FAMILIES AND SPRINGER’S CORRESPONDENCE

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INTRODUCTION

0.1. Let $G$ be a connected reductive algebraic group over an algebraically closed field $k$ of characteristic $p$. Let $W$ be the Weyl group of $G$; let $\text{Irr}W$ be a set of representatives for the isomorphism classes of irreducible representations of $W$ over $\mathbf{Q}_l$, an algebraic closure of the field of $l$-adic numbers ($l$ is a fixed prime number $\neq p$).

Now $\text{Irr}W$ is partitioned into subsets called families as in [L1, Sec.9], [L3, 4.2]. Moreover to each family $\mathcal{F}$ in $\text{Irr}W$, a certain set $\mathbf{X}_\mathcal{F}$, a pairing $\{,\} : \mathbf{X}_\mathcal{F} \times \mathbf{X}_\mathcal{F} \rightarrow \mathbf{Q}_l$, and an imbedding $\mathcal{F} \rightarrow \mathbf{X}_\mathcal{F}$ was canonically attached in [L1],[L3, Ch.4]. (The set $\mathbf{X}_\mathcal{F}$ with the pairing $\{,\}$, which can be viewed as a nonabelian analogue of a symplectic vector space, plays a key role in the classification of unipotent representations of a finite Chevalley group [L3] and in that of unipotent character sheaves on $G$). In [L1],[L3] it is shown that $\mathbf{X}_\mathcal{F} = M(\mathcal{G}_\mathcal{F})$ where $\mathcal{G}_\mathcal{F}$ is a certain finite group associated to $\mathcal{F}$ and, for any finite group $\Gamma$, $M(\Gamma)$ is the set of all pairs $(g, \rho)$ where $g$ is an element of $\Gamma$ defined up to conjugacy and $\rho$ is an irreducible representation over $\mathbf{Q}_l$ (up to isomorphism) of the centralizer of $g$ in $\Gamma$; moreover $\{,\}$ is given by the “nonabelian Fourier transform matrix” of [L1, Sec.4] for $\mathcal{G}_\mathcal{F}$.

In the remainder of this paper we assume that $p$ is not a bad prime for $G$. In this case a uniform definition of the group $\mathcal{G}_\mathcal{F}$ was proposed in [L3, 13.1] in terms of special unipotent classes in $G$ and the Springer correspondence, but the fact that this leads to a group isomorphic to $\mathcal{G}_\mathcal{F}$ as defined in [L3, Ch.4] was stated in [L3, (13.1.3)] without proof. One of the aims of this paper is to supply the missing proof.

To state the results of this paper we need some definitions. For $E \in \text{Irr}W$ let $a_E \in \mathbf{N}, b_E \in \mathbf{N}$ be as in [L3, 4.1]. As noted in [L2], for $E \in \text{Irr}W$ we have

(a) $a_E \leq b_E$;

we say that $E$ is special if $a_E = b_E$.

For $g \in G$ let $Z_G(g)$ or $Z(g)$ be the centralizer of $g$ in $G$ and let $A_G(g)$ or $A(g)$ be the group of connected components of $Z(g)$. Let $C$ be a unipotent conjugacy class in $G$ and let $u \in C$. Let $\mathcal{B}_u$ be the variety of Borel subgroups of $G$ that contain

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a; this is a nonempty variety of dimension, say, $e_C$. The conjugation action of $Z(a)$ on $B_a$ induces an action of $A(a)$ on $S_a := H^{2e_C}(B_a, Q_l)$. Now $W$ acts on $S_a$ by Springer’s representation [Spr]; however here we adopt the definition of the $W$-action on $S_a$ given in [L4] which differs from Springer’s original definition by tensoring by sign. The $W$-action on $S_a$ commutes with the $A(a)$-action. Hence we have canonically $S_a = \bigoplus_{E \in \text{Irr}W} E \otimes V_E$ (as $W \times A(a)$-modules) where $V_E$ are finite dimensional $Q_l$-vector spaces with $A(a)$-action. Let $\text{Irr}_C W = \{ E \in \text{Irr}W; V_E \neq 0 \}$; this set does not depend on the choice of $u$ in $C$. By [Spr], the sets $\text{Irr}_C W$ (for $C$ variable) form a partition of $\text{Irr}W$; also, if $E \in \text{Irr}_C W$ then $V_E$ is an irreducible $A(a)$-module and, if $E \neq E'$ in $\text{Irr}_C W$, then the $A(a)$-modules $V_E, V_{E'}$ are not isomorphic. By [BM] we have

\[(b) \text{ } e_C \leq b_E \text{ for any } E \in \text{Irr}_C W\]

and the equality $b_E = e_C$ holds for exactly one $E \in \text{Irr}_C W$ which we denote by $E_C$ (for this $E$, $V_E$ is the unit representation of $A(a)$).

Following [L3, (13.1.1)] we say that $C$ is special if $E_C$ is special. (This concept was introduced in [L2, Sec.9] although the word “special” was not used there.) From (b) we see that $C$ is special if and only if $a_{E_C} = e_C$.

Now assume that $C$ is special. We denote by $\mathcal{F} \subset \text{Irr}W$ the family that contains $E_C$. (Note that $C \rightarrow \mathcal{F}$ is a bijection from the set of special unipotent classes in $G$ to the set of families in $\text{Irr}W$.) We set $\text{Irr}^*_C W = \{ E \in \text{Irr}_C W; E \in \mathcal{F} \}$ and

$$\mathcal{K}(u) = \{ a \in A(a); a \text{ acts trivially on } V_E \text{ for any } E \in \text{Irr}^*_C W \}.$$ 

This is a normal subgroup of $A(a)$. We set $\bar{A}(u) = A(a)/\mathcal{K}(u)$, a quotient group of $A(a)$. Now, for any $E \in \text{Irr}^*_C W$, $V_E$ is naturally an (irreducible) $\bar{A}(a)$-module. Another definition of $\bar{A}_u$ is given in [L3, (13.1.1)]. In that definition $\text{Irr}^*_C W$ is replaced by $\{ E \in \text{Irr}_C W; a_E = e_C \}$ and $\mathcal{K}(u), \bar{A}(u)$ are defined as above but in terms of this modified $\text{Irr}^*_C W$. However the two definitions are equivalent in view of the following result.

**Proposition 0.2.** Assume that $C$ is special. Let $E \in \text{Irr}_C W$.

\[(a) \text{ We have } a_E \leq e_C.\]

\[(b) \text{ We have } a_E = e_C \text{ if and only if } E \in \mathcal{F}.\]

This follows from [L8, 10.9]. Note that (a) was stated without proof in [L3, (13.1.2)] (the proof I had in mind at the time of [L3] was combinatorial).

**0.3.** The following result is equivalent to a result stated without proof in [L3, (13.1.3)].

**Theorem 0.4.** Let $C$ be a special unipotent class of $G$, let $u \in C$ and let $\mathcal{F}$ be the family that contains $E_C$. Then we have canonically $X_{\mathcal{F}} = M(\bar{A}(u))$ so that the pairing $\{,\}$ on $X_{\mathcal{F}}$ coincides with the pairing $\{,\}$ on $M(\bar{A}(u))$. Hence $G_{\mathcal{F}}$ can be taken to be $\bar{A}(u)$.

This is equivalent to the corresponding statement in the case where $G$ is adjoint, which reduces immediately to the case where $G$ is adjoint simple. It is then enough
to prove the theorem for one $G$ in each isogeny class of semisimple, almost simple algebraic groups; this will be done in §3 after some combinatorial preliminaries in §1, §2. The proof uses the explicit description of the Springer correspondence: for type $A_n, G_2$ in [Spr]; for type $B_n, C_n, D_n$ in [S1] (as an algorithm) and in [L4] (by a closed formula); for type $F_4$ in [S2]; for type $E_n$ in [AL], [Sp1].

An immediate consequence of (the proof of) Theorem 0.4 is the following result which answers a question of R. Bezrukavnikov.

**Corollary 0.5.** In the setup of 0.4 let $E \in \text{Irr}_C^\ast W$ and let $\mathcal{V}_E$ be the corresponding $A(u)$-module viewed as an (irreducible) $\overline{A(u)}$-module. The image of $E$ under the canonical imbedding $\mathcal{F} \to \mathbf{X}_\mathcal{F} = M(\overline{A(u)})$ is represented by the pair $(1, \mathcal{V}_E) \in M(\overline{A(u)})$. Conversely, if $E \in \mathcal{F}$ and the image of $E$ under $\mathcal{F} \to \mathbf{X}_\mathcal{F} = M(\overline{A(u)})$ is represented by the pair $(1, \rho) \in M(\overline{A(u)})$ where $\rho$ is an irreducible representation of $\overline{A(u)}$, then $E \in \text{Irr}_C^\ast W$ and $\rho \cong \mathcal{V}_E$.

**0.6.** Corollary 0.5 has the following interpretation. Let $Y$ be a (unipotent) character sheaf on $G$ whose restriction to the regular semisimple elements is $\neq 0$; assume that in the usual parametrization of unipotent character sheaves by $\cup_{\mathcal{F} \subseteq \mathcal{F}_x} \mathbf{X}_\mathcal{F}$, $Y$ corresponds to $(1, \rho) \in M(\overline{A(u)})$ where $C$ is the special unipotent class corresponding to a family $\mathcal{F}$, $u \in C$ and $\rho$ is an irreducible representation of $\overline{A(u)}$. Then $Y|_C$ is (up to shift) the irreducible local system on $C$ defined by $\rho$.

A parametrization of unipotent character sheaves on $G$ in terms of restrictions to various conjugacy classes of $G$ is outlined in §4.

**0.7.** **Notation.** If $A, B$ are subsets of $N$ we denote by $A \cup B$ the union of $A$ and $B$ regarded as a multiset (each element of $A \cap B$ appears twice). For any set $\mathcal{X}$, we denote by $\mathcal{P}(\mathcal{X})$ the set of subsets of $\mathcal{X}$ viewed as an $F_2$-vector space with sum given by the symmetric difference. If $\mathcal{X} \neq \emptyset$ we note that $\{\emptyset, \mathcal{X}\}$ is a line in $\mathcal{P}(\mathcal{X})$ and we set $\mathcal{P}_\emptyset(\mathcal{X}) = \mathcal{P}(\mathcal{X})/\{\emptyset, \mathcal{X}\}$, $\mathcal{P}_{\text{ev}}(\mathcal{X}) = \{L \in \mathcal{P}(\mathcal{X}); |L| = 0 \mod 2\}$; let $\mathcal{P}_{\text{ev}}(\mathcal{X})$ be the image of $\mathcal{P}_{\text{ev}}(\mathcal{X})$ under the obvious map $\mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ (thus $\mathcal{P}_{\text{ev}}(\mathcal{X}) = \mathcal{P}(\mathcal{X})$ if $|\mathcal{X}|$ is odd and $\mathcal{P}_{\text{ev}}(\mathcal{X})$ is a hyperplane in $\mathcal{P}(\mathcal{X})$ if $|\mathcal{X}|$ is even). Now if $\mathcal{X} \neq \emptyset$, the assignment $L, L' \mapsto |L \cap L'| \mod 2$ defines a symplectic form on $\mathcal{P}_{\text{ev}}(\mathcal{X})$ which induces a nondegenerate symplectic form $(,)$ on $\mathcal{P}_{\text{ev}}(\mathcal{X})$ via the obvious linear map $\mathcal{P}_{\text{ev}}(\mathcal{X}) \to \mathcal{P}_{\text{ev}}(\mathcal{X})$.

For $g \in G$ let $g_s$ (resp. $g_w$) be the semisimple (resp. unipotent) part of $g$.

For $z \in (1/2)\mathbf{Z}$ we set $[z] = z$ if $z \in \mathbf{Z}$ and $[z] = z - (1/2)$ if $z \in \mathbf{Z} + (1/2)$.

**Erratum to [L3].** On page 86, line -6 delete: ”$b' < b$” and on line -4 before ”In the language...” insert: ”The array above is regarded as identical to the array obtained by interchanging its two rows.”

On page 343, line -5, after ”respect to $M'$” insert: ”and where the group $\mathcal{G}_\mathcal{F}$ defined in terms of $(u', M)$ is isomorphic to the group $\mathcal{G}_\mathcal{F}$ defined in terms of $(u, G)$”.

**Erratum to [L4].** In the definition of $A_n, B_\alpha$ in [L4, 11.5], the condition $I \in \alpha$ should be replaced by $I \in \alpha'$ and the condition $I \in \alpha$ should be replaced by $I \in \alpha$. 


1. Combinatorics

1.1. Let $N$ be an even integer $\geq 0$. Let $a := (a_0, a_1, a_2, \ldots, a_N) \in \mathbb{N}^{N+1}$ be such that $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_N$, $a_0 < a_2 < a_4 < \ldots$, $a_1 < a_3 < a_5 < \ldots$. Let $\mathcal{J} = \{i \in [0, N]; a_i$ appears exactly once in $a\}$. We have $\mathcal{J} = \{i_0, i_1, \ldots, i_{2M}\}$ where $M \in \mathbb{N}$ and $i_0 < i_1 < \cdots < i_{2M}$ satisfy $i_s = i \mod 2$ for $s \in [0, 2M]$. Hence for any $s \in [0, 2M - 1]$ we have $i_{s+1} = i_s + 2m_s + 1$ for some $m_s \in \mathbb{N}$. Let $\mathcal{E}$ be the set of $b := (b_0, b_1, b_2, \ldots, b_N) \in \mathbb{N}^{N+1}$ such that $b_0 < b_2 < b_4 < \ldots$, $b_1 < b_3 < b_5 < \ldots$ and such that $[b] = [a]$ (we denote by $[b], [a]$ the multisets $\{b_0, b_1, \ldots, b_N\}, \{a_0, a_1, \ldots, a_N\}$). We have $a \in \mathcal{E}$. For $b \in \mathcal{E}$ we set

\[ \hat{b} = (\hat{b}_0, \hat{b}_1, \hat{b}_2, \ldots, \hat{b}_N) = (b_0, b_1+1, b_2+1, b_3+2, b_4+2, \ldots, b_{N-1}+(N/2), b_N+(N/2)). \]

Let $[\hat{b}]$ be the multiset $\{\hat{b}_0, \hat{b}_1, \hat{b}_2, \ldots, \hat{b}_N\}$. For $s \in \{1, 3, \ldots, 2M-1\}$ we define $a_s = (a_0^{[s]}, a_1^{[s]}, a_2^{[s]}, \ldots, a_N^{[s]}) \in \mathcal{E}$ by

\[ (a_0^{[s]}, a_1^{[s]}, a_2^{[s]}, \ldots, a_N^{[s]}) = (a_{i_0}, a_{i_1}, a_{i_2}, a_{i_3}+1, \ldots, a_{i_s}, a_{i_s+1}, \ldots, a_{i_s+2m_s+1}, a_{i_s+2m_s+1}) \]

and $a_i^{[s]} = a_i$ if $i \in [0, N] - [i_s, i_{s+1}]$. More generally for $X \subset \{1, 3, \ldots, 2M-1\}$ we define $a_X = (a_0^X, a_1^X, a_2^X, \ldots, a_N^X) \in \mathcal{E}$ by $a_i^X = a_i^{[s]}$ if $s \in X$, $i \in [i_s, i_{s+1}]$, and $a_i^X = a_i$ for all other $i \in [0, N]$. Note that $[a^X] = [\hat{a}]$. Conversely, we have the following result.

**Lemma 1.2.** Let $b \in \mathcal{E}$ be such that $[\hat{b}] = [\hat{a}]$. There exists $X \subset \{1, 3, \ldots, 2M-1\}$ such that $b = a^X$.

The proof is given in 1.3-1.5.

1.3. We argue by induction on $M$. We have

\[ a = (y_1 = y_1 < y_2 = y_2 < \cdots < y_{r'} = y_{r'} < a_{i_0} < \ldots) \]

for some $r$. Since $[b] = [a]$, we must have

\[ (b_0, b_2, b_4, \ldots) = (y_1, y_2, \ldots, y_{r}, \ldots), (b_1, b_3, b_5, \ldots) = (y_1, y_2, \ldots, y_{r}, \ldots). \]

Thus,

\[ (a) \ b_i = a_i \text{ for } i < i_0. \]

We have $a = (\cdots > a_{2M} > y_1' = y_1' < y_2' = y_2' < \cdots < y_{r'}' = y_{r}'')$ for some $r'$. Since $[b] = [a]$, we must have

\[ (b_0, b_2, b_4, \ldots) = (\cdots, y_1', y_2', \ldots, y_{r}''), (b_1, b_3, b_5, \ldots) = (\cdots, y_1', y_2', \ldots, y_{r}''). \]

Thus,

\[ (b) \ b_i = a_i \text{ for } i > i_{2M}. \]

If $M = 0$ we see that $b = a$ and there is nothing further to prove. In the rest of the proof we assume that $M \geq 1$. 

1.4. From 1.3 we see that
\[
(a_0, a_1, a_2, \ldots, a_{i_{2M}}) = (\ldots, a_{i_{2M} - 1} < x_1 = x_2 = x_2 \cdots < x_q = x_q < a_{i_{2M}})
\]
(for some \(q\)) has the same entries as \((b_0, b_1, b_2, \ldots, b_{i_{2M}})\) (in some order). Hence the pair
\[
(\ldots, b_{i_{2M} - 5}, b_{i_{2M} - 3}, b_{i_{2M} - 1}), (\ldots, b_{i_{2M} - 4}, b_{i_{2M} - 2}, b_{i_{2M}})
\]
must have one of the following four forms.

\[
(\ldots, a_{i_{2M} - 1}, x_1, x_2, \ldots, x_q), (\ldots, x_1, x_2, \ldots, x_q, a_{i_{2M}}),
\]

\[
(\ldots, x_1, x_2, \ldots, x_q, a_{i_{2M}}), (\ldots, a_{i_{2M} - 1}, x_1, x_2, \ldots, x_q),
\]

\[
(\ldots, a_{i_{2M} - 1}, x_1, x_2, \ldots, x_q).
\]

Hence \((\ldots, b_{i_{2M} - 2}, b_{i_{2M} - 1}, b_{i_{2M}})\) must have one of the following four forms.

(I) \((\ldots, a_{i_{2M} - 1}, x_1, x_2, x_2, \ldots, x_q, x_q, a_{i_{2M}})\),

(II) \((\ldots, x_1, x_{i_{2M} - 1}, x_2, x_1, x_3, x_2, \ldots, x_q, x_q - 1, a_{i_{2M}}, x_q)\),

(III) \((\ldots, a_{i_{2M} - 1}, z, x_1, x_2, x_2, \ldots, x_q, x_q, a_{i_{2M}})\),

(IV) \((\ldots, a_{i_{2M} - 1}, x_1, x_1, x'_2, x_1, x_3, x_2, \ldots, x_q, x_q - 1, a_{i_{2M}}, x_q)\),

where \(a_{i_{2M} - 1} > z, a_{i_{2M} - 1} > z'' > z'\) and all entries in \(\ldots\) are \(< a_{i_{2M} - 1}\). Correspondingly, \((\ldots, \hat{b}_{i_{2M} - 2}, \hat{b}_{i_{2M} - 1}, \hat{b}_{i_{2M}})\) must have one of the following four forms.

(I) \((\ldots, a_{i_{2M} - 1} + h - q, x_1 + h - q, x_1 + h - q + 1, x_2 + h - q + 1, x_2 + h - q + 2, \ldots, x_q + h - 1, x_q + h, a_{i_{2M} + h})\),

(II) \((\ldots, x_1 + h - q, a_{i_{2M} - 1} + h - q, x_2 + h - q + 1, x_1 + h - q + 1, x_3 + h - q + 2, x_2 + h - q + 1, \ldots, x_q + h - 1, x_q - 1 + h - 1, a_{i_{2M} + h}, x_q + h)\),

(III) \((\ldots, a_{i_{2M} - 1} + h - q - 1, z + h - q, x_1 + h - q, x_1 + h - q + 1, x_2 + h - q + 1, x_2 + h - q + 2, \ldots, x_q + h - 1, x_q + h, a_{i_{2M} + h})\),

(IV) \((\ldots, a_{i_{2M} - 1} + h - q - 1, z' + h - q - 1, x_1 + h - q, z'' + h - q, x_2 + h - q + 1, a_{i_{2M} + h}, x_q + h)\)

where \(h = i_{2M}/2\) and in case (III) and (IV), \(a_{i_{2M} - 1} + h - q\) is not an entry of \((\ldots, \hat{b}_{i_{2M} - 2}, \hat{b}_{i_{2M} - 1}, \hat{b}_{i_{2M}})\).

Since \((\ldots, \hat{a}_{i_{2M} - 2}, \hat{a}_{i_{2M} - 1}, \hat{a}_{i_{2M}})\) is given by (I) we see that \(a_{i_{2M} - 1} + h - q\) is an entry of \((\ldots, \hat{a}_{i_{2M} - 2}, \hat{a}_{i_{2M} - 1}, \hat{a}_{i_{2M}})\). Using 1.3(b) we see that

\[
\{\ldots, \hat{a}_{i_{2M} - 2}, \hat{a}_{i_{2M} - 1}, \hat{a}_{i_{2M}}\} = (\ldots, b_{i_{2M} - 2}, b_{i_{2M} - 1}, b_{i_{2M}})
\]
as multisets. We see that cases (III) and (IV) cannot arise. Hence we must be in case (I) or (II). Thus

we have either

\[
(b_{i_{2M} - 1}, b_{i_{2M} - 1} + 1, \ldots, b_{i_{2M} - 2}, b_{i_{2M} - 1}, b_{i_{2M}}) = (a_{i_{2M} - 1}, a_{i_{2M} - 1} + 1, \ldots, a_{i_{2M} - 2}, a_{i_{2M} - 1}, a_{i_{2M}})
\]

or

\[
(b_{i_{2M} - 1}, b_{i_{2M} - 1} + 1, \ldots, b_{i_{2M} - 2}, b_{i_{2M} - 1}, b_{i_{2M}}) = (a_{i_{2M} - 1} + 1, a_{i_{2M} - 1}, a_{i_{2M} - 1} + 3, a_{i_{2M} - 1} + 2, \ldots, a_{i_{2M}}, a_{i_{2M} - 1}).
\]
1.5. Let $a' = (a_0, a_1, a_2, \ldots, a_{i_2M-1})$, $b' = (b_0, b_1, b_2, \ldots, b_{i_2M-1})$, $\hat{a}' = (a_0, a_1 + 1, a_2 + 1, a_3 + 2, a_4 + 2, \ldots, a_{i_2M-1} + (i_2M-1)/2)$, $\hat{b}' = (b_0, b_1 + 1, b_2 + 1, b_3 + 2, b_4 + 2, \ldots, b_{i_2M-1} + (i_2M-1)/2)$.

From $[\hat{b}] = [\hat{a}]$ and 1.3(b), 1.4(a),(b) we see that the multiset formed by the entries of $\hat{a}'$ coincides with the multiset formed by the entries of $\hat{b}'$. Using the induction hypothesis we see that there exists $X' \subseteq \{1, 3, \ldots, 2M-3\}$ such that $b' = a'X'$ where $a'X'$ is defined in terms of $a', X'$ in the same way as $aX$ was defined (see 1.1) in terms of $a, X$. We set $X = X'$ if we are in case 1.4(a) and $X = X' \cup \{2M-1\}$ if we are in case 1.4(b). Then we have $b = aX$ (see 1.4(a),(b)), as required. This completes the proof of Lemma 1.2.

1.6. We shall use the notation of 1.1. Let $\mathcal{I}$ be the set of all unordered pairs $(\mathcal{A}, \mathcal{B})$ where $\mathcal{A}, \mathcal{B}$ are subsets of $\{0, 1, 2, \ldots\}$ and $\mathcal{A} \cup \mathcal{B} = (a_0, a_1, a_2, \ldots, a_N)$ as multisets. For example, setting $\mathcal{A}_0 = (a_0, a_2, a_4, \ldots, a_N), \mathcal{B}_0 = (a_1, a_3, \ldots, a_{N-1})$, we have $(\mathcal{A}_0, \mathcal{B}_0) \in \mathcal{I}$. For any subset $a$ of $\mathcal{J}$ we consider

$$\mathcal{A}_a = ((\mathcal{J} - a) \cap \mathcal{A}_0) \cup (a \cap \mathcal{B}_0) \cup (\mathcal{A}_0 \cap \mathcal{B}_0),$$

$$\mathcal{B}_a = ((\mathcal{J} - a) \cap \mathcal{B}_0) \cup (a \cap \mathcal{A}_0) \cup (\mathcal{A}_0 \cap \mathcal{B}_0).$$

Then $(\mathcal{A}_a, \mathcal{B}_a) \in \mathcal{I}$ and the map $a \mapsto (\mathcal{A}_a, \mathcal{B}_a)$ induces a bijection $\hat{\mathcal{P}}(\mathcal{J}) \leftrightarrow \mathcal{I}$. (Note that if $a = \emptyset$ then $(\mathcal{A}_a, \mathcal{B}_a)$ agrees with the earlier definition of $(\mathcal{A}_0, \mathcal{B}_0)$.)

Let $\mathcal{I}'$ be the set of all $(\mathcal{A}, \mathcal{B}) \in \mathcal{I}$ such that $|\mathcal{A}| = |\mathcal{A}_0|, |\mathcal{B}| = |\mathcal{B}_0|$.

Let $\mathcal{P}(\mathcal{J})_0$ be the subspace of $\mathcal{P}_{ev}(\mathcal{J})$ spanned by the 2-element subsets

$$\{a_{i_0}, a_{i_1}\}, \{a_{i_2}, a_{i_3}\}, \ldots, \{a_{i_{2M-2}}, a_{i_{2M-1}}\}$$

of $\mathcal{J}$. Let $\mathcal{P}(\mathcal{J})_1$ be the subspace of $\mathcal{P}_{ev}(\mathcal{J})$ spanned by the 2-element subsets

$$\{a_{i_1}, a_{i_2}\}, \{a_{i_3}, a_{i_4}\}, \ldots, \{a_{i_{2M-1}}, a_{i_{2M}}\}$$

of $\mathcal{J}$.

Let $\hat{\mathcal{P}}(\mathcal{J})_0$ (resp. $\hat{\mathcal{P}}(\mathcal{J})_1$) be the image of $\mathcal{P}(\mathcal{J})_0$ (resp. $\mathcal{P}(\mathcal{J})_1$) under the obvious map $\mathcal{P}(\mathcal{J}) \rightarrow \hat{\mathcal{P}}(\mathcal{J})$. Note that

(a) $\hat{\mathcal{P}}(\mathcal{J})_0$ and $\hat{\mathcal{P}}(\mathcal{J})_1$ are opposed Lagrangian subspaces of the symplectic vector space $\hat{\mathcal{P}}(\mathcal{J}), (,)$, (see 0.7); hence $(,)$ defines an identification $\hat{\mathcal{P}}(\mathcal{J})_0 = \hat{\mathcal{P}}(\mathcal{J})_1^*$ where $\hat{\mathcal{P}}(\mathcal{J})_1^*$ is the vector space dual to $\hat{\mathcal{P}}(\mathcal{J})_1$.

Let $\mathcal{I}_0$ (resp. $\mathcal{I}_1$) be the subset of $\mathcal{I}$ corresponding to $\hat{\mathcal{P}}(\mathcal{J})_0$ (resp. $\hat{\mathcal{P}}(\mathcal{J})_1$) under the bijection $\hat{\mathcal{P}}(\mathcal{J}) \leftrightarrow \mathcal{I}$. Note that $\mathcal{I}_0 \subset \mathcal{I}', \mathcal{I}_1 \subset \mathcal{I}', |\mathcal{I}_0| = |\mathcal{I}_1| = 2^M$.

For any $X \subset \{1, 3, \ldots, 2M-1\}$ we set $a_X = \bigcup_{s \in X} \{a_{i_s}, a_{i_{s+1}}\} \in \mathcal{P}(\mathcal{J})$. Then $(\mathcal{A}_{a_X}, \mathcal{B}_{a_X}) \in \mathcal{I}_1$ is related to $a^X$ in 1.1 as follows:

$$\mathcal{A}_{a_X} = \{a_0^X, a_2^X, a_4^X, \ldots, a_N^X\}, \mathcal{B}_{a_X} = \{a_1^X, a_3^X, \ldots, a_{N-1}^X\}.$$
1.7. We shall use the notation of 1.1. Let $T$ be the set of all ordered pairs $(A, B)$ where $A$ is a subset of $\{0, 1, 2, \ldots\}$, $B$ is a subset of $\{1, 2, 3, \ldots\}$, $A$ contains no consecutive integers, $B$ contains no consecutive integers, and $A \cup B = (\hat{a}_0, \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N)$ as multisets. For example, setting $A_\emptyset = (\hat{a}_0, \hat{a}_2, \hat{a}_4, \ldots, \hat{a}_N)$, $B_\emptyset = (\hat{a}_1, \hat{a}_3, \ldots, \hat{a}_{N-1})$, we have $(A_\emptyset, B_\emptyset) \in T$.

For any $(A, B) \in T$ we define $(A^-, B^-)$ as follows: $A^-$ consists of $x_0 < x_1 - 1 < x_2 - 2 < \cdots < x_p - p$ where $x_0 < x_1 < \cdots < x_p$ are the elements of $A$; $B^-$ consists of $y_1 - 1 < y_2 - 2 < \cdots < y_q - q$ where $y_1 < y_2 < \cdots < y_q$ are the elements of $B$.

We can enumerate the elements of $T$ as in [L4, 11.5]. Let $J$ be the set of all $c \in \mathbb{N}$ such that $c$ appears exactly once in the sequence $(\hat{a}_0, \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N) = (a_0, a_1+1, a_2+1, a_3+2, a_4+2, \ldots, a_{N-1}+(N/2), a_N+(N/2))$.

A nonempty subset $I$ of $J$ is said to be an interval if it is of the form $\{i, i+1, i+2, \ldots, j\}$ with $i - 1 \notin J$, $j + 1 \notin J$ and with $i \neq 0$. Let $\mathcal{I}$ be the set of intervals of $J$. For any $s \in \{1, 3, \ldots, 2M - 1\}$, the set $I_s := \{\hat{a}_i, \hat{a}_{i+1}, \hat{a}_{i+2}, \ldots, \hat{a}_{i+2m_s+1}\}$ is either a single interval or a union of intervals $I_s^1 \sqcup I_s^2 \sqcup \ldots \sqcup I_s^t$ ($t_s \geq 2$) where $\hat{a}_{i_s} \in I_s^1$, $\hat{a}_{i_s+2m_s+1} \in I_s^t$, $|I_s^1|, |I_s^t|$ are odd, $|I_s^h|$ are even for $h \in [2, t_s - 1]$ and any element in $I_s^e$ is $< \alpha$ than any element in $I_s^{e'}$ for $e < e'$. Let $\mathcal{I}_s$ be the set of all $I \in \mathcal{I}$ such that $I \subset I_s$. We have a partition $\mathcal{I} = \sqcup_{s \in \{1, 3, \ldots, 2M - 1\}} \mathcal{I}_s$. Let $H$ be the set of elements of $c \in J$ such that $c < a_{i_1}$ (that is such that $c$ does not belong to any interval). For any subset $\alpha \subset \mathcal{I}$ we consider

$$A_\alpha = \bigcup_{I \in \mathcal{I} - \alpha} (I \cap A_\emptyset) \cup \bigcup_{I \in \alpha} (I \cap B_\emptyset) \cup (H \cap A_\emptyset) \cup (A_\emptyset \cap B_\emptyset),$$

$$B_\alpha = \bigcup_{I \in \mathcal{I} - \alpha} (I \cap B_\emptyset) \cup \bigcup_{I \in \alpha} (I \cap A_\emptyset) \cup (H \cap B_\emptyset) \cup (A_\emptyset \cap B_\emptyset).$$

Then $(A_\alpha, B_\alpha) \in T$ and the map $\alpha \mapsto (A_\alpha, B_\alpha)$ is a bijection $\mathcal{P}(\mathcal{I}) \leftrightarrow T$. (Note that if $\alpha = \emptyset$ then $(A_\alpha, B_\alpha)$ agrees with the earlier definition of $(A_\emptyset, B_\emptyset)$.)

Let $T' = \{(A, B) \in T; |A| = |A_\emptyset|, |B| = |B_\emptyset|\}$, $T_1 = \{(A, B) \in T'; A^- \cup B^- = A^- \cup B^-\}$. Let $\mathcal{P}(\mathcal{I})'$ (resp. $\mathcal{P}(\mathcal{I}_1)$) be the subset of $\mathcal{P}(\mathcal{I})$ corresponding to $T'$ (resp. $T_1$) under the bijection $\mathcal{P}(\mathcal{I}) \leftrightarrow T$.

Now let $X$ be a subset of $\{1, 3, \ldots, 2M - 1\}$. Let $\alpha_X = \sqcup_{s \in X} \mathcal{I}_s \in \mathcal{P}(\mathcal{I})$. From the definitions we see that

(a) $A^-_{\alpha_X} = \mathfrak{A}_{\alpha_X}, B^-_{\alpha_X} = \mathfrak{B}_{\alpha_X}$

(b) $|T_1| = 2^M$ and $T_1$ consists of the pairs $(A_\alpha, B_\alpha)$ with $X \subset \{1, 3, \ldots, 2M - 1\}$.

Using (a),(b) we deduce:

(c) The map $T_1 \to \mathcal{I}_1$ given by $(A, B) \mapsto (A^-, B^-)$ is a bijection.

2. Combinatorics (continued)

2.1. Let $N \in \mathbb{N}$. Let

$$a := (a_0, a_1, a_2, \ldots, a_N) \in \mathbb{N}^{N+1}$$
be such that \( a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_N \), \( a_0 < a_2 < a_4 < \ldots \), \( a_1 < a_3 < a_5 < \ldots \) and such that the set \( \mathcal{J} := \{ i \in [0, N]; a_i \text{ appears exactly once in } a \} \) is nonempty. Now \( \mathcal{J} \) consists of \( \mu + 1 \) elements \( i_0 < i_1 < \ldots < i_\mu \) where \( \mu \in \mathbb{N}, \mu = N \mod 2 \). We have \( i_s = s \mod 2 \) for \( s \in [0, \mu] \). Hence for any \( s \in [0, \mu - 1] \) we have \( i_{s+1} = i_s + 2m_s + 1 \) for some \( m_s \in \mathbb{N} \). Let \( \mathcal{E} \) be the set of \( b := (b_0, b_1, b_2, \ldots, b_N) \in \mathbb{N}^{N+1} \) such that \( b_0 < b_2 < b_4 < \ldots, b_1 < b_3 < b_5 < \ldots \) and such that \( [b] = [a] \) (we denote by \([b], [a]\) the multisets \( \{b_0, b_1, \ldots, b_N\}, \{a_0, a_1, \ldots, a_N\} \)). We have \( a \in \mathcal{E} \). For \( b \in \mathcal{E} \) we set

\[
\tilde{b} = (\tilde{b}_0, b_1, \tilde{b}_2, \ldots, \tilde{b}_N) = (b_0, b_1, b_2 + 1, b_3 + 1, b_4 + 2, b_5 + 2, \ldots) \in \mathbb{N}^{N+1}.
\]

Let \( [\tilde{b}] \) be the multiset \( \{\tilde{b}_0, b_1, \tilde{b}_2, \ldots, \tilde{b}_N\} \). For any \( s \in [0, \mu - 1] \in 2\mathbb{N} \) we define \( a^{(s)} = (a_0^{(s)}, a_1^{(s)}, a_2^{(s)}, \ldots, a_N^{(s)}) \in \mathcal{E} \) by

\[
(a_i^{(s)}, a_{i+s+1}^{(s)}, a_{i+s+2}^{(s)}, a_{i+s+3}^{(s)}, \ldots, a_{i+s+2m_s}^{(s)}, a_{i+s+2m_s+1}^{(s)}) = (a_{i+1}, a_i, a_{i+3}, a_{i+2}, \ldots, a_{i+2m_s+1}, a_{i+2m_s})
\]

and \( a_i^{(s)} = a_i \) if \( i \in [0, N] - [i_s, i_{s+1}] \). More generally for a subset \( X \) of \([0, \mu - 1] \cap 2\mathbb{N} \) we define \( a^X = (a_0^X, a_1^X, a_2^X, \ldots, a_N^X) \in \mathcal{E} \) by \( a_i^X = a_i^{(s)} \) if \( s \in X \), \( i \in [i_s, i_{s+1}] \), and \( a_i^X = a_i \) for all other \( i \in [0, N] \). Note that \([\tilde{a}^X] = [\tilde{a}] \). Conversely, we have the following result.

**Lemma 2.2.** Let \( b \in \mathcal{E} \) be such that \([\tilde{b}] = [\tilde{a}] \). Then there exists \( X \subset [0, \mu - 1] \cap 2\mathbb{N} \) such that \( b = a^X \).

The proof is given in 2.3-2.5.

**2.3.** We argue by induction on \( \mu \). By the argument in 1.3 we have

(a) \( b_i = a_i \) for \( i < i_0 \),

(b) \( b_i = a_i \) for \( i > i_\mu \).

If \( \mu = 0 \) we see that \( b = a \) and there is nothing further to prove. In the rest of the proof we assume that \( \mu \geq 1 \).

**2.4.** From 2.3 we see that \((a_{i_0}, a_{i_0+1}, \ldots, a_N) = (a_{i_0} < x_1 = x_1 < x_2 = x_2 < \cdots < x_p = x_p < a_{i_1} < \ldots) \) (for some \( p \)) has the same entries as \((b_{i_0}, b_{i_0+1}, \ldots, b_N) \) (in some order). Hence the pair \((b_{i_0}, b_{i_0+2}, b_{i_0+4}, \ldots), (b_{i_0+1}, b_{i_0+3}, b_{i_0+5}, \ldots) \) must have one of the following four forms.

\[
(a_{i_0}, x_1, x_2, \ldots, x_p, \ldots), (x_1, x_2, \ldots, x_p, a_{i_1}, \ldots), (x_1, x_2, \ldots, x_p, a_{i_1}, \ldots), (a_{i_0}, x_1, x_2, \ldots, x_p, a_{i_1}, \ldots) \),
\]

\[
(x_1, x_2, \ldots, x_p, a_{i_1}, \ldots), (a_{i_0}, x_1, x_2, \ldots, x_p, a_{i_1}, \ldots), (a_{i_0}, x_1, x_2, \ldots, x_p, a_{i_1}, \ldots), (x_1, x_2, \ldots, x_p, a_{i_1}, \ldots) \).
\]

Hence \((b_{i_0}, b_{i_0+1}, b_{i_0+2}, \ldots, b_N) \) must have one of the following four forms.

(I) \((a_{i_0}, x_1, x_1, x_2, \ldots, x_p, x_p, a_{i_1}, \ldots) \),
(II) \((x_1, a_{i_0}, x_2, x_1, x_3, x_2, \ldots, x_p, x_{p-1}, a_{i_1}, x_p, \ldots)\),

(III) \((a_{i_0}, x_1, x_2, x_2, \ldots, x_p, x_p, z, a_{i_1}, \ldots)\),

(IV) \((x_1, a_{i_0}, x_2, x_1, x_3, x_2, \ldots, x_p, x_{p-1}, z', x_p, z'', a_{i_1}, \ldots)\)

where \(a_{i_1} < z\), \(a_{i_1} < z' \leq z''\) and all entries in \(\ldots\) are \(> a_{i_1}\). Correspondingly,

\((b_{i_0}, b_{i_0+1}, b_{i_0+2}, \ldots, b_N)\) must have one of the following four forms.

(I) \((a_{i_0} + h, x_1 + h, x_1 + h + 1, x_2 + h + 1, x_2 + h + 2, \ldots, x_p + h + p - 1, x_p + h + p, a_{i_1} + h + p, \ldots)\),

(II) \((x_1 + h, a_{i_0} + h, x_2 + h + 1, x_1 + h + 1, x_3 + h + 2, x_2 + h + 2, \ldots, x_p + h + p - 1, x_p + h + p - 1, a_{i_1} + h + p, x_p + h + p, \ldots)\),

(III) \((a_{i_0} + h, x_1 + h, x_1 + h + 1, x_2 + h + 1, x_2 + h + 2, \ldots, x_p + h + p - 1, x_p + h + p, z + p, a_{i_1} + h + p + 1, \ldots)\),

\((x_1 + h, a_{i_0} + h, x_2 + h + 1, x_1 + h + 1, x_3 + h + 2, x_2 + h + 2, \ldots, x_p + h + p - 1, x_p + h + p - 1, x_p + h + p - 1, x_p + h + p - 1, a_{i_1} + h + p + 1, \ldots)\)

where \(h = i_0/2\) and in case (III) and (IV) \(a_{i_1} + h + p\) is not an entry of

\((b_{i_0}, b_{i_0+1}, b_{i_0+2}, \ldots)\).

Since \((a_{i_0}, a_{i_0+1}, a_{i_0+2}, \ldots)\) is given by (I) we see that \(a_{i_1} + h + p\) is an entry of \((a_{i_0}, a_{i_0+1}, a_{i_0+2}, \ldots)\). Using 2.3 we see that

\[\{\tilde{a}_{i_0}, \tilde{a}_{i_0+1}, \tilde{a}_{i_0+2}, \ldots\} = \{\tilde{b}_{i_0}, \tilde{b}_{i_0+1}, \tilde{b}_{i_0+2}, \ldots\}\]

as multisets. We see that cases (III) and (IV) cannot arise. Hence we must be in case (I) or (II). Thus

we have either

(a) \((b_{i_0}, b_{i_0+1}, b_{i_0+2}, \ldots, b_{i_1}) = (a_{i_0}, a_{i_0+1}, a_{i_0+2}, \ldots, a_{i_1})\)

or

(b) \((b_{i_0}, b_{i_0+1}, b_{i_0+2}, \ldots, b_{i_1}) = (a_{i_0+1}, a_{i_0}, a_{i_0+3}, a_{i_0+2}, \ldots, a_{i_1}, a_{i_1-1})\).

. From 2.3 and (a),(b) we see that if \(\mu = 1\) then Lemma 2.2 holds. Thus in the rest of the proof we can assume that \(\mu \geq 2\).

2.5. Let \(a' = (a_{i_1+1}, a_{i_1+2}, \ldots, a_N)\), \(b' = (b_{i_1+1}, b_{i_1+2}, \ldots, b_N)\),

\[\tilde{a}' = (a_{i_1+1}, a_{i_1+2}, a_{i_1+3} + 1, a_{i_1+4} + 1, a_{i_1+5} + 2, a_{i_1+6} + 2, \ldots),\]

\[\tilde{b}' = (b_{i_1+1}, b_{i_1+2}, b_{i_1+3} + 1, b_{i_1+4} + 1, b_{i_1+5} + 2, b_{i_1+6} + 2, \ldots).\]
From \( \hat{b} = [\hat{a}] \) and 2.3(a), 2.4(a), (b) we see that the multiset formed by the entries of \( \hat{a}' \) coincides with the multiset formed by the entries of \( b' \). Using the induction hypothesis we see that there exists \( X' \subset [2, \mu - 1] \cap 2\mathbb{N} \) such that \( b' = a'_{X'} \) where \( a'_{X'} \) is defined in terms of \( a', X' \) in the same way as \( a^X \) (see 2.1) was defined in terms of \( a, X \). We set \( X = X' \) if we are in case 2.4(a) and \( X = \{0\} \cup X' \) if we are in case 2.4(b). Then we have \( b = a^X \) (see 2.4(a), (b)), as required. This completes the proof of Lemma 2.2.

2.6. We shall use the notation of 2.1. Let \( \mathcal{X} \) be the set of all unordered pairs \((\mathcal{A}, \mathcal{B})\) where \( \mathcal{A}, \mathcal{B} \) are subsets of \( \{0, 1, 2, \ldots\} \) and \( \mathcal{A} \cup \mathcal{B} = \{a_0, a_1, a_2, \ldots, a_N\} \) as multisets. For example, setting \( \mathcal{A}_0 = \{a_i; i \in [0, N] \cap 2\mathbb{N}\}, \mathcal{B}_0 = \{a_i; i \in [0, N] \cap (2\mathbb{N} + 1)\} \), we have \((\mathcal{A}_0, \mathcal{B}_0) \in \mathcal{X}\). For any subset \( a \) of \( J \) we consider

\[
\mathcal{A}_a = ((J - a) \cap \mathcal{A}_0) \cup (a \cap \mathcal{B}_0) \cup (\mathcal{A}_0 \cap \mathcal{B}_0),
\]

\[
\mathcal{B}_a = ((J - a) \cap \mathcal{B}_0) \cup (a \cap \mathcal{A}_0) \cup (\mathcal{A}_0 \cap \mathcal{B}_0).
\]

Then \((\mathcal{A}_a, \mathcal{B}_a) = (\mathcal{A}_{J-a}, \mathcal{B}_{J-a}) \in \mathcal{X}\) and the map \( a \mapsto (\mathcal{A}_a, \mathcal{B}_a) \) induces a bijection \( \tilde{\mathcal{P}}(J) \leftrightarrow \mathcal{X} \). (Note that if \( a = \emptyset \) then \((\mathcal{A}_a, \mathcal{B}_a)\) agrees with the earlier definition of \((\mathcal{A}_\emptyset, \mathcal{B}_\emptyset)\).

Let \( \mathcal{X}' \) be the set of all \((\mathcal{A}, \mathcal{B})\) in \( \mathcal{X}\) such that \( |\mathcal{A}| = |\mathcal{A}_0|, |\mathcal{B}| = |\mathcal{B}_0|\). Let \( \mathcal{P}(J)_1 \) be the subspace of \( \mathcal{P}(J) \) spanned by the following 2-element subsets of \( J \):

\[
\{a_{i_1}, a_{i_2}\}, \{a_{i_3}, a_{i_4}\}, \ldots, \{a_{i_{\mu-2}}, a_{i_{\mu-1}}\} \text{ (if } N \text{ is odd)}
\]
or

\[
\{a_{i_1}, a_{i_2}\}, \{a_{i_3}, a_{i_4}\}, \ldots, \{a_{i_{\mu-1}}, a_{i_{\mu}}\} \text{ (if } N \text{ is even)}
\]

Let \( \mathcal{P}(J)_0 \) be the subspace of \( \mathcal{P}(J) \) spanned by the following 2-element subsets of \( J \):

\[
\{a_{i_0}, a_{i_1}\}, \{a_{i_2}, a_{i_3}\}, \ldots, \{a_{i_{\mu-1}}, a_{i_{\mu}}\} \text{ (if } N \text{ is odd)}
\]
or

\[
\{a_{i_0}, a_{i_1}\}, \{a_{i_2}, a_{i_3}\}, \ldots, \{a_{i_{\mu-2}}, a_{i_{\mu-1}}\} \text{ (if } N \text{ is even)}
\]

Let \( \tilde{\mathcal{P}}(J)_0 \) (resp. \( \tilde{\mathcal{P}}(J)_1 \)) be the image of \( \mathcal{P}(J)_0 \) (resp. \( \mathcal{P}(J)_1 \)) under the obvious map \( \mathcal{P}(J) \to \tilde{\mathcal{P}}(J) \).

Note that

(a) \( \tilde{\mathcal{P}}(J)_0 \) and \( \tilde{\mathcal{P}}(J)_1 \) are opposed Lagrangian subspaces of the symplectic vector space \( \mathcal{P}_{ev}(J), (, ) \), (see 0.7); hence \((, )\) defines an identification \( \tilde{\mathcal{P}}(J)_1 = \mathcal{P}(J)_0^* \)

where \( \mathcal{P}(J)_0^* \) is the vector space dual to \( \tilde{\mathcal{P}}(J)_0 \).

Let \( \mathcal{X}_0 \) (resp. \( \mathcal{X}_1 \)) be the subset of \( \mathcal{X}\) corresponding to \( \tilde{\mathcal{P}}(J)_0 \) (resp. \( \tilde{\mathcal{P}}(J)_1 \)) under the bijection \( \tilde{\mathcal{P}}(J) \leftrightarrow \mathcal{X} \). Note that \( \mathcal{X}_0 \subset \mathcal{X}', \mathcal{X}_1 \subset \mathcal{X}' \), \( |\mathcal{X}_0| = |\mathcal{X}_1| = 2^{\lfloor \mu/2 \rfloor} \).

For any \( X \subset [0, \mu - 1] \cap 2\mathbb{N} \) we set \( a^X = \cup_{s \in X} \{a_{i_s}, a_{i_{s+1}}\} \in \mathcal{P}(J) \). Then \((\mathcal{A}_{a^X}, \mathcal{B}_{a^X})\) is related to \( a^X \) in 2.1 as follows:

\[
\mathcal{A}_{a^X} = \{a^X_i; i \in [0, N] \cap 2\mathbb{N}\}, \mathcal{B}_{a^X} = \{a^X_i; i \in [0, N] \cap (2\mathbb{N} + 1)\}.
\]
2.7. We shall use the notation of 2.1. Let \( T \) be the set of all unordered pairs \((A, B)\) where \( A \) is a subset of \( \{0, 1, 2, \ldots\} \), \( B \) is a subset of \( \{1, 2, 3, \ldots\} \), \( A \) contains no consecutive integers, \( B \) contains no consecutive integers, and \( A \cup B = (\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_N) \) as multisets. For example, setting

\[
A_\emptyset = \{\tilde{a}_i; i \in [0, N] \cap 2\mathbb{N}\}, \quad B_\emptyset = (\tilde{a}_i; i \in [0, N] \cap (2\mathbb{N} + 1)),
\]

we have \((A_\emptyset, B_\emptyset) \in T\).

For any \((A, B) \in T\) we define \((A^-, B^-)\) as follows: \( A^- \) consists of \( x_1 < x_2 < x_3 < \cdots < x_p - p + 1 \) where \( x_1 < x_2 < \cdots < x_p \) are the elements of \( A \); \( B^- \) consists of \( y_1 < y_2 < \cdots < y_q - q + 1 \) where \( y_1 < y_2 < \cdots < y_q \) are the elements of \( B \).

We can enumerate the elements of \( T \) as in [L4, 11.5]. Let \( J \) be the set of all \( c \in \mathbb{N} \) such that \( c \) appears exactly once in the sequence

\[
(\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_N) = (a_0, a_1, a_2 + 1, a_3 + 1, a_4 + 2, a_5 + 2, \ldots).
\]

A nonempty subset \( I \) of \( J \) is said to be an interval if it is of the form \( \{i, i + 1, i + 2, \ldots, j\} \) with \( i - 1 \notin J, j + 1 \notin J \). Let \( \mathcal{I} \) be the set of intervals of \( J \). For any \( s \in [0, \mu - 1] \cap 2\mathbb{N} \), the set \( I_s := \{\tilde{a}_{i_s}, \tilde{a}_{i_s+1}, \tilde{a}_{i_s+2}, \ldots, \tilde{a}_{i_s+2m_s+1}\} \) is either a single interval or a union of intervals \( I_s^1 \sqcup I_s^2 \sqcup \ldots \sqcup I_s^{t_s} \) \((t_s \geq 2)\) where \( \tilde{a}_{i_s} \in I_s^1 \), \( \tilde{a}_{i_s+2m_s+1} \in I_s^{t_s} \), \( |I_s^1|, |I_s^{t_s}| \) are odd, \( |I_s^h| \) are even for \( h \in [2, t_s - 1] \) and any element in \( I_s^e \) is \( e \) than any element in \( I_s^{e'} \) for \( e < e' \). Let \( I_s \) be the set of all \( I \in \mathcal{I} \) such that \( I \subset I_s \). We have a partition \( \mathcal{I} = \sqcup_{s \in [0, \mu - 1] \cap 2\mathbb{N}} I_s \). For any subset \( \alpha \subset I \) we consider

\[
A_\alpha = \bigcup_{I \in \mathcal{I} - \alpha} (I \cap A_\emptyset) \cup \bigcup_{I \in \alpha} (I \cap B_\emptyset) \cup (A_\emptyset \cap B_\emptyset),
\]

\[
B_\alpha = \bigcup_{I \in \mathcal{I} - \alpha} (I \cap B_\emptyset) \cup \bigcup_{I \in \alpha} (I \cap A_\emptyset) \cup (A_\emptyset \cap B_\emptyset).
\]

Then \((A_\alpha, B_\alpha) \in T \) and the map \( \alpha \mapsto (A_\alpha, B_\alpha) \) is a bijection \( \mathcal{P}(\mathcal{I}) \leftrightarrow T \). (Note that if \( \alpha = \emptyset \) then \((A_\alpha, B_\alpha)\) agrees with the earlier definition of \((A_\emptyset, B_\emptyset)\).)

Let \( T' = \{(A, B) \in T; |A| = |A_\emptyset|, |B| = |B_\emptyset|\} \), \( T_1 = \{(A, B) \in T'; A^- \cup B^- = A_\emptyset^- \cup B_\emptyset^-\} \). Let \( \mathcal{P}(\mathcal{I})' \) (resp. \( \mathcal{P}(\mathcal{I}_{\mathcal{T}_1}) \)) be the subset of \( \mathcal{P}(\mathcal{I}) \) corresponding to \( T' \) (resp. \( T_1 \)) under the bijection \( \mathcal{P}(\mathcal{I}) \leftrightarrow T \).

Now let \( X \) be a subset of \([0, \mu - 1] \cap 2\mathbb{N} \). Let \( \alpha_X = \cup_{s \in X} I_s \in \mathcal{P}(\mathcal{I}) \). From the definitions we see that

(a) \( A_{\alpha_X} = \mathfrak{A}_{\alpha_X}, \quad B_{\alpha_X} = \mathfrak{B}_{\alpha_X} \)

(Notation of 2.6). In particular we have \((A_{\alpha_X}, B_{\alpha_X}) \in T_1 \). Thus \(|T_1| \geq 2^{[\mu/2]} \).

Using Lemma 2.2 we see that

(b) \(|T_1| = 2^{[\mu/2]} \) and \( T_1 \) consists of the pairs \((A_{\alpha_X}, B_{\alpha_X})\) with \( X \subset [0, \mu - 1] \cap 2\mathbb{N} \).

Using (a),(b) we deduce:

(c) The map \( T_1 \to \mathcal{S}_1 \) given by \((A, B) \mapsto (A^-, B^-)\) is a bijection.
3. Proof of Theorem 0.4 and of Corollary 0.5

3.1. If $G$ is simple adjoint of type $A_n$, $n \geq 1$, then 0.4 and 0.5 are obvious: we have $A(u) = \{1\}$, $\hat{A}(u) = \{1\}$.

3.2. Assume that $G = Sp_{2n}(k)$ where $n \geq 2$. Let $N$ be a sufficiently large even integer. Now $u : k^{2n} \rightarrow k^{2n}$ has $i_v$ Jordan blocks of size $e$ ($e = 1, 2, 3, \ldots$). Here $i_1, i_2, i_3, \ldots$ are even. Let $\Delta = \{e \in \{2, 4, 6, \ldots \}; i_e \geq 1\}$. Then $A(u)$ can be identified in the standard way with $\mathcal{P}(\Delta)$. Hence the group of characters $\hat{A}(u)$ of $A(u)$ (which may be canonically identified with the $F_2$-vector space dual to $\mathcal{P}(\Delta)$) may be also canonically identified with $\mathcal{P}(\Delta)$ itself (so that the basis given by the one element subsets of $\Delta$ is self-dual).

To the partition $1i_1 + 2i_2 + 3i_3 + \ldots$ of $2n$ we associate a pair $(A, B)$ as in [L4, 11.6] (with $N, 2m$ replaced by $2n, N$). We have $A = (a_0, a_2, a_4, \ldots, a_N)$, $B = (a_1, a_3, a_5, \ldots, a_{N-1})$, where $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_N$ is obtained from a sequence $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_N$ as in 1.1. (Here we use that $C$ is special.) Now the definitions and results in §1 are applicable. As in [L3, 4.5] the family $\mathcal{F}$ is in canonical bijection with $\mathcal{T}'$ in 1.6.

We arrange the intervals in $\mathcal{I}$ in increasing order $I_{(1)}, I_{(2)}, \ldots, I_{(f)}$ (any element in $I_{(1)}$ is smaller than any element in $I_{(2)}$, etc.). We arrange the elements of $\Delta$ in increasing order $e_1 < e_2 < \cdots < e_p$; then $f' = f'$ and we have a bijection $\mathcal{I} \leftrightarrow \Delta$, $I_{(h)} \leftrightarrow e_h$; moreover we have $|I_{(h)}| = i_{e_h}$ for $h \in [1, f]$; see [L4, 11.6]. Using this bijection we see that $A(u)$ and $\hat{A}(u)$ are identified with the $F_2$-vector space $\mathcal{P}(\mathcal{I})$ with basis given by the one element subsets of $\mathcal{I}$. Let $\pi : \mathcal{P}(\mathcal{I}) \rightarrow \mathcal{P}(\mathcal{I})^*_1$ (with $\mathcal{P}(\mathcal{I})^*_1$ as in 1.7(c)) be the (surjective) $F_2$-linear map which to $X \subset \mathcal{I}$ associates the linear form $L \mapsto |X \cap L|$ mod 2 on $\mathcal{P}(\mathcal{I})_1$. We will show that

(a) $\ker \pi = \mathcal{K}(u)$ ($\mathcal{K}(u)$ as in 0.1).

We identify $\text{Irr}_C W$ with $T'$ (see 1.7) via the restriction of the bijection in [L4, (12.2.4)] (we also use the description of the Springer correspondence in [L4, 12.3]). Under this identification the subset $\mathcal{I}_c^* W$ of $\text{Irr}_C W$ becomes the subset $T_1$ (see 1.7) of $T'$. Via the identification $\mathcal{P}(\mathcal{I})' \leftrightarrow T'$ in 1.7 and $\hat{A}(u) \leftrightarrow \mathcal{P}(\mathcal{I})$ (see above), the map $E \mapsto \mathcal{V}_E$ from $T'$ to $\hat{A}(u)$ becomes the obvious imbedding $\mathcal{P}(\mathcal{I})' \rightarrow \mathcal{P}(\mathcal{I})$ (we use again [L4, 12.3]). By definition, $\mathcal{K}(u)$ is the set of all $X \in \mathcal{P}(\mathcal{I})$ such that for any $L \in \mathcal{P}(\mathcal{I})_1$ we have $|X \cap L| = 0$ mod 2. Thus, (a) holds.

Using (a) we have canonically $\hat{A}(u) = \mathcal{P}(\mathcal{I})^*_1$ via $\pi$. We define an $F_2$-linear map $\mathcal{P}(\mathcal{I})_1 \rightarrow \mathcal{P}(\mathcal{J})_1$ (see 1.6) by $I_s \mapsto \{a_{i_s}, a_{i_{s+1}}\}$ for $s \in \{1, 3, \ldots, 2M - 1\}$ ($I_s$ as in 1.7). This is an isomorphism; it corresponds to the bijection 1.7(c) under the identification $T_1 \leftrightarrow \mathcal{P}(\mathcal{I})_1$ in 1.7 and the identification $\mathcal{T}_1 \leftrightarrow \mathcal{P}(\mathcal{J})_1$ in 1.6. Hence we can identify $\mathcal{P}(\mathcal{I})^*_1$ with $\mathcal{P}(\mathcal{J})^*_1$ and with $\mathcal{P}(\mathcal{J})_0$ (see 1.6(a)). We obtain an identification $\hat{A}(u) = \mathcal{P}(\mathcal{J})_0$.

By [L3, 4.5] we have $X_F = \mathcal{P}(\mathcal{J})$. Using 1.6(a) we see that $\mathcal{P}(\mathcal{J}) = M(\mathcal{P}(\mathcal{J})_0) = M(\hat{A}(u))$ canonically so that 0.4 holds in our case. From the arguments above we see that in our case 0.5 follows from 1.7(c).

3.3. Assume that $G = SO_n(k)$ where $n \geq 7$. Let $N$ be a sufficiently large integer
such that $N = n \mod 2$. Now $u : k^n \to k^n$ has $i_e$ Jordan blocks of size $e$ ($e = 1, 2, 3, \ldots$). Here $i_2, i_4, i_6, \ldots$ are even. Let $\Delta = \{ e \in \{1, 3, 5, \ldots \}; i_e \geq 1 \}$. If $\Delta = \emptyset$ then $A(u) = \{1\}$, $\hat{A}(u) = \{1\}$ and $\mathcal{G}_F = \{1\}$ so that the result is trivial.

In the remainder of this subsection we assume that $\Delta \neq \emptyset$. Then $A(u)$ can be identified in the standard way with the $F_2$-subspace $\mathcal{P}_{ev}(\Delta)$ of $\mathcal{P}(\Delta)$ and the group of characters $\hat{A}(u)$ of $A(u)$ (which may be canonically identified with the $F_2$-vector space dual to $A(u)$) becomes $\hat{\mathcal{P}}(\Delta)$; the obvious pairing $A(u) \times \hat{A}(u) \to F_2$ is induced by the inner product $L, L' \mapsto |L \cap L'| \mod 2$ on $\mathcal{P}(\Delta)$.

To the partition $1i_1 + 2i_2 + 3i_3 + \ldots$ of $n$ we associate a pair $(A, B)$ as in [L4, 11.7] (with $N, M$ replaced by $n, N$). We have $A = \{ \alpha_i; i \in [0, N] \cap 2\mathbb{N} \}$, $B = \{ \hat{\alpha}_i; i \in [0, N] \cap (2\mathbb{N} + 1) \}$ where $\hat{\alpha}_0 \leq \hat{\alpha}_1 \leq \hat{\alpha}_2 \leq \cdots \leq \hat{\alpha}_N$ is obtained from a sequence $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_N$ as in 2.1. (Here we use that $C$ is special.) Now the definitions and results in §2 are applicable. As in [L3, 4.5] (if $N$ is even) or [L3, 4.6] (if $N$ is odd) the family $\mathcal{F}$ is in canonical bijection with $\mathcal{S}'$ in 2.6.

We arrange the intervals in $\mathcal{I}$ in increasing order $I_{(1)}, I_{(2)}, \ldots, I_{(f)}$ (any element in $I_{(1)}$ is smaller than any element in $I_{(2)}$, etc.). We arrange the elements of $\Delta$ in increasing order $e_1 < e_2 < \cdots < e_{\mu'}$; then $f = f'$ and we have a bijection $\mathcal{I} \leftrightarrow \Delta$, $I_{(h)} \leftrightarrow e_h$; moreover we have $|I_{(h)}| = i_{e_h}$ for $h \in [1, f]$; see [L4, 11.7]. Using this bijection we see that $A(u)$ is identified with $\mathcal{P}_{ev}(\mathcal{I})$ and $\hat{A}(u)$ is identified with $\hat{\mathcal{P}}(\mathcal{I})$. For any $X \in \mathcal{P}_{ev}(\mathcal{I})$, the assignment $L \mapsto |X \cap L| \mod 2$ can be viewed as an element of $\mathcal{P}(\mathcal{I})^*_1$ (the dual space of $\hat{\mathcal{P}}(\mathcal{I})_1$ in 2.7 which by 2.7(b) is an $F_2$-vector space of dimension $2^{\lfloor \mu/2 \rfloor}$). This induces a (surjective) $F_2$-linear map $\pi : \mathcal{P}_{ev}(\mathcal{I}) \to \mathcal{P}(\mathcal{I})^*_1$. We will show that

(a) $\ker \pi = \mathcal{K}(u)$ ($\mathcal{K}(u)$ as in 0.1).

We identify $\text{Irr}_C W$ with $T'$ (see 2.7) via the restriction of the bijection in [L4, (13.2.5)] if $N$ is odd or [L4, (13.2.6)] if $N$ is even (we also use the description of the Springer correspondence in [L4, 13.3]). Under this identification the subset $\text{Irr}_C^\mu W$ of $\text{Irr}_C W$ becomes the subset $T_1$ (see 2.7) of $T'$. Via the identification $\hat{\mathcal{P}}(\mathcal{I})' \leftrightarrow T'$ in 2.7 and $\hat{A}(u) \leftrightarrow \mathcal{P}(\mathcal{I})$ (see above), the map $E \mapsto \mathcal{V}_E$ from $T'$ to $\hat{A}(u)$ becomes the obvious imbedding $\hat{\mathcal{P}}(\mathcal{I})_0 \to \mathcal{P}(\mathcal{I})$ (we use again [L4, 13.3]). By definition, $\mathcal{K}(u)$ is the set of all $X \in \mathcal{P}_{ev}(\mathcal{I})$ such that for any $L \in \mathcal{P}(\mathcal{I})$ representing a vector in $\mathcal{P}(\mathcal{I})_1$ we have $|X \cap L| = 0 \mod 2$. Thus, (a) holds.

Using (a) we have canonically $\hat{A}(u) = \mathcal{P}(\mathcal{I})^*_1$ via $\pi$. We have an $F_2$-linear map $\mathcal{P}(\mathcal{I})_1 \to \mathcal{P}(\mathcal{J})_0$ (see 2.6) induced by $I_s \mapsto \{a_{is}, a_{is+1}\}$ for $s \in [0, \mu - 1] \cap 2\mathbb{N}$ ($I_s$ as in 2.7). This is an isomorphism; it corresponds to the bijection 2.7(c) under the identification $T_1 \leftrightarrow \mathcal{P}(\mathcal{I})_1$ in 2.7 and the identification $\Sigma_1 \leftrightarrow \mathcal{P}(\mathcal{J})_0$ in 2.6. Hence we can identify $\mathcal{P}(\mathcal{I})^*_1$ with $\mathcal{P}(\mathcal{J})^*_0$ and with $\mathcal{P}(\mathcal{J})_1$ (see 2.6(a)). We obtain an identification $\hat{A}(u) = \mathcal{P}(\mathcal{J})_1$.

By [L3, 4.6] we have $\mathbf{X}_\mathcal{F} = \mathcal{P}_{ev}(\mathcal{J})$. Using 2.6(a) we see that $\mathcal{P}(\mathcal{J}) = M(\mathcal{P}(\mathcal{J})_1) = M(\hat{A}(u))$ canonically so that 0.4 holds in our case. From the arguments above we see that in our case 0.5 follows from 2.7(c).

3.4. In 3.5-3.9 we consider the case where $G$ is simple adjoint of exceptional type.
In each case we list the elements of the set \( \text{Irr}_C W \) for each special unipotent class \( C \) of \( G \); the elements of \( \text{Irr}_C W - \text{Irr}^* CW \) are enclosed in \([-]\). (The notation for the various \( C \) is as in [Sp2]; the notation for the objects of \( \text{Irr} W \) is as in [Sp2] (for type \( E_n \)) and as in [L3, 4.10] for type \( E_4 \).) In each case the structure of \( A(u), \bar{A}(u) \) (for \( u \in C \)) is indicated; here \( S_n \) denotes the symmetric group in \( n \) letters. The order in which we list the objects in \( \text{Irr}_C W \) corresponds to the following order of the irreducible representations of \( A(u) = S_n \):

1, \( \epsilon \) (\( n = 2 \)); 1, \( r, \epsilon \) (\( n = 3, G \neq G_2 \)); 1, \( r \) (\( n = 3, G = G_2 \)); 1, \( \lambda^1, \lambda^2, \sigma \) (\( n = 4 \));

1, \( \nu, \lambda^1, \nu', \lambda^2, \lambda^3 \) (\( n = 5 \))

(note of [L3, 4.3]). Now 0.4 and 0.5 follow in our case from the tables in 3.5-3.9 and the definitions in [L3, 4.8-4.13]. (In those tables \( S_n \) is the symmetric group in \( n \) letters.)

3.5. Assume that \( G \) is of type \( E_8 \).

\( \text{Irr}_{E_8} W = \{1\}_0 \}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \n\text{Irr}_{E_8(a_1)} W = \{8\}_1 \}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \n\text{Irr}_{E_8(a_2)} W = \{352\}_1 \}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \n\text{Irr}_{E_8(A_1)} W = \{1123, 288\}_1 \}; \ A(u) = S_2, \bar{A}(u) = S_2 \n\text{Irr}_{D_4} W = \{2104, 160\}_1 \}; \ A(u) = S_2, \bar{A}(u) = S_2 \n\text{Irr}_{E_7(A_1)} W = \{560, 50\}_1 \}; \ A(u) = S_2, \bar{A}(u) = \{1\} \n\text{Irr}_{E_7(A_1)} W = \{567\}_1 \}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \n\text{Irr}_{D_4(a_1)} W = \{700, 300\}_1 \}; \ A(u) = S_2, \bar{A}(u) = S_2 \n\text{Irr}_{E_7(a_2)} W = \{1400, 1008, 561\}_1 \}; \ A(u) = S_3, \bar{A}(u) = S_3 \n\text{Irr}_{E_6} W = \{1400, 1575, 350\}_1 \}; \ A(u) = S_3, \bar{A}(u) = S_3 \n\text{Irr}_{D_5(a_1)} W = \{3240, 1050\}_1 \}; \ A(u) = S_2, \bar{A}(u) = \{1\} \n\text{Irr}_{D_5(a_2)} W = \{2240, 175, 840\}_1 \}; \ A(u) = S_3, \bar{A}(u) = S_2 \n\text{Irr}_{D_6} W = \{2268, 1296\}_1 \}; \ A(u) = S_2, \bar{A}(u) = S_2 \n\text{Irr}_{E_6(a_1)} W = \{4096, 11, 4096\}_1 \}; \ A(u) = S_2, \bar{A}(u) = S_2 \n\text{Irr}_{E_6} W = \{525\}_1 \}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \n\text{Irr}_{D_7(a_2)} W = \{4200, 3360\}_1 \}; \ A(u) = S_2, \bar{A}(u) = S_2 \n\text{Irr}_{E_6(a_1)} W = \{2800, 2100\}_1 \}; \ A(u) = S_2, \bar{A}(u) = S_2 \n\text{Irr}_{D_5} W = \{4536, [840\}_1 \}; \ A(u) = S_2, \bar{A}(u) = \{1\} \n\text{Irr}_{D_6(a_1)} W = \{6075, 700\}_1 \}; \ A(u) = S_2, \bar{A}(u) = \{1\} \n\text{Irr}_{A_5} W = \{2835\}_1 \}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \n\text{Irr}_{A_5} W = \{4200\}_1 \}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \n\text{Irr}_{D_6(a_1)} W = \{5600, 2400\}_1 \}; \ A(u) = S_2, \bar{A}(u) = S_2 \n\text{Irr}_{A_5} W = \{4480, 4536, 5670\}_1 \}; \ A(u) = S_5, \bar{A}(u) = S_5 \n\text{Irr}_{D_7} W = \{2100\}_1 \}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \n\text{Irr}_{A_5(a_1)} W = \{5600\}_1 \}; \ A(u) = S_2, \bar{A}(u) = S_2 \n\text{Irr}_{A_5(a_1)} W = \{4200, 160\}_1 \}; \ A(u) = S_2, \bar{A}(u) = \{1\} \n\text{Irr}_{A_5(a_2)} W = \{4536\}_1 \}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \n\text{Irr}_{D_5} W = \{2800, 2100\}_1 \}; \ A(u) = S_2, \bar{A}(u) = S_2 \n
3.6. Assume that $G$ is adjoint of type $E_7$.

\[
\begin{align*}
\text{Irr}_{A_2 A_1} W &= \{4200_{24}, 3360_{25}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{D_4} W &= \{525_{36}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{A_4 A_1} W &= \{4096_{26}, 4096_{27}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{A_4} W &= \{2268_{30}, 1296_{33}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{D_4(a_1) A_2} W &= \{2240_{28}, 840_{31}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{A_5 A_3} W &= \{3240_{31}, [972_{32}]\}; \ A(u) = S_2, \bar{A}(u) = \{1\} \\
\text{Irr}_{D_4(a_1) A_1} W &= \{1400_{32}, 1575_{34}, 350_{38}\}; \ A(u) = S_3, \bar{A}(u) = S_3 \\
\text{Irr}_{D_4(a_1)} W &= \{1400_{37}, 1008_{39}, 56_{49}\}; \ A(u) = S_3, \bar{A}(u) = S_3 \\
\text{Irr}_{2 A_3} W &= \{700_{42}, 300_{44}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{A_3} W &= \{567_{46}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{A_2 A_1} W &= \{560_{47}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{A_2 A_1} W &= \{210_{52}, 160_{55}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{A_5} W &= \{112_{63}, 28_{68}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{2 A_1} W &= \{35_{74}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{A_3} W &= \{8_{91}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{0} W &= \{1_{120}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\}
\end{align*}
\]

\[
\begin{align*}
\text{Irr}_{E_7} W &= \{1_{10}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{E_7(a_1)} W &= \{7_{1}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{E_7(a_2)} W &= \{27_{2}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{D_6 A_1} W &= \{563_{5}, 21_{6}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{D_6} W &= \{21_{3}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{E_6(a_1)} W &= \{120_{4}, 105_{5}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{D_6(a_1) A_1} W &= \{189_{5}, [15_{7}]\}; \ A(u) = S_2, \bar{A}(u) = \{1\} \\
\text{Irr}_{D_6(a_1)} W &= \{210_{6}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{A_6} W &= \{105_{6}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{D_5 A_1} W &= \{168_{6}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{D_5} W &= \{189_{7}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{D_6(a_2) A_1} W &= \{315_{7}, 280_{9}, 35_{13}\}; \ A(u) = S_3, \bar{A}(u) = S_3 \\
\text{Irr}_{(A_5 A_1)_V} W &= \{405_{8}, 189_{10}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{D_5(a_1) A_1} W &= \{378_{9}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{A_4 A_1} W &= \{210_{10}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{D_5(a_1)} W &= \{420_{10}, 336_{11}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{A_3} W &= \{105_{12}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{A_4 A_1} W &= \{512_{11}, 512_{12}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{D_5} W &= \{105_{15}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{A_4} W &= \{420_{13}, 336_{14}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{A_3 A_2 A_1} W &= \{210_{13}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{A_4 A_2} W &= \{378_{14}, [84_{15}]\}; \ A(u) = S_2, \bar{A}(u) = \{1\} \\
\text{Irr}_{D_3(a_1) A_1} W &= \{405_{15}, 189_{17}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{D_3(a_1)} W &= \{315_{16}, 280_{18}, 35_{22}\}; \ A(u) = S_3, \bar{A}(u) = S_3 \\
\text{Irr}_{(A_3 A_1)_V} W &= \{189_{20}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\}
\end{align*}
\]
\( \text{Irr}_{2A_2} W = \{168_{21}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_3 A_1} W = \{105_{21}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_4} W = \{210_{21}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_2} W = \{189_{22}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_2 A_1} W = \{120_{25}, 105_{26}\}; A(u) = S_2, \bar{A}(u) = S_2 \)
\( \text{Irr}_{3A_1'} W = \{21_{36}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_2} W = \{56_{30}, 21_{33}\}; A(u) = S_2, \bar{A}(u) = S_2 \)
\( \text{Irr}_{2A_1} W = \{27_{37}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_1} W = \{7_{46}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_0 W = \{1_{63}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)

3.7. Assume that \( G \) is adjoint of type \( E_6 \).
\( \text{Irr}_{E_6} W = \{10\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{E_6(a_1)} W = \{61\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{D_4} W = \{20_{2}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_6} W = \{30_{3}, 15_{5}\}; A(u) = S_2, \bar{A}(u) = S_2 \)
\( \text{Irr}_{D_5(a_1)} W = \{64_{4}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_4} W = \{60_{5}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_4} W = \{81_{6}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{D_4} W = \{24_{6}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{D_4(a_1)} W = \{80_{7}, 90_{8}, 20_{10}\}; A(u) = S_3, \bar{A}(u) = S_3 \)
\( \text{Irr}_{2A_2} W = \{24_{12}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_3} W = \{81_{10}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_2 A_1} W = \{60_{11}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_2} W = \{64_{13}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_1} W = \{30_{15}, 15_{17}\}; A(u) = S_2, \bar{A}(u) = S_2 \)
\( \text{Irr}_{2A_1} W = \{20_{20}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_1} W = \{62_{5}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_0 W = \{1_{36}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)

3.8. Assume that \( G \) is of type \( F_4 \).
\( \text{Irr}_{F_4} W = \{11\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{F_4(a_1)} W = \{42_{2}, 23\}; A(u) = S_2, \bar{A}(u) = S_2 \)
\( \text{Irr}_{F_4(a_2)} W = \{9_{1}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{B_4} W = \{81\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{C_4} W = \{83\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{F_4(a_3)} W = \{12_{1}, 9_{3}, 6_{2}, 1_{3}\}; A(u) = S_4, \bar{A}(u) = S_4 \)
\( \text{Irr}_{A_2} W = \{82\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_2} W = \{84, [1_{2}]\}; A(u) = S_2, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_1 A_1} W = \{9_{4}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
\( \text{Irr}_{A_1} W = \{45, 2_{2}\}; A(u) = S_2, \bar{A}(u) = S_2 \)
\( \text{Irr}_0 W = \{1_{4}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \)
3.9. Assume that $G$ is of type $G_2$.

$\text{Irr}_{G_2} W$ is the unit representation; $A(u) = \{1\}, \bar{A}(u) = \{1\}$

$\text{Irr}_{G_2(a_1)} W$ consists of the reflection representation and the one dimensional representation on which the reflection with respect to a long (resp. short) simple coroot acts nontrivially (resp. trivially); $A(u) = S_3, \bar{A}(u) = S_3$

$\text{Irr}_0 W = \{\text{sgn}\}; A(u) = \{1\}, \bar{A}(u) = \{1\}$

3.10. This completes the proof of Theorem 0.4 and that of Corollary 0.5.

We note that the definition of $G_{\mathcal{F}}$ given in [L3] (for type $C_n, B_n$) is $\mathcal{P}(\mathcal{J})_1$ (in the setup of 3.2) and $\mathcal{P}(\mathcal{J})_0$ (in the setup of 3.3) which is noncanonically isomorphic to $\bar{A}(u)$, unlike the definition adopted here that is, $\mathcal{P}(\mathcal{J})_0$ (in the setup of 3.2) and $\mathcal{P}(\mathcal{J})_1$ (in the setup of 3.3) which makes $G_{\mathcal{F}}$ canonically isomorphic to $\bar{A}(u)$.

4. Character sheaves

4.1. Let $\hat{G}$ be a set of representatives for the isomorphism classes of character sheaves on $G$. For any conjugacy class $D$ in $G$ let $D_\omega := \{g_\omega; g \in D\}$, a unipotent class in $G$. For any unipotent class $C$ in $G$ let $S_C$ be the set of conjugacy classes $D$ of $G$ such that $D_\omega = C$. It is likely that the following property holds.

(a) Let $K \in \hat{G}$. There exists a unique unipotent class $C$ of $G$ such that
   - for any $D \in S_C$, $K|_D$ is a local system (up to shift);
   - for some $D \in S_C$, we have $K|_D \neq 0$;
   - for any unipotent class $C'$ of $G$ such that $\dim C' \geq \dim C$, $C' \neq C$ and any $D \in S_{C'}$, we have $K|_D = 0$.

We say that $C$ is the unipotent support of $K$.

(The uniqueness part is obvious.) Note that [L8, 10.7] provides some support (no pun intended) for (a).

We shall now try to make (a) more precise in the case where $K \in \hat{G}^\text{un}$, the subset of $\hat{G}$ consisting of unipotent character sheaves (that is $\hat{G}^\text{un} = \hat{G}_{\text{Q}}^\text{un}$ with the notation of [L7, 4.2]). As in [L7, 4.6] we have a partition $\hat{G}^\text{un} = \sqcup_{\mathcal{F}} \hat{G}_{\mathcal{F}}^\text{un}$ where $\mathcal{F}$ runs over the families of $W$.

In the remainder of this section we fix a family $\mathcal{F}$ of $W$ and we denote by $C$ the special unipotent class of $G$ such that $E_C \in \mathcal{F}$, see 0.1; let $u \in C$. Let $\Gamma = \bar{A}(u)$ and let $Z(u) \xrightarrow{j'} A(u) \xrightarrow{h} \Gamma$ be the obvious (surjective) homomorphisms; let $j = hj' : Z(u) \to \Gamma$. Let $[\Gamma]$ be the set of conjugacy classes in $A(u)$. For $D \in S_C$ let $\phi(D)$ be the conjugacy class of $j(g_s)$ in $\Gamma$ where $g \in D$ is such that $g_\omega = u$; clearly such $g$ exists and is unique up to $Z(u)$-conjugacy so that the conjugacy class of $j(g_s)$ is independent of the choice of $g$. Thus we get a (surjective) map $\phi : S_C \to [\Gamma]$. For $\gamma \in [\Gamma]$ we set $S_{C, \gamma} = \phi^{-1}(\gamma)$. We now select for each $\gamma \in [\Gamma]$ an element $x_\gamma \in \gamma$ and we denote by $\text{Irr} Z_{\Gamma}(x_\gamma)$ a set of representatives for the isomorphism classes of irreducible representations of $Z_{\Gamma}(x_\gamma) := \{y \in \Gamma; y x_\gamma = x_\gamma y\}$ (over $\mathbb{Q}_l$). Let $D \in S_{C, \gamma}$, $\mathcal{E} \in \text{Irr} Z_{\Gamma}(x_\gamma)$. We can find $g \in D$ such that $g_\omega = u, j(g_s) = x_\gamma$ (and another choice for such $g$.
must be of the form $bgb^{-1}$ where $b \in Z(u)$, $j(b) \in Z_{\Gamma}(x_\gamma)$. Let $\mathcal{E}^D$ be the $G$-equivariant local system on $D$ whose stalk at $g_1 \in D$ is $\{z \in G; zgz^{-1} = g_1\} \times \mathcal{E}$ modulo the equivalence relation $(z, e) \sim (zh^{-1}, j(h)e)$ for all $h \in Z(g)$. If $g$ is changed to $g_1 = bgb^{-1}$ (as above) then $\mathcal{E}^D$ is changed to the $G$-equivariant local system $\mathcal{E}_1^D$ on $D$ whose stalk at $g' \in D$ is $\{z' \in G; z'g_1z'^{-1} = g'\} \times \mathcal{E}$ modulo the equivalence relation $(z', e') \sim (z'h'^{-1}, j(h')e)$ for all $h' \in Z(g_1)$. We have an isomorphism of local systems $\mathcal{E}^D \sim \mathcal{E}_1^D$ which for any $g' \in D$ maps the stalk of $\mathcal{E}^D$ at $g'$ to the stalk of $\mathcal{E}_1^D$ at $g'$ by the rule $(z, e) \mapsto (zb^{-1}, j(b)e)$. (We have $zb^{-1}g_1bz^{-1} = zgz^{-1} = g'$.) This is compatible with the equivalence relations. Thus the isomorphism class of the local system $\mathcal{E}^D$ does not depend on the choice of $g$.

The properties (b),(c) below appear to be true (\[ denotes a shift).

(b) Let $K \in \hat{G}_{\mathcal{F}}^{\text{un}}$. There exists a unique $\gamma \in [\Gamma]$ and a unique $E \in \text{Irr} Z_{\Gamma}(x_\gamma)$ such that

(i) if $D \in S_{C, \gamma}$, we have $K|_D \cong \mathcal{E}^D[\gamma]$;

(ii) if $D \in S_{C, \gamma}$ with $\gamma \in [\Gamma] - \{\gamma\}$, we have $K|_D = 0$;

(iii) for any unipotent class $C'$ of $G$ such that $\dim C' \geq \dim C$, $C' \neq C$ and any $D \in S_{C'}$, we have $K|_D = 0$.

(c) $K \mapsto (\gamma, E)$ in (b) defines a bijection $\hat{G}_{\mathcal{F}}^{\text{un}} \sim M(\Gamma)$.

Note that (b)(iii) follows from [L8, 10.7], at least if $p$ is sufficiently large or 0.

In the case where $G$ is of type $E_8$ and $\mathcal{F}$ contains the irreducible representation of degree 4480 (so that $\Gamma = S_5$), (b)(i),(b)(ii),(c) have been already stated (without proof) in [L7, 4.7].

For any finite dimensional representation $E$ of $W$ (over $\mathbb{Q}$) let $\mathcal{E}$ be the intersection cohomology complex on $G$ with coefficients in the local system with monodromy given by the $W$-module $E$ on the open set of regular semisimple elements. We have an imbedding $\mathcal{F} \to \hat{G}_{\mathcal{F}}^{\text{un}}$, $E \mapsto E[\gamma]$. Composing this imbedding with the map $\hat{G}_{\mathcal{F}}^{\text{un}} \sim M(\Gamma)$ in (c) (which we assume to hold) we obtain an imbedding $\mathcal{F} \to M(\Gamma)$. We expect that:

(d) The imbedding $\mathcal{F} \to M(\Gamma)$ defined above coincides with the imbedding $\mathcal{F} \to M(\Gamma)$ in [L3, Sec.4].

Note that 0.6 can be regarded as evidence for the validity of (b),(c),(d). Further evidence is given in 4.2-4.5.

4.2. Assume that $G$ is simply connected. Let $D \in S_C$. Let $s$ be a semisimple element of $G$ such that $su \in D$. Let $C_0$ be the conjugacy class of $u$ in $Z(s)$. Let $W'$ be the Weyl group of $Z(s)$ regarded as a subgroup of $W$. For any finite dimensional $W'$-module $E'$ over $\mathbb{Q}$, let $\mathcal{E}'$ be the intersection cohomology complex on $Z(s)$ defined in terms of $Z(s), E'$ in the same way as $\mathcal{E}$ was defined in terms of $G, E$. Using [L5, (8.8.4)] and the $W$-equivariance of the isomorphism in loc.cit. we see that:

(a) $\mathcal{E}|_{sC_0} \cong (\mathcal{E}|_{W'})|_{sC_0}$.

Now, if $K \in \hat{G}_{\mathcal{F}}^{\text{un}}$ is of the form $E[\gamma]$ for some $E \in \mathcal{F}$ then the computation of $K|_D$
is reduced by (a) to the computation of $E'_0|_{sC_0}$ for any irreducible $W'$-module $E'$ such that $(E' : E_{W'}) > 0$ (here $(E' : E_{W'})$ is the multiplicity of $E'$ in $E|_{W'}$). If for such $E'$ we define a unipotent class $C_{E'}$ of $Z(s)$ by $E' \in \text{Irr}_{C_{E'}} W'$ then, by a known property of $E'_0$, we have (with notation of 0.1 with $G$ replaced by $Z(s)$):

(b) if $C_0 = C_{E'}$ then $E'_0|_{sC_0}$ is the irreducible $Z(s)$-equivariant local system corresponding to $V_{E'}$;

(c) if $C_0 \neq C_{E'}$ and dim $C_0 \geq \text{dim} C_{E'}$ then $E'_0|_{sC_0} = 0$.

We say that $D$ is $E$-negligible if for any $E' \in \text{Irr} W'$ such that $(E' : E|_{W'}) > 0$ we have dim $C_0 > \text{dim} C_{E'}$.

(d) We say that $D$ is $E$-relevant if

- there is a unique $E'_0 \in \text{Irr} W'$ such that $(E'_0 : E|_{W'}) = 1$ and $C_{E'} = C_0$ (we then write $E_l = E'_0$);

- for any $E' \in \text{Irr} W'$ such that $(E' : E|_{W'}) > 0$, $E' \neq E_l$ we have dim $C_0 > \text{dim} C_{E'}$.

It is likely that $D$ is always $E$-negligible or $E$-relevant. If $D$ is $E$-negligible then $E_0|_{sC_0} = 0$ (hence $K_D = 0$); if $D$ is $E$-relevant then $E_0|_{sC_0}$ (hence $K_D$) can be explicitly computed using (b),(c).

In the remainder of this subsection we assume in addition that $G$ is almost simple of exceptional type and that $C$ is a distinguished unipotent class. In these cases one can verify that $D$ is $E$-negligible or $E$-relevant for any $E \in \mathcal{F}$ hence $K_D$ can be explicitly computed and we can check that 4.1(b) holds. Moreover, we can compute $K_D$ for any $K \in \hat{G}_{\mathcal{F}}^\mathfrak{un}$ (not necessarily of form $E[0]$) using an appropriate analogue of (a) (coming again from [L5, (8.8.4)]) and the appropriate analogues of (b),(c) (given in [L4]). We see that 4.1(b) holds again. Moreover we see that 4.1(c),(d) hold in these cases.

4.3 In this subsection we assume that $G$ is of type $E_8$ and $C$ is distinguished. In this subsection we indicate for each $D \in S_C$ the set $\mathcal{F}_D = \{ E \in \mathcal{F}; D$ is $E$-relevant $\}$ and we describe the map $E \mapsto E_l$ (see 4.2(d)). (Note that if $E \in \mathcal{F} - \mathcal{F}_D$, $D$ is $E$-negligible.) The notation is as in [Sp2]. We denote by $g_i$ an element of order $i$ of $A(u)$ (except that if $A(u) = S_5$, $g_2$ denotes a transposition and we denote by $g'_2$ an element of $A(u)$ whose centralizer has order 8). For each $g_i$ we denote by $\hat{g}_i$ a semisimple element of $Z(u)$ that represents $g_i$; similarly when $A(u) = S_5$, we denote by $\hat{g}'_2$ a semisimple element of $Z(u)$ that represents $g'_2$. We write $\mathcal{F}_{g_i}$ (resp. $\mathcal{F}_{g'_2}$) instead of $\mathcal{F}_D$ where $D$ is the $G$-conjugacy class of $u\hat{g}_i$ (resp. of $u\hat{g}'_2$). We write $H_{g_i}$ (resp. $H_{g'_2}$) for the set of all $E_l \in \text{Irr} W'$ where $E$ runs through $\mathcal{F}_{g_i}$ (resp. $\mathcal{F}_{g'_2}$); here $W' \subset W$ is the Weyl group of $Z(\hat{g}_i)$ (resp. $Z(\hat{g}'_2)$) and $E_l$ is as in 4.2(d). We write $C_{g_i}$ (resp. $C_{g'_2}$) for the conjugacy class of $u$ in $Z(\hat{g}_i)$ (resp. $Z(\hat{g}'_2)$).

Assume that $C$ is the regular unipotent class. Then $A(u) = \{ 1 \}$, $\mathcal{F}_{g_1} = H_{g_1} = \{ 1_0 \}$.

Assume that $C$ is the subregular unipotent class. Then $A(u) = \{ 1 \}$, $\mathcal{F}_{g_1} = H_{g_1} = \{ 8_1 \}$. 
Assume that $C = E_8(a_2)$. Then $A(u) = \{1\}$, $F_g = \mathcal{H} = \{352\}$.

Assume that $C = E_7A_1$. Then $A(u) = S_2$, $Z(\hat{g}_1) = G$, $Z(\hat{g}_2)$ is of type $E_7A_1$, $F_g = \mathcal{H} = \{1123, 28s\}$, $F_g = \{844\}$, $\mathcal{H} = \{10\}$, $C_{g_2} =$ regular unipotent class.

Assume that $C = D_8$. Then $A(u) = S_2$, $Z(\hat{g}_1) = G$, $Z(\hat{g}_2)$ is of type $D_8$, $F_g = \mathcal{H} = \{2104, 160\tau\}$, $F_g = \{50s\}$, $\mathcal{H} = \{1\}$, $C_{g_2} =$ regular unipotent class.

Assume that $C = E_7(a_1)A_1$. Then $A(u) = S_2$, $Z(\hat{g}_1) = G$, $Z(\hat{g}_2)$ is of type $E_7A_1$, $F_g = \mathcal{H} = \{560\}$, $F_g = \{560\}$, $\mathcal{H} = \{7, 1\}$, $C_{g_2} =$ subregular unipotent class in $E_7$ factor times regular unipotent class in $A_1$ factor.

Assume that $C = D_8(a_1)$. Then $A(u) = S_2$, $Z(\hat{g}_1) = G$, $Z(\hat{g}_2)$ is of type $D_8$, $F_g = \mathcal{H} = \{700s, 300s\}$, $F_g = \{400\}$, $\mathcal{H} = \{1\}$, $C_{g_2} =$ subregular unipotent class.

Assume that $C = E_7(a_2)A_1$. Then $A(u) = S_3$, $Z(\hat{g}_1) = G$, $Z(\hat{g}_2)$ is of type $E_7A_1$, $Z(\hat{g}_3)$ is of type $E_7A_2$, $F_g = \mathcal{H} = \{1400s, 1008g, 5619\}$, $F_g = \{1344s\}$, $\mathcal{H} = \{27, 2\}$, $C_{g_2} =$ subregular unipotent class in $E_7$-factor times regular unipotent class in $A_1$ factor, $F_g = \{448s\}$, $\mathcal{H} = \{1\}$, $C_{g_3} =$ regular unipotent class.

Assume that $C = A_8$. Then $A(u) = S_3$, $Z(\hat{g}_1) = G$, $Z(\hat{g}_2)$ is of type $D_8$, $Z(\hat{g}_3)$ is of type $A_8$, $F_g = \mathcal{H} = \{1400s, 157510, 350\}$, $F_g = \{1050\}$, $\mathcal{H} = \{28 - \text{dimensional repres.}\}$, $C_{g_2} =$ unipotent class with Jordan blocks of size $5, 11$, $F_g = \{17512\}$, $\mathcal{H} = \{1\}$, $C_{g_3} =$ regular unipotent class.

Assume that $C = D_8(a_3)$. Then $A(u) = S_3$, $Z(\hat{g}_1) = G$, $Z(\hat{g}_2)$ is of type $D_8$, $Z(\hat{g}_3)$ is of type $E_6A_2$, $F_g = \mathcal{H} = \{2240s, 840\}$, $F_g = \{1400\}$, $\mathcal{H} = \{56 - \text{dimensional repres.}\}$, $C_{g_2} =$ unipotent class with Jordan blocks of size $7, 9$, $F_g = \{2240s\}$, $\mathcal{H} = \{6, 1\}$, $C_{g_3} =$ subregular unipotent class in $E_6$-factor times regular unipotent class in $A_1$ factor.

Assume that $C = 2A_4$. Then $A(u) = S_5$, $Z(\hat{g}_1) = G$, $Z(\hat{g}_2)$ is of type $E_7A_1$, $Z(\hat{g}_2)$ is of type $D_8$, $Z(\hat{g}_3)$ is of type $E_6A_2$, $Z(\hat{g}_4)$ is of type $D_5A_3$, $Z(\hat{g}_5)$ is of type $A_4A_4$, $Z(\hat{g}_6)$ is of type $A_5A_2A_1$, $F_g = \mathcal{H} = \{4480s, 4536, 5670s, 1400s, 1680s, 7032\}$, $F_g = \{7168, 5600, 4482\}$, $\mathcal{H} = \{3157 \otimes 1, 280s, 1, 3513 \otimes 1\}$, $C_{g_2} = D_6(a_1)A_1$ in $E_7$-factor times regular unipotent class in $A_1$-factor, $F_g = \{4200s, 2688\}$, $\mathcal{H} = \{\text{repres. with symbol } (2 < 5; 0 < 3), (2 > 3; 0 < 5), (0 < 1, 4 < 5)\}$, $C_{g_2} =$ unipotent class with Jordan blocks of sizes $1, 3, 5, 7$, $F_g = \{3150, 113420\}$, $\mathcal{H} = \{303 \otimes 1, 155 \otimes 1\}$, $C_{g_3} = A_5A_1$ in $E_6$-factor times regular unipotent class in $A_2$-factor, $F_g = \{1344\}$, $\mathcal{H} = \{5 - \text{dimensional repres.}\}$, $C_{g_4} =$ subregular unipotent class in $D_5$-factor times regular unipotent class in $A_3$-factor, $F_g = \{420s\}$, $\mathcal{H} = \{1\}$, $C_{g_5} =$ regular unipotent class.
\( \mathcal{F}_{g_6} = \{2016_{19}\}, \mathcal{H}_{g_6} = \{1\}, C_{g_6} = \text{regular unipotent class.} \)

In each case the \( i \)-th member of a list \( \mathcal{F}_g \) and the \( i \)-th member of the corresponding list \( \mathcal{H}_g \) are related by the map \( E \mapsto E_1 \). Note that the members of the list \( \mathcal{F}_{g_2} \) (when \( C = 2A_4 \)) are not all in the same family. But in all cases, the members of the list \( \mathcal{F}_g \) form exactly the subset of \( \text{Irr} W' \) corresponding to the unipotent class \( C_g \) under Springer’s correspondence for \( Z(\dot{g}) \); thus they can be indexed by certain irreducible representations of the group of components of the centralizer of \( u \) in \( Z(\dot{g}) \) modulo the centre of \( Z(\dot{g}) \). (Here \( g \) is \( g_1 \) or \( g_2 \).) From this one recovers the imbedding \( \mathcal{F} \to M(\tilde{A}(u)) \) in geometric terms.

4.4. In this subsection we assume that \( G = Sp_4(k) \) and that \( \mathcal{F} \) is the family in \( \text{Irr} W \) containing the reflection representation so that \( C \) is the subregular unipotent class in \( G \). Let \( D \) be the conjugacy class in \( G \) containing \( sv \) where \( s \) is semisimple with \( Z(s) \cong SL_2(k) \times SL_2(k) \) and \( v \) is a regular unipotent element of \( Z(s) \) so that \( v \in C \). Let \( D' \) be a conjugacy class in \( G \) containing \( s'v' \) where \( s' \) is semisimple with \( Z(s') \cong GL_2(k) \) and \( v' \) is a regular unipotent element of \( Z(s') \) so that \( v' \in C \). In this case \( \mathcal{G}^\text{un}_\mathcal{F} \) consists of four character sheaves \( K_1, K_2, K_3, K_4 \), the last one being cuspidal. They can be characterized as follows.

\[
\begin{align*}
K_1|_C &= Q_1[], K_1|_D = 0, K_1|_{D'} = Q_1[]; \\
K_2|_C &= L[], K_2|_D = 0, K_2|_{D'} = Q_1[]; \\
K_3|_C &= 0, K_3|_D = Q_1[], K_3|_{D'} = 0; \\
K_4|_C &= 0, K_4|_D = L[], K_4|_{D'} = 0.
\end{align*}
\]

Here \( L \) is a nontrivial \( G \)-equivariant local system of rank 1 on \( C \), \( L' \) is the inverse image of \( L \) under the obvious map \( D \to C \). We see that 4.1(b) holds for all \( K \in \mathcal{G}^\text{un}_\mathcal{F} \) and 4.1(c),(d) hold.

4.5. Assume that \( \mathcal{F} \) is the family containing the unit representation of \( W \). Then \( C \) is the regular unipotent class of \( G \) and \( \mathcal{G}^\text{un}_\mathcal{F} \) consists of a single character sheaf, namely \( Q_1[] \). Clearly, 4.1(b),(c),(d) hold in this case.

Next we assume that \( \mathcal{F} \) is the family containing the sign representation of \( W \). Then \( C = \{1\} \) and \( \mathcal{G}^\text{un}_\mathcal{F} \) consists of a single character sheaf, namely \( K = \text{sgn}[] \). Note that for any semisimple class \( D \) of \( G \) we have \( K|_D = Q_1[] \) so that 4.1(b),(c),(d) hold in this case.

References


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