segment that was removed is given by Equation (2.16).

\[ A_{\text{circular segment}} = \frac{R^2}{2} (\theta - \sin \theta) \]  

Thus the volume of interest is given by Equation (2.17).

\[ V = b \left( Lh - 2R^2 (\theta - \sin(\theta)) \right) \]  

The stiffness equations are more complicated, so first the functions \( f, g, H \) are defined in Equations (2.18) - (2.20) [2].

\[ f(\beta) \equiv \left[ \frac{1}{2\beta + \beta^2} \right] \left[ \frac{3 + 4\beta + 2\beta^2}{(1 + \beta)(2\beta + \beta^2)} + \frac{6(1 + \beta)}{(2\beta + \beta^2)^{3/2}} \tan^{-1}\left( \sqrt{\frac{2 + \beta}{\beta}} \right) \right] \]  

\[ g(\beta) \equiv \frac{2(1 + \beta)}{\sqrt{2\beta + \beta^2}} \tan^{-1}\left( \sqrt{\frac{2 + \beta}{\beta}} \right) - \frac{\pi}{2} \]  

\[ H(\beta) \equiv 2 + 2\beta + \frac{\pi}{2} (1 + 4\beta + 2\beta^2) - 4(1 + \beta) \sqrt{2\beta + \beta^2} \tan^{-1}\left( \sqrt{\frac{2 + \beta}{\beta}} \right) \]  

The torsional stiffness is given by Equation (2.21).

\[ k_t = K_{\theta_x M_x} = \frac{2Eba_x^2}{3\varepsilon^3 f(\varepsilon x)} \]  

The axial stiffness of just the notch is given by Equation (2.22).

\[ k_{a,1} = K_{\delta_x F_x} = \frac{Eb}{g(\varepsilon x)\varepsilon} \]
The axial stiffness of the bar is given by Equation (2.23).

\[ k_{a,2} = \frac{Ebh}{L - 4a_x} \]  

(2.23)

The total axial stiffness is given by Equation (2.24).

\[ k_a = \left( \frac{2}{k_{a,1}} + \frac{1}{k_{a,2}} \right)^{-1} \]  

(2.24)

The out-of-plane stiffness is given by the inverse of the sum of three compliances: the bending compliance of the notch, the shear compliance of the notch [2], and the bending compliance of the link, given in Equation (2.25).

\[ k_{op} = \left( \frac{E b^3}{12a_x^2 \epsilon \left[ g(\epsilon \beta_x) + \frac{H(\epsilon \beta_x)}{2} \right]} + \frac{G b}{\epsilon g(\epsilon \beta_x)} \right)^{-1} + \left( \frac{E h b^3}{L^3} \right)^{-1} \]  

(2.25)

The critical buckling load for this structure may be found by energy methods or finite element analysis, but the notched flexure may be approximated as a rigid rod supported by torsion joints. The critical buckling load for a rod supported by two torsional joints is given by Equation (2.26).

\[ P_{cr} = \frac{2k_t}{L_0} \]  

(2.26)

The combination of these allows for \( \lambda \) as a function of \( t \), the critical dimension for various values of \( \epsilon \), is shown in Figure 2.11. This figure essentially shows that by making the notch larger, \( \lambda \) gets smaller. As the notch becomes a greater fraction of the length \( \lambda \) decreases. As \( \epsilon \) decreases, the notch takes on a larger fraction of the overall width, which also causes \( \lambda \) to decrease. The figure also shows that the \( \lambda \sim 1/t^5 \), which is similar to what was found for a straight beam. As \( \epsilon \) increases to infinity, the notched flexure becomes a beam flexure, which suggests that the lines should tend towards the line shown in Figure 2.9. This discrepancy could be due to the approximation given by Equation (2.26): as \( \epsilon \) increases, the middle portion decreases and becomes less rigid, which decreases the accuracy of the approximation.
Figure 2.11: Notched Flexure $\lambda$ as a function of the $t$ for various values of $\varepsilon$, plotted on a log-log scale. The slope is $\sim -5$, which means $\lambda \sim 1/t^5$. 
2.3.3 Flexural Pivot

A typical flexural pivot is shown in Figure 2.12. This offers the advantage that the flexures enable the linkage pivots to be projected where desired.

![Figure 2.12: Flexural Pivot Nomenclature.](image)

The torsional stiffness of this joint is given by Equation (2.27), where \( n \) is the number of flexure elements (Figure 2.12 has \( n = 2 \)) [24].

\[
k_t = 4n \frac{EbI^3}{12l} \left[ 3 \left( \frac{r}{l} \right)^2 + 3 \left( \frac{r}{l} \right) + 1 \right]
\]  

Equation (2.28) gives the radius about which the torsion spring rotates.

\[
r = \frac{\Delta x}{2 \cos \theta}
\]

The width of the bar is modeled as in Equation (2.29).
The length of the flexures ($l$) may be varied, but for a fixed overall length ($L$), and a fixed angle ($\theta$), the maximum length ($l_{\text{max}}$) is given by Equation (2.30).

$$l_{\text{max}} = \frac{L}{2 \sin \theta}$$  \hspace{1cm} (2.30)

This allows the linkage length $L_0$ to be given by Equations (2.31) and (2.32) for the “cross” and the “diamond” shape respectively. Equation (2.31) illustrates the interesting property of the “cross” shape, which is that the linkage length may be smaller or larger than the actual length of the transmission element. This occurs because the point about which the torsion spring rotates may be projected. The “diamond” shape is configured such that the linkage length is greater than or equal to the actual length. The trade-off is that the torsional stiffness also increases.

$$L_0 = |L - 2(r + l) \sin \theta|$$  \hspace{1cm} (2.31)

$$L_0 = L + 2(r + l) \sin \theta$$  \hspace{1cm} (2.32)

The axial stiffness of just the torsional joint is given by Equation (2.33).

$$k_{a,1} = 2 \frac{Ebt}{l} \sin \theta + 2 \frac{Ebt^3}{l^3} \cos \theta$$  \hspace{1cm} (2.33)

The axial stiffness of the bar is given by Equation (2.34).

$$k_{a,2} = \frac{Ebh}{L - 2l \sin \theta}$$  \hspace{1cm} (2.34)

The total axial stiffness of the flexural transmission is given by Equation (2.35).

$$k_a = \left( \frac{2}{k_{a,1}} + \frac{1}{k_{a,2}} \right)^{-1}$$  \hspace{1cm} (2.35)
The total volume is given by Equation (2.36).

\[ V = (L - 2l \cos \theta) \left( \Delta x + 2 \frac{t}{\sin \theta} \right) b + 4ltb \]  

\hspace{1cm} (2.36)

Equation (2.37) gives the out-of-plane stiffness that was derived using Castigliano’s Theorem.

\[ k_{op} = \frac{4Eb^3}{\frac{1}{t} (L^3 + (l \sin \theta)^3 - (L - l \sin \theta)^3) + \frac{1}{\Delta x + \frac{2t}{\sin \theta}} ((L - l \sin \theta)^3 - (l \sin \theta)^3)} \]  

\hspace{1cm} (2.37)

The buckling load is given by Equation (2.38).

\[ P_{cr} = \frac{2k_t}{L_0} \]  

\hspace{1cm} (2.38)

Figure 2.13 shows plots for all of the tunable parameters. The figure shows that there are some values that allow the flexural pivot to be greater than a beam at imitating a massless, rigid link with pin joints. The general trends show that the gap \( \Delta x \) should be as small as possible, the flexure thickness \( t \) should be as small as possible, the flexure length \( l \) should be as long as possible, and the angle should be \( \sim 51.5^\circ \) for a 5x improvement in \( \lambda \).

2.3.4 “Diamond” configuration

The “Diamond” configuration exhibits many of the same characteristics as the “cross” configuration. All of the stiffness formulas are the same, but the geometry is slightly different, as described by Equations (2.32) and (2.39).

\[ h = \Delta x + \frac{2t}{\sin \theta} + 2l \cos \theta \]  

\hspace{1cm} (2.39)

The results of computing \( \lambda \) for a parameter sweep showed that the values of \( \lambda \) were lower than even the notched flexure.
\[ \theta = 51.5^\circ \]

\[ \theta = 80^\circ \]

\[ \theta = 20^\circ \]

\[ \theta = 60^\circ \]

\[ \Delta x / L = 0.1 \]

\[ \gamma \]

\[ t/L \]

\[ l / l_{\text{max}} = 1 \]

\[ l / l_{\text{max}} = 0.75 \]

\[ l / l_{\text{max}} = 0.5 \]

\[ l / l_{\text{max}} = 0.25 \]

\[ l / l_{\text{max}} = 0.1 \]
Figure 2.13: Notched Flexure $\lambda$ as a function of the $t$ for various values of $l$, $\theta$, and $\Delta x$, plotted on a log-log scale
2.3.5 Optimization constraints

The goal of this section is to impose constraints to find the best shape from the options presented. The transmission element needs to transmit a load without buckling, so $P_{cr}$ is constant across the shapes. Also they are the same material (i.e. the same Young’s Modulus $E$), and same thickness ($b$) and length ($L$). They also have the same manufacturing limits on the minimum feature size ($w_{mfg}$). Equation (2.10) shows that $\lambda \sim 1/t^5$ for the straight beam. This implies that the optimal $\lambda$ occurs when $t$ is infinitesimally small. The buckling load $P_{cr} \sim t^3$, so the infinitesimally thin beam will not be able to transmit any load without buckling. Thus, the optimal $\lambda$ is given by the $\lambda$ that occurs at the maximum of the minimum manufacturing width and the buckling width as shown in the following equations.

$$w_{buckling} = \sqrt[3]{\frac{12P_{cr}K^2L^2}{\pi^2 Eb}}$$

$$\lambda_{optimal, beam} = \lambda(w_{min}) = \lambda(\max(w_{mfg}, w_{buckling}))$$

The procedure for finding the optimal $\lambda$ depends on these limits as well as the function of interest. There may also be a maximum feature size ($w_{max}$). For a beam, once $w/L > 1/10$, then the beam equations presented here break down and are no longer appropriate. Thus, the optimal $\lambda$ is generally found by Equation (2.42).

$$\lambda_{optimal} = \max_{w \in [w_{min}, w_{max}]} \lambda(w)$$

Typically $\lambda(w)$ is monotonically decreasing because the decreased torsional stiffness is more important than the increased axial and out of plane stiffnesses. Thus $w_{max}$ typically does not need to be calculated.
2.4 Summary and Contributions

This chapter presented develops design rules for designing flexural transmissions that include directionality and decoupled mode shapes. The design rules are essentially to separately design the bi-directional motion of the input and output stage(s), and then connect them with an intermediate stage that has a flexural transmission element. The quality of the flexural transmission element is explored through a linkage parameter that was derived from the Buckingham Π theorem. The parameter provides a means of comparing and quantitatively deciding on a design and tuning the design for optimal performance.
MEMS TFGs provide an excellent case study for the application of the flexural transmissions discussed in Chapter 2. Although the final product is meant to be a MEMS device, this case study is investigated at the meso-scale. This chapter provides analytical models of the flexures for the drive and sense modules. These models are then validated with finite element analysis (FEA).

3.1 Functional Requirements and Constraints

This section covers functional requirements of the gyroscope and the physical parameters that govern them. This enables the design of a tuning fork gyroscope that has decoupled mode shapes and coupled mass motions.

3.1.1 Required Kinematics

The block diagram of the required mass motions is given Figure 3.1: . The kinematic requirements result from the TFG physics described in Chapter 1.

![Figure 3.1: Required kinematics. a) Block diagram b) Expanded subsystem diagram.](image)
3.1.2 Dynamics: Mode Separation

The mode ordering is critical to the performance of the gyroscope. The desired frequency of the drive mode is 18-20kHz to filter out acoustic noise from the environment. Then the frequency of sense mode needs to be 5-15% away from drive mode. The gain is too small to be measured if the sense mode is further than 15% away in either direction. Making the sense mode less than 5% away from the drive mode makes it too sensitive to frequency shifts due to fabrication errors.

3.1.3 Fabrication Limits

The final flexure configuration must be at most 2.5 D to be amenable to MEMS. This means that the flexure systems that are constructed must be planar. The constraint space for a screw is a circular hyperboloid (3D), so there must be no screws on the input, output, or intermediate stage(s) in this mechanism. The device is meant for MEMS, so the final device footprint must be ~1mm². If the device is smaller, then the fabrication tolerances become an issue. The typical tolerance that sets the minimum feature size for MEMS is about 1 micron. The aspect ratio that may be micromachined is ~10:1, which sets the aspect ratios of the flexures.

3.1.4 Elasto-mechanics

A functional requirement of the flexures is that they deform elastically; the maximum stress is less than or approximately equal to 1/3 of the yield strength.

3.2 Concept generation based on design rules

The design rules presented in Chapter 2 may be used to rapidly synthesize flexures that have decoupled mode shapes and coupled stage motions.

3.2.1 Drive Module

The drive module requires the two of the stages to translate coaxially in opposite directions. FACT shows that translation requires the constraint space of two parallel planes. The design rules suggest that the stages are designed independently, and then linked together with an intermediate stage or multiple stages.
3.2.2 Sense Module

The sense module requires two stages to translate on different axes in opposite directions. Figure 3.3 shows two different concepts in their natural and deformed configurations.

Figure 3.2: Drive Flexure Concepts: a) Drive concept 1, b) Drive concept 2. From left to right: the block diagrams, the flexures, and the first mode shapes (the color in the mode shapes represent resultant displacement).
3.2.3 Module Synthesis

The modules were combined to form three example designs shown in Figure 3.2. These concepts may be combined serially to maximize the decoupling of the mode shapes. The drive module shown in Figure 3.2a was chosen due to its large open area in the center. The coupling flexures were chosen such that they were either in the overlap of the modules’ constraint space (Figure 3.2b and Figure 3.2c) or they were such that they did not interfere with the constraint space of each other (Figure 3.2a). The design chosen was Figure 3.2a because the intermediate masses could be made relatively small (as opposed to making the “speed holes” in the sense module intermediate stage in Figure 3.2b), the device would be easier to fabricate than Figure 3.2c and the device would be easier to model as shown in the subsequent section.
Figure 3.4: Modally decoupled TFG Concepts generated based on design rules.
3.3 Kinematic Modeling

This section develops a model for the stiffness of the drive and sense modules. The goal of this section is to convert the flexure shown in Figure 3.5a into a lumped parameter model of equivalent compression/extension springs and torsion springs (shown in Figure 3.5b), then reduce the lumped parameter model in Figure 3.5b to one equivalent spring, shown in Figure 3.5c. The symmetry of the drive and sense modules permit the modeling of one corner of the flexure as sufficient for modeling the entire flexure.

The structure shown in Figure 3.5b is a nonlinear arrangement of springs that cannot be broken down into series/parallel components. Applying a force $F$ to block 1 causes the mass to displace by an amount $\delta_1$, as well as compressing the linkage flexure by an amount $\delta_a$, and pushing block 2 up by amount $\delta_2$. The angle formed by the linkage and block 1 ($\theta_0$ in Figure 3.5b) changes as the blocks displace, which changes the direction of the force applied to block 2. This angular change also displaces the torsion springs. Rigid, frictionless walls shown in blue in Figure 3.5b represent the ideal constraints imposed by the flexures.

The stiffness of the overall structure is derived from the principle of virtual work. First, the kinematics must be modeled so the linear and angular displacements are put into the energy function. Then, the energy may be differentiated to give the force as a function of displacement. The force may be differentiated to give the stiffness as a function of displacement.
The energy of the system is found from the virtual work of displacing block 1 by an amount $\delta_1$, and is given in Equation (3.1).

$$U = \frac{1}{2} k_1 \delta_1^2 + \frac{1}{2} k_2 \delta_2^2 + \frac{1}{2} k_a \delta_a^2 + \frac{1}{2} \left( \bar{k}_1 + \bar{k}_2 \right) \Delta \theta^2$$  \hspace{1cm} (3.1)

The stiffness of the overall flexure is given by the second derivative of the energy, which shown in Equation (3.2).
The kinematics of the flexure is reduced to the geometry shown in Figure 3.6.

\[ K = \frac{\partial^2 U}{\partial \delta_1^2} = k_1 + k_2 \left[ \delta_2 \frac{\partial^2 \delta_2}{\partial \delta_1^2} + \left( \frac{\partial \delta_2}{\partial \delta_1} \right)^2 \right] + k_a \left[ \delta_a \frac{\partial^2 \delta_a}{\partial \delta_1^2} + \left( \frac{\partial \delta_a}{\partial \delta_1} \right)^2 \right] + \left( \tilde{k}_1 + \tilde{k}_2 \right) \left[ \Delta \theta \frac{\partial^2 \Delta \theta}{\partial \delta_1^2} + \left( \frac{\partial \Delta \theta}{\partial \delta_1} \right)^2 \right] \]  

(3.2)

Figure 3.6: Kinematics of the flexure.

It is sufficient to limit the displacement of the blocks such that the bottom of the linkage does not travel beyond block 2 for the purposes of this work. That is, \( \theta_0 \) is between 0 and 90 degrees. These limitations are described by Equations (3.3) and (3.4). The displacements are normalized by \( L_0 \), the length of the flexural transmission.

\[ \left( \frac{\delta_1}{L_0} \right)_{\text{max}} = \cos \theta_0 \]  

(3.3)

\[ \left( \frac{\delta_2}{L_0} \right)_{\text{max}} = 1 - \sin \theta_0 \]  

(3.4)
3.3.1 Definitions

The compression of the flexure linkage is defined as $\delta_a$, and Equation (3.5) gives its expression (derived from a force balance).

$$\delta_a = \frac{k_2/k_a}{\sin \theta_0} \delta_2$$  \hspace{1cm} (3.5)

The compressed length of the linkage flexure is given by the $L_1$, defined in Equation (3.6).

$$L_1 \equiv L_0 - \delta_a$$  \hspace{1cm} (3.6)

The angular change is denoted by $\Delta \theta$, and is defined by Equation (3.7).

$$\Delta \theta \equiv \theta - \theta_0$$  \hspace{1cm} (3.7)

The axial stiffness of the linkage of the flexure is often several orders of magnitude larger than the bearing stiffness of the linear flexure ($k_2$). Thus, the linkage compression $\delta_a$ is often negligible. The displacement formulas are presented herein for completeness.

3.3.2 Displacements

Equation (3.8) gives the displacement of block 2 as a function of block 1 if the link is assumed to be rigid.

$$\frac{\delta_2}{L_0} = - \sin \theta_0 + \sqrt{1 - \left( \cos \theta_0 - \frac{\delta_1}{L_0} \right)^2}$$  \hspace{1cm} (3.8)

Equation (3.9) gives the displacement of block 2 as a function of block 1 if the link is assumed to be compliant.

$$\frac{\delta_2}{L_0} = \frac{- \left( \frac{k_2/k_a}{\sin \theta_0} + \sin \theta_0 \right) + \sqrt{\left( \frac{k_2/k_a}{\sin \theta_0} + \sin \theta_0 \right)^2 + \left[ 1 - \left( \frac{k_2/k_a}{\sin \theta_0} \right)^2 \right] \left( 2 \frac{\delta_1}{L_0} \cos \theta_0 - \left( \frac{\delta_1}{L_0} \right)^2 \right)}}{1 - \left( \frac{k_2/k_a}{\sin \theta_0} \right)^2}$$  \hspace{1cm} (3.9)
Equation (3.10) gives a limiting case for Equation (3.9).

\[
\lim_{k_2 \to \infty \text{ as } \sin \theta_0} \frac{\delta_2}{L_0} = \frac{2 \frac{\delta_1}{L_0} \cos \theta_0 - \left( \frac{\delta_1}{L_0} \right)^2}{(2 - \sin \theta_0)(1 + \sin \theta_0)}
\]  

The displacements are shown graphically in Figure 3.7 to the limits of \( \delta_1 \) and \( \delta_2 \).

![Graph showing \( \delta_2 \) as a function of \( \delta_1 \) for various initial angles.](image)

**Figure 3.7:** Block 2 displacement as a function of block 1 for various initial angles.

A more complete picture is given by the 3D graph of \( \delta_2 \) as a function of \( \delta_1 \) and \( \theta_0 \), shown in Figure 3.8.
Figure 3.8: Block 2 displacement as a function of block 1 displacement for various initial angles, shown with various views including, a) isometric view, b) top view, c) left view, d) right view.
The displacement of block 2 is nonlinearly related to block 1. Performing a Taylor series expansion of Equations (3.8) and (3.9) about $\delta_1 = 0$ linearizes the model for small displacements. Equation (3.11) gives the linearized model for a rigid link. Equation (3.12) gives the linearized model for a compliant link.

\[
\frac{\delta_2}{L_0} \approx \frac{\delta_1}{L_0} \cot \theta_0 \tag{3.11}
\]

\[
\frac{\delta_2}{L_0} \approx \left( \frac{k_2}{k_a + \sin^2 \theta_0} \right) \frac{\delta_1}{L_0} \tag{3.12}
\]

The percent error in displacement from using the approximation for a rigid link is shown in Figure 3.9. If the displacement is less than 1% of the linkage length, then the error will be less than 2% for an initial angle of 45°. A more complete picture is given by the 3D graph shown in Figure 3.10.

![Error in Displacement](image_url)

**Figure 3.9:** Percent error in displacement by using linear approximations for various initial angles.
Figure 3.10: Percent error in displacement by using linear approximations for various initial angles shown with various views including, a) 3D view, b) top view, c) left view, d) right view.
3.3.3 Angular Changes

Equation (3.13) gives the angular change if the link is assumed to be rigid.

\[ \Delta \theta = \cos^{-1} \left( \cos \theta_0 - \frac{\delta_1}{L_0} \right) - \theta_0 \]  

(3.13)

Equation (3.14) gives a reduced form of the angular change if the link is assumed to be compliant; Equation (3.15) gives the full form.

\[ \Delta \theta = \cos^{-1} \left( \frac{\cos \theta_0 - \frac{\delta_1}{L_0}}{1 - \frac{\delta_1}{L_0}} \right) - \theta_0 \]  

(3.14)

\[
\Delta \theta = \cos^{-1} \left( \frac{\sin \theta_0 \left( \cos \theta_0 - \frac{\delta_1}{L_0} \right)}{\sin \theta_0 + \frac{k_2}{k_a} / \left( \sin \theta_0 + \frac{k_2}{k_a} \right)} - \left( \frac{k_2}{k_a} / \left( \sin \theta_0 + \frac{k_2}{k_a} \right) \right) + \left[ \left( \frac{k_2}{k_a} / \sin \theta_0 + \sin \theta_0 \right)^2 + 1 - \left( \frac{k_2}{k_a} / \sin \theta_0 \right)^2 \right] \left( 2 \frac{\delta_1}{L_0} \cos \theta_0 - \left( \frac{\delta_1}{L_0} \right)^2 \right) \right) - \theta_0
\]  

(3.15)

The angular changes given by Equation (3.13) are shown in Figure 3.11. Figure 3.12 shows a more complete picture of the angular changes. The angular change is nonlinearly related to the displacement of block 1. Taylor expanding Equations (3.14) and (3.15) about \( \delta_1 = 0 \) linearizes the model for small displacements. Equation (3.16) gives the linearized model for a rigid link; Equation (3.17) gives the linearized model for a compliant link. Figure 3.13 shows the percent error as a function of displacement. Figure 3.14 shows the more complete 3D picture.

\[ \Delta \theta \approx \left( \frac{1}{\sin \theta_0} \right) \frac{\delta_1}{L_0} \]  

(3.16)

\[ \Delta \theta_0 \approx \frac{1}{\sin \theta_0} \left[ 1 - \frac{k_2}{k_a} \left( \frac{\cos^2 \theta_0}{\left( \frac{k_2}{k_a} + \sin^2 \theta_0 \right) \left( \frac{k_2}{k_a} \right)} \right) \right] \frac{\delta_1}{L_0} \]  

(3.17)
The percent error in displacement from using the approximation for a rigid link is shown in Figure 3.13. If the displacement is less than 1% of the linkage length, then the error will be less than 2% for an initial angle of $45^\circ$.

Figure 3.11: Angular changes as a function of block 1 displacement for various initial angles.
Figure 3.12: Angular changes as a function of block 1 displacement for various initial angles shown with various views including, a) 3D view, b) top view, c) left view, d) right view.
Figure 3.13: Percent Error in angular changes as a function of block 1 displacement for various initial angles.
Figure 3.14: Percent Error in angular changes as a function of block 1 displacement for various initial angles shown with various views including, a) isometric view, b) top view, c) left view, d) right view.
3.4 Stiffness Modeling

The individual models of the lumped stiffnesses and the linearized displacement models presented in sections 3.3.2 and 3.3.3 yield an equivalent spring expressed in Equation (3.18). The overconstraint in the Drive mode flexures introduces significant strain stiffening that practically prohibits the motions of the flexure beyond one beam width.

\[ K \approx k_1 + k_2 \left( \frac{\cos^2 \theta_0}{k_a + \sin^2 \theta_0} \right) + \left( k_1 + k_2 \right) \left( \frac{1}{L_0 \sin \theta_0} \left[ 1 - \frac{k_2}{k_a} \left( \frac{\cos^2 \theta_0}{k_a + \sin^2 \theta_0} \right) \right] \right)^2 \]  
\[ \text{(3.18)} \]

The stiffness of the beams may be lumped into spring elements as in section 2.3.

3.4.1 Principle of Virtual Work

The principle of virtual work is used to derive the stiffness of nonlinear springs. The transverse stiffness of one fixed-clamped beam has been derived in [17] according to Equations (3.19) - (3.23), with the parameters of \( \gamma = 0.85 \) and \( K_\theta = 2.65 \) [1/rad].

\[ K_T = 2 \frac{\gamma}{L_0} K_\theta (EI)_{\text{bending}} \]  
\[ \text{(3.19)} \]

\[ \Delta L(\delta) = \sqrt{\delta^2 + (\gamma L_0)^2} - \gamma L_0 \]  
\[ \text{(3.20)} \]

\[ A(\delta) = \frac{A_0 L_0 \left( 1 + \frac{\Delta L(\delta)}{L_0} (1 - 2 \nu) \right)}{L_0 + \Delta L(\delta)} \]  
\[ \text{(3.21)} \]

\[ K_A(\delta) = \frac{(EA(\delta))_{\text{axial}}}{\gamma L_0 + \Delta L(\delta)} \]  
\[ \text{(3.22)} \]

\[ F_y(\delta) = \frac{K_A(\delta) \Delta L(\delta) \delta}{\Delta L(\delta) + \gamma L_0} + \frac{2K_T \gamma L_0 \tan^{-1} \left( \frac{\delta}{\gamma L_0} \right)}{(\Delta L(\delta) + \gamma L_0)^2} \]  
\[ \text{(3.23)} \]

These equations are implemented in the model because they account for the strain stiffening in the flexures. The strain-stiffening model is shown in Figure 3.17.
3.4.2 Axial Stiffness

The axial stiffness of the transmission element must be included for the stiffness analysis. The axial stiffness of a deformed beam has been shown [81] to be given by Equation (3.24), and the axial stiffness of a straight beam \( k_{a0} \) is given by Equation (3.25), where \( \Delta x \) represents a transverse initial displacement, \( h \) represents the beam width, \( E \) represents the Young’s modulus, \( A \) represents the cross-sectional area, and \( L \) represents the beam length.

\[
k_a(\Delta x) = k_{a0} \frac{1}{1 + \frac{12}{700} \left( \frac{\Delta x}{h} \right)^2}
\]

(3.24)

\[
k_{a0} = \frac{EA}{L}
\]

(3.25)

The deformation of the transmission element is shown in Figure 3.15. Only \( \delta_1 \) and \( \delta_2 \) are known, but it is convenient to convert these displacements into a combination of a displacement that is normal to the beam and one that is parallel to the beam. The geometry is shown in Figure 3.15.

![Figure 3.15: Axial stiffness modeling, including a) a beam that has been transversely displaced and loaded axially, b) the deformed transmission element, and c) a close-up of the geometry shown in b.](image)

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Equation (3.26) relates $\Delta x$ and $\Delta y$ to $\delta_1$ and $\delta_2$.

\[
\begin{bmatrix}
\Delta x \\
\Delta y
\end{bmatrix} = 
\begin{bmatrix}
\sin \theta_0 & \cos \theta_0 \\
\cos \theta_0 & -\sin \theta_0
\end{bmatrix}
\begin{bmatrix}
\delta_1 \\
\delta_2
\end{bmatrix}
\tag{3.26}
\]

The direction of the axial displacement $\delta_a$ changes direction as the masses displace, so the component of the force is given by Equation (3.27).

$$\delta_a = \Delta y \cos(\Delta \theta)$$  

A force balance for block 2 yields Equation (3.28).

$$F_y(\delta_2) = k_{a0} \left( \frac{\delta_1 \cos \theta_0 - \delta_2 \sin \theta_0}{1 + \frac{3(\delta_1 \sin \theta_0 + \delta_2 \cos \theta_0)^2}{175h^2}} \right) \sin \theta_0$$

The simple case of $F_y(\delta_2) = k_2 \delta_2$ yields a third-order equation in $\delta_2$ given by Equation (3.29).

$$A \delta_2^3 + B \delta_2^2 + C \delta_2 + D = 0$$

$$A = \frac{3k_2 \cos^2 \theta_0}{175h^2}$$

$$B = \frac{3k_2 \delta_1 \sin(2\theta_0)}{175h^2}$$

$$C = k_2 \left( 1 + \frac{3}{175} \left( \frac{\delta_1 \sin \theta_0}{h} \right)^2 \right) + k_{a0} \sin^2 \theta_0$$

$$D = -\frac{1}{2} k_{a0} \delta_1 \sin(2\theta_0)$$

This axial compliance is included in the model of the flexures.

### 3.5 Finite Element Analysis

This subsection covers the finite element analysis of the gyroscope modules. The finite element models verify that the models agree with FEA to within 12%. Finite element analysis was also used to investigate the sensitivity of the gyroscope to manufacturing errors and revealed that the drive mode exhibited a 4x reduction to quadrature error.
3.5.1 Stiffness modeling verification

The goal of this section is to confirm the models with finite element analysis. The drive and sense module may be modeled with quarter models to capture their behavior. First, the appropriate boundary conditions must be imposed. The drive module is composed of flexures that impose a roller-guided boundary condition on the input and output stages of the quarter model. The sense module is composed of flexures that have a free boundary condition on the input and output stages. The FEA and the models are compared in Figure 3.17.

![Figure 3.16: FEA model of a flexure quarter, including a) the converged mesh, and b) the flexure deformed under a load.](image)

The FEA model has a fine mesh on the flexures and a coarse mesh on the ground and stages. The mesh was made increasingly fine until it converged, meaning that making a finer mesh produced the same results within 2%. A force was applied to one stage and the displacement was recorded for increasing force until the onset of plastic deformation. The surface of the ground was held fixed, as well as the two holes in the ground. The appropriate boundary conditions were applied, and then nonlinear FEA was applied.
Figure 3.17: Drive module force vs. dimensionless displacement example, including a) small deformations, i.e. displacements less than 1 beam width ($w$), and b) large deformations with plastic deformation occurring at the highest FEA point plotted.

Sense module force vs. dimensionless displacement example, including c) Small deformations, i.e. displacements less than 1 beam width ($w$), and d) Large deformations with plastic deformation occurring at the highest FEA point plotted.
The drive module exhibits significant strain stiffening that results in a highly nonlinear force vs. displacement curve. The strain stiffening overwhelms the strain softening that occurs due to the kinematics. The maximum error (~12%) between the strain-stiffening model and the FEA results occurs at the onset of plastic deformation. The slight difference between the model and the FEA could be due to the nonlinear material changes that occur as the material deforms.

### 3.5.2 Sensitivity to fabrication errors

Current TFGs are bulk micromachined and then trimmed with a laser to get the beam widths within tolerance. The heads of the lasers may be cocked by up to one degree, and this flexure taper causes the proof masses to undesirably move out-of-plane [25]. Thus, the sensitivity to fabrication errors must be investigated on this device. Figure 3.18 illustrates the taper in the flexures.

![Figure 3.18: 1° wall taper illustration.](image)

The total derivative of capacitance shows that the percent change in capacitance is the negative of the percent change of out-of-plane displacement (when the area is not changing). Equation (3.30) is gives the capacitance of a parallel-plate capacitor, where $\varepsilon$ is the dielectric constant, $A$ is the area of the parallel plates, and $d$ is the separation distance. Equation (3.31) gives the variation of capacitance due to geometry variations. Equation (3.32) expresses Equation (3.31) in terms of percent changes in capacitance as a result of percent changes in geometry.
This sensitivity analysis corresponds to a component of bias, i.e. a change in capacitance when the device is not rotating. Figure 3.19 shows that the out of plane motion is ~2% of the in plane input motion. This is a 4x reduction to sensitivity to quadrature error from the Draper-Honeywell TFG [25].

![Diagram](image)

**Figure 3.19:** FEA for 1° wall taper and 1.5mm input displacement for a ~150mm meso-scale model of the drive model. Block 1 displaces downward (into the page) ~28µm; Block 2 displaces upward (out of the page) ~30µm. Blocks 3 and 4 displace on the order of 1µm.
Figure 3.20 shows the experimental setup to measure the out-of-plane displacements.

![Experimental schematic of measuring sensitivity to quadrature error](image)

**Figure 3.20:** Experimental schematic of measuring sensitivity to quadrature error.

### 3.5.3 Modal Analysis

Modal analysis was performed via FEA for the MEMS TFG. The analysis presented in earlier sections of this chapter was used to generate a design that meets the functional requirements, shown in Figure 3.21. The masses were hollowed out to maximize the ratio of the proof masses to the intermediate masses. The TFG was modeled as being made of silicon. The thickness of the device is $\sim 60 \, \mu m$, the minimum beam thickness is $\sim 6 \, \mu m$, which meets the 10:1 aspect ratio requirement. The footprint of this device is 3mm x 3mm.
The mode shapes of the device are presented in Figure 3.22. Mode 1 and mode 2 are in the 18-20 kHz range (18.001 kHz, and 19.377 kHz respectively). The frequency of mode 2 is 7.6% greater than that of mode 1; this meets the requirement that the drive mode is within 5-15% of the sense mode. The lowest undesirable mode is 49% greater than the drive mode; this meets the requirement that the undesired modes are >10% away from the desired modes.
Figure 3.22: First six mode shapes of TFG design. a) The sense mode, b) the drive mode, c) a spurious mode, d) - f) buckling modes.
### 3.5.4 Range of motion

The range of motion gives the scale factor, which ultimately determines the range of detectable rotation rates. The gyroscope was pushed in the drive direction on both sides by a uniform load. Figure 3.23 shows that a 1µm deflection that induces a stress less than 1/3 of the yield strength.

![Figure 3.23: Range of motion for the drive mode.](image)

**Figure 3.23:** Range of motion for the drive mode.

**Figure 3.24** shows that for a 1µm displacement of the proof masses in the sense mode, the induced stress is less than 1/3 of the yield strength.

![Figure 3.24: Range of motion for the sense mode.](image)
3.6 Summary

Chapter 3 presents a design for a modally decoupled TFG that meets functional requirements. The flexure topology is generated from the design rules, and the kinematics and stiffness are investigated analytically and with finite element analysis. The range of motion is ~1 µm, the ratio of Coriolis mass to sense mass is ~1/2, the quality factor has been reported to be as high as ~10^5, and the ratio of the required inertial grade rotation rate to the sense mode frequency is ~10^{-14}. Thus the scale factor is ~ 0.001 picometers. This means that the capacitive change that needs to be measured is ~0.001aF. This capacitance change presents a difficult challenge, which means that the quality factor should be maximized, the range should be maximized, and the fabrication techniques need to have greater precision.
The purpose of this chapter is to investigate sensitivity to manufacturing defects on displacements and resonant modes.

4.1 Displacement sensitivity to 1° wall taper

A meso-scale version of the drive module was fabricated with 1° wall taper on the flexures to simulate fabrication errors that would be seen when micro-fabricating the gyroscope. The MEMS gyroscope is first bulk micromachined, and then the flexures are trimmed with a laser to achieve the desired thicknesses. The head of the laser may be tilted by up to 1°, which introduces tapered flexure sidewalls. Tapered sidewalls create undesired out-of-plane motions of the proof masses by 8% of the in-plane motion [25].

4.1.1 Instrumentation and setup

The experimental setup was fabricated from 6061-T6 aluminum and is shown in Figure 4.1a. The expanded setup is shown in Figure 4.1b. The input displacements are imparted to the flexure by a micrometer head. Capacitance probes are used to measure the out-of-plane displacement that results from in-plane input displacements. The capacitance probes were held in place with ¼-20 set screws with nylon tips to avoid damaging the probes. The capacitance probe holder is connected to ground via a kinematic coupling with a 1kg weight used as a preload.
Figure 4.1: Taper sensitivity experimental setup, shown a) collapsed, and b) expanded.
The drive module was made on a NMV1500DCG Mori Seiki 5-axis mill shown in Figure 4.2.

Figure 4.2: 5-Axis mill setup, including a) a kinematic coupling used for a fixture, b) the fixture assembly including kinematic coupling preloaded with a zip-tie, the c) stock bolted to fixture plate mounted in the 5-axis mill.

A fixture plate was held in a vise and the stock was bolted to the plate via four bolts. A kinematic coupling was used to repeatedly align the workpiece to the fixture. This allowed the flexure to be made with 1° of taper. The rest of the parts were made on a ProtoTrak CNC Mill.

Great care must be taken when assembling these parts, as any flatness deviation may cause the parts to warp when they are bolted down to the optical table. Typical tolerances when milling a part are $\pm 25.4 \, \mu\text{m}$ ($\pm 0.001"$). These tolerances are unacceptable when attempting to measure 10s of microns, but fabrication techniques that have tighter tolerances are more expensive and time-consuming. The compromise that was reached was to machine the parts on a mill and use shim stock to make up for any planar deviation.
4.1.2 Data Acquisition

The data was acquired with a LabVIEW script that saves the voltages read by the capacitance probes as a text file, and then a MATLAB script converts the voltages into displacements. The displacement data points are gathered from steps in the graph that represent leaving the micrometer head in the same position for approximately 3 seconds, then moving the micrometer head by 127µm (0.005”), then repeating the process until the flexure reached the hard stop, then backing off entirely. This entire process was repeated in a “round-robin” style, which allows averaging and allowing the full range of motion as opposed to half (micrometer heads only push). Each data point in Figure 4.3 was taken by the average of the average of each step for the four measurements.

4.1.3 Experimental results of displacement tests

The results of the displacement experiment are shown in Figure 4.4.

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Figure 4.3: Experimental displacement data (also published in [82]).
4.1.4 Discussion of Results

The results exhibit the right trends, but the wrong values. Mass 1 should move ±28 µm over the full range of motion. Mass 2 should move ±30 µm. The other two masses should move ±3 µm. The displacement from mass 1 is about 50% too small, and displacement 3 is about 300% too large. These results may be due to the micrometer having a rotating head that grabs the proof mass and moves it up and down by several microns. Also, bolting the flexure down at four points over-constrains the flexure, which causes it to warp. Any flatness error from the table combined with the height variation of the pads where the flexures were bolted to the table may cause this warping. For future work, the ground of the flexures should be one continuous loop that allows the flexure to be clamped to the table at three points.

4.2 Resonant mode separation

Resonant mode separation is critical to the performance of the gyroscope, so the resonance frequency was measured for the flexure with 1° taper. An elegant way to get the natural frequencies is to take an FFT of the impulse response. There are many other ways to acquire dynamic information using actuators (e.g. a sine sweep) but care must be taken as the actuators (e.g. voice coils) may add mass and thereby change the resonant frequencies.

4.2.1 Instrumentation and Setup

The resonant modes of the flexure were analyzed via an impulse response of the system. A true impulse excites all frequencies with unity magnitude. Striking a proof mass with a 3/16” hex wrench and measuring the in-plane and out-of-plane motions with capacitance probes generated the impulse. The results are averaged over the four tests (an impulse was imparted to each proof mass). The setup and instrumentation is the same as the setup shown in Figure 4.1. The natural frequency of the first mode and the damping ratio were calculated using two data points from the impulse response.

4.2.2 Data Acquisition

The data was acquired using the same instrumentation and LabView script as described in section 4.1.2. Caution was taken to avoid hitting the flexure so hard that the proof masses bounced off the capacitance probes. The MATLAB script shifts the time axis to where the
an impulse occurred, selects the data point of the first peak \((A_1 \text{ at time } t_1)\), and then selects a data point of another peak \((A_2 \text{ at time } t_2)\). The relevant details for analysis are shown in Figure 4.4.

![Figure 4.4: Ideal Impulse Response.](image)

The natural frequency is then calculated along with the damping ratio. The natural log decrement is given in Equation (4.1).

\[
\Delta_N = \ln \left( \frac{A_1}{A_2} \right) \tag{4.1}
\]

The damping ratio is given by Equation (4.2).
\[ \zeta = \frac{\Delta N}{2\pi N} \sqrt{1 + \left( \frac{\Delta N}{2\pi N} \right)^2} \] (4.2)

Also, the damped period of oscillation is related to the damped natural frequency in Equation (4.3).

\[ T_d = \frac{t_2 - t_1}{N} = \frac{2\pi}{\omega_d} \] (4.3)

The natural frequency is related to the damped natural frequency in Equation (4.4).

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \] (4.4)

Also, the impulse happens at a time \( t_0 \), so Figure 4.4 uses the time shift defined in Equation (4.5).

\[ t' = t - t_0 \] (4.5)

The higher modes of the system die out faster than the lower modes because of the exponential term. The fast Fourier transform (FFT) algorithms from MATLAB are used to see higher order modes. The FFT is calculated up to the Nyquist frequency (half the sampling rate).

### 4.2.3 Experimental Results

The experimental results for one test are shown in Figure 4.6. The natural frequency \( \omega_n \) is 628 rad/s (\( f_n = 100 \) Hz) and the damping ratio \( \zeta \) is 0.008 from the log decrement method. The natural frequency is 141 Hz from the FFT. FEA predicts a natural frequency of 150 Hz.
4.2.4 Discussion of Results

The natural frequency of the impulse response method disagrees significantly from the FEA (~50%), but the damping ratio is typical of a flexure $0.001 < \zeta_{\text{typical}} < 0.1$ [2]. The actual flexure has the outer flexures 127 µm (0.005”). The next highest modes are ~1kHz and they are from the buckling of the transmission elements, which are unobservable from the capacitance probes. Although the impulse response has minimal leakage and it is quick and easy to implement, it has low signal to noise ratio and is unable to characterize nonlinearity. Future work should involve a burst sine signal because of it has a higher signal to noise ratio and more repeatable.
CONCLUSIONS AND FUTURE WORK

The purpose of this work was to create flexure design rules for coupled motions and decoupled mode shapes. This work is important because it illustrates how to design flexures that cannot be broken down into serial/parallel components. These flexures allow coupled motions that impact MEMS gyroscopes for the inertial guidance of submarines and UAVs, etc.

5.1 Synopsis

Chapter 1 covers the previous work and shortcomings in the field of flexure design. Chapter 2 illustrates and derives the design rules for coupled motions and decoupled mode shapes. Chapter 2 also investigates different flexural transmission elements and a procedure for choosing an optimal design from a set of possible designs. Chapter 3 provides the application of the theory developed in Chapter 2 to a case study of the MEMS TFG. The design rules allow the rapid synthesis of several concepts that achieve the required kinematics by first designing the modules, then combining the modules that make decoupled mode shapes. Then the stiffness of a subsection of the model was derived that included strain-stiffening. This model was verified with finite element analysis. Chapter 4 investigates the sensitivity to fabrication errors. A device was fabricated on a 5-axis mill that introduces a 1° taper to the flexure sidewalls. Then the proof masses were displaced in plane (desired motion) with a micrometer head and the out-of-plane (undesired) displacement was measured with capacitance probes. Then the resonant mode was measured by taking an FFT of the impulse response. Chapter 5 provides a summary and explores the future of this research.
5.2 Future Work

Future work entails a deeper investigation of flexure design and further development of the MEMS gyroscope.

5.2.1 Flexural transmission design

The future work of the flexural design is to push beyond the limitations listed in Chapters 1 and 2. The work may be expanded to include actuation spaces [21] and buckling behavior. Also, as mentioned in Chapter 3, there are practical limitations to small displacements. There are additional mechanisms that must be designed to maintain linear stiffness for large displacements. The flexure design techniques illustrated in this work may be used to design MEMS devices, including energy harvesters and grippers as well as for meso-scale precision machine design.

5.2.2 MEMS Gyroscope

An essential factor for testing the MEMS gyroscope is fabricating the gyroscope on a micro-scale rather than a meso-scale. There are inherently different phenomena that occur on a small-scale that are nearly impossible to measure on a meso-scale (see Appendix A). Also, this work only covered experimental testing of the drive module, but the sense module also needs to be analyzed before successful implementation of the gyroscope. Further, the full gyroscope needs to be tested to see the actual modal leakage from one mode to another. Using a setup similar to the one in Chapter 4 becomes less feasible for the full gyroscope because the proof masses are embedded in other masses and flexures. A better setup for the full gyroscope would be to use a 3D scanning laser vibrometer after imparting an impact at several points.
REFERENCES


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This appendix discusses the general modeling of MEMS TFGs, including modeling the drive mode, the Coriolis response (sense mode), the quality factor, and actuation to understand the trade-offs that occur with the scale factor. This appendix is heavily influenced by [24], and is placed here for convenience.

A.1 Drive Mode

The drive mode is modeled as a mass-spring-dashpot system that is driven by a harmonic load. The equation of motion is given by Equation (A.1), where \( m_d \) is the mass that is involved in the drive motion, \( c_d \) is the effective dashpot constant for the drive mode, and \( k_d \) is the stiffness that is involved in the drive motion, \( F_d \) is the driving force amplitude, and \( \omega \) is the driving force frequency.

\[
m_d \ddot{x} + c_d \dot{x} + k_d x = F_d \sin \omega t
\]  
(A.1)

Equation (A.2) gives the solution to Equation (A.1), where \( x_0 \) is the amplitude of the \( x \)-motion, and \( \phi_d \) is the phase lag between the driving force and the \( x \)-motion.

\[
x = x_0 \sin(\omega t + \phi_d)
\]  
(A.2)

Equation (A.3) gives the amplitude of the \( x \)-motion, where \( \omega_d \) is the resonant frequency of the drive mode, and \( Q_d \) is the quality factor of the drive mode.
\[ x_0 = \frac{F_d}{k_d \sqrt{\left(1 - \left(\frac{\omega}{\omega_d}\right)^2\right)^2 + \left[\frac{1}{Q_d \omega_d}\right]^2}} \]  \hfill (A.3)

Equation (A.4) gives the resonant frequency of the drive mode.

\[ \omega_d = \sqrt{\frac{k_d}{m_d}} \]  \hfill (A.4)

Equation (A.5) gives the quality factor of a resonator.

\[ Q_d = \frac{m_d \omega_d}{c_d} \]  \hfill (A.5)

The quality factor is a measure of how sharp the resonant peak is; a high quality factor means a large resonance response, and low damping (the converse is also true). Equation (A.6) gives the maximum amplitude of \( x_0 \).

\[ x_{0_{\text{res}}} = Q_d \frac{F_d}{m_d \omega_d^2} \]  \hfill (A.6)

Equation (A.7) is the phase lag.

\[ \phi_d = -\tan^{-1} \left(\frac{\frac{1}{Q_d \omega_d}}{1 - \left(\frac{\omega}{\omega_d}\right)^2}\right) \]  \hfill (A.7)

The phase lag goes to \(-90^\circ\) as the driving frequency, \( \omega \), approaches the resonant frequency, \( \omega_d \).

### A.2 Coriolis Response

The response of the system in the \( y \)-direction (i.e. the sense mode) is a similar mass-spring-dashpot system. Equation (A.8) gives the governing differential equation, where \( m_s \) is the mass that participates in the sense mode, \( c_s \) is the effective dashpot constant for the sense mode, and \( k_s \) is the stiffness that is involved in the sense mode, \( F_C \) is the Coriolis force.
\( m_s \ddot{y} + c_s \dot{y} + k_s y = F_C \) \hspace{1cm} (A.8)

Equation (A.9) gives the Coriolis acceleration.

\[ \vec{a}_C = 2\Omega_z \times \vec{v} \] \hspace{1cm} (A.9)

Equation (A.10) results if the system is driven at the resonant frequency of the drive mode, \( \omega_d \).

\[ F_C = -2m_C \Omega_z \dot{x} = -2m_C \Omega_z x_0 \omega_d \cos(\omega_d t + \phi_d) \] \hspace{1cm} (A.10)

Equation (A.11) gives the amplitude of the Coriolis response, which is called the scale factor.

\[ y_0 = \Omega_z \frac{2x_0}{m_s \omega_s^2 \sqrt{1 - \left(\frac{\omega_d}{\omega_s}\right)^2 + \left[\frac{1}{Q_s} \frac{\omega_d}{\omega_s}\right]^2}} \] \hspace{1cm} (A.11)

Equation (A.12) gives the resonant frequency of the sense mode.

\[ \omega_s = \sqrt{\frac{k_s}{m_s}} \] \hspace{1cm} (A.12)

Equation (A.13) gives the quality factor of the sense mode.

\[ Q_s = \frac{m_s \omega_s}{c_s} \] \hspace{1cm} (A.13)

Equation (A.14) gives the maximum sense response, which is when \( \omega = \omega_d = \omega_s \).

\[ y_{0_{\text{max}}} = \Omega_z \frac{2Q_s x_0 m_C}{m_s \omega_s} \] \hspace{1cm} (A.14)

Expanding the \( x_0 \) term yields Equation (A.15): the expression for the maximum scale factor.
\[ y_0 = 2Q_s Q_d \frac{m_C}{m_d m_s} \frac{\Omega_z}{\omega_s \omega_d^2} F_d \]  \hspace{1cm} (A.15)

This response should be maximized to sense a minimum rotation rate. There are four distinct parts of this equation that may be maximized independently: quality factors, a mass ratio, a frequency ratio, and the driving force. These parts are analyzed independently in the subsequent sections.

### A.3 Quality Factor

The quality factor depends on several factors that combine some meso-scale and nano-scale effects. Thermoelastic dissipation sets the upper bound on the quality factor \( Q_{TED} \). The inverse quality factor has been shown by Zener [83] to be Equation (A.16), where \( T_0 \) is the initial temperature, \( E \) is the Young’s modulus, \( \alpha \) is the coefficient of thermal expansion, \( c_p \) is the specific heat capacity at constant pressure, \( \omega \) is the driving frequency, and \( \tau_z \) is the relaxation time, given by Equation (A.17).

\[ Q_z^{-1} = T_0 \frac{E \alpha}{c_p} \frac{\omega \tau_z}{1 + (\omega \tau_z)^2} \]  \hspace{1cm} (A.16)

\[ \tau_z = \frac{\hbar^2}{\pi^2 \chi} \]  \hspace{1cm} (A.17)

There are many dissipation mechanisms in MEMS devices, including phonon-phonon scattering \( (Q_{ph-ph}) \), anchor loss \( (Q_{anch}) \), electrical loss \( (Q_{elec}) \), and others \( (Q_{other}) \) including surface losses, and phonon-electron scattering. The total quality factor is given by Equation (A.18).

\[ \frac{1}{Q_{total}} = \frac{1}{Q_{TED}} + \frac{1}{Q_{Ph-ph}} + \frac{1}{Q_{anch}} + \frac{1}{Q_{elec}} + \frac{1}{Q_{other}} \]  \hspace{1cm} (A.18)
A.4 Mass ratio

The mass ratio compares the mass that contributes to the Coriolis force to the rest of the masses. Essentially, the mass ratio takes the form of Equation (A.19), where \( a, b \geq 0 \).

\[
\frac{m_C}{m_d m_s} = \frac{m_C}{(m_C + a)(m_C + b)}
\]  

(A.19)

This mass ratio is maximized when \( a \) and \( b \) equal 0, i.e. when there are no intermediate masses. Thus, the gyroscope should minimize the mass of the intermediate stages. The masses may be used to tune the resonant frequencies.

A.5 Frequency ratio

The requirement for strategic inertial guidance is that \( \Omega_z \) is \(~0.001\) deg/hr (\(4.84 \times 10^{-9}\) rad/s or about 1 revolution per 40 years). The resonant frequencies of the drive and sense modes are about 18-20 kHz (\(1.13 - 1.26 \times 10^5\) rad/s) to filter out acoustic noise. Thus, the frequency ratio for inertial guidance is \(~10^{-24}\) (s/rad)^2.

A.6 Actuation force

There are many different ways to actuate this device. Comb drives are ubiquitous in TFGs because of their low power, fast switching, simple fabrication, and the ability to sense and actuate. Thermal actuators generally have higher power, but they also can push harder, have a large travel range, simple fabrication techniques, and lower voltages. Piezoelectric actuators have similar advantages as electrostatic comb drives, but the micro-fabrication process becomes much more complex, which precludes their practical use. Similarly, magnetic actuators have complicated fabrication processes that preclude their practical use in TFGs.