Quantum State Reconstruction and Tomography
using Phase-Sensitive Light Detection

by

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Abstract

In this thesis we present an optical and electronic setup that is capable of performing coherent state tomography. We fully characterize it in order to verify whether or not it will be capable to perform non-demolition homodyne detection of squeezed light in a high-finesse cavity QED setup with an ensemble of Cesium atoms coupled to the cavity. After quantifying sources of noise, the photodiode efficiency, we perform a series of measurements of low photon number coherent states and compare them against the standard quantum limit. We discuss a variety of technical challenges encountered in such systems and some methods to overcome them. Lastly, we test the apparatus' ability to do quantum state tomography and quantum state reconstruction by reconstructing the density matrix and Wigner functions for low photon-number coherent states.

Thesis Supervisor: Vladan Vuletic
Title: Professor
Acknowledgments

Working in the Vuletic group has easily been the highlight of my undergraduate experience. I can’t help but to constantly admire the incredibly talented, down-to-earth, and friendly faculty and students who work in the CUA. I have endless gratitude for both Kristin Beck and Dr. Mahdi Hosseini who did so much to help me develop the requisite technical knowledge needed to do this experiment, and to perform effective research in general. I cannot thank them enough for their kindness and patience in mentoring me. Of course, I have as many thanks for Prof. Vladan Vuletic in his help and assistance in developing this project.

As I graduate and begin an internship at Raytheon BBN Technologies Quantum Information Processing division, I feel incredibly lucky that I’ve had this experience. I’m ecstatic to use my newly developed skills and confidence as I begin my internship, and when I enroll in graduate school next year.
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Chapter 1

Introduction

The long-term goal of this project is to develop and test a photodetection setup that is capable of doing measurements of squeezed states. In order to perform measurements on quantum states of light, however, we must understand some of their fundamental properties. This chapter will cover the requisite information in order to understand the most commonly studied quantum states of light, basic representations of quantum optical states, and the experimental means of studying these states as well as their applications. We will briefly overview these states, but a more thorough investigation is easily accessible in [1],[2], and [3].

1.1 Brief Introduction to Quantum Optics

The fundamental concept behind quantum optics is the quantization of the electromagnetic field. Thus, we have a variety of quantum states of light, the most straightforward of which is the Fock state, or photon number state $|n\rangle$. Fock states are simply eigenstates of $\hat{N} = \hat{a}^\dagger \hat{a}$, which implies that they are also eigenstates of the quantum harmonic oscillator. Just as with the simple harmonic oscillator eigenstates, applying a creation operator $\hat{a}^\dagger$ increases the photon number of a state $|n\rangle$ by 1, $|n + 1\rangle$. Applying the annihilation operator decreases the photon number of the state by 1. Fock states are crucial to understanding the statistics behind other states of light, as we will see.
1.1.1 Coherent States

At the most intuitive level, coherent states are quantum states whose behavior is most similar to the classical harmonic oscillator. They are a very particular example of the regular quantum harmonic oscillator, noticeable when written as a superposition of states $|n\rangle$.

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{\frac{1}{2}}} |n\rangle$$

(1.1)

When written in the Fock state basis, it is clear that coherent states are superpositions of photon number states and are eigenstates of the annihilation operator. We can now write the coherent state $|\alpha\rangle$ in terms of the vacuum state $|0\rangle$.

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{n=0}^{\infty} \frac{(\alpha \hat{a})^n}{n!} |0\rangle = \exp(\alpha \hat{a}^\dagger - \frac{1}{2}|\alpha|^2) |0\rangle$$

(1.2)

We introduce the displacement operator, $\hat{D}(\alpha)$, where $|\alpha\rangle = \hat{D}(\alpha) |0\rangle$ and $\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$. A coherent state is a displaced vacuum state, and we will shortly see what this displacement in phase space means.

With the mathematical background of coherent states, we can now calculate some statistics about how photons in these states behave. Calculating the mean photon number of a coherent state, we get that

$$\langle n \rangle = \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = |\alpha|^2,$$

(1.3)

and that

$$\langle n^2 \rangle = \langle \alpha | n^2 | \alpha \rangle = |\alpha|^4 + |\alpha|^2 = \langle n \rangle^2 + \langle n \rangle.$$  

(1.4)

Thus, the variance $(\Delta n)^2 = |\alpha|^2 = \langle n \rangle$ is the same as the mean.

We can calculate the probability of $n$ photons being in a given coherent state $|\alpha\rangle$ by taking the norm squared of the inner product of a Fock state $|n\rangle$ and a coherent state $|\alpha\rangle$.

$$P(n) = |\langle n | \alpha \rangle|^2 = \exp(-|\alpha|^2) \frac{|\alpha|^{2n}}{n!} = \exp(-\langle n \rangle) \frac{(\langle n \rangle)^n}{n!}$$

(1.5)
Notice that $P(n)$ is a Poisson distribution, just how shot noise is a Poisson distribution in standard photon counting statistics. This makes sense considering the variance is the same as the mean. This will become a crucial feature in future chapters when we interpret our data.

Using the properties we just discussed, we can calculate the expectation values of the canonically conjugate quadrature operators $\hat{X}$ and $\hat{Y}$ (alternatively $\hat{X}$ and $\hat{P}$) as

$$
\langle \alpha | \hat{X} | \alpha \rangle = \frac{1}{2} \langle \alpha | \hat{a}^\dagger + \hat{a} | \alpha \rangle = \frac{1}{2(\alpha^* + \alpha)} = \text{Re} \alpha = |\alpha| \cos \theta, \quad (1.6)
$$

and

$$
\langle \alpha | \hat{Y} | \alpha \rangle = \text{Im} \alpha = |\alpha| \sin \theta. \quad (1.7)
$$

After calculating the expectation values of the squares of the quadrature operators, we can calculate their variances

$$
(\Delta X)^2 = (\Delta Y)^2 = \frac{1}{4} \quad (1.8)
$$

From this, we can see that a coherent state is a minimum uncertainty state. We can also calculate the coherent electric signal by taking the expectation value of the electric field operator:

$$
S = \langle \alpha | \hat{E}(\chi) | \alpha \rangle = |\alpha| \cos (\chi - \theta) \quad (1.9)
$$

This relation will prove useful for us in future chapters. The field variance, or noise, is $\mathcal{N} = (\Delta E(\chi))^2$. The field noise for a coherent state is independent of the phase, and thus will be evenly distributed around the value for $S$. In phase space, a coherent state traces out a circle with diameter $\mathcal{N}^{1/2} = \frac{1}{2}$ displaced from the origin with a distance of magnitude $|\alpha|^2 = \langle n \rangle^{1/2}$. Coherent states obey a photon-number and phase uncertainty relation, $\Delta n \Delta \phi = \frac{1}{2}$. Using the probability distribution in equation 1.1.1,
we can calculate a coherent state's phase probability distribution.

\[ P(\phi) = \frac{1}{2\pi} \left| \sum_n \exp(-\frac{1}{2}|\alpha|^2) \frac{\alpha^n}{(n!)^{\frac{1}{2}}} e^{i n(\theta - \phi)} \right|^2 \]  

(1.10)

Both the equation above and the uncertainty relation for phase and photon-number govern a standard quantum limit for coherent state measurement. We will only be able to measure phase with a finite degree of uncertainty depending on the mean photon number for a coherent state. In future chapters we will get a clearer idea of what this means in an experimental setting.

1.1.2 Squeezed States

As we saw in the previous section, a coherent state electric field has phase-independent noise. A quadrature squeezed state has noise that not only varies according to phase, but also lies within the range \(0 < (\Delta E(\chi))^2 < \frac{1}{4}\) for a given value of the angle \(\chi\). Let us define the squeezed vacuum as

\[ |\zeta\rangle = \hat{S}(\zeta)|0\rangle, \]  

(1.11)

where the squeezing operator \(\hat{S}(\zeta)\) is

\[ \hat{S}(\zeta) = \exp \left( \frac{1}{2} \zeta^* \hat{a}^2 - \frac{1}{2} \zeta (\hat{a}^d)^2 \right). \]  

(1.12)

\(\zeta\) is the complex squeezing parameter, where \(\zeta = se^{i\theta}\). \(s\) here represents the amplitude and \(\theta\) the phase. Exploiting the unitarity of the squeezing operator, we can calculate the mean photon number of a squeezed state in the Fock state basis.

\[ \langle n \rangle = \langle \zeta | \hat{a}^d \hat{a} | \zeta \rangle = \langle 0 | \hat{S}^+ \hat{a}^d \hat{S} \hat{a}^d \hat{S}^+ | 0 \rangle \]  

(1.13)

Using the value for \(\zeta = se^{i\theta}\) and the Baker Campbell Hausdorff formula, we can calculate how \(\hat{S}\) and \(\hat{S}^+\) transform the \(\hat{a}\) and \(\hat{a}^d\) operators.
\[ \hat{S}^\dagger \hat{a} \hat{S}^\dagger = \hat{a} \cosh s - \hat{a}^\dagger e^{-i\theta} \sinh s \] (1.14)

Considering this relation, we can use it in equation 1.1.2 to define the mean photon number as \( \langle n \rangle = \sinh^2 s \). As \( s \) approaches 0, the squeezed vacuum state becomes the regular vacuum state \( |0\rangle \), and the mean photon number for the vacuum state becomes 0 once again. We can calculate the photon number variance as \( (\Delta n)^2 = 2\langle n \rangle (\langle n \rangle + 1) \).

We can calculate the variances of the quadrature operators \( X \) and \( Y \) as

\[ (\Delta X)^2 = \frac{1}{4} \left[ e^{2s} \sin^2 \left( \frac{1}{2} \theta \right) + e^{-2s} \cos^2 \left( \frac{1}{2} \theta \right) \right], \] (1.15)

and

\[ (\Delta Y)^2 = \frac{1}{4} \left[ e^{2s} \cos^2 \left( \frac{1}{2} \theta \right) + e^{-2s} \sin^2 \left( \frac{1}{2} \theta \right) \right] \] (1.16)

[1] shows an excellent pictorial representation of the quadrature operator means and uncertainties for the squeezed state. Unlike the circle that the coherent state traces out in phase space, the squeezed state traces an ellipse with major and minor axes of \( \exp(s)/2 \) and \( \exp(-s)/2 \), respectively, and is inclined at an angle \( \theta/2 \) with respect to the quadrature axes. The mean field coherent signal \( S \) vanishes and the signal becomes entirely due to the field variance \( \mathcal{N} \) which is phase dependent. We will not be experimentally detecting squeezed states in this thesis, but their detection is a future goal for the experimental setup described in future chapters.

1.1.3 The Density Matrix and the P Function

The density matrix \( \rho \) describes a statistical ensemble of quantum states. If we are concerned with a statistical ensemble of photon number states, we can write the density matrix in terms of the following expression.

\[ \rho = \sum P_n |n\rangle \langle n| \] (1.17)
$P_n$, here is the $P$-function, a probability distribution denoting the probability of finding $n$ photons in a given mode. For a field with a Poisson distribution of photons such as a coherent state, the $P$ function would take on the form of equation 1.1.1. Coherent states $|\alpha\rangle$ are linear superpositions of photon number states and they themselves form a complete (in fact, overcomplete) set of states although they are not orthogonal. For this reason, we can write the density operator $\rho$ in terms of a set of states $|\alpha\rangle$.

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha$$

(1.18)

We will use this information in future chapters to analyze our data.

The Wigner Quasiprobability Distribution

The Wigner Distribution was first introduced in 1932 by Eugene Wigner in order to examine quantum mechanical corrections to statistical mechanics. [4] The Wigner Function is useful because typically we do not have a probability distribution of a quantum state simultaneously in position and momentum, or in the conjugate quadrature operators of $\hat{X}$ and $\hat{Y}$. The Wigner Function remedies this by generating a probability density in phase space. Wigner’s original goal with the development of the Wigner Function was to draw parallels between quantum and classical mechanics, as quantum mechanics generally deals with probabilities and classical mechanics uses descriptions in phase space. As coherent states have many behaviors akin to the classical harmonic oscillator, one can gain a considerable amount of information about a state’s behavior by looking to the Wigner Function. In other cases, there are states with distinctly nonclassical behavior such as squeezed states or Schrödinger cat states where one can visually observe nonclassical behavior. [5] Smithey and Beck were the first to measure the Wigner function experimentally in 1994.[6] Figure 1-1 shows the contour plot and three-dimensional plot of a coherent state Wigner Function, and figure 1-2 shows similar plots for a squeezed state with $s = 3$. Both plots were generated in Mathematica.
\[ W(X, Y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{-i\alpha Y} (X + x|\rho|X - x) \tag{1.19} \]

\[ W(\alpha) = \frac{1}{\pi^2} \int d^2\eta \exp(\eta^* \alpha - \eta \alpha^*) \text{Tr} \rho e^{\eta^* \eta} \tag{1.20} \]

where \( \text{Tr} \rho e^{\eta^* \eta} \) is the characteristic function for \( \rho \). For a coherent state \( \mid \alpha \rangle = \mid \frac{1}{2}(X + iY) \rangle \) the corresponding Wigner Function is

\[ W(X, Y) = \frac{1}{2\pi} \exp \left[ -\frac{1}{2} (X'^2 + Y'^2) \right], \tag{1.21} \]

where \( X' = X - x_0 \) and \( Y' = Y - y_0 \). Thus, the Wigner function for a coherent state is a Gaussian centered at the point \((x_0, y_0)\) in phase space or the Wigner plane. For a vacuum state, the Wigner function would be centered at the origin. For coherent states with greater mean photon numbers, the Wigner function would be displaced somewhere in the plane.

\section*{1.2 Fundamentals of Photodetection}

One of the most basic yet important fundamental tools in optical and atomic physics is photodetection. There are several schemes of photodetection which are suited to particular physical setups. Here we will primarily discuss photodetection from a quantum mechanical viewpoint, however a more extensive discussion including semiclassical approaches is in Prof. Jeffrey Shapiro’s notes for his course on quantum optical communication. \cite{8}

\subsection*{1.2.1 Direct Photodetection}

Photodetectors are square-law devices that generate an electric current when photons are incident upon it. We can determine the number of photons \( N \) from a time \( t \) to \( T \) with the following.
Here, \( q \) is the electron charge and \( i(u) \) is the photocurrent. Direct photodetection, however, gives no information about the phase of the light. To retrieve information about phase, we must turn to interferometric methods such as homodyne and heterodyne detection.

### 1.2.2 Phase-Sensitive Photodetection

Figure 1-3 shows the most basic layout for these two forms of detection. Two beams, one signal beam and the other a much stronger reference beam called a local oscillator or LO, are allowed to interfere with one another when they meet at a beam splitter. Two photodiodes detect the two interfered beams, and subtraction electronics subtract the two photocurrents to get the difference. The primary difference between homodyne and heterodyne detection is that with heterodyne detection, the frequency of the local oscillator and the signal are different, and thus the resulting subtracted signal is not DC.

#### Homodyne Detection

For the quantum theory of homodyne detection, we can write the signal and local oscillator fields as operators. Thus,

\[
E_S(x, y, t) = \frac{a_S e^{-i\omega t}}{\sqrt{A T}} + \text{other terms} \tag{1.23}
\]

is the signal field, and

\[
E_{LO}(x, y, t) = \frac{a_{LO} e^{-i\omega t}}{\sqrt{AT}} + \text{other terms} \tag{1.24}
\]

is the local oscillator field. \( A \) is some spatial distance to maintain proper dimensions and \( T \) is a time scale. The "other terms" in both expressions represent unexcited
modes of the electric field. The signal field is much weaker than the local oscillator field, that is \((\hat{a}_s^\dagger \hat{a}_s) \ll (\hat{a}_{LO}^\dagger \hat{a}_{LO})\), and the local oscillator is assumed to be in a coherent state \(|\sqrt{N_{LO}}e^{i\theta}\rangle\). \(N_{LO}\) is a factor for normalization. The 50/50 beam splitter in figure 2-1 combines the two signals such that

\[
\hat{E}_\pm (x, y, t) = \frac{a_\pm e^{-i\omega t}}{\sqrt{A}} + \text{other terms},
\]  

(1.25)

where \(a_\pm \equiv \frac{a_{LO} \pm a_s}{\sqrt{2}}\). For homodyne and heterodyne detection, we desire to measure an output which is the difference between the two signals received at the photodiodes. Classically, this would be

\[
\alpha_\theta = \frac{N_+ - N_-}{2\sqrt{N_{LO}}}. \tag{1.26}
\]

Where the + and − symbols represent the two different photodiodes. The photon numbers at both ports \(N_+\) and \(N_-\) can be expressed quantum mechanically in terms of individual creation and annihilation operators, such that

\[
\alpha_\theta = \frac{a_+^\dagger a_+ - a_-^\dagger a_-}{2\sqrt{N_{LO}}}. \tag{1.27}
\]

In the limit of infinite local oscillator strength, we get that \(\alpha_\theta\) approaches \(\text{Re}(\hat{a}_s e^{-i\theta})\). Thus, as the local oscillator angle \(\theta\) scans to both 0 and \(\frac{\pi}{2}\), we measure an observable value.

**Heterodyne Detection**

For the quantum mechanical description of heterodyne detection, we consider a signal field and local oscillator field that are different by some intermediate frequency. Thus,

\[
\hat{E}_{LO} (x, y, t) = \frac{a_{LO} e^{-i(\omega - \omega_{IF}) t}}{\sqrt{A}} + \text{other terms}, \tag{1.28}
\]

where \(\omega_{IF}\) is the intermediate frequency. The photocurrent from either of the photodiodes is

\[
\hat{i}_\pm (t) = q \int_0^T dx dy \hat{E}_\pm^*(x, y, t) \hat{E}_\pm (x, y, t), \tag{1.29}
\]
and we can calculate the operator $\hat{a}$ as

$$\hat{a} = \lim_{N_{\text{LO}} \to +\infty} \frac{1}{q\sqrt{N_{\text{LO}}}} \int_0^T dt [\hat{i}_+(t) - \hat{i}_-(t)]. \tag{1.30}$$

Balanced homodyne or heterodyne detection mean that the beam splitter is a 50/50 beam splitter, so the signals are split evenly. It is important to note in all of these circumstances that we are considering ideal photodetectors. The assumptions of ideal photodetection systems are as follows.

1.) The responsivity of the photodiodes covers all frequencies.

2.) The photodiodes are 100% in their conversion of photons to electrons.

3.) There is no "dark" current (current generated from electronic noise).

4.) There is amplification of the resulting photocurrent.

5.) The preamplification and detection setup has no noise and infinite bandwidth.

6.) For balanced detection, there is perfect rejection of common modes.

With the setup we will be discussing in this thesis, each of the above will be quantified in order to determine if we have sufficiently good enough detection and low-enough noise to perform measurements and tomography of quantum states of light. Although measurement of squeezed states has not yet been performed with this detection setup, we must characterize these aspects of the photodetection setup's performance to ensure that it will be capable of measuring squeezing in future experiments.

1.3 Initial Concept Designs

1.3.1 RF Electronic Layout

For the original design we were considering a heterodyne setup constructed almost entirely out of off the shelf RF devices and lab-constructed avalanche photodiode
electronics. At the heart of this setup was an I/Q Demodulator. An I/Q demodulator is generally a four port device including a port for a local oscillator signal, an output port, and two ports designated I and Q. The I and Q ports, both located between the LO and RF ports, have mixers that can subtract and add frequencies coming in from the LO or either of the I and Q ports. In this sense, the LO signal of homodyne setup can be used as an input to the LO port. The two signals from the avalanche photodiodes, or APDs, can be used as the inputs to the I and Q ports, where the mixers can be used to subtract the interfered signals. Thus with this setup may be able to perform quadrature measurements.

We intended on using a Minicircuits ZFMIQ-70P I/Q demodulator as well as RF Bay's LNA-580 low-noise amplifiers. To supply power to the avalanche photodiodes we constructed two linear voltage supplies. Preliminary electronic tests included using a voltage-controlled oscillator to simulate electric signals from a local oscillator, and combining this signal with a Gaussian pulse sent in from an arbitrary waveform generator using a switch. The combined Gaussian pulse and sinusoid from the VCO would be the input to the I and Q ports, and we would add some attenuation on either port. We would then proceed to measure power differences from the RF port to see how efficiently signals from the I and Q ports would be subtracted.

1.4 Noise Sources in Phase-Sensitive Photodetection

We eventually stopped proceeding with this design because of its innate inability to measure squeezing well. When one quadrature of an optical field is squeezed, the other quadrature must consequentially be anti-squeezed as governed the Heisenberg Uncertainty Principle. Conventional homodyne detection measures either one of these quadratures by scanning the phase of the local oscillator, which directs the local oscillator along this quadrature in phase space. The best-possible noise reduction when measuring this quadrature is when the local oscillator is parallel to the squeezed
quadrature. This is the only opportunity to perform measurements of a squeezed quadrature below the typical shot noise level. Typical heterodyne detection with a single local oscillator is incapable of doing this, as the constantly varying signal has a phase moving continuously in time relative to the squeezed quadrature.\[9\] [10]

Double balanced heterodyne schemes remedy this by using two local oscillators of equal strength. Although the superposition of the both of them will still oscillate, the phase remains constant in time and thus fixed in one position in phase space, allowing one to choose the position in phase space to perform quadrature measurements. One can view this from the light intensity at the output port of the beamplitter,

\[
I(t) = \left[ E^+(t) + \mathcal{E}_1(t) + \mathcal{E}_2(t) \right] \times \left[ E^+(t) + \mathcal{E}_1(t) + \mathcal{E}_2(t) \right] \\
= \left[ e^{i\omega_0 t} (\hat{a}^\dagger + \mathcal{E}e^{i\Omega t - i\phi_1} + \mathcal{E}e^{-i\Omega t - i\phi_2}) \right] \\
\times \left[ e^{-i\omega_0 t} (\hat{a} + \mathcal{E}e^{-i\Omega t + i\phi_1} + \mathcal{E}e^{i\Omega t + i\phi_2}) \right] \\
= \hat{a}^\dagger \hat{a} + 2\mathcal{E}\cos(\Omega t + \delta\phi)\hat{X}(\bar{\phi}) + 4\mathcal{E}^2\cos^2(\Omega t + \delta\phi). \tag{1.34}
\]

Here, \(E^+(t) = \hat{a}e^{-i\omega_0 t}, \mathcal{E}_1(t) = \mathcal{E}e^{-i(\omega_0 + \Omega)t + i\phi_1}, \) and \(\mathcal{E}_2(t) = \mathcal{E}e^{-i(\omega_0 - \Omega)t + i\phi_2}\) are the signal optical field and the two local oscillators, respectively. \(\delta\phi \equiv \frac{\phi_2 - \phi_1}{2}\) is the difference in phase between the two local oscillators. \(\hat{X}(\bar{\phi}) = \hat{X}\cos(\bar{\phi}) + \hat{P}\sin(\bar{\phi})\) is the new quadrature operator for two local oscillators, where \(\bar{\phi}\) is the average of \(\phi_1\) and \(\phi_2\). The first term in the last expression in equation 1.4 is a DC signal that is weak compared to the LO-dependent signal, and the last term is a classical signal. In the limit of a strong local oscillator the middle term dominates,

\[
\Delta I(t) \approx 2\mathcal{E}\cos(\Omega t + \delta\phi)\Delta \hat{X}(\bar{\phi}). \tag{1.35}
\]

Already one can see that the intensity for this double balanced scheme is on the order of twice as large compared to the traditional scheme. Calculating the exact amount of additional noise requires one to calculate the spectral density of
photocurrent fluctuations,

\[ \chi(\omega) = \frac{1}{T} \int_{0}^{T} dt \int_{-\infty}^{\infty} e^{i\omega \tau} \langle \Delta J(t) \Delta J(t + \tau) \rangle, \]  

(1.36)

where \( \langle \Delta J(t) \Delta J(t + \tau) \rangle \) is the auto-correlation function of photocurrent fluctuations dependent on \( \Delta \hat{I}(t) \). Auto-correlation functions measure how much a signal cross-correlates with itself. Calculating the resulting spectral density is highly involved and is done in completion in [10]. However, the calculated spectral density in this case differs by a global factor of two from the conventional balanced homodyne scheme. Intuitively, the factor of two arrives from using two local oscillators, which will thus generate twice the shot noise in measurements. With a factor of two greater in the spectral density, this corresponds to an additional 3 decibels of noise in the quadrature one wants to make a measurement in, greatly reducing the observed squeezing. In [10] they discuss possible methods of reducing this noise, but these methods have not been experimentally verified.

Once we realized this significant hindrance to our eventual goals we sought out a different setup. However, this design concept may remain a viable idea for those seeking to do measurements solely with heterodyne detection.
Figure 1-1: The Wigner Function of a coherent state. b.) The corresponding contour plot of the Wigner Function. Both plots generated in Mathematica.
Figure 1-2: a.) The Wigner Function of a squeezed state with a squeezing parameter of $s=3$. b.) The corresponding contour plot of the Wigner Function. Both plots generated in Mathematica.
Figure 1-3: The basic layout for either homodyne or heterodyne detection. PD denotes the photodiodes, BS denotes the beam splitter, LO and SIG denote the local oscillator and signal beam, respectively.
Chapter 2

Detection Setup and Specifications

The high-finesse cavity setup is located in 26-228. However, we characterized the optical setup one floor above in 26-340 due to greater available space. We will discuss each aspect of the setup, its purpose, and how we characterized its ability to perform low noise measurements.

2.0.1 Optics and Electronics

Instead of using individual RF electronics as mentioned in the previous section, we decided on a commercial balanced photodetector, which includes two photodiodes, electronic amplification of the photocurrents, as well as the subtraction electronics to subtract the two signals. The main specifications the balanced photodetectors must have are low electronic noise, high quantum efficiency within the range of wavelengths we will be using, a reasonable damage threshold of the electronics and photodiodes, and a high common mode rejection ratio which we will be measuring in this chapter. Many of the performance tests and characterizations we will be demonstrating were inspired by other experiments with slightly different electronic and optical setups but with similar experimental goals. [12] [13] [14] [15]

The quantum efficiency of photodiodes quantifies how many electrons are converted from the photons incident to the photodiodes. This is closely related to the responsivity of the photodiodes, which measures the electrical output per optical in-
put of a device in Amperes per Watt. Spectral responsivity in the case of photodiodes is almost always dependent on the frequency of incident light, so it is crucial to choose photodiodes that have a high responsivity and efficiency in the wavelength range you will be performing measurements in. Responsivity is measured as

\[ R = \eta \frac{q}{hf}, \]  

(2.1)

where \( \eta \) is the quantum efficiency, \( q \) is the elementary charge, \( h \) is Planck's constant, and \( f \) is the frequency of light. Thus, we can measure quantum efficiency as

\[ \eta = \frac{N_e}{N_\nu} = \frac{hf}{q}. \]  

(2.2)

A big constraint in searching for an adequate set of balanced photodiodes is finding a pair with sufficiently high quantum efficiency. This is because many experiments seek to characterize systems where there are a few or only one photon, and thus we would like to ensure that we are capable of detecting them most of the time.

We eventually decided upon purchasing the Thorlabs PDB210A Large-Area Balanced Amplified Photodetectors. The specifications quoted a decently high CMRR, low enough noise, and the highest reported responsivity and quantum efficiency of the photodiodes at 852 nm for any set of balanced photodiodes we were able to find on the market [11]. The responsivity of the photodiodes is 0.58A/W at 852 nm, or a quantum efficiency of 84.4%.

Figure 2-1 shows the actual setup for coherent state tomography. Figure 2-2 shows the laser and optics before entering the fiber it is coupled to. We used a Toptica DL100 Grating Stabilized Tunable Single Mode Diode Laser as both the local oscillator and the signal beam. The wavelength we used was 852nm, as that will be the frequency of light used in the actual experiment located in 26-228. As a coherent state describes a stable oscillating electric field, coherent laser light is often idealized as a coherent state.

The acousto optic modulators, or AOMs, are pieces of quartz connected to a piezoelectric transducer. An RF signal is sent to the PZT, which generates soundwaves in
the quartz. These soundwaves cause periodic fluctuations in the index of refractions of the material, causing Bragg diffraction of the incoming light. The angle at which the light is diffracted is

\[
\sin(\theta) = \frac{m\lambda}{\Lambda},
\]

where \(\Lambda\) is the speed of sound in the medium, \(\theta\) is the angle of diffraction, \(\lambda\) is the wavelength of the light, and \(m\) is the order of diffraction. AOMs are employed frequently in optical physics because of their ability to modulate both power and frequency of an input laser beam. For this particular experiment, we sent RF signals from either the Stanford Research Systems DS345 function generator or the Analog Devices AD9959 four channel direct digital synthesizer, often abbreviated to DDS. In most cases the SRS and the DDS served similar purposes, however the SRS has a maximum output of 40MHz and there is less ability to control the output power. In addition, the DDS has four channels that can output different frequencies compared to the one output of the SRS. However the SRS can output a variety of waveforms, including arbitrary waveforms specified by the user. In Figure 2-1, notice that there are two AOMs both on the paths of the signal beam and the local oscillator before they interfere. The irises I1 and I2 are used to block the 0th order of both diffracted beams. It is also important to mode-match the beam on the signal path with the beam on the local oscillator path so they interfere properly. We will be explaining how we use the unique properties of the AOM to modulate our laser in the next two chapters.

### 2.0.2 Measuring Common Mode Rejection Ratio

The common mode rejection ratio, or CMRR, of an electronic device measures how well an electronic device with amplification is able to reject noise or unwanted signals common to two input leads. In the case of a balanced photodetection system, the CMRR measures how well the electronics are capable to cancel out frequencies and modes of light common to both of the input ports. The Thorlabs website quotes the CMRR of the PDB210A to have a minimum of 30 dB within its specified bandwidth.
Figure 2-1: The actual setup for homodyne and heterodyne tomography. BS denotes regular 50/50 beam splitters, AOM represents the two acousto-optics modulators, QWP represents the quarter wave plates, and M represents the mirrors in the setup. I1 and I2 denote the irises, making sure only the first order diffraction from the AOM enter the following optics. The PZT can be used to scan the LO phase, however it ended up being used as a mirror in favor of an alternative method. The PBS is a variable attenuating polarizing beam splitter that can be used to adjust the signal beam power. Right after BS 1 is an additional quarter wave plate.

of 1 MHz. However, it is important to know what the exact CMRR is exactly at several frequency ranges of interest.

In order to measure the CMRR exactly, we used the SRS function generator to generate a sinusoidal signal sent to the second AOM in the setup from 0 kHz to 1 MHz in increments of 100 kHz. We connected the balanced (subtraction) port of the balanced photodetector to a spectrum analyzer. Averaging 10 traces of the spectrum analyzer, we recorded traces for when both photodiodes were exposed to the laser light and when one of the photodiodes was blocked off. When one photodiode is
Figure 2-2: The optical setup ensuring that the laser entering the homodyne setup is well-coupled.

blocked from the light, the frequency being sent to the AOM will show a prominent Lorentzian peak in frequency space on the spectrum analyzer with additional higher harmonics. Measuring the difference in the height of the Lorentzian peaks in the spectrum analyzer traces for when one of the photodiodes is blocked and when both of them are unblocked yield the CMRR for that frequency. To measure the difference in the peak heights with precision and to yield some uncertainties, we fitted Lorenzian peaks to both the blocked and unblocked traces.

Figure 2-3 shows the CMRR over the measured frequency ranges. The noticeable yet small "zig-zag" dips and increases in the CMRR may possibly be due to phase-matched photocurrents at certain frequencies or other sources of RF phase shifts such
2.0.3 Putting a Lower Bound on Quantum Efficiency

Although we have a quoted value of responsivity from Thorlabs, it is still important to estimate the quantum efficiency in a working setup. To do this, we need to measure the power of the detected laser light as well as the photocurrent from one of the photodiodes. Oftentimes traditional power meters are either inaccurate or difficult to fit in a small setup, so we used an avalanche photodiode that we connected to a 1.025
MΩ resistive load before having the signal read out on the oscilloscope. Similarly, we read out the voltage of one of the PDB210A photodiodes on the oscilloscope. Adjusting the quarter wave plate to get 10 values of laser signal power, we measured the voltage from both the PDB210 and the power from the APD and resistive load.

Calculating the photocurrent from the PDB210A requires dividing the measured voltage by the trans-impedance gain. The Thorlabs PDB210A has a trans-impedance gain of 500 Volts/Ampere. To measure the power of the APD, one needs divide the measured voltage by the product of the resistance and the photodiode responsivity. The APD has a responsivity of .565 Amperes/Watt. These ten values, shown in Figure 2-4, follow a linear trend. The slope of the fitted line is 0.514 ± .052, yielding an estimated quantum efficiency of 75.6% ± 5.18%. One of the causes of low efficiency is reflection of light off of the photodiode; this can be remedied by putting a mirror near the photodiode as to reflect back the light without blocking the path of the original beam. By back-reflecting some of the light, we were able to increase the quantum efficiency of 8%. This measurement is only a lower-bound estimate; it is not exact as the amplification behind the APD and the amplification of the photodiodes involve different electronics, and thus have different gains and noise figures.

This quantum efficiency is sufficient to perform tomography, however even higher quantum efficiency photodiodes would be desirable to perform measurements for weak signals with as little noise as possible. For our original concept of a balanced heterodyne detection setup, we purchased a series of high-efficiency photodiodes from Hamamatsu including the Hamamatsu S5971 Si PIN diodes, with a responsivity greater than 0.62 at 852 nm. For the actual experiment it may be an option to consider replacing the photodiodes on the Thorlabs PDB210A with the Hamamatsu S5971 photodiodes.
Figure 2-4: Plot of the measured current from one of the vs. the measured power from the APD. The slope of the linear fit provides an estimation of the responsivity.

### 2.0.4 Measurements of Shot Noise, Electronic Noise, and Detector Linearity

For the amplification behind photodiodes, particularly balanced photodetectors, it is important that they are *shot noise limited*. That is, the main source of noise in the detection setup is due to shot noise from photon counts rather than electronic noise. To quantify the amount of shot noise the balanced photodiodes generate, we turn again to the spectrum analyzer and measure the noise trace with the signal beam blocked off. Blocking off the signal beam requires blocking off the optical path before either the PZT or the first iris. The trace measured here has both shot noise and electronic noise included. To measure only the electronic noise, simply
blocking off both of the photodiodes and measuring the spectrum analyzer trace will suffice. Oftentimes, the electronic noise of a device such as this will vary according to frequency. Figure 2-5 shows how this noise varies over a range of frequencies.

![Figure 2-5: A plot of the frequency-dependent electronic noise produced by the balanced photodetectors without any light input.](noise.pdf)

Notice how as the frequency reaches the detector bandwidth of 1 MHz, the variations in electronic noise quickly even out. Measuring the shot noise generated by the device is as simple as subtracting the electronic noise from the signal-blocked trace at different frequency ranges. We performed shot noise measurements at both 1V and 0.5V of local oscillator power, as the actual experiment will be using low LO power to probe low photon number states. Table 2.1 shows the shot noise measurements over a range of frequencies and LO voltages. The averaged electronic noise in the setup is
Table 2.1: Shot noise measurements at 0.5 V and 1.0 V over various frequency ranges.

<table>
<thead>
<tr>
<th>LO Power</th>
<th>0.5-0.6 MHz</th>
<th>&gt;0.6 MHz</th>
<th>All ranges averaged</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0 V</td>
<td>12.99± 3.48 dB</td>
<td>10.85± 3.25 dB</td>
<td>10.60 ± 2.6 dB</td>
</tr>
<tr>
<td>0.5 V</td>
<td>9.93 ± 3.23 dB</td>
<td>7.73 ± 3.10 dB</td>
<td>7.46 ± 3.10 dB</td>
</tr>
</tbody>
</table>

-77.60 ± 0.84 dB. With the low amount of electronic noise compared to shot noise, we have verified that our device is shot-noise limited. For photodetection systems that operate at much lower frequencies than ours, such as those used in gravitational wave detection, effects from Johnson noise may be considerable. Johnson noise is noise due to thermal fluctuations, and can often be present in some electronics.

**Shot Noise Linearity**

Another important trait of this photodetection system is that the relationship between input RF power and the resulting shot noise is linear or sub-linear. [16] [12] If the shot noise increases faster than a linear rate this may suggest the presence of additional noise in the electronic setup. To measure the relationship between input power and measured shot noise, we added a series of attenuators from 1 dB to 9 dB directly after the RF electronic setup. We used a series of attenuators instead of altering the LO power directly as it provides a greater range of measurement. We measured the shot noise just as we did in the previous part of the section, subtracting the electronic noise from the spectrum analyzer trace.

Figure 2-6 shows the linear trend of measured shot noise versus the input attenuation. We represented this in terms of attenuation instead of input power because the greater difficulty of measuring the input power compared with the known attenuation. Visible in the figure is the linear decrease of shot noise as the power decreases. We calculated the uncertainty in shot noise by taking the standard deviation across all frequency ranges.

With this seemingly promising information about our detection setup, we can now proceed to do quantum measurements and quantify how accurately we perform them.
Figure 2-6: A plot of linear trend of the measured shot noise versus the attenuation after the input RF power. The resulting linear fit was $y = -0.6981x + 10.94$ with a reduced chi square value $\chi^2_{p-1} = 1.1$. 
Chapter 3

Coherent State Tomography using Heterodyne Detection

We begin our quantum measurements by doing coherent state tomography with low-frequency heterodyne tomography at 500 kHz. We generate the 500 kHz signal by sending an 80.5 MHz to one AOM and an 80.0 MHz signal to the other AOM using the DDS. The irises before the beam splitter in Figure 2-1 block the 0th order so that one AOM will diffract the $m = -1$ order and the other diffracts the $m = 1$ order. When the two beams meet at the beam splitter, they interfere destructively, yielding the difference of the two signals. In this sense, a beam splitter is the optical analog of a mixer; depending on whether the beams interfere constructively or destructively, one receives the sum or the difference of the two frequencies respectively. We chose an 80 MHz signal because not only will the balanced photodetector not be able to detect it as it is far out of its bandwidth, but also because the AOM will better respond to frequencies in this range.
3.1 Simultaneous Measurement of Vacuum and Coherent Signal

The goal for this section of the experiment is to measure quadrature values of coherent states and see how capable we are of detecting states with low photon numbers. When measuring quadrature values from the balanced port of the detector, they are not normalized against the corresponding vacuum state. We can measure the vacuum state by blocking off the signal portion of the optical setup, either right before or after the second iris in Figure 2-1. The vacuum state only contains information from the local oscillator, quantifying how much shot noise we have for each value of local oscillator power. To normalize the quadrature values we only need to divide them by the standard deviation of the vacuum.

We used the PBS with variable attenuation in Figure 2-1 to vary the signal power, and in turn vary the mean photon number of the coherent state. It is additionally important to read out the signals coming from the individual photodiodes and check that they have equal power. If not, we may adjust the quarter wave plate directly after the first beamsplitter to equalize the two beams and ensure they are balanced. We recorded each trace from the balanced detector from an oscilloscope with the highest possible resolution on the y-axis (typically 5-10 mV per division), and the maximum data depth of 1,048,576 points. A problem arises when there are small fluctuations in local oscillator power. If data acquisition were instantaneous, this would not necessarily be an issue because the local oscillator generally does not drift over a period of seconds. However, recording this many points on an oscilloscope requires enough time such that the local oscillator may drift during this period of time.

We resolve this issue by adding a switch in the RF electronic setup where the DDS is only outputting the 80.5 MHz and 80 MHz signals for only half of the time, and the other half of the time the laser light is unmodulated, resulting in a DC signal where we can measure the vacuum. Thus we can record data for both the vacuum state and the unnormalized coherent state in a single oscilloscope trace so that the
data remains consistent.

To analyze the data, we divided it into the region with the vacuum signal and the region with the coherent signal, each containing roughly 500,000 points. We tracked the phase of the portion of the trace with the 500 kHz signal, and normalized the voltages in this region with the standard deviation of the vacuum signal. We performed this analysis in Mathematica.

Figure 3-1: A plot of the noise trace of the vacuum state as a function of the local oscillator phase via heterodyne detection.

Figure 3-1 shows the electric field and distribution of quantum noise for the vacuum state, and figure 3-2 shows the phase-dependent electric field and quantum noise for a coherent state. Notice that for the vacuum state, the mean value for the electric field and the noise distributed around it do not depend on the local oscillator phase. The coherent state, in contrast, has quadrature values that vary sinusoidally with the local oscillator phase, following equation 1.1.1in Chapter 1. Another important feature of these two plots is that the noise is always uniformly distributed about a
3.2 Measurements against the Standard Quantum Limit

We would like to measure the mean photon numbers of each coherent state akin to the photon statistics of coherent states defined in Chapter 1. As an estimate, we can sort the normalized quadrature values from least to greatest, take the variance, and
subtract 1. As equations ?? and 1.1.1 show how the uncertainty of phase depends on the mean photon number \( \langle n \rangle \). We calculated the uncertainty in phase from our measured coherent states by performing sinusoidal fits to the coherent state noise traces, as shown in figure 3-2, and calculated the uncertainty in the phase fitting parameter from a nonlinear least squares routine in Mathematica.

![Graph](image)

**Figure 3-3:** Measurement of the phase variance of the coherent state trace versus the estimated mean photon number \( \langle n \rangle \). Superimposed is the numerically evaluated standard quantum limit based on the number of measurements. The point at \( \langle n \rangle = 0.0018 \) is difficult to see due to being so close to the vertical axis.

Figure 3-3 shows the variance in phase versus the estimated photon number for a variety of weak coherent states. Superimposed is a plot of a numerical evaluation based on equation 1.1.1. This numerical evaluation in actual measurements translates to a plot of

\[
\frac{1}{\text{number of measurements} \times \langle n \rangle}
\]

(3.1)

Rodney Loudon shows a similar numerically evaluated plot on page 198 [1]. This plot
gives some intuition on the amount of certainty you can make a measurement with as determined by Heisenberg uncertainty. Although figure 3-3 depends on estimations of mean photon number, it shows that we are capable of making measurements with relatively low uncertainty and quantum noise given the standard quantum limit. With standard coherent light we cannot exceed the standard quantum limit, that is, perform quantum measurements below the shot noise level. One can only exceed the standard quantum limit using amplitude-squeezed light. [1] [3]
Chapter 4

Quantum State Reconstruction with Pulsed Homodyne Detection

4.1 Foray into Quantum State Reconstruction

The Maximum Likelihood Estimation Method

Maximum-likelihood estimation is a commonly used method in statistics and probability to estimate parameters in a statistical model given a set of data. In the case of reconstructing the density matrix, maximum likelihood estimates other quadrature values in the distribution to have more complete information about the quantum state in order to infer $\rho$. Here we will briefly overview a maximum likelihood estimation iteration algorithm first presented by A. I. Lvovsky [18] and notes from [7]. Consider a set of measurements $|y_i\rangle$ representing a sampling of a quantum state. If $f_j$ is the frequency of each outcome, and with the system being in a state $\rho$, we can calculate the likelihood of a particular data set $f_i$, as $L(\rho)$.

$$L(\rho) = \prod_j \text{Tr}(|y_j\rangle\langle y_j|\rho)^{f_j}$$  \hspace{1cm} (4.1)

$\text{Tr}(|y_j\rangle\langle y_j|\rho = \langle y_j|\rho|y_j\rangle$ is the $P$ function described in Chapter 1. We desire to calculate the density matrix $\rho$ which maximizes the likelihood. To do this, we introduce
another operator \( R \).

\[
R(\rho) = \sum_j \frac{f_j}{\text{Tr}(\langle y_j \rangle \langle y_j | \rho \rangle)} |y_j \rangle \langle y_j |
\]

(4.2)

Notice that for an ensemble \( \rho_0 \) that is most likely to generate the given data set, \( f_j \propto \text{Tr}(\langle y_j \rangle \langle y_j | \rho \rangle) \). We can use the resolution of the identity, \( \sum_j |y_j \rangle \langle y_j | \propto \hat{I} \), to see that \( R(\rho_0) \propto \hat{I} \). Thus,

\[
R(\rho_0) \rho_0 = \rho_0 R(\rho_0) \propto \rho_0,
\]

(4.3)

and

\[
R(\rho_0) \rho_0 R(\rho_0) \propto \rho_0.
\]

(4.4)

With equation 4.1 we will begin the iteration algorithm to find the density matrix \( \rho \). After choosing an initial density matrix as \( \rho^0 = N[\hat{I}] \), we repeat many successive iterations,

\[
\rho^{k+1} = N[R(\rho^k) \rho^k R(\rho^k)],
\]

(4.5)

where \( N \) is a constant for normalization. For each iteration the likelihood associated with the current estimate for the density matrix will increase, and \( \rho \) will asymptotically approach the maximum likelihood density matrix \( \rho_0 \).

With direct application to optical homodyne tomography, we need to consider that we perform quadrature measurements at various local oscillator phases. Each quadrature measurement is an observable \( \hat{X}_\theta = \hat{X} \cos \theta + \hat{P} \sin \theta \). \( \hat{X} \) and \( \hat{P} \) are the canonically conjugate position and momentum, and \( \theta \) is the local oscillator phase. The \( P \) function in this case is

\[
P_\theta(x) = \text{Tr}(|\theta, x \rangle \langle \theta, x | \rho),
\]

(4.6)

where \( |\theta, x \rangle \langle \theta, x | \) is the operator projecting onto the quadrature eigenstate. When we reconstruct the Wigner Function, it will be helpful to construct it in terms of the Fock (photon number or quantum harmonic oscillator) state basis. This projection operator, represented in the Fock state basis, is \( \langle m | \theta, x \rangle \langle \theta, x | n \rangle \). The overlap of Fock
and quadrature eigenstates result in the stationary solutions of the eigenvalue problem for the harmonic oscillator,

\[ \langle n|\theta, x \rangle = e^{i\theta} \left( \frac{2}{\pi} \right)^{\frac{1}{4}} \frac{H_n(\sqrt{2x})}{\sqrt{2^n n!}} \exp(-x^2). \]  

(4.7)

In implementing this scheme, which we will do later in this chapter, we need to bin the data into regular sets of \( x \) quadratures and \( \theta \) local oscillator phases. We will then count \( f_{\theta,x} \), the number of events belonging in each bin. Using \( f_{\theta,x} \), we can construct histograms that represent marginal distributions of the desired Wigner Function. An additional method of quantum state reconstruction is the inverse Radon transform, used in [15] and in conventional optical coherence tomography (OCT).

### 4.2 Pulsed Homodyne Detection Scheme

In order to do full quantum state tomography, we need some way of scanning the phase as well as effectively stabilizing and tracking it. Traditional methods of scanning the phase in homodyne tomography most commonly include scanning a PZT, however we will show here that other methods are effective. Oftentimes in such a setup, the laser will encounter some thermal drift which can cause fluctuations in quadrature and phase. Although this can be used to retrieve some information about amplitude and phase, this can add a source of noise on top of actively scanning the phase with a device such as a PZT. In order to avoid thermal drift when using a PZT one can scan the PZT at a fast enough rate to effectively average out the thermal drift, or use an active method of tracking the phase as will be described below.

The primary constraint on our device in this case is its relatively short bandwidth of 1 MHz. With this bandwidth, traditional laser phase locking techniques may become difficult if there is introduction of frequencies higher than the bandwidth. To overcome this we use a pulsed homodyne scheme where for a certain amount of time we have a short pulse including a sine wave where we can measure the phase, and another period of time with a DC pulse that stores the quadrature value. Figure
4-1 shows the electronic setup before the optics that mixes an 80 MHz signal with a square wave at 100 kHz. This signal is switched with an 81 MHz signal and combined at a power splitter. When the $m = 1$ and the $m = -1$ orders of the AOMs cancel the 80 MHz signal out after the second beam splitter, the resulting signal at the balanced output is one where for roughly 3μs of time there is a DC signal and for the next 3μs there is a pulse with a frequency of 1 MHz.

Using the variable attenuating PBS in Figure 2-1, we adjusted it < 5 degrees for 13 values of signal power. Small adjustments in the PBS generate large differences in signal power and thus mean photon number. Even small fluctuations in local oscillator power and polarization in the fiber can make noticeable differences in the detected signal. For this reason, we need to monitor the voltage of the local oscillator from one of the two unbalanced ports and make minute adjustments of the quarter wave plate after the LO beam to ensure the power remains at the same level. Additionally we need to adjust the fiber after coupling it to the laser so it does not cause power fluctuations. This was done by adjusting it manually, but for larger polarization fluctuations one can use polarizing optics.

For each of the signal powers we recorded an oscilloscope trace with maximum data depth, that 1,048,576 points per trace. Additionally, we recorded a trace of the vacuum state for each signal power by blocking off the signal route after the PZT and before the iris. Analyzing the data requires averaging between 10-50 points for each DC segment of the trace to get a quadrature value, and retrieving the phase by performing a sinusoidal fit to each segment of pulsed data. For each data trace we calculated 1040 quadrature values by averaging 20 points in each DC region, and calculated 1040 phase values by performing least squares fitting on each pulsed region. Of course, we must normalize the quadrature values by dividing them by the standard deviation of the vacuum in the corresponding area in phase space.

With the current setup we just described, the only source of changes in quadrature or phase values would be from thermal fluctuations. This caused the phase to drift only 0.5 radians throughout the entire trace, which is not sufficient for full state tomography or Wigner function reconstruction. As mentioned previously, we can use
the PZT to scan the local oscillator phase. Alternatively, we can add a small sinusoidal RF signal to the 80 MHz mixed with the square wave from another channel on the DDS. We added a 5 kHz signal to the LO from another channel in order to scan the phase. Figure 4-2 shows a small section of one oscilloscope trace for a coherent state with a photon number \( \langle n \rangle \approx 30 \). Each segment of DC value and sinusoidal pulse correspond to an unnormalized quadrature value and phase value, respectively. For Figure 4-2, there is a noticeable difference in the DC value throughout the trace. If we were not scanning the LO, the DC values would not fluctuate nearly as much.

![Figure 4-1: The RF electronic layout that mixes a square wave with a 1 MHz signal, which is then combined with an 80 MHz signal. The combination of the \( m = -1 \) and \( m = 1 \) modes from the AOMs at the beamsplitter then result in one area with a DC pulse and another area with a 1MHz pulse.](image)

Plotting these values for phase and quadrature yield figures very similar to Figure 3-2, but for a homodyne scheme. Figure 4-3 shows an example of the phase-dependent noise trace for a coherent state with a mean photon number \( \langle n \rangle = 3.52 \). Notice again that the noise is distributed evenly across the entire trace.
4.3 Density Matrix and Wigner Function Reconstruction

With the help of postdoctoral researcher Dr. Mahdi Hosseini in the Vuletic Group, we were able to implement a series of MATLAB codes to reconstruct both the density matrices and the Wigner functions of a series of coherent states. We will provide a brief overview of how we implemented the code here.
Figure 4-3: A plot of the phase-dependent noise as a function of local oscillator phase for a coherent state with a mean photon number estimated at $\langle n \rangle = 3.52$. A sinusoidal fit is superimposed onto the trace.
4.3.1 Overview of Implementation in MATLAB

1.) Sort the quadrature values from least to greatest, placing them with their corresponding phase values. It is at this point that one should determine the size of the Hilbert Space. The Hilbert Space should be larger than the mean photon number in order to be able to have an accurate depiction of matrix elements representing probabilities of having a higher photon number. For example, a state with mean photon number \( \langle n \rangle \approx 3 \) should have a Hilbert Space of at least 10. A much larger Hilbert Space, however, will only yield a density matrix with many empty elements, making it difficult to see entries with high probabilities.

2.) Rotate the table of quadrature values so that we always begin with a quadrature value with phase 0. This was done using the circshift function in MATLAB. One should now define the number of points being used to reconstruct the density matrix (in this case, 1000), the number of phases being used to generate the density matrix \( n_0 \), and the incremental angle \( \Delta \theta \). \( n_0 \) is used to bin the data as described earlier in the chapter.

3.) Calculate the \( P \)-function as a function of the size of the Hilbert Space, the number of bins, and \( n_0 \). The number of bins is given by the total number of points (the number of quadrature measurements) divided by the number of phases \( n_0 \). \( P \) is calculated iteratively, going through the quadrature values based on how they are binned.

4.) Initialize the density matrix, first as an identity matrix with the same dimension as the Hilbert Space size defined previously.

5.) Calculate \( R \) and \( \rho \) using the iterative method in equation 4.1.

6.) Now that we have \( \rho \), we can reconstruct the Wigner function. We do this using a method presented in [19], where one creates a representation of the Wigner function expressed as a decomposition of Fock states.
Figure 4-4: The resulting density matrix for a coherent state with a mean photon number $< n > = 3.52$.

With this, we can finally present the reconstructed density matrix and Wigner Function for coherent states of varying intensity. The amount of angles we used in our calculations was 100, giving us 10 bins.

4.3.2 Presentation of Data

Density Matrices

Figure 4-4 shows the reconstructed density matrix of a coherent state with a mean photon number of $3.52 \pm 0.35$. With the heterodyne methods presented in the pre-
Density matrix diagonal representing Poissonian Photon Distribution

\[ <n> = 3.76 \pm 0.35 \]

Figure 4-5: A plot of the probability of photon numbers that follows a Poisson distribution as determined by the photon statistics of coherent states. These values are directly from the diagonal of the density matrix.

Previous chapter, we estimated the mean photon number by taking the variance of the normalized quadrature values and subtracting 1. Here, we have a far more precise way of measuring the mean photon number. The diagonal of the density matrix yields a probability distribution of the coherent state having a certain photon number. In fact, the probability distribution takes on the Poissonian form in equation 1.1.1 from Chapter 1. Thus calculating the mean photon number is as simple as calculating the variance of the Poisson distribution along the diagonal through fitting routines. Figure 4-5 shows the diagonal values from the density matrix in figure 4-4 with a Poisson fit superimposed on top.
Wigner Functions

Figures 4-6, 4-7, and 4-8 are a sample of the Wigner functions produced for a varied range of photon numbers. To the side of the Wigner functions along the $X$ and $Y$ quadratures are the plotted marginal distributions. Just as we determined in equation 1.1.3, each of the reconstructed Wigner functions take on a Gaussian form, although with some noise near the base of each of them. Figure 4-6 shows most notably the displacement of the center of the Wigner Function from the origin due to the mean photon number of 3.52. As the mean photon number becomes smaller in the two following figures, the center of the Wigner Function approaches closer to the origin. An important feature of all these Wigner functions is that they are symmetric in the $X$ and $Y$ quadratures apart from some additional noise either due to how we sampled the Wigner function or from experimental parameters. We will describe in more detail some of these effects in the next chapter.

Measurements Against the Standard Quantum Limit

As we did in the previous chapter, we would like to quantify how much quantum noise each of our measurements has. We will again compare our measurements against the standard quantum limit by fitting a sinusoid to the traces of our coherent states’ electric field and noise, and calculating the variance of the phase fitting parameter. Here, we use nonlinear fitting routines in MATLAB to find the phase. To calculate the phase uncertainty, however, we can write a chi squared function based on the fit and parameters, including the raw data. If we tweak the phase enough, the chi squared will change by 1. We can use the amount we changed the phase fitting parameter as the uncertainty. Figure 4-9 shows the phase variance (the uncertainty squared), plotted against the mean photon number measured by the Poisson fits of the density matrix diagonals. We evaluated the standard quantum limit the same way as the previous chapter, although with the reduced amount of measurements the SQL is larger than in the previous chapter.

In spite of the larger SQL the uncertainties generally match well compared against
Figure 4-6: The Wigner function for a coherent state with a mean photon number of $\langle n \rangle = 3.52$. 

Wigner Function for low Photon Number Coherent State
Figure 4-7: A plot of the phase-dependent noise as a function of local oscillator phase for a coherent state with a mean photon number estimated at $\langle n \rangle = 3.52$. A sinusoidal fit is superimposed onto the trace.
Figure 4-8: The Wigner function for a coherent state with a very low photon number near the vacuum.
Figure 4-9: A graph of the phase variance from fitting parameters versus the measured photon number from the corresponding diagonals of the density matrices.
it, particularly for states with very low photon number. This compares well against
the SQL in the previous chapter, especially because we now have a more exact mea-
surement of mean photon number.
Chapter 5

Conclusion

This thesis serves as an overview of some experimental methods for performing shot-noise limited measurements of a quantum state of light, and reviews some methods of data analysis to extract photon statistics as well as reconstruct as much information of the quantum state as possible. We have presented an optical and electronic setup to do so as well as several ways of characterizing such a setup to determine if it will be capable of doing low-noise quantum measurements. The successful reconstruction of the density matrices and Wigner functions of several coherent states serve as capstone measurements and a proof-of-concept that this setup is effective in completing our end goals.

5.1 Additional Sources of Noise and Error

Although we were capable of performing relatively low-noise quantum measurements, a more in-depth discussion of noise sources is necessary to understand where noise may arise in such a setup. For coherent states with mean photon numbers near $\langle n \rangle \approx 10$ and greater we noticed some peculiar characteristics in the coherent state noise trace, density matrix, and Wigner function. The maximum amplitude of the phase-dependent electric field for larger photon coherent states had noticeably less noise than the rest of the distribution, appearing somewhat narrower. The more points averaged for the coherent state quadratures, the more this effect was noticeable.
Notice how points at the extrema in the phase-dependent electric field in Figure 5-1 bunch slightly closer together with a smaller overall distribution of noise. Similarly, in Figure 5-2, the distribution of probability is notably narrower in one quadrature. The ripples on the side of the distributions in part a.) and b.) of Figure 5-2 are an artifact of only having 1000 points to estimate the distribution.

The most probable cause of this effect is due to phase-dependent noise from the laser. There are a variety of causes of phase noise, but a significant one is due to thermal noise. With weak coherent states with \( \langle n \rangle \approx 10 \) or less, the issue of phase noise was not a noticeable issue in Wigner function or density matrix reconstruction. This may be due to the fact that at lower intensities effects from thermal noise and drift, which may cause phase noise, are likely smaller in magnitude. In addition, with quantum noise being more noticeable for smaller mean photon numbers, the effect of this noise may be less noticeable. Another reason this is far more likely attributed to the laser itself rather than the setup or the analysis is that the vacuum state exhibits none of these effects, nor do the Wigner functions of vacuum or near-vacuum states.

Observing the Wigner functions in the previous chapter for \( \langle n \rangle = 3.520 \), \( \langle n \rangle = 0.638 \), and \( \langle n \rangle = 0.024 \) you can notice a successive broadening of the Wigner function as well as some increased noise in some regions at the bottom of the Gaussian profile. This is especially noticeable in the 2D contour plot below the figure. This may be due to the noise mentioned above, or as an artifact of only having 1000 points to estimate and sample the density matrix and Wigner function with. We only have 1000 quadrature values because each of the pulsed and DC regions only have 1000 points each, and the oscilloscope can only record a maximum of 1,048,576 points. In an ideal situation one could record multiple traces for the same setting of signal power. However, over the course of the 10 minutes it takes to record a trace the power of the LO may drift leading to a measurement of a coherent state with a different photon number. An easy solution to this in the actual setup is using a data acquisition system (DAQ) with a very high acquisition rate. Such systems can be purchased from companies such as National Instruments.
Figure 5-1: The phase-dependent electric field of a coherent state with $\langle n \rangle = 16.40$.

5.2 Outlook

We can easily remedy the errors mentioned above in the final experimental setup. A data acquisition card with an especially high acquisition rate can resolve problems in data such as fluctuating signal and local oscillator power, and under-sampling the Wigner distribution. The issue of phase noise coming from the laser is also an issue easily remedied when we implement the actual setup in 26-228.

The actual setup involves an ensemble of cold Cesium atoms coupled to a high-finesse optical cavity with a magneto-optical trap. With the entirely different physical parameters, much care will be taken to minimize sources of systematic error and noise.
One constraint we might encounter is due to the quantum efficiency of the photodiodes being below 85%. We may consider re-outfitting the balance photodetector with higher quantum efficiency diodes such as the Hamamatsu S5971 Si PIN diodes if possible. With most of the error not coming from the detection system itself, there is a probable outlook of performing quantum non-demolition measurements of squeezing with this device.
Figure 5-2: a.) The resulting contour plot of a coherent state for $<n> = 16.40$ with additional noise and b.) The 3-dimensional representation of the same density matrix. The ripples on the side of the regions of greatest probability are due to under-sampling of the state.
Bibliography


