On Moduli Stacks of Finite Group Schemes

by

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Abstract

Let $\mathcal{X}_n$ be the moduli stack of commutative finite locally free group schemes of order p^n annihilated by p. In this thesis I determined the local singularities of $\mathcal{X}_n$ over a perfect field of characteristic p. Moreover, given n, there are finitely isomorphic classes of such local singularities and up to a power series ring, they are isomorphic to the complete local ring of $\mathcal{X}_r$ at α_p^r for some r. We also showed that $\mathcal{X}_n$ is reduced, Cohen-Macaulay, and flat over $\mathbf{Z}_p$.

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Chapter 1

Introduction

The purpose of my thesis is to the study of deformations of finite commutative locally free group schemes that is killed by p. Let $\mathcal{X}_n$ be the stack of groupoids whose category of sections over a scheme T is the category of such group schemes of order p^n. We know that $\mathcal{X}_n$ is an algebraic stack in the sense of Artin [1]. We would like to ask the following natural questions : where are the singularities of this stack? How do the local rings look like? Is this stack flat?

To answer these questions we need to study the deformations of finite group schemes. Our approach is to first embed a group scheme into a p-divisible group. Then we use Dieudonné cristalline theory to compute the deformations of the p-divisible groups. The properties of deformations of finite group schemes can be derived then. This technique has been applied in De Jong's article [6] on abelian schems with $\Gamma_0(p)$-level structures.

In the first three chapters we focus on determining the deformation rings of group schemes. The main result (see Theorem 13, Corollary 15) is

THEOREM A. *Let k be perfect field of characteristic p. Let N_0 be a finite commutative locally free group scheme killed by p. Then the deformation ring of N_0 is isomorphic to the completion of*

$$W(k)[\underline{\mathbf{X}}, \underline{\mathbf{Y}}]/(\underline{\mathbf{XY}} - pI_n, \underline{\mathbf{YX}} - pI_n),$$

at a certain point P_N, where $W(k)$ is the Witt ring of k and $\underline{\mathbf{X}} = (X_{ij})$, $\underline{\mathbf{Y}} = (Y_{ij})$ are written as $n \times n$ matrices. Moreover, this ring is smooth if and only if N_0 is a truncated Barsotti-Tate group scheme.

Furthermore, we can determine at which point P_N we should complete the localization. The point P_N basically corresponds to the matrix forms of the Frobenius operator $\mathbf{F}$ and the Verschiebung operator $\mathbf{V}$ on the Dieudonné module of N_0.

It is well known that in many cases [4] the worst deformations of group schemes occur at the group scheme α_p^n. We discovered that in fact all the deformation ring over α_p^r include all types of deformations that ever occur. To be precise, we have the following result (see Theorem 17)

THEOREM B. *Given n, there are only finitely many number of isomorphic classes of deformation rings of our group schemes. In fact the deformation ring of a group scheme N_0 is isomorphic to, up to a power series ring, the deformation ring of the group scheme α_p^r, Here $r = n - r_F - r_V$ where r_F and r_V are the ranks of* $\mathbf{F}$ *and* $\mathbf{V}$ *of the Dieudonné module of N_0.*

The last part of my thesis is on more abstract properties of the moduli stack $\mathcal{X}_n$. We will prove the Cohen-Maucaulayness of $\mathcal{X}_n$ based on our previous explicit computation of the complete local rings. Our approach uses the theory of Hodge algebras extensively following the ideas in [8] and [4]. we find that the complete local rings of this moduli stack are Hodge algebras on certain posets which display interesting combinatoric properties. As a consequence, we have the following results for the stack $\mathcal{X}_n$

THEOREM C. *Let k be a perfect field of characteristic p*

1. *(Theorem 22) The local rings of $\mathcal{X}_n$ over k are reduced, Cohen-Macaulay, and have normal irreducible components.*

2. *(Proposition 24) $\mathcal{X}_n$ is not Gorenstein when $n > 1$.*

3. *(Theorem 25) $\mathcal{X}_n$ is flat over $\mathbf{Z}_p$.*

Chapter 2

Deformations of finite group schemes and p-divisible groups

2.1 Deformation theory

Fix a prime number p and a natural number n. In this article a group scheme is a finite locally free commutative group scheme whose order is a power of p.

Let $\mathcal{X}_n$ be the stack of groupoids over *Spec* $\mathbf{Z}$ whose category of sections over a scheme S is the category of group schemes of order p^n over S that are killed by p, with isomorphisms as morphisms.

LEMMA 1 *$\mathcal{X}_n$ is an algebraic stack in the sense of Artin [1].*

Proof. Consider the functor

$$\begin{array}{rcl} (Sch/\mathbf{Z}) & \longrightarrow & (Sets) \\ T & \mapsto & (\Gamma, \iota : \Gamma \approx O_T^{p^n}) \end{array}$$

where Γ is a sheaf of Hopf algebras over T coming from a group scheme of order p^n killed by p over T. This functor is readily seen to be representable by an affine scheme of finite type over *Spec* $\mathbf{Z}$. Let us call this scheme X_n. The algebraic group $\mathbf{GL}_{p^n}(\mathbf{Z})$ acts on X_n by changing the choice of basis. We leave it to the reader to show that $\mathcal{X}_n = [X_n/\mathbf{GL}_{p^n}(\mathbf{Z})]$. ∎

For certain stacks we can speak of their local rings. These rings, which we will define precisely next, come from deformation theory over local artinian algebras. This framework was developped in Schlessinger's thesis [19].

Let k be a perfect field of characteristic p and $W(k)$ the Witt ring of k. Let $\mathcal{C}$ denote the category of local artinian $W(k)$-algebras with residue field k. For any stack of groupoids $\mathcal{S}$ with a suitable deformation theory (in particular for any algebraic stack in the sense of Artin). We define the complete local rings of the stack as follows. Let x : *Spec* $k \longrightarrow \mathcal{S}$ be a point. We denote by $\mathrm{Def}_k(x)$ the functor

$$\mathrm{Def}_k(x) : \mathcal{C} \longrightarrow Set$$

given by

$$\mathrm{Def}_k(x)(R) = \{y : Spec\ R \longrightarrow \mathcal{S}, \varphi(y) = x\}/\sim$$

Here φ is the inclusion $Spec\ k \longrightarrow Spec\ R$. The equivalence relation $\sim$ is defined by $y \sim y'$ if and only if there is an morphism ψ in the category of sections $\mathcal{S}(R)$ between y and y' such that ψ induces the identity morphism on x.

With "suitable deformation theory" we mean that we want $\mathrm{Def}_k(x)$ to have a pro-representation hull (or simply a hull) [19, Definition 2.7]. We denote by $\hat{O}_{\mathcal{S},x}$ a complete local ring representing this hull. According to the theory of Artin [1], we know that $\mathrm{Def}_k(x)$ has a hull when $\mathcal{S}$ is an Artin stack. For example, in the case $\mathcal{S} = \mathcal{X}_n$, $\mathrm{Def}_k(x)$ may not be pro-representable (e.g. when x corresponds to the group scheme α_p) but does have a hull by Lemma 1.

2.2 Embedding into p-divisible groups

We start with two lemmas which would enable us to embed a group scheme over k into p-divisible groups in a certain way.

LEMMA 2 *For any group scheme N_0 of order p^n over k, there exist p-divisible groups G_0, H_0 over k and an exact sequence of $fppf$-sheaves*

$$0 \longrightarrow N \longrightarrow G_0 \xrightarrow{\varphi_0} H_0 \longrightarrow 0.$$

Moreover, if N_0 is killed by p, then there exists a unique morphism ψ_0 of p-divisible groups : $H_0 \longrightarrow G_0$ such that $\varphi_0\psi_0 = p_{G_0}$ and $\psi_0\varphi_0 = p_{H_0}$.

Proof. As a result of Raynaud [2, Theorem 3.1.1], we can embed N_0 in some p-divisible group G_0. It is a fact [2, Lemma 3.3.12] that the fppf-sheaf cokernel $H_0 = G_0/N$ is a p-divisible group. The condition N_0 is killed by p will assure the existence of the morphism ψ_0. ∎

This lemma also holds for more general rings. As an unpublished result, Oort proved the case when k is a local artinian algebra with perfect residue field of characteristic p [18], which we will use later.

Over a perfect field k of characteristic p, we are able to construct G_0 and H_0 explicitly using Dieudonné theory. We will see this later in the proof of Lemma 3.

2.3 p-factorizations

The lemmas in the previous section enables us to embed a group scheme into p-divisible groups in a certain way.

DEFINITION 1 *Let S be a scheme. We call a sequence of morphisms of p-divisible groups*

$$G \xrightarrow{\varphi} H \xrightarrow{\psi} G$$

a p-factorization of degree p^n over S if

1. *G and H are p-divisible groups of height $2n$ over S.*

2. *$\psi\varphi = p_G$ and $\varphi\psi = p_H$.*

3. *$\ker(\varphi)$ and $\ker(\psi)$ are both group schemes of order p^n.*

Let N be a group scheme over S that is killed by p. In the above, we call $(G \xrightarrow{\varphi} H \xrightarrow{\psi} G)$ a p-factorization for N if $\ker(\varphi) \tilde{\longrightarrow} N$.

LEMMA 3 *Let N_0 be a group scheme of order p^n over k, annihilated by p. Then there exists a p-factorization of degree p^n for N_0.*

Proof. We postpone the proof to Chapter 3 after the summary of Dieudonné theory. ∎

DEFINITION 2 *We denote by $\mathcal{A}_n$ the stack of groupoids whose category of sections over a scheme T is the category whose objects are p-factorizations of degree p^n over T. The morphisms are isomorphisms of sequences of p-divisible groups compatible with φ's and ψ's.*

Note that $\mathcal{A}_n$ is not an algebraic stack of finite type. The reason is that for any element $I = (G_0 \to H_0 \to G_0)$ in $\mathcal{A}_n(\mathbf{F}_p)$, the functor $\underline{Aut}(I)$ is representable by a group scheme $Aut(I)$ if $\mathcal{A}_n$ is an algebraic stack of finite type. As a consequene, $Aut(I)(\bar{\mathbf{F}}_p)$ would be countable. However, we know that $\mathbf{Z}_p^*$ can be embedded in $Aut(I)(\bar{\mathbf{F}}_p)$ as the scalar multiplication in the corresponding chain on p-divisible groups which contradicts the uncountability of $Aut(I)(\bar{\mathbf{F}}_p)$. In our study we are not affected by this technical difficulty. Indeed, the deformation theory of p-divisible groups induces a suitable deformation theory for $\mathcal{A}_n$, which enables us to study its local behavior.

We can define a morphism of stacks

$$\gamma : \mathcal{A}_n \longrightarrow \mathcal{X}_n$$

as follows: for every scheme T we associate to the p-factorization

$$(G \xrightarrow{\varphi} H \xrightarrow{\psi} G)$$

the group scheme $\ker(\varphi)$. This is compatible with morphism and with pullback. To see it is compatible with pullbacks, we consider a group scheme N over a ring R which is killed by p. Let $(G \xrightarrow{\varphi} H \xrightarrow{\psi} G)$ be a p-factorization for N. Let $R' \longrightarrow R$ be any ring homomorphism and

$$(G' \xrightarrow{\varphi'} H' \xrightarrow{\psi'} G')$$

be a lift of $(G \xrightarrow{\varphi} H \xrightarrow{\psi} G)$ over R' as sequence of p-divisible groups. Then $N' = \ker(G' \xrightarrow{\varphi'} H')$ is a group scheme killed by p. N' is a lift of N due to right exactness of the tensor product.

Let k be a perfect field and $\mathcal{C}$ be as defined before. Let $y : Spec\ k \longrightarrow \mathcal{A}_n$ be a point that corresponds to $(G_0 \xrightarrow{\varphi_0} H_0 \xrightarrow{\psi_0} G_0)$. Then the point $x = \gamma(y)$ corresponds to the group scheme $N_0 = \ker(\varphi_0)$. The morphism γ induces a transformation $\gamma_y^{\#}$ on the local deformation functors

$$\gamma_y^{\#} : \mathrm{Def}_k(G_0 \xrightarrow{\varphi_0} H_0 \xrightarrow{\psi_0} G_0) \longrightarrow \mathrm{Def}_k(N_0).$$

PROPOSITION 4 *The functor $\gamma_y^{\#}$ is formally smooth.*

Proof. This was proved by Oort in an unpublished article [18] but we will give a proof here using a result in the literature. Let $R' \longrightarrow R$ be a small surjection in $\mathcal{C}$. Let $(G \xrightarrow{\varphi} H \xrightarrow{\psi} G)$ be a p-factorization over R and $i : N = \ker(\varphi) \hookrightarrow G$ be the injection. Let N' be a group scheme over R' killed by p such that $N' \otimes R \xrightarrow{\sim} N$. We have to show that we can find a p-factorization

$$(G' \xrightarrow{\varphi'} H' \xrightarrow{\psi'} G')$$

for N' over R' that lifts $(G \xrightarrow{\varphi'} H \xrightarrow{\psi} G)$. The existence of G' and $i' : N' \hookrightarrow G'$ lifting i is actually a result by Grothendieck in Illusie's monograph [16, Theorem 4.11]. We complete the proof by taking $H' = \mathrm{coker}(i')$. ∎

Chapter 3

System of modules associated to deformation of p-divisible groups

Let k be a perfect field of characteristic p. In this chapter we use Dieudonné and crystalline theory to analyze the local behaviour of the stack $\mathcal{A}_n$ around a k-valued point.

3.1 Review of Dieudonné crystalline theory

In this chapter we will summarize the results we have from Dieudonné and crystalline theory.

Dieudonné theory [12] gives us a contravariant functor :

$$\mathbf{M} : \left\{ \begin{array}{l} \text{extensions of } p\text{-divisible groups} \\ \text{by group schemes over } k \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{left } D_k\text{-modules} \\ \text{ of finite type over W(k)} \end{array} \right\}$$
$$G \longmapsto \mathbf{M}(G)$$

which is an anti-equivalence of categories. Here $W(k)$ the Witt ring over k and $D_k = W(k)[F, V]$ is the Dieudonné ring as usual. Moreover, putting $M = \mathbf{M}(G)$, we have

1. M is a free k-module when G is a finite group scheme killed by p. Moreover,

$$p^{\text{rank } M} = \text{order}(G).$$

2. M is a free $W(k)$-module when G is a p-divisible group, with

$$\begin{aligned} \dim_k(M/pM) &= \text{height}(G) \\ \dim_k(M/FM) &= \dim(G). \end{aligned}$$

For any scheme S where p is locally nilpotent, crystalline theory (see [17], [2]) gives a contravariant functor

$$\mathbf{D} : p\text{-divisible groups over } S \longrightarrow \text{locally free crystals on } NCris(S).$$

Moreover, for any p-divisible group G over S, we have a locally free direct summand of O_S-modules

$$\omega_{G/S} \subset \mathbf{D}(G)_S.$$

The following theorem by Grothendieck [14, p. 116-118] gives an equivalence between deformations of p-divisible group over a ring with nilpotent divided power structures and filtered Dieudonné modules.

THEOREM 5 *Let $S \hookrightarrow S'$ be a closed immersion defined by an ideal with locally divided nilpotent powers. Then there is an anti-equivalence of the categories:*

$$\begin{array}{ccc} p\text{-divisible groups over } S' & \longrightarrow & \left\{ \begin{array}{l} \text{pairs } (G, Fil^1) \\ \text{where } G \text{ is a } p\text{-divisible group over } S \\ \text{and } Fil^1 \subset \mathbf{D}(G)_{S'} \text{ is locally free direct} \\ \text{summands that lifts } w_G \subset \mathbf{D}(G)_S \end{array} \right\} \\ G' & \longmapsto & (G' \times_{S'} S, w_{G'}) \end{array}$$

Given a p-divisible group G over S, this theorem gives an equivalence between deformations of p-divisible groups over $S' \hookrightarrow S$ and locally free summand of $\mathbf{D}(G)_{S'}$.

The connection between Dieudonné theory and crystalline theory over the perfect field k is as follows [2, Section 4.2]. Let G be a p-divisible group over $S_0 = Spec\ k$. Put

$$D(G) = \varprojlim \Gamma(\mathbf{D}(G), S_0 \hookrightarrow S_n),$$

where $S_n = Spec\ W_n(k)$, the n^{th} Witt vector ring of k. Then $D(G)$ is a $W(k)$-module and there is a canonical isomorphism [2, Theorem 4.2.14]

$$\mathbf{M}^{\sigma}(G) = D(G).$$

For any local artinian algebra $R \in \mathcal{C}$, there is natually a $W(k)$-algebra structure on R. When the kernel I of $R \twoheadrightarrow k$ has a given nilpotent divided power structure, the value of the crystal $\mathbf{D}(G)$ on $Spec\ R$ can be extracted from $D(G)$ by a canonical isomorphism

$$\mathbf{D}(G)_R = D(G) \otimes_{W(k)} R.$$

3.2 Standardization of p-factorizations in modules

Now we get back to deformations of group schemes. For any ring R, we can define the category of *p-factorization in modules.* A p-factorization in modules is a system of morphisms of free R-modules,

$$(A \xrightarrow{\varphi} B \xrightarrow{\psi} A)$$

such that $\varphi\psi = p_B$, $\psi\varphi = p_A$. Morphisms are isomorphisms of systems as usual.

Now consider a ring $R \in \mathcal{C}$ and N a group scheme over R with order p^n over k, killed by p. Let $(G \xrightarrow{\varphi} H \xrightarrow{\psi} G)$ be a p-factorization for N. Because G and H are both defined over R, $\mathbf{D}(G)_R$ and $\mathbf{D}(H)_R$ are both free R-modules and the p-factorization $(G \xrightarrow{\varphi} H \xrightarrow{\psi} G)$ induces a p-factorization in modules

$$\mathbf{D}(H)_R \xrightarrow{\mathbf{D}(\psi)_R} \mathbf{D}(G)_R \xrightarrow{\mathbf{D}(\varphi)_R} \mathbf{D}(H)_R.$$

Because $\varphi\psi = p$, $\psi\varphi = p$ when N is killed by p, we can standardize this p-factorization in modules as in the following theorem. It turns out that this standardization works quite well with lifting problems where there are nilpotent divided powers.

Denote by $St.$ the sequence of $\mathbf{Z}$-modules

$$St_1 \xrightarrow{\alpha} St_2 \xrightarrow{\beta} St_1$$

where $St_i = \mathbf{Z}^{2n}$, and

$$\alpha = \begin{pmatrix} I_n & 0 \\ 0 & pI_n \end{pmatrix}, \beta = \begin{pmatrix} pI_n & 0 \\ 0 & I_n \end{pmatrix}.$$

THEOREM 6 *Let* $(G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_1)$ *be a* p*-factorization of degree* p^n *for* N. *Let* R *be any ring where* p *is locally nilpotent. Let*

$$R' \twoheadrightarrow R$$

be a surjection such that p *is locally nilpotent in* R' *and the kernel* J *has a nilpotent divided power structure. Denote by* $\mathbf{D}(G.)_R$ *and* $\mathbf{D}(G.)_{R'}$ *the* p*-factorization in modules by taking the values of crystals on the corresponding rings. Then we have,*

1. *If* R *is local, then there exists an isomorphism of* p*-factorization of* R'*-modules*

$$\chi' : St. \otimes R' \xrightarrow{\sim} \mathbf{D}(G.)_{R'}$$

2. *Any isomorphism*

$$\chi : St. \otimes R \xrightarrow{\sim} \mathbf{D}(G.)_R$$

can be lifted to an isomorphism

$$\chi' : St. \otimes R' \xrightarrow{\sim} \mathbf{D}(G.)_{R'}.$$

Proof. When J has a nilpotent divided power, we can take values of $\mathbf{D}(G.)$ on R'. Moreover, we know that $\mathbf{D}(G_1)_{R'}$ and $\mathbf{D}(G_2)_{R'}$ are both locally free R'-modules of rank $2n$ [2,]. As R' is local if R is local, they are free R'-modules. Since N is killed by p, by [2, Proposition 4.3.1], $\operatorname{coker}(\mathbf{D}(\psi)_{R'} : \mathbf{D}(G_1)_{R'} \to \mathbf{D}(G_2)_{R'})$ and $\operatorname{coker}(\mathbf{D}(\varphi)_{R'} : \mathbf{D}(G_2)_{R'} \to \mathbf{D}(G_1)_{R})$ are both locally free R'/pR'-module of rank n, hence free. Apply the following linear algebra lemma and the proof is complete. ∎

REMARK 7 *In the case when we allow deformations by group schemes not killed by p, but by p^r for some $r > 1$, we will still have a p^r-factorization. However, we will not be able to standardize the p^r-factorization in this nice manner because the cokernels are not necessarily R'/pR'-moduels.*

LEMMA 8 *Let R be any ring such that p is nilpotent in R. Let*

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} A$$

be a p-factorization of R-modules such that $B/\varphi(A)$ and $A/\psi(B)$ are both free R/pR-modules of rank n. Then the sequence $(A \xrightarrow{\varphi} B \xrightarrow{\psi} A)$ is isomorphic to $St. \otimes R$. Moreover, if $J \subset R$ is any ideal contained in the Jacobson radical of R, then any isomorphism

$$(A/JA \xrightarrow{\bar{\varphi}} B/JB \xrightarrow{\bar{\psi}} A/JA) \tilde{\longrightarrow} St. \otimes_R R/J$$

can be lifted to an isomorphism

$$(A \xrightarrow{\varphi} B \xrightarrow{\psi} A) \tilde{\longrightarrow} St. \otimes R.$$

Proof. Pick a basis of $B/\varphi(A)$ and let $x_1, \dots, x_n \in A$ be a lift of it. Pick a basis of $A/\psi(B)$ and let $y_1, \dots, y_n \in B$ be a lift. Then $x_1, \dots, x_n, \psi y_1, \dots, \psi y_n$ is a basis for A and $\varphi x_1, \dots, \varphi x_n, y_1, \dots, y_n$ is a basis for B. An isomorphism is then obtained by this choice of bases.

For the lifting statement, let

$$\bar{A} = A/JA, \bar{B} = B/JB,$$

then that the sequence $(\bar{A} \xrightarrow{\bar{\varphi}} \bar{B} \xrightarrow{\bar{\psi}} A/JA)$ is also a p-factorization , hence by the argument above, we can find $\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_n$ which span $\bar{B}/\bar{\varphi}(\bar{A})$ and $\bar{A}/\bar{\psi}(\bar{B})$ respectively. Pick any lift $x_1, \dots, x_n \in A$ for $\bar{x}_1, \dots, \bar{x}_n$, and $y_1, \dots, y_n \in B$ for $\bar{y}_1, \dots, \bar{y}_n$, then by Nakayama's lemma, $x_1, \dots, x_n$ generate $B/\varphi(A)$ and $y_1, \dots, y_n$ span $A/\varphi(B)$. Given any isomorphism

$$\bar{\iota} : (A/JA \xrightarrow{\bar{\varphi}} B/JB \xrightarrow{\bar{\psi}} A/JA) \otimes_R R/J \tilde{\longrightarrow} St. \otimes_R R/J,$$

let $(\bar{e}_1, \dots, \bar{e}_{2n})$ be the image of $(\bar{x}_1, \dots, \bar{x}_n, \bar{\psi}\bar{y}_1, \dots, \bar{\psi}\bar{y}_n)$ under ι. Fix a lift $e_1, \dots, e_{2n}$ of $(\bar{e}_1, \dots, \bar{e}_{2n})$, then by setting

$$x_1 \mapsto e_1, \dots, x_n \mapsto e_n, \psi y_1 \mapsto \psi e_{n+1}, \dots \psi y_n \mapsto e_{2n},$$

we obtain an isomorphism $\iota_1 : St_1 \xrightarrow{\sim} A$. Similarly we get an another isomorphism $\iota_2 : St_2 \xrightarrow{\sim} B$ by lifting images of $\varphi x_1, \dots, \varphi x_n, y_1, \dots, y_n$. It is easy to verify they give an isomorphism as sequences of R-modules that lifts $\bar{\iota}$. ∎

3.3 The local rings

The standardization of the p-factorizations in modules leads us to introduce the following moduli stack which turns out to inherit the same singularities. As we will see later in this chapter, this new moduli stack behaves as an intermediate for the computation of the local rings.

DEFINITION 3 *Let $NSch_p$ be the category of schemes T such that p is locally nilpotent in O_T. We denote by $\mathcal{W}$ the stack of groupoids over the category $NSch_p$ whose category of sections over a scheme T is the category of the pairs*

$$(G \xrightarrow{\varphi} H \xrightarrow{\psi} G, \chi)$$

where $I = (G \xrightarrow{\varphi} H \xrightarrow{\psi} G)$ is a p-factorization of degree p^n over T and

$$\chi : St. \otimes O_T \xrightarrow{\sim} \mathbf{D}(I)_T$$

is an isomorphism .

Let π be the obvious forgetful functor $\mathcal{W} \to \mathcal{A}_n$. By Theorem 6, π is a formally smooth morphism of stacks over $NSch_p$. Combining this with Proposition 4, to study singularities of $\mathcal{X}_n$ at a point over a perfect field, we only need to study the singularities of the stack $\mathcal{W}$.

DEFINITION 4 *Let $\mathcal{M}$ over Spec $\mathbf{Z}$ be the scheme representing the functor $\underline{\mathcal{M}}$:*

$$\begin{array}{rcl} Sch & \longrightarrow & Sets \\ T & \longmapsto & \left\{ \begin{array}{l} \text{pairs } (V_1, V_2) \text{ such that} \\ V_i \subset St_i \otimes O_T \text{ locally free direct summand}, i = 1,2 \\ \text{and } \alpha(V_1) \subset V_2, \beta(V_2) \subset V_1. \end{array} \right\} \end{array}$$

Let $\underline{\mathcal{M}}^{r,s}$ be the subfunctor of $\underline{\mathcal{M}}$ in which $\mathrm{rank}(V_1) = r$, $\mathrm{rank}(V_2) = s$. $\underline{\mathcal{M}}^{r,s}$ is represented by a closed subscheme $\mathcal{M}^{r,s}$ of two Grassmanians and $\underline{\mathcal{M}}$ is represented by the disjoint union of all $\mathcal{M}^{r,s}$, $0 \le r, s \le n$.

For any $(G \xrightarrow{\varphi} H \xrightarrow{\psi} G) \in \mathcal{W}(T)$, we have

$$\begin{aligned} \omega_{G/T} &\subset \mathbf{D}(G)_T \\ \omega_{H/T} &\subset \mathbf{D}(H)_T \\ \mathbf{D}(\psi)(\omega_G) &\subset \omega_H \\ \mathbf{D}(\varphi)(\omega_H) &\subset \omega_G. \end{aligned}$$

By the functority of ω and $\mathbf{D}$, we have a transformation of functors which is also a morphism of stacks $f : \mathcal{W} \longrightarrow \mathcal{M}$ given by

$$\begin{aligned} f_T : \mathcal{W}(T) &\longrightarrow \mathcal{M}(T) \\ (T, (G \to H \to G), \chi) &\longmapsto (\chi^{-1} w_G, \chi^{-1} w_H) \end{aligned}$$

THEOREM 9 *f is formally smooth.*

Proof. Let R' be a ring and $I \in R'$ be an ideal such that $I^2 = 0$. Put $R = R/I$ and $S = Spec\ R$. We need to show that for every diagram

$$\begin{array}{ccc} x : S & \longrightarrow & \mathcal{W} \\ \downarrow & & \downarrow \\ y : S' & \longrightarrow & \mathcal{M} \end{array}$$

we can find a morphism $S' \longrightarrow W$ to make it commutative.

We can impose a divided power structure on I by letting all divided powers of order higher than 1 to be zero. Let the point x correspond to

$$(G_1 \longrightarrow G_2 \longrightarrow G_1, \chi : St. \otimes R \xrightarrow{\sim} \mathbf{D}(G.)_R)$$

and the point y correspond to

$$V_i \subset St_i \otimes R', i = 1, 2$$

such that $\alpha(V_1) \subset V_2$, $\beta(V_2) \subset V_1$. By the commutativity of the diagram, $(V. \subset St. \otimes R')$ lifts $(\chi^{-1} w_{G.} \subset \chi^{-1}\mathbf{D}(G.)_R)$.

Then apply Theorem 6, we can find an isomorphism

$$\chi' : St. \otimes R' \xrightarrow{\sim} \mathbf{D}(G.)_{R'}$$

which lifts $\chi : St. \otimes R \xrightarrow{\sim} \mathbf{D}(G.)_R$.

By Theorem 5, the sequence $(\chi'(V.) \subset \mathbf{D}(G.)_{R'})$ correspond to a sequence of p-divisible groups $(G.')$ which is a deformation of $(G.)$ over R. Then

$$(G_1' \longrightarrow G_2' \longrightarrow G_1', \chi')$$

defines a morphism $S' \longrightarrow \mathcal{W}$ as desired. ∎

We have now defined the stacks $\mathcal{X}_n$, $\mathcal{A}_n$, $\mathcal{W}$ and $\mathbf{M}$, and morphisms of stacks over

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbf{M} \\
\mathcal{A}_n \arrow[d, "\gamma"'] & \\
\mathcal{X}_n &
\end{tikzcd}$$

We have proved that over $NSch_p$, the morphisms f,π are formally smooth. We also know that γ is formally smooth at a point over a perfect field of characteristic p. Therefore, we can calculate the complete local rings of $\mathcal{X}_n$ at such points the using complete local rings of the scheme $\mathbf{M}$ at corresponding points. This strategy is has been used in De Jong's article on $\Gamma_0(p)$ structures of abelian varieties [6]

COROLLARY 10 *Let k be a perfect field.*

1. *Let $y : Spec\ k \longrightarrow \mathcal{A}_n$ be a point in $\mathcal{A}_n$ Then there exists a point $z : Spec\ k \longrightarrow \mathcal{M}$ such that*

$$\hat{O}_{\mathcal{A}_n,y} \simeq \hat{O}_{\mathcal{M},z}.$$

   In fact, the isomorphism holds for all $z = f(w)$ where $w : Spec\ k \longrightarrow \mathcal{W}$ is a point in $\mathcal{W}$ such that $\pi(w) = y$.

2. *Let $x : Spec\ k \longrightarrow \mathcal{X}_n$ be a point in $\mathcal{X}_n$. Then there exists a point $z : Spec\ k \longrightarrow \mathcal{M}$ such that*

$$\hat{O}_{\mathcal{X}_n,x}[[t_1, t_2, \dots, t_r]] \simeq \hat{O}_{\mathcal{M},z}.$$

   for some $r = r(x)$. In fact z can be chosen as any $z = f(w)$ where $w : Spec\ k \longrightarrow \mathcal{W}$ is a point in $\mathcal{W}$ such that $\gamma\pi(w) = x$.

Proof.

(1) By Theorem 6 in the case $R = k$, we can take a point $w : Spec\ k \longrightarrow \mathcal{W}$ such that $\pi(w) = y$. Let w correspond to

$$(G_0 \to H_0 \to G_0, \chi_0).$$

Put $z = f(w)$.

We have two formally smooth morphisms: $f : \mathcal{W} \longrightarrow \mathcal{M}$ and $\gamma : \mathcal{W} \longrightarrow \mathcal{A}_n$. By Schlessinger's results stated as Lemma 11, we know that $\hat{O}_{\mathcal{M},z}$ and $\hat{O}_{\mathcal{A}_n,y} = \mathrm{Def}_k(G_0 \to H_0 \to G_0)$ differ only by a power series ring. It suffices to show that the dimension of the tangent spaces of $\mathcal{M}$ and $\mathcal{A}_n$ are equal. By the comment after Theorem 5, for any nilpotent pd-thickening R of k (in particular on $k[\varepsilon]/(\varepsilon^2)$ for tangent dimension computation) and a p-divisible group G over k, we know that

$$\mathbf{D}(G)_R \approx D(G) \otimes_{W(k)} R.$$

Now apply Theorem 5 to $S = Spec\ k$, $S' = Spec\ R$. We get that the value of $\mathrm{Def}_k(y)$ on R is the set of locally free direct summands of $D(G_0) \otimes R$ and $D(H_0) \otimes R$ compatible with the morphisms

$$D(G_0) \otimes R \longrightarrow D(H_0) \otimes R \longrightarrow D(G_0) \otimes R,$$

and reduces to ω_{G_0} and ω_{H_0}. Using χ_0, this correponds exactly to elements of $\mathrm{Def}_k(z)$.

As for (2), we can find a point $y : Spec\ k \longrightarrow \mathcal{W}_n$ such that $\gamma(y) = x$ by Lemma 3 whose proof will be given in the next chapter. By Proposition 4, $\gamma : A_n \longrightarrow \mathcal{X}_n$ is formally smooth at y. Now we just need to apply (1) and part (3) of the following lemma by Schlessinger. ∎

LEMMA 11 *Let $\mathcal{F} : \mathcal{C} \longrightarrow Sets$ be a covariant functor which has a pro-representable hull $h_R \longrightarrow \mathcal{F}$. Then we have*

1. *If $h_{R'} \longrightarrow \mathcal{F}$ is formally smooth, then this morphism factors through a formally smooth morphism $h_{R'} \longrightarrow h_R$.*

2. *If $h_{R'} \longrightarrow h_R$ is a formally smooth morphism, then $R \longrightarrow R'$ makes R' a power series ring over R.*

3. *Let R_1 and R_2 are two complete local Noetherian rings. If $R_1[[t_1, t_2, \ldots, t_r]] \simeq R_2[[t_1, t_2, \ldots, t_r]]$ then $R_1 \simeq R_2$.*

Proof.

1. This is the universal property of a pro-representation hull.

2. See [19, Remark 2.10.].

3. See [6, Lemma 4.7] for (3).

∎

Chapter 4

Calculation of local singularities

In this chapter k is a perfect field of characteristic p. Let $x : Spec\ k \longrightarrow \mathcal{X}_n$ be a point. we will compute explicitly the local singularity at x. Let N be the group scheme over k that corresponds to the point x, where k is a perfect field of characteristic p. We first use Dieudonné cristalline theory to find a p-factorization $(G \xrightarrow{\varphi} H \xrightarrow{\psi} G)$ for N over k. Then by Corollary 10, the complete local ring of $\mathcal{X}_n$ at the point is isomorphic to some complete local ring of the scheme $\mathcal{M}$.

4.1 The actual embedding

We start with the postponed proof for Lemma 3.

Proof. Let N be a group scheme over k annihilated by p and of order p^n. Let σ be the Frobenius automorphism of k. According to contravariant Dieudonné theory, $\mathbf{M}(N)$ is a k-vector space of dimension n with a σ-linear endomorphisms $\mathbf{F}$ and a σ^{-1}-linear endomorphism $\mathbf{V}$, such that

$$\mathbf{F} \circ \mathbf{V} = \mathbf{V} \circ \mathbf{F} = 0.$$

Fix a basis $\{e_1, \ldots, e_n\}$ of $\mathbf{M}(N)$, write

$$\begin{aligned} \mathbf{F}(e_i) &= \sum F_{ij} e_j \\ \mathbf{V}(e_i) &= \sum V_{ij} e_j \end{aligned}$$

and put $F = (F_{ij})$, $V = (V_{ij})$ as matrices in k The condition above is then equivalent to the following in matrix forms:

$$FV^{\sigma} = V^{\sigma} F = 0.$$

According to Dieudonné theory over a perfect field, p-divisible groups correspond to finite free $W(k)$-modules with operators $\mathbf{F}$ and $\mathbf{V}$. These operators in the Dieudonné modules can be described in matrix forms. We start with a lemma in linear algebra.

LEMMA 12 *Let F, V be two $n \times n$ matrices in k such that $FV^\sigma = V^\sigma F = 0$. Then there exist matrices $\tilde{F}$ and $\tilde{V}$ in $W(k)$ that lift F and V respectively such that*

$$\tilde{F}\tilde{V}^\sigma = \tilde{V}^\sigma \tilde{F} = 0.$$

Proof. By linear algebra, we can always write $M = M_b \oplus M_n$ where

1. $F : M_b \xrightarrow{\sim} M_b$ is bijective.

2. $F : M_n \longrightarrow M_n$ is nilpotent.

Then there exists $\{e_1, e_2, \dots, e_s\} \subset M_n$ such that

$$\{e_1, F(e_1), \dots, F^{r_1}(e_1), e_2, F(e_2), \dots, F^{r_2}e_2, \dots, e_s, F(e_s), F^{r_s}(e_s)\}$$

is a basis of M_n and $F^{r_i+1}(e_i) = 0$ for all i. Then using the hypothesis that $FV^\sigma = V^\sigma F = 0$, we have

- $V^\sigma(F^j(e_i)) = 0, \forall i,\ \forall j \geq 1$
- $V^\sigma(e_i) = \sum_{j=0}^{s} \alpha_{ij} F^{r_i}(e_j)$

Now we set $\tilde{F}$ and $\tilde{V}$ as follows

- for $\tilde{F}$ on M_n, lift F such that a new basis of M_n over $W(k)$ is

$$\{\tilde{F}^j(e_i)|\ 1 \leq i \leq s, 0 \leq j \leq r_i\}$$

- for $\tilde{V}$ on M_n, lift α_{ij} to elements $\tilde{\alpha}_{ij} \in W(k)$ and put

$$\tilde{V}(e_i) = \sum_{j=1}^{s} \tilde{\alpha}_{ij} \tilde{F}^{r_i}(e_j)$$

- for $\tilde{F}$ and $\tilde{V}$ on M_b, use any lift F such that $\tilde{F}(M_b) \subset M_b$, and set $\tilde{V}(M_b) = 0$.

It is easy to verify the resulting $\tilde{F}$ and $\tilde{V}$ are as desired. ∎

Choose some $\tilde{F}$ and $\tilde{V}$ as in the previous lemma. Consider the p-divisible groups G and H over k with Dieudonné modules as follows: $\mathbf{M}(G) = W(k)^{2n}$ and F_G, V_G are given in the following matrix forms for some basis:

$$F_G = \begin{pmatrix} \tilde{F} & pI_n \\ I_n & -\tilde{V}^\sigma \end{pmatrix},\ V_G = \begin{pmatrix} \tilde{V} & pI_n \\ I_n & -\tilde{F}^{\sigma^{-1}} \end{pmatrix}.$$

$\mathbf{M}(H) = W(k)^{2n}$ and F_H, V_H are given in the following matrix forms for some basis:

$$F_H = \begin{pmatrix} \tilde{F} & I_n \\ pI_n & -\tilde{V}^\sigma \end{pmatrix},\ V_H = \begin{pmatrix} \tilde{V} & I_n \\ pI_n & -\tilde{F}^{\sigma^{-1}} \end{pmatrix}.$$

We have an exact sequence of Dieudonné modules:

$$0 \longrightarrow \mathbf{M}(H) \xrightarrow{\varphi_M} \mathbf{M}(G) \xrightarrow{\pi_M} \mathbf{M}(N) \longrightarrow 0$$

where π_M is the projection map to the first n coordinates followed by taking mod p and φ_M is given by the matrix

$$\varphi_M = \begin{pmatrix} pI_n & 0 \\ 0 & I_n \end{pmatrix},$$

and gives a corresponding exact sequence

$$0 \longrightarrow N \longrightarrow G \xrightarrow{\pi} H \longrightarrow 0$$

as desired. ∎

4.2 Determining the local rings

Let y be the point $Spec\ k \longrightarrow \mathcal{A}_n$ corresponding to the p-factorization $(G \longrightarrow H \longrightarrow G)$ we chose above. Let $\mathbf{M}(G)$, $\mathbf{M}(H)$ be as in the previous lemma. The Frobenius operators $\mathbf{F}_G$ and $\mathbf{F}_H$ induce linear maps

$$\bar{\mathbf{F}}_G : (D(G)/pD(G)) = (\mathbf{M}(G)/p\mathbf{M}(G))^\sigma \longrightarrow M(G)/pM(G) \tag{4.1}$$
$$\bar{\mathbf{F}}_H : (D(H)/pD(H)) = (\mathbf{M}(H)/p\mathbf{M}(H))^\sigma \longrightarrow M(H)/pM(H) \tag{4.2}$$

By [2, Proposition 4.3.10], We have

$$\begin{aligned} \omega_G = \ker \bar{\mathbf{F}}_G &= \left\{ \begin{bmatrix} V^\sigma u \\ u \end{bmatrix}, u \text{ any } k\text{-vector of length } n \right\} \subset D(G)/pD(G) \\ \omega_H = \ker \bar{\mathbf{F}}_H &= \left\{ \begin{bmatrix} v \\ -Fv \end{bmatrix}, v \text{ any } k\text{-vector of length } n \right\} \subset D(H)/pD(H) \end{aligned}$$

and $\dim_k(\omega_G) = \dim_k(\omega_H) = n$.

By computing the deformation of the p-factorization of our choice using cristalline theory, we get the following result.

THEOREM 13 *With the notation as above, we have*

$$\hat{O}_{\mathcal{A}_n,y} = W(k)[[X_{ij}, Y_{ij}]]/I$$

where with $X = (X_{ij})$, $Y = (Y_{ij})$, and I is the ideal generated by the entries of the $n \times n$ matrices

$$\begin{aligned} &(X - \tilde{F})(Y + \tilde{V}^\sigma) - pI_n, \\ &(Y + \tilde{V}^\sigma)(X - \tilde{F}) - pI_n. \end{aligned}$$

Proof. By Corollary 10, it suffices to compute a hull for the functor $\mathcal{C} \longrightarrow Sets$

$$R \longrightarrow \left\{ \begin{array}{l} (V_H, V_G) \\ V_H \subset D(H) \otimes R \text{ free submodule lifting } \omega_H \\ V_G \subset D(G) \otimes R \text{ free submodule lifting } \omega_G \\ \alpha(V_G) \subset V_H, \beta(V_H) \subset V_G \end{array} \right\}$$

Let $R \in \mathcal{C}$ and $(V_H, V_G) \in \mathcal{M}(R)$. By applying the base changes

$$\sigma_1 = \begin{pmatrix} \tilde{F} & I_n \\ I_n & 0 \end{pmatrix}$$

on $D(H)$ and

$$\sigma_2 = \begin{pmatrix} I_n & -\tilde{V}^\sigma \\ 0 & I_n \end{pmatrix}$$

on $D(G)$, we can assume that

$$\begin{aligned} \omega_G &= \left\{ \begin{bmatrix} 0 \\ u \end{bmatrix} ; u \text{ any } k\text{-vectors of length} n \right\} \\ \omega_H &= \left\{ \begin{bmatrix} 0 \\ v \end{bmatrix} ; v \text{ any } k\text{-vectors of length} n \right\} \end{aligned}$$

The maps α, β under the new bases are

$$\begin{aligned} \alpha &= \sigma_1 \begin{pmatrix} I_n & 0 \\ 0 & pI_n \end{pmatrix} \sigma_2^{-1} = \begin{pmatrix} \tilde{F}^\sigma & pI_n \\ I_n & \tilde{V}^\sigma \end{pmatrix} \\ \beta &= \sigma_2 \begin{pmatrix} pI_n & 0 \\ 0 & I_n \end{pmatrix} \sigma_1^{-1} = \begin{pmatrix} -\tilde{V}^\sigma & pI_n \\ I_n & -\tilde{F} \end{pmatrix}. \end{aligned}$$

Since V_G is a lift of ω_G as above, we can find a basis of V_G in the following form:

$$\begin{pmatrix} Y \\ I_n \end{pmatrix},$$

and similiarly a basis for V_H:

$$\begin{pmatrix} X \\ I_n \end{pmatrix}$$

where X and Y are $n \times n$ matrices over $W(k)$.

Now the conditions $\alpha(V_G) \subset V_H$ and $\beta(V_H) \subset V_G$ mean that

$$\begin{pmatrix} \tilde{F} & pI_n \\ I_n & \tilde{V}^\sigma \end{pmatrix} \begin{pmatrix} Y \\ I_n \end{pmatrix} = \begin{pmatrix} X \\ I_n \end{pmatrix} T_2$$
$$\begin{pmatrix} -\tilde{V}^\sigma & pI_n \\ I_n & -\tilde{F} \end{pmatrix} \begin{pmatrix} X \\ I_n \end{pmatrix} = \begin{pmatrix} Y \\ I_n \end{pmatrix} T_1$$

for some $n \times n$ matrices T_1 and T_2.

Solve the matrix equations we get $T_1 = X - \tilde{F}$, $T_2 = Y + \tilde{V}^\sigma$ and

$$pI_n = (X - \tilde{F})(Y + \tilde{V}^\sigma) = (Y + \tilde{V}^\sigma)(X - \tilde{F}).$$

∎

4.3 Determining the singular points

To determine at which points $\mathcal{X}_n$ is singular , we would like to compute the dimension of the tangent spaces $\mathcal{T}_{F,V}$ of the scheme

$$\mathcal{M}_{F,V} = Spec\ k[\underline{\mathbf{X}}, \underline{\mathbf{Y}}]/((\underline{\mathbf{X}} - F)(\underline{\mathbf{Y}} + V^\sigma),\ (\underline{\mathbf{Y}} + V^\sigma)(\underline{\mathbf{X}} - F)).$$

at the point $(0, 0)$. By definition,

$$\mathcal{T}_{F,V} = \{\phi : k[\underline{\mathbf{X}}, \underline{\mathbf{Y}}]/I \longrightarrow k[\varepsilon]/\varepsilon^2 \text{ s.t. } \phi \mod \varepsilon = (F, V^\sigma)\}$$

where $I = ((\underline{\mathbf{X}} - F)(\underline{\mathbf{Y}} + V^\sigma),\ (\underline{\mathbf{Y}} + V^\sigma)(\underline{\mathbf{X}} - F))$. Any such ϕ maps

$$X_{ij} \mapsto F_{ij}\varepsilon x_{ij}, Y_{ij} \mapsto V^\sigma_{ij}\varepsilon y_{ij},$$

such that the matrices (x_{ij}) and (y_{ij}) satisfy the condition in the following linear algebra lemma, which gives

$$\dim_k \mathcal{T}_{F,V} = n^2 + (n - \text{rank}(F) - \text{rank}(V))^2.$$

LEMMA 14 *Let $F, V \in Mat_n(k)$ satisfy that $FV = VF = 0$. Then*

1. *The dimension of the k-vector space consisting of pairs $(X, Y) \in Mat_n(k)^2$ such that*

$$FY = XV, YF = VX$$

is $n^2 + (n - \text{rank}(F) - \text{rank}(V))^2$.

2. *The dimension of the k-vector space of matrices $X \in Mat_n(k)$ such that*

$$FX = XV = 0$$

is $(n - \text{rank}(F))(n - \text{rank}(V))$.

Proof. Let $g, h \in \mathbf{GL}_n(k)$ be any invertible matrices. Note that if we replace F and V with gFh^{-1} and hVg^{-1} respectively, the dimension of the vector space described in (1) and (2) will not change as we can replace X and Y with gXh^{-1} and hYg^{-1} in accordance. Therefore, put $r_F = \mathrm{rank}(F)$ and $r_V = \mathrm{rank}(V)$, we can assume that

$$F = \begin{pmatrix} I_{r_F} & \\ & 0 \end{pmatrix}.$$

Using the fact that $FV = VF = 0$, we know V has to be of the form

$$V = \begin{pmatrix} 0 & \\ & V' \end{pmatrix}.$$

Repeat the same process in V, we can assume that

$$F = \begin{pmatrix} I_{r_F} & & \\ & 0 & \\ & & 0 \end{pmatrix}, V = \begin{pmatrix} 0 & & \\ & I_{r_V} & \\ & & 0 \end{pmatrix}.$$

Then it is easy to verify that the dimension of the solution space is

$$n^2 + (n - \mathrm{rank}(F) - \mathrm{rank}(V))^2,$$

and similarly for (2). ∎

The geometric properties of the scheme

$$\mathcal{M}_{0,0} = Spec\ k[\underline{\mathbf{X}}, \underline{\mathbf{Y}}]/(\underline{\mathbf{XY}}, \underline{\mathbf{YX}})$$

are well known. We list a few of them without proof as follows.

1. $\mathcal{M}_{0,0}$ has dimension n^2.
2. The irreducible components of $\mathcal{M}_{0,0}$ are $\mathcal{M}_{0,0}^{r,s}$ $(r+s=n)$, defined by the Zariski closure of the points (X, Y) such that $\mathrm{rank}(X) = r, \mathrm{rank}(Y) = s$.

Now let x be a k-valued point of $\mathcal{X}_n$. Let N be the of the group scheme x that corresponds to. Suppose that F and V are matrices obtained from the Dieudonné module of N just before. We know that x is a nonsingular point (i.e. $\hat{O}_{\mathcal{X}_n,x}$ is regular) if and only if the tangent dimension $\dim_k \mathcal{T}_{F,V}$ equals to the dimension of the scheme $\mathcal{M}_{F,V}$. Because the dimension of $\mathcal{M}_{F,V}$ is n^2, this means $n - \mathrm{rank}(F) - \mathrm{rank}(V) = 0$. As a consequence, on the Dieudonné module of the group scheme N, we must have

$$\ker(\mathbf{F}) = \mathrm{im}(\mathbf{V}),\ \ker(\mathbf{V}) = \mathrm{im}(\mathbf{F}).$$

COROLLARY 15 *Let $x : Spec\ k \longrightarrow \mathcal{X}_n$ be a point. Then x is a nonsingular point of $\mathcal{X}_n$ if and only if x corresponds to a Barsotti-Tate group scheme.*

We are now ready to compute the dimension of the tangent space at a point $x : Spec\ k \to \mathcal{X}_n$. Let N be the group scheme that x corresponds to. It suffices to compute the dimension of deformations of N over the ring $k[\varepsilon]/(\varepsilon^2)$. According to a result by DeJong [7, Remark 10.3], group schemes of order p^n over $R = k[\varepsilon]/(\varepsilon^2)$ that are killed by p are equivalent to the category of triples (M, F, V) where

1. M is a free R-module of rank n.
2. $F : M^{(p)} \longrightarrow M$ and $V : M \longrightarrow M^{(p)}$ are R-linear maps such that $F \circ V = V \circ F = 0$.

This equivalence is functorial and enables us to compute deformations of group schemes via linear algebra. Let N correspond to the triple (M_0, F_0, V_0), then we have the following result

PROPOSITION 16 *The tangent space at x has k-dimension $(n - r_{F_0} - r_{V_0})^2 + (n - r_{F_0})(n - r_{V_0})$.*

Proof. we know that the deformations over $k[\varepsilon]$ corresponds to quadruples (M, F, V, α) where $(M, F, V) = (M_0, F_0, V_0) \pmod{\varepsilon M}$ and $\alpha : M/\varepsilon M \xrightarrow{\sim} M_0$. If we change M by an linear isomorphism X, we get that

$$(M, F, V, \alpha) \approx (M, X^\sigma F X^{-1}, X V X^{-\sigma}, \alpha X^{-1}).$$

Fixing a basis for M and M_0, we can think all the linear maps as matrices. Because $(F, V) = (F_0, V_0) \pmod{\varepsilon M}$, we can write

$$F = F_0 + \varepsilon A,\ V = V_0 + \varepsilon B,$$

where A, B are matrices in $Mat_n(k)$. In order that $FV = VF = 0$, we must have

$$AV_0 + F_0 B = V_0 A + BF_0 = 0.$$

By the linear algebra Lemma 14, the dimension of space of such (A, B) is $n^2 + (n - r_{F_0} - r_{V_0})^2$. To count the dimension of isomorphism classes, first we choose an X so that $\alpha X^{-1} = id$ thus reduce the problem to counting isomorphism classes of the form (M, F, V, id). An isomorphism X between quadruples of this type must satisfy that $X = id \pmod{\varepsilon M}$ thus $X^\sigma = id$. We get that $(M, F, V, id) \approx (M, F', V', id)$ if and only if $F' = FX^{-1}, V' = XV$. It remains to calculate the dimension of the kernel of the map

$$X \mapsto (F - FX^{-1}, V - XV)$$

which is a linear space as we can write $X = id + \varepsilon Y$ where Y is any $n \times n$ matrix in k. Subsituting with

$$F = F_0 + \varepsilon A,\ V = V_0 + \varepsilon B,\ X = id + \varepsilon Y$$

shows this space is just

$$\{Y \in Mat_n(k) \mid F_0 Y = Y V_0 = 0\},$$

which has dimension $(n - r_{F_0})(n - r_{V_0})$ by Lemma 14. Therefore, the dimension of the isomorphism classes is

$$n^2 + (n - r_{F_0} - r_{V_0})^2 - (n^2 - (n - r_{F_0})(n - r_{V_0})) = \\ (n - r_{F_0} - r_{V_0})^2 + (n - r_{F_0})(n - r_{V_0}).$$

∎

4.4 Classification of singularities

It is suspected that the worst singularities occur over the point that corresponds to the group scheme α_p^n. Deeper study of the complete local rings shows that any local singularities look like the same as the singularity over some α^r in the moduli stack $\mathcal{X}_r$, by a linear algebra result as follows

THEOREM 17 *Let*

$$R_{F,V} = W(k)[[\underline{\mathbf{X}}, \underline{\mathbf{Y}}]]/((\underline{\mathbf{X}} - F)(\underline{\mathbf{Y}} + V) - pI, (\underline{\mathbf{Y}} + V)(\underline{\mathbf{X}} - F) - pI).$$

Let $r = n - r_F - r_V$. Put

$$R_r = W(k)[[\underline{x}, \underline{y}]](\underline{x}\underline{y} - p, \underline{y}\underline{x} - p),$$

where $\underline{x} = (x_{ij})$,$\underline{y} = (y_{ij})$ are coordinates of $r \times r$ matrices. Then

$$R_{F,V} \tilde{\longrightarrow} R_r[[t_1, t_2, \dots, t_{n^2 - r^2}]].$$

Proof. As in Lemma 14, we can replace

$$F \longrightarrow gFh^{-1}, V \longrightarrow hVg^{-1},$$

and assume that

$$F = \begin{pmatrix} I_{r_F} & & \\ & 0 & \\ & & 0 \end{pmatrix}, V = \begin{pmatrix} 0 & & \\ & I_{r_V} & \\ & & 0 \end{pmatrix}.$$

Rewriting in block matrices, let $\underline{\mathbf{X}} - F = (X_{ij})$ and $\underline{\mathbf{Y}} + V = (Y_{ij})$, $1 \le i, j \le 3$. Consider the set S which consists of all the variables that occur in the blocks

$$X_{11} \cup X_{12} \cup X_{13} \cup X_{21} \cup X_{31} \cup Y_{22} \cup Y_{23} \cup Y_{32}.$$

Obviously S has $n^2 - r^2$ elements. We claim that there exist invertible matrices $G, H \in Mat_n(R)$ whose entries are power series on S. such that

$$\begin{aligned}
G(\underline{\mathbf{X}} - F)H^{-1} &= \begin{pmatrix} I & & \\ & pI & \\ & & U(\underline{\mathbf{X}}) \end{pmatrix} \\
H(\underline{\mathbf{Y}} + V)G^{-1} &= \begin{pmatrix} pI & & \\ & I & \\ & & V(\underline{\mathbf{Y}}) \end{pmatrix} \\
\text{and } G^{-1}\begin{pmatrix} I & & \\ & pI & \\ & & \underline{x} \end{pmatrix} H &= \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & u(\underline{x}, S) \end{pmatrix} \\
H^{-1}\begin{pmatrix} pI & & \\ & I & \\ & & \underline{y} \end{pmatrix} G &= \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & v(\underline{y}, S) \end{pmatrix}.
\end{aligned}$$

where $u(\underline{x}, S)$ (resp. $v(\underline{y}, S)$) is a power series on S and $\underline{x}$ (resp. $\underline{y}$).

Once the claim is established, we can construct morphisms between $R_r[[S]]$ and $R_{F,V}$ which are inverse of each other:

$$\begin{aligned}
\phi : R_r[[S]] &\longrightarrow R_{F,V} \\
\underline{x} &\mapsto U(\underline{\mathbf{X}}) \\
\underline{y} &\mapsto V(\underline{\mathbf{Y}})
\end{aligned}$$

and the $n^2 - r^2$ variables in S are just sent under phi to the same variables in $R_{F,V}$. The inverse transformation is defined by

$$\begin{aligned}
\psi : R_{F,V} &\longrightarrow R_r[[S]] \\
\underline{\mathbf{X}} - F &\mapsto G^{-1}\begin{pmatrix} I & & \\ & pI & \\ & & \underline{x} \end{pmatrix} H \\
\underline{\mathbf{Y}} + V &\mapsto H^{-1}\begin{pmatrix} pI & & \\ & I & \\ & & \underline{y} \end{pmatrix} G
\end{aligned}$$

Note that since only variables in S appear in G and H, ψ does map into the ring $R_r[[S]]$. The map ψ is well defined since $\psi((\underline{\mathbf{X}} - F)(\underline{\mathbf{Y}} + V)) = \psi((\underline{\mathbf{Y}} + V)(\underline{\mathbf{X}} - F)) = p$ as $\underline{x}\underline{y} = \underline{y}\underline{x} = p$ in $R_r[[S]]$.

Now let us get back to the proof of the claim. For any matrix in block forms

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

if A_{11} is invertible, we can find invertible matrices in the form

$$B = \begin{pmatrix} A_{11}^{-1} & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix}, C = \begin{pmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix},$$

so that

$$BAC^{-1} = \begin{pmatrix} I & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}.$$

There are two important properties for the matrices B and C:

1. B and C do not depend on A_{22}.
2. The transformation

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \mapsto CDB^{-1} = \begin{pmatrix} * & * \\ * & D_{22} \end{pmatrix}$$

   keeps the block D_{22} untouched.

Keep this linear algebra lemma in mind. We consider the matrix $\underline{\mathbf{X}} + F$. The first diagonal block X_{11} is invertible and we can apply the linear algebra procedure above. We can find invertible matrices G_0, H_0 such that

$$G_0(\underline{\mathbf{X}} - F)H_0^{-1} = \begin{pmatrix} I_{r_F} & 0 \\ 0 & * \end{pmatrix}.$$

Using the fact that $(\underline{\mathbf{X}} - F)(\underline{\mathbf{Y}} + V) = (\underline{\mathbf{Y}} + V)(\underline{\mathbf{X}} - F) = p$ in the ring $R_{F,V}$, we know that $H_0(\underline{\mathbf{Y}} + V)G_0^{-1}$ must be of the form $\begin{pmatrix} pI_{r_F} & 0 \\ 0 & Y' \end{pmatrix}$, where G_0 and H_0 only use variables in S. By property (2), we know that

$$Y' = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}.$$

Since Y_{22} is invertible, we can also apply the procedure to Y'. By using matrices G_1 and H_1 of the form $\begin{pmatrix} I_{r_F} & 0 \\ 0 & * \end{pmatrix}$, we can further standardize the matrices $\underline{\mathbf{X}} - F$ and $\underline{\mathbf{Y}} + V$ as desired. The actions $G = G_1G_0, H = H_1H_0$ only depend on the variables that appear in S. ∎

Chapter 5

Cohen-Macaulayness

The goal of this chapter is to show the the local ring

$$\Lambda = W(k)[[\underline{\mathbf{X}}, \underline{\mathbf{Y}}]](\underline{\mathbf{XY}} - p, \underline{\mathbf{YX}} - p)$$

is Cohen-Macaulay. A result in [4, Proposition 7.1] allows us to take reduction *mod p* and consider the ring $\Lambda/p\Lambda$. Similar rings are discovered Cohen-Macaulay: De Concini and Strickland [9] showed that the ring

$$R[\underline{\mathbf{X}}, \underline{\mathbf{Y}}]/(\underline{\mathbf{XY}})$$

is Cohen-Macaulay if R is; in the case of symmetric matrices, Chai and Norman [5] proved that

$$R[\underline{\mathbf{X}}, \underline{\mathbf{Y}}]/(\underline{\mathbf{XY}}, \underline{\mathbf{X}}^T - \underline{\mathbf{X}}, \underline{\mathbf{Y}}^T - \underline{\mathbf{Y}})$$

is Cohen-Macaulay. Their methods are based on the theory of Hodge algebras [8], which we will use extensively. Once we prove our rings are Hodge algebras on a certain poset, whether they are Cohen-Macaulay and Gorenstein would follow from the combinatoric properties of the poset that defines the Hodge algebra structure.

5.1 The Hodge algebra structure

We start with a ring R that is universally catanery. Put

$$A = R[\underline{\mathbf{X}}, \underline{\mathbf{Y}}]/(\underline{\mathbf{XY}}, \underline{\mathbf{YX}}).$$

We first give a number of definitions on Young tableaux.

A *Young diagram* λ is a finite sequence of rows of nondecreasing length and each row consists of finite number of elements. A Young diagram is a subset of $\mathbf{N} \times \mathbf{N}$ and can be considered as a sequence of rows of empty boxes. A *Young tableau* T on $1, 2, \ldots, n$ is a map from a Young diagram $\lambda \subset \mathbf{N} \times \mathbf{N}$ to the set $\{1, 2, \ldots, n\}$. We write the underlying Young diagram $\lambda = |T|$ and call it the *shape* of T. T can be considered as filling in the boxes of λ with numbers in $\{1, 2, \ldots, n\}$. In this chapter we fix the integer n and all Young tableaux are assumed to be on the set $\{1, \ldots, n\}$.

For two Young tableaux T_1 and T_2, if the length of the last row of T_1 is no less than the length of the first row of T_2, we write $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ as the Young tableau obtained by putting T_1 on top of T_2.

We call a Young tableau T *standard* if the entries of T are strictly increasing in each row and nondecreasing in each column. There is an involution for standard tableaux: let T be a standard tableau with rows $\mathbf{rr}_1, \ldots, \mathbf{rr}_l$, then $\hat{T}$ is the tableau with rows $\hat{\mathbf{rr}}_l, \ldots, \hat{\mathbf{rr}}_1$, where the row $\hat{\mathbf{rr}}$ is obtained by removing numbers appeared in $\mathbf{rr}$ from $\{1, \ldots, n\}$ and forming the leftovers in an increasing manner. It is readily seen that $\hat{T}$ is also standard.

We call $(T_1|T_2)$ a *bitableau* if T_1 and T_2 are tableaux with the same shape. A bitableau $(T_1|T_2)$ is *standard* if both T_1 and T_2 are standard tableaux. Fixing the integer n, we can associate a standard bitableau $(T_1|T_2)$ with products of minors, written as $[T_1|T_2]$, in the polynomial ring $R[\mathbf{Z}]$, as follows:

1. if $T_1 = \mathbf{u}$ and $T_2 = \mathbf{v}$ consist only one row, let $(u_1, \ldots, u_s)$ and $(v_1, \ldots, v_s)$ be the elements of $\mathbf{u}$ and $\mathbf{v}$, then the corresponding element in $R[\mathbf{Z}]$ is the determinant of the minor $(Z_{u_i,v_j})_{i,j=1,\ldots,s}$.

2. $(T_1|T_2)$ corresponds to the product of the determinants associated with each pair of individual rows (numbered the same in T_i).

We call $[(T_1|T_2)(T_3|T_4)]$ a *quartableau* if both $(T_1|T_2)$ and $(T_3|T_4)$ are bitableaux. A quartableau can be associated with the product of the two minors: $(T_1|T_2)$ in variables $\underline{\mathbf{X}}$ and $(T_3|T_4)$ in variables $\underline{\mathbf{Y}}$. We write $[T_1|T_2|T_3|T_4]$ as the associated element in A. Note that the two bitableaux do not necessarily have the same shape and are possibly empty. A quartableaux $[T_1|T_2|T_3|T_4]$ is *standard* if both $(T_1|T_2)$ and $(T_3|T_4)$ are standard bitableaux and $\begin{pmatrix} \hat{T}_2 \\ T_3 \end{pmatrix}$ and $\begin{pmatrix} \hat{T}_4 \\ T_1 \end{pmatrix}$ are standard tableaux.

Using the same method as in the article [9, Proposition 1.4] by De Concini and Strickland on the variety of complexes, we will show that the standard quartableaux form an R-basis for A. We denote by $\mathcal{E}$,$\mathcal{E}_{xy}$,$\mathcal{E}_{yx}$ the ideals $(\underline{\mathbf{XY}}, \underline{\mathbf{YX}})$, $(\underline{\mathbf{XY}})$, and $(\underline{\mathbf{YX}})$ respectively.

PROPOSITION 18 *Every element in A can be written as a linear combination of standard quartableaux.*

Proof. It suffices to show that every nonstandard quartableau $[T_1|T_2|T_3|T_4]$ can be written as a linear combination of the standard quartableaux.

From the standard basis theorem for the polynomial ring [10], every bitableau can be written as a linear combination of the standard bitableaux, therefore we can assume that both $(T_1|T_2)$ and $(T_3|T_4)$ are standard bitableaux.

Let $(s_1 \ldots s_h|\widehat{j_1 \ldots j_{n-h}})$ be the last row of $(T_1|T_2)$ and $(t_1 \ldots t_k|\widehat{q_1 \ldots q_{n-k}})$ be the last row of $(T_3|T_4)$. Suppose $\begin{pmatrix} \hat{T}_2 \\ T_3 \end{pmatrix}$ or equivalently, $\begin{pmatrix} \hat{T}_4 \\ T_1 \end{pmatrix}$ is not a Young tableau, i.e., $h+k>n$, then apply [9, Proposition 1.3(i)], we get that $[T_1|T_2|T_3|T_4] \in \mathcal{E}_{xy}$ and $\in \mathcal{E}_{yx}$.

Now we can assume $h+k \le n$. Suppose that $1 \le u \le k$ is the smallest index such that $j_u > t_u$, apply [9, Proposition 1.3(ii)], we get

$$
\begin{aligned}
&[s_1 \dots s_h|\widehat{j_1 \dots j_{n-h}}][t_1 \dots t_k|\widehat{q_1 \dots q_{n-k}}] \\
&\qquad = \sum_{\sigma \ne id} [s_1 \dots s_h|j_1 \dots j_{u-1}\widehat{\sigma(j_u) \dots \sigma(j_{n-h})}] \\
&\qquad\qquad [\sigma(t_1) \dots \sigma(t_u)t_{u+1} \dots t_k|\widehat{q_1 \dots q_{n-k}}] \pmod{\mathcal{E}_{xy}}
\end{aligned}
$$

where σ runs through all permutations in $S_{n-h+1}/(S_u \times S_{n-h+1-u})$ which acts on $\{j_u, \dots, j_{n-h}, t_1, \dots, t_u\}$ in the obvious way . Suppose $1 \le v \le h$ is the smallest index such that $q_v > s_v$. Apply the previous procedure to the $s-q$ pair in each item on the RHS above, we get

$$
\begin{aligned}
&[s_1 \dots s_h|\widehat{j_1 \dots j_{n-h}}][t_1 \dots t_k|\widehat{q_1 \dots q_{n-k}}] \\
&\qquad = \sum_{\tau \ne id} \sum_{\sigma \ne id} [\tau(s_1) \dots \tau(s_{v-1})s_v \dots s_h|j_1 \dots j_{u-1}\widehat{\sigma(j_u) \dots \sigma(j_{n-h})}] \\
&\qquad\qquad [\sigma(t_1) \dots \sigma(t_u)t_{u+1} \dots t_k|q_1 \dots q_{v-1}\widehat{\tau(q_v) \dots \tau(q_{n-k})}] \pmod{\mathcal{E}_{yx}}
\end{aligned}
$$

where τ runs through all classes of permutations in $S_{n-k+1}/(S_v \times S_{n-k+1-v})$ with the action on $\{q_v, \dots, q_{n-k}, s_1, \dots, s_v\}$.

Let $\lhd$ be the lexicographical order in [9, p. 66], it is easy to verify that each item (after reordering of the entries) on the RHS is standard, and is lexicographically strictly smaller than the LHS, i.e.,

$$
\begin{aligned}
(\tau(s_1) \dots \tau(s_{v-1})s_v \dots s_h|j_1 \dots j_{u-1}\widehat{\sigma(j_u) \dots \sigma(j_{n-h})}) &\lhd (s_1 \dots s_h|\widehat{j_1 \dots j_{n-h}}) \\
(\sigma(t_1) \dots \sigma(t_{u-1})t_u \dots t_k|q_1 \dots q_{v-1}\widehat{\tau(q_v) \dots \tau(q_{n-k})}) &\lhd (t_1 \dots t_k|\widehat{q_1 \dots q_{n-k}}).
\end{aligned}
$$

To sum up, we can write

$$
[T_1|T_2|T_3|T_4] = \sum a_i[(T_{i1}|T_{i2})(T_{i3}|T_{i4})] \pmod{\mathcal{E}}
$$

where $(T_{i1}|T_{i2}) \lhd (T_1|T_2)$ and $(T_{i3}|T_{i4}) \lhd (T_3|T_4)$. Our claim is then achieved by induction on the lexicographic ordering. ∎

Before we proceed to show that the standard quartableaux are linearly independent, let us recall some general theory on polynomial representations of $\Gamma = \mathbf{GL}_n$.

Let K be an infinite field. A finite dimensional $\mathbf{GL}_n(K)$-representation V is a polynomial representation if, for any basis $\{V_b\}$, we have

$$g(V_b) = \sum p_{ab}(g)V_a, \forall g \in \Gamma$$

where p_{ab} is a polynomial on the entries (g_{ij}) of g. Let $M_K(n,r)$ be the category of polynomial representations of $\mathbf{GL}_n$ such that the polynomials p_{ab} defined above are homogeneous of degree r. Let λ be a Young diagram. We say that λ has range n if each row of λ has at most length n. We denote by $|\lambda|$ the degree of λ, i.e., the sum of the length of all rows. Let λ^T denote the transpose Young diagram of λ, i.e., the shape obtained by swapping rows and columns. For example, if $\lambda = (4,3,1)$, then $\lambda^T = (3,2,2,1)$.

Let $\Lambda^+(n,r)$ denote the set of all Young diagrams of degree r such that each row has at most n entries. If $\lambda \in \Lambda^+(n,r)$, then λ^T can be considered as a partition of r into n parts by counting the length of each row (zero allowed). Therefore we have established a correspondance between Young diagrams and weights:

$$\lambda \mapsto (\lambda_1^T, \lambda_2^T, \dots, \lambda_n^T).$$

For each Young diagram $\lambda = (\lambda_1, \lambda_2, \dots \lambda_l) \in \Lambda^+(n,r)$, we denote by C_λ the "canonical tableau" associated to λ

$$\begin{pmatrix} 1 & 2 & \dots\dots\dots\dots & \lambda_1 \\ 1 & 2 & \dots\dots\dots \; \lambda_2 & \\ \dots\dots & \dots & \dots\dots\dots\dots & \\ 1 & 2 & \dots \; \lambda_l & \end{pmatrix}.$$

The main result of polynomial representations of $\mathbf{GL}_n$ [13, Theorem 3.5a, 4.5,5.4] says that,

THEOREM 19 *For every $\lambda \in \Lambda^+(n,r)$, there is a unique irreducible $L_\lambda \in M_K(n,r)$ whose maximal weight corresponds to λ. Every irreducible representation $V \in M_K(n,r)$ is isomorphic to L_λ for exactly one $\lambda \in \Lambda^+(n,r)$. Moreover, if* $\mathrm{char}(K) = 0$, *L_λ can be constructed as the left $\mathbf{GL}_n$-representation of the K-span of all the standard bitableaux of the form $(T|C_\lambda)$, whose dimension is the number of standard tableaux of shape λ.*

Consider the case of polynomial representations of $G = \mathbf{GL}_n \times \mathbf{GL}_n$. As any irreducible polynomial representation of G is the tensor product of two polynomial $\mathbf{GL}_n$-representations, we can easily extend the results above.

Now let us turn to our problem. Let $R_{p,q}$ be the space of homogeneous elements in R of bidegree $(p, n-q)$. Let $d_{p,q}$ be the number of standard quartableaux of all shapes $[\sigma, \tau]$ such that $|\sigma| = p$, $|\hat{\tau}| = q$. To show that the standard quartableaux are linearly independent, it suffices to prove that the $\mathrm{rank}(R_{p,q}) = d_{p,q}$. From Proposition 18, we know that the standard quartableaux span R, hence we have

$$\mathrm{rank}\, R_{p,q} \le d_{p,q}.$$

In order to apply Theorem 19 which works the best in characteristic zero, we replace R with $A = \mathbf{Q}[\underline{\mathbf{X}}, \underline{\mathbf{Y}}]/(\underline{\mathbf{XY}}, \underline{\mathbf{YX}})$. We can do that since all the ranks are equal to those of $\mathbf{Z}[\underline{\mathbf{X}}, \underline{\mathbf{Y}}]/(\underline{\mathbf{XY}}, \underline{\mathbf{YX}})$.

Consider A as a G-module induced by the G-action on the space of matrices:

$$(g, h) \circ (X, Y) = (gXh^{-1}, hYg^{-1})$$

where $(g, h) \in G$, X, Y are $n \times n$ matrices. As a representation of G, A is not a polynomial representation because of the multiplication by g^{-1} and h^{-1}. However, let $\det^m$ be the G-representation

$$det_{\mathbf{GL}} \otimes det_{\mathbf{GL}}.$$

Then $A_{p,q} \otimes \det^m$ is a polynomial representation for some m sufficiently large.

For any Young diagrams σ, τ, let $T_{\sigma,\tau} = [(C_\sigma, C_\sigma)|(\hat{C}_\tau, \hat{C}_\tau)]$ be the standard quartableau. Let $V_{\sigma,\tau}$ be the $\mathbf{Q}$-span of the sub-G-module generated by $T_{\sigma,\tau}$. Claim that

1. $T_{\sigma,\tau} \neq 0$.

2. $T_{\sigma,\tau}$ is a weight vector for $T = \mathbf{T}_n \times \mathbf{T}_n$.

3. $\dim_{\mathbf{Q}} A_{p,q} \geq d_{p,q}$

where $\mathbf{T}_n$ is subgroup of diagonal matrices in $\mathbf{GL}_n$.

Proof. To show (1), we just evaluate the bideterminate of $T_{\sigma,\tau}$ at

$$P = \left(\begin{pmatrix} I_{\sigma_1} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & I_{\tau_1} \end{pmatrix} \right)$$

We have $T_{\sigma,\tau}(P) = 1$. Since $\sigma_1 + \tau_1 \leq n$, P is a $\mathbf{Q}$-valued point of *Spec* A. Therefore, $T_{\sigma,\tau} \neq 0$.

For (2), let $T = (diag(s_1, \ldots, s_n), diag(t_1, \ldots, t_n))$. Then we get

$$\begin{aligned} T \circ C_{\sigma,\tau} &= (s_1 t_1^{-1})^{\lambda_1^T} \ldots (s_n t_n^{-1})^{\lambda_n^T} (s_n^{-1} t_n)^{\tau_1^T} (s_{n-1}^{-1} t_{n-1}^{-1})^{\tau_2^T} \ldots (s_1^{-1} t_1)^{\tau_n^T} \\ &= (s_1 t_1^{-1})^{\mu_1^T} \ldots (s_n t_n^{-1})^{\mu_n^T} (s_1 \ldots s_n t_1^{-1} \ldots t_n^{-1})^{\tau_1^T} \end{aligned}$$

where μ is the bitableau $\begin{pmatrix} \hat{\tau} \\ \sigma \end{pmatrix}$. Hence if we put $m = \tau_1^T$, then $V_{\sigma,\tau} \otimes \det^m$ is a polynomial representation of G with weight μ. Because we just proved $V_{\sigma,\tau} \neq 0$, by Theorem 19, $V_{\sigma,\tau} \otimes \det^m$ must contain a copy of the irreducible representation L_μ.

Let $W_{\sigma,\tau}$ be a submodule of $V_{\sigma,\tau}$ such that $W_{\sigma,\tau} \otimes \det^m \approx L_\mu$. Note that τ_1^T is the smallest integer m such that $V_{\sigma,\tau} \otimes \det^m$ is a polynomial representation, and $[\sigma, \tau]$ is determined by the pair (μ, τ_1^T). It follows that

$$W_{\sigma,\tau} \not\approx W_{\sigma',\tau'} \text{ if } [\sigma, \tau] \neq [\sigma', \tau'].$$

Hence, we have established that

$$\dim A_{p,q} \geq \sum_{|\sigma|=p,|\hat{\tau}|=q} \dim W_{\sigma,\tau} = \sum_{\mu,\tau_1^T} \dim L_\mu$$

On the other hand, consider the following map for any standard quartableaux $[T_1|T_2|T_3|T_4]$ of shape $[\sigma, \hat{\tau}]$

$$[T_1|T_2|T_3|T_4] \mapsto (\tau_1^T, \left(\begin{array}{c} \hat{T}_3|\hat{T}_4 \\ T_2|T_1 \end{array} \right)).$$

This gives a bijection from standard quartableaux of bidegree $(p, n-q)$ to standard bitableaux of shape $\left(\begin{array}{c} \hat{\tau} \\ \sigma \end{array} \right)$ together with a marked row. Therefore,

$$d_{p,q} = \sum_{\mu,\tau_1^T} \dim L_\mu,$$

and finally,

$$\dim_{\mathbf{Q}} A_{p,q} \geq d_{p,q}.$$

∎

As a consequence, we have

Proposition 20 *The standard quartableaux in A are linearly independent.*

Now we will construct a poset on which A is an ordinal Hodge algebra. Note that the two bitaleaux in a standard quartableau do not necessarily have the same number of rows and we can not simply choose the set of standard quartableau that consists of a single row.

Let $L = \{(\mathbf{rr}_1|\mathbf{rr}_2)\}$ be the poset of standard bitableaux with a single row, with the partial order that $(\mathbf{rr}_1|\mathbf{rr}_2) \leq (\mathbf{rr}'_1|\mathbf{rr}'_2)$ if and only if $\left(\begin{array}{c} \mathbf{rr}_1 \\ \mathbf{rr}'_1 \end{array} \right)$ and $\left(\begin{array}{c} \mathbf{rr}_2 \\ \mathbf{rr}'_2 \end{array} \right)$ are both standard tableaux. It is readily seen that [8, p.52] L is a distributive lattice. Consider the poset $H = L_x \cup L_y$, where L_x and L_y are two disjoint copies of the poset L. Define the partial order in H such that $h \leq h'$ if and only if one of the following happens:

$$\begin{aligned} h, h' \in L_x, &\quad \underline{h} \leq \underline{h}' \\ h, h' \in L_y, &\quad \underline{h} \leq \underline{h}' \\ h \in L_x, h' \in L_y, &\quad \underline{h} \leq \underline{h}' \end{aligned}$$

where $\underline{\text{h}}$ denotes the projection of h into L. H is again a distributive lattice. A chain in H corresponds to an ordered pair of standard bitableaux

$$((S_1|S_2), (S_3|S_4)), S_1 \leq S_3, S_2 \leq S_4,$$

where the tableau $(S_1|S_2)$ is the subchain in $L \times \{0\}$ and $(S_3|S_4)$ the subchain in $L \times \{1\}$.

Now consider the map from standard quartableaux to the ordered pairs of standard bitableaux we just defined:

$$[T_1|T_2|T_3|T_4] \mapsto ((\hat{T}_3|\hat{T}_4), (T_2|T_1)).$$

which is a bijection. Under this bijection quartableaux can be regarded as monomials on H and standard quartableaux as chains in H. Let Σ be the set of nonstandard quartableaux. Then we have

PROPOSITION 21 *A is a Hodge algebra on H governed by Σ. In particular, A is reduced if R is reduced.*

Remark: see [8, p.14] for definition and Hodge algebras and [8, Proposition 5.1] for the second statement.

Put

$$A^{\#} = R[H]/\Sigma R[H],$$

then $A^{\#}$ is an ordinal Hodge algebra on H. As a general result of Hodge algebras [8, Corollary 3.2], the reducedness, Cohen-Macaulayness, Gorensteinness of A are equivalent to those of $A^{\#}$ as one can explicitly establish a stepwise flat deformation whose most special fiber is $A^{\#}$ and whose most general fiber is A.

The combinatoric properties of H will enable us to determine the Cohen-Macaulayness and Gorensteinness of A. The poset H is a distributive lattice hence a wonderful (or locally semi-modular) poset [8, p.40]. The property of ordinal Hodge algebra over a wonderful poset [8, Theorem 8.1] gives us

THEOREM 22 *If R is Cohen-Macaulay, then A is Cohen-Macaulay.*

5.2 Gorensteinness

Since H is a distributive lattice, the Gorensteinness of A can be extracted from the combinatoric property of Δ_H, defined as the poset of the join-irreducible elements [3, p.20, p.139] of H. A result by Hibi [15, p.105] tells us that A is Gorenstein if and only if Δ_H is pure, i.e., all maximal chains of Δ_H have the same length. We will check this in the following lemma.

LEMMA 23 *Let L and H be the distributive lattice above. Then Δ_L is pure with length $2n-1$ but Δ_H is not pure when $n > 1$.*

Proof. Let $\left\{ \begin{array}{c} 2n \\ n \end{array} \right\}$ be the set of vectors $[i_1, \ldots, i_n]$ of length n such that $1 \leq i_1 < i_2 < \ldots < i_n \leq 2n$. We impose a partial order on $\left\{ \begin{array}{c} 2n \\ n \end{array} \right\}$ by setting

$$[i_1, \ldots, i_n] \leq [j_1, \ldots, j_n]$$

if and only if

$$i_k \leq j_k \text{ for all } k = 1, \ldots, n.$$

Then by [8, p.52], we have that

$$L \xrightarrow{\sim} \left\{ \begin{array}{c} 2n \\ n \end{array} \right\} - \{[n+1, \ldots, 2n]\}.$$

Note that $[n+1, n+2, \ldots, 2n]$ is the unique maximal element in $\left\{ \begin{array}{c} 2n \\ n \end{array} \right\}$, and $[1, 2, \ldots, n]$ is the unique minimal element. Recall that the join-irreducible elements can be classified as the elements with at most one immediate child. It is easy to verify that the poset of join-irreducible elements of $\left\{ \begin{array}{c} 2n \\ n \end{array} \right\}$ is the subposet consisting of elements of the following form

$$[1, 2, \ldots, n]$$

or

$$[1, 2, \ldots, i, i+j+1, i+j+2, \ldots, j+n],$$

where $0 \leq i \leq n-1$, $1 \leq j \leq n$. Such elements correspond to pairs (i, j) with the partial order

$$(i, j) \leq (i', j') <=> i \geq i', j \leq j'.$$

We set the minimal element $[1, 2, \ldots, n]$ to correspond to $(n, 0)$, which is consistent with the ordering we just defined. When we remove $[n+1, \ldots, 2n]$ (or the correponding pair $(0, n)$, we get that the maximal elements of $\left\{ \begin{array}{c} 2n \\ n \end{array} \right\} - [n+1, \ldots, 2n]$ correspond to the pairs $(1, n)$ and $(0, n-1)$. It is readily seen that $\left\{ \begin{array}{c} 2n \\ n \end{array} \right\} - [n+1, \ldots, 2n]$ is pure because all maximal chains starting from $[i, j]$ have length $n-i+j-1$. Therefore Δ_L is pure with length $2n-1$.

As for H, we can see that Δ_H is simply 2 copies of Δ_L. The partial order on $\Delta_H = \Delta_L^1 \cup \Delta_L^2$ is such that if $x \in \Delta_L^1$ and $x' \in \Delta_L^2$ and they project to the same element in Δ_L, then x is an immediate child of x'. This implies that the only join-irreducible element in Δ_L^2 is its minimal element. Therefore, the set of join-irreducible elements in H consists of the join-irreducible elements in L plus the element $\{([1, 2, \ldots, n], 1)\}$. H is not pure when $n > 1$ since the maximal chain starting from this extra element has length 1. ∎

Therefore, we have a negative answer for Gorensteinness.

PROPOSITION 24 *If $n > 1$, then A is not Gorenstein.*

5.3 An application: flatness of the moduli stack

After this extensive study of the ring A, we are ready to come back to the orginal local ring

$$\Lambda = W(k)[[\underline{\mathbf{X}}, \underline{\mathbf{Y}}]]/(\underline{\mathbf{XY}} - p, \underline{\mathbf{YX}} - p).$$

We have proved that $A = \Lambda/p\Lambda$ is Cohen-Macaulay, reduced. We also know A has exactly $n + 1$ irreducible components of equal dimension n^2. Going through the proof of [4, Proposition 7.1], we conclude that

THEOREM 25 *Λ is a domain, flat over $W(k)$, and Cohen-Macaulay. In particular, the stack $\mathcal{X}_n$ is flat over $\mathbf{Z}_p$.*

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