Classification of finite irreducible modules over the 
Lie conformal superalgebra $CK_6$

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Abstract

We classify all continuous degenerate irreducible modules over the exceptional linearly compact Lie superalgebra $E(1,6)$, and all finite degenerate irreducible modules over the exceptional Lie conformal superalgebra $CK_6$, for which $E(1,6)$ is the annihilation algebra.

1 Introduction

Lie conformal superalgebras encode the singular part of the operator product expansion of chiral fields in two-dimensional quantum field theory [8].

A complete classification of finite simple Lie conformal superalgebras was obtained in [7]. The list consists of current Lie conformal superalgebras $\text{Cur} \mathfrak{g}$, where $\mathfrak{g}$ is a simple finite-dimensional Lie superalgebra, four series of “Virasoro like” Lie conformal superalgebras $W_n(n \geq 0)$, $S_{n,b}$ and $\tilde{S}_n(n \geq 2, b \in \mathbb{C})$, $K_n(n \geq 0, n \neq 4)$, $K'_4$, and the exceptional Lie conformal superalgebra $CK_6$.

All finite irreducible representations of the simple Lie conformal superalgebras $\text{Cur} \mathfrak{g}$, $K_0 = \text{Vir}$ and $K_1$ were constructed in [3], and those of $S_{2,0}, W_1 = K_2$, $K_3$, and $K_4$ in [6]. More recently, the problem has been solved for all Lie conformal superalgebras of the three series $W_n$, $S_{n,b}$, and $\tilde{S}_n$ in [1], and for all Lie conformal superalgebras of the remaining series $K_n(n \geq 4)$ in [2]. The construction in all cases relies on the observation that the representation theory of a Lie conformal superalgebra $R$ is controlled by the representation theory of the associated (extended) annihilation algebra $\mathfrak{g} = (\text{Lie} R)_+$ [3], thereby reducing the problem to the construction of continuous irreducible modules with discrete topology over the linearly compact Lie superalgebra $\mathfrak{g}$.

The construction of the latter modules consists of two parts. First one constructs a collection of continuous $\mathfrak{g}$-modules $\text{Ind}(F)$, associated to all finite-dimensional

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irreducible $g_0$-modules $F$, where $g_0$ is a certain subalgebra of $g(= gl(1|n))$ or $\mathfrak{sl}(1|n)$ for the $W$ and $S$ series, and $= \mathfrak{so}_n$ for the $K_n$ series and $CK_6$).

The irreducible $g$-modules $\text{Ind}(F)$ are called non-degenerate. The second part of the problem consists of two parts: (A) classify the $g_0$-modules $F$, for which the $g$-modules $\text{Ind}(F)$ are non-degenerate, and (B) construct explicitly the irreducible quotients of $\text{Ind}(F)$, called degenerate $g$-modules, for reducible $\text{Ind}(F)$.

Both problems have been solved for types $W$ and $S$ in [1], and it turned out, remarkably, that all degenerate modules occur as cokernels of the super de Rham complex or their duals. More recently both problems have been solved for type $K$ in [2], and it turned out that again, all degenerate modules occur as cokernels of a certain complex or their duals. This complex is a certain reduction of the super de Rham complex, called in [2] the super contact complex (since it is a “super” generalization of the contact complex of M. Rumin).

The present paper is the first in the series of three papers on construction of all finite irreducible representations of the Lie conformal superalgebra $CK_6$. In this paper we find all singular vectors of the $g$-modules $\text{Ind}(F)$ for $g = E(1, 6)$, where $F$ is a finite-dimensional irreducible representation of the Lie algebra $\mathfrak{so}_6$. In particular, we find the list of all finite degenerate irreducible modules over $CK_6$. In our second paper we give a proof of the key Lemma 4.4, and in the third paper construct the complexes, consisting of all degenerate $E(1, 6)$-modules $\text{Ind}(F)$, providing thereby an explicit construction of all finite irreducible degenerate modules over $CK_6$.

All degenerate $E(1, 6)$-modules $\text{Ind}(F)$ can be represented by the diagram below (very similar to that for $E(5, 10)$ in [10]), with the point $(4, 0)$ excluded, where the nodes represent the highest weights of the modules $\text{Ind}(F)$, and arrows represent the morphisms between these modules. Here $\lambda_2, \lambda_1, \lambda_3$ are the fundamental weights of $\mathfrak{so}_6 = A_3$ (where $\lambda_1$ is attached to the middle node of the Dynkin diagram).

In the subsequent publication we shall compute cohomology of these complexes, providing thereby an explicit construction of all degenerate continuous irreducible $E(1, 6)$-modules, hence of all degenerate finite irreducible $CK_6$-modules.

This work is organized as follows: In section 2 we introduce notations and definitions of formal distributions, Lie conformal superalgebras and their modules. In section 3 we introduce the Lie conformal algebra $CK_6$, the annihilation Lie algebra $E(1, 6)$ and the induced modules. In section 4 we classify the singular vectors of the induced modules (Theorem 4.1), and in Theorem 4.3 we present the list of highest weights of degenerate irreducible modules. The last part of this section and Appendix A are devoted to their proofs through several lemmas. More precisely, we used the software Macaulay2 to simplify the computations, and Appendix A contains the notations in the complementary files that use Macaulay2 and the reduction procedure to find simplified conditions on singular vectors of small degree. All these simplified and equivalent list of equations, obtained with the software as explained in Appendix A are analyzed in details in the proofs of Lemmas 4.11, 4.10 in Section 4.
2 Formal distributions, Lie conformal superalgebras and their modules

In this section we introduce the basic definitions and notations in order to have a more or less self-contained work, for details see [2], [7], [8] and references there in.

Definition 2.1. A Lie conformal superalgebra $R$ is a left $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R \otimes R \rightarrow \mathbb{C}[\lambda] \otimes R, a \otimes b \mapsto [a_\lambda b]$, called the $\lambda$-bracket, and satisfying the following axioms ($a, b, c \in R$):

Conformal sesquilinearity $[\partial a_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda \partial b] = (\lambda + \partial)[a_\lambda b],$

Skew-symmetry $[a_\lambda b] = -(-1)^{p(a)p(b)}[b_{-\lambda-\partial} a],$

Jacobi identity $[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + (-1)^{p(a)p(b)}[b_\mu [a_\lambda c]].$

Here and further $p(a) \in \mathbb{Z}/2\mathbb{Z}$ is the parity of $a.$

A Lie conformal superalgebra is called finite if it has finite rank as a $\mathbb{C}[\partial]$-module. The notions of homomorphism, ideal and subalgebras of a Lie conformal superalgebra are defined in the usual way. A Lie conformal superalgebra $R$ is simple if $[R_\lambda R] \neq 0$ and contains no ideals except for zero and itself.
Definition 2.2. A module $M$ over a Lie conformal superalgebra $R$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R \otimes M \rightarrow \mathbb{C}[\lambda] \otimes M$, $a \otimes v \mapsto a\lambda v$, satisfying the following axioms $(a, b \in R, v \in M)$,

(M1)$_\lambda$ \quad $(\partial a)^M_\lambda v = [\partial^M, a^M_\lambda]v = -\lambda a^M_\lambda v,$

(M2)$_\lambda$ \quad $[a^M_\lambda, b^M_\mu]v = [a\lambda b]^M_{\lambda+\mu}v.$

An $R$-module $M$ is called finite if it is finitely generated over $\mathbb{C}[\partial]$. An $R$-module $M$ is called irreducible if it contains no non-trivial submodule, where the notion of submodule is the usual one.

Given a Lie conformal superalgebra $R$, let $\check{R} = R[t, t^{-1}]$ with $\check{\partial} = \partial + \partial t$ and define the bracket \[ [at^n, bt^m] = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} [a(j)b]t^{m+n-j}. \]

Observe that $\check{\partial}\check{R}$ is an ideal of $\check{R}$ with respect to this bracket, and consider the Lie superalgebra $\text{Alg}R = \check{R}/\check{\partial}\check{R}$ with this bracket.

An important tool for the study of Lie conformal superalgebras and their modules is the (extended) annihilation superalgebra. The annihilation superalgebra of a Lie conformal superalgebra $R$ is the subalgebra $A(R)$ (also denoted by $\text{Alg}R_+$) of the Lie superalgebra $\text{Alg}R$ spanned by all elements $at^n$, where $a \in R, n \in \mathbb{Z}_+$. It is clear from (2.1) that this is a subalgebra, which is invariant with respect to the derivation $\partial = -\partial t$ of $\text{Alg}R$. The extended annihilation superalgebra is defined as

$A(R)^e = (\text{Alg}R)^+ := \mathbb{C}\partial \ltimes (\text{Alg}R)_+.$

Introducing the generating series

$a_\lambda = \sum_{j \in \mathbb{Z}_+} \frac{\lambda^j}{j!} (at^j), \ a \in R, \quad (2.2)$

we obtain from (2.1):

$[a_\lambda, b_\mu] = [a\lambda b]_{\lambda+\mu}, \quad \partial(a_\lambda) = (\partial a)_\lambda = -\lambda a_\lambda. \quad (2.3)$

Formula (2.3) implies the following important proposition relating modules over a Lie conformal superalgebra $R$ to certain modules over the corresponding extended annihilation superalgebra $(\text{Alg}R)^+.$

Proposition 2.3. \[ A \text{ module over a Lie conformal superalgebra } R \text{ is the same as a module over the Lie superalgebra } (\text{Alg}R)^+ \text{ satisfying the property} \]

$a_\lambda m \in \mathbb{C}[\lambda] \otimes M \text{ for any } a \in R, m \in M. \quad (2.4)$

(One just views the action of the generating series $a_\lambda$ of $(\text{Alg}R)^+$ as the $\lambda$-action of $a \in R$.)
The problem of classifying modules over a Lie conformal superalgebra $R$ is thus reduced to the problem of classifying a class of modules over the Lie superalgebra $(\text{Alg} R)^{+}$.

Let $\mathfrak{g}$ be a Lie superalgebra satisfying the following three conditions (cf. [6], p.911):

(L1) $\mathfrak{g}$ is $\mathbb{Z}$-graded of finite depth $d \in \mathbb{N}$, i.e. $g = \bigoplus_{j \geq -d} g_j$ and $[g_i, g_j] \subset g_{i+j}$.

(L2) There exists a semisimple element $z \in g_0$ such that its centralizer in $g$ is contained in $g_0$.

(L3) There exists an element $\partial \in g_{-d}$ such that $[\partial, g_i] = g_{i-d}$, for $i \geq 0$.

Some examples of Lie superalgebras satisfying (L1)-(L3) are provided by annihilation superalgebras of Lie conformal superalgebras.

If $g$ is the annihilation superalgebra of a Lie conformal superalgebra, then the modules $V$ over $g$ that correspond to finite modules over the corresponding Lie conformal superalgebra satisfy the following conditions:

(1) For all $v \in V$ there exists an integer $j_0 \geq -d$ such that $g_{j} v = 0$, for all $j \geq j_0$.

(2) $V$ is finitely generated over $\mathbb{C}[\partial]$.

Motivated by this, the $g$-modules satisfying these two properties are called finite conformal modules.

We have a triangular decomposition

$$g = g_{<0} \oplus g_0 \oplus g_{>0}, \quad \text{with} \quad g_{<0} = \bigoplus_{j<0} g_j, \quad g_{>0} = \bigoplus_{j>0} g_j. \quad (2.5)$$

Let $g_{\geq 0} = \bigoplus_{j \geq 0} g_j$. Given a $g_{\geq 0}$-module $F$, we may consider the associated induced $g$-module

$$\text{Ind}(F) = \text{Ind}_{g_{\geq 0}}^g F = U(g) \otimes_{U(g_{\geq 0})} F,$$

called the generalized Verma module associated to $F$. We shall identify $\text{Ind}(F)$ with $U(g_{<0}) \otimes F$ via the PBW theorem.

Let $V$ be a $g$-module. The elements of the subspace

$$\text{Sing}(V) := \{ v \in V | g_{>0} v = 0 \}$$

are called singular vectors. For us the most important case is when $V = \text{Ind}(F)$. The $g_{\geq 0}$-module $F$ is canonically a $g_{\geq 0}$-submodule of $\text{Ind}(F)$, and $\text{Sing}(F)$ is a subspace of $\text{Sing}(\text{Ind}(F))$, called the subspace of trivial singular vectors. Observe that $\text{Ind}(F) = F \oplus F_{+}$, where $F_{+} = U_{+}(g_{<0}) \otimes F$ and $U_{+}(g_{<0})$ is the augmentation ideal of the algebra $U(g_{<0})$. Then non-zero elements of the space

$$\text{Sing}_{+}(\text{Ind}(F)) := \text{Sing}(\text{Ind}(F)) \cap F_{+}$$

are called non-trivial singular vectors. The following simple key result will be used in the rest of the paper, see [9, 6].
Theorem 2.4. Let \( g \) be a Lie superalgebra that satisfies (L1)-(L3).

(a) If \( F \) is an irreducible finite-dimensional \( g_{\geq 0} \)-module, then the subalgebra \( g_{>0} \) acts trivially on \( F \) and \( \text{Ind}(F) \) has a unique maximal submodule.

(b) Denote by \( \text{Ir}(F) \) the quotient by the unique maximal submodule of \( \text{Ind}(F) \). Then the map \( F \mapsto \text{Ir}(F) \) defines a bijective correspondence between irreducible finite-dimensional \( g_{0} \)-modules and irreducible finite conformal \( g \)-modules.

(c) A \( g \)-module \( \text{Ind}(F) \) is irreducible if and only if the \( g_{0} \)-module \( F \) is irreducible and \( \text{Ind}(F) \) has no non-trivial singular vectors.

In the following section we will describe the Lie conformal superalgebra \( CK_{6} \) and its annihilation superalgebra \( E(1, 6) \). In the remaining sections we shall study the induced \( E(1, 6) \)-modules and its singular vectors in order to apply Theorem 2.4 to get the classification of irreducible finite modules over the Lie conformal algebra \( CK_{6} \).

3 Lie conformal superalgebra \( CK_{6} \), annihilation Lie superalgebra \( E(1, 6) \) and the induced modules

Let \( \Lambda(n) \) be the Grassmann superalgebra in the \( n \) odd indeterminates \( \xi_{1}, \xi_{2}, \ldots, \xi_{n} \). Let \( t \) be an even indeterminate, and let \( \Lambda(1, n)_{+} = \mathbb{C}[t] \otimes \Lambda(n) \). The Lie conformal superalgebra \( K_{n} \) can be identified with

\[
K_{n} = \mathbb{C}[\partial] \otimes \Lambda(n), \quad (3.1)
\]

the \( \lambda \)-bracket for \( f = \xi_{i_{1}} \ldots \xi_{i_{r}}, g = \xi_{j_{1}} \ldots \xi_{j_{s}} \) being as follows [2]:

\[
[f_{\lambda}g] = \left((r - 2)\partial(fg) + (-1)^{r}\sum_{i=1}^{n}(\partial_{i}f)(\partial_{i}g)\right) + \lambda(r + s - 4)fg. \quad (3.2)
\]

The annihilation Lie superalgebra of \( K_{n} \) can be identified with (see [2])

\[
\mathcal{A}(K_{n}) = K(1, n)_{+} = \Lambda(1, n)_{+}, \quad (3.3)
\]

with the corresponding Lie bracket for elements \( f, g \in \Lambda(1, n) \) being

\[
[f, g] = \left(2f - \sum_{i=1}^{n}\xi_{i}\partial_{i}f\right)(\partial_{i}g) - (\partial_{i}f)\left(2g - \sum_{i=1}^{n}\xi_{i}\partial_{i}g\right) + (-1)^{p(f)}\sum_{i=1}^{n}(\partial_{i}f)(\partial_{i}g).
\]

The extended annihilation superalgebra is

\[
\mathcal{A}(K_{n})^{e} = K(1, n)^{+} = \mathbb{C}\partial \ltimes K(1, n)_{+},
\]
where \( \partial \) acts on it as \(-\text{ad} \partial \). Note that \( \mathcal{A}(K_n)^e \) is isomorphic to the direct sum of \( \mathcal{A}(K_n) \) and the trivial 1-dimensional Lie algebra \( \mathbb{C}(\partial + \frac{1}{2}) \).

We define in \( K(1, n)_+ \) a gradation by putting
\[
\deg(t^m \xi_{i_1} \cdots \xi_{i_k}) = 2m + k - 2,
\]
making it a \( \mathbb{Z} \)-graded Lie superalgebra of depth 2: \( K(1, n)_+ = \oplus_{j \geq -2}(K(1, n)_+) j \).

It is easy to check that \( K(1, n)_+ \) satisfies conditions (L1)-(L3).

We introduce the following notation:
\[
\xi_I := \xi_{i_1} \cdots \xi_{i_k}, \quad \text{if} \quad I = \{i_1, \ldots, i_k\},
\]
\[
|f| := k \quad \text{if} \quad f = \xi_{i_1} \cdots \xi_{i_k}.
\]

For a monomial \( \xi_I \in \Lambda(n) \), we let \( \xi_I^* \) be its Hodge dual, i.e. the unique monomial in \( \Lambda(n) \) such that \( \xi_I \xi_I^* = \xi_1 \cdots \xi_n \).

**Warning:** this definition corresponds to the one in [3] or [6], pp. 922, but in [2] Theorem 4.3, the Hodge dual was defined in a different way.

The Lie conformal superalgebra \( CK_6 \) is defined as the subalgebra of \( K_6 \) given by (cf. [4], Theorem 3.1)
\[
CK_6 = \mathbb{C}[\partial]-\text{span } \{ f - i(-1)^{|f|}(d/dt)^{3-|f|}f^* : f \in \Lambda(6), 0 \leq |f| \leq 3 \}.
\]

Now, we define a linear operator \( A : K(1, 6)_+ \to K(1, 6)_+ \) by (cf. [5], p.267)
\[
A(f) = (-1)^{\frac{d(d+1)}{2}} (\frac{d}{dt})^{3-d} f^*,
\]

where \( f \) is a monomial in \( K(1, 6)_+ \), \( d \) is the number of odd indeterminates in \( f \), the operator \( (\frac{d}{dt})^{-1} \) indicates integration with respect to \( t \) (i.e. it sends \( t^n \) to \( t^{n+1}/(n+1) \)), and \( f^* \) is the Hodge dual of \( f \). Then, the annihilation Lie superalgebra \( E(1, 6)_+ \) of \( CK_6 \) is identified with the subalgebra of \( K(1, 6)_+ \) given by the image of the operator \( I - iA \). Since the linear map \( A \) preserve the \( \mathbb{Z} \)-gradation, the subalgebra \( E(1, 6)_+ \) inherits the \( \mathbb{Z} \)-gradation.

Using Theorem 2.4, the classification of finite irreducible \( CK_6 \)-modules can be reduced to the study of induced modules for \( E(1, 6)_+ \). Observe that the graded subspaces of \( E(1, 6)_+ \) and \( K(1, 6)_+ \) with non-positive degree are the same. Namely,
\[
E(1, 6)_{-2} = \langle \{1\} \rangle,
E(1, 6)_{-1} = \langle \{\xi_i : 1 \leq i \leq 6\} \rangle
E(1, 6)_0 = \langle \{t\} \cup \{\xi_i \xi_j : 1 \leq i < j \leq 6\} \rangle
\]

We shall use the following notation for the basis elements of \( E(1, 6)_0 \) (cf. [2]):
\[
E_{00} = t, \quad F_{ij} = -\xi_i \xi_j.
\]
Observe that $E(1,6)_0 \simeq \mathbb{C}E_{00} \oplus \mathfrak{so}(6) \simeq \mathfrak{cso}(6)$. Take
\[
\partial := -\frac{1}{2} \mathbf{1}
\] as the element that satisfies (L3) in section 2.

For the rest of this work, $\mathfrak{g}$ will be $E(1,6)$. Let $F$ be a finite-dimensional irreducible $\mathfrak{g}_0$-module, which we extend to a $\mathfrak{g}_{\geq 0}$-module by letting $\mathfrak{g}_j$ with $j > 0$ acting trivially. Then we shall identify, as above:
\[
\text{Ind}(F) \simeq \Lambda(1,6) \otimes F \simeq \mathbb{C}[\partial] \otimes \Lambda(6) \otimes F
\] as $\mathbb{C}$-vector spaces.

Since the non-positive graded subspaces of $E(1,6)$ are the same as those of $K(1,6)_+$, the $\lambda$-action is given by restricting the $\lambda$-action in Theorem 4.1 in [2]. In the following theorem, we describe the $\mathfrak{g}$-action of $K(1,6)_+$ on $\text{Ind}(F)$ using the $\lambda$-action notation in (2.2), i.e.
\[
f_{\lambda}(g \otimes v) = \sum_{j \geq 0} \frac{\lambda^j}{j!} (t^j f) \cdot (g \otimes v)
\] for $f, g \in \Lambda(6)$ and $v \in F$.

**Theorem 3.1.** For any monomials $f, g \in \Lambda(6)$ and $v \in F$, where $F$ is a $\mathfrak{cso}(6)$-module, we have the following formula for the $\lambda$-action of $K(1,6)_+$ on $\text{Ind}(F)$:
\[
f_{\lambda}(g \otimes v) = (-1)^{p(f)} (|f| - 2) \partial (\partial f g) \otimes v + \sum_{i=1}^{6} \partial (\partial_i f g) \otimes v + \sum_{r<s} (-1)^{p(f) + p(g)} \partial (\partial_r \partial_s f g) \otimes F_{rs} v
\]
\[+ \lambda \left[ (-1)^{p(f)} (\partial f g) \otimes E_{00} v + (-1)^{p(f) + p(g)} \sum_{i=1}^{6} \partial (\partial_i f g) \otimes v + \sum_{i \neq j} \partial (\partial_i f g) \otimes F_{ij} v \right]
\]
\[+ \lambda^2 (-1)^{p(f)} \sum_{i<j} \partial f (\partial_i \partial_j g) \otimes F_{ij} v.
\]

In the last part of this section we shall state an easier formula for the $\lambda$-action in the induced module (see Theorem 4.3, [2]). This is done by taking the Hodge dual of the basis modified by a sign, since we are using the definition of Hodge dual given in [3], instead of the one used in [2]. Namely, let $T$ be the vector space automorphism of $\text{Ind}(F)$ given by $T(g \otimes v) = (-1)^{|g|} g^* \otimes v$, then the following theorem gives the formula for the composition $T \circ (f_{\lambda} \cdot) \circ T^{-1}$.
**Theorem 3.2.** Let \( F \) be a \( \mathfrak{cs}(6) \)-module. Then the \( \lambda \)-action of \( K(1,6)_+ \) in \( \text{Ind}(F) = \mathbb{C}[\partial] \otimes \Lambda(6) \otimes F \), given by Theorem [3.1], is equivalent to the following one:

\[
\begin{align*}
    f_{\lambda}(g \otimes v) &= (-1)^{\frac{|f|(f+1)}{2} + |f||g|} \times \\
    &\times \left\{(|f| - 2)\partial(fg) \otimes v - (-1)^{p(f)} \sum_{i=1}^{6} (\partial_i f)(\partial_i g) \otimes v - \sum_{r<s} (\partial_r \partial_s f)g \otimes F_{rs}v \\
    &+ \lambda \left[ fg \otimes E_{00}v - (-1)^{p(f)} \sum_{i=1}^{6} \partial_i (f_{\xi_i}g) \otimes v + (-1)^{p(f)} \sum_{i \neq j} (\partial_i f)\xi_j g \otimes F_{ij}v \right] \\
    &- \lambda^2 \sum_{i<j} f_{\xi_i} \xi_j g \otimes F_{ij}v \right\}.
\end{align*}
\]

4 Singular vectors

By Theorem [2.4] the classification of irreducible finite modules over the Lie conformal superalgebra \( CK_6 \) reduces to the study of singular vectors in the induced modules \( \text{Ind}(F) \), where \( F \) is an irreducible finite-dimensional \( \mathfrak{cs}(6) \)-module. This section will be devoted to the classification of singular vectors.

When we discuss the highest weight of vectors and singular vectors, we always mean with respect to the upper Borel subalgebra in \( E(1,6) \) generated by \( (E(1,6))_+ \) and the elements of the Borel subalgebra of \( \mathfrak{so}(6) \) in \( E(1,6)_0 \). More precisely, recall [3.6], where we defined \( F_{ij} = -\xi_i \xi_j \in E(1,6)_0 \simeq \mathbb{C} E_{00} \oplus \mathfrak{so}(6) \). Observe that \( F_{ij} \) corresponds to \( E_{ij} - E_{ji} \in \mathfrak{so}(6) \), where \( E_{ij} \) are the elements of the standard basis of matrices. Consider the following (standard) notation for \( \mathfrak{so}(6,\mathbb{C}) \) (cf. [11], p.83):

We take

\[
    H_j = i\ F_{2j-1,2j}, \quad 1 \leq j \leq 3,
\]

(4.1)

a basis of a Cartan subalgebra \( \mathfrak{h}_0 \). Let \( \varepsilon_j \in \mathfrak{h}_0^\ast \) be given by \( \varepsilon_j(H_k) = \delta_{jk} \). Let

\[
    \Delta = \{ \pm \varepsilon_i \pm \varepsilon_j \ | \ i < j \}
\]

be the set of roots. The root space decomposition is

\[
    \mathfrak{g} = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \quad \text{with} \quad \mathfrak{g}_\alpha = \mathbb{C} E_\alpha
\]

where, for \( 1 \leq l < j \leq 3 \),

\[
\begin{align*}
    E_{\varepsilon_l - \varepsilon_j} &= F_{2l-1,2j-1} + F_{2l,2j} + i(F_{2l-1,2j} - F_{2l,2j-1}), \\
    E_{\varepsilon_l + \varepsilon_j} &= F_{2l-1,2j-1} - F_{2l,2j} - i(F_{2l-1,2j} + F_{2l,2j-1}), \\
    E_{-(\varepsilon_l - \varepsilon_j)} &= F_{2l-1,2j-1} + F_{2l,2j} - i(F_{2l-1,2j} - F_{2l,2j-1}), \\
    E_{-(\varepsilon_l + \varepsilon_j)} &= F_{2l-1,2j-1} - F_{2l,2j} + i(F_{2l-1,2j} + F_{2l,2j-1}),
\end{align*}
\]

(4.2)
Let $\Pi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_2 + \varepsilon_3\}$ and $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j \mid i < j\}$, be the simple and positive roots respectively. Consider

$$\alpha_{ij} := F_{2i-1,2j-1} - iF_{2i,2j-1} = \frac{1}{2}(E_{\varepsilon_i - \varepsilon_j} + E_{\varepsilon_i + \varepsilon_j})$$

$$\beta_{ij} := F_{2i,2j} + iF_{2i-1,2j} = \frac{1}{2}(E_{\varepsilon_i - \varepsilon_j} - E_{\varepsilon_i + \varepsilon_j})$$

(4.3)

Then, the Borel subalgebra is

$$B_{\mathfrak{so}(6)} = \langle \{\alpha_{ij}, \beta_{ij} \mid 1 \leq i < j \leq 3\} \rangle .$$

(4.4)

Recall that the Cartan subalgebra $\mathfrak{h}$ in $(CK(1,6)_0 \simeq \mathbb{C}E_{00} \oplus \mathfrak{so}(6) \simeq \mathfrak{cso}(6)$ is spanned by the elements $E_{00}, H_1, H_2, H_3$. Let $e_0 \in \mathfrak{h}^*$ be the linear functional given by $e_0(E_{00}) = 1$ and $e_0(H_i) = 0$ for all $i$. Let

$$\begin{align*}
\lambda_1 &= \varepsilon_1 \\
\lambda_2 &= \frac{\varepsilon_1 + \varepsilon_2 - \varepsilon_3}{2} \\
\lambda_3 &= \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{2}
\end{align*}$$

(4.5)

be the fundamental weights of $\mathfrak{so}(6)$, extended to $\mathfrak{h}^*$ by letting $\lambda_i(E_{00}) = 0$. That is $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$, where $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \alpha_3 = \varepsilon_2 + \varepsilon_3$. Then the highest weight $\mu$ of the finite irreducible $\mathfrak{cso}(6)$-module $F_\mu$ can be written as

$$\mu = n_0 e_0 + n_1 \lambda_1 + n_2 \lambda_2 + n_3 \lambda_3,$$

(4.6)

where $n_1, n_2$ and $n_3$ are non-negative integers. In order to write explicitly weights for vectors in $CK(1,6)_+$-modules, we will consider the notation (4.6).

Consider a singular vector $\vec{m}$ in the $CK(1,6)_+$-module $\operatorname{Ind}(F) = \mathbb{C}[\partial] \otimes \Lambda(6) \otimes F$, where $F$ is an irreducible $\mathfrak{cso}(6)$-module, and the $\lambda$-action of $CK(1,6)_+$ in $\operatorname{Ind}(F)$ is given by restricting the $\lambda$-action of $K(1,6)_+$ in Theorem 3.2

Using (3.3), (3.9), and the description of the upper Borel subalgebra of $E(1,6)$, we obtain that a vector $\vec{m}$ in the $E(1,6)$-module $\operatorname{Ind}(F)$ is a singular vector if and only if the following conditions are satisfied

(S1) For all $f \in \Lambda(6)$, with $0 \leq |f| \leq 3$,

$$\frac{d^2}{d\lambda^2} \left( f^* \vec{m} - i(-1)^{|f|(|f|+1)} \frac{\lambda^{3-|f|} \lambda^{|f|}(f^* \lambda \vec{m})}{2} \right) = 0.$$
(S2) For all $f \in \Lambda(6)$, with $1 \leq |f| \leq 3$,
\[
\left. \frac{d}{d\lambda} \left( f_\lambda \vec{m} - i(-1)^{|f|(|f|+1)} \lambda^{3-|f|} (f^* \lambda \vec{m}) \right) \right|_{\lambda=0} = 0.
\]

(S3) For all $f$ with $|f| = 3$ or $f \in B_{so(6)}$,
\[
\left. \left( f_\lambda \vec{m} - i(-1)^{|f|(|f|+1)} \lambda^{3-|f|} (f^* \lambda \vec{m}) \right) \right|_{\lambda=0} = 0.
\]

In order to classify the finite irreducible $CK_6$-modules we should solve the equations (S1-S3) to obtain the singular vectors.

Observe that the $\mathbb{Z}$-gradation in $E(1,6)$, translates into a $\mathbb{Z}_{\leq 0}$-gradation in $\text{Ind}(F) = \mathbb{C}[\partial] \otimes \Lambda(6) \otimes F$ where $\text{wt} \ \partial = -2$ and $\text{wt} \ \xi_i = -1$, but we shall work with the $\lambda$-action given by Theorem 3.2 where we considered the (modified) Hodge dual basis, therefore from now on the $\mathbb{Z}_{\leq 0}$-gradation in $\text{Ind}(F)$ is given by $\text{wt} \ \partial = -2$ and $\text{wt} \ \xi_I = 6 - |I|$.

The next theorem is one of the main result of this section and gives us the complete classification of singular vectors:

**Theorem 4.1.** Let $F_\mu$ be an irreducible finite-dimensional $\mathfrak{_so}(6)$-module with highest weight $\mu$. Then, there exists a non-trivial singular vector $\vec{m}$ in the $E(1,6)$-module $\text{Ind}(F_\mu)$ if and only if the highest weight $\mu$ is one of the following:

(a) $\mu = \left( \frac{1}{2} + 4 \right) e_0 + \lambda_2$, where (up to scalar) $\vec{m} = \partial^2 g_5 + \partial g_3 + g_1$ of degree $-5$, given by (4.40), with singular weight
\[
\left( -\frac{1}{2} \right) e_0 + \lambda_3,
\]

(b) $\mu = \left( \frac{n}{2} + 4 \right) e_0 + n\lambda_2$, with $n \geq 2$, where (up to scalar) $\vec{m} = \sum_{|I|=3} \xi_I \otimes v_I$ of degree $-3$, given by (4.181), with singular weight
\[
\left( \frac{n}{2} + 1 \right) e_0 + (n-2)\lambda_2,
\]

(c) $\mu = \left( -\frac{n}{2} + 2 \right) e_0 + n\lambda_3$, with $n \geq 0$, where (up to scalar) $\vec{m} = \sum_{|I|=3} \xi_I \otimes v_I$ of degree $-3$, given by (4.183), with singular weight
\[
\left( -\frac{n}{2} - 1 \right) e_0 + (n+2)\lambda_3,
\((d)\) \(\mu = \left( n_1 + \frac{n_2}{2} + 4 \right) e_0 + n_1 \lambda_1 + n_2 \lambda_2, \) with \(n_1 \geq 1, n_2 \geq 0,\) where (up to scalar)
\[
\bar{m} = \sum_{|I|=5} \xi_I \otimes v_I \text{ of degree } -1, \text{ given by (4.249), with singular weight}
\]
\[
\mu = \left( n_1 + \frac{n_2}{2} + 3 \right) e_0 + (n_1 - 1) \lambda_1 + n_2 \lambda_2,
\]

\((e)\) \(\mu = \left( \frac{n_2}{2} - \frac{n_3}{2} + 2 \right) e_0 + n_2 \lambda_2 + n_3 \lambda_3, \) with \(n_2 \geq 1, n_3 \geq 0,\) where (up to scalar)
\[
\bar{m} = \sum_{|I|=5} \xi_I \otimes v_I \text{ of degree } -1, \text{ given by (4.254), with singular weight}
\]
\[
\mu = \left( \frac{n_2}{2} - \frac{n_3}{2} + 1 \right) e_0 + (n_2 - 1) \lambda_2 + (n_3 + 1) \lambda_3,
\]

\((f)\) \(\mu = -\left( n_1 + \frac{n_3}{2} \right) e_0 + n_1 \lambda_1 + n_3 \lambda_3, \) with \(n_1 \geq 0, n_3 \geq 0.\) where (up to scalar)
\[
\bar{m} = (\xi_\{2\c\} - i \xi_\{1\c\}) \otimes v_\mu \text{ of degree } -1, \text{ with singular weight}
\]
\[
\mu = -\left( n_1 + \frac{n_3}{2} + 1 \right) e_0 + (n_1 + 1) \lambda_1 + n_3 \lambda_3,
\]

**Remark 4.2.** (a) We have the explicit expression of all singular vectors in terms of the highest weight vector \(v_\mu\) in each case.

(b) The family (f) of singular vectors with \(n_3 = 0\) corresponds to the first family of singular vectors in \(K_6,\) computed in Theorem 5.1 [2].

(c) The family (d) of singular vectors with \(n_2 = 0\) corresponds to the second family of singular vectors in \(K_6,\) computed in Theorem 5.1 [2].

(d) The family (e) is new, with no analog in \(K_6.\)

(e) The highest weight where we have a singular vector of degree -5 correspond to the case \(k = \frac{1}{2}\) in the family (b), and the one parameter families (b) and (c) are the (not present) cases \(k = l\) in the families (d) and (e) respectively.

Using Theorem 4.1, we obtain the following theorem that is the main result of this work and gives us the complete list of highest weights of degenerate irreducible modules:

**Theorem 4.3.** Let \(F_\mu\) be an irreducible finite-dimensional \(\mathfrak{so}(6)\)-module with highest weight \(\mu.\) Then the \(E(1,6)\)-module \(\text{Ind}(F_\mu)\) is degenerate if and only if \(\mu\) is one of the following:

\((a)\) \(\mu = \left( n_1 + \frac{n_2}{2} + 4 \right) e_0 + n_1 \lambda_1 + n_2 \lambda_2, \) with \(n_1 \geq 1, n_2 \geq 0\) or \(n_1 = 0, n_2 \geq 1,\)
(b) \( \mu = \left( \frac{n_2}{2} - \frac{n_3}{2} + 2 \right) e_0 + n_2 \lambda_2 + n_3 \lambda_3 \), with \( n_2 \geq 0, n_3 \geq 0 \),

(c) \( \mu = -\left( n_1 + \frac{n_3}{2} \right) e_0 + n_1 \lambda_1 + n_3 \lambda_3 \), with \( n_1 \geq 0, n_3 \geq 0 \).

The rest of this section together with the Appendix A are devoted to the proof of this theorem. The proof will be done through several lemmas.

Recall that the Cartan subalgebra \( \mathfrak{h} \) in \( (CK(1,6))_0 \simeq \mathbb{C}E_{00} \oplus \mathfrak{so}(6) \simeq \mathfrak{cso}(6) \) is spanned by the elements

\[
E_{00}, H_1, H_2, H_3,
\]

and, for technical reasons as in our work [2], from now on we shall write the weights of an eigenvector for the Cartan subalgebra \( \mathfrak{h} \) as an \( 1+3 \)-tuple for the corresponding eigenvalues of this basis:

\[
\mu = (\mu_0; \mu_1, \mu_2, \mu_3).
\]

(4.7)

Let \( \mu = n_0 e_0 + n_1 \lambda_1 + n_2 \lambda_2 + n_3 \lambda_3 \) be the highest weight of the finite irreducible \( \mathfrak{cso}(6) \)-module \( F_\mu \), where \( n_1, n_2 \) and \( n_3 \) are non-negative integers, as in (4.6). Using the notation (4.7), this highest weight can be written as the \( 1+3 \)-tuple

\[
\mu = \left( n_0; n_1 + \frac{n_2}{2} + \frac{n_3}{2} , \frac{n_2}{2} + \frac{n_3}{2} , -\frac{n_2}{2} + \frac{n_3}{2} \right).
\]

(4.8)

Let \( \vec{m} \in \text{Ind}(F) = \mathbb{C}[\partial] \otimes \Lambda(6) \otimes F \) be a singular vector, then

\[
\vec{m} = \sum_{k=0}^N \sum_I \partial^k (\xi_I \otimes v_{I,k}), \quad \text{with } v_{I,k} \in F.
\]

Lemma 4.4. If \( \vec{m} \in \text{Ind}(F) \) is a singular vector, then the degree of \( \vec{m} \) in \( \partial \) is at most 2. Moreover, any singular vector have this form:

\[
\vec{m} = \partial^2 \sum_{|I|\geq 5} \xi_I \otimes v_{I,2} + \partial \sum_{|I|\geq 3} \xi_I \otimes v_{I,1} + \sum_{|I|\geq 1} \xi_I \otimes v_{I,0}.
\]

Proof. the proof of this lemma will be published in a second paper of the series of three papers.

The \( \mathbb{Z} \)-gradation in \( E(1,6) \), translates into a \( \mathbb{Z}_{\leq 0} \)-gradation in \( \text{Ind}(F) \):

\[
\text{Ind}(F) \simeq \Lambda(1,6) \otimes F \simeq \mathbb{C}[\partial] \otimes \Lambda(6) \otimes F
\]

\[
\simeq \mathbb{C} \mathfrak{1} \otimes F \oplus \mathbb{C}^6 \otimes F \oplus \left( \mathcal{C} \partial \otimes F \oplus \left( \mathcal{L}^2 (\mathbb{C}^6) \otimes F \right) \right) \oplus \cdots
\]

deg 0 deg -1 deg -2
Therefore, in the previous lemma, we have proved that any singular vector must have degree at most -5.

Recall that in the theorem that gives us the $\lambda$-action, we considered the Hodge dual of the natural bases in order to simplify the formula of the action. Hence, any singular vector must have one of the following forms:

\begin{align*}
\vec{m} = \partial^2 &\sum_{|I|=5} \xi_I \otimes v_{I,2} + \partial \sum_{|I|=3} \xi_I \otimes v_{I,1} + \sum_{|I|=1} \xi_I \otimes v_{I,0}, \quad \text{(Degree -5).} \\
\vec{m} = \partial^2 &\sum_{|I|=6} \xi_I \otimes v_{I,2} + \partial \sum_{|I|=4} \xi_I \otimes v_{I,1} + \sum_{|I|=2} \xi_I \otimes v_{I,0}, \quad \text{(Degree -4).} \\
\vec{m} = &\partial \sum_{|I|=5} \xi_I \otimes v_{I,1} + \sum_{|I|=3} \xi_I \otimes v_{I,0}, \quad \text{(Degree -3).} \\
\vec{m} = &\partial \sum_{|I|=6} \xi_I \otimes v_{I,1} + \sum_{|I|=4} \xi_I \otimes v_{I,0}, \quad \text{(Degree -2).} \quad \text{(4.9)} \\
\vec{m} = &\sum_{|I|=5} \xi_I \otimes v_{I,0}, \quad \text{(Degree -1).}
\end{align*}

Now, we shall introduce a very important notation. Observe that the formula for the action given by Theorem 3.2 have the form

\[ f_\lambda(g \otimes v) = \partial a + b + \lambda (B + \lambda^2 \ C) = (\lambda + \partial) (a + b + \lambda (B - a) + \lambda^2 \ C), \]

by taking the coefficients in $\partial$ and $\lambda^i$. Using it, we can write the $\lambda$-action on the singular vector $\vec{m} = \partial^2 \ m_2 + \partial \ m_1 + m_0$ of degree 2 in $\partial$, as follows

\[ f_\lambda \vec{m} = \left[ (\lambda + \partial) \ a_0 + b_0 + \lambda (B_0 - a_0) + \lambda^2 \ C_0 \right] \]
\[ + (\lambda + \partial) \left[ (\lambda + \partial) \ a_1 + b_1 + \lambda (B_1 - a_1) + \lambda^2 \ C_1 \right] \quad \text{(4.10)} \]
\[ + (\lambda + \partial)^2 \left[ (\lambda + \partial) \ a_2 + b_2 + \lambda (B_2 - a_2) + \lambda^2 \ C_2 \right]. \]

Obviously, these coefficients depend also in $f$ and $m$, and sometimes we shall write for example $a_2(f)$ or $a(f, m_2)$, instead of $a_2$, to emphasize the dependance, but we will keep it implicit in the notation if no confusion may arise. In a similar way, for the $\lambda$-action of $f^*$ on $\vec{m} = \partial^2 \ m_2 + \partial \ m_1 + m_0$ we use the notation

\[ f^*_\lambda \vec{m} = \left[ (\lambda + \partial) \ ad_0 + bd_0 + \lambda (Bd_0 - ad_0) + \lambda^2 \ Cd_0 \right] \]
\[ + (\lambda + \partial) \left[ (\lambda + \partial) \ ad_1 + bd_1 + \lambda (Bd_1 - ad_1) + \lambda^2 \ Cd_1 \right] \quad \text{(4.11)} \]
\[ + (\lambda + \partial)^2 \left[ (\lambda + \partial) \ ad_2 + bd_2 + \lambda (Bd_2 - ad_2) + \lambda^2 \ Cd_2 \right]. \]
As before, these coefficients depend also in \( f^* \) and \( m \), and sometimes we shall write for example \( ad_2(f) \) or \( ad(f, m_2) \), instead of \( ad_2 \), to emphasize the dependance, but we will keep it implicit in the notation if no confusion may arise.

**Lemma 4.5.** Let \( \vec{m} = \partial^2 m_2 + \partial m_1 + m_0 \) be a vector of degree at most -5. The conditions (S1)-(S3) on \( \vec{m} \) are equivalent to the following list of equations

- For \(|f| = 0\):
  
  \begin{align*}
  C_0 &= -B_1, \quad \text{(4.12)} \\
  2B_2 &= -a_2 - C_1, \quad \text{(4.13)} \\
  2bd_0 &= ia_2 - iC_1. \quad \text{(4.14)}
  \end{align*}

- For \(|f| = 1\):
  
  \begin{align*}
  3B_2 &= -2i bd_1 - 2i ad_0 - 2C_1, \quad \text{(4.15)} \\
  2C_0 &= a_1 - B_1 - 2bd_0 i, \quad \text{(4.16)} \\
  2a_2 &= -B_2, \quad \text{(4.17)} \\
  3Bd_0 &= iC_1 - bd_1 + 2ad_0, \quad \text{(4.18)} \\
  2b_2 &= -a_1 - B_1, \quad \text{(4.19)} \\
  b_1 &= -B_0. \quad \text{(4.20)}
  \end{align*}

- For \(|f| = 2\):
  
  \begin{align*}
  2C_0 &= -2Bd_0 i - B_1 + iad_0 - ibd_1, \quad \text{(4.21)} \\
  2b_2 &= -iad_0 - ibd_1 - B_1, \quad \text{(4.22)} \\
  bd_0 &= b_1 i + B_0 i. \quad \text{(4.23)}
  \end{align*}

- For \(|f| = 3\):
  
  \begin{align*}
  C_0 &= Cd_0 i, \quad \text{(4.24)} \\
  bd_0 &= -i b_0, \quad \text{(4.25)} \\
  B_1 &= Bd_1 i + a_1 - ad_1 i, \quad \text{(4.26)} \\
  b_2 &= bd_2 i - a_1 + ad_1 i, \quad \text{(4.27)} \\
  bd_1 &= -Bd_0 - B_0 i - b_1 i, \quad \text{(4.28)} \\
  ad_0 &= -a_0 i + Bd_0 + B_0 i. \quad \text{(4.29)}
  \end{align*}

- For \( f \in B_{20(6)} \):
  
  \begin{align*}
  b_2 &= 0, \quad \text{(4.30)} \\
  b_1 &= 0, \quad \text{(4.31)} \\
  b_0 &= 0. \quad \text{(4.32)}
  \end{align*}
Remark 4.6. The equations in Lemma 4.5 are written using the previously introduced notation. For example, strictly speaking, if \( \vec{m} = \partial^2 m_2 + \partial m_1 + m_0 \) then equation (4.23) for an element \( f = \xi_j \xi_k \) means

\[
bd(\xi_j\xi_k, m_0) = b(\xi_j\xi_k, m_1) i + B(\xi_j\xi_k, m_0) i. \tag{4.33}
\]

Proof. Using this notation, by taking coefficients in \( \partial^i \lambda^j \), conditions (S1)-(S3) translate into the following list, for \( f \in \Lambda(6) \) (see file ”equations.mws” where the computations were done using Maple, this file is located in the link written at the beginning of the Appendices):

- For \( |f| = 0 \):
  \[
  C_0 = -B_1 - b_2, \\
  B_2 = -\frac{a_2}{2} - \frac{C_1}{2}, \\
  bd_0 = \frac{1}{2} i \ a_2 - \frac{1}{2} i \ C_1, \\
  ad_0 = Bd_0, \\
  C_2 = 0, \ Cd_2 = 0, \ ad_2 = 0, \ Bd_2 = 0, \ Cd_1 = 0, \ Cd_0 = 0, \\
  bd_2 = -Bd_1, \\
  bd_1 = -Bd_0, \\
  ad_1 = Bd_1,
  \]

- For \( |f| = 1 \):
  \[
  B_2 = -\frac{2}{3} i \ bd_1 - \frac{2}{3} i \ ad_0 - \frac{2}{3} \ C_1, \\
  a_2 = \frac{1}{3} i \ bd_1 + \frac{1}{3} i \ ad_0 + \frac{C_1}{3}, \\
  C_0 = \frac{a_1}{2} - \frac{B_1}{2} - bd_0 i, \\
  C_2 = -Bd_1 i - bd_2 i, \\
  Bd_0 = \frac{1}{3} i \ C_1 - \frac{bd_1}{3} + \frac{2 \ ad_0}{3}, \\
  b_2 = -\frac{a_1}{2} - \frac{B_1}{2}, \\
  b_1 = -B_0,
  \]
$C_{d_2} = 0$, $a_{d_2} = 0$, $B_{d_2} = 0$, $C_{d_1} = 0$, $C_{d_0} = 0$, 
$a_{d_1} = B_{d_1}$.

- For $|f| = 2$:

  
  \begin{align*}
  B_2 &= -b_{d_2} i - \frac{1}{3} i a_{d_1} - \frac{2}{3} C_{d_1} - \frac{2}{3} i B_{d_1}, \\
  C_{d_0} &= \frac{1}{3} i C_{d_1} - \frac{B_{d_1}}{3} + \frac{a_{d_1}}{3}, \\
  C_0 &= -B_{d_0} i - \frac{B_{d_1}}{2} - \frac{1}{2} i a_{d_0} - \frac{1}{2} i b_{d_1} + \frac{a_{d_1}}{2}, \\
  a_2 &= \frac{C_{d_1}}{3} + \frac{1}{3} i B_{d_1} - \frac{1}{3} i a_{d_1}, \\
  b_2 &= -\frac{1}{2} i a_{d_0} - \frac{1}{2} i b_{d_1} - \frac{a_{d_1}}{2} - \frac{B_{d_1}}{2}, \\
  b_{d_0} &= b_1 i + B_0 i, \\
  C_2 &= -i B_{d_2}, \\
  C_{d_2} &= 0, a_{d_2} = 0, C_{d_1} = 0.
  \end{align*}

- For $|f| = 3$:

  \begin{align*}
  C_0 &= C_{d_0} i, \\
  C_{d_1} &= C_{d_1} i, \\
  C_2 &= C_{d_2} i, \\
  b_{d_0} &= -i b_0, \\
  a_2 &= a_{d_2} i, \\
  B_{d_1} &= B_{d_1} i + a_1 - a_{d_1} i, \\
  b_2 &= b_{d_2} i - a_1 + a_{d_1} i, \\
  b_{d_1} &= -B_{d_0} - B_0 i - b_1 i, \\
  a_{d_0} &= -a_0 i + B_{d_0} + B_0 i, \\
  B_2 &= B_{d_2} i,
  \end{align*}

- For $f \in B_{\theta_0(0)}$:

  \begin{align*}
  0 &= b_0, \\
  0 &= b_1 + a_0, \\
  0 &= b_2 + a_1, \\
  0 &= a_2.
  \end{align*}
Now, taking care of the length of the elements $\xi_i$ involved in the expression of vectors in $\text{Ind}(F)$ of degree at most -5, we observe that some equations are always zero, getting the list in the statement of the lemma.

In order to classify the singular vectors, we should impose equations (4.12)-(4.32) to the 5 possible forms of singular vectors listed in (4.9), depending on the degree. The following lemmas describe the result in each case.

**Lemma 4.7.** All the singular vectors of degree -5 are listed in the theorem.

*Proof.* Using the softwares Macaulay2 and Maple, the conditions of Lemma 4.5 on the singular vector $\bar{m}_5$ were simplified in several steps. First, the conditions of Lemma 4.5 were reduced to a linear system of equations with a $992 \times 544$ matrix. After the reduction of this linear system, we obtained in the middle of the file "m5-macaulay-2" a simplified list of 542 equations (see Appendix A for the details of this reduction). In particular, we obtained the following identities:

\begin{align*}
0 &= v_1 + v_{1,3,4,5,6} \\
0 &= v_3 + v_{1,2,3,5,6} \\
0 &= v_5 + v_{1,2,3,4,5} \\
0 &= v_{1,2,3} - v_{1,2,3,5,6} i \\
0 &= v_{1,2,5} - v_{1,2,3,4,5} i \\
0 &= v_{1,3,4} - v_{1,3,4,5,6} i \\
0 &= v_{1,3,6} + v_{2,4,6} \\
0 &= v_{1,4,6} - v_{2,4,6} i \\
0 &= v_{2,3,4} - v_{1,3,4,5,6} \\
0 &= v_{2,3,6} - v_{2,4,6} i \\
0 &= v_{2,5,6} - v_{1,3,4,5,6} \\
0 &= v_{3,4,6} - v_{1,2,3,4,5} \\
0 &= v_{4,5,6} - v_{1,2,3,5,6} \\
0 &= v_{1,2,4,5,6} + v_{1,2,3,5,6} i \\
0 &= v_2 - v_{1,3,4,5,6} i \\
0 &= v_4 - v_{1,2,3,5,6} i \\
0 &= v_6 - v_{1,2,3,4,5} i \\
0 &= v_{1,2,4} - v_{1,2,3,5,6} \\
0 &= v_{1,2,6} - v_{1,2,3,4,5} \\
0 &= v_{1,3,5} + v_{2,4,6} i \\
0 &= v_{1,4,5} + v_{2,4,6} i \\
0 &= v_{1,5,6} - v_{1,3,4,5,6} i \\
0 &= v_{2,3,5} + v_{2,4,6} \\
0 &= v_{2,4,5} - v_{2,4,6} i \\
0 &= v_{3,4,5} - v_{1,2,3,4,5} i \\
0 &= v_{3,5,6} - v_{1,2,3,5,6} i \\
0 &= v_{2,3,4,5,6} + v_{1,3,4,5,6} i \\
0 &= v_{1,2,3,4,6} + v_{1,2,3,4,5} i
\end{align*}

In particular, all the vectors $v_I$ can be written in terms of $v_1, v_3, v_5$ and $v_{1,3,5}$. By imposing this identities, we obtained at the end of the file "m5-macaulay-2" the following simplified list of 64 equations (see Appendix A for the details of this reduction):
\begin{align*}
0 &= H_1 v_1 + 1/2 v_1 \\
0 &= H_3 v_1 - 1/2 v_1 \\
0 &= H_2 v_3 + 1/2 v_3 \\
0 &= H_1 v_5 - 1/2 v_5 \\
0 &= H_3 v_5 + 1/2 v_5 \\
0 &= H_2 v_{1,3,5} + 1/2 v_{1,3,5} \\
0 &= E_{00} v_1 - 9/2 v_1 \\
0 &= E_{00} v_5 - 9/2 v_5 \\
\end{align*}

\begin{align*}
0 &= E_{-(\varepsilon_1-\varepsilon_2)} v_1 \\
0 &= E_{-(\varepsilon_1-\varepsilon_2)} v_5 \\
0 &= E_{-(\varepsilon_1-\varepsilon_3)} v_1 \\
0 &= E_{-(\varepsilon_1-\varepsilon_3)} v_5 - 2 v_1 \\
0 &= E_{-(\varepsilon_2-\varepsilon_3)} v_1 \\
0 &= E_{-(\varepsilon_2-\varepsilon_3)} v_5 - 2 v_3 \\
0 &= E_{-(\varepsilon_1+\varepsilon_2)} v_1 \\
0 &= E_{-(\varepsilon_1+\varepsilon_2)} v_{5} - 2 v_{1,3,5} \\
0 &= E_{-(\varepsilon_1+\varepsilon_3)} v_1 \\
0 &= E_{-(\varepsilon_1+\varepsilon_3)} v_5 \\
0 &= E_{-(\varepsilon_2+\varepsilon_3)} v_1 - 2 v_{1,3,5} \\
0 &= E_{-(\varepsilon_2+\varepsilon_3)} v_5 \\
\end{align*}

\begin{align*}
0 &= H_2 v_1 - 1/2 v_1 \\
0 &= H_1 v_3 - 1/2 v_3 \\
0 &= H_3 v_3 - 1/2 v_3 \\
0 &= H_2 v_5 - 1/2 v_5 \\
0 &= H_1 v_{1,3,5} + 1/2 v_{1,3,5} \\
0 &= H_3 v_{1,3,5} + 1/2 v_{1,3,5} \\
0 &= E_{00} v_3 - 9/2 v_3 \\
0 &= E_{00} v_{1,3,5} - 9/2 v_{1,3,5} \\
\end{align*}

\begin{align*}
0 &= E_{-(\varepsilon_1-\varepsilon_2)} v_3 - 2 v_1 \\
0 &= E_{-(\varepsilon_1-\varepsilon_2)} v_{1,3,5} \\
0 &= E_{-(\varepsilon_1-\varepsilon_3)} v_3 \\
0 &= E_{-(\varepsilon_1-\varepsilon_3)} v_{1,3,5} \\
0 &= E_{-(\varepsilon_2-\varepsilon_3)} v_3 \\
0 &= E_{-(\varepsilon_2-\varepsilon_3)} v_{1,3,5} \\
0 &= E_{-(\varepsilon_2-\varepsilon_3)} v_{1,3,5} \\
0 &= E_{-(\varepsilon_1+\varepsilon_2)} v_3 \\
0 &= E_{-(\varepsilon_1+\varepsilon_2)} v_{1,3,5} \\
0 &= E_{-(\varepsilon_1+\varepsilon_3)} v_3 \\
0 &= E_{-(\varepsilon_1+\varepsilon_3)} v_{1,3,5} \\
0 &= E_{-(\varepsilon_2+\varepsilon_3)} v_3 \\
0 &= E_{-(\varepsilon_2+\varepsilon_3)} v_{1,3,5} \\
\end{align*}

\begin{align*}
0 &= E_{-(\varepsilon_1+\varepsilon_2)} v_{1,3,5} - 2 v_{1,3,5} \\
0 &= E_{-(\varepsilon_1+\varepsilon_3)} v_3 + 2 v_{1,3,5} \\
0 &= E_{-(\varepsilon_1+\varepsilon_3)} v_{1,3,5} \\
0 &= E_{-(\varepsilon_2+\varepsilon_3)} v_3 \\
0 &= E_{-(\varepsilon_2+\varepsilon_3)} v_{1,3,5} \\
\end{align*}

\text{and}
\begin{align*}
0 &= E_{\varepsilon_1 - \varepsilon_2} v_1 + 2v_3 & 0 &= E_{\varepsilon_1 - \varepsilon_2} v_3 \\
0 &= E_{\varepsilon_1 - \varepsilon_2} v_5 & 0 &= E_{\varepsilon_1 - \varepsilon_2} v_{1,3,5} \\
0 &= E_{\varepsilon_1 - \varepsilon_3} v_1 + 2v_5 & 0 &= E_{\varepsilon_1 - \varepsilon_3} v_3 \\
0 &= E_{\varepsilon_1 - \varepsilon_3} v_5 & 0 &= E_{\varepsilon_1 - \varepsilon_3} v_{1,3,5} \\
0 &= E_{\varepsilon_2 - \varepsilon_3} v_1 & 0 &= E_{\varepsilon_2 - \varepsilon_3} v_3 + 2v_5 \\
0 &= E_{\varepsilon_2 - \varepsilon_3} v_5 & 0 &= E_{\varepsilon_2 - \varepsilon_3} v_{1,3,5} \\
0 &= E_{\varepsilon_1 + \varepsilon_2} v_1 & 0 &= E_{\varepsilon_1 + \varepsilon_2} v_3 \\
0 &= E_{\varepsilon_1 + \varepsilon_2} v_5 & 0 &= E_{\varepsilon_1 + \varepsilon_2} v_{1,3,5} + 2v_5 \\
0 &= E_{\varepsilon_1 + \varepsilon_3} v_1 & 0 &= E_{\varepsilon_1 + \varepsilon_3} v_3 \\
0 &= E_{\varepsilon_1 + \varepsilon_3} v_5 & 0 &= E_{\varepsilon_1 + \varepsilon_3} v_{1,3,5} - 2v_3 \\
0 &= E_{\varepsilon_2 + \varepsilon_3} v_1 & 0 &= E_{\varepsilon_2 + \varepsilon_3} v_3 \\
0 &= E_{\varepsilon_2 + \varepsilon_3} v_5 & 0 &= E_{\varepsilon_2 + \varepsilon_3} v_{1,3,5} + 2v_1 \\
\end{align*}

Therefore, a vector

\[
\vec{m}_5 = \partial^2 \sum_{|I|=5} \xi_I \otimes v_I + \partial \sum_{|I|=3} \xi_I \otimes v_I + \sum_{|I|=1} \xi_I \otimes v_I
\]

satisfies conditions of Lemma 4.5 if and only if equations (4.34-4.37) hold. We divided the final analysis of these equations in several cases:

- **Case** \( v_5 \neq 0 \): Using (4.37) we obtain that the Borel subalgebra of \( \mathfrak{so}(6) \) annihilates \( v_5 \). Hence, it is a highest weight vector in the irreducible \( \mathfrak{so}(6) \)-module \( F_{\mu} \), and by (4.35), the highest weight is

\[
\mu = \left( \frac{9}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right).
\]

Using (4.36), the other vectors are completely determined by the highest weight vector \( v_5 \), namely

\[
\begin{align*}
    v_1 &= \frac{1}{2} E_{-(\varepsilon_1 - \varepsilon_3)} v_5, \\
    v_3 &= \frac{1}{2} E_{-(\varepsilon_2 - \varepsilon_3)} v_5, \\
    v_{1,3,5} &= \frac{1}{2} E_{-(\varepsilon_1 + \varepsilon_2)} v_5.
\end{align*}
\]

After lengthy computation, it is easy to see that (4.35,4.37) hold by using (4.38).
and (4.39). Therefore, the vector
\[
\vec{m}_5 = \partial^2 \left[ \sum_{l=1}^{3} (\xi_{2l})^c - i \xi_{(2l-1) c} \right] \otimes v_{2l-1}
\]
\[
+ \partial \left[ (i \xi_{134} + \xi_{234} + i \xi_{156} + \xi_{256}) \otimes v_1 + (i \xi_{123} + \xi_{124} + i \xi_{356} + \xi_{456}) \otimes v_3 + (i \xi_{125} + \xi_{126} + i \xi_{345} + \xi_{346}) \otimes v_5 + (i \xi_{136} + \xi_{236} + i \xi_{145} + \xi_{146} + i \xi_{235} + \xi_{245} - i \xi_{246} - \xi_{135}) \otimes v_{1,3,5} \right]
\]
\[
+ \sum_{l=1}^{3} (\xi_{2l} + i \xi_{2l-1}) \otimes v_{2l-1}
\]
is a singular vector of \(\text{Ind}(F_{\mu})\), where \(v_5\) is a highest weight vector in \(F_{\mu}\), \(\mu = (9/2; 1/2, 1/2, -1/2)\) and \(v_1, v_3, v_{1,3,5}\) are given by (4.39). By computing
\[
E_{00} \cdot \vec{m}_5 = \text{coefficient of } \lambda^1 (1 \chi \vec{m}_5)
\]
\[
H_1 \cdot \vec{m}_5 = \text{coefficient of } \lambda^0 (-i \xi_1 \xi_2 \chi \vec{m}_5)
\]
\[
H_2 \cdot \vec{m}_5 = \text{coefficient of } \lambda^0 (-i \xi_3 \xi_4 \chi \vec{m}_5)
\]
\[
H_3 \cdot \vec{m}_5 = \text{coefficient of } \lambda^0 (-i \xi_5 \xi_6 \chi \vec{m}_5),
\]
one can prove that
\[
\text{wt } \vec{m}_5 = \left( \frac{9}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right),
\]
finishing this case.

- Case \(v_5 = 0\) and \(v_3 \neq 0\): In this case, using (4.37) and (4.35), we have that \(v_3\) is a highest weight vector in \(F_{\mu}\), with \(\mu = (9/2; 1/2, 1/2, -1/2, 1/2)\) that is not dominant integral, getting a contradiction.

- Case \(v_5 = 0, v_3 = 0\) and \(v_1 \neq 0\): In this case, using (4.37) and (4.35), we have that \(v_1\) is a highest weight vector in \(F_{\mu}\), with \(\mu = (9/2; -1/2, 1/2, 1/2)\) that is not dominant integral, getting a contradiction.

- Case \(0 = v_5 = v_3 = v_1\) and \(v_{1,3,5} \neq 0\): In this case, using (4.37) and (4.35), we have that \(v_{1,3,5}\) is a highest weight vector in \(F_{\mu}\), with \(\mu = (9/2; -1/2, -1/2, -1/2)\) that is not dominant integral, getting a contradiction and finishing the proof.

Lemma 4.8. There is no singular vector of degree -4.

Proof. The proof of this lemma was done entirely with the softwares Macaulay2 and Maple. The conditions on the singular vector \(\vec{m}_4\) were reduced to a linear systems of equations with a \(1104 \times 527\) matrix, whose rank is 527 (see Appendix A for details). Therefore, there is no non-trivial solution of this linear system, proving that there is no singular vector of degree -4, finishing the lemma.
Lemma 4.9. All the singular vectors of degree -3 are listed in the theorem.

Proof. Using the softwares Macaulay2 and Maple, the conditions of Lemma 4.5 on the singular vector $\vec{m}_3$ were simplified in several steps. First, the conditions of Lemma 4.5 were reduced to a linear system of equations with a $694 \times 442$ matrix. After the reduction of this linear system, we obtained at the end of the file "m3-macaulay-1" a simplified list of 397 equations (see Appendix A for the details of this reduction). In particular, we obtained the following identities:

\[
\begin{align*}
0 &= v_{1,2,3} - v_{4,5,6} i \\
0 &= v_{1,2,5} - v_{3,4,6} i \\
0 &= v_{1,3,4} - v_{2,5,6} i \\
0 &= v_{1,3,6} - v_{2,4,5} i \\
0 &= v_{1,4,6} + v_{2,3,5} i \\
0 &= v_{2,3,4,5,6} \\
0 &= v_{1,2,4,5,6} \\
0 &= v_{1,2,3,4,6}
\end{align*}
\]

Now, we have to impose the identities (4.42) to reduce the number of variables. Observe that everything can be written in terms of $v_{1,j,k}$ with $2 \leq j < k \leq 6$.

Unfortunately, the result is not enough to obtain in a clear way the possible highest weight vectors. For example, after the reduction and some extra computations it is possible to see that

\[
(v_{1,3,6} - v_{1,4,5}) + i (v_{1,3,5} + v_{1,4,6})
\]

is annihilated by the Borel subalgebra. Hence, it is necessary to impose (4.42) and make a change of variables. We produced an auxiliary file where we imposed (4.42), and after the analysis of the results, we found that the following change of variable is convenient:

\[
\begin{align*}
u_1 &= v_{1,2,3} - i v_{1,2,4} \\
u_2 &= v_{1,2,3} + i v_{1,2,4} \\
u_3 &= v_{1,2,5} - i v_{1,2,6} \\
u_4 &= v_{1,2,6} + i v_{1,2,6} \\
u_5 &= v_{1,3,4} - v_{1,5,6} \\
u_6 &= v_{1,3,4} + v_{1,5,6} \\
u_7 &= v_{1,3,5} - v_{1,4,6} + i (v_{1,3,6} + v_{1,4,5}) \\
u_8 &= v_{1,3,5} + v_{1,4,6} - i (v_{1,3,6} - v_{1,4,5}) \\
u_9 &= v_{1,3,5} - v_{1,4,6} - i (v_{1,3,6} + v_{1,4,5}) \\
u_{10} &= v_{1,3,5} + v_{1,4,6} + i (v_{1,3,6} - v_{1,4,5})
\end{align*}
\]
Observe that all the equations will be written in terms of $u_i$ with $1 \leq i \leq 10$. By imposing this identities, we obtained at the end of the file "m3-macaulay-2" the following simplified list of 125 equations (see Appendix A for the details of this reduction):

\begin{align*}
0 &= H_1 u_1 - 1/4 E_{-(e_1-e_3)} u_{10} i + u_1 \\
0 &= -H_1 u_1 + H_2 u_1 - u_1 \\
0 &= H_1 u_1 + H_3 u_1 \\
0 &= H_1 u_2 - 1/4 E_{-(e_1+e_2)} u_5 - 1/4 E_{-(e_1+e_3)} u_8 i + 3/2 u_2 \\
0 &= H_1 u_2 + H_2 u_2 - 1/2 E_{-(e_1+e_2)} u_5 + 2 u_2 \\
0 &= -H_1 u_2 + H_3 u_2 + 1/2 E_{-(e_1+e_2)} u_5 - u_2 \\
0 &= H_1 u_3 - 1/4 E_{-(e_1-e_3)} u_5 + 1/4 E_{-(e_1-e_2)} u_8 i + 3/2 u_3 \\
0 &= H_1 u_3 + H_2 u_3 - 1/2 E_{-(e_1-e_3)} u_5 + u_3 \\
0 &= -H_1 u_3 + H_3 u_3 + 1/2 E_{-(e_1-e_3)} u_5 - 2 u_3 \\
0 &= H_1 u_4 + H_3 u_4 + u_4 \\
0 &= H_2 u_4 + H_3 u_4 + u_4 \\
0 &= H_3 u_4 - 1/4 E_{-(e_1+e_2)} u_{10} i \\
0 &= H_1 u_5 + 1/4 E_{-(e_2-e_3)} u_{10} i \\
0 &= -H_1 u_5 + H_2 u_5 + u_5 \\
0 &= H_2 u_5 + H_3 u_5 \\
0 &= 1/2 E_{-(e_1+e_3)} u_3 + H_1 u_6 + H_3 u_6 + 2 u_6 \\
0 &= 1/2 E_{-(e_1+e_2)} u_1 + H_1 u_6 + H_2 u_6 + 2 u_6 \\
0 &= H_2 u_6 + H_3 u_6 + 1/2 E_{-(e_2+e_3)} u_9 i + 2 u_6 \\
0 &= -E_{-(e_1+e_2)} u_4 i + H_1 u_7 + H_2 u_7 - H_3 u_7 + E_{00} u_7 - 2 u_7 \\
0 &= E_{-(e_1+e_3)} u_2 i + H_1 u_7 - H_2 u_7 + H_3 u_7 + E_{00} u_7 - 2 u_7 \\
0 &= -E_{-(e_2+e_3)} u_6 i - H_1 u_7 + H_2 u_7 + H_3 u_7 + E_{00} u_7 - 2 u_7
\end{align*}
\[ 0 = H_1u_8 - E_{00}u_8 + 4u_8 \] 
\[ 0 = -1/2E_{-}(e_2-e_3)u_5 i + H_2u_8 + E_{00}u_8 - 3u_8 \] 
\[ 0 = H_2u_8 + H_3u_8 \] 
\[ 0 = H_1u_9 + H_2u_9 + H_3u_9 - E_{00}u_9 + 4u_9 \] 
\[ 0 = 1/2E_{-}(e_1-e_3)u_3 i + H_2u_9 - E_{00}u_9 + 3u_9 \] 
\[ 0 = -1/2E_{-}(e_1-e_3)u_1 i + H_3u_9 - E_{00}u_9 + 3u_9 \] 
\[ 0 = H_1u_{10} - E_{00}u_{10} + 4u_{10} \] 
\[ 0 = H_2u_{10} - E_{00}u_{10} + 4u_{10} \] 
\[ 0 = H_3u_{10} + E_{00}u_{10} - 4u_{10} \] 
\[ 0 = -H_1u_1 + E_{00}u_1 - 5u_1 \] 
\[ 0 = -H_1u_2 + E_{00}u_2 - 5u_2 \] 
\[ 0 = -H_1u_3 + E_{00}u_3 - 5u_3 \] 
\[ 0 = H_3u_4 + E_{00}u_4 - 4u_4 \] 
\[ 0 = -H_3u_5 + E_{00}u_5 + 1/2E_{-}(e_2-e_3)u_{10} i - 3u_5 \] 
\[ 0 = -H_1u_6 + E_{00}u_6 + 1/2E_{-}(e_2+e_3)u_9 i - 4u_6 \]

together with

\[ 0 = E_{-}(e_1-e_2)u_1 \] 
\[ 0 = E_{-}(e_2-e_3)u_1 + E_{-}(e_1-e_3)u_5 \] 
\[ 0 = E_{-}(e_1-e_2)u_2 + E_{-}(e_1+e_3)u_3 \] 
\[ 0 = E_{-}(e_1-e_3)u_2 - E_{-}(e_1+e_3)u_3 \] 
\[ 0 = E_{-}(e_2-e_3)u_2 + E_{-}(e_1+e_3)u_8 i \] 
\[ 0 = E_{-}(e_1-e_3)u_3 + E_{-}(e_2-e_3)u_9 i \] 
\[ 0 = E_{-}(e_2-e_3)u_3 + E_{-}(e_1-e_3)u_8 i \] 
\[ 0 = E_{-}(e_1-e_2)u_4 \] 
\[ 0 = E_{-}(e_1+e_2)u_1 + E_{-}(e_1-e_3)u_4 \] 
\[ 0 = E_{-}(e_2-e_3)u_4 - E_{-}(e_1+e_3)u_5 \] 
\[ 0 = E_{-}(e_1-e_2)u_5 + 2u_1 \] 
\[ 0 = E_{-}(e_1-e_2)u_6 - E_{-}(e_1+e_3)u_9 i \] 
\[ 0 = E_{-}(e_1-e_3)u_6 + E_{-}(e_1+e_3)u_9 i \] 
\[ 0 = -E_{-}(e_1+e_2)u_3 + E_{-}(e_2-e_3)u_6 \] 
\[ 0 = E_{-}(e_1+e_3)u_6 i + E_{-}(e_1-e_3)u_7 \] 
\[ 0 = -E_{-}(e_1+e_2)u_6 i + E_{-}(e_1-e_3)u_7 \] 
\[ 0 = E_{-}(e_1-e_2)u_{10} \] 
\[ 0 = E_{-}(e_1+e_3)u_1 \] 
\[ 0 = E_{-}(e_2+e_3)u_1 + 2u_4 \]
\[ 0 = E_{-(\varepsilon_2 + \varepsilon_3)} u_2 \]  
\[ 0 = E_{-(\varepsilon_2 + \varepsilon_3)} u_3 - 2u_2 \]  
\[ 0 = E_{-(\varepsilon_1 + \varepsilon_3)} u_4 \]  
\[ 0 = E_{-(\varepsilon_2 + \varepsilon_3)} u_4 \]  
\[ 0 = E_{-(\varepsilon_1 + \varepsilon_3)} u_5 + 2u_4 \]  
\[ 0 = E_{-(\varepsilon_2 + \varepsilon_3)} u_5 \]  
\[ 0 = E_{-(\varepsilon_2 + \varepsilon_3)} u_8 \]  
\[ 0 = E_{-(\varepsilon_1 + \varepsilon_3)} u_{10} \]  
\[ 0 = E_{-(\varepsilon_2 + \varepsilon_3)} u_{10} \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_1 - 2u_5 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_1 + 2u_{10} \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_2 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_2 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_2 + 2u_4 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_3 - 2u_{8} \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_3 + 2u_5 \]  
\[ 0 = E_{\varepsilon_2 - \varepsilon_3} u_3 - 2u_1 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_4 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_4 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_4 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_5 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_5 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_5 - 2u_{10} \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_6 + 2u_2 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_6 + 2u_4 \]  
\[ 0 = E_{\varepsilon_2 - \varepsilon_3} u_6 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_7 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_7 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_7 \]  
\[ 0 = E_{\varepsilon_2 - \varepsilon_3} u_8 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_8 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_8 \]  
\[ 0 = E_{\varepsilon_2 - \varepsilon_3} u_8 + 4u_5 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_9 + 4u_3 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_9 - 4u_1 \]  
\[ 0 = E_{\varepsilon_2 - \varepsilon_3} u_9 \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_{10} \]  
\[ 0 = E_{\varepsilon_1 - \varepsilon_3} u_{10} \]  
\[ 0 = E_{\varepsilon_2 - \varepsilon_3} u_{10} \]
Therefore, a singular vector of degree -3 must have the simplified form

\[
\vec{m}_3 = \sum_{|I|=3} \xi_I \otimes v_I
\]

and it satisfies conditions of Lemma 4.5 if and only if equations (4.42), (4.44) and (4.145-4.169) hold. We divided the final analysis of these equations in several cases:
Case $u_{10} \neq 0$: Using (4.137, 4.139) and (4.167, 4.169) we obtain that the Borel subalgebra of $\mathfrak{so}(6)$ annihilates $u_{10}$. Hence, it is a highest weight vector in the irreducible $\mathfrak{cso}(6)$-module $F_\mu$, and by (4.72, 4.74), the highest weight is

$$\mu = (k + 4 ; k, -k), \quad \text{with } 2k \in \mathbb{Z}_{\geq 0}. \quad (4.170)$$

Then we shall prove that the cases $k = 0$ and $k = 1/2$ are not possible. Using (4.111), (4.127), (4.149) and other similar equations, we deduce that if $u_{10} \neq 0$ then $u_i \neq 0$ for all $i$. Now, we shall see that all $u_i$ are completely determined by the highest weight vector $u_{10}$. If $k = 0$, we are working with the trivial $\mathfrak{so}(6)$ representation, and using (4.111) we obtain $u_{10} = 0$ getting a contradiction. Assume that $k \neq 0$.

Now, applying $E_{\epsilon_1 - \epsilon_3}$ to (4.45), we can prove that

$$u_1 = \frac{i}{4k} E_{-(\epsilon_1 - \epsilon_3)} u_{10}. \quad (4.171)$$

Similarly, using (4.56) and (4.60), we have

$$u_4 = -\frac{i}{4k} E_{-(\epsilon_1 + \epsilon_2)} u_{10}, \quad (4.172)$$
$$u_5 = -\frac{i}{4k} E_{-(\epsilon_2 - \epsilon_3)} u_{10}. \quad (4.173)$$

Applying $E_{\epsilon_1 + \epsilon_2}$ to (4.49) and using (4.143), we can prove that

$$2(2k - 1)u_2 = E_{-(\epsilon_1 + \epsilon_2)} u_5. \quad (4.174)$$

If $k \neq \frac{1}{2}$, we have

$$u_2 = \frac{1}{2(2k - 1)} E_{-(\epsilon_1 + \epsilon_2)} u_5$$
$$= \frac{-i}{8k(2k - 1)} E_{-(\epsilon_1 + \epsilon_2)} E_{-(\epsilon_2 - \epsilon_3)} u_{10} \quad (4.175)$$
$$= \frac{1}{2(2k - 1)} E_{-(\epsilon_2 - \epsilon_3)} u_4.$$
Using (4.58) and (4.152), we have

\[ u_6 = -\frac{1}{2(2k-1)} E_{-(\varepsilon_1+\varepsilon_2)} u_1 \]
\[ = -\frac{i}{8k(2k-1)} E_{-(\varepsilon_1+\varepsilon_2)} E_{-(\varepsilon_1-\varepsilon_3)} u_{10} \]
\[ = \frac{1}{2(2k-1)} E_{-(\varepsilon_1-\varepsilon_3)} u_4. \]  

(4.177)

By (4.63), (4.149), (4.158) and (4.172), we have

\[ u_7 = \frac{i}{2(2k-1)} E_{-(\varepsilon_1+\varepsilon_2)} u_4 \]
\[ = \frac{1}{8k(2k-1)} E_{-(\varepsilon_1+\varepsilon_2)} E_{-(\varepsilon_1+\varepsilon_2)} u_{10}. \]  

(4.178)

Using (4.67), (4.127), (4.133) and (4.173), we have

\[ u_8 = \frac{i}{2(2k-1)} E_{-(\varepsilon_2-\varepsilon_3)} u_5 \]
\[ = \frac{1}{8k(2k-1)} E_{-(\varepsilon_2-\varepsilon_3)} E_{-(\varepsilon_2-\varepsilon_3)} u_{10}. \]  

(4.179)

By (4.71), (4.111), (4.135) and (4.171), we have

\[ u_9 = \frac{-i}{2(2k-1)} E_{-(\varepsilon_1-\varepsilon_3)} u_1 \]
\[ = \frac{1}{8k(2k-1)} E_{-(\varepsilon_1-\varepsilon_3)} E_{-(\varepsilon_1-\varepsilon_3)} u_{10}. \]  

(4.180)

After a lengthy computation, it is possible to see that (4.45)-(4.169) hold by using (4.170) with \( k \neq 1/2 \), and the expressions of \( u_i \) obtained in (4.171)-(4.180). Therefore,
using the expressions of \( v_{k,l,j} \)'s given in terms of \( u_i \) as in (A.14), the vector

\[
\tilde{m}_3 = 2 \left[ (\xi_{1,2,3} - i \xi_{1,2,3} \xi) - (\xi_{3,5,6} - i \xi_{3,5,6} \xi) \right] \otimes u_1, \\
+ 2 \left[ (\xi_{1,2,3} - i \xi_{1,2,3} \xi) + (\xi_{3,5,6} - i \xi_{3,5,6} \xi) \right] \otimes u_2, \\
+ 2 \left[ (\xi_{1,2,5} - i \xi_{1,2,5} \xi) - (\xi_{3,4,5} - i \xi_{3,4,5} \xi) \right] \otimes u_3, \\
+ 2 \left[ (\xi_{1,2,5} - i \xi_{1,2,5} \xi) + (\xi_{3,4,5} - i \xi_{3,4,5} \xi) \right] \otimes u_4, \\
+ 2 \left[ (\xi_{1,3,4} - i \xi_{1,3,4} \xi) - (\xi_{1,5,6} - i \xi_{1,5,6} \xi) \right] \otimes u_5, \\
+ 2 \left[ (\xi_{1,3,4} - i \xi_{1,3,4} \xi) + (\xi_{1,5,6} - i \xi_{1,5,6} \xi) \right] \otimes u_6, \\
+ \left[ (\xi_{1,3,5} + i \xi_{1,3,5} \xi) - (\xi_{2,4,5} + i \xi_{2,4,5} \xi) \right] \otimes u_7, \\
+ \left[ (\xi_{2,3,6} + i \xi_{2,3,6} \xi) - (\xi_{2,4,5} + i \xi_{2,4,5} \xi) \right] \otimes u_8, \\
+ \left[ (\xi_{2,3,6} + i \xi_{2,3,6} \xi) + (\xi_{4,1,6} + i \xi_{4,1,6} \xi) \right] \otimes u_9, \\
+ \left[ (\xi_{2,3,6} + i \xi_{2,3,6} \xi) - (\xi_{4,1,6} + i \xi_{4,1,6} \xi) \right] \otimes u_{10}.
\]

(4.181)

\[
\text{is a singular vector of } \text{Ind}(F_\mu), \text{ where the } u_i \text{'s are written in (4.174-4.180) in terms of } u_{10}, \text{ where } u_{10} \text{ is a highest weight vector in } F_\mu, \text{ and } \mu = (k + 4; k, -k) \text{ with } 2k \in \mathbb{Z}_{>0} \text{ and } k \neq \frac{1}{2}. \text{ Now using (4.11), one can prove that}
\]

\[
\text{wt } \tilde{m}_3 = (k + 1 ; k - 1, k - 1, -k + 1)
\]

finishing this case.

- **Case** \( u_{10} = 0 \) and \( u_5 \neq 0 \): In this case, using (4.122-4.124) and (4.152-4.154), we have that \( u_5 \) is a highest weight vector in \( F_\mu \). Considering (4.57-4.59) and (4.79), we have \( \mu = (4; 0, -1, 1) \) that is not dominant integral, getting a contradiction.

- **Case** \( u_{10} = u_5 = 0 \) and \( u_1 \neq 0 \): In this case, using (4.116-4.118) and (4.146-4.148), we have that \( u_1 \) is a highest weight vector in \( F_\mu \). Considering (4.45-4.47) and (4.75), we have \( \mu = (4; -1, 0, 1) \) that is not dominant integral, getting a contradiction.

- **Case** \( u_{10} = u_5 = u_1 = 0 \) and \( u_8 \neq 0 \): In this case, using (4.131-4.133) and (4.161-4.163), we have that \( u_8 \) is a highest weight vector in \( F_\mu \). Considering (4.66-4.68) we have \( \mu = (k + 4; k, -k - 1, k + 1) \) that is not dominant integral, getting a contradiction.

- **Case** \( u_{10} = u_5 = u_1 = u_8 = 0 \) and \( u_4 \neq 0 \): In this case, using (4.119-4.121) and (4.149-4.151), we have that \( u_4 \) is a highest weight vector in \( F_\mu \). Considering (4.54-4.56) and (4.78), we have \( \mu = (4; -1, -1, 0) \) that is not dominant integral, getting a contradiction.

- **Case** \( u_{10} = u_5 = u_1 = u_8 = u_4 = 0 \) and \( u_3 \neq 0 \): In this case, using (4.116-4.118) and (4.146-4.148), we have that \( u_3 \) is a highest weight vector in \( F_\mu \). Considering (4.51-
and \([1.77]\), we have \(\mu = (7/2; -3/2, 1/2, 1/2)\) that is not dominant integral, getting a contradiction.

- Case \(u_{10} = u_5 = u_1 = u_8 = u_4 = u_3 = 0\) and \(u_2 \neq 0\): In this case, using \([1.13][4.115]\) and \([4.143][4.145]\), we have that \(u_2\) is a highest weight vector in \(F_\mu\). Considering \([4.48][4.50]\) and \([4.76]\), we have \(\mu = (7/2; -3/2, -1/2, -1/2)\) that is not dominant integral, getting a contradiction.

- Case \(u_{10} = u_5 = u_1 = u_8 = u_4 = u_3 = u_2 = u_9 = 0\) and \(u_6 \neq 0\): In this case, using \([1.13][4.136]\) and \([4.164][4.166]\), we have that \(u_6\) is a highest weight vector in \(F_\mu\). Considering \([4.69][4.71]\), we have \(\mu = (-k + 2; k, -(k + 1), -(k + 1))\) that is not dominant integral, getting a contradiction.

- Case \(u_{10} = u_5 = u_1 = u_8 = u_4 = u_3 = u_2 = u_9 = 0\) and \(u_6 \neq 0\): In this case, using \([4.128][4.130]\) and \([4.158][4.160]\), we have that \(u_7\) is a highest weight vector in \(F_\mu\). Considering \([4.63][4.65]\), we have \(\mu = (-k + 2; k, k, k)\) which is a multiple of the spin representation with \(2k \in \mathbb{Z}_{\geq 0}\). In this case \(u_i = 0\) for all \(i \neq 7\) and most of the equations \([4.45][4.169]\) are trivial, and it is easy to check that the remaining equations hold in this case. Therefore, using \([4.14]\), we have that the vector

\[
\bar{m}_3 = (\xi_{\{1,3,5\}} - \xi_{\{1,4,6\}} - i (\xi_{\{1,3,6\}} + \xi_{\{1,4,5\}})) \otimes u_7 \\
+ \left( i (\xi_{\{1,3,5\}} c - \xi_{\{1,4,6\}} c) - (\xi_{\{1,3,6\}} c + \xi_{\{1,4,5\}} c) \right) \otimes u_7
\]

is a singular vector in \(\text{Ind}(F_\mu)\), where \(\mu = (-k + 2; k, k, k)\) with \(2k \in \mathbb{Z}_{\geq 0}\) and \(u_7\) is a highest weight vector in \(F_\mu\). Now using \([4.41]\), one can prove that

\[\text{wt } \bar{m}_3 = (-k - 1; k + 1, k + 1, k + 1)\]

finishing the classification of singular vectors of degree -3.

**Lemma 4.10.** There is no singular vector of degree -2.

**Proof.** Using the softwares Macaulay2 and Maple, the conditions of Lemma [1.3] on the singular vector \(\bar{m}_2\) were reduced to a linear system of equations with a \(268 \times 272\) matrix. After the reduction of this linear system, we obtained at the end of the file ”m2-macaulay” or in file ”m2-ecuations.pdf” a simplified list of 192 equations (see Appendix [A] for the details of this reduction). The 15th and 16th equations of this list are the following

\[
0 = -i * F_{1,2} * v_{3,4,5,6} + E * v_{3,4,5,6} - 5 * v_{3,4,5,6} + i * v_{1,2,3,4,5,6} \quad (4.184)
\]

\[
0 = E * v_{1,2,3,4,5,6} - 3 * v_{1,2,3,4,5,6} \quad (4.185)
\]
and at the end of this list we have the conditions

\[
0 = F_{1,2} * v_{1,2,3,4,5,6} - v_{3,4,5,6} \\
0 = F_{1,3} * v_{1,2,3,4,5,6} + v_{2,4,5,6} \\
0 = F_{1,4} * v_{1,2,3,4,5,6} - v_{2,3,5,6} \\
0 = F_{1,5} * v_{1,2,3,4,5,6} + v_{2,3,4,6} \\
0 = F_{1,6} * v_{1,2,3,4,5,6} - v_{2,3,4,5} \\
0 = F_{2,3} * v_{1,2,3,4,5,6} - i * v_{2,4,5,6} \\
0 = F_{2,4} * v_{1,2,3,4,5,6} + i * v_{2,3,5,6} \\
0 = F_{2,5} * v_{1,2,3,4,5,6} - i * v_{2,3,4,6} \\
0 = F_{2,6} * v_{1,2,3,4,5,6} + i * v_{2,3,4,5} \\
0 = F_{3,4} * v_{1,2,3,4,5,6} - v_{1,2,5,6} \\
0 = F_{3,5} * v_{1,2,3,4,5,6} + v_{1,2,4,6} \\
0 = F_{3,6} * v_{1,2,3,4,5,6} - v_{1,2,4,5} \\
0 = F_{4,5} * v_{1,2,3,4,5,6} - i * v_{1,2,4,6} \\
0 = F_{4,6} * v_{1,2,3,4,5,6} + i * v_{1,2,4,5} \\
0 = F_{5,6} * v_{1,2,3,4,5,6} - v_{1,2,3,4}.
\]

(4.186)

Therefore, if \( \vec{m}_2 = \partial \sum_{|I|=6} \xi_I \otimes v_I + \sum_{|I|=4} \xi_I \otimes v_I \) is a singular vector in \( \text{Ind}(F_\mu) \), using equations (4.186), we prove that \( v_{1,2,3,4,5,6} \in F_\mu \) is annihilated by the Borel subalgebra of \( \mathfrak{so}(6) \) (see (4.14)), and using that \( F_\mu \) is irreducible, we get that \( v_{1,2,3,4,5,6} \) is a highest weight vector. Now, we shall compute the corresponding weight \( \mu \). Recall (4.11) and observe that using (4.185) and (4.186), the equation (4.184) is equivalent to the following

\[
(H_1^2 + 2H_1 + 1)v_{1,2,3,4,5,6} = 0,
\]

(4.187)

obtaining that \( H_1v_{1,2,3,4,5,6} = -v_{1,2,3,4,5,6} \). Therefore, the weight \( \mu \) is not dominant integral, getting a contradiction and finishing the proof. \( \square \)

**Lemma 4.11.** All the singular vectors of degree -1 are listed in the theorem.

**Proof.** Since the singular vectors found in [2] for \( K_6 \) are also singular vectors for \( CK_6 \), using (B42-B43) in [2], we have that it is convenient to introduce the following notation:

\[
\vec{m}_1 = \sum_{i=1}^{6} \xi_{\{i\}} \circ v_{\{i\}} \\
= \sum_{l=1}^{3} \left[ (\xi_{\{2l\}} \circ + i\xi_{\{2l-1\}}) \otimes w_l + (\xi_{\{2l\}} \circ - i\xi_{\{2l-1\}}) \otimes \overrightarrow{w} \right]
\]

(4.188)
that is, for $1 \leq l \leq 3$

$$v_{(2l)} = w_l + \overline{w}_l, \quad v_{(2l-1)} = i(w_l - \overline{w}_l)$$  \hspace{1cm} (4.189)$$

or equivalently, for $1 \leq l \leq 3$

$$w_l = \frac{1}{2}(v_{(2l)} - i v_{(2l-1)}), \quad \overline{w}_l = \frac{1}{2}(v_{(2l)} + i v_{(2l-1)}).$$  \hspace{1cm} (4.190)$$

We applied the change of variables (4.189), and using the softwares Macaulay2 and Maple, the conditions of Lemma 4.5 on the singular vector $\hat{m}_1$ were simplified in several steps. First, the conditions of Lemma 4.5 were reduced to a linear system of equations with a $62 \times 102$ matrix. After the reduction of this linear system, we obtained at the end of the file "m1-macaulay" a simplified list of 51 equations (see Appendix A for the details of this reduction). More precisely, we obtained the following identities:

$$0 = H_1 w_1 - E_{00} w_1 + 4w_1$$  \hspace{1cm} (4.191)$$

$$0 = H_2 w_1 + H_3 w_1$$  \hspace{1cm} (4.192)$$

$$0 = \frac{1}{2} E_{-(\epsilon_1 - \epsilon_2)} w_1 + H_1 w_2 + H_3 w_2 + w_2$$  \hspace{1cm} (4.193)$$

$$0 = -\frac{1}{2} E_{-(\epsilon_1 - \epsilon_2)} w_1 + H_2 w_2 - E_{00} w_2 + 3w_2$$  \hspace{1cm} (4.194)$$

$$0 = \frac{1}{2} E_{-(\epsilon_1 - \epsilon_3)} w_1 + \frac{1}{2} E_{-(\epsilon_2 - \epsilon_3)} w_2 + H_1 w_3 + H_2 w_3 + 2w_3$$  \hspace{1cm} (4.195)$$

$$0 = -\frac{1}{2} E_{-(\epsilon_1 - \epsilon_3)} w_1 - \frac{1}{2} E_{-(\epsilon_2 - \epsilon_3)} w_2 + H_3 w_3 - E_{00} w_3 + 2w_3$$  \hspace{1cm} (4.196)$$

$$0 = \frac{1}{2} E_{-(\epsilon_1 + \epsilon_2)} w_1 + \frac{1}{2} E_{-(\epsilon_1 + \epsilon_3)} w_3 + H_1 \overline{w}_1 + E_{00} \overline{w}_1 - \frac{1}{2} E_{-(\epsilon_1 - \epsilon_2)} \overline{w}_2 - \frac{1}{2} E_{-(\epsilon_1 - \epsilon_3)} \overline{w}_3$$  \hspace{1cm} (4.197)$$

$$0 = \frac{1}{2} E_{-(\epsilon_1 + \epsilon_2)} w_1 - \frac{1}{2} E_{-(\epsilon_1 + \epsilon_3)} w_3 + H_2 \overline{w}_2 + \frac{1}{2} E_{-(\epsilon_1 - \epsilon_2)} \overline{w}_2 - \frac{1}{2} E_{-(\epsilon_1 - \epsilon_3)} \overline{w}_3$$  \hspace{1cm} (4.198)$$

$$0 = -\frac{1}{2} E_{-(\epsilon_1 + \epsilon_2)} w_1 + \frac{1}{2} E_{-(\epsilon_2 + \epsilon_3)} w_3 + H_3 \overline{w}_3 + \frac{1}{2} E_{-(\epsilon_1 - \epsilon_2)} \overline{w}_3 - \frac{1}{2} E_{-(\epsilon_2 - \epsilon_3)} \overline{w}_3$$  \hspace{1cm} (4.199)$$

$$0 = -\frac{1}{2} E_{-(\epsilon_1 + \epsilon_2)} w_1 + \frac{1}{2} E_{-(\epsilon_2 + \epsilon_3)} w_3 + H_2 \overline{w}_2 + E_{00} \overline{w}_2 - \frac{1}{2} E_{-(\epsilon_2 - \epsilon_3)} \overline{w}_3 - \overline{w}_2$$  \hspace{1cm} (4.200)$$

$$0 = -\frac{1}{2} E_{-(\epsilon_1 + \epsilon_3)} w_1 + \frac{1}{2} E_{-(\epsilon_2 + \epsilon_3)} w_2 + H_1 \overline{w}_3 - H_2 \overline{w}_3$$  \hspace{1cm} (4.201)$$

$$0 = -\frac{1}{2} E_{-(\epsilon_1 + \epsilon_3)} w_1 - \frac{1}{2} E_{-(\epsilon_2 + \epsilon_3)} w_2 + H_3 \overline{w}_3 + E_{00} \overline{w}_3 - 2\overline{w}_3$$  \hspace{1cm} (4.202)$$

$$0 = -E_{-(\epsilon_1 + \epsilon_2)} w_3 + E_{-(\epsilon_2 - \epsilon_3)} \overline{w}_1 - E_{-(\epsilon_1 - \epsilon_3)} \overline{w}_2$$  \hspace{1cm} (4.203)$$

$$0 = E_{-(\epsilon_1 + \epsilon_3)} w_2 + E_{-(\epsilon_1 - \epsilon_2)} \overline{w}_3$$  \hspace{1cm} (4.204)$$

$$0 = E_{-(\epsilon_2 + \epsilon_3)} w_1$$  \hspace{1cm} (4.205)$$

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and

\begin{align}
0 &= E_{e_1-e_2} w_1 \\
0 &= E_{e_1-e_3} w_1 \\
0 &= E_{e_2-e_3} w_1 \\
0 &= E_{e_1-e_2} w_2 - 2w_1 \\
0 &= E_{e_1-e_3} w_2 \\
0 &= E_{e_2-e_3} w_2 \\
0 &= E_{e_1-e_3} w_3 \\
0 &= E_{e_2-e_3} w_3 - 2w_1 \\
0 &= E_{e_1-e_2} w_3 \\
0 &= E_{e_1-e_3} w_3 - 2w_2 \\
0 &= E_{e_1-e_2} w_1 + 2w_2 \\
0 &= E_{e_2-e_3} w_1 + 2w_3 \\
0 &= E_{e_1-e_2} w_2 \\
0 &= E_{e_1-e_3} w_2 \\
0 &= E_{e_1-e_2} w_3 \\
0 &= E_{e_1-e_3} w_3 \\
0 &= E_{e_2-e_3} w_2 + 2w_3 \\
0 &= E_{e_1-e_2} w_3 \\
0 &= E_{e_1-e_3} w_3 \\
0 &= E_{e_2-e_3} w_3 \\
0 &= E_{e_1+e_2} w_1 \\
0 &= E_{e_1+e_3} w_1 \\
0 &= E_{e_2+e_3} w_1 \\
0 &= E_{e_1+e_2} w_2 \\
0 &= E_{e_1+e_3} w_2 \\
0 &= E_{e_2+e_3} w_2 \\
0 &= E_{e_1+e_2} w_3 \\
0 &= E_{e_1+e_3} w_3 \\
0 &= E_{e_2+e_3} w_3 \\
0 &= E_{e_1+e_2} w_1 - 2w_2 \\
0 &= E_{e_1+e_3} w_1 - 2w_3 \\
0 &= E_{e_2+e_3} w_1 \\
0 &= E_{e_1+e_2} w_2 + 2w_1 \\
0 &= E_{e_1+e_3} w_2 \\
0 &= E_{e_2+e_3} w_2 \\
0 &= E_{e_1+e_2} w_3 - 2w_3 \\
0 &= E_{e_1+e_3} w_3 \\
0 &= E_{e_2+e_3} w_3 + 2w_1 \\
0 &= E_{e_2+e_3} w_3 + 2w_2 
\end{align}
We divided the final analysis of these equations in several cases:

- Case $w_1 \neq 0$: Using (4.206-4.208) and (4.224-4.226) we obtain that the Borel subalgebra of $\mathfrak{so}(6)$ annihilates $w_1$. Hence, it is a highest weight vector in the irreducible $\mathfrak{cso}(6)$-module $F_\mu$, and by (4.191-4.192), the (dominant integral) highest weight is

$$\mu = (k+4; k, l, -l), \quad \text{with } 2k \in \mathbb{Z}_{\geq 0}, 2l \in \mathbb{Z}_{\geq 0} \text{ and } k-l \in \mathbb{Z}_{\geq 0}. \quad (4.242)$$

Then we shall prove that the case $k = l$ is not possible. Using (4.209), (4.213), (4.233) and other similar equations, we deduce that if $w_1 \neq 0$ then $w_i \neq 0 \neq w_i$ for all $i$. Now, we shall see that all $w_i$’s and $w_i$’s are completely determined by the highest weight vector $w_1$. More precisely, using (4.208), we have that $\text{wt}_{w_2} = (k+4; k-1, l+1, -l)$. Hence, from (4.193) we can prove that

$$2(l-k)w_2 = E_{-(\epsilon_1-\epsilon_2)}w_1. \quad (4.243)$$

If $k \neq l$, we have

$$w_2 = \frac{1}{2(l-k)}E_{-(\epsilon_1-\epsilon_2)}w_1. \quad (4.244)$$

If $k = l$, using (4.208), we have $w_2 \in [F_\mu]_{\mu-(-\epsilon_1-\epsilon_2)}$ that has dimension 0 if $k = l$, which is a contradiction since $w_2 \neq 0$.

Therefore, from now on we shall assume that $k \neq l$. Similarly, using (4.213), we have that $w_3 = (k+4; k-1, l, -l+1)$. Hence, from (4.195) we can prove that

$$w_3 = -\frac{1}{2(k+l+1)} \left( E_{-(\epsilon_1-\epsilon_3)}w_1 + E_{-(\epsilon_2-\epsilon_3)}w_2 \right). \quad (4.245)$$

Using (4.240), we have that $\text{wt}_{\overline{w}_3} = (k+4; k-1, l, -l-1)$. Hence, from the sum of (4.197) and (4.198) we can prove that

$$\overline{w}_3 = \frac{1}{2(k-l)}E_{-(\epsilon_1+\epsilon_3)}w_1. \quad (4.246)$$

Using (4.236), we have that $\text{wt}_{\overline{w}_2} = (k+4; k-1, l-1, -l)$. Hence, from the sum of (4.199) and (4.200) we can prove that

$$\overline{w}_2 = \frac{1}{2(k+l+1)} \left( E_{-(\epsilon_1+\epsilon_2)}w_1 + E_{-(\epsilon_2-\epsilon_3)}\overline{w}_3 \right). \quad (4.247)$$

Using (4.233), we have that $\text{wt}_{\overline{w}_1} = (k+4; k-2, l, -l)$. Hence, from the sum of (4.197) and (4.198) we can prove that

$$\overline{w}_1 = \frac{1}{2(k+l+1)} \left( E_{-(\epsilon_1-\epsilon_3)}\overline{w}_3 - E_{-(\epsilon_1+\epsilon_2)}w_2 \right). \quad (4.248)$$
Now, we have an explicit expression of all \( w_i \)'s and \( \overline{w}_j \)'s in terms of \( w_1 \). After some lengthy computations it is possible to prove that equations (4.1-4.24) hold. Hence, the vector

\[
\vec{m}_1 = \sum_{l=1}^{3} \left[ (\xi_{(2l)} + i\xi_{(2l-1)}) \otimes w_l + (\xi_{(2l)} - i\xi_{(2l-1)}) \otimes \overline{w}_l \right] \tag{4.249}
\]

is a singular vector, where \( w_1 \) is a highest weight vector of \( F_\mu \), \( \mu = (k + 4 ; k, l, -l) \), with \( 2k \in \mathbb{Z}_{\geq 0}, 2l \in \mathbb{Z}_{\geq 0}, k - l \in \mathbb{Z}_{> 0} \), and all \( w_i \)'s and \( \overline{w}_j \)'s are written in terms of \( w_1 \) in (4.244), (4.245), (4.248), (4.247) and (4.246). Now using (4.41), one can prove that

\[
\text{wt } \vec{m}_1 = (k + 3 ; k - 1, l, -l)
\]

finishing this case.

- **Case** \( w_1 = 0 \) and \( w_2 \neq 0 \): Using (4.209-4.211) and (4.227-4.229) we obtain that the Borel subalgebra of \( \mathfrak{so}(6) \) annihilates \( w_2 \). Hence, it is a highest weight vector in the irreducible \( \mathfrak{so}(6) \)-module \( F_\mu \), and considering (4.193-4.194), we have

\[
\mu = (l + 3 ; k, l, -k - 1), \quad \text{with } 2k \in \mathbb{Z}_{\geq 0}, 2l \in \mathbb{Z}_{\geq 0},
\]

that is not dominant integral, getting a contradiction.

- **Case** \( w_1 = w_2 = 0 \) and \( w_3 \neq 0 \): Using (4.212-4.214) and (4.230-4.232) we obtain that the Borel subalgebra of \( \mathfrak{so}(6) \) annihilates \( w_3 \). Hence, it is a highest weight vector in the irreducible \( \mathfrak{so}(6) \)-module \( F_\mu \), and considering (4.195-4.196), we have

\[
\mu = (l + 2 ; k, -k - 2, l),
\]

that is not dominant integral, getting a contradiction.

- **Case** \( w_1 = w_2 = w_3 = 0 \) and \( \overline{w}_3 \neq 0 \): Using (4.221-4.223) and (4.239-4.241) we obtain that the Borel subalgebra of \( \mathfrak{so}(6) \) annihilates \( \overline{w}_3 \). Hence, it is a highest weight vector in the irreducible \( \mathfrak{so}(6) \)-module \( F_\mu \), and considering (4.201-4.202), we have

\[
\mu = (-l + 2 ; k, k, l), \quad \text{with } 2k \in \mathbb{Z}_{\geq 0}, 2l \in \mathbb{Z}, k + l \in \mathbb{Z}_{\geq 0}, k - l \in \mathbb{Z}_{\geq 0}. \tag{4.250}
\]

Then we will see that the case \( k = l \) is not possible. Using (4.216) and (4.220), we have \( \overline{w}_1 \neq 0 \neq \overline{w}_2 \).

Now, we shall see that all \( \overline{w}_j \)'s are completely determined by the highest weight vector \( \overline{w}_3 \). More precisely, applying \( E_{\epsilon_2 - \epsilon_3} \) to (4.193), we can prove that

\[
2(k - l)\overline{w}_2 = E_{-(\epsilon_2 - \epsilon_3)} \overline{w}_3. \tag{4.251}
\]

If \( k \neq l \), we have

\[
\overline{w}_2 = \frac{1}{2(k - l)} E_{-(\epsilon_2 - \epsilon_3)} \overline{w}_3. \tag{4.252}
\]
If \( k = l \), using (4.220), we have \( 0 \neq \overline{w}_2 \in [F_\mu]_{\mu-(\epsilon_2-\epsilon_3)} \), but \( \dim [F_\mu]_{\mu-(\epsilon_2-\epsilon_3)} = 0 \), getting a contradiction.

Therefore, from now on we shall assume that \( k \neq l \). Similarly, applying \( E_{\epsilon_1-\epsilon_3} \) to the sum of (4.197) and (4.198), we can prove that

\[
\overline{w}_1 = \frac{1}{2(k-l)} E_{-(\epsilon_1-\epsilon_3)} \overline{w}_3.
\] (4.253)

In this case, equations (4.191-4.241) collapse to a few ones and it is easy to see that all of them hold. Hence, the vector

\[
\overline{m}_1 = \frac{1}{2(k-l)} \left( (\xi(2)^{\epsilon} - i\xi(1)^{\epsilon}) \otimes E_{-(\epsilon_1-\epsilon_3)} \overline{w}_3 + \right.
\]

\[
\left. \frac{1}{2(k-l)} (\xi(4)^{\epsilon} - i\xi(3)^{\epsilon}) \otimes E_{-(\epsilon_2-\epsilon_3)} \overline{w}_3 + (\xi(6)^{\epsilon} - i\xi(5)^{\epsilon}) \otimes \overline{w}_3 \right)
\]

is a singular vector, where \( \overline{w}_3 \) is a highest weight vector of \( F_\mu \), and \( \mu = (-l+2 ; k, k, l) \), with \( 2k \in \mathbb{Z}_{\geq 0}, 2l \in \mathbb{Z}, k + l \in \mathbb{Z}_{\geq 0}, k - l \in \mathbb{Z}_{> 0} \). Now using (4.41), one can prove that

\[
\text{wt } \overline{m}_1 = (-l + 1 ; k, k, l + 1)
\]

- Case \( w_1 = w_2 = w_3 = \overline{w}_3 = 0 \) and \( \overline{w}_2 \neq 0 \): Using (4.218-4.220) and (4.236-4.238) we obtain that the Borel subalgebra of \( \mathfrak{so}(6) \) annihilates \( \overline{w}_2 \). Hence, it is a highest weight vector in the irreducible \( \mathfrak{so}(6) \)-module \( F_\mu \), and considering (4.199-4.200), we have

\[
\mu = (-k + 1 ; l, k, l + 1),
\]

that is not dominant integral, getting a contradiction.

- Case \( w_1 = w_2 = w_3 = \overline{w}_3 = \overline{w}_2 = 0 \) and \( \overline{w}_1 \neq 0 \): Using (4.215-4.217) and (4.233-4.235) we obtain that the Borel subalgebra of \( \mathfrak{so}(6) \) annihilates \( \overline{w}_1 \). Hence, it is a highest weight vector in the irreducible \( \mathfrak{so}(6) \)-module \( F_\mu \), and considering (4.197-4.198), we have

\[
\mu = (-k ; k, l, l), \quad \text{with } 2k \in \mathbb{Z}_{> 0}, 2l \in \mathbb{Z}_{\geq 0}, \text{ and } k - l \in \mathbb{Z}_{\geq 0},
\] (4.255)

which is dominant integral. In this case, the conditions (4.191-4.241) reduces to the equation \( E_{-(\epsilon_2-\epsilon_3)} \overline{w}_1 = 0 \), that holds for the highest weight (4.253). Therefore, the vector

\[
\overline{m}_1 = (\xi(2)^{\epsilon} - i\xi(1)^{\epsilon}) \otimes \overline{w}_1
\] (4.256)

is a singular vector of \( \text{Ind}(F_\mu) \) with \( \mu \) as in (4.255). Now using (4.41), one can prove that

\[
\text{wt } \overline{m}_1 = (-k - 1 ; k + 1, l, l)
\]

finishing the proof.
Appendices

Link to the folder with the files described below:

https://docs.google.com/leaf?id=0BvKQC9Agle4YYjQ0NTk3N2YtMmVhMC00OTY2LWI4MmEtNWVkJmMyOTVkJOWY4&hl=en_US

A Notations in the files that use Macaulay2

This appendix contains the explanations of notations used in the files written for Macaulay2 in order to classify singular vectors in $CK_6$-induced modules of degree $-1, \ldots, -5$. These notations are the link between this paper and the files that use Macaulay2.

As we have seen in (4.9), the possible forms of the singular vectors are the following:

\[
\bar{m} = \partial^2 \sum_{|I|=3} \xi_I \otimes v_{I,1} + \partial \sum_{|I|=1} \xi_I \otimes v_{I,0}, \quad \text{(Degree -5)}.
\]

\[
\bar{m} = \partial^2 \sum_{|I|=6} \xi_I \otimes v_{I,2} + \partial \sum_{|I|=4} \xi_I \otimes v_{I,1} + \sum_{|I|=2} \xi_I \otimes v_{I,0}, \quad \text{(Degree -4)}.
\]

\[
\bar{m} = \partial \sum_{|I|=5} \xi_I \otimes v_{I,1} + \sum_{|I|=3} \xi_I \otimes v_{I,0}, \quad \text{(Degree -3)}.
\]

\[
\bar{m} = \partial \sum_{|I|=6} \xi_I \otimes v_{I,1} + \sum_{|I|=4} \xi_I \otimes v_{I,0}, \quad \text{(Degree -2)}.
\]

\[
\bar{m} = \sum_{|I|=5} \xi_I \otimes v_{I,0}, \quad \text{(Degree -1)}.
\]

In order to abbreviate and capture the length of the elements $\xi_I$ in the summands of the possible singular vectors $\bar{m}$, and the degree of $\bar{m}$, we introduce the following notation that will be used in the software

\[
g_i = \sum_{|I|=i} \xi_I \otimes v_{I,-} \quad \text{(A.2)}
\]
so they can be rewritten as follows:

\[ \vec{m}_5 = \partial^2 g_5 + \partial g_3 + g_1, \quad \text{(Degree -5)} \]
\[ \vec{m}_4 = \partial^2 g_6 + \partial g_4 + g_2, \quad \text{(Degree -4)} \]
\[ \vec{m}_3 = \partial g_5 + g_3, \quad \text{(Degree -3)} \]
\[ \vec{m}_2 = \partial g_6 + g_4, \quad \text{(Degree -2)} \]
\[ \vec{m}_1 = g_5, \quad \text{(Degree -1)} \]

We have done a file (or a serie of files in the case of \(\vec{m}_5, \vec{m}_3 \) and \(\vec{m}_1\)) for each possible singular vector of type (A.3). The first part of all the files have the same structure, and the idea is to impose the equations given in Lemma 4.5 to each \(\vec{m}_i\). From these equations, we constructed a matrix by taking the coefficients of these equations in terms of a natural basis, getting in this way a homogeneous linear system that is solve in order to get a simplified list of conditions. Unfortunately we are not expert in Macaulay2 or Maple, therefore it is not done in the optimal or simpler way.

**Description of the inputs:**

- **Input 1:** We define \( R_0 = \mathbb{Q}[z]/(z^2 + 1) \cong \mathbb{Q} + i\mathbb{Q} \). We defined \( R_0 \) because the scalars involved in the equations of Lemma 4.5 and in the formula of the \(\lambda\)-action belongs to this field.

- **Input 2:** We define the polynomial ring \( R \) with coefficients in \( R_0 \), in the skew-commutative variables \( x_1, \ldots, x_6 \) and the commutative variables \( F_{(i,j)} \) (\( 1 \leq i < j \leq 6 \)), \( E, v_I \) (\( 1 \leq |I| \leq 6 \)). Observe that the variables \( x_i \) correspond to the variables \( \xi_i \) in the paper and \( E \) corresponds to the operator \( E_{00} \). All the other variables are the same as in the paper. Note that in this case the software considers the term \( F_{(1,2)}v_3 \) as a monomial in the polynomial ring, not as the element \( F_{(1,2)} \in \mathfrak{so}(6) \) acting in \( v_3 \in F \).

**Remark A.1.** Observe that the command “\( \text{diff}(x, f) \)” in Macaulay2 is the derivative of \( f \) on the right with respect to \( x \). Since we work with skew-commutative variables \( x_i \) and we need to compute the left derivative \( \partial_{x_i} \) (see the formula of the \(\lambda\)-action on induced modules). In our case, we have

\[ \partial_{x_i}(f) = (-1)^{|J|-1}(\text{diff}(x_i, f)), \quad \text{(A.4)} \]

and

\[ \partial_{x_i}\partial_{x_j}(f) = -(\text{diff}(x_i, \text{diff}(x_j, f))). \quad \text{(A.5)} \]
• Input 3: We define \( f_\omega(0) = 1_R, f_\omega(I) = x_I = \xi_I \) for \( 1 \leq |I| \leq 3 \), and \( f_\omega(I)* = \xi_I^* \) for \( 0 \leq |I| \leq 3 \). Observe that we used “diff” in the definition of \( f_d \).

• Input 4: For \( 1 \leq i \leq 6 \), we define \( g_\omega(i) = \sum_{|I|=i} x_I * v_\omega(I) \) as in (A.2) and (A.3).

• Inputs 5-10: Now we write the terms used in the notation introduced in (4.10) and (4.11). Namely, we define the terms

\[
\begin{align*}
a_\omega(I, k) &:= a(\xi_I, g_k), & a_\omega(I, k) &:= a(\xi_I, g_k), \\
b_\omega(I, k) &:= b(\xi_I, g_k), & b_\omega(I, k) &:= b(\xi_I, g_k), \\
B_\omega(I, k) &:= B(\xi_I, g_k), & B_\omega(I, k) &:= B(\xi_I, g_k), \\
C_\omega(I, k) &:= C(\xi_I, g_k), & C_\omega(I, k) &:= C(\xi_I, g_k),
\end{align*}
\]

for all \( 1 \leq k \leq 6 \) and \( 0 \leq |I| \leq 3 \). Observe that in order to write the terms that appear in the \( \lambda \)-action, we have to take care of the sign in the derivative by using (A.4) and (A.5).

• Inputs 11-17: According to Lemma 4.5, the conditions (S1)-(S3) on a vector \( \vec{m} \), of degree at most -5, are equivalent to the following list of equations

* For \( |f| = 0 \):

\[
\begin{align*}
0 &= C_0 + B_1, & ec_\omega((0), 1) \\
0 &= 2 B_2 + a_2 + C_1, & ec_\omega((0), 2) \\
0 &= 2 bd_0 - i a_2 + i C_1. & ec_\omega((0), 3)
\end{align*}
\]

* For \( f = \xi_i \):

\[
\begin{align*}
0 &= 3 B_2 + 2 i bd_1 + 2 i ad_0 + 2 C_1, & ec_\omega((i), 1) \\
0 &= 2 C_0 - a_1 + B_1 + 2 bd_0 i, & ec_\omega((i), 2) \\
0 &= 2 a_2 + B_2, & ec_\omega((i), 3) \\
0 &= 3 Bd_0 - i C_1 + bd_1 - 2 ad_0, & ec_\omega((i), 4) \\
0 &= 2 b_2 + a_1 + B_1, & ec_\omega((i), 5) \\
0 &= b_1 + B_0. & ec_\omega((i), 6)
\end{align*}
\]

* For \( f = \xi_i \xi_j \ (i < j) \):

\[
\begin{align*}
0 &= 2 C_0 + 2 Bd_0 i + B_1 - i ad_0 + i bd_1, & ec_\omega((i, j), 1) \\
0 &= 2 b_2 + i ad_0 + i bd_1 + B_1, & ec_\omega((i, j), 2) \\
0 &= bd_0 + b_1 i - B_0 i. & ec_\omega((i, j), 3)
\end{align*}
\]
For $f = \xi_i \xi_j \xi_k$ ($i < j < k$):

\begin{align*}
0 &= C_0 - C d_0 i, & ec_\bot((i, j, k), 1) \\
0 &= b d_0 + i b_0, & ec_\bot((i, j, k), 2) \\
0 &= B_1 - B d_1 i - a_1 + a d_1 i, & ec_\bot((i, j, k), 3) \\
0 &= b_2 - b d_2 i + a_1 - a d_1 i, & ec_\bot((i, j, k), 4) \\
0 &= b d_1 + B d_0 + B_0 i + b_1 i, & ec_\bot((i, j, k), 5) \\
0 &= a d_0 + a_0 i - B d_0 - B_0 i. & ec_\bot((i, j, k), 6)
\end{align*}

For $f = \alpha_{ij}$ or $\beta_{ij} \in B_{so(6)}$ (1 ≤ $i < j$ ≤ 3):

\begin{align*}
b_1(\alpha_{ij}) &= 0, & ec_{borel}(i, j), 1) \\
b_1(\beta_{ij}) &= 0, & ec_{borel}(i, j), 2) \\
b_2(\alpha_{ij}) &= 0, & ec_{borel}(i, j), 3) \\
b_2(\beta_{ij}) &= 0, & ec_{borel}(i, j), 4) \\
b_0(\alpha_{ij}) &= 0, & ec_{borel}(i, j), 5) \\
b_0(\beta_{ij}) &= 0, & ec_{borel}(i, j), 6) \\
\end{align*}

The right column of the previous list of conditions contains the name that is used in the Macaulay file of $\vec{m}_5$ for each equation. Observe that for each vector $\vec{m}_i$, the equations are implemented in a different way, taking care of the elements $g_k$. Namely, if we work with $\vec{m}_4 = \partial^2 g_6 + \partial g_4 + g_2$, then equation $ec_{\bot}(0, 3)$ is written in Macaulay file as

\begin{equation}
2 \; bd_{\bot}(0, 2) - z \; a_{\bot}(0, 6) + z \; C_{\bot}(0, 4) = 0,
\end{equation}

where $z$ corresponds to the complex number $i$. And for $\vec{m}_5 = \partial^2 g_5 + \partial g_3 + g_1$, then equation $ec_{\bot}(0, 3)$ is written in the corresponding Macaulay file as

\begin{equation}
ec_{\bot}(0, 3) = 2 \; bd_{\bot}(0, 1) - z \; a_{\bot}(0, 5) + z \; C_{\bot}(0, 3) = 0.
\end{equation}

Not all the equations are non-trivial for the different $\vec{m}_i$, since the length of the monomial $\xi_I$ may be greater than 6. For example, in (A.7), the length of $\xi_I$ is 6, but this equation is trivial when it is implemented for $\vec{m}_3$ since the length of $\xi_I$ is 7. In the following table we indicate which equations appear for the different $\vec{m}_i$ and we give the length of $\xi_I$ that is present in each case. Therefore, the name and number of the equations is modified for the file of each $\vec{m}_i$. 
\[ |f| = 0 : \]
\[
\begin{align*}
C_0 + B_1 & : \quad 3 \quad 4 \quad 5 \quad 6 \quad - \\
2 B_2 + a_2 + C_1 & : \quad 5 \quad 6 \quad - \quad - \quad - \\
2 bd_0 - i a_2 + i C_1 & : \quad 5 \quad 6 \quad - \quad - \quad - 
\end{align*}
\]

\[ |f| = 1 : \]
\[
\begin{align*}
3 B_2 + 2 i bd_1 + 2 i ad_0 + 2 C_1 & : \quad 6 \quad - \quad - \quad - \quad - \\
2 C_0 - a_1 + B_1 + 2 bd_0 i & : \quad 4 \quad 5 \quad 6 \quad - \quad - \\
2 a_2 + B_2 & : \quad 6 \quad - \quad - \quad - \quad - \\
3 Bd_0 - i C_1 + bd_1 - 2 ad_0 & : \quad 6 \quad - \quad - \quad - \quad - \\
2 b_2 + a_1 + B_1 & : \quad 4 \quad 5 \quad 6 \quad - \quad - \\
B_1 + b_1 i - B_0 i & : \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 
\end{align*}
\]

\[ |f| = 2 : \]
\[
\begin{align*}
2 C_0 + 2 Bd_0 i + B_1 - i ad_0 + i bd_1 & : \quad 5 \quad 6 \quad - \quad - \quad - \\
2 b_2 + i ad_0 + i bd_1 + B_1 & : \quad 5 \quad 6 \quad - \quad - \quad - \\
b_0 + b_1 i - B_0 i & : \quad 3 \quad 4 \quad 5 \quad 6 \quad - 
\end{align*}
\]

\[ |f| = 3 : \]
\[
\begin{align*}
C_0 - Cd_0 i & : \quad 6 \quad - \quad - \quad - \quad - \\
b_0 + i b_0 & : \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \\
B_1 - Bd_1 i - a_1 + ad_1 i & : \quad 6 \quad - \quad - \quad - \quad - \\
b_2 - bd_2 i + a_1 - ad_1 i & : \quad 6 \quad - \quad - \quad - \quad - \\
b_1 + Bd_0 - B_0 i + b_1 i & : \quad 4 \quad 5 \quad 6 \quad - \quad - \\
ad_0 - ad_1 i - Bd_0 - B_0 i & : \quad 4 \quad 5 \quad 6 \quad - \quad - 
\end{align*}
\]

\[ f \in \text{Borel:} \]
\[
\begin{align*}
b_2 & : \quad 5 \quad 6 \quad - \quad - \quad - \\
b_1 & : \quad 3 \quad 4 \quad 5 \quad 6 \quad - \\
b_0 & : \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 
\end{align*}
\]

- Input 18-21: We denote by \( A_{\neq}(I, k) \) a one column matrix whose entries are the coefficients in the monomials \( x_I \) of the equation \( ec_{\neq}(I, k) \). We should impose that the equation of each entry must be zero.

- Input 22: We denote by \( M_{\neq}(I, k) \) a one column matrix whose entries are the coefficients in the monomials \( x_I \) of the equation \( ec_{\text{borel}_I}(I, k) \). We should impose that the equation of each entry must be zero.
• Input 23: The previously defined matrices \( A(I,k) \) and \( M(I,k) \) are one column matrices whose entries are R0-linear combinations of the monomials \( v(I), F(i,j) \) and \( E \cdot v(I) \). Each entry must be zero, for that reason we define the lists \( zvari = \{ v(I), F(i,j), E \} \) and \( wvari = \{ v(I), F(i,j) \cdot v(I), E \cdot v(I) \} \), in this order, with the auxiliary lists avari1, avari2 and avari3.

• Input 28-31: We take the transpose of \( A(I,k) \) getting a one row matrix. Then for each entry in this one row matrix, we produce a column formed by the coefficients of this entry with respect to the variables in wvari, obtaining in this way a matrix with coefficients in R0 whose transpose is called \( D(I,k) \). If we consider wvari as a one column matrix, then we have \( A(I,k) = D(I,k) \cdot wvari \) and it must be zero. Therefore we obtained a homogeneous linear system that must be solved.

• Input 32: With the same procedure, using the matrices \( M(I,k) \), we define the matrices \( N(I,k) \) that complete the linear system.

• Input 33: The matrices \( D(I,k) \) and \( N(I,k) \) are put together into one matrix that is called \( X \) whose coefficients are in R0. So, we need to solve the homogeneous linear system associated to \( X \).

Observe that with this procedure, we consider the elements \( F(i,j) \cdot v(I) \) as a monomial in the ring \( R \), not as an element in \( \mathfrak{so}(6) \) acting in \( v(I) \). Since the software (at least from our knowledge) does not work with Lie theory, we first solve the linear system, and then we impose the Lie setting by hand. The description of the inputs that we gave is essentially the structure of all the Macaulay files associated to the vectors \( \vec{m}_5, \vec{m}_4, \vec{m}_2 \). The files associated to \( \vec{m}_3 \) and \( \vec{m}_1 \) have a modification: before the definition of the matrices "D" and "M" all the variables \( F(i,j) \) are written as linear combinations of the more natural basis of \( \mathfrak{so}(6) \) given by the \( H_i \) and \( E_\alpha \). Therefore the list of monomials in wvari is written in terms of them.

Now, we describe in details the list of files associated to each \( \vec{m}_i \).

Files associated to \( \vec{m}_5 \)

File "m5-macaulay-1"

With the list of inputs previously described, we get a 1952 × 544 matrix \( X \) of rank 540 (see inputs 33-40). This matrix \( X \) is constructed by joining together the list of matrices \( l_0, l_1, \ldots, l_4 \). In order to reduce the size of the matrix, we study the rank of these matrices and we found that the 992 × 544 matrix, called \( Y_{25} \), formed with the matrices \( l_0, l_1, l_2, l_4 \) also has rank 540. Unfortunately, the software
Macaulay2 can solve a linear system if the matrix is over $\mathbb{Z}_p, \mathbb{R}$ or $\mathbb{C}$, and it must be a non-singular square matrix in the cases $\mathbb{R}$ or $\mathbb{C}$. Therefore, we exported the matrix $Y25$ and we used Maple, see the file "m5-maple-1", to find the row-reduced echelon matrix of $Y25$, that is called $C$ in that file.

- File "m5-macaulay-2"

If we try to copy the matrix $C$ in the file "m5-macaulay-1" the software run out of memory. Therefore, we continue the work in this NEW Macaulay file "m5-macaulay-2". Now, we describe the inputs in details:

* Input 1-7: The rings $R0$ and $R$, and the list of variables $wvari$ are copied from the file "m5-macaulay-1". We need $wvari$ because, in input 19, we reconstruc the (reduced) equations as linear combinations of the monomials $v_{-}(i), F_{-}(i,j) * v_{-}(I)$ and $E * v_{-}(I)$.

* Input 8-15: The matrix $C$ that is produced in the file "m5-maple-1", which is the row-reduced echelon matrix of $Y25$, is introduced in this NEW Macaulay file "m5-macaulay-2" divided in several parts, called $X1, \ldots, X7$. These parts are put together to reconstruct the matrix $C$ and it is called $X11$ (input 15). Observe that $Y25$ was a $992 \times 544$ matrix of rank 540. For this reason, we copied the first 542 rows of $C$ (the row-reduced echelon matrix of $Y25$). Therefore $X11$ is a $542 \times 544$ matrix with zero in the last two rows.

* Input 16-19: We obtain a reduced (and equivalent) list of equations in a one column matrix $X28 = X27 * wvari$ (whose size is $542 \times 1$), where $X27$ is $X11$ viewed with entries in the ring $R$. Each entry must be zero.

At the end of this list of equations, we observe the following conditions:

\[
\begin{align*}
0 = & \quad v_1 + v_{1,3,4,5,6} \\
0 = & \quad v_3 + v_{1,2,3,5,6} \\
0 = & \quad v_5 + v_{1,2,3,4,5} \\
0 = & \quad v_{1,2,3} - v_{1,2,3,5,6} i \\
0 = & \quad v_{1,2,5} - v_{1,2,3,4,5} i \\
0 = & \quad v_{1,3,4} - v_{1,3,4,5,6} i \\
0 = & \quad v_{1,3,6} + v_{2,4,6} \\
0 = & \quad v_{1,4,6} - v_{2,4,6} i \\
0 = & \quad v_{2,3,4} - v_{1,3,4,5,6} \\
0 = & \quad v_{2,3,6} - v_{2,4,6} i \\
0 = & \quad v_{2,5,6} - v_{1,3,4,5,6} \\
0 = & \quad v_{3,4,6} - v_{1,2,3,4,5} \\
0 = & \quad v_{4,5,6} - v_{1,2,3,5,6} \\
0 = & \quad v_{1,2,4,5,6} + v_{1,2,3,5,6} i \\
0 = & \quad v_2 - v_{1,3,4,5,6} i \\
0 = & \quad v_4 - v_{1,2,3,5,6} i \\
0 = & \quad v_6 - v_{1,2,3,4,5} i \\
0 = & \quad v_{1,2,4} - v_{1,2,3,5,6} \\
0 = & \quad v_{1,2,6} - v_{1,2,3,4,5} \\
0 = & \quad v_{1,3,5} + v_{2,4,6} i \\
0 = & \quad v_{1,4,5} + v_{2,4,6} \\
0 = & \quad v_{1,5,6} - v_{1,3,4,5,6} i \\
0 = & \quad v_{2,3,5} + v_{2,4,6} \\
0 = & \quad v_{2,4,5} - v_{2,4,6} i \\
0 = & \quad v_{3,4,5} - v_{1,2,3,4,5} i \\
0 = & \quad v_{3,5,6} - v_{1,2,3,5,6} i \\
0 = & \quad v_{2,3,4,5,6} + v_{1,3,4,5,6} i \\
0 = & \quad v_{1,2,3,4,6} + v_{1,2,3,4,5} i \\
\end{align*}
\]
In order to simplify the 540 equations, we need to impose conditions (A.8). Observe that all the vectors \(v_i\) can be written in terms of the set \(\{v_1, v_3, v_5, v_{(1,3,5)}\}\). This is done in the following inputs.

* Input 20-21: We define a ring \(P\) that is isomorphic to \(R\). In this case, \(P\) is the polynomial ring with coefficients in \(R_0\), in the skew-commutative variables \(t_1, \ldots, t_6\) and the commutative variables \(h_{i,j}, e_{(i,j)}, em_{(i,j)}, me_{(i,j)}, mem_{(i,j)}\) (1 \(\leq i < j \leq 3\)). \(E_0, u_q\) (1 \(\leq |l| \leq 6\)). The idea is to replace the basis \(F_{(i,j)} \in \mathfrak{so}(6)\) by the basis given by \(H_i\) and \(E_\alpha\). We are using the following notation, for 1 \(\leq i < j \leq 3\):

\[
\begin{align*}
    e_{\pm}(i,j) &= E_{\varepsilon_i - \varepsilon_j} \\
    em_{\pm}(i,j) &= E_{\varepsilon_i + \varepsilon_j} \\
    me_{\pm}(i,j) &= E_{-(\varepsilon_i - \varepsilon_j)} \\
    mem_{\pm}(i,j) &= E_{-(\varepsilon_i + \varepsilon_j)}
\end{align*}
\] (A.9)

* Input 22: We define a map \(Q : R \rightarrow P\), that impose conditions (A.8) and change the basis in \(\mathfrak{so}(6)\) using the notation (A.9). The definition of \(Q\) is the following:

**Input 22: Qvari = \(\{x \cdot i \Rightarrow t_i,\)**

\(F_{(1,2)} \Rightarrow -z \cdot h_{1}, F_{(3,4)} \Rightarrow -z \cdot h_{2}, F_{(5,6)} \Rightarrow -z \cdot h_{3},\)

\(F_{(2 \cdot i - 1, 2 \cdot j - 1)} \Rightarrow (e_{\pm}(i,j) + em_{\pm}(i,j) + me_{\pm}(i,j) + mem_{\pm}(i,j))/4,\)

\(F_{(2 \cdot i, 2 \cdot j)} \Rightarrow (e_{\pm}(i,j) - em_{\pm}(i,j) + me_{\pm}(i,j) - mem_{\pm}(i,j))/4,\)

\(F_{(2 \cdot i - 1, 2 \cdot j)} \Rightarrow -z \cdot (e_{\pm}(i,j) - em_{\pm}(i,j) - me_{\pm}(i,j) + mem_{\pm}(i,j))/4,\)

\(F_{(2 \cdot i, 2 \cdot j - 1)} \Rightarrow -z \cdot (e_{\pm}(i,j) - em_{\pm}(i,j) + me_{\pm}(i,j) + mem_{\pm}(i,j))/4,\)

\(E \Rightarrow E_0,\)

\(v_1 \Rightarrow u_1, v_2 \Rightarrow -z \cdot u_1, v_3 \Rightarrow u_3,\)

\(v_4 \Rightarrow -z \cdot u_3, v_5 \Rightarrow u_5, v_6 \Rightarrow -z \cdot u_5,\)

\(v_{(i,j)} \Rightarrow u_{(i,j)},\)

\(v_{(1,2,3)} \Rightarrow -z \cdot u_3, v_{(1,2,4)} \Rightarrow -u_3, v_{(1,2,5)} \Rightarrow -z \cdot u_5,\)

\(v_{(1,2,6)} \Rightarrow -u_5, v_{(1,3,4)} \Rightarrow -z \cdot u_1, v_{(1,3,5)} \Rightarrow u_{(1,3,5)},\)

\(v_{(1,3,6)} \Rightarrow -z \cdot u_{(1,3,5)}, v_{(1,4,5)} \Rightarrow -z \cdot u_{(1,3,5)}, v_{(1,4,6)} \Rightarrow -u_{(1,3,5)},\)

\(v_{(1,5,6)} \Rightarrow -z \cdot u_{(1,3,5)}, v_{(2,3,4)} \Rightarrow -u_{(1,3,5)}, v_{(2,3,5)} \Rightarrow -z \cdot u_{(1,3,5)},\)

\(v_{(2,3,6)} \Rightarrow -u_{(1,3,5)}, v_{(2,4,5)} \Rightarrow -u_{(1,3,5)}, v_{(2,4,6)} \Rightarrow z \cdot u_{(1,3,5)},\)

\(v_{(2,5,6)} \Rightarrow -u_{(1,3,5)}, v_{(3,4,5)} \Rightarrow -z \cdot u_5, v_{(3,4,6)} \Rightarrow -u_5,\)

\(v_{(3,5,6)} \Rightarrow -z \cdot u_3, v_{(4,5,6)} \Rightarrow -u_5,\)

\(v_{(i,j,k,l)} \Rightarrow u_{(i,j,k,l))},\)

\(v_{(2,3,4,5,6)} \Rightarrow z \cdot u_1, v_{(1,3,4,5,6)} \Rightarrow -u_{(1,2,4,5,6)} \Rightarrow z \cdot u_3,\)

\(v_{(1,2,3,5,6)} \Rightarrow -u_3, v_{(1,2,3,4,6)} \Rightarrow z \cdot u_5, v_{(1,2,3,4,5)} \Rightarrow -u_5,\)

\(v_{(1,2,3,4,5,6)} \Rightarrow u_{(1,2,3,4,5,6)},\)

\(Q = map(P, R, Qvari);\)
* Input 23-27: We define a new 'wvari', which is the list of variables that will appear in the equations. More precisely, wvari=list{\(u(I)\), \(h_k * u(I)\), \(e_{(i,j)} * u(I)\), \(em_{(i,j)} * u(I)\), \(me_{(i,j)} * u(I)\), \(mem_{(i,j)} * u(I)\), \(E_0 * u(I)\)}, where \(u(I)\) is restricted in this case to the set \{\(u_1\), \(u_3\), \(u_5\), \(u_{(1,3,5)}\)\}.

* Input 28-31: We apply the map \(Q\) to the equations in the matrix \(X_{28}\), and then we obtain a \(542 \times 68\) matrix, called \(X_{29}\), given by the coefficients in the monomials of 'wvari' that appear in the equations of \(Q(X_{28})\). The rank of \(X_{29}\) is 64. The matrix \(X_{29}\) is exported in order to reduce the linear system.

* Input 33: We use Maple to reduce the matrix \(X_{29}\). The matrix \(C\) that is produced in the file "m5-maple-2", which is the row-reduced echelon matrix of \(X_{29}\), is introduced in this input and it is called \(X_{30}\). Observe that \(X_{29}\) was a \(542 \times 68\) matrix of rank 64. For this reason, we copied the first 70 rows of \(C\) (the row-reduced echelon matrix of \(X_{29}\)). Therefore \(X_{30}\) is a \(70 \times 68\) matrix with zero in the last six rows.

* Input 34-38: We obtain a reduced (and equivalent) list of equations in a one column matrix \(X_{34} = X_{33} * \text{wvari}\) (whose size is \(70 \times 1\)), where \(X_{33}\) is \(X_{30}\) viewed with entries in the ring \(P\). Each entry must be zero. This final list of 64 simplified and equivalent equations is copied in the proof of Lemma 4.7.

**Files associated to \(\vec{m}_4\)**

• File "m4-macaulay"

With the list of inputs previously described, except that we do not need to impose Borel equations (hence the matrices "M" and "N" are not needed), we get a \(1104 \times 527\) matrix \(X\) of rank 527. Therefore, there is no non-trivial solution of this linear system, proving that there is no singular vector of degree -4.

**Files associated to \(\vec{m}_3\)**

• File "m3-macaulay-1"

With the list of inputs previously described, we get a \(694 \times 442\) matrix \(X\) of rank 397. This matrix \(X\) is exported to a file and using Maple, see the file "m3-maple-1", we obtain the row-reduced echelon matrix of \(X\), that is called \(C\). This matrix \(C\) is introduced as the matrix \(X_{11}\) in the Macaulay file (see input 43-44), to reconstruct the (reduced) equations as linear combinations of the monomials \(v(I), F_{(i,j)} * v(I)\) and \(E * v(I)\). In fact, the matrix \(X_{11}\) is \(400 \times 442\) because we removed the last zero rows of the row-reduced echelon matrix, therefore it has zero in the last three rows.

* Input 45-48: We obtain a reduced (and equivalent) list of equations in a one column matrix \(X_{14} = X_{12} * \text{wvari}\) (whose size is \(400 \times 1\)), where \(X_{12}\) is \(X_{11}\)
viewed with entries in the ring \( R \). Each entry must be zero.

At the end of this list of equations, we observe the following conditions:

\[
\begin{align*}
0 &= v_{1,2,3} - v_{4,5,6} \ i \\
0 &= v_{1,2,5} - v_{3,4,6} \ i \\
0 &= v_{1,3,4} - v_{2,5,6} \ i \\
0 &= v_{1,3,6} - v_{2,4,5} \ i \\
0 &= v_{1,4,6} + v_{2,3,5} \ i \\
0 &= v_{2,3,4,5,6} \\
0 &= v_{1,2,4,5,6} \\
0 &= v_{1,2,3,4,6}
\end{align*}
\]

Observe that (A.10) can be written as

\[
v_I = 0 \text{ if } |I| = 5, \text{ and } v_{\{a,b,c\}} = (-1)^{a+b+c} \ i \ v_{\{a,b,c\}c} \text{ for } a < b < c. \quad (A.11)
\]

- File "m3-macaulay-2"

Now, we have to impose the identities (A.10) to reduce the number of variables. Using (A.11), everything can be written in terms of

\[
v_{1,j,k} \text{ with } 2 \leq j < k \leq 6.
\]

Unfortunately, the result is not enough to obtain in a clear way the possible highest weight vectors. For example, after the reduction and some extra computations it is possible to see that

\[
(v_{1,3,6} - v_{1,4,5}) + i \ (v_{1,3,5} + v_{1,4,6})
\]

is annihilated by the Borel subalgebra. Hence, it is necessary to impose (A.10) and make a change of variables. We produced an auxiliary file where we imposed (A.10), and after the analysis of the results, we found that the following change of variable is convenient:

\[
\begin{align*}
&u_1 = v_{1,2,3} - i \ v_{1,2,4} \\
&u_2 = v_{1,2,3} + i \ v_{1,2,4} \\
&u_3 = v_{1,2,5} - i \ v_{1,2,6} \\
&u_4 = v_{1,2,6} + i \ v_{1,2,6} \\
&u_5 = v_{1,3,4} - v_{1,5,6} \\
&u_6 = v_{1,3,4} + v_{1,5,6} \\
&u_7 = v_{1,3,5} - v_{1,4,6} + i \ (v_{1,3,6} + v_{1,4,5}) \\
&u_8 = v_{1,3,5} + v_{1,4,6} - i \ (v_{1,3,6} - v_{1,4,5}) \\
&u_9 = v_{1,3,5} - v_{1,4,6} - i \ (v_{1,3,6} + v_{1,4,5}) \\
&u_{10} = v_{1,3,5} + v_{1,4,6} + i \ (v_{1,3,6} - v_{1,4,5})
\end{align*}
\]
or equivalently

\[ v_{1,2,3} = \frac{1}{2}(u_1 + u_2) = i v_{4,5,6} \]
\[ v_{1,2,4} = \frac{i}{2}(u_1 - u_2) = -i v_{3,5,6} \]
\[ v_{1,2,5} = \frac{1}{2}(u_3 + u_4) = i v_{3,4,6} \]
\[ v_{1,2,6} = \frac{i}{2}(u_3 - u_4) = -i v_{3,4,5} \]
\[ v_{1,3,4} = \frac{1}{2}(u_5 + u_6) = i v_{2,5,6} \]
\[ v_{1,5,6} = -\frac{1}{2}(u_5 - u_6) = i v_{2,3,4} \]
\[ v_{1,3,5} = \frac{1}{4}(u_7 + u_8 + u_9 + u_{10}) = -i v_{2,4,6} \]
\[ v_{1,4,6} = \frac{i}{4}(-u_7 + u_8 - u_9 + u_{10}) = -i v_{2,3,5} \]
\[ v_{1,3,6} = \frac{i}{4}(-u_7 + u_8 + u_9 - u_{10}) = i v_{2,4,5} \]
\[ v_{1,4,5} = \frac{i}{4}(-u_7 - u_8 + u_9 + u_{10}) = i v_{2,3,6} \]  

\[ (A.14) \]

We also need to replace the basis \( F_{(i,j)} \in \mathfrak{so}(6) \) by the basis given by \( H_t \) and \( E_\alpha \). We are using the notation introduced in \((A.9)\). The identities \((A.10)\), the change of variables \((A.14)\) and the new basis of \( \mathfrak{so}(6) \) are implemented with the definition of a ring \( P \) that is isomorphic to \( R \) and a map \( Q : R \to P \). More precisely,

* Input 1-22: They are the same inputs in the file ”m3-macaulay-1”.

* Input 23: We define a ring \( P \) as the polynomial ring with coefficients in \( R_0 \), in the skew-commutative variables \( t_1, \ldots, t_6 \) and the commutative variables \( u_i \) (\( 1 \leq i \leq 10 \)), \( h, e_{(i,j)} \), \( em_{(i,j)} \), \( me_{(i,j)} \), \( mem_{(i,j)} \) (\( 1 \leq i < j \leq 3 \)), \( E_0 \).

* Input 24: We define a map \( Q : R \to P \), that impose conditions \((A.10)\), the change of variables \((A.14)\) and change the basis in \( \mathfrak{so}(6) \) using the notation \((A.9)\). The definition of \( Q \) is the following:

\[ \text{Quasi} = \{ x_i \Rightarrow t_i, \]
\[ F_{\bot}(1, 2) \Rightarrow -z \ast h_1, F_{\bot}(3, 4) \Rightarrow -z \ast h_2, F_{\bot}(5, 6) \Rightarrow -z \ast h_3, \]
\[ F_{\bot}(2 \ast i - 1, 2 \ast j - 1) \Rightarrow (e_{\bot}(i, j) + em_{\bot}(i, j) + me_{\bot}(i, j) + mem_{\bot}(i, j))/4, \]
\[ F_{\bot}(2 \ast i, 2 \ast j) \Rightarrow (e_{\bot}(i, j) - em_{\bot}(i, j) + me_{\bot}(i, j) - mem_{\bot}(i, j))/4, \]
\[ F_{\bot}(2 \ast i - 1, 2 \ast j) \Rightarrow -z \ast (e_{\bot}(i, j) - em_{\bot}(i, j) - me_{\bot}(i, j) + mem_{\bot}(i, j))/4, \]
\[ F_{\bot}(2 \ast i, 2 \ast j - 1) \Rightarrow -z \ast (-e_{\bot}(i, j) - em_{\bot}(i, j) + me_{\bot}(i, j) + mem_{\bot}(i, j))/4, \]
\[ E \Rightarrow E_0, \]
\[ v_i \Rightarrow u_i, \]
\[ v_{\bot}(i, j) \Rightarrow u_{\bot}(i, j), \]

\[ 47 \]
\[ v_\cdot(1, 2, 3) \Rightarrow (u_1 + u_2)/2, v_\cdot(1, 2, 4) \Rightarrow z \cdot \frac{u_1 - u_2}{2}, \]
\[ v_\cdot(1, 2, 5) \Rightarrow (u_3 + u_4)/2, v_\cdot(1, 2, 6) \Rightarrow z \cdot \frac{u_3 - u_4}{2}, \]
\[ v_\cdot(1, 3, 4) \Rightarrow (u_5 + u_6)/2, v_\cdot(1, 5, 6) \Rightarrow -(u_5 - u_6)/2, \]
\[ v_\cdot(1, 3, 5) \Rightarrow (u_7 + u_8 + u_9 + u_{10})/4, \]
\[ v_\cdot(1, 3, 6) \Rightarrow z \cdot \frac{-u_7 + u_8 + u_9 - u_{10}}{4}, \]
\[ v_\cdot(1, 4, 5) \Rightarrow z \cdot \frac{-u_7 - u_8 + u_9 + u_{10}}{4}, \]
\[ v_\cdot(1, 4, 6) \Rightarrow -(u_7 + u_8 - u_9 + u_{10})/4, \]
\[ v_\cdot(2, 3, 4) \Rightarrow z \cdot \frac{(u_5 - u_6)}{2}, \]
\[ v_\cdot(2, 3, 5) \Rightarrow z \cdot \frac{-u_7 + u_8 - u_9 + u_{10}}{4}, \]
\[ v_\cdot(2, 3, 6) \Rightarrow -(u_7 - u_8 + u_9 + u_{10})/4, \]
\[ v_\cdot(2, 4, 5) \Rightarrow -(u_7 + u_8 + u_9 - u_{10})/4, \]
\[ v_\cdot(2, 4, 6) \Rightarrow z \cdot \frac{(u_7 + u_8 + u_9 + u_{10})}{4}, \]
\[ v_\cdot(2, 5, 6) \Rightarrow -z \cdot \frac{(u_5 + u_6)}{2}, v_\cdot(3, 4, 5) \Rightarrow -(u_3 - u_4)/2, \]
\[ v_\cdot(3, 4, 6) \Rightarrow -z \cdot \frac{(u_3 + u_4)}{2}, v_\cdot(3, 5, 6) \Rightarrow -(u_1 - u_2)/2, \]
\[ v_\cdot(4, 5, 6) \Rightarrow -z \cdot \frac{(u_1 + u_2)}{2}, \]
\[ v_\cdot(i, j, k, l) \Rightarrow u_\cdot(i, j, k, l), \]
\[ v_\cdot(1..i - 1|i + 1.6) \Rightarrow 0_p, \]
\[ v_\cdot(1, 2, 3, 4, 5, 6) \Rightarrow u_\cdot(1, 2, 3, 4, 5, 6), \]

\[ Q = \text{map}(P, R, Q\text{vari}). \]

* Input 25-41: With the same list of inputs as in the file "m3-macaulay-1", but applying the map \( Q \), we get a 694 \times 170 matrix \( X \) of rank 125. This matrix \( X \) is constructed by joining together the list of matrices \( l0, l1, \ldots, l4 \).

* Input 42-47: In order to reduce the size of the matrix, we studied the rank of these matrices and we found that the 354 \times 170 matrix, called \( X2 \), formed with the matrices \( l0, l1, l2, l4 \) also has rank 125. We exported the matrix \( X2 \) and we used Maple, see the file "m3-maple-2", to find the row-reduced echelon matrix of \( X2 \), that is called \( C \) in that file.

* Input 48-49: The matrix \( C \) that is produced in the file "m3-maple-2", which is the row-reduced echelon matrix of \( X2 \), is introduced in this file and it is called \( X11 \). Observe that \( X2 \) was a 354 \times 170 matrix of rank 125. For this reason, we copied the first 127 rows of \( C \) (the row-reduced echelon matrix of \( X2 \)). Therefore \( X11 \) is a 127 \times 170 matrix with zero in the last two rows.

* Input 50-55: We obtain a reduced (and equivalent) list of equations in a one column matrix \( X14 = X12 \ast\text{vari} \) (whose size is 127 \times 1), where \( X12 \) is \( X11 \) viewed with entries in the ring \( P \). Each entry must be zero.

* Input 56-60: In order to simplify the list of equations, we define \( Z_\cdot i = \text{row} i \) of \( X14 \), and then we consider the following list of linear combinations of these rows:
for $i$ in $0..124$ do $Z_i = X_{14}(i, 0)$;

$i57:
X25 = \{Z_0, Z_1 - Z_0, Z_2 + Z_0, Z_3, Z_4 + Z_3, Z_5 - Z_3, Z_6, Z_7 + Z_6,$
$Z_8 - Z_6, Z_9 + Z_{11}, Z_{10} + Z_{11}, Z_{11}, Z_{12}, Z_{13} - Z_{12}, Z_{14} + Z_{13},$
$Z_{15} + Z_{17}, Z_{16} + Z_{15}, Z_{17} + Z_{16}, Z_{18} + Z_{19} - Z_{20}, Z_{18} - Z_{19} + Z_{20},$
$- Z_{18} + Z_{19} + Z_{20}, Z_{21}, Z_{22}, Z_{23} + Z_{22}, Z_{24} + Z_{25} + Z_{26},$

for $i$ in $25..29$ list $Z_i$,
$Z_{30} - Z_0, Z_{31} - Z_3, Z_{32} - Z_6, Z_{33} + Z_{11}, Z_{34} - Z_{14}, Z_{35} - Z_{15},$

for $i$ in $36..124$ list $Z_i$\};

obtaining an equivalent list of equations given by the rows of the $127 \times 1$ matrix $X29$ (all of them must be zero). These equations are copied in the proof of Lemma 4.9 together with the final analysis of them, see (4.45-4.169).

Files associated to $\vec{m}_2$

• File "m2-macaulay"

With the list of inputs previously described, we get a $268 \times 272$ matrix $X$ of rank 192. This matrix $X$ is exported to a file and using Maple (see the file "m2-maple.mws") we obtain the row-reduced echelon matrix of $X$, that is called $X11$. In fact, the matrix $X11$ is $195 \times 272$ because we removed the last zero rows of the row-reduced echelon matrix. This matrix $X11$ is introduced in the Macaulay file "m2-macaulay" as the input 43. In order to reconstruct the reduced system of equations as linear combinations of the monomials $v(I), F_{(i, j)} \ast v(I)$ and $E \ast v(I)$ we multiply $X11 \ast \text{wvari}$, obtaining a one column matrix, called $X14$, with the list of equations that must be zero (see inputs 45-52). This matrix $X14$ is exported into a latex-pdf file "m2-ecuations.pdf", and the analysis of these equations is done in the paper (see the proof of the lemma 4.10 corresponding to $\vec{m}_2$).

Files associated to $\vec{m}_1$

• File "m1-macaulay"

Since the singular vectors found in [2] for $K_6$ are also singular vectors for $CK_6$, using (B42-B43) in [2], we have that it is convenient to introduce the following
notation:
\[
\bar{m}_1 = \sum_{i=1}^{6} \xi_{(i)}c \otimes v_{(i)c}
\]
\[
= \sum_{l=1}^{3} \left( (\xi_{(2l)}c + i\xi_{(2l-1)}c) \otimes w_l + (\xi_{(2l)}c - i\xi_{(2l-1)}c) \otimes \bar{w}_l \right)
\]
that is, for \(1 \leq l \leq 3\)
\[
v_{(2l)c} = w_l + \bar{w}_l, \quad v_{(2l-1)c} = i(w_l - \bar{w}_l)
\]
or equivalently, for \(1 \leq l \leq 3\)
\[
w_l = \frac{1}{2}(v_{(2l)c} - i v_{(2l-1)c}), \quad \bar{w}_l = \frac{1}{2}(v_{(2l)c} + i v_{(2l-1)c}).
\]

Now, with the usual list of inputs previously described, we have to impose the identities (A.16) to change the variables. We also need to replace the basis \(F_{(i,j)} \in so(6)\) by the basis given by \(H_i\) and \(E_\alpha\). We are using the notation introduced in (A.9). The change of variables (A.16) and the new basis of \(so(6)\) are implemented with the definition of a ring \(P\) that is isomorphic to \(R\) and a map \(Q : R \rightarrow P\). More precisely,

* Input 1-19: They are the usual inputs, for example as in the file "m3-macaulay-1".

* Input 20-21: We define a ring \(P\) as the polynomial ring with coefficients in \(R0\), in the skew-commutative variables \(t_1, \ldots, t_6\) and the commutative variables \(u_i\) \((1 \leq i \leq 10)\), \(omega_1, omega_2, omega_3, domega_1, domega_2, domega_3\), \(h_{\cdot \cdot}, e_{(i,j)}, em_{(i,j)}, me_{(i,j)}, mem_{(i,j)}\) \((1 \leq i < j \leq 3)\), \(E_0\).

* Input 22: We define a map \(Q : R \rightarrow P\), that change of variables (A.16) and change the basis in \(so(6)\) using the notation (A.9). The definition of \(Q\) is the following:

\[
Qvari = \{x_{i \cdot} \Rightarrow t_{i \cdot}, F_{\cdot}(1, 2) \Rightarrow -z \ast h_{\cdot 1}, F_{\cdot}(3, 4) \Rightarrow -z \ast h_{\cdot 2}, F_{\cdot}(5, 6) \Rightarrow -z \ast h_{\cdot 3}, F_{\cdot}(2 \ast i - 1, 2 \ast j - 1) \Rightarrow (e_{\cdot}(i, j) + em_{\cdot}(i, j) + me_{\cdot}(i, j) + mem_{\cdot}(i, j))/4, F_{\cdot}(2 \ast i, 2 \ast j) \Rightarrow (e_{\cdot}(i, j) - em_{\cdot}(i, j) + me_{\cdot}(i, j) - mem_{\cdot}(i, j))/4, F_{\cdot}(2 \ast i - 1, 2 \ast j) \Rightarrow -z \ast (e_{\cdot}(i, j) - em_{\cdot}(i, j) - me_{\cdot}(i, j) + mem_{\cdot}(i, j))/4, F_{\cdot}(2 \ast i, 2 \ast j - 1) \Rightarrow -z \ast (-e_{\cdot}(i, j) - em_{\cdot}(i, j) + me_{\cdot}(i, j) + mem_{\cdot}(i, j))/4, E \Rightarrow E_0, v_{\cdot \cdot} \Rightarrow u_{\cdot \cdot}, v_{\cdot}(i, j) \Rightarrow u_{\cdot}(i, j), v_{\cdot}(i, j, k) \Rightarrow u_{\cdot}(i, j, k), v_{\cdot}(i, j, k, l) \Rightarrow u_{\cdot}(i, j, k, l),
\]
\begin{align*}
v_\perp(2, 3, 4, 5, 6) &= z \ast (\omega_1 - d\omega_1), \\
v_\perp(1, 2, 4, 5, 6) &= z \ast (\omega_2 - d\omega_2), \\
v_\perp(1, 2, 3, 4, 6) &= z \ast (\omega_3 - d\omega_3), \\
v_\perp(1, 3, 4, 5, 6) &= (\omega_1 + d\omega_1), \\
v_\perp(1, 2, 3, 5, 6) &= (\omega_2 + d\omega_2), \\
v_\perp(1, 2, 3, 4, 5) &= (\omega_3 + d\omega_3), \\
v_\perp(1, 2, 3, 4, 5, 6) &= u_\perp(1, 2, 3, 4, 5, 6), \\
Q &= \text{map}(P, R, Q\text{vari}).
\end{align*}

* Input 23-37: With the usual list of inputs, but applying the map \(Q\) to the equations, we get a \(62 \times 102\) matrix \(X\) of rank 51. We exported the matrix \(X\) and we used Maple, see the file "m1-maple", to find the row-reduced echelon matrix of \(X\), that is called \(C\) in that file.

* Input 39: The matrix \(C\) that is produced in the file "m1-maple", which is the row-reduced echelon matrix of \(X\), is introduced in this file and it is called \(X_{11}\). Observe that \(X\) was a \(62 \times 102\) matrix of rank 51. Therefore \(X_{11}\) is a \(62 \times 102\) matrix with zero in the last 11 rows.

* Input 40-45: We obtain a reduced (and equivalent) list of equations in a one column matrix \(X_{12} = X_{11} \ast \text{wvari}\) (whose size is \(62 \times 1\)), where \(X_{12}\) is \(X_{11}\) viewed with entries in the ring \(P\). Each entry must be zero.

These equations are copied in the proof of Lemma 4.11 in the equations (4.191-4.241), and the final analysis of them is done in that proof.

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**References**


