OPTIMAL FOUR IMPULSE TRANSFERS
BETWEEN TWO NEARBY ORBITS

by
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ABSTRACT

This thesis considers the problem of finding minimum
fuel, fixed time, four-impulse transfers between two coplanar
orbits. The problem is solved once the magnitudes, the
directions, and the times of application of the impulses are
known.

Under the assumption that the orbits are close together,
the linearized equations are used. It is further assumed
that the first and fourth impulses are applied at known
initial and final positions and velocities.

The necessary conditions of the optimization problem are
formulated, and, based on the assumption that a solution
exists, an iterative technique is developed for finding the
optimum thrust strategy.

It is shown that optimum four-impulse solutions exist
for transfers whose initial impulse is applied at the perigee
of the reference orbit and whose final impulse is applied at
eccentric anomalies up to and including 540°. These solutions
are computed for eccentricities varying between 0.0 and 0.7.
It is recommended that the analysis be extended beyond $540^\circ$ eccentric anomaly and above $e = 0.7$. It is further recommended that solutions be obtained for transfers whose initial impulse is located at a point other than perigee.

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter No.</th>
<th>Title</th>
<th>Page No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>II</td>
<td>Determination of the Necessary Conditions</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.1 Introduction</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>2.2 Equations of Motion</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>2.3 Equations of the Propulsive System</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>2.4 Development of the Necessary Conditions</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>2.5 Solution of the Necessary Conditions</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>2.6 Determination of the Impulsive Magnitudes</td>
<td>16</td>
</tr>
<tr>
<td>III</td>
<td>Solution by Numerical Techniques</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3.1 Introduction</td>
<td>19</td>
</tr>
<tr>
<td></td>
<td>3.2 The Coordinate System</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>3.3 The State Transition Matrix</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>3.4 Review of the Necessary Conditions</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>3.5 Development of the Numerical Procedure</td>
<td>23</td>
</tr>
<tr>
<td>Chapter No.</td>
<td>Page No.</td>
<td>Description</td>
</tr>
<tr>
<td>------------</td>
<td>----------</td>
<td>-------------</td>
</tr>
<tr>
<td>IV</td>
<td></td>
<td>Results for a Special Class of Transfers</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.1 Introduction 29</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.2 Generation of Solutions 30</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.3 Discussion of Results 32</td>
</tr>
<tr>
<td>V</td>
<td></td>
<td>Conclusions and Recommendations</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5.1 Conclusions 37</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5.2 Recommendations for Further Study 38</td>
</tr>
</tbody>
</table>

**Appendices**

<table>
<thead>
<tr>
<th>Appendix</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.</td>
<td>Partitioned Matrices of the State Transition Matrix 39</td>
</tr>
<tr>
<td>B.</td>
<td>Lambda Components 45</td>
</tr>
<tr>
<td>C.</td>
<td>Time of Second &amp; Third Impulses 55</td>
</tr>
<tr>
<td>D.</td>
<td>Direction of Impulse Application 65</td>
</tr>
</tbody>
</table>

**Figures**

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Geometrical Properties of the Necessary Conditions 14</td>
</tr>
<tr>
<td>2.</td>
<td>The p, q, f Coordinate System 20</td>
</tr>
</tbody>
</table>

**References**

| References | 75 |
A central problem which arises in orbital mechanics is one in which it is desired to determine the optimum thrust program for a fixed time transfer when the initial and the final positions and velocities are known. The optimization problem consists of finding the thrust program that results in the minimum expenditure of fuel. When the time of thrusting, its direction of application and its magnitude are known, the problem is solved.

The problem to be investigated will assume that the transfer is executed by using a number of impulsive maneuvers. The question that immediately comes to mind is: "How many impulses will be necessary to transfer from one orbit to another in an optimum fashion?"

Neustadt and Stern and Potter have found an important result for the linearized problem which places an upper bound on the number of impulses for an optimum solution. In order to use this result, it is necessary to restrict the class of allowable transfers to those which may be linearized about a reference or nominal path.
To quote Neustadt

"...if the equations of motion of the space vehicle can be approximated by equations of a special form, and if \( n(n<6) \) components of the vehicle's position and velocity vectors must take on given values at a given terminal time, then a minimum-fuel maneuver for achieving these end values will consist of not more than \( n \) impulsive corrections...."

The special form of the equations referred to above is the linearized equations of motion in a conservative gravitational field.

The above quote, however, does not state how many impulse (\( n \) or less) will yield the absolute minimum impulsive transfer—it only defines a way of prescribing the maximum number of impulses allowable for an optimum program. In order to determine the absolute minimum solution, it is necessary to compare different solutions which use up to \( n \) impulses.

Let the control problem be restricted even further by considering only two (2) dimensional transfers. With the initial and final positions and velocities known, for a given fixed time, the result of Neustadt indicates that the minimum fuel thrust program consists of four (4), or fewer, impulses. In order to determine the absolute minimum, it must be necessary to have the solutions of four (4) and fewer impulses.

An extensive amount of work has been done on solutions using two (2) impulse transfers\(^3\), \(^{14}\). For the linearized time fixed case, these solutions may be obtained in closed
form. Relatively little has been done on three (3) impulse maneuvers and, so far as the author is aware, hardly anything has been done on solutions using four (4) impulses.

Although Neustadt's result indicates that four (4) impulses are the maximum number in impulses for an optimum solution, it does not provide a method of determining these solutions. This thesis will develop a technique for determining optimum four (4) impulse transfer solutions. The importance of this study is that the optimum four (4) impulse transfer is the limiting two (2) dimensional case.

It is the intent of this work to:

i. develop the necessary conditions for an optimum four (4) impulse transfer (Chapter II)

ii. develop a technique for solving the optimum thrust program based on the assumption that a four (4) impulse solution exists (Chapter III)

iii. demonstrate that optimum four (4) impulse solutions do exist at least for one (1) class of trajectories (Chapter IV).
CHAPTER II

DETERMINATION OF THE NECESSARY CONDITIONS

2.1 Introduction: Statement of the Problem

In a given fixed time, in a two dimensional space, it is desired to transfer from a given position and velocity in one orbit to a given position and velocity in another orbit, using a four-impulse maneuver in such a manner that the total fuel consumption is minimized. The thrust strategy will assume that one impulse is applied at the initial time, another impulse at the final time, and two additional impulses at intermediate times on the transfer.

It is assumed that the only forces acting on the vehicle are due to an inverse-square gravitational field and the vehicle's propulsion system. The vehicle's propulsion system is assumed to be an ideal chemical rocket.

It is the purpose of this analysis to develop a method by which the impulse magnitudes, their directions, and the times of the two intermediate impulses may be determined. Once this information is known, the problem can be considered solved.
It is the purpose of this chapter to determine the necessary conditions of the analysis that will satisfy the above requirements.

2.2 Equations of Motion

In order to properly define the maneuver requirements, the equations of motion along the transfer path must be known. Since these equations are non-linear, the method of linear perturbations will be used to describe the transfer. The equations are linearized about a reference or nominal path. The selection of this nominal path is constrained such that the equations are linearized along the entire transfer. Therefore, the reference path may be the original orbit, a two-impulse transfer orbit, or any selected intermediate orbit between the initial and final orbits. It may be that in the time specified for the transfer that no reference orbit exists about which the equations remain linear. In this case, the following analysis does not hold and higher orders of approximation will be necessary.

Since the linearized equations of motion are available in the literature, \(^1,^3,^8\), they will be presented here without a formal derivation.

Define the variation in position as \(\delta r\), the variation in velocity as \(\delta v\), and the acceleration due to the propulsive system as \(a_r\). The equations of motion are then given by
\[
\frac{d}{dt} (\delta r) = \delta \dot{r} = \delta v \tag{2.2.1}
\]
\[
\frac{d}{dt} (\delta v) = \delta \dot{v} = G \delta r - a_T \tag{2.2.2}
\]

where \( G = \frac{\partial g}{\partial r} \) is evaluated along the nominal path. \( g \) designates the gravitational force.

Since the equations of motion have been linearized, transfers can only take place between two nearby orbits.

### 2.3 Equations of the Propulsive System

Again, since the equations of the propulsive system are available in the literature, \( 6, 7, 12 \), they will not be derived. It is assumed that the propulsive system has a constant exhaust velocity, \( c \). The thrust equation is given by

\[
T = -C \frac{dm}{dt} u \tag{2.3.1}
\]

where \( T \) is the thrust, \( m \) the mass of the vehicle, and \( u \) the unit cosine vector defining the thrust direction. Let

\[
\frac{dm}{dt} \equiv \dot{m} = -\sigma \tag{2.3.2}
\]

Thus

\[
T = C \sigma u \tag{2.3.3}
\]

The acceleration due to the thrust, \( a_T \), is then given by

\[
a_T = \frac{1}{m} T \tag{2.3.4}
\]
and hence,

$$\alpha_T = \frac{c \sigma}{m} u$$

(2.3.5)

Since $u$ is a unit cosine, then it must satisfy the condition that

$$u^T u - 1 = 0$$

(2.3.6)

One more equation will be developed to fully describe the propulsive system. For many propulsive systems, the maximum thrust is limited by a bound on $\sigma$. Define the maximum value of $\sigma$ as $\bar{\sigma}$. Therefore

$$\bar{\sigma} \geq \sigma \geq 0$$

(2.3.7)

describes the capabilities of the propulsive system. Alternatively this condition may be written in a form suggested by Valentine$^{10}$.

$$\sigma(\bar{\sigma} - \sigma) - \alpha^2 = 0$$

(2.3.8)

$\alpha^2 > 0$ is a quantity which must be determined during the optimization problem.

### 2.4 Development of the Necessary Conditions

The optimization problem will be set up as a calculus of variations problem. The approach to be taken is similar to Lawden$^7$ in that the optimal control will be found for a
continuous thrust control and then restricted to thrusts of short duration which in the limit of the time of duration become impulses. It is assumed the reader is familiar with the elements of the calculus of variations 4, 5, 6.

The optimization problem is designed to minimize the total fuel consumption. This can be represented as an attempt to minimize a function $J$, where

$$J = \int_{t_0}^{t_f} \| \dot{a}_r \| \, dt$$  \hspace{1cm} (2.4.1)$$

The magnitude of the thrust acceleration is given by equation (2.3.5). Hence $J$ becomes

$$J = \int_{t_0}^{t_f} \frac{c \sigma}{m} \, dt$$  \hspace{1cm} (2.4.2)$$

Since the above function must be minimized over the transfer path subject to the constraints of the propulsive system, equations (2.2.1), (2.2.2), (2.3.6) and (2.3.8) are adjoined to $J$ by Lagrange multipliers to form a new function $\bar{J}$.

$$\bar{J} = \int_{t_0}^{t_f} \left[ \frac{c \sigma}{m} + \lambda_v^T (G \delta \tau - \frac{c \sigma}{m} u - \delta \dot{v}) + \lambda_r^T (\delta v - \delta \dot{r}) + \psi_1 (u^T u - 1) + \psi_2 (\sigma (\bar{\sigma} - \sigma - \alpha^2) \right] \, dt \hspace{1cm} (2.4.3)$$

It is now desired to minimize $\bar{J}$ with the additional constraints that $\delta r$ and $\delta v$ are given at $t_0$ and $t_f$ and that $m$ is given at $t_0$. 
Integrating $\bar{J}$ by parts and allowing the first variation of $\bar{J}$ to go to zero subject to the end point constraints yields

$$0 = \int_{t_0}^{t_f} \left\{ \left[ \frac{c}{m}(1-\lambda_v^T \lambda_r) + u_2(\bar{\sigma} - 2\sigma) \right] \Delta \sigma + \left[-2v_2 \alpha \right] \Delta \alpha 

+ \left[2v_1 u^T - \lambda_v^T \frac{c\sigma}{m} \right] \Delta u + \left[ \lambda_v^T G + \lambda_r^T \delta \right] \Delta (\delta r) \right\} dt$$

where $\Delta$ implies the first variation.

In order for the integral to be zero for arbitrary $\Delta \sigma, \Delta \alpha, \Delta u, \Delta (\delta r)$ and $\Delta (\delta \nu)$ the quantities in square brackets must be zero.

$$\dot{\lambda}_r = -G \lambda_v$$

$$\dot{\lambda}_v = -\lambda_r$$

$$0 = -2v_2 \alpha$$

$$0 = -v_2 (\bar{\sigma} - 2\sigma) + \frac{c}{m} (\lambda_v^T u - 1)$$

$$0 = 2v_1 u^T - \lambda_v^T \frac{c\sigma}{m}$$
From equation (2.4.7), if \( a = 0 \), \( \sigma = 0 \) or \( \sigma = \bar{\sigma} \); if \( \nu_2 = 0 \), \( a \) is arbitrary and therefore \( \sigma < \bar{\sigma} \). This allows three types of transfers.

i) \( \sigma = 0 \) or a null thrust transfer

ii) \( \sigma < \bar{\sigma} \) or an intermediate thrust transfer

iii) \( \sigma = \bar{\sigma} \) or a maximum thrust transfer

Equation (2.4.9) has special significance when rewritten

\[
\underline{u} = \frac{c\sigma}{2\nu m} \underline{\lambda}_v
\]  

(2.4.10)

This equation shows that \( \underline{u} \) (the control direction) is a linear function of \( \underline{\lambda}_v \). Using the constraint \( \underline{u}^T\underline{u} = 1 \), \( \nu_1 \) may be solved for and substituted back into (2.4.10). This yields

\[
\underline{u} = \pm \frac{\underline{\lambda}_v}{|\underline{\lambda}_v|}
\]  

(2.4.11)

The proper choice of signs will follow from the Weierstrass condition.

This condition may be given as

\[
[\lambda_v^T \underline{u} - 1] \sigma \geq [\lambda_v^T \underline{u}^* - 1] \sigma^*
\]  

(2.4.12)

where the control parameters without an asterisk signify the control along an optimal path while the control parameters with an asterisk signify any other control that satisfies the constraint equations.
For the null thrust condition \( \sigma = 0 \) and equation (2.4.12) results in

\[
\lambda_v^T \mathbf{u}^* \leq 1
\]

(2.4.13)

This implies that the maximum value of the left hand side must be less than or equal to unity for any \( \mathbf{u}^* \). The maximum value of \( \mathbf{u}^* \) is attained when it is aligned with \( \lambda_v \) and hence \( \mathbf{u}^* \) becomes \( \mathbf{u} \) with the positive sign. As a result

\[
\lambda_v^T \mathbf{u} = |\lambda_v| \leq 1
\]

(2.4.14)

For the maximum thrust case \( \sigma = \bar{\sigma} \) and hence \( \sigma^* = \bar{\sigma} \), equation (2.4.12) yields

\[
\lambda_v^T \mathbf{u} \geq \lambda_v^T \mathbf{u}^*
\]

(2.4.15)

For this relationship to be true for all choices of \( \mathbf{u}^* \), \( \mathbf{u} \) must be aligned with and in the direction \( \lambda_v \). Therefore \( \mathbf{u} \) takes the positive sign. Since \( \sigma^* \leq \bar{\sigma} \) the maximum thrust condition may be written as

\[
\lambda_v^T \mathbf{u} - 1 \geq 0
\]

(2.4.16)

For the intermediate thrust region \( \nu_2 \) equals zero and hence

\[
\lambda_v^T \mathbf{u} - 1 = 0
\]

(2.4.17)

This condition may only be satisfied when \( |\lambda_v| = 1 \) and \( \mathbf{u} = \lambda_v \).
Therefore it has been shown that $\mathbf{u}$ is always aligned with and in the direction of $\lambda_v$. Equation (2.4.11) is now written

$$\mathbf{u} = \frac{1}{|\lambda_v|} \lambda_v$$  \hspace{1cm} (2.4.18)

Equations (2.4.14), (2.4.16), and (2.4.17) may be summarized in terms of a function $\kappa$.

$$\kappa = \left( \lambda_v^T \mathbf{u} - 1 \right)$$  \hspace{1cm} (2.4.19)

where $\kappa \leq 0$ over a null thrust region and $\kappa > 0$ over a thrust region.

Define

$$\rho \equiv \lambda_v^T \mathbf{u}$$  \hspace{1cm} (2.4.20)

Thus

$$\kappa = \rho - 1$$  \hspace{1cm} (2.4.21)

$$\dot{\kappa} = \dot{\rho}$$  \hspace{1cm} (2.4.22)

Since $\rho$ must be continuous then $\kappa$ must also be continuous.

The values of $\kappa$ will now be analyzed at the times of an impulse. First the conditions for the intermediate impulses are determined and then the conditions for the initial and final impulses will be determined.
On the thrust arc $\kappa > 0$ while on either side of the thrust arc a region of null thrust exists where $\kappa < 0$. Therefore as the region of thrust decreases to a very short period of time and the null region extends an impulse is approximated. Hence the value of $\kappa$ at the time of the impulse must be zero and also at this point $\kappa$ must be a maximum.

$$\kappa = \dot{\kappa} = 0$$ (2.4.23)

Therefore at an intermediate impulse where $\kappa = 0$ equation (2.4.21) implies that $\rho = 1$, and since $\dot{\kappa} = 0$ at this point also then $\rho$ must be a maximum.

For the initial and final impulses the condition again becomes $\kappa = 0$ at the points of application; however, $\dot{\kappa} = 0$ need not be satisfied at these points. Again from equation (2.4.21) $\rho = 1$ at these times.

Finally equations (2.4.18) and 2.4.20) yield

$$\rho = |\lambda|$$ (2.4.24)

where $\rho = 1$ at the times of impulse application and $\rho < 1$ at all other times. Equation (2.4.23) implies that $\dot{\rho} = 0$ at the two intermediate impulses.

As a final summary, the necessary conditions of a four-impulse solution are:

i) $\lambda_v$ and its first derivative are continuous everywhere

ii) at the time of impulse application $\rho$ must be unity, and the thrust must be directed in the direction of $\lambda_v$
iii) \( \rho \leq 1 \) over any null thrust region

iv) \( \dot{\rho} = 0 \) at the second and third impulses.

Geometrically a plot of \( \rho \) versus time can be represented as

![Figure 1](image_url)

GEOMETRICAL PROPERTIES OF THE NECESSARY CONDITIONS

2.5 Solution of the Necessary Conditions

In section 2.4 only the requirements of determining the optimal thrust strategy were determined without showing how this strategy may be computed. In this section, a method will be determined for computing \( \lambda_v \) and hence \( \rho \) such that the four necessary conditions described in section 2.4 are satisfied.

Using equations (2.4.5) and (2.4.6) one may write

\[
\dot{\lambda} (t) = - F^T (t) \lambda (t) \tag{2.5.1}
\]

where

\[
\lambda (t) = \begin{pmatrix} \lambda_r (t) \\ \lambda_v (t) \end{pmatrix} \tag{2.5.2}
\]
\[ F(t) = \begin{bmatrix} 0 & I \\ G(t) & 0 \end{bmatrix} \] (2.5.3)

Equation (2.5.1) is commonly called the adjoint or co-state equation and its solution is given by the adjoint or co-state vector. The solution is given in terms of the state transition matrix that relates the state at time \( s \) to the state at time \( t \). This may be written as

\[ \delta x(t) = \Phi(t, s) \delta x(s) \] (2.5.4)

where

\[ \delta x(t) = \begin{pmatrix} \delta r(t) \\ \delta y(t) \end{pmatrix} \] (2.5.5)

and \( \Phi(t, s) \) is the state transition matrix. The state transition matrix has the following properties

\[ \frac{d}{dt} \Phi(t, s) = F(t) \Phi(t, s) \] (2.5.6)

and

\[ \Phi(s, t) = \Phi^{-1}(t, s) \] (2.5.7)

The solution of the adjoint equation is now given as

\[ \lambda(t) = \Phi^T(s, t) \lambda(s) \] (2.5.8)
If $s$ is now defined as the final time $t_f$, then $\lambda_v(t)$ will be given by

$$\lambda_v(t) = B^T(t_s,t) \Lambda(t_s)$$

(2.5.9)

where

$$B(t_s,t) = \Phi(t_s,t) M$$

(2.5.10)

$$M = \begin{pmatrix} 0 \\ I \end{pmatrix}$$

(2.5.11)

Therefore equation (2.5.9) forms the basic equation to be investigated. The cornerstone of the analysis is based upon determining a four dimensional vector of constants (i.e. the co-state at the final time) such that $\lambda_v(t)$ and $\rho$ satisfy the necessary conditions of section 2.4 when propagated from the initial time to the final time.

Chapter 3 will describe a method of determining these constants and the times of the second and third impulses for a selected family of transfers.

2.6 Determination of the Impulse Magnitudes

From section 2.5 it will be assumed now that for a fixed time transfer the reference orbit has been selected and the thrust directions and their times are known. To complete the analysis, the thrust magnitudes must be determined. The problem may be formulated as in Stern and Potter².
\[ \delta x(t_f) = \delta x'(t_f) + \sum_k B(t_f, t_k) c_k u_k \]  

(2.6.1)

\( \delta x(t_f) \) is the desired variation in the final state from the reference orbit. \( \delta x'(t_f) \) is the variation from the reference orbit if no impulses had been applied. This term is zero if the reference is the initial orbit; otherwise, it is non-zero. The \( t_k \) are the times of the impulse applications; \( c_k \) are the impulse magnitudes; and the \( u_k \) are the impulse directions.

The summation may be written in a matrix-vector form as follows:

\[ \sum_k B(t_f, t_k) c_k u_k = Q \mathbf{C} \]  

(2.6.2)

where

\[ Q = \left[ B(t_f, t_i) \, u_i : \ldots : B(t_f, t_f) \, u_f \right] \]  

(2.6.3)

and

\[ \mathbf{C} = \begin{pmatrix} c_{t_i} \\ \vdots \\ c_{t_f} \end{pmatrix} \]  

(2.6.4)
Equation (2.6.1) now becomes

$$\delta x(t_4) = \delta x'(t_4) + Q \mathcal{C}$$

(2.6.5)

Solving for $\mathcal{C}$ and hence completing the analysis

$$\mathcal{C} = Q^{-1} \left( \delta x(t_4) - \delta x'(t_4) \right)$$

(2.6.6)

Our problem is entirely solved, assuming that the co-state at the final time may be found.
CHAPTER III

SOLUTION BY A NUMERICAL TECHNIQUE

3.1 Introduction

This chapter will develop a method of determining $\lambda(t_f)$ and the times of the second and third impulses. The method will be numerical rather than analytical because of the complexity of the equations. The numerical procedure will be a Newton-Raphson iteration such that the properties of $\rho$ as defined in section 2.4 are satisfied.

The state transition matrix chosen is that developed by Stern. As of now no mention has been made of the coordinate system in which $\lambda_v$ will be determined. Since the matrices of Stern are computed in a rotating coordinate system, this system will become the one in which the analysis will be done.
3.2 The Coordinate System

The coordinate system in which the perturbation matrices have been determined is a rotating sun-centered system, whose axes are designated $p$, $q$, $z$. The $p$-$q$ plane is in the nominal trajectory plane. The positive $q$ axis is parallel to and in the direction of the nominal velocity vector. The positive $p$ axis is perpendicular to the positive $q$ axis, and is directed radially outward. The positive $z$ axis is perpendicular to the $p$-$q$ plane, and is directed in such a manner as to complete a right-handed system. The system rotates about the $z$ axis such that $q$ always remains parallel to the velocity vector.

![The p,q,z Coordinate System](image)

**Figure 2.**

**THE p,q,z COORDINATE SYSTEM**
3.3 The State Transition Matrix

In terms of Stern's notation, the state transition matrix may be partitioned as follows:

\[
\Phi_{ji} = \begin{bmatrix} M_{ji} & N_{ji} \\ S_{ji} & T_{ji} \end{bmatrix}
\]  

(3.3.1)

where \(\phi_{ji} = \phi(t_j, t_i)\). Since Stern has chosen the eccentric anomaly rather than time as the independent variable of the analysis, it is necessary to relate the time to the eccentric anomaly. Once a reference orbit has been selected, the eccentric anomaly may be determined from the time through Kepler's equation. The eccentric anomaly at time \(t_j\) will be designated as \(E_j\) and the eccentric anomaly corresponding to time \(t_i\) will be designated \(E_i\). Matrices \(M_{ji}, N_{ji}, S_{ji}, T_{ji},\) and \(G_i\) are found in appendix A. In these matrices, \(e\) is the eccentricity, \(n\) is the mean angular motion,

\[
E_p = \frac{1}{2} (E_j + E_i)
\]

(3.3.2)

\[
E_m = \frac{1}{2} (E_j - E_i)
\]

(3.3.3)

In the analysis \(E_j\) will be defined as the eccentric anomaly at the final time of the transfer and will be designated as \(E_4\). The eccentric anomaly at the initial time of the transfer will be defined as \(E_1\) and the times of the second and third impulses will be given in terms of \(E_2\) and \(E_3\) respectively.
Therefore, the transfer can be considered as starting at $E_1$ and terminating at $E_4$. It is important to point out that these eccentric anomalies are measured with respect to the reference orbit.

To facilitate the notation, the subscript "$j_i$" will be dropped and henceforth will be implied throughout the rest of the analysis. Also since a two dimensional problem has been assumed, the third row and column of the partitioned matrices will be ignored.

### 3.4 Review of the Necessary Conditions

The necessary conditions will be stated once more in a functional form. Define a function $f$ as

$$f = \frac{1}{2} \lambda \lambda^T = \lambda^T BB^T \lambda$$  \hspace{1cm} (3.4.1)$$

and the components of $\lambda$ as

$$\lambda = \left( \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{array} \right)$$  \hspace{1cm} (3.4.2)$$

In terms of the eccentric anomaly $f$ has the following properties:

$$f(E_i) \leq 1 \hspace{1cm} E_1 \leq E_i \leq E_4 \hspace{1cm} (3.4.3)$$

$$f(E_k) = 1 \hspace{1cm} k = 1, 2, 3, 4 \hspace{1cm} (3.4.4)$$

$$\frac{d}{dE} [f(E_n)] = f'(E_n) = 0 \hspace{1cm} n = 2, 3 \hspace{1cm} (3.4.5)$$
Equations (3.4.3), (3.4.4), (3.4.5) are equivalent to the necessary conditions of section 2.4.

Equation (3.4.4) represents four equations and equation (3.4.5) represents two equations.

Six equations are now available in terms of the six unknowns, \( \lambda, E_2, \) and \( E_3 \). However, in order to solve the unknowns, it is necessary to solve six second order transcendental equations. It is the nature of the equations described above that motivates a numerical approach rather than an analytical solution.

The numerical procedure to be used in the analysis is a Newton-Raphson iteration. The use of this method trades the requirement of solving the equations analytically to the requirement of guessing the six unknowns in the region of a solution such that the iteration will converge.

3.5 Development of the Numerical Procedure

In terms of the partition matrices equation (2.5.10) may be written

\[
B = \begin{pmatrix} N \\ T \end{pmatrix}
\]  

(3.5.1)

It is necessary to determine the first derivative of \( B \) with respect to \( E \) as a result of equation (3.4.5). In addition, the Newton-Raphson iteration will require the second derivative of \( f \) with respect to \( E \) in order to iterate the solutions of equation (3.4.5).
Since the computation of the state transition matrix is in a rotating coordinate system, the derivatives of the partition matrices with respect to a variable in a non-rotating system will have an additional term due to the relative motion of the two coordinate systems.

The derivatives of $M$, $N$, $S$, and $T$ as taken from Stern are:

\[
\frac{d}{dE_i} (N) = \frac{1 - e \cos E_i}{n} \left[ -M + N W \right] \quad (3.5.2)
\]

\[
\frac{d}{dE_i} (T) = \frac{1 - e \cos E_i}{n} \left[ -S + T W \right] \quad (3.5.3)
\]

\[
\frac{d}{dE_i} (S) = \frac{1 - e \cos E_i}{n} \left[ -T G + S W \right] \quad (3.5.4)
\]

\[
\frac{d}{dE_i} (M) = \frac{1 - e \cos E_i}{n} \left[ -N G + M W \right] \quad (3.5.5)
\]

The matrix $W$ is a skew symmetric (i.e. $W^T = -W$) rotation matrix which relates the relative motion of the rotating coordinate system. Knowledge of the elements of $W$ is not essential to the analysis, since the derivatives of $f$ will be independent of $W$. In addition, some other useful properties of the matrices are

\[
ST^T = T S^T 
\]

\[
MN^T = N M^T 
\]

\[
MT^T - NS^T = I = TM^T - SN^T 
\]
Before the final equations necessary for the iteration are presented one more simplifying observation may be made.

Notice that matrix $N$ has a multiplier $\frac{2}{n}$ which remains constant for any given reference trajectory. However, if the parameter were included in the constants $\lambda_1$ and $\lambda_2$, it is possible to obtain a new set of constants which allow us to solve the problem independent of the knowledge of the mean angular motion. Let it be inherently assumed, now, that $\frac{2}{n}$ is contained in $\lambda_1$ and $\lambda_2$ and that $B$ may be written

$$B = \begin{pmatrix} N^* \\ T \end{pmatrix}$$

(3.5.9)

where

$$N^* = \frac{n}{2} N$$

(3.5.10)

With this simplification in mind the equations for $f$, $f'$, and $f''$ are
\[ f(E_i) = \lambda^T \left[ \begin{array}{cc} N^* & N^T \\ T^* & T^T \end{array} \right] \lambda \]  \hspace{1cm} (3.5.11)

\[ f'(E_i) = - (1 - e \cos E_i) \lambda^T \left[ \frac{1}{2} \left( \begin{array}{cc} O & I \\ I & O \end{array} \right) + \left( \begin{array}{cc} M^* & 4 N^* \\ 4 N^T & 4 T^* \end{array} \right) \right] \lambda \]  \hspace{1cm} (3.5.12)

\[ f''(E_i) = \frac{e \sin E_i}{1 - e \cos E_i} f'(E_i) + 2 (1 - e \cos E_i)^2 \lambda^T \left[ \begin{array}{cc} M G N^* + \frac{1}{4} M M^T & M G T^* + M S^* \\ T G N^* + S M^T & T G T^* + 4 S S^* \end{array} \right] \lambda \]  \hspace{1cm} (3.5.13)

\[ \tilde{S} = \frac{1}{2n} S \]  \hspace{1cm} (3.5.14)

\[ \tilde{G} = \frac{1}{n^2} G \]  \hspace{1cm} (3.5.15)
Although the equations are very complicated, some simple analytical results can be obtained when \( E_i = E_4 \). At this point, the matrices \( N \) and \( S \) are zero matrices, and the matrices \( M \) and \( T \) are identity matrices. Equation (3.5.11) gives

\[
\left( E_4 \right) = \lambda_3^2 + \lambda_4^2 \equiv 1
\]  

(3.5.16)

which is unity by the necessary conditions applied to the end time.

By equation (3.4.3) it is apparent that in the neighborhood of \( E_4 \), the slope of \( f \) must be greater than or equal to zero. Evaluating equation (3.5.12) at \( E_4 \) yields a necessary condition associated with the \( \lambda \)'s.

\[
\lambda_1 \lambda_3 + \lambda_2 \lambda_4 \leq 0
\]  

(3.5.17)

The foundation has now been set for developing the iteration equations. Let

\[
\begin{align*}
\vec{f} &= \begin{bmatrix} f(E_1) \\ f(E_2) \\ f(E_3) \\ f(E_4) \end{bmatrix}, \\
\vec{h} &= \begin{bmatrix} \lambda \\ E_1 \\ E_2 \\ E_3 \end{bmatrix}, \\
\vec{S} &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}
\end{align*}
\]  

(3.5.18a)
The Newton-Raphson iteration equation is then given as

\[
\begin{bmatrix}
\frac{\partial b}{\partial \lambda} & \frac{\partial b}{\partial E_2} & \frac{\partial b}{\partial E_3} \\
\end{bmatrix}
\]

(3.5.18c)

\[
\dot{\lambda}_n = \dot{\lambda}_{n-1} - D_{n-1}^{-1} (b_{n-1} - s)
\]

(3.5.19)

The basis of determining a solution to the four impulse problem now rests on determining six initial values \((\lambda, E_1, E_2)\) in the region of a solution.
CHAPTER IV

RESULTS FOR A SPECIAL CLASS OF TRANSFERS

4.1 Introduction

There has been no guarantee as yet that four impulse solutions exist. The necessary conditions have been found and then, based on the assumption that a solution exists, an iterative method was developed. It is the purpose of this chapter to show that at least for one class of transfers, optimal four impulse solutions exist.

Because of the infinite number of transfers that may be considered, attention was focused to a class of transfers in which it was assumed that the first impulse is applied at perigee (measured in the chosen reference path.) Let the control transfer angle be defined as the angle in eccentric anomaly measured from perigee through which the vehicle must travel to reach the desired terminal state.
4.2 Generation of Solutions

With the class of transfers defined as those in which the transfer angle starts at \( E_1 = 0.0^\circ \) (assume that all angles are in degrees eccentric anomaly) and ends at a designated \( E_4 \) depending upon the specified time of the transfer, it is now necessary to determine whether optimal four impulse solutions exist.

Our problem now is to guess a set of initial \( \lambda \)'s and \( E_2 \) and \( E_3 \) for a given \( E_4 \). It would be ideal if a point may be found from which solutions may be propagated by taking step-wise increments in \( E_4 \) and computing the new solution at this point. Through experience gained in determining the solutions for the case of transfers between two nearby circles a very useful analytic result was obtained.

Consider a transfer where \( E_1 = 0.0^\circ \) and \( E_4 = 540^\circ \) and assume that \( E_2 = 180^\circ \) and \( E_3 = 360^\circ \). Assume \( \lambda \) of the form

\[
\lambda = \begin{pmatrix} \alpha \\ 0 \\ 0 \\ -1 \end{pmatrix}
\]

(4.2.1)

Evaluating equation (3.4.1) at \( E_1 = 0.0^\circ \) and solving for \( \alpha \), the only acceptable solution is

\[
\alpha = -\frac{(1-e)}{(1+e)(1-e^2)^{1/2}}
\]

(4.2.2)
and $\lambda_v$ becomes

$$\lambda_v = \left( -\frac{(1-e\cos E_i)^{1/2}}{2(1+e\cos E_i)} \sin E_i \right) - \frac{(1-e^2)^{1/2}}{(1-e^2\cos^2 E_i)^{1/2}} \cos E_i.$$  
(4.2.3)

It is apparent that this solution satisfies the original assumptions and the necessary conditions of section 2.4.

The value of the above analysis is immediately apparent. A final time, $E_4$, has been found which has similar properties for all eccentricities less than one. For a given eccentricity this point may then be used as a starting point for propagating a field of solutions which start at $E_1 = 0.0^\circ$ and terminate at a given $E_4$.

The results of carrying out this analysis is presented graphically in appendices B, C, and D. The computation was done on a CDC 3600 (equivalent to an IBM 7094) with the plotting done by an SC-4020 (a cathode-ray tube) plotter. Appendix B contains plots of the $\lambda$ components, Appendix C contains plots of the time of application of the second and third impulses, and Appendix D contains plots of the directions of application of the impulses.
Plots are given for eccentricity, e, of 0.0 to 0.7. Above e = 0.7, numerical difficulties were encountered which caused computation time to become excessive and hence these results are not presented. This problem will be discussed further in section 4.3.

4.3 Discussion of Results

This section will discuss the results of the plots found in appendices B, C, and D. The plots have been set up in such a manner that their use is relatively simple.

Given a reference orbit and the final time of the transfer, the eccentricity and $E_4$ are known. It is necessary now to go to the plot with the desired eccentricity, locate $E_4$ on the abscissa and all the necessary information will be found on the ordinate.

Lambda Components Plots

Plots of the lambda components are found in appendix B. The numbers on the lines correspond to that component of $\lambda$ (e.g. line 1 corresponds to $\lambda_1$, etc.)

As was pointed out in section 4.2 numerical difficulties were encountered above $e = 0.7$. This difficulty was due primarily to the sensitivity of the solution to the initial inputs. Notice on the plot $e = 0.7$, $\lambda_1$ and $\lambda_2$ have regions where the slope becomes large. In this region, small steps in $E_4$, give large steps in these $\lambda$'s. For the computation of $e = 0.7$
steps of $0.1^\circ$ were taken and the computation time became excessive. In order to compute solutions for larger eccentricities, the step size would have to be decreased even further. It was felt that the information to be gained at these higher eccentricities didn't warrant the use of a great deal of computer time. Therefore, the analysis was terminated at $e = 0.7$.

Knowledge of the solution at $E_4 = 540^\circ$ did not indicate the direction that steps in $E_4$ should be made. Therefore, $E_4$ was propagated in both directions. However, if it is assumed that $\lambda_2$ and $\lambda_3$ are continuous above $540^\circ$, their slopes would indicate that they will enter the negative region. Clearly this would violate the condition at the terminal time given by equation (3.5.17). Therefore, for this family of solutions, $E_4 = 540^\circ$ is the maximum transfer angle for an optimal four impulse transfer.

The boundary conditions for the smallest four impulse transfer have different properties and will be discussed under the section discussing the times of the second and third impulses.

Edelbaum\textsuperscript{13} has shown that there are other families of solutions beyond $540^\circ$ in the case of a transfer between two nearly circular orbits. However, an attempt to obtain these solutions through the iteration was unsuccessful. It may be pointed out that work currently being carried out independently by
Mr. J. Prussing of M.I.T. has verified the validity of the solution for \( e = 0 \), the circular case.

A final statement may be made about the \( \lambda \) vector. Our analysis has been concerned primarily with the magnitude of \( \lambda v \). Therefore, a positive \( \lambda \) or a negative \( \lambda \) are equally valid solutions. The only difference is that the directions of thrust applications of the positive sign will be in the opposite direction of that of the negative sign. The positive sign is chosen when it is desired to move into a larger orbit and the negative sign is chosen for moving into a smaller orbit.

**Time of Second and Third Impulse Plots**

It has been assumed that the first impulse is applied at \( E_1 = 0.0^\circ \). Therefore, by locating \( E_4 \) on the abscissa, the points of applying the second and third impulses may be located on the ordinate.

The case of \( e = 0 \) inhibits several interesting properties. The first is that the times of the second and third impulses are symmetric about the first and fourth impulses, respectively. This is

\[
E_2 - E_1 = E_4 - E_3
\]  

(4.3.1)

A second property which may be considered as resulting from the first is that as the final time is decreased the four impulse transfer becomes a two impulse

transfer.

One of the most surprising features is that four impulse transfers exist below a transfer angle of $180^\circ$ for the circular case.

Another feature which may warrant further investigation is the almost linear nature of the plots. This may suggest an empirical guidance scheme for transferring between two nearby circular orbits.

Transfers of eccentricity greater than zero exhibit different properties in the region of small transfer angles. Rather than approaching a two impulse transfer, the solution approaches a three impulse transfer where the third and fourth impulse coincide. Also with increasing eccentricity the plots lose their linear nature. Therefore, a linear guidance scheme would not be valid for large eccentricities.

**Direction of Impulse Application Plots**

Before making any general comments about these plots it is necessary to point out again that these thrust directions are measured in the rotating coordinate system. Once the central transfer angle has been located on the abscissa, for a given eccentricity, the desired thrust direction of the $n^{th}$ impulse where $n$ is the number of the line, will be given on the ordinate. In order to determine the thrust direction with respect to the non-rotating system,
it is only necessary to add the eccentric anomaly at the time of application to the thrust direction obtained from the plots.
5.1 Conclusions

For a time fixed, two dimensional transfer, it has been shown that minimum fuel four impulse transfers exist. The assumptions of the problem were that the transfer may be linearized about a nominal or reference path and that the first and fourth impulses are applied at known positions and velocities.

The solution to the problem depended upon guessing the components of the four dimensional costate vector evaluated at the final time such that the necessary conditions of the optimization problem were satisfied.

Based on the assumption that a solution existed, a Newton-Raphson iteration program was developed. This program not only iterated the four components of the costate but also the times of the second and third impulses.

Solutions for the case in which the first impulse was applied at perigee were computed for eccentricities from 0.0 to 0.7. The family of solutions obtained was valid for
transfers whose central transfer angle was less than $540^\circ$ eccentric anomaly. An analytical result obtained for one of these solutions allowed the entire family to be generated by a step-wise process. It is possible now to use this same process and this family of solutions to compute other solutions with different properties.

5.2 Recommendations for Further Study

The following extensions to this thesis are suggested for further study.

1. Extend the analysis to eccentricities greater than 0.7.

2. Extend the transfer angle to greater than $540^\circ$ eccentric anomaly to verify the result of Edelbaum.

3. Determine solutions for other classes of transfers.

4. Extend the analysis to three dimensions in which case six impulses become the limiting number.

5. Determine solutions (if they exist) to the time open case.

6. Determine solutions for optimum three impulse transfers and compare these with the two and four impulse case.

As is evident, a great deal of work lies ahead before the absolute minimum-fuel impulsive transfer problem is solved.
APPENDIX A

PARTITIONED MATRICES OF THE STATE TRANSITION MATRIX
\[
M_{jl} = \left\{ \begin{array}{ccc}
\frac{1 + e \cos E_l}{(1 - e^2 \cos^2 E_l)^{1/2}} (1 - e \cos E_j) & \frac{2(1 - e^2)^{1/2}(1 - e \cos E_j)}{(1 - e \cos E_j)^2} \sin E_M (\cos E_M - e \cos E_P) \\
+ \frac{2 \sin E_M (1 - e \cos E_j)}{(1 - e \cos E_j)^2} \left[(1 - e^2)^{1/2} \sin E_M - \sin E_M (\cos E_M - e \cos E_P)\right] \\
- (1 - e \cos E_j) e \sin E_M \cos E_M \\
\end{array} \right.
\]

\[
\left( \begin{array}{ccc}
1 + e \cos E_l & \frac{2(1 - e^2)^{1/2}(1 - e \cos E_j)}{(1 - e \cos E_j)^2} \sin E_M (\cos E_M - e \cos E_P) \\
+ \frac{2 \sin E_M (1 - e \cos E_j)}{(1 - e \cos E_j)^2} \left[(1 - e^2)^{1/2} \sin E_M - \sin E_M (\cos E_M - e \cos E_P)\right] \\
- (1 - e \cos E_j) e \sin E_M \cos E_M \\
\end{array} \right).
\]
\[ G_i = \frac{n^2}{(1-e \cos E_i)^3} \left\{ \begin{array}{ccc} 1-e^2 & (1-e^2) \frac{1}{2} e \sin E_i & 0 \\ 3 \frac{(1-e^2) \frac{1}{2} e \sin E_i}{1-e^2 \cos^2 E_i} & e^2 \sin^2 E_i & 0 \\ 0 & 0 & 0 \end{array} \right\} -I \]
APPENDIX B

LAMBDA COMPONENTS
ECCENTRICITY = 0.0

TRANSFER ANGLE IN DEGREES ECCENTRIC ANOMALY

LAMBDA COMPONENTS
ECCENTRICITY = 0.05

TRANSFER ANGLE IN DEGREES ECCENTRIC ANOMALY

LAMBDA COMPONENTS

COMPONENTS
ECCENTRICITY = 0.3
ECCENTRICITY = 0.4

TRANSFER ANGLE IN DEGREES
ECCENTRIC ANOMALY
ECCENTRICITY = 0.7

TRANSFER ANGLE IN DEGREES ECCENTRIC ANOMALY

LAMBDA COMPONENTS
APPENDIX C

TIME OF SECOND AND THIRD IMPULSES
ECCENTRICITY = 0.0
ECCENTRICITY = 0.05

![Graph showing the relationship between transfer angle in degrees and eccentric anomaly, with time of second and third impulses on the y-axis and transfer angle on the x-axis.](image-url)
ECCENTRICITY = 0.1

TIME OF SECOND AND THIRD IMPULSES

TRANSFER ANGLE IN DEGREES ECCENTRIC ANOMALY
ECCENTRICITY = 0.2

TRANSFER ANGLE IN DEGREES ECCENTRIC ANOMALY
ECCENTRICITY = 0.4

TRANSFER ANGLE IN DEGREES ECCENTRIC ANOMALY

TIME OF SECOND AND THIRD IMPULSES
ECCENTRICITY = 0.5

TIME OF SECOND AND THIRD IMPULSES

TRANSFER ANGLE IN DEGREES ECCENTRIC ANOMALY
ECCENTRICITY = 0.6

TIME OF SECOND AND THIRD IMPULSES (CIRCULAR ECCENTRIC ANOMALY)

TRANSFER ANGLE IN DEGREES ECCENTRIC ANOMALY

Graph showing the relationship between transfer angle in degrees eccentric anomaly and time of second and third impulses for an eccentricity of 0.6.
ECCENTRICITY = 0.7

TIME OF SECOND AND THIRD IMPULSES

TRANSFER ANGLE IN DEGREES ECCENTRIC ANOMALY
APPENDIX D

DIRECTION OF IMPULSE APPLICATION
ECCENTRICITY = 0.0

DIRECTION OF IMPULSE APPLICATION

TRANSFER ANGLE IN DEGREES ECCENTRIC ANOMALY
ECCENTRICITY = 0.05

[Diagram showing the relationship between direction of impulse application and transfer angle in degrees eccentric anomaly.]
ECCENTRICITY = 0.1

TRANSFER ANGLE IN DEGREES ECCENTRIC ANOMALY

DIRECTION OF IMPULSE APPLICATION

1
2
3
4
ECCENTRICITY = 0.2
ECCENTRICITY = 0.3

TRANSFER ANGLE IN DEGREES ECCENTRIC ANOMALY

DIRECTION OF IMPULSE APPLICATION
ECCENTRICITY = 0.4
ECCENTRICITY = 0.5

[Graph showing the relationship between transfer angle in degrees and eccentric anomaly, with four curves labeled 1, 2, 3, and 4.]
REFERENCES


