ON THE STEINBERG CHARACTER
OF A SEMISIMPLE $p$-ADIC GROUP

Ju-Lee Kim and George Lusztig

Dedicated to Robert Steinberg on the occasion of his 90-th birthday

1. Introduction

1.1. Let $K$ be a nonarchimedean local field and let $\overline{K}$ be a maximal unramified field extension of $K$. Let $O$ (resp. $\overline{O}$) be the ring of integers of $K$ (resp. $\overline{K}$) and let $p$ (resp. $\overline{p}$) be the maximal ideal of $O$ (resp. $\overline{O}$). Let $\overline{K}^* = \overline{K} - \{0\}$. We write $O/p = \mathbb{F}_q$, a finite field with $q$ elements, of characteristic $p$.

Let $G$ be a semisimple almost simple algebraic group defined and split over $K$ with a given $O$-structure compatible with the $K$-structure.

If $V$ is an admissible representation of $G(K)$ of finite length, we denote by $\phi_V$ the character of $V$ in the sense of Harish-Chandra, viewed as a $C$-valued function on the set $G(K_{rs}) := G_{rs} \cap G(K)$. (Here $G_{rs}$ is the set of regular semisimple elements of $G$ and $C$ is the field of complex numbers.)

In this paper we study the restriction of the function $\phi_V$ to:

(a) a certain subset $G(K)_{vr}$ of $G(K)_{rs}$, that is to the set of very regular elements in $G(K)$ (see 1.2), in the case where $V$ is the Steinberg representation of $G(K)$ and

(b) a certain subset $G(K)_{svr}$ of $G(K)_{vr}$, that is to the set of split very regular elements in $G(K)$ (see 1.2), in the case where $V$ is an irreducible admissible representation of $G(K)$ with nonzero vectors fixed by an Iwahori subgroup.

In case (a) we show that $\phi_V(g)$ with $g \in G(K)_{rs}$ is of the form $\pm q^n$ with $n \in \{0, -1, -2, \ldots\}$ (see Corollary 3.4) with more precise information when $g \in G(K)_{svr}$ (see Theorem 2.2) or when $g \in G(K)_{cvr}$ (see Theorem 3.2); in case (b) we show (with some restriction on characteristic) that $\phi_V(g)$ with $G(K)_{svr}$ can be expressed as a trace of a certain element of an affine Hecke algebra in an irreducible module (see Theorem 4.3).

Note that the Steinberg representation $S$ is an irreducible admissible representation of $G(K)$ with a one dimensional subspace invariant under an Iwahori

Both authors are supported in part by the National Science Foundation
subgroup on which the affine Hecke algebra acts through the “sign” representation, see [MA], [S]. This is a p-adic analogue of the Steinberg representation [St] of a reductive group over $F_q$. In [R], it is proved that $\phi_S(g) \neq 0$ for any $g \in G(K)_s$. 

1.2. Let $g \in G_{rs} \cap G(K)$. Let $T' = T_g'$ be the maximal torus of $G$ that contains $g$. We say that $g$ is very regular (resp. compact very regular) if $T'$ is split over $K$ and for any root $\alpha$ with respect to $T'$ viewed as a homomorphism $T'(K) \to K^*$ we have

$$\alpha(g) \notin (1 + \mathfrak{p}) \ (\text{resp. } \alpha(g) \notin \mathfrak{O}, \alpha(g) \notin (1 + \mathfrak{p})).$$

Let $G(K)_{vr}$ (resp. $G(K)_{c vr}$) be the set of elements in $G(K)$ which are very regular (resp. compact very regular). We write $G(K)_{vr} = G(K)_{vr} \cap G(K)$, $G(K)_{c vr} = G(K)_{c vr} \cap G(K)$. Let $G(K)_{svr}$ be the set of all $g \in G(K)_{vr}$ such that $T'_g$ is split over $K$.

1.3. Notation. Let $K^* = K - \{0\}$ and let $v : K^* \to \mathbb{Z}$ be the unique (surjective) homomorphism such that $v(p^n - p^{n+1}) = n$ for any $n \in \mathbb{N}$. For $a \in K^*$ we set $|a| = q^{-v(a)}$.

We fix a maximal torus $T$ of $G$ defined and split over $K$. Let $Y$ (resp. $X$) be the group of cocharacters (resp. characters) of the algebraic group $T$. Let $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{Z}$ be the obvious pairing. Let $R \subset X$ be the set of roots of $G$ with respect to $T$, let $R^+$ be a set of positive roots for $R$ and let $\Pi$ be the set of simple roots of $R$ determined by $R^+$. We write $\Pi = \{\alpha_i ; i \in I_0\}$. Let $R^- = R - R^+$. Let $Y^+$ (resp. $Y^{++}$) be the set of all $y \in Y$ such that $\langle y, \alpha \rangle \geq 0$ (resp. $\langle y, \alpha \rangle > 0$) for all $\alpha \in R^+$. We define $2\rho \in X$ by $2\rho = \sum_{\alpha \in R^+} \alpha$.

We have canonically $T(K) = K^* \otimes Y$; we define a homomorphism $\chi : T(K) \to Y$ by $\chi(\lambda \otimes y) = v(\lambda)y$ for any $\lambda \in K^*$, $y \in Y$. For any $y \in Y$ we set $T(K)_y = \chi^{-1}(y)$. For $y \in Y$ let $T(K)_y^\bullet = T(K)_y \cap G(K)_{svr}$. Note that if $y \in Y^{++}$ then $T(K)_y^\bullet = T(K)_y$.

For each $\alpha \in R$ let $U_\alpha$ be the corresponding root subgroup of $G$.

Let $G(K)'/T$ be the derived subgroup of $G(K)$.

2. Calculation of $\phi_S$ on $G(K)_{svr}$

2.1. Let $\mathcal{W} \subset \text{Aut}(T)$ be the Weyl group of $G$ regarded as a Coxeter group; for $i \in I_0$ let $s_i$ be the simple reflection in $\mathcal{W}$ determined by $\alpha_i$. We can also view $\mathcal{W}$ as a subgroup of $\text{Aut}(Y)$ or $\text{Aut}(X)$. Let $w = w_0$ be the longest element of $\mathcal{W}$. For any $J \subset I_0$ let $\mathcal{W}_J$ be the subgroup of $\mathcal{W}$ generated by $\{s_i ; i \in J\}$ and let $R_J$ be the set of $\alpha \in R$ such that $\alpha = w(\alpha_i)$ for some $w \in \mathcal{W}_J$, $i \in J$. Let $R^+_J = R_J \cap R^+$, $R^-_J = R_J - R^+_J$.

Let $\mathfrak{g}$ be the Lie algebra of $G$; let $\mathfrak{t} \subset \mathfrak{g}$ be the Lie algebra of $T$. For any $J \subset I_0$ let $\mathfrak{t}_J$ be the Lie subalgebra of $\mathfrak{g}$ spanned by $\mathfrak{t}$ and by the root spaces corresponding to roots in $R_J$; let $\mathfrak{n}_J$ be the Lie subalgebra of $\mathfrak{g}$ spanned by the root spaces corresponding to roots in $R^+ - R^+_J$.

According to [C1], $\phi$ is an alternating sum of characters of representations induced from one dimensional representations of various parabolic subgroups of $G$. 

defined over $K$. From this one can deduce that, if $t \in T(K) \cap G(K)_{rs}$, then

$$\phi_S(t) = \sum_{J \subseteq I} (-1)^{2J} \sum_{w \in J^W} \delta_J(w(t))^{1/2} D_{I,J}(w(t))^{-1/2}$$

where for any $J \subseteq I$ and $t' \in T(K) \cap G(K)_{rs}$ we set

$$D_{I,J}(t') = |\det(1 - \text{Ad}(t')|_{g/I,J})|,$$

$$\delta_J(t') = |\det(\text{Ad}(t')|_{n,J})|,$$

and $^JW$ is a set of representatives for the cosets $W_j \backslash W$. (It will be convenient to assume that $^JW$ is the set of representatives of minimal length for the cosets $W_j \backslash W$.) Here for a real number $a \geq 0$ we denote by $a^{1/2}$ or $\sqrt{a}$ the $\geq 0$ square root of $a$. We have the following result. (We write $\phi$ instead of $\phi_S$.)

**Theorem 2.2.** Let $y \in Y^+$ and let $t \in T(K)_y^\bullet$. Then $\phi(t) = q^{-(y,2p)}$.

2.3. More generally let $t \in T(K)^\bullet_y$ where $y \in Y$. By a standard property of Weyl chambers there exists $w \in W$ such that $w(y) \in Y^+$. Let $t_1 = w(t)$. Then the theorem is applicable to $t_1$ and we have $\phi(t) = \phi(t_1) = q^{-(w(y),2p)}$.

2.4. Let $y' = w_0(y), t' = w_0(t)$. We have $\phi_S(t) = \phi_S(t'), t' \in T(K)^\bullet_{y'}, -y' \in Y^+$. We show:

(a) if $\beta \in R^+$ then $v(1 - \beta(t')) = v((\beta(t'))$; if $\beta \in R^-$ then $v(1 - \beta(t')) = 0$.
Assume first that $\beta \in R^+$. If $v(\beta(t')) \neq 0$ then $v(\beta(t')) < 0$ (since $\langle y', \beta \rangle \neq 0$, $\langle y', \beta \rangle \leq 0$) hence $v(1 - \beta(t')) = v((\beta(t')))$. If $v(\beta(t')) = 0$ then $\beta(t') - 1 \in O - p$ hence $v(1 - \beta(t')) = 0 = v((\beta(t'))$ as required.

Assume next that $\beta \in R^-$. If $v(\beta(t')) \neq 0$ then $v(\beta(t')) > 0$ (since $\langle y', \beta \rangle \neq 0$, $\langle y', \beta \rangle \geq 0$) hence $v(1 - \beta(t')) = 0$. If $v(\beta(t')) = 0$ then $\beta(t') - 1 \in O - p$ hence $v(1 - \beta(t')) = 0$ as required.

For any $w \in W, J \subseteq I$ we have:

$$D_{I,J}(w(t')) = \prod_{\alpha \in R - R_J} q^{-v(1-\alpha(w(t')))} = \prod_{\alpha \in R - R_J; w^{-1}\alpha \in R^+} q^{-v(\alpha(w(t'))) = \prod_{\alpha \in R - R_J; w^{-1}\alpha \in R^+} q^{-\langle y', w^{-1}\alpha \rangle},$$

$$\delta_J(w(t')) = \prod_{\alpha \in R^+ - R_J} q^{-v(\alpha(w(t')))} = \prod_{\alpha \in R^+ - R_J} q^{-\langle y', w^{-1}\alpha \rangle},$$

$$D_I(t') = \prod_{\alpha \in R^+} q^{-\langle y', \alpha \rangle},$$
(We have used (a) with $\beta = w^{-1}(\alpha)$. We see that

$$\phi(t) = \phi(t') = \sum_{J \subset I} (-1)^{\sharp J} \sum_{w \in J \mathcal{W}} \sqrt{q}^{-\langle y', x_w, \cdot \rangle}$$

where for $w \in J \mathcal{W}$ we have

$$x_{w,J} = \sum_{\alpha \in R^+ - R^+_J} w^{-1} \alpha - \sum_{\alpha \in R^-} w^{-1} \alpha
= \sum_{\alpha \in R^+ - R^+_J; w^{-1}(\alpha) \in R^-} w^{-1} \alpha - \sum_{\alpha \in R^-; w^{-1}(\alpha) \in R^+} w^{-1} \alpha
= 2 \sum_{\alpha \in R^+ - R^+_J; w^{-1}(\alpha) \in R^-} w^{-1} \alpha \in X.$$

For $w \in J \mathcal{W}$ we have $\alpha \in R^+_J \implies w^{-1} \alpha \in R^+$ hence

$$\sum_{\alpha \in R^+ - R^+_J; w^{-1}(\alpha) \in R^-} w^{-1} \alpha = \sum_{\alpha \in R^+; w^{-1}(\alpha) \in R^-} w^{-1} \alpha$$

so that $x_{w,J} = x_w$ where

$$x_w = 2 \sum_{\alpha \in R^+; w^{-1}(\alpha) \in R^-} w^{-1} \alpha \in X.$$

Thus we have

$$\phi(t) = \sum_{J \subset I} (-1)^{\sharp J} \sum_{w \in J \mathcal{W}} \sqrt{q}^{-\langle y', x_w \rangle} = \sum_{w \in \mathcal{W}} c_w \sqrt{q}^{-\langle y', x_w \rangle}$$

where for $w \in \mathcal{W}$ we set

$$c_w = \sum_{J \subset I; w \in J \mathcal{W}} (-1)^{\sharp J}.$$

For $w \in \mathcal{W}$ let $\mathcal{L}(w) = \{ i \in I; s_i w > w \}$ where $<$ is the standard partial order on $\mathcal{W}$. For $J \subset I$ we have $w \in J \mathcal{W}$ if and only if $J \subset \mathcal{L}(w)$. Thus,

$$c_w = \sum_{J \subset \mathcal{L}(w)} (-1)^{\sharp J}$$

and this is 0 unless $\mathcal{L}(w) = \emptyset$ (that is $w = w_0$) when $c_w = 1$. Note also that $x_{w_0} = -4\rho$. Thus we have

$$\phi(t) = c_{w_0} \sqrt{q}^{-\langle y', x_{w_0} \rangle} = q^{\langle y', 2\rho \rangle} = q^{-\langle y, 2\rho \rangle}.$$

Theorem 2.2 is proved.
2.5. Assume now that \( \tau \in T(K) \) satisfies the following condition: for any \( \alpha \in R \) we have \( \alpha(\tau) - 1 \in p - \{0\} \) so that \( \alpha(\tau) - 1 \in p^{n_{\alpha}} - p^{n_{\alpha} + 1} \) for a well defined integer \( n_{\alpha} \geq 1 \). Note that \( n_{-\alpha} = n_{\alpha} \) and \( v(1 - \alpha(\tau)) = n_{\alpha} \geq 1 \) for all \( \alpha \in R \). Hence
\[
\phi(\tau) = \sum_{J \subset I} (-1)^{|J|} \sum_{w \in J} q^{\sum_{\alpha \in R} n_{\alpha}/2 - \sum_{\alpha \in R, J} n_{\alpha - 1}/2}.
\]
Thus,
\[
\phi(\tau) = \sharp(W)q^{\sum_{\alpha \in R} n_{\alpha}/2} + \text{strictly smaller powers of } q.
\]
In the case where \( K \) is the field of power series over \( F_q \), the leading term
\[
\sharp(W)q^{\sum_{\alpha \in R} n_{\alpha}/2}
\]
is equal to \( \sharp(W)q^m \) where \( m \) is the dimension of the “variety” of Iwahori subgroups of \( G(K) \) that contain the topologically unipotent element \( \tau \) (see [KL2]).

3. Calculation of \( \phi_s \) on \( G(K)_{cvr} \)

3.1. We will again write \( \phi \) instead of \( \phi_s \). In this section we assume that we are given \( \gamma \in G(K)_{cvr} \). Let \( T' = T'_{\gamma} \). Note that \( T' \) is defined over \( K \); let \( A' \) be the largest \( K \)-split torus of \( T' \). For any parabolic subgroup \( P \) of \( G \) defined over \( K \) such that \( \gamma \in P \) we set \( \delta_P(\gamma) = |\det(\Ad(\gamma)|_n|) \) where \( n \) is the Lie algebra of the unipotent radical of \( P \).

Let \( \mathcal{X} \) be the set of all pairs \((P,A)\) where \( P \) is a parabolic subgroup of \( G \) defined over \( K \) and \( A \) is the unique maximal \( K \)-split torus in the centre of some Levi subgroup of \( P \) defined over \( K \); then that Levi subgroup is uniquely determined by \( A \) and is denoted by \( M_A \). Let \( \mathcal{X}' = \{(P,A) \in \mathcal{X}; A \subset A'\} \). According to Harish-Chandra [H] we have
\[
(a) \quad \phi(\gamma) = (-1)^{\dim T} \sum_{(P,A) \in \mathcal{X}'} (-1)^{\dim A} \delta_P(\gamma)^{1/2} D_{G/M_A}(\gamma)^{-1/2}
\]
where \( D_{G/M_A}(\gamma) = |\det(1 - \Ad(\gamma)|_{g/1}| \) (we denote by \( I \) the Lie algebra of \( M_A \)).

**Theorem 3.2.** Assume in addition that \( \gamma \in G(K)_{cvr} \). Then
\[
\phi(\gamma) = (-1)^{\dim T - \dim A'}.
\]

From our assumptions we see that for any \((P,A) \in \mathcal{X}'\) we have \( \delta_P(\gamma) = 1 = D_{G/M_A}(\gamma) \). Hence 3.1(a) becomes
\[
\phi(\gamma) = (-1)^{\dim T} \sum_{(P,A) \in \mathcal{X}'} (-1)^{\dim A}.
\]

Let \( \mathcal{Y} \) be the group of cocharacters of \( A' \) and let \( \mathfrak{h} = \mathcal{Y} \otimes \mathbb{R} \). The real vector space \( \mathfrak{h} \) can be partitioned into facets \( F_{P,A} \) indexed by \((P,A) \in \mathcal{X}'\) such that \( F_{P,A} \) is homeomorphic to \( \mathbb{R}^{\dim A} \). Note that the Euler characteristic with compact support of \( F_{P,A} \) is \( (-1)^{\dim A} \) and the Euler characteristic with compact support of \( \mathfrak{h} \) is \( (-1)^{\dim \mathfrak{h}} \mathfrak{h} = (-1)^{\dim A'} \). Using the additivity of the Euler characteristic with compact support we see that \( \sum_{(P,A) \in \mathcal{X}'} (-1)^{\dim A} = (-1)^{\dim A'} \). Thus, \( \phi(\gamma) = (-1)^{\dim T - \dim A'} \), as required. \( \Box \)
3.3. In the setup of 3.1 let $P_\gamma$ be the parabolic subgroup of $G$ associated to $\gamma$ as in [C2]. Note that $P_\gamma$ is defined over $K$. The following result can be deduced by combining Theorem 3.2 with the results in [C2] and with Proposition 2 of [R].

**Corollary 3.4.** We have $\phi(\gamma) = (-1)^{\dim T - \dim A'} \delta_{P_\gamma}(\gamma)$.

4. Iwahori spherical representations: split elements

4.1. Let $B$ be the subgroup of $G(K)$ generated by $U_\alpha(\mathcal{O}), (\alpha \in R^+)$, $U_\alpha(p), (\alpha \in R^-)$ and $T(K)_0$. (The subgroups $U_\alpha(\mathcal{O}), U_\alpha(p)$ of $U_\alpha$ are defined by the $\mathcal{O}$-structure of $G$. We have $B \in B$ where $B$ is the set of Iwahori subgroups of $G(K)$. Note that $B \subset G(K)'$. For any $\alpha \in R$ we choose an isomorphism $x_\alpha : K \xrightarrow{\sim} U_\alpha(K)$ (the restriction of an isomorphism of algebraic groups from the additive group to $\mathcal{O}$) which carries $\mathcal{O}$ onto $U_\alpha(\mathcal{O})$ and $p$ onto $U_\alpha(p)$. We set $W := Y \cdot W$ with $Y$ normal in $W$ (recall that $W$ acts naturally on $Y$). Let $Y'$ be the subgroup of $Y$ generated by the coroots. Then $W' := Y' \cdot W$ is naturally a subgroup of $W$. According to [IM], $W$ is an extended Coxeter group (the semidirect product of the Coxeter group $W'$ with the finite abelian group $Y/Y'$) with length function

$$l(yw) = \sum_{\alpha \in R^+: w^{-1}(\alpha) \in R^+} ||(y, \alpha)|| + \sum_{\alpha \in R^+: w^{-1}(\alpha) \in R^-} ||(y, \alpha) - 1||$$

where $||a|| = a$ if $a \geq 0$, $||a|| = -a$ if $a < 0$. According to [IM], the set of double cosets $B \backslash G(K)/B$ is in bijection with $W$; to $yw$ (where $y \in Y, w \in W$) corresponds the double coset $\Omega_{yw}$ containing $T(K)_{yw}$ (here $\dot{w}$ is an element in $G(\mathcal{O})$ which normalizes $T(K)_0$ and acts on it in the same way as $w$); moreover, $\sharp(\Omega_{yw}/B) = \sharp(B/\Omega_{yw}) = q^{l(yw)}$ for any $y \in Y, w \in W$. For example, if $y \in Y^{++}$ then $l(y) = (y, 2\rho)$.

Let $H$ be the algebra of $B$-biinvariant functions $G(K) \to \mathbb{C}$ with compact support with respect to convolution (we use the Haar measure $dg$ on $G(K)$ for which $vol(B) = 1$). For $y, w$ as above let $\tau_{yw} \in H$ be the characteristic function of $\Omega_{yw}$. Then the functions $\tau_w, w \in W$, form a $\mathbb{C}$-basis of $H$ and according to [IM] we have

$$\tau_w \tau_{w'} = c_{ww'} \tau_{ww'} \text{ if } w, w' \in W \text{ satisfy } l(ww') = l(w) + l(w'),$$

$$(\tau_w + 1)(\tau_w - q) = 0 \text{ if } w \in W', l(w) = 1.$$ 

In other words, $H$ is what now one calls the Iwahori-Hecke algebra of the (extended) Coxeter group $W$ with parameter $q$.

4.2. Let $C_0^\infty(G(K))$ be the vector space of locally constant functions with compact support from $G(K)$ to $\mathbb{C}$. Let $(V, \sigma)$ be an irreducible admissible representation of $G(K)$ such that the space $V^B$ of $B$-invariant vectors in $V$ is nonzero. If $f \in C_0^\infty(G(K))$ then there is a well defined linear map $\sigma_f : V \to V$ such that for any $x \in V$ we have $\sigma_f(x) = \int_G f(g)\sigma(g)(x) dg$. This linear map has finite rank hence it has a well defined trace $\text{tr}(\sigma_f) \in \mathbb{C}$. From the definitions we see that for $f, f' \in C_0^\infty(G(K))$ we have $\sigma_{f*f'} = \sigma_f \sigma_{f'} : V \to V$ where $*$ denotes convolution.
If \( f \in H \), then \( \sigma_f \) maps \( V \) into \( V^B \) and \( \text{tr}(\sigma_f) = \text{tr}(\sigma_f|_{V^B}) \). (Recall that \( \dim V^B < \infty \).) We see that the maps \( \sigma_f|_{V^B} \) define a (unital) \( H \)-module structure on \( V^B \). It is known \([BO]\) that the \( H \)-module \( V^B \) is irreducible. Moreover for \( w \in W \) we have \( \text{tr}(\sigma_{\tau^w}) = \text{tr}(\sigma^w) \) where the trace in the right side is taken in the \( H \)-module \( V^B \). We have the following result.

**Theorem 4.3.** Assume that \( K \) has characteristic zero and that \( p \) is sufficiently large. Let \( y \in Y^+ \) and let \( t \in T(K)^\bullet \). We have

\[
\phi_V(t) = q^{-\langle y, \rho \rangle} \text{tr}(\tau^y)
\]

where the trace in the right side is taken in the irreducible \( H \)-module \( V^B \).

An equivalent statement is that

\[
\phi_V(t) = \text{tr}(\sigma^y)/\text{vol}(\Omega_y).
\]

(Recall that \( \tau^y \) in the right hand side is the characteristic function of \( \Omega_y = BT(K)_yB \).)

The assumption on characteristic in the theorem is needed only to be able to use a result from \([AK]\), see 5.1(†). We expect that the theorem holds without that assumption.

In the case where \( y = 0 \) the theorem becomes:

(a) If \( t \in T(K) \cap G_{cvr} \) then \( \phi_V(t) = \dim(V^B) \).

As pointed out to us by R. Bezrukavnikov and S. Varma, in the special case where \( y \in Y^{++} \), Theorem 4.3 can be deduced from results in \([C2]\).

### 4.4.
In the case where \( V = S \), see 1.1, for any \( y \in Y^+ \), \( \tau^y \) acts on the one dimensional vector space \( V^B \) as the identity map so that \( \phi_V(t) = q^{-\langle y, \rho \rangle} \); we thus recover Theorem 2.2 (which holds without assumption on the characteristic).

#### 5. Proof of Theorem 4.3

5.1. Let \( B = B_0, B_1, B_2, \ldots \) be the strictly decreasing Moy-Prasad filtration of \( B \). In \([MP]\), this is a sequence associated to a point \( x \) in the building such that \( B = G_{x,0} \). Note that each \( B_i/B_{i+1} \) is abelian. Let \( T_n := T(K) \cap B_n \). Applying Corollary 12.11 in \([AK]\) to \( \phi_V \), we have

(†) \( \phi_V \) is constant on the \( \text{Ad}(G) \)-orbit \( G(tT_1) \) of \( tT_1 \).

**Lemma 5.2.** Let \( n \geq 1 \). For any \( t' \in T(K)^\bullet \) and \( z \in B_n \), there exist \( g \in B_n \), \( t'' \in T_n \) and \( z' \in B_{n+1} \) such that \( \text{Ad}(g)(t'z) = t''z' \).

**Proof.** Let \( Z = \{ \alpha \in R | U_\alpha \cap B_n \supseteq U_\alpha \cap B_{n+1} \} \). If \( Z = \emptyset \), \( B_n = T_nB_{n+1} \). Hence, \( z = t''z' \) for some \( t'' \in T_n \) and \( z' \in B_{n+1} \) and one can take \( g = 1 \). If \( Z \neq \emptyset \), there are \( a_\alpha \in K, \alpha \in Z \) such that \( x_\alpha(a_\alpha) \in B_n \) and \( z \equiv \prod_{\alpha \in Z} x_\alpha(a_\alpha) \) (mod \( T_nB_{n+1} \)). Such \( a_\alpha \) can be chosen independent of the order of \( \prod \) since
$B_n/T_nB_{n+1}$ is abelian. Take $g = \prod_{\alpha \in \mathbb{Z}} x_\alpha((1 - \alpha(t' - 1))^{-1}a_\alpha)$. Then, we have $t' - 1gt'g^{-1} \equiv z^{-1} \pmod{T_nB_{n+1}}$. Moreover, since $y \in Y^+$, we have $|1 - \alpha(t' - 1)| \geq 1$ and thus $g \in B_n$. (We argue as in 2.4(a).) Assume first that $\alpha \in R^+$. If $v(\alpha(t' - 1)) \neq 0$ then $v(\alpha(t' - 1)) \leq 0$ (since $\langle y, \alpha \rangle \neq 0$, $\langle y, \alpha \rangle \geq 0$) hence $v(1 - \alpha(t' - 1)) = v((\alpha(t' - 1)) < 0$ and $|1 - \alpha(t' - 1)| > 1$. If $v(\alpha(t' - 1)) = 0$ then $\alpha(t' - 1) - 1 \in \mathcal{O} - \mathfrak{p}$ hence $v(1 - \alpha(t' - 1)) = 0$ and $|1 - \alpha(t' - 1)| = 1$ as required. Assume next that $\alpha \in R^-$. If $v(\alpha(t' - 1)) \neq 0$ then $v(\alpha(t' - 1)) > 0$ (since $\langle y, \alpha \rangle \neq 0$, $\langle y, \alpha \rangle \leq 0$) hence $v(1 - \alpha(t' - 1)) = 0$ and $|1 - \alpha(t' - 1)| = 1$ as required. If $v(\alpha(t' - 1)) = 0$ then $\alpha(t' - 1) - 1 \in \mathcal{O} - \mathfrak{p}$ hence $v(1 - \alpha(t' - 1)) = 0$ and $|1 - \alpha(t' - 1)| = 1$ as required.)

Writing $Ad(g)(t'z) = t' \cdot (t' - 1gt'g^{-1}) \cdot (gzg^{-1})$, we observe that $gzg^{-1} \equiv z \pmod{B_{n+1}}$ and $t' - 1gt'g^{-1} \in T_nB_{n+1}$. Hence $Ad(g)(t'z)$ can be written as $t't'z'$ with $t' \in T_n$ and $z' \in B_{n+1}$. □

**Lemma 5.3.** $B_1tB_1 \subset G(tT_1)$.

**Proof.** It is enough to show that $tB_1 \subset G(tT_1)$. Let $t_0z_1 \in tB_1$ with $t_0 = t$ and $z_1 \in B_1$. We will construct inductively sequences $g_1, g_2, \ldots, t_1, t_2, \ldots$ and $z_1, z_2, \ldots$ such that $Ad(g_k \cdots g_2g_1)(t_0z_1) = Ad(g_k)(t_0t_1 \cdots t_{k-1}z_k) = (t_0t_1 \cdots t_k)z_{k+1}$ with $g_i \in B_i$, $t_i \in T_i$ and $z_i \in B_i$.

Applying Lemma 5.2 to $n = 1$, $t' = t_0$ and $z = z_1$, we find $t_1 \in T_1$ and $z_2 \in B_2$ such that $g_1t_0z_1g_1^{-1} = t_0t_1z_2$ with $t_1 \in T_1$ and $z_2 \in B_2$. Suppose we found $g_i \in B_i$, $z_{i+1} \in B_{i+1}$ and $t_i \in T_i$ for $i = 1, \ldots k$ where $k \geq 1$. Applying Lemma 5.2 to $n = k + 1$, $t' = t_0t_1 \cdots t_k$ and $z = z_{k+1}$, we find $g_{k+1} \in B_{k+1}$, $t_{k+1} \in T_{k+1}$ and $z_{k+2} \in B_{k+2}$ so that $g_{k+1}t_0t_1 \cdots t_kz_{k+1}g_{k+1}^{-1} = Ad(g_k \cdots g_2g_1)(t_0z_1) = t_0t_1t_2 \cdots t_{k+1}z_{k+2}$. (To apply Lemma 5.2 we note that $t' \in T(K)^\bullet$ since $t_0 \in T(K)^\bullet$ and $t_1 \cdots t_k \in T_1$ so that for any $\alpha \in R$ we have $\alpha(t_1 \cdots t_k) \in 1 + \mathfrak{p}$.)

Taking $g \in B_1$ be the limit of $g_k \cdots g_2g_1$ as $k \to \infty$, we have $Ad(g)(t_0z_1) \in tT_1$. □

**5.4.** Continuing with the proof of Theorem 4.3, we note that by Lemma 5.3 and 5.1(†), for the characteristic function $f_t$ of $B_1tB_1$ we have

$$tr(\sigma_{f_t}) = \int_G f_t(g)\phi_V(g) \, dg = \int_{B_1tB_1} \phi_V(t) \, dg = vol(B_1tB_1)\phi_V(t).$$

Thus it remains to show that

$$tr(\sigma_{f_t})/vol(B_1tB_1) = tr(\sigma_{\pi_y})/vol(BtB).$$

Since $B_1$ is normalized by $B$, $B$ acts on $V^{B_1}$. Moreover, since $V$ is irreducible and $V^B \neq 0$, $B$ acts trivially on $V^{B_1}$ (otherwise, there would exist a nonzero subspace of $V$ on which $B$ acts through a nontrivial character of $B/B_1$; since $V^B \neq 0$ we see that $(V, \sigma)$ would have two distinct cuspidal supports, a contradiction). Thus
we have $V^{B_1} = V^B$. Since $\sigma_t$ and $\sigma_{\Xi_y}$ have image contained in $V^{B_1} = V^B$, it is enough to show that

(a) $\text{tr}(\sigma_t|_{V^B})/\text{vol}(B_1tB_1) = \text{tr}(\sigma_{\Xi_y}|_{V^B})/\text{vol}(BtB)$.

We can find a finite subset $L$ of $T(K)_0$ such that $BtB = \sqcup_{\tau \in L} B_1tB_1\tau$. It follows that

(b) $\text{vol}(BtB) = \text{vol}(B_1tB_1)\sharp(L)$

and $\sigma_{\Xi_y} = \sum_{\tau \in L} \sigma_t, \sigma(\tau)$ as linear maps $V \to V$. Restricting this equality to $V^B$ and using the fact that $\sigma(\tau)$ acts as identity on $V^B$ we obtain

(c) $\sigma_{\Xi_y}|_{V^B} = \sharp(L)\sigma_t|_{V^B}$

as linear maps $V^B \to V^B$. Clearly, (a) follows from (b) and (c). This completes the proof of Theorem 4.3.

The following result will not be used in the rest of the paper.

**Proposition 5.5.** If $y \in Y^{++}$ and $t \in T(K)_y$ then $BtB \subset ^G T(K)_y$.

**Proof.** It is enough to show that $tz \subset ^G T(K)_y$ for any $z \in B$. We can write $z = t_0z'$ where $t_0 \in T(K)_0, z' \in B_1$. We have $tz = tt_0z'$ where $tt_0 \in T(K)_y = T(K)_y^\bullet$ (here we use that $y \in Y^{++}$). Using Lemma 5.3 we have $tt_0z' \in ^G (tt_0T_1) \subset ^G T(K)_y$. This completes the proof. □

**5.6.** In the remainder of this section we assume that $G$ is adjoint. In this case the irreducible representations $(V, \sigma)$ as in 4.2 (up to isomorphism) are known to be in bijection with the irreducible finite dimensional representations of the Hecke algebra $H$ (see [BO]) by $(V, \sigma) \mapsto V^B$. The irreducible finite dimensional representations of $H$ have been classified in [KL1] in terms of geometric data. Moreover in [L] an algorithm to compute the dimensions of the (generalized) weight spaces of the action of the commutative semigroup $\{\Xi_y; y \in Y^{++}\}$ on any tempered $H$ module is given. In particular the right hand side of the equality in Theorem 4.3 (hence also $\phi_V(t)$ in that Theorem) is computable when $V$ is tempered.

**References**


[St] R. Steinberg, A geometric approach to the representations of the full linear group over a Galois field, Trans.Amer.Math.Soc. 71 (1951), 274282.