Axisymmetric Equilibrium and Stability Analysis in Alcator C-Mod, Including Effects of Current Profile, Measurement Noise and Power Supply Saturation

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by

Marco Ferrara

S.M. Nuclear Science and Engineering (2005)

Submitted to the Department of Nuclear Science and Engineering in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY

February 2009

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Abstract

The vertical position of elongated tokamak plasmas is unstable on the time scale of the eddy currents in the axisymmetric conducting structures. In the absence of feedback control, the plasma would drift vertically and quench on the wall, a situation known as Vertical Displacement Event (VDE), with serious consequences for machine integrity. As tokamaks approach reactor regimes, VDE’s cannot be tolerated: vertical feedback control must be robust against system uncertainty and the occurrence of noise and disturbances. At the same time, adaptive routines should be in place to handle unexpected events. The problem of robust control of the vertical position can be formulated in terms of identifying which variables affect vertical stability and which ones are not directly controlled/controllable; identifying the physical region of these variables, and the corresponding most unstable equilibria; and designing the control system to stabilize all equilibria with sufficient margin. The margin should be enough to allow the system to tolerate realistic scenarios of noise and disturbances. A set of metrics is introduced to characterize the problem of vertical stability: the stability margin describes the plasma-wall interaction and the open-loop growth rate; the maximum controllable displacement looks at the vertical stabilization power supplies and their ability to handle noise and off-normal events; the gain and phase margins quantify the linear stability of the feedback control loop. The dependence of these metrics on relevant plasma parameters is proven with analytic calculations and numerical simulations: in particular, it is shown that the stability margin is a decreasing function of the plasma internal inductance, for a given plasma elongation. An upper bound of the value of the internal inductance is derived and validated with database analysis, which describes the most unstable equilibrium for given values of the external elongation and the edge safety factor. The stability metrics are evaluated for typical and ITER-like C-Mod plasmas to give an example of
the C-Mod operational space and of feasible control conditions. The vertical stabilization system should be able to tolerate realistic scenarios of noise and disturbances. The main sources of noise and pick-ups in Alcator C-Mod are identified and their effects on the measurement and control of the vertical position are evaluated. Broadband noise may affect controllability of C-Mod plasmas at limit elongations and may become an issue with high-order controllers, therefore two applications of Kalman filters are investigated. A Kalman filter is compared to a state observer based on the pseudo-inverse of the measurement matrix and proves to be a better candidate for state reconstruction for vertical stabilization, provided adequate models of the system, the inputs, the intrinsic and measurement noise and an adequate set of diagnostic measurements are available. A single-input single-output application of the filter for the vertical observer rejects high frequency noise without destabilizing high-elongation plasmas, however does not match the performance of an optimized low-pass filter. Aggressive control targets and large off-normal events can cause a control current to rail. The magnetic topology is consequently perturbed and the plasma might become uncontrollable. An adaptive anti-saturation control routine is demonstrated which avoids an impending saturation by interpolating in real-time to a safe equilibrium. This approach becomes necessary when poor redundancy of control coils may require mid-shot pulse rescheduling, as opposed to an adaptation in control.

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Chapter 1

Introduction

Alcator C-Mod is a tokamak experiment at MIT for the investigation of magnetic confinement fusion. Tokamaks are toroidal devices that use a combination of toroidal and poloidal magnetic fields to confine a plasma\(^1\).

The toroidal field is produced by coils situated around the torus, while the poloidal field is produced by the plasma current itself and by a number of axisymmetric coils. These coils have two main functions: they are used to inductively drive the plasma current (the ohmic coils OH1, OH2U, OH2L) and to control the plasma shape and position (OH2U, OH2L and the equilibrium field coils EF1U, EF1L, EF2U, EF2L, EF3U, EF3L, EF4U, EF4L). Figure 1-1 shows a cross-section of Alcator C-Mod.

The plasma is formed inside the vacuum vessel by strong heating of a gaseous species. In a limited-plasma configuration, a physical limiter breaks the outermost flux surfaces and determines the spatial extension of the plasma; in a diverted-plasma configuration, a field null, also known as X-point, is created inside the vacuum vessel. For example, a lower-single-null plasma is illustrated in figure 1-1. The shape of the plasma cross section is described in terms of parameters by which the last closed flux surface, also known

\(^1\) A plasma is a highly ionized gas satisfying special conditions: the e-folding length of the electrostatic effect of a charge in the plasma, also known as Debye length, is much shorter than the size of the plasma; the characteristic frequency of oscillation of the plasma electrons, also known as plasma frequency, is much larger than any frequency in the problem; and long range collective interactions dominate over short range Coulomb collisions [1].
Figure 1-1: Cross-section of Alcator C-Mod.
as separatrix, deviates from a circular cross-section: these parameters are elongation, triangularity, squareness, etc. [2].

The plasma shape and position are reconstructed in real-time from measurements of the magnetic field, the plasma current and the control currents. Figure 1-2 shows the magnetic diagnostics in C-Mod, consisting of a large set of flux loops and poloidal pick-up coils.

![Figure 1-2: Flux loops (left panel) and poloidal pick-up coils (right panel) in Alcator C-Mod.](image)

Different control currents are more effective for changing different shape and position parameters: for example, OH2U and OH2L in anti-phase for the vertical position of the plasma; EF1U and EF1L for the vertical position of the upper and lower X-point (i.e. the elongation); EF2U and EF2L for the radial position of the upper and lower X-point (i.e. the top and bottom triangularity); EF3U and EF3L, connected in series, for the radial
position of the plasma; etc. However it is really a combination of all the currents that produces the magnetic configuration necessary to obtain the desired shape. Therefore the control of the plasma shape and position is a Multiple-Input Multiple-Output (MIMO) problem.

In C-Mod the Digital Plasma Control System (DPCS, [3]) reads the inputs for the diagnostics, calculates relevant descriptors of the plasma shape and position and their differences from the desired targets, processes these error signals through PID controllers and applies appropriate demands to the control power supplies. Figure 1-3 shows a simplified schematic of the C-Mod feedback control system.

![Simplified schematic of the C-Mod feedback control system.](image)

Figure 1-3: Simplified schematic of the C-Mod feedback control system.

The digital architecture provides significant flexibility to test linear and non-linear control strategies. Moreover, C-Mod has significant similarities with the international experiment ITER [4] in terms of machine design, plasma parameters and available control
coils. Therefore it provides excellent opportunities for control design for next-generation fusion machines.

The cross-section of tokamak plasmas is usually vertically elongated. Some of the advantages of this configuration are the favorable scaling of the magneto-hydro-dynamic stability limit with the elongation\(^2\), and the larger cross-section, and therefore total plasma current, for a given major radius. Some of the disadvantages are the complicated shape of the vacuum vessel and structures, with non-uniform stress distribution during machine operation, and the need for active stabilization of the plasma vertical position, which is unstable. With next generation machines approaching reactor regimes, loss of vertical control and subsequent plasma disruptions are not tolerable. Closed-loop stability has to be guaranteed for all possible equilibria and in the presence of realistic conditions of noise and disturbances.

The problem is then readily formulated in terms of identifying which variables affect vertical stability and which ones are not directly controlled/controllable; identifying the physical region of these variables, and the corresponding most unstable equilibria; and designing the control system to stabilize all equilibria with sufficient margin. The margin should be enough to allow the system to tolerate realistic scenarios of noise and disturbances.

Chapter 2 introduces the analytical tools; a set of metrics is discussed that characterizes the problem of plasma vertical stability by looking at each contribution: the stability margin \(m_s\) describes the plasma-wall interaction and the open-loop growth rate of the unstable mode; the maximum controllable displacement \(\delta z_{\text{max}}\) looks at the vertical stabilization power supplies and their ability to handle noise and off-normal events; the gain and phase margins \(m_g, m_\phi\) quantify the linear stability of the feedback control loop. A tokamak machine will be able to operate safely only above certain values of these stability metrics.

\(^2\)This result was first derived by Troyon and is known as Troyon scaling [5]. The external elongation is defined as the elongation of the last closed magnetic surface.
As may be illustrated by a simple analytical model, the stability margin decreases at larger values of the plasma internal inductance \( l_i \) [6]. Because the current profile is not well controlled in modern tokamak machines, it is important to identify the envelope of possible current profiles. An upper bound of the value of the plasma internal inductance may be derived in the form of a relationship between the internal inductance, which is not directly controlled, and the external elongation and the edge safety factor, which can be accurately controlled. In chapter 3 this result is validated with the analysis of a large database of C-Mod plasmas. It is also shown, through calibrated numerical simulations, that the stability margin of C-Mod plasmas indeed decreases at larger values of the internal inductance. This suggests that the control system should be designed to stabilize the highest-\( l_i \) plasmas obtainable under otherwise identical conditions.

Noise enters the control loop at various points and limits the measurement resolution and the control precision. A general evaluation of noise in tokamak machines is not possible, because noise is very dependent on the specific hardware, its operating conditions and its surrounding environment. However, a collection of data from existing machines may help to predict noise contributions and their implications in large-scale reactors. Chapter 4 discusses the main sources of noise and pick-ups in Alcator C-Mod and their effects on the measurement and control of the vertical position. White noise originating from the plasma and the magnetic diagnostics limit the resolution of the vertical observer and potentially affect the controllability of C-Mod plasmas at limit elongations; pick-ups at the output of the power supplies drive large oscillations of high-elongation equilibria with destabilizing effects.

Other forms of perturbations originating from the physics of tokamak plasmas are generally referred to as disturbances: they can be step-wise perturbations, such as beta drops, large ejections, etc., or periodic perturbations, such as tearing modes. The loop response to noise and disturbances can be optimized by assigning the poles of the closed-loop system with full-state feedback control. Full-state feedback control has also been proposed for the ITER vertical stabilization loop [7]. The first stage of a full-state
controller is a state observer: chapter 5 discusses the design of a state observer based on a linear Kalman filter. The filter contains a reduced-order model of the system and uses knowledge of its inputs and outputs and of the intrinsic and measurement noise to reconstruct the states of the system. Chapter 6 illustrates a single-input single-output application of the filter to reject noise from the vertical observer and improve plasma controllability at limit elongations.

Finally, chapter 7 considers one type of large-signal perturbation, i.e. the saturation of control currents. Aggressive targets and large off-normal events can cause a control current to rail. The magnetic topology is consequently perturbed and the plasma might become uncontrollable. An adaptive anti-saturation control routine is demonstrated which avoids an impending saturation by interpolating in real-time to a safe equilibrium. This approach becomes necessary when poor redundancy of control coils may require mid-shot pulse rescheduling, as opposed to an adaptation in control.
Chapter 2

Linear Model of the Plasma Vertical Position. Stability Metrics

In this chapter we discuss the equilibrium and stability conditions of a filament plasma and develop the linear model of the evolution of the vertical position, including the interaction of plasma and axisymmetric tokamak structures.

We introduce a set of metrics that characterize the problem of vertical stability: the stability margin $m_s$, the maximum controllable displacement $\delta z_{\text{max}}$ and the gain and phase margins $m_g$, $m_{\phi}$. The dependence of the stability margin on the plasma internal inductance is proven with a simple analytical model.

We also discuss the linear piece-wise simulator Alcasim, which is used at C-Mod for the simulation of full plasma discharges and the evaluation of stability metrics.

2.1 Equilibrium and Stability of a Filament Plasma

Figure 2-1 shows a section of a toroidal plasma together with the reference coordinate system.

The plasma experiences a radial expansion force given by the superposition of the hoop force, the $1/R$ force and the "tire-tube" force. The hoop force is a consequence of
Figure 2-1: Reference coordinate system. The toroidal angle $\varphi$ selects a poloidal plane. A point in the poloidal plane is uniquely identified by the Cartesian coordinates based in the center of the torus $(R, z)$ or by the poloidal coordinates based in the geometric center of the plasma cross section $(r, \vartheta)$.

the gradient of the intensity of the poloidal field, which is stronger inside the ring. The $1/R$ force is also due to a magnetic field gradient, but it is the external toroidal field that decays as $1/R$. The "tire-tube" force is caused by the kinetic and magnetic pressure inside the plasma. In the case of a low aspect-ratio ($r/R \ll 1$) circular cross-section plasma the radial expansion force is given by [8]:

$$F_R = \frac{\mu_0 I_p^2}{2} \left[ \beta_\vartheta + \frac{l_i}{2} + \ln \left( \frac{8 R_0}{a} \right) - \frac{3}{2} \right] \quad (2.1)$$

where $I_p$ is the total toroidal current, $R_0$ is the radius of the toroidal current centroid, $a$ is the plasma minor radius, $l_i$ is the plasma internal inductance and $\beta_\vartheta$ is the poloidal beta. The internal inductance is defined as $l_i \equiv \langle B_{\vartheta}^2 \rangle / B_{\vartheta}^2(a)$, where $B_\vartheta$ denotes the poloidal field and the point brackets stand for the average over the plasma cross-section. The poloidal beta is defined as $\beta_\vartheta \equiv 2 \mu_0 \langle p \rangle / B_{\vartheta}^2(a)$, where $p$ is the plasma kinetic pressure. The poloidal beta is a measure of how much field is needed to confine a plasma at a
certain pressure. It is therefore a metric of the efficiency of a magnetic configuration.

A useful approximation of a spatially extended plasma is a rigid set of filaments, each carrying a portion of the total current \( I_p \). In the presence of an external magnetic field, the plasma is subject to a Lorentz force in addition to the radial expansion force; the equilibrium conditions are:

\[
2\pi I_p \sum_{i=1}^{\text{numfil}} w_i R_i B_{zi} = -\frac{\mu_0 I_p^2}{2} \left[ \beta_\phi + \frac{l_i}{2} + \ln \left( \frac{R_0}{a} \right) \right] - \frac{3}{2} = 2\pi I_p R_0 B_{eq} \tag{2.2}
\]

\[-2\pi I_p \sum_{i=1}^{\text{numfil}} w_i R_i B_{Ri} = 0 \tag{2.3}\]

where \( w_i \) is the current fraction in the \( i \)-th filament, \( R_i \) is the radial position of the \( i \)-th filament, \( B_{zi} \) and \( B_{Ri} \) are the vertical and radial field at the location of the \( i \)-th filament and the equilibrium field is defined as:

\[
B_{eq} \equiv -\frac{\mu_0 I_p}{4\pi R_0} \left[ \beta_\phi + \frac{l_i}{2} + \ln \left( \frac{R_0}{a} \right) \right] - \frac{3}{2} \tag{2.4}
\]

Depending on the curvature of the magnetic field, the equilibrium described by equations 2.2 and 2.3 can be stable or unstable. For example, the vertical Lorentz force is expanded around the equilibrium position:

\[
F_z = \frac{\partial F_z}{\partial z} dz \tag{2.5}
\]

The plasma is vertically stable if:

\[
\frac{\partial F_z}{\partial z} = -2\pi I_p \sum_{i=1}^{\text{numfil}} w_i R_i \left( \frac{\partial B_R}{\partial z} \right)_i = -2\pi I_p \sum_{i=1}^{\text{numfil}} w_i R_i \left( \frac{\partial B_z}{\partial R} \right)_i < 0 \tag{2.6}
\]

The last identity in 2.6 is true because the magnetic field is curl-free. Introducing the curvature index of the magnetic field:
the condition for vertical stability becomes:

\[ n > 0 \]  

Although the filament model is fairly simple, it is a good approximation for studying the linear dynamics of the vertical position [9], [10]. Moreover, any residual error can be compensated by calibrating the filament position to match experimental data, as described in section 2.5.

### 2.2 Circuit Equations

Elongated shapes are obtained by adding a quadrupole field that pulls and pushes the plasma in orthogonal directions. Figure 2-2 shows the magnetic surfaces formed by the superposition of the quadrupole field, vertical field and filament plasma in a toroidal geometry. Figure 2-3 shows the external field in the same simulation. The curvature index is \( n < 0 \): if the plasma is displaced by a small distance from its equilibrium, it will continue to drift in the same direction, until it disrupts against the wall. In order to avoid a disruption, a vertical stabilization loop is designed to detect the vertical motion of the plasma and push back through magnetic fields generated by special stabilization coils.

A dynamic model of the vertical position is needed in order to design the vertical stabilization loop. The axisymmetric structures of the tokamak are discretized in toroidal elements, each one characterized by a resistance, a self-inductance and mutual inductances with the other elements. A difference is made between active elements, i.e. control coils fed by external voltages, and passive elements. The circuit equations are:

\[ n \equiv - \frac{1}{B_{eq}} \sum_{i=1}^{numfil} w_i R_i \left( \frac{\partial B_z}{\partial R} \right)_i \] (2.7)

the condition for vertical stability becomes:

\[ n > 0 \] (2.8)
Figure 2-2: Magnetic surfaces produced by the superposition of quadrupole field, vertical field and filament plasma. The intensity of the currents is arbitrary.

Figure 2-3: External field in the example in figure 2-2.
\[ \mathbf{M}_{cvu} \delta \mathbf{I}_{cv} + \dot{\mathbf{\Psi}}_{cvp} + \mathbf{R}_{cv} \delta \mathbf{I}_{cv} = \mathbf{V}_c \]  

(2.9)

where \( \delta \mathbf{I}_{cv} \equiv [\delta \mathbf{I}_c; \delta \mathbf{I}_v] \) is the vector of perturbations of active and passive currents, \( \mathbf{M}_{cvu} \) is the mutual inductance matrix of the discretized model of the tokamak, \( \mathbf{\Psi}_{cvp} \) is the magnetic flux from the plasma current coupled with active and passive elements and \( \mathbf{V}_c \) is the vector of external voltages (its coefficients are zero for passive elements). For convenience of notation \( \dot{\mathbf{F}} \equiv \frac{\partial \mathbf{F}}{\partial t} \) and \( \mathbf{F}' \equiv \frac{\partial \mathbf{F}}{\partial z} \).

In the case of a rigid vertical displacement of the plasma:

\[ \dot{\mathbf{\Psi}}_{cvp} = \mathbf{M}'_{cvp} \dot{\mathbf{z}} (\mathbf{w} \mathbf{I}_p) + \mathbf{M}_{cvp} (\mathbf{w} \mathbf{I}_p) \]  

(2.10)

where \( \mathbf{M}_{cvp} \) is the matrix of mutual inductances of plasma filaments and tokamak structures and \( \mathbf{w} \) is the vector of current weights. Furthermore, it has been shown [9], that a rigid constant current shift is never more stable or further from the exact energy minimizing eigenmode than a rigid constant flux shift, therefore it is convenient to use the approximation of constant current:

\[ \dot{\mathbf{\Psi}}_{cvp} = \mathbf{M}'_{cvp} \mathbf{w} (\mathbf{z} \mathbf{I}_p) \]  

(2.11)

The quantity \( (\mathbf{z} \mathbf{I}_p) \) is derived from the equation of motion of the plasma. If a rigid multi-filament plasma is displaced by a small distance \( dz \), it will experience a Lorentz force given by:

\[ F_z \simeq -2\pi \left( \sum_{i=1}^{\text{numfil}} w_i R_i \left( \frac{\partial B_R}{\partial z} \right) \right) * (dz \mathbf{I}_p) = 2\pi n B_{eq} (dz \mathbf{I}_p) \]  

(2.12)

The plasma will also experience a restoring force, because of the change of magnetic energy:

\[ F_{mag} = -\frac{\partial W_{pcv}}{\partial z} = -\frac{\partial}{\partial z} (\mathbf{I}_p \mathbf{w}^T \mathbf{M}_{pcv} \delta \mathbf{I}_{cv}) = -\mathbf{I}_p \mathbf{w}^T \mathbf{M}'_{pcv} \delta \mathbf{I}_{cv} \]  

(2.13)
The total vertical force is zero for a massless plasma:

\[ F_p = F_z + F_{mag} = 0 \Rightarrow 2\pi n B_{eq} (dz I_p) - I_p w^T M'_{pcv} \delta I_{cv} = 0 \] (2.14)

Differentiating equation 2.14:

\[ (dz I_p) = \frac{I_p}{2\pi n B_{eq}} \left( w^T M'_{pcv} \delta I_{cv} \right) \] (2.15)

and substituting into equations 2.11, 2.9:

\[
\begin{bmatrix} M_{cvv} + \frac{I_p}{2\pi n B_{eq}} \left( M'_{cvp} w \right) \left( w^T M'_{pcv} \right) \end{bmatrix} \delta I_{cv} + R_{cv} \delta I_{cv} = V_c
\] (2.16)

\[
M^*_{cvv} \frac{d(\delta I_{cv})}{dt} + R_{cv} \delta I_{cv} = V_c
\] (2.17)

where \( M^*_{cvv} \) is the total mutual matrix inclusive of plasma-mediated effects. Finally, equation 2.17 can be put in the familiar state-space form:

\[
\frac{d(\delta I_{cv})}{dt} = A \delta I_{cv} + B V_c
\] (2.18)

where:

\[
A = -(M^*_{cvv})^{-1} R_{cv}
\] (2.19)

\[
B = (M^*_{cvv})^{-1}
\] (2.20)

It can be shown that the linearized model remains valid during an evolving equilibrium if the normalized toroidal distribution of the plasma current and the vacuum field within the plasma do not change [11].
2.3 Plasma-wall Interaction. Stability Margin

Equation 2.16 is easy to analyze in the case of a single-filament plasma and a single wall mode:

\[
\left[ 1 + \frac{I_p}{2\pi n B_{eq} L_c} \left( M'_{cp} \right)^2 \right] L_c \delta I_c + R_c \delta I_c = 0
\]  
(2.21)

The eigenvalue of the mode is:

\[
\lambda^* = \frac{-R_c/L_c}{\frac{I_p}{2\pi n B_{eq} L_c} \left( M'_{cp} \right)^2 + 1} = -\frac{\lambda_c}{\left[ \frac{n_c}{n} - 1 \right]}
\]  
(2.22)

where \( \lambda_c \equiv -R_c/L_c \). The critical index \( n_c \) is given by:

\[
n_c \equiv -\frac{I_p}{2\pi B_{eq} L_c} \left( M'_{cp} \right)^2 = -\frac{1}{\left[ \beta_\phi + \frac{l_i}{2} + \ln \left( \frac{8R_0}{a} \right) - \frac{3}{2} \right]} \frac{2R_0 \left( M'_{cp} \right)^2}{\mu_0 L_c}
\]  
(2.23)

where equation 2.4 was used (a similar definition is in [12]). Figure 2-4 shows \( \lambda^*/|\lambda_c| \) as a function of \( n/n_c \).

When \( n > 0 \), \( \lambda^* < 0 \) and the plasma is vertically stable. When \( n_c < n < 0 \), \( \lambda^* > 0 \) and the plasma is vertically unstable. Because the plasma motion is slowed by the eddy currents in the tokamak structures, which decay with their resistive time constants, the plasma is said to be resistively unstable. In the limit case \( n \to n_c \), \( \lambda^* \to +\infty \), the walls do not provide passive stabilization and the time of the instability is the Alfvén time\(^1\): the plasma is said to be ideally unstable.

The stability margin is defined as:

\(^1\)The Alfvén time is \( \tau_A \equiv a/v_{A\phi} \), where \( a \) is the plasma minor radius and \( v_{A\phi} \) is the poloidal Alfvén speed; \( v_{A\phi} \equiv \sqrt{B_\phi^2/\mu_0 n_i m_i} \), where \( B_\phi \) is the poloidal field, \( \mu_0 \) is the vacuum magnetic permeability, \( n_i \) is the main ion density and \( m_i \) is the main ion mass. The Alfvén time is of the order of 1\( \mu \)s for typical tokamaks.
m_s \equiv -\frac{\lambda_c}{\lambda^*} = \left[ \frac{n_c}{n} - 1 \right] \quad (2.24)

m_s decreases for larger negative \( n \) and it becomes zero for an ideally unstable plasma.

The definition of \( m_s \) can be readily extended to the case of multi-mode systems [13]:

\[
m_s \equiv -\max \left\{ \text{eig} \left( \left( M_{cucv}^{-1} R_{cv} \right) \cdot \left( R_{cv}^{-1} M_{cucv}^* \right) \right) \right\} = -\max \left\{ \text{eig} \left( M_{cucv}^{-1} \cdot M_{cucv}^* \right) \right\} \quad (2.25)
\]

Because the resistance matrix \( R_{cv} \) cancels out, this metric is independent from external loads added to the control coils.
2.3.1 Plasma Inductance and Stability Margin

Physical intuition suggests that, for a given elongation, the stability margin should decrease with the internal inductance. The external quadrupole field has to pull harder in order to produce the same target elongation when the current profile is narrower.

This result can be derived analytically for an infinite aspect-ratio plasma with top-hat current profile [14]. The top-hat profile is illustrated in figure 2-5. It is uniform inside some radius $r_h$ and zero outside.

![Top-hat current profile](image)

Figure 2-5: Top-hat current profile with elongation $\kappa$.

In the case of circular cross-section, the internal inductance is easily calculated\(^2\):

$$l_i \equiv \frac{\langle B_\theta^2 \rangle}{B_{\theta e}^2} = \frac{1}{2} + 2 \ln \left( \frac{r_e}{r_h} \right)$$

(2.26)

\(^2\)Here and in the following the ITER definition of the internal inductance is used: $l_i = l_i(3) \equiv \langle B_\theta^2 \rangle / B_{\theta e}^2$, where $\langle B_\theta^2 \rangle / 2\mu_0$ is the volume average poloidal field energy density, $B_{\theta e}^2 / 2\mu_0$ is the poloidal field energy density at the edge of the plasma and $B_{\theta e} \equiv \mu_0 I_p / \sqrt{2V_p/R}$, where $V_p$ is the plasma volume and $R$ is the plasma major radius.
The correction due to the elongation can be derived analytically only for a uniform-current ellipse, as shown in Appendix A. Such correction is $2\kappa/(1 + \kappa^2)$, smaller than 20% at practical elongations, and it will be neglected in the present context.

Because there is no vertical field, the stability margin is expressed in terms of the external component of the quadrupole field $e_2$:

$$m_s \equiv \frac{e_{2c}}{e_2} - 1$$  \hfill (2.27)

where $e_{2c}$ is the critical quadrupole field at which the plasma column is ideally unstable. The quantity $e_2$ can be expressed in terms of $r_e/r_h$. Firstly, the poloidal flux is expressed in terms of the zeroth and second order moments as in equation 2.28:

$$\psi \approx \frac{\mu_0 I_p}{2\pi} \ln \left( \frac{r}{r_h} \right) + i_2 r^{-2} \cos(2\theta) + e_2 r^2 \cos(2\theta)$$  \hfill (2.28)

Secondly, the solution for a cylindrical, uniform current, infinite aspect-ratio plasma is used (see e.g. [15], [16]):

$$\psi = \psi_a \left( \frac{x^2}{a_h^2} + \frac{y^2}{b_h^2} \right) = \psi_a r^2 \left( \frac{\cos^2(\theta)}{a_h^2} + \frac{\sin^2(\theta)}{b_h^2} \right)$$  \hfill (2.29)

$$\psi_a = \frac{\mu_0 I_p}{2\pi} \frac{a_h b_h}{a_h + b_h}$$  \hfill (2.30)

where $a_h$ and $b_h$ are the minor and major axes, $x$ and $y$ are the Cartesian coordinates relative to the center of the plasma, $r = \sqrt{x^2 + y^2}$ is the minor radius and $I_p$ is the total plasma current.

Lastly, the flux and the radial field gradient at the extrema of the elliptical current region are matched. It is easily found that $i_2 = a_h^2 b_h^2 \ln(b_h/a_h) / [2 (a_h^2 + b_h^2)]$, $e_2 = \ln(b_h/a_h) / [2 (a_h^2 + b_h^2)]$, $i_2/e_2 = a_h^2 b_h^2$. In the context of this approximate model one can take $a_h^2 b_h^2 \sim r_h^4$. The elongation $\kappa$ is directly related to the strength of the total quadrupole component at $r_e$, namely $i_2 r_e^{-2} + e_2 r_e^2 = e_2 r_e^2 [1 + (r_h/r_e)^4]$. In the
limit of a narrow filamentary plasma \((r_h = 0, \ l_i = \infty)\), the external component is denoted \(e_{20}\), which depends on the target elongation. Then for that fixed elongation \(e_{20}r_e^2 = e_{2r_e^2}[1 + (r_h/r_e)^4]\). Therefore:

\[
e_2 = \frac{e_{20}}{[1 + (r_h/r_e)^4]}, \ r_h \leq r_e
\]  

(2.31)

and finally:

\[
m_s = \frac{e_{2r}}{e_{20}}[1 + (r_h/r_e)^4] - 1 = (m_{s0} + 1) [1 + (r_h/r_e)^4] - 1
\]  

(2.32)

From equations 2.26 and 2.32 one can see how the stability margin depends on the internal inductance for a given elongation. Examples are illustrated in figure 2-6.

![Figure 2-6: Stability margin as a function of the internal inductance calculated from the top-hat plasma model.](image)
2.4 Maximum Controllable Displacement. Gain and Phase Margins

The stability margin is a linear passive stabilization metric, it is based on a linear model of the plasma and the surrounding structures. Any vertical stabilization (VS) system includes two other essential components, i.e. one or more power supplies to drive the vertical stabilization coils and a control algorithm, which processes the measured vertical position to feed back to the power supplies.

A metric that includes also the VS power supply is the maximum controllable displacement \( \delta z_{\text{max}} \) [17], [18]. This is the maximum free excursion of the plasma, when the poloidal shape control (SC) is frozen and the VS loop is off, that can be turned around by a saturated command to the VS power supplies. In other words, the plasma is left free to drift at time \( t_0 \), when the inputs to the SC are frozen and the VS loop is turned off, then at time \( t_0 + \delta t \) a saturated command of the right sign is applied to the VS power supplies and the vertical motion of the plasma turns around or does not.

\( \delta z_{\text{max}} \) is a non-linear metric which could be used to characterize the system ability to handle off-normal events or to quantify the relative importance of measurement noise. In other words, disturbances such as drops in plasma beta, large ejections from edge localized modes, etc., may cause sudden vertical displacements of the plasma and \( \delta z_{\text{max}} \) gives an idea of the maximum amplitude of the disturbance from which the system can recover. \( \delta z_{\text{max}} \) also sets a threshold for the system noise: if noise is larger than the maximum controllable displacement, then the plasma is uncontrollable at the system noise level. \( \delta z_{\text{max}} \) depends on many variables, i.e. the saturation limits and the slew rate of the VS power supplies, the penetration time of the control fields through the walls, the amplitude of the noise and the growth rate of the instability.

The gain and phase margins \( m_g \) and \( m_\phi \) are linear stability metrics summarizing all the elements in the feedback loop, i.e. the plasma-wall system, the power supplies and the control system. They are calculated by opening the feedback loop, linearizing
non-linear blocks and calculating the amplitude and phase responses of the loop transfer function. These responses are also known as Bode plots. The gain margin is the value of the amplitude response at zero phase\(^3\). The phase margin is the value of the phase response at unit amplitude (zero dB). The gain and phase margins measure how far the feedback loop is from instability, i.e. they describe how tolerant the closed-loop stability is to errors in the blocks’ amplitude and phase responses. The meaning of these metrics is quite intuitive: when the loop gain is unitary at a certain frequency, a sine wave at that frequency reproposes itself unchanged after a full path around the loop, i.e. the loop oscillates. The control system is designed to maximize the gain and phase margins, while meeting other performance specifications [22].

### 2.5 Alcasim Simulator

At C-Mod the simulator Alcasim is used to simulate the evolution of plasma discharges including the current ramp-up, flattop and ramp-down [19], [21]. Alcasim is built around a Simulink model including tokamak and plasma, magnetic diagnostics, control system and power supplies, as illustrated in figure 2-7.

All power supplies are represented with known gains, low-order linear approximations to their frequency response, and current and voltage limits. The vertical stabilization power supply differs from all the others, being a chopper. Its highly non-linear input-output characteristic was derived from the analysis of a full model of the supply, as reported in appendix B.

The passive structures in Alcasim are represented by discrete axisymmetric conducting elements, as shown in figure 2-8. The parameters of this structure are based on the known geometrical and physical properties of the machine, but are calibrated by observations of coil-current tests (without plasma).

\(^3\)Here and in the following we adopt the convention of including the negative sign of the feedback loop in the phase response. When the sign is not included, the gain margin is defined as the value of the amplitude at 180 deg.
Figure 2-7: Simulink model of Alcator C-Mod. The blocks 60HzEF2L, AlternatorEF1U and AlternatorOH2U model the pick-ups at the output of the power supplies connected to the corresponding coils.
Figure 2-8: The passive structures and the active coils are modeled by discrete axisymmetric conducting elements. The plasma is modeled by eight filaments symmetrically located on a current surface.
The plasma model in Alcasim consists of eight filaments locked to a current surface: the filaments follow the evolution of the position and elongation of the current surface during the discharge. At any time point a linear set of equations is solved comprising the circuit equations, the force balance equations 2.2, 2.3 and the toroidal flux conservation, simulations are linear piece-wise [19]. The trajectories of some physical quantities used in simulations are imported from real shots or drawn from scratch with the help of specific graphic widgets. These quantities are the electron temperature on axis, for the plasma resistance; the toroidal field on axis, for the plasma resistance and the toroidal field pickup; the elongation and the top and bottom triangularity, for the position of the filaments; the safety factors on axis and at the edge of the plasma, for the plasma resistance and the current profile. The internal inductance is by default calculated from the current profile. The loop voltage is by default calculated from the plasma resistance and the plasma current, but it can be imported from EFIT reconstructions.

The DPCS block in figure 2-7 contains write and read FIFO operations, so the real-time IDL [20] control code and the Simulink model are synchronized: the DPCS routine that normally does I/O from/to the input and output cards is replaced by an equivalent routine (in the sense of having the same arguments) which reads and writes through the FIFO’s.

The minimum set of parameters to adjust before a simulation are the resistivity anomaly of the plasma $\mathcal{Z}_\sigma$, which affects the plasma resistance, and the current surface on which the filaments sit, which is identified by a scalar parameter $j/j_0$, i.e. the ratio of the toroidal current density on the surface and the peak toroidal current density. These quantities are calibrated to match respectively the loop voltage and the average curvature index in experimental shots as determined by EFIT reconstructions. Calibrated simulations have been able to correctly model the overall coil-current trajectories and to reproduce the marginal stability of high-elongation C-Mod plasmas, the vertical oscillation frequency and the time constant of VDEs. The calibration essentially compensates for the uncertainties and inaccuracies in the models of the plasma and the structures
so that the ratio of the passive structure response \((n_c)\) and the effective destabilizing curvature \((n)\) represents the actual machine realistically. The importance of this calibration approach is that it enables us then to extract from the model accurate experimental values of stability parameters.

The linear model of the plant can be output at any instant during a discharge and the stability metrics \(m_s, m_g, m_\phi\) can be calculated. Furthermore, non-linear time-domain simulations can reproduce the experiments to measure the maximum controllable displacement \(\delta z_{\text{max}}\).

### 2.6 Conclusions

We discussed the linear model of the plasma vertical stability and introduced a set of metrics to characterize each aspect of the problem: the stability margin \(m_s\) describes the plasma-wall interaction and is directly related to the open-loop growth rate of the unstable mode; the maximum controllable displacement \(\delta z_{\text{max}}\) describes the ability of the vertical stabilization power supplies to handle noise and off-normal events; the gain margin \(m_g\) and the phase margin \(m_\phi\) describe how far the feedback loop is from instability. The metrics need to be evaluated for conservative and for marginally stable equilibria in C-Mod in order to describe the safe operational region. The dependence of the stability margin on the current profile, proven with a simple analytical model, suggests that high-\(\kappa\) high-\(l_t\) equilibria are the most unstable.
Chapter 3

Plasma Inductance and Stability
Metrics on Alcator C-Mod

In section 2.3.1 we showed that the stability margin decreases with the plasma internal inductance. Unfortunately this quantity is not well controlled, therefore the entire physical space of $l_i$ should be identified and the most unstable accessible equilibria selected for robust control design.

In this chapter we explore the physical space of $l_i$, comparing a theoretical upper bound with the analysis of a large database of C-Mod plasmas. We also use Alcasim to evaluate the stability metrics for a variety of C-Mod equilibria to give an example of the C-Mod operational space and of feasible control conditions.

3.1 Physical Space of the Plasma Internal Inductance

Based on the top-hat current profile model, a simple theoretical upper bound can be derived for $l_i$. The top-hat profile corresponds to the highest possible $l_i$ for given $q_a$ and
$q_0$, where $q_a$ and $q_0$ are the safety factor at the edge and at the center of the plasma. Therefore, for a circular cross-section plasma, it can be generally written:

$$l_i \leq \frac{1}{2} + 2 \ln \left( \frac{r_a}{r_h} \right)$$

(3.1)

The dependence from $q_a$ and $q_0$ becomes explicit after taking into account the identity:

$$\left( \frac{r_e}{r_h} \right)^2 = \frac{q_a}{q_0}$$

(3.2)

Therefore:

$$l_i \leq \frac{1}{2} + \ln \left( \frac{q_a}{q_0} \right)$$

(3.3)

The effect of the external elongation, $\kappa_a$, can be incorporated as an approximate correction factor:

$$l_i \leq \left[ \frac{1}{2} + \ln \left( \frac{q_a}{q_0} \right) \right] \frac{2\kappa_a}{1 + \kappa_a^2}$$

(3.4)

The correction factor is exact only for a uniform-current ellipse, for the ITER $l_i$ definition, but in any case deviates from unity by < 20% for relevant plasmas. In general $q_0 = 1$, therefore equation 3.4 is a good approximation of the relationship between external elongation, edge safety factor and current profile, in the form of an upper bound on the value of the internal inductance.

Figures 3-1 and 3-2 show scatter plots of $l_i$ vs $q_a = q_{95}$ from equilibria of non-disrupting C-Mod plasmas in the years 2004-2007$^2$. The values are obtained from EFIT reconstructions using 52 flux loops and poloidal field sensors. The color coding in figure 3-1 represents the time during the discharge. Times earlier than about 0.35s can be considered in C-Mod to be part of the current ramp-up, while times after 1.6s are usually in the current ramp-down. The color coding in figure 3-2 represents the elongation of these equilibria.

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$^1$This is true under the assumption that the toroidal current density does not change sign.

$^2$q_{95} is defined as the safety factor at 95% of the flux of the last closed flux surface.
The dotted curves are maximum allowable values of $l_i$ calculated from equation 3.4, for two different values of the elongation. The agreement of observed range of equilibria and the theoretical scaling is quite good.

Figure 3-1: Scatter plot of $l_i$ vs $q_{95}$ from equilibria of non-disrupting C-Mod plasmas. The color is the time during the discharge. The dotted curves are maximum allowable values of $l_i$ calculated from equation 3.4.

Figures 3-3 and 3-4 show the scatter plots of $l_i$ vs $\kappa_a$. The color coding in figure 3-3 represents the time during the discharge, while the color coding in figure 3-4 represents the value of $q_{95}$ for these equilibria. The dotted curves are maximum allowable values of $l_i$ calculated from equation 3.4, for special cases of the safety factor $q_{95}$. The trends of the database are in reasonable agreement with the analytic curves.

The database analysis shows that at high values of $q_{95}$ narrow current-channel plasmas readily reach $l_i > 1.0$. Recent Corsica and TSC simulations of the ITER startup
Figure 3-2: Scatter plot of $l_i$ vs $q_{95}$ from equilibria of non-disrupting C-Mod plasmas. The color is the elongation of these equilibria. The dotted curves are maximum allowable values of $l_i$ calculated from equation 3.4.
Figure 3-3: Scatter plot of $l_i$ vs $\kappa_a$ from equilibria of non-disrupting C-Mod plasmas. The color is the time during the discharge. The dotted curves are maximum allowable values of $l_i$ calculated from equation 3.4.
Figure 3-4: Scatter plot of $l_i$ vs $\kappa_\alpha$ from equilibria of non-disrupting C-Mod plasmas. The color is the safety factor $q_{95}$ of these equilibria. The dotted curves are maximum allowable values of $l_i$ calculated from equation 3.4.
phase confirm this prediction [23], [24]. These values of $l_i$ exceed the original design specifications of the ITER vertical stabilization system [17], and lead to plasmas with lower margins of stability.

### 3.2 Stability Metrics on Alcator C-Mod

Calibrated Alcasim simulations were used to evaluate the stability metrics of a variety of C-Mod plasmas. Figure 3-5 shows the stability margin $m_s$ as a function of the internal inductance $l_i$ for C-Mod plasmas with $q_{95}/q_0 = 3.0 \div 3.7$, similar to the ITER targets. The vertical error bars represent the calibration uncertainty, while the horizontal error bars represent the uncertainty in estimating the value of $l_i$ from EFIT reconstructions. Results are shown for three different elongations $\kappa_a \simeq 1.6, 1.7, 1.85$. The higher elongation cases ended in VDEs caused by the current saturation of the VS power supply. The large range of values of $l_i$ for $\kappa_a \simeq 1.7$ correlates with the large range of values of $\beta_p$ considered in this case. The smaller stability margin at larger values of the internal inductance is consistent with the analytical derivation in 2.3.1.

Table 3.1 summarizes the equilibria analyzed for the curves in figure 3-5.

---

3 The horizontal error bar represents the scatter in estimating the value of $l_i$ from EFIT reconstructions of various nominally identical plasmas. The bottom extremities of the vertical error bars are calculated with filament position such that the average field curvature seen by the filaments matches the curvature index at the current centroid output by EFIT. The top extremities are with filament position such that the average field curvature seen by the filaments matches the average over the current profile calculated from the EFIT reconstruction. We have shown by analysis of a variety of plasmas that the field index that gives agreement with machine behavior lies between these two extremes.
Figure 3-5: The stability margin $m_s$ of ITER-95 plasmas plotted against $l_i$. The vertical error bars are the uncertainty of the linear model used to calculate $m_s$. The horizontal error bar is the uncertainty of the reconstructed $l_i$. 
<table>
<thead>
<tr>
<th>Shot</th>
<th>Time [s]</th>
<th>$\beta_p$ [%]</th>
<th>$q_{95}$</th>
<th>$\kappa_a$</th>
<th>$l_i$</th>
<th>$m_a$</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1050725003$^4$</td>
<td>1.0</td>
<td>38</td>
<td>3.2</td>
<td>1.60</td>
<td>0.960</td>
<td>0.91 $\div$ 1.1</td>
<td>$H^5$(RF$^6$)</td>
</tr>
<tr>
<td>1050718022$^7$</td>
<td>1.0</td>
<td>23</td>
<td>3.4</td>
<td>1.62</td>
<td>1.03</td>
<td>0.67 $\div$ 0.82</td>
<td>$L^8$(RF)</td>
</tr>
<tr>
<td>1050725022$^9$</td>
<td>1.3</td>
<td>18</td>
<td>3.4</td>
<td>1.58</td>
<td>1.06</td>
<td>0.71 $\div$ 0.83</td>
<td>$L$ (RF)</td>
</tr>
<tr>
<td>1050329011$^{10}$</td>
<td>1.3</td>
<td>22</td>
<td>3.5</td>
<td>1.65</td>
<td>1.12</td>
<td>0.55 $\div$ 0.66</td>
<td>$L$ (RF)</td>
</tr>
<tr>
<td>1050804018$^{11}$</td>
<td>0.80</td>
<td>37</td>
<td>3.7</td>
<td>1.70</td>
<td>0.922</td>
<td>0.63 $\div$ 0.78</td>
<td>$L \rightarrow H$</td>
</tr>
<tr>
<td>1050804018</td>
<td>1.0</td>
<td>33</td>
<td>3.5</td>
<td>1.70</td>
<td>0.873</td>
<td>0.76 $\div$ 0.91</td>
<td>$H$ (RF)</td>
</tr>
<tr>
<td>1050406016$^{12}$</td>
<td>0.90</td>
<td>22</td>
<td>3.2</td>
<td>1.68</td>
<td>0.973</td>
<td>0.47 $\div$ 0.60</td>
<td>$H$ (RF)</td>
</tr>
<tr>
<td>1050406016</td>
<td>1.1</td>
<td>16</td>
<td>3.2</td>
<td>1.68</td>
<td>1.03</td>
<td>0.37 $\div$ 0.48</td>
<td>$H \rightarrow L$</td>
</tr>
<tr>
<td>1050406016</td>
<td>1.3</td>
<td>13</td>
<td>3.2</td>
<td>1.70</td>
<td>1.07</td>
<td>0.34 $\div$ 0.44</td>
<td>$L$</td>
</tr>
<tr>
<td>1040413021$^{13}$</td>
<td>1.2</td>
<td>9.0</td>
<td>3.3</td>
<td>1.67</td>
<td>1.11</td>
<td>0.37 $\div$ 0.42</td>
<td>$L$ (Ohmic$^{14}$)</td>
</tr>
<tr>
<td>1040413021</td>
<td>1.3</td>
<td>8.0</td>
<td>3.3</td>
<td>1.66</td>
<td>1.16</td>
<td>0.36 $\div$ 0.40</td>
<td>$L$ (Ohmic)</td>
</tr>
<tr>
<td>1040408026$^{15}$</td>
<td>0.90</td>
<td>18</td>
<td>3.2</td>
<td>1.66</td>
<td>1.20</td>
<td>0.38 $\div$ 0.45</td>
<td>$L$ (Ohmic, high-density)</td>
</tr>
<tr>
<td>1040408026</td>
<td>1.0</td>
<td>20</td>
<td>3.4</td>
<td>1.68</td>
<td>1.23</td>
<td>0.37 $\div$ 0.43</td>
<td>$L$ (Ohmic, high-density)</td>
</tr>
<tr>
<td>1040127010$^{16}$</td>
<td>0.83</td>
<td>15</td>
<td>3.0</td>
<td>1.85</td>
<td>0.857</td>
<td>0.42 $\div$ 0.64</td>
<td>$L$ (Ohmic)</td>
</tr>
<tr>
<td>1040127010</td>
<td>0.88</td>
<td>11</td>
<td>3.0</td>
<td>1.85</td>
<td>0.890</td>
<td>0.38 $\div$ 0.57</td>
<td>$L$ (Ohmic)</td>
</tr>
<tr>
<td>1040127010</td>
<td>0.92</td>
<td>10</td>
<td>3.0</td>
<td>1.85</td>
<td>0.920</td>
<td>0.33 $\div$ 0.50</td>
<td>$L$ (Ohmic)</td>
</tr>
</tbody>
</table>

Table 3.1: Equilibria analyzed for the curves in figure 3.5.

$^4$ Alcasim simulation settings: loop voltage from EFIT reconstruction, $j/j_0 = 0.65 \div 0.55$, $q_0$ calculated from $l_i$.

$^5$ High-confinement mode.

$^6$ Radio frequency heating of the plasma.

$^7$ Alcasim simulation settings: $Z_\sigma = 1.2$, $j/j_0 = 0.65 \div 0.55$, $q_0$ calculated from $l_i$.

$^8$ Low-confinement mode.

$^9$ Alcasim simulation settings: $Z_\sigma = 1.4$, $j/j_0 = 0.60 \div 0.50$, $q_0$ calculated from $l_i$.

$^{10}$ Alcasim simulation settings: $Z_\sigma = 1.4$, $j/j_0 = 0.62 \div 0.52$, $q_0$ calculated from $l_i$.

$^{11}$ Alcasim simulation settings: $Z_\sigma = 1.0$, $j/j_0 = 0.60 \div 0.50$, $q_0$ calculated from $l_i$.

$^{12}$ Alcasim simulation settings: $Z_\sigma = 1.2$, $j/j_0 = 0.70 \div 0.60$, $q_0$ calculated from $l_i$.

$^{13}$ Alcasim simulation settings: $Z_\sigma = 0.90$, $j/j_0 = 0.85 \div 0.75$, $q_0$ calculated from $l_i$.

$^{14}$ Ohmic heating only.

$^{15}$ Alcasim simulation settings: $Z_\sigma = 0.50$, $j/j_0 = 0.80 \div 0.70$, $q_0$ calculated from $l_i$.

$^{16}$ Alcasim simulation settings: $Z_\sigma = 1.4$, $j/j_0 = 0.85 \div 0.75$, $q_0$ calculated from $l_i$. 

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Figure 3-6 compares $m_s$ between ITER-$q_{95}$ ($q_{95} = 3.2 \div 3.7$, $\kappa_a = 1.66 \div 1.70$, $\beta_p = 8.0 \div 37\%$) and typical C-Mod plasmas ($q_{95} = 4.5 \div 5.6$, $\kappa_a = 1.65 \div 1.72$, $\beta_p = 18 \div 75\%$).

ITER-$q_{95}$ equilibria have lower values of the poloidal beta and lower values of the stability margin. The dependence of the stability margin on the poloidal beta results from various competing effects:

1. the $(\beta_p + l_i/2)$ dependence of the denominator of the critical index $n_c$ in equation 2.23;

2. the net effect of an outward shift of the current centroid (Shafranov shift) at larger values of beta, comprising: the stronger coupling with the outer wall (stabilizing effect); the weaker coupling with the inner wall (destabilizing effect). The coupling appears through $M'_{cp}$ in equation 2.23.

3. the smaller curvature index $n$ sampled by the plasma at larger values of beta because of the outward shift of the current centroid (Shafranov shift).

In cases where the curvature index changes significantly with the radial position, 3. may dominate over 1. and 2. (if destabilizing) and the plasma is more stable at larger values of beta: this seems to be the case in C-Mod, according to calibrated Alcasim simulations\(^{17}\).

Table 3.2 summarizes the equilibria analyzed for the curves in figure 3-6.

---

\(^{17}\)Alcasim simulations account for an increasing radial shift at larger values of beta via the radial force-balance equation.
Comparison of $m_s$ between ITER-$q_{95}$ and typical C-Mod plasmas

Figure 3-6: The stability margin $m_s$ of C-Mod typical plasmas plotted against $l_i$. 
Table 3.2: Equilibria analyzed for the curves in figure 3.6.

Figure 3-7 shows $m_s$ as a function of the elongation $\kappa_a$ ($q_{95} \simeq 5, l_i \simeq 1.1$). The simulator was calibrated to reproduce the experimental condition of marginal stability of this plasma at peak elongation. Stability margin $m_s = 0.26$ is the observed limit for vertical stabilization on C-Mod, with the fast controller currently used (mostly derivative).

Table 3.3 summarizes the equilibria analyzed for the curves in figure 3-7.

18 Alcasim simulation settings: $Z_\sigma = 1.2, j/j_0 = 0.75 \div 0.65, q_0$ calculated from $l_i$.

19 Alcasim simulation settings: $Z_\sigma = 1.2, j/j_0 = 0.50 \div 0.40, q_0$ calculated from $l_i$.

20 Alcasim simulation settings: $Z_\sigma = 1.2, j/j_0 = 0.40 \div 0.30, q_0$ calculated from $l_i$.

21 Alcasim simulation settings: $Z_\sigma = 0.90, j/j_0 = 0.50 \div 0.40, q_0$ calculated from $l_i$.

22 Alcasim simulation settings: $Z_\sigma = 0.70, j/j_0 = 0.50 \div 0.42, q_0$ calculated from $l_i$.

23 Alcasim simulation settings: $Z_\sigma = 0.70, j/j_0 = 0.50 \div 0.42, q_0$ calculated from $l_i$.

24 Alcasim simulation settings: $Z_\sigma = 1.0, j/j_0 = 0.45 \div 0.35, q_0$ calculated from $l_i$. 
The stability margin $m_s$ calculated for shot 1070406015 as a function of $\kappa_a$. The observed C-Mod control limit is $m_s \simeq 0.26$, everything below is closed-loop unstable.

<table>
<thead>
<tr>
<th>Shot</th>
<th>Time [s]</th>
<th>$q_{95}$</th>
<th>$\kappa_a$</th>
<th>$l_i$</th>
<th>$m_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1070406015$^{25}$</td>
<td>0.75</td>
<td>4.6</td>
<td>1.72</td>
<td>1.12</td>
<td>0.35 ÷ 0.41</td>
</tr>
<tr>
<td>1070406015</td>
<td>0.90</td>
<td>4.7</td>
<td>1.76</td>
<td>1.10</td>
<td>0.28 ÷ 0.35</td>
</tr>
<tr>
<td>1070406015</td>
<td>1.1</td>
<td>4.9</td>
<td>1.80</td>
<td>1.08</td>
<td>0.26 ÷ 0.32</td>
</tr>
<tr>
<td>1070406015</td>
<td>1.3</td>
<td>5.2</td>
<td>1.85</td>
<td>1.06</td>
<td>N.A. ÷ 0.28</td>
</tr>
</tbody>
</table>

Table 3.3: Equilibria analyzed for the curves in figure 3.7.

The gain and phase margins are obtained from the Bode plots of the linearized model of the vertical control loop, which is shown in figure 3-8.

The chopper is a highly non-linear device and is described in appendix B. Its linear approximation consists of a DC gain of 80 and a 4-th order Butterworth filter with cut-off at 800Hz. The former is calibrated to reproduce the condition of marginal stability of

$^{25}$Alcasim simulation settings: $Z_{\sigma} = 0.70$, $j/j_0 = 0.50 \div 0.42$, $q_0$ calculated from $l_i$. 

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Figure 3-8: Linearized model of the vertical control loop. The fast ZCUR controller stabilizes the vertical position of the plasma and is mostly derivative. The slow ZCUR controller controls the vertical position of the plasma and is mostly proportional and integral.

1070406015 at maximum elongation. The latter is used to model the filter at the output of the AFOL link between the control system and the input stage of the supply. Figures 3-9, 3-10 show the Bode plots for the uppermost equilibria of the error bars in figure 3-7.

Table 3.4 summarizes $m_s$, $m_g$ and $m_\phi$ for the equilibria in figure 3-7.

<table>
<thead>
<tr>
<th>$\kappa_a$</th>
<th>$\lambda$ [rad/s]</th>
<th>$m_s$</th>
<th>$m_g$ [dB]</th>
<th>$m_\phi$ [deg]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.72</td>
<td>210</td>
<td>0.35 \div 0.41</td>
<td>6.5 \div 9.5</td>
<td>14 \div 23</td>
</tr>
<tr>
<td>1.76</td>
<td>260</td>
<td>0.28 \div 0.35</td>
<td>3.3 \div 7.9</td>
<td>8.0 \div 17</td>
</tr>
<tr>
<td>1.80</td>
<td>310</td>
<td>0.26 \div 0.32</td>
<td>1.7 \div 6.2</td>
<td>4.0 \div 14</td>
</tr>
<tr>
<td>1.85</td>
<td>410</td>
<td>N.A. \div 0.28</td>
<td>N.A. \div 3.2</td>
<td>N.A. \div 7.5</td>
</tr>
</tbody>
</table>

Table 3.4: Stability metrics for standard and high-elongation C-Mod plasmas.

The linear stability metrics were not measured by explicit perturbative experiments. However, the calibration procedure described in 2.5 should correctly compensate for
Figure 3-9: Bode plots for four target elongations. The low frequency phase response is shifted by 90 deg because the plasma responds primarily to the time derivative of the magnetic flux. The upward bend starting at 20Hz is largely determined by the fast derivative controller. The frequencies of zero-phase crossing are thus moved away from the frequencies where the magnitude is close to unity, and the feedback loop is stabilized.
Figure 3-10: Details of Bode plots.
errors in the models of the structures and the plasma and therefore the simulation-based
determinations are good estimates of the experimental parameters.

3.2.1 Experimental Measurement of the Maximum Controllable
Displacement

The maximum controllable displacement is an important metric to describe the ability
of actuators (vertical stabilization coils) and power supplies to control off-normal events
or simply to cope with the system noise. A database of values of $\delta z_{\text{max}}$ at which existing
machines operate can be compared with ITER simulations in order to understand if the
design margin of the ITER vertical stabilization system is sufficient at target elongations.

The maximum controllable displacement $\delta z_{\text{max}}$ was experimentally measured in C-
Mod for different elongations and compared with non-linear Alcasim simulations. The
experimental procedure was slightly different from what is described in section 2.4 in that
the control gains of the vertical stabilization loop were zeroed at $t_0$ to allow the plasma
to drift freely in one direction, then the proportional gain was set to a very large value at
$t_0 + \delta t$, to assure that a saturated command was sent to the chopper with the right sign.
This also implies that the plasma could oscillate a few times before finally disrupting, as
shown in figure 3-12. Figures 3-11, 3-12 show the vertical position, the chopper demand
and the chopper current during two experiments.

The experimental measurement of $\delta z_{\text{max}}$ is significantly complicated by the lack of
control of the initial kick to the plasma and by the presence of large pick-ups on the
power supplies that drive vertical oscillations, therefore we can only indicate confidence
intervals. The simulated values of $\delta z_{\text{max}}$ are calculated for the uppermost equilibria of
the error bars in figure 3-7, therefore they represent an upper bound for the maximum
controllable displacement. The experimental and simulated values of $\delta z_{\text{max}}$, denoted
respectively $\delta z_{\text{exp}}$ and $\delta z_{\text{sim}}$, are summarized in table 3.5 and plotted in figure 3-13.
Figure 3-11: Vertical position, chopper demand and chopper current during the $\delta z_{\text{max}}$ experiment 1071211017. The control gains are zeroed at 0.8s when $\kappa_a = 1.72$. The initial perturbation of the plasma vertical position is small and the plasma drifts only $\sim 7\text{mm}$ by the time the proportional gain is back on at 0.808s.
Figure 3-12: Vertical position, chopper demand and chopper current during the $\delta z_{\text{max}}$ experiment 1071211016. The control gains are zeroed at 0.8s when $\kappa_a = 1.72$. There are large oscillations starting at 0.805s when the proportional gain is back on. From the last cycle before disruption it is possible to conclude that $\delta z_{\text{max}} \sim 28\text{mm}$ at this elongation.
\[ \kappa_a \quad \lambda \quad \delta z_{\text{exp}} \quad \delta z_{\text{exp}} / \alpha \quad \delta z_{\text{exp}} / \sigma_z \quad \delta z_{\text{sim}} \]

<table>
<thead>
<tr>
<th>\kappa_a</th>
<th>\lambda \quad \text{rad/s}</th>
<th>\delta z_{\text{exp}} \quad \text{cm}</th>
<th>\delta z_{\text{exp}} / \alpha \quad %</th>
<th>\delta z_{\text{exp}} / \sigma_z</th>
<th>\delta z_{\text{sim}} \quad \text{cm}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.72</td>
<td>210</td>
<td>3 \div 2</td>
<td>14 \div 10</td>
<td>30 \div 20</td>
<td>&lt; 4.1</td>
</tr>
<tr>
<td>1.76</td>
<td>260</td>
<td>2 \div 1</td>
<td>10 \div 5</td>
<td>20 \div 10</td>
<td>&lt; 3.6</td>
</tr>
<tr>
<td>1.80</td>
<td>310</td>
<td>\lesssim 1</td>
<td>\lesssim 5</td>
<td>\lesssim 10</td>
<td>&lt; 2.1</td>
</tr>
<tr>
<td>1.85</td>
<td>410</td>
<td>N.A. \textsuperscript{26}</td>
<td>N.A.</td>
<td>N.A.</td>
<td>&lt; 1.0</td>
</tr>
</tbody>
</table>

Table 3.5: Maximum controllable displacement for standard and high-elongation C-Mod plasmas.

- In table 5, \( \alpha \) is the minor radius of the plasma (21 cm) and \( \sigma_z \) is the RMS value of the measurement noise, typically \( \sigma_z \sim 1 \text{mm} \) for C-Mod. The ratio \( \delta z_{\text{exp}} / \sigma_z \) is a useful metric.

\textsuperscript{26}At elongations \( \kappa_a > 1.80 \) it was not possible to measure the characteristic \( \delta z_{\text{max}} \) evolution and have a convincing estimate for this metric: this is mostly because the effects of noise and pick-ups become comparable with \( \delta z_{\text{max}} \) at these elongations.
to assess the importance of measurement noise. At elongations \( \kappa_a \leq 1.80 \) broadband noise is not a major concern in C-Mod (\( \delta z_{\text{exp}} / \sigma_z \geq 10 \)), however, at elongations approaching the experimental limit \( \kappa_a \sim 1.85 \), it becomes \( \delta z_{\text{exp}} \sim 1 mm \) (extrapolating from the results in figure 3-13) and \( \delta z_{\text{exp}} / \sigma_z \sim 1 \): the system noise might cause a displacement that is not controllable.

C-Mod operates safely at \( \delta z_{\text{exp}} / a \geq 5\% \), \( \delta z_{\text{exp}} / \sigma_z \geq 10 \), similar values have been reported for DIII-D [28].

### 3.3 Conclusions

We showed that there are physical limits to the values of the internal inductance, given the external elongation and the edge safety factor. A large database of C-Mod plasmas spans almost the full range of possible inductances and shows that low-current, high-\( q_{95} \) equilibria have values of the internal inductance \( l_i > 1.0 \) for target elongations \( \kappa_a > 1.7 \). It seems prudent that the ITER design specification should take this full range into account.

We also found that the safe operational region of C-Mod is described by \( m_s \geq 0.3 \), \( m_\varphi \geq 10 \) deg, \( \delta z_{\text{max}} / a \geq 5\% \), \( \delta z_{\text{exp}} / \sigma_z \geq 10 \). The observed limit for vertical stabilization is \( m_s = 0.26 \): higher-order controllers may help lower the stability margin of controllable plasmas nearer to the ideal instability limit, however noise may become an issue with these controllers and appropriate filtering might become necessary.

The stability metrics discussed are dimensionless and quite generally applicable to tokamak machines, however it cannot be assumed that their numerical values immediately transfer to a different tokamak: the linear stability metrics are likely to be affected by a number of machine-specific features, such as the exact wall and coil geometry and materials; similarly, the maximum controllable displacement depends on power supply features, such as the slew rate and current limits. Rather, they should be regarded as examples of practically feasible control.
Chapter 4

Noise and Pick-ups in Alcator C-Mod

The control of the vertical position of tokamak plasmas is adversely affected by noise and pick-ups entering at various points in the feedback control loop: noise reduces measurement resolution and control precision and may compromise plasma control when its magnitude approaches the maximum controllable displacement; the high frequency components are amplified by high-order controllers and propagate to the power supplies to increase power consumption and AC losses.

A collection of noise data from existing tokamaks may help to predict noise contributions and their implications in large-scale reactors. In this chapter we discuss the main sources of noise and pick-up in Alcator C-Mod and identify their effects on the measurement and control of the vertical position.

4.1 Measurement of Noise and Pick-ups in Alcator C-Mod

Generally speaking, noise is any spurious signal that leaks into the control loop. In the following, noise will be used for broadband random processes and pick-up will be used
for narrowband interferences. Plasma noise refers to spurious signals arising from the plasma itself. Figure 4-1 shows the sources of noise and pick-up identified in Alcator C-Mod.

![Diagram of noise sources in Alcator C-Mod](image)

Figure 4-1: Sources of noise and pick-up in Alcator C-Mod. The main sources of noise are indicated in red: the current noise of the power supplies, the white noise of the magnetic diagnostics and the plasma noise. The main pick-ups are indicated in blue: the alternator and line frequency at the power supplies and the line frequency in the magnetic diagnostics. The chopper bang-bang behavior is mostly negligible because of its high frequency.

The main sources of noise are indicated in red: the current noise of the power supplies, the white noise of the magnetic diagnostics and the plasma noise. The main pick-ups are indicated in blue: the alternator and line frequency at the power supplies and the line frequency in the magnetic diagnostics. The chopper bang-bang behavior is mostly negligible because of its high frequency\(^1\). The focus is on how noise and pick-ups affect the control of the plasma vertical position. On C-Mod, this is reconstructed from magnetic

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\(^{1}\) The chopping frequency at zero input demand is \(\sim 3.5kHz\), however large-signal modulation during chopper operation might bring this frequency into the band of interest.
measurements through a well validated linear observer \cite{ZCUR Observer in figure 4-1}, whose output is the product of the vertical position of the current centroid and the plasma current. A first feedback loop $PID \text{ Slow}$, with proportional, derivative and integral gains, controls the vertical position on the time scale of the poloidal equilibrium (i.e. fractions of a second); the output of the PID controller generates a demand to all of the equilibrium field power supplies with the vector of coefficients $Slow \ Z \ Controller$. A second loop $PD \text{ Fast}$, with proportional and derivative gains, stabilizes the vertical position on the time scale of the vertical instability (i.e. milliseconds); the output of the PD controller drives only the chopper with the gain $Fast \ Z \ Controller$. The chopper is connected to the anti-series EFC coil pair, which is the actuator for vertical stabilization. The bandwidth of the fast loop is cut off at $800Hz$ by a 4-th order Butterworth filter at the output of the AFOL link of the chopper.

The noise and pick-up of the magnetic diagnostics were estimated from open-loop tests with no power supplies and no plasma (for example, shot 1060321015). The typical $60Hz$ pick-up is $\sim 2mV$ and the broadband contribution is $\sim 2.3mV$ on each channel. In physical units, the latter figure is equivalent to $\sim 1.2mWb$ on the flux loops and $\sim 1.2mT$ on the pick-up coils. The total pick-up on the vertical observer is $\sim 13mV$ and the broadband contribution is approximately the same value $\sim 13mV$. In physical units, $13mV \equiv 0.3mm \cdot MA$.

The current noise generated by the fluctuations of the output of the power supplies was estimated from power supply test shots at constant demand. The baseline from shot 1060321015 was subtracted from the variance of the DPCS inputs corresponding to the active currents in order to isolate the power supply contribution. Table 4.1 summarizes our results.
<table>
<thead>
<tr>
<th>PLC test</th>
<th>DPCS input</th>
<th>Shot #</th>
<th>$\sigma_{\text{Physical}}$ [A]</th>
</tr>
</thead>
<tbody>
<tr>
<td>OH1 forward</td>
<td>66</td>
<td>1060207022</td>
<td>29</td>
</tr>
<tr>
<td>OH1 reverse</td>
<td>66</td>
<td>1060207023</td>
<td>29</td>
</tr>
<tr>
<td>EFC$^2$</td>
<td>77</td>
<td>1060208005</td>
<td>24</td>
</tr>
<tr>
<td>OH2U forward</td>
<td>67</td>
<td>1060207024</td>
<td>42</td>
</tr>
<tr>
<td>OH2U reverse$^3$</td>
<td>67</td>
<td>1060207025</td>
<td>120$^4$</td>
</tr>
<tr>
<td>OH2L forward</td>
<td>68</td>
<td>1060207026</td>
<td>52</td>
</tr>
<tr>
<td>OH2L reverse</td>
<td>68</td>
<td>1060207027</td>
<td>55</td>
</tr>
<tr>
<td>EF1U forward$^5$</td>
<td>69</td>
<td>1060207017</td>
<td>16</td>
</tr>
<tr>
<td>EF1U reverse$^6$</td>
<td>69</td>
<td>1060207019</td>
<td>13</td>
</tr>
<tr>
<td>EF1L forward$^7$</td>
<td>70</td>
<td>1060207020</td>
<td>15</td>
</tr>
<tr>
<td>EF1L reverse$^8$</td>
<td>70</td>
<td>1060207021</td>
<td>13</td>
</tr>
<tr>
<td>EF2U</td>
<td>71</td>
<td>1060207010</td>
<td>5</td>
</tr>
<tr>
<td>EF2L$^9$</td>
<td>72</td>
<td>1060207011</td>
<td>18</td>
</tr>
<tr>
<td>EF3</td>
<td>73</td>
<td>1060207013</td>
<td>78</td>
</tr>
<tr>
<td>EF4 forward$^{10}$</td>
<td>75</td>
<td>1060207014</td>
<td>10</td>
</tr>
<tr>
<td>EF4 reverse$^{11}$</td>
<td>75</td>
<td>1060208003</td>
<td>11</td>
</tr>
</tbody>
</table>

Table 4.1: Noise and pick-ups in C-Mod power supplies.

$^2$Shot 1060208005 was used to evaluate the broadband noise of the chopper. The input is set at zero, the large tone at 3.6kHz is removed and the power spectrum is integrated. The EFC current has a large 3.6kHz component. In general this frequency changes as a function of the DC input and broadens if the input signal is modulated. However, a close look at the spectra of the DPCS input signals 1 ÷ 57 shows that the tone doesn’t significantly couple to the magnetics, it is screened by the vessel and structures. This high frequency is unlikely to affect the plasma itself, so it can be neglected to first order.

$^3$Large tone at the output of the supply of amplitude $\sim 23V$ at $56 \pm 2Hz$.

$^4$Most of this current comes from a large tone at $56 \pm 2Hz$.

$^5$Large tones at the output of the supply of amplitude $\sim 29V$ at $56 \pm 2Hz$ and $\sim 14V$ at $170 \pm 2Hz$.

$^6$Large tone at the output of the supply of amplitude $\sim 23V$ at $56 \pm 2Hz$.

$^7$Large tone at the output of the supply of amplitude $\sim 26V$ at $170 \pm 2Hz$.

$^8$Large tone at the output of the supply of amplitude $\sim 38V$ at $170 \pm 2Hz$.

$^9$Large tone at the output of the supply of amplitude $\sim 60V$ at $60 \pm 2Hz$.

$^{10}$Large tones at the output of the supply of amplitude $\sim 38V$ at $56 \pm 2Hz$ and $\sim 44V$ at $170 \pm 2Hz$.

$^{11}$Large tones at the output of the supply of amplitude $\sim 40V$ at $56 \pm 2Hz$ and $\sim 50V$ at $170 \pm 2Hz$. 

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The signals produced in the magnetic diagnostics by these currents can exceed the diagnostics noise in a few specific channels, but the overall effect on the vertical observer is negligible. To prove this point, figure 4-2 shows the spectra of the vertical observer during some power supply test shots and compares it with the baseline from shot 1060321015.

![Comparison of ZCUR power spectra during power supply test shots.](image-url)

Figure 4-2: Comparison of ZCUR power spectra during power supply test shots.

The output of the EF2L power supply shows significant pick-up at the line frequency $60\,Hz$ and harmonics. Similarly, the output of the EF1U and OH2U power supplies show significant pick-up at the alternator frequency $\sim 54\,Hz$ and harmonics\[^{12}\]. The direct coupling of the pick-ups with the vertical observer is negligible, as can be seen from figure 4-2, however they do limit the performance of C-Mod by driving large vertical oscillations of high-elongation plasmas.

\[^{12}\]In Alcator C-Mod some of the equilibrium field supplies are powered directly from the grid, others are powered through an alternator whose operating frequency varies from $\sim 57\,Hz$ to $\sim 54\,Hz$ during a plasma discharge.
4.2 Effects of Noise and Pick-ups on the Vertical Position

Figure 4-3 shows the spectra of the ZCUR observer from different target plasmas.

![Comparison of PSDs from Various Plasma Discharges](image)

Figure 4-3: Comparison of the spectral power densities of ZCUR from different target plasmas.

The spectra are calculated with the DFT algorithm discussed in appendix C. The spectral analysis is conducted on samples taken during 0.5s of the flattop with an interval \( T = 100\mu s \), the spectral resolution is \( \Delta f = 2Hz \). Most of the information is at low frequency \( 0 \div 200Hz \). The main features in these spectra are the components at \( \sim 54Hz \) and harmonics (\( \sim 108Hz, \sim 162Hz \)) and \( 60Hz \) and harmonics (\( 120Hz, 180Hz \)), and
the closed loop resonances at \(\sim 20\,\text{Hz}\) and \(\sim 110\,\text{Hz}\). These features are real plasma oscillations driven by the pick-ups on some of the power supplies. Independent X-ray emission measurements show that their amplitude can be several millimeters in high-elongation shots (in shot 1070406015 the amplitude of the 54\,\text{Hz} component is \(\sim 7\,\text{mm}\)).

In order to calculate the broadband contributions, the signal peaks at 54\,\text{Hz} and harmonics and 60\,\text{Hz} and harmonics and the closed loop resonances are removed with digital 20\,\text{Hz}-stopband filters\(^{14}\) centered at 20\,\text{Hz}, 54\,\text{Hz}, 60\,\text{Hz}, 110\,\text{Hz}, 120\,\text{Hz}, 160\,\text{Hz} and 180\,\text{Hz}: what is left is the "noise band". The PSDs are then integrated in the 800\,\text{Hz} bandwidth of the C-Mod fast vertical channel and the square root is taken. The results are further multiplied by an appropriate correction factor, in order to compensate approximately for the removal of the signal bands\(^{15}\).

The noise level seems to be slightly dependent on the elongation, as shown in figure 4-4, however this might be an artefact deriving from the method used to separate noise from useful signal. Additional features appear at higher values of the elongation and spread into the noise band, as seems to be the case at \(\kappa = 1.85\): the corresponding spectrum in figure 4-3 shows broadening of the features at \(\sim 54\,\text{Hz}\) and \(\sim 110\,\text{Hz}\) with significant overlap in the noise band 70\,\text{Hz} \div 100\,\text{Hz}.

For a typical elongation \(\kappa = 1.7\), it is \(\sigma_{\text{ZCUR}} \sim 1\,\text{mm} \times \text{MA}\). Recalling that the diagnostics noise amounts to \(\sigma_{\text{Diag}} \sim 0.3\,\text{mm} \times \text{MA}\) and that the effect of the current noise in the active coils is negligible, one can infer that the plasma is contributing most of the content in the noise band of the vertical observer:

\[
\sigma_{\text{Plasma}} = \sqrt{\sigma_{\text{ZCUR}}^2 - \sigma_{\text{Diag}}^2} \lesssim 1\,\text{mm} \times \text{MA}
\]  

(4.1)

Figure 4-5 shows the spectra of the time derivative of \(\text{ZCUR}\) in two cases. This is

\(^{13}\)The vertical position feedback loop has two resonances, at \(\sim 20\,\text{Hz}\) and \(\sim 110\,\text{Hz}\). The former mode is coupled with the slow poloidal equilibrium, while the latter corresponds to the fast vertical stabilization system. With the nominal control gains the "slow mode" is mostly suppressed.

\(^{14}\)The width of the stop-band is heuristically derived from the width of the main features in the spectra.

\(^{15}\)The correction factor used is the ratio of the useful bandwidth, assumed to be 200\,\text{Hz}, and the unfiltered portion of this bandwidth.
relevant because vertical stabilization uses derivative feedback. The RMS fluctuation is calculated by filtering the signal bands and integrating in the 800 Hz bandwidth of the C-Mod fast vertical channel. The typical velocity fluctuation is $\sigma_{dZCUR/dt} \sim 1 m/s \times MA$. Higher-order derivatives would amplify the high frequency noise and further reduce the control precision.

### 4.3 Conclusions

The vertical resolution on C-Mod is $\sigma_{ZCUR} \sim 1 mm \times MA$ and $\sigma_{dZCUR/dt} \sim 1 m/s \times MA$, limited by broadband noise. The main contribution comes directly from the plasma, therefore it seems unlikely that significant noise reduction could be achieved by improving the diagnostic circuitry. On the other hand, an optimized filter on the vertical observer may help reduce the broadband noise and improve plasma controllability at
Figure 4-5: PSD of the digital derivative of ZCUR for two different plasmas.

limit elongations $\kappa \sim 1.85$, where $\delta z_{\text{max}}/\sigma_z \sim 1$.

We also discussed a larger, and potentially destabilizing, perturbation, represented by the pick-ups at the output of the power supplies, which can drive vertical oscillations of high-elongation plasmas.

At moderate elongations $\kappa \leq 1.80$, noise does not seems to be a major concern in C-Mod. However, if controllers with high-order derivatives were implemented, control precision might decrease because of higher noise levels: an accurate assessment of the trade-off between improvements of theoretical stability margins and loss of vertical resolution would then become necessary.
Chapter 5

State Observer for Vertical Stabilization Based on a Kalman Filter

If a generic system \( \{A, B, C\} \) is controllable and if the states of the system \( x \) are known in real time, it is possible to place the poles of the closed loop anywhere in the Laplace plane with a feedback law of the form \( u = -Gx \), where \( u \) is the control input of the system [22]. Because of the relationship between the location of the poles and the frequency and the time response of the system, there is large freedom for meeting performance specifications, such as time response to step-wise perturbations, while guaranteeing closed-loop stability. In other words, the poles are not constrained to lie on a specific root locus as it is the case with a simple PD controller.

The first stage of a full-state controller is a state observer that reconstructs the relevant states of the system from the measurements, while rejecting measurement noise, which may represent a problem with high-order controllers. A widely used state observer is the Kalman filter [25]. Other groups have shown that a Kalman filter is effective at discriminating Resistive Wall Modes from noise and interferences [26], [27]. In this chapter we discuss the design of a Kalman filter based on a reduced-order model of the system.
which uses knowledge of the inputs and outputs of the system and of the intrinsic and measurement noise to reconstruct a number of states sufficient for the problem of vertical stability.

5.1 System Model Reduction

The evolution of active and passive currents is described by equation 2.18:

\[
\frac{d (\delta I_{cv})}{dt} = A \delta I_{cv} + BV_c \tag{5.1}
\]

In the case of C-Mod, the problem of vertical stability is well approximated by a single-input multiple-output (SIMO) system, which evolves on a time-scale much faster than the poloidal equilibrium. The SIMO’s input is the demand to the vertical stabilization coil \(V_{vs}\). The state equation is obtained by setting all the inputs to zero, except \(V_{vs}\).

\[
\frac{d (\delta I_{cv})}{dt} = A \delta I_{cv} + bV_{vs} \tag{5.2}
\]

Note that the matrix \(B\) is now a vector \(b\). The SIMO’s output is the vector of diagnostic measurements \(y\), comprising the flux loops, the poloidal field coils, the Rogowski coil and the active currents. For a given plasma position, the output is a linear combination of the currents by the diagnostic matrix \(Y\):

\[
y = Y \begin{bmatrix} I_{cv} \\ I_p \end{bmatrix} \tag{5.3}
\]

The output equation is obtained by linearizing around the equilibrium:

\[
\delta y = Y_0 \begin{bmatrix} \delta I_{cv} \\ \delta I_p \end{bmatrix} + \left( \frac{\partial Y}{\partial z} dz \right) \begin{bmatrix} I_{cv} \\ I_p \end{bmatrix} \tag{5.4}
\]

The derivative with respect to the vertical position is identically zero except for the plasma current elements. Furthermore, on the time-scale of interest, the plasma current
is approximately constant $\delta I_p = 0$. Therefore, using equation 2.14:

$$
\delta y = \left( Y_0 + \frac{I_p}{2\pi n B_{eq}} \frac{\partial Y}{\partial z} w^T \cdot M_{pw}' \right) \delta I_{cv} \equiv C \delta I_{cv}
$$

(5.5)

Our full state-space model of tokamak and plasma is of order 200 and needs to be reduced for real-time computation of the filter. In our custom implementation of model reduction, the plant is firstly diagonalized and the eigenvalues ordered so that the unstable mode is in the lower right corner of the state matrix. The stable part of the plant is then reduced with Schur balanced truncation, as described in appendix D. Finally, the system and the truncation matrices are augmented with the unstable mode and the reduced system is diagonalized, in order to preserve the identity of the modes. The Matlab code for model reduction is described in appendix D. A high-elongation equilibrium with growth rate $\lambda = 370 rad/s$ is used here and in the following for numerical studies and simulations. The corresponding matrices are extracted from calibrated Alcasim simulations of shot 1080430028 at 1.5s during the pulse and are reported in appendix E.

One problem is to determine how many modes are enough to reproduce the input-output response of the system. For this purpose, we introduce the relative error bound:

$$
\varepsilon \equiv \frac{2 \cdot \sum_{i: \text{excluded}} \sigma_i}{\sum_{i: \text{all}} \sigma_i}
$$

(5.6)

where the Hankel Singular Values (HSV’s) $\sigma_i$ measure the contribution of the modes of the system to the energy of the input-output response. Figure 5-1 shows the relative error bound for model reduction of our reference equilibrium. Because the error plateaus above five modes, this number seems to be sufficient to reproduce the stable part of the system.

Figure 5-2 shows the eigenvalues of the reduced model for different orders $N_{\text{red}} = 1, 2, 3, 5, 9$. This is a convenient way to understand which modes are kept in the reduced model.

Blue diamonds show some of the stable eigenvalues for order $N_{\text{red}} = 9$. There are
Figure 5-1: Error bound of the Schur balanced truncation of the stable part of a high-elongation equilibrium as a function of the number of modes retained in the model.
Figure 5-2: Eigenvalues of the reduced model of the stable part of the system for $N_{\text{red}} = 1, 2, 3, 5, 9$. The eigenvalues of the model of order 5 are labeled for future reference.
some slow modes with small negative eigenvalues, which are mainly localized in the equilibrium field coils. There is also the EFC mode, which is localized in the EFC coils and the nearest wall, and has a decay constant $\sim -200 \text{rad/s}$. The remaining modes have even smaller time constants. As the order is reduced to $N_{\text{red}} = 5$ (magenta circles), only two of the slow modes, the EFC mode and a couple of faster modes are kept, however the fastest mode at $\sim -2600 \text{rad/s}$ should be a result of the synthesis of the two fastest modes of $N_{\text{red}} = 9$. The identity of the modes at lower orders can be analyzed by similar arguments.

The field topology of the EFC mode is shown in figure 5-3. By comparison, figure 5-4 shows the unstable mode, which is localized in the wall nearest the EFC coils and in the inner wall of the vacuum vessel.

The tokamak and plasma’s reduced state-space $\{A_r, b_r, C_r\}$ is augmented with the vertical stabilization power supply’s state-space $\{A_{ps}, b_{ps}, c_{ps}\}$ to obtain equations 5.7 and 5.8:

$$\frac{dx}{dt} = \begin{bmatrix} A_{ps} & 0 \\ b_{ps} & A_r \end{bmatrix} x + \begin{bmatrix} b_{ps} \\ 0 \end{bmatrix} u = A_{tot}x + b_{tot}u$$

(5.7)

$$\delta y = \begin{bmatrix} 0 & C_r \end{bmatrix} x = C_{tot}x$$

(5.8)

where $x \equiv [x_{ps}; \delta I_{cv}]$ is the augmented state vector and $x_{ps}$ stands for the internal states of the supply.

Finally, the model is discretized on the sampling time $T$ of the C-Mod digital plasma control system, leading to equations 5.9 and 5.10:

$$x(k) = A_{d}x(k - 1) + b_{d}u(k - 1) + w(k)$$

(5.9)

$$y(k) = C_{d}x(k) + r(k)$$

(5.10)
Figure 5-3: Magnetic flux surfaces of the stable EFC mode. The amplitude of the eigenmode is normalized.
Figure 5-4: Magnetic flux surfaces of the unstable vertical mode. The amplitude of the eigenmode is normalized.
where $A_d = \exp(A_{tot}T)$, $b_d = \int_0^T \exp(A_{tot}\tau) d\tau b_{tot}$, $C_d = C_{tot}$, $w$ is the intrinsic noise process and $r$ is the measurement noise process. The matrices of the reference equilibrium are reported in appendix E.

5.2 Kalman Filter Equations

A Kalman filter comprises a set of time-update equations (equations 5.11 and 5.12) and a set of measurement-update equations (equations 5.13 - 5.15) [25]:

\[
\hat{x}^-(k) = A_d\hat{x}(k - 1) + b_d u(k - 1) \tag{5.11}
\]

\[
P^-(k) = A_d P(k - 1) A^T_d + Q(k) \tag{5.12}
\]

\[
K(k) = P^-(k) C_d^T \left[ C_d P^-(k) C_d^T + R(k) \right]^{-1} \tag{5.13}
\]

\[
\hat{x}(k) = \hat{x}^-(k) + K(k) [y_{exp}(k) - C_d\hat{x}^-(k)] \tag{5.14}
\]

\[
P(k) = [I - K(k)C_d] P^-(k) \tag{5.15}
\]

where $\hat{x}^-(k)$ is the a priori estimate of the state at time $k$; $P(k)$ is the error covariance matrix at time $k$: $P(k) \equiv E \left\{ (\delta x(k) - E\{\delta x(k)\}) \cdot (\delta x(k) - E\{\delta x(k)\})^T \right\}$, $\delta x \equiv x(k) - \hat{x}(k)$ ($P^-(k)$ is the corresponding a priori estimate); $Q(k)$ is the intrinsic noise covariance matrix: $Q(k) \equiv E \left\{ (w(k) - E\{w(k)\}) \cdot (w(k) - E\{w(k)\})^T \right\}$; $K(k)$ is the Kalman gain at time $k$; $R$ is the measurement noise covariance matrix: $R(k) \equiv E \left\{ (r(k) - E\{r(k)\}) \cdot (r(k) - E\{r(k)\})^T \right\}$; $y_{exp}(k)$ is the output measurements at time $k$. 

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If the intrinsic and measurement noise are stationary \((Q(k) \equiv Q, R(k) \equiv R)\), the Kalman gain and the error covariance matrix can be calculated off-line. The error covariance matrix is initialized to the intrinsic noise covariance matrix \(P(0) \equiv Q\) and equations 5.12, 5.13, 5.15 are solved iteratively until convergence. The only equations to compute in real time are 5.11 and 5.14, which can be written in compact form:

\[
\hat{x}(k) = A'_d \hat{x}(k - 1) + b'_d u(k - 1) + K y_{\text{exp}}(k) \tag{5.16}
\]

where \(A'_d = (I - KC_d) A_d\), \(b'_d = (I - KC_d) b_d\).

The intrinsic noise covariance \(Q\) and the measurement covariance \(R\) are external parameters: their relative magnitude sets the trade-off between filter bandwidth and noise rejection. In the simplest case, the intrinsic noise covariance matrix \(Q\) is initialized assuming that the six modes in the reduced model are uncorrelated with the same variance \(\sigma_w^2\) and that the internal states of the power supply have negligible fluctuations. The measurement covariance matrix \(R\) is initialized assuming that all the channels are uncorrelated with the same variance \(\sigma_R^2\). The ratio \(\sigma_R^2/\sigma_w^2\) is an external parameter available for tuning: when \(\sigma_R^2/\sigma_w^2 \to 0\), the filter assumes that all the fluctuations of the measurements are a direct consequence of the fluctuations of the states and no filtering occurs. This is readily seen in the simple case when \(C_d\) is square and invertible. Solving equations 5.12, 5.13, 5.15 gives \(K = C_d^{-1}\). Then, from equation 5.14, \(x(k) = C_d^{-1} y_{\text{exp}}(k)\). Conversely, when \(\sigma_R^2/\sigma_w^2 \to \infty\), the filter assumes that all the fluctuations of the measurements are noise and the measurements are discarded entirely. In fact, \(K = 0\) in this case.

\(\sigma_w^2\) can be calculated from the fluctuations that intrinsic noise would produce in the vertical observer \(\sigma_{\text{ZCUR,intrinsic}}^2\):

\[
\sigma_{\text{ZCUR,intrinsic}}^2 = \|A_{\text{out}} * 3 * C_d\|^2 \sigma_w^2 \tag{5.17}
\]
\[ \sigma_w^2 = \frac{\sigma_{ZCUR, intrinsic}^2}{\|A_{out\_3} \cdot C_d\|^2} \] (5.18)

where \( A_{out\_3} \) is the vector of coefficients of the vertical observer and \( \|a\|^2 \) is the sum of the squares of the components of \( a \). Similarly, \( \sigma_R^2 \) can be calculated from the total experimental fluctuations of the vertical observer:

\[ \sigma_{ZCUR}^2 = \|A_{out\_3}\|^2 \sigma_R^2 \] (5.19)

\[ \sigma_R^2 = \frac{\sigma_{ZCUR}^2}{\|A_{out\_3}\|^2} = \frac{\sigma_{ZCUR}^2}{\|A_{out\_3}\|^2} \] (5.20)

The advantage of this approach is that now:

\[ \frac{\sigma_R^2}{\sigma_w^2} = \frac{\sigma_{ZCUR}^2}{\sigma_{ZCUR,intrinsic}^2} \frac{\|A_{out\_3} \cdot C_d\|^2}{\|A_{out\_3}\|^2} \] (5.21)

where the noise ratio is expressed in terms of \( \alpha^2 \equiv \sigma_{ZCUR}^2/\sigma_{ZCUR,intrinsic}^2 \), i.e. the relative magnitude of their effect on the vertical trace, an easier quantity to estimate.

### 5.3 Simulated Behavior of the Filter

Figure 5-5 shows the linear model to test the behavior of the filter.

The red path shows the vertical stabilization loop which comprises: the full model of the plant, augmented with the dynamic models of the power supplies (State-Space), and the linearized diagnostics (Ctotalpc); the digitizer (Zero-Order Hold), with the DPCS sampling time \( T = 100 \mu s \); the vertical observer (A\_out\_3), which is the same as in the reference plasma discharge 1080430028; the fast PD controller, whose gains are the same as in the reference discharge: \( P = 4, D = 0.006 \); and the chopper gain, which is calibrated to reproduce the condition of marginal stability of high-elongation plasmas (see the discussion of the Bode plots in section 3.2).
Figure 5-5: The linear model used to test the behavior of the Kalman filter. The red path shows the vertical stabilization loop. The blue path shows the section of the Kalman filter. The green path shows the section of a simple pseudo-inverse state observer used to benchmark the behavior of the Kalman filter. The blocks labeled $K^*u$ perform matrix multiplications.
The blue path shows the Kalman filter, which is implemented with a Matlab sFunction (VerticalFilter6).

The green path shows a simple pseudo-inverse state observer used to benchmark the behavior of the Kalman filter. The observer is preceded by an optimized low-pass filter in order to reject part of the measurement noise.

Intrinsic Noise is a vector of six uncorrelated white noise processes designed to drive the six modes of the reduced model. The processes have the same amplitude, which is calibrated in order to produce RMS vertical fluctuations $\sim 0.5\text{mm}$ in the absence of all other noise sources. $0.5\text{mm}$ is an arbitrary choice.

Diagnostics Noise is white noise calibrated in order to obtain RMS vertical fluctuations $\sim 40\text{mV}$ in DPCS units, equivalent to the resolution of the vertical observer $\sim 1\text{mm} \ast MA$. 60Hz Typical is the 60Hz pick-up of amplitude 2mV.

54Hz Pick-up models the large pick-up on OH2U and was calibrated in order to produce large 54Hz vertical oscillations of peak-to-peak amplitude $\sim 1.2\text{cm}$, as observed in experiments: it can be turned on or off to study how the filter behaves in the presence of measurement noise only or when a large input is present in the system that is not included in the SIMO model.

The purpose of the linear model is to determine if the Kalman filter can reconstruct the states of a reduced-order system, while rejecting noise, better than a simple pseudo-inverse observer with an optimized low-pass filter, in realistic scenarios of noise and pick-ups\footnote{In a real implementation of the filter, the slow poloidal equilibrium would be superposed to the fast SIMO signals and would have to be separated. However, this task can be easily realized by a high-pass filter, given the large time-scale separation of the SIMO system and the poloidal equilibrium.}. Moreover, the Kalman filter can be tested for robustness against parameter setting.

The pseudo-inverse observer is built by calculating the pseudo-inverse of the output matrix of the reduced-order system $C_r$ with the Matlab pinv routine. The observer does not reject most of the measurement noise, therefore it is necessary to add an optimized low-pass filter. This is designed with the Digital Filter Design toolbox of Simulink, with
the goal of rejecting most of the noise, without introducing enough phase lag to destabilize the feedback loop. The best solution appears to be a 5-th order FIR Equiripple with 800Hz pass-band, 5kHz stop-band, −40dB stop-band attenuation, 16 density factor and phase lag < 10 deg at 200Hz.

The error of the state observers is calculated by subtracting the reconstructed states from the real states, which are extracted from the full model with the truncation matrix $slbig$. The power spectrum is calculated for each one of the error signals and integrated in order to have both a spectral distribution and a RMS estimate of the error. The latter is normalized by the RMS values of the real states².

With the simple model of uniform intrinsic noise, the only knob available for tuning is the noise ratio $\alpha^2$, which should be large enough to reject most of the measurement noise from the reconstructed states, without introducing significant distortion. In order to determine the optimal setting of this parameter, some system evolution should be present that appears in the measurements but cannot be predicted by the filter from its knowledge of the plant and the input alone: this evolution will be tracked by the filter with some distortion, depending on the value of $\alpha^2$. A reasonable choice is to inject a sine wave of appropriate frequency at the input of the plant, like Test Sine in figure 5-5. The frequency can be determined by looking at the Bode plots of a similar high-elongation equilibrium, figure 3-10. At 100Hz the phase margin is $\sim 7.4$ deg, so this is about as much phase lag as can be tolerated at this frequency. Figure 5-6 shows a comparison of the real and reconstructed unstable state with different values of $\alpha^2$. Figure 5-7 shows a comparison of the linear observer ZCUR and the vertical position calculated from the reconstructed states from the same set of simulations.

An interesting feature is that the states have different amplitude and phase distortion (not shown here) and that the total effect on the calculated vertical position is not identical to the distortion of the unstable state: in particular, the vertical position is

²The RMS values are only coarse zeroth-order estimates of the errors in state reconstruction, lumping together deterministic errors, such as amplitude and phase errors, and random fluctuations. A careful inspection of the time traces is always needed to separate these two cases.
Figure 5-6: Comparison of the real and reconstructed unstable state for different values of $\alpha^2$. Amplitudes are normalized by the amplitude of the real state. All noise sources were turned off for this test.
Figure 5-7: Comparison of the linear observer and the vertical position calculated from the reconstructed states for different values of $\alpha^2$. Amplitudes are normalized by the amplitude of the linear observer. All noise sources were turned off for this test.
not significantly attenuated at these values of $\alpha^2$. From these simulations it follows that the range of $\alpha^2$ should be $\alpha^2 \leq 4$. Moreover, the resolution of the vertical observer on C-Mod is $\sim 1\text{mm}$ and the intrinsic noise is calibrated to obtain oscillations of amplitude $\sim 0.5\text{mm}$, so $\alpha^2 = 4$ is a consistent choice. We also tried to estimate the range of $\alpha^2$ from simulations where the loop was closed on the vertical position estimated by the Kalman filter and $\alpha^2$ was changed until the loop would become unstable; however, these tests were not reliable, because the filter’s frequency response changes when it operates inside the feedback loop. The reason for this is the presence of a second feedback loop, consisting of the filter and the PD controller (Kalman Filter and Fast Controller in figure 5-5). The frequency response in this case is discussed in details in chapter 6. All the following data are from simulations where the loop is closed on the linear vertical observer (red path in figure 5-5) and the values of noise and pick-up are consistent with real experimental values.

In order to have a reference of how much error is tolerable, a simple argument can be made. The coefficients of the vector $c_z \equiv A_{\text{out}}.3 * C_d$ represent the contribution of individual states to the vertical observer. The ratio of the vertical resolution $\sigma_{\text{ZCUR}}$ and any coefficient of $c_z$ represents an error that, by itself, would produce vertical fluctuations equal to the vertical resolution:

$$\sigma_{\varepsilon,i} = \frac{\sigma_{\text{ZCUR}}}{|c_{z,i}|}, \ i = 1, \ldots, 6$$

(5.22)

If the state fluctuations are uncorrelated, then the state observer does not allow a more accurate representation of the vertical position than the C-Mod vertical observer if any of the errors is larger than the corresponding value $\sigma_{\varepsilon,i}$. In other words, the errors of all reconstructed states should be well below the corresponding $\sigma_{\varepsilon,i}$.

Figure 5-8 shows the superposition of real and reconstructed states with the pseudo-inverse observer and the Kalman filter. The 54Hz pick-up is off in order to allow a true SIMO test. Figure 5-9 and table 5.1 show the RMS values of the error signals together with "Max Error", i.e. the vector $\sigma_\varepsilon$ normalized by the RMS values of the states.
Figure 5-8: Comparison of real (red) and reconstructed states (blue). Vertical axes are Amperes. The 54Hz pick-up is off. The model of intrinsic noise is uniform with $\alpha^2 = 4$. 
Figure 5-9: RMS value of the error signals between real and reconstructed states. The 54Hz pick-up is off. The model of intrinsic noise is uniform with $\alpha^2 = 4$. 
<table>
<thead>
<tr>
<th>Mode</th>
<th>Kalman [%]</th>
<th>Pinv [%]</th>
<th>Max [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (Fast 3)</td>
<td>49</td>
<td>250</td>
<td>7900</td>
</tr>
<tr>
<td>2 (Fast 2)</td>
<td>22</td>
<td>71</td>
<td>510</td>
</tr>
<tr>
<td>3 (Fast EFC)</td>
<td>23</td>
<td>66</td>
<td>160</td>
</tr>
<tr>
<td>4 (Slow 2)</td>
<td>110</td>
<td>100</td>
<td>670</td>
</tr>
<tr>
<td>5 (Slow 1)</td>
<td>120</td>
<td>110</td>
<td>440</td>
</tr>
<tr>
<td>6 (Unstable)</td>
<td>47</td>
<td>66</td>
<td>150</td>
</tr>
</tbody>
</table>

Table 5.1: RMS errors in figure 5.9.

All of the errors are well below the dashed line: both state observers allow a representation of the vertical position with lower noise than the C-Mod vertical observer, as shown in figure 5-10. However, in the case of the pseudo-inverse observer, this is due to the optimized low-pass filter at its input.

Figure 5-10: Both the Kalman filter and the pseudo-inverse observer allow a representation of the vertical position with lower noise than the C-Mod vertical observer.

The performance of the Kalman filter on the slow modes can be improved by increasing
the amount of filtering, which does not affect the low-frequency signal content. Equation 5.21 is generalized as follows:

\[
\frac{\sigma^2_R}{\sigma^2_{w,i}} = \frac{\sigma^2_{ZCUR}}{\sigma^2_{ZCUR,intrinsic,i}} \frac{\|A_{\text{out} 3} \cdot C_d\|^2}{\|A_{\text{out} 3}\|^2} \tag{5.23}
\]

and \( \alpha^2 \) is now a vector with components \( \alpha^2_i \equiv \frac{\sigma^2_{ZCUR}}{\sigma^2_{ZCUR,intrinsic,i}} \). Figures 5-11, 5-12 and table 5.2 show results with \( \alpha^2 = [4, 4, 4, 64, 64, 4] \).

Figure 5-11: Comparison of real (red) and reconstructed states (blue). Vertical axes are Amperes. The 54Hz pick-up is off. The model of intrinsic noise is non-uniform with \( \alpha^2 = [4, 4, 4, 64, 64, 4] \).
Figure 5-12: RMS value of the error signals between real and reconstructed states. The 54Hz pick-up is off. The model of intrinsic noise is non-uniform with $\alpha^2 = [4, 4, 4, 64, 64, 4]$. 
<table>
<thead>
<tr>
<th>Mode</th>
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<th>Pinv [%]</th>
<th>Max [%]</th>
</tr>
</thead>
<tbody>
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<td>250</td>
<td>8300</td>
</tr>
<tr>
<td>2</td>
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<tr>
<td>6</td>
<td>42</td>
<td>63</td>
<td>150</td>
</tr>
</tbody>
</table>

Table 5.2: RMS errors in figure 5.12.

Finally, the 54Hz pick-up is introduced in simulations. Figures 5-13, 5-14 and table 5.3 show results in this case.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Kalman [%]</th>
<th>Pinv [%]</th>
<th>Max [%]</th>
</tr>
</thead>
<tbody>
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<td>1</td>
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<td>170</td>
<td>3600</td>
</tr>
<tr>
<td>2</td>
<td>7.1</td>
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<td>3</td>
<td>6.4</td>
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<td>4</td>
<td>18</td>
<td>26</td>
<td>93</td>
</tr>
<tr>
<td>5</td>
<td>56</td>
<td>71</td>
<td>260</td>
</tr>
<tr>
<td>6</td>
<td>37</td>
<td>28</td>
<td>34</td>
</tr>
</tbody>
</table>

Table 5.3: RMS errors in figure 5.14.

There are significant amplitude and phase errors in the reconstructed states by both the pseudo-inverse observer and the Kalman filter. In the latter case, a large set of values of $\alpha$ was tried without significant improvements. This result is not surprising: the pseudo-inverse observer is the pseudo-inverse of the measurement matrix of a reduced-order model of the SIMO system; similarly, the Kalman filter is simulating a reduced-order model of the SIMO plant. In order to correctly reproduce the effects of the large pick-up a full MIMO formulation may be necessary.
Figure 5-13: Comparison of real (red) and reconstructed states (blue). Vertical axes are Amperes. The 54Hz pick-up is on. The model of intrinsic noise is non-uniform with $\alpha^2 = [4, 4, 4, 64, 64, 4]$. 
Figure 5-14: RMS value of the error signals between real and reconstructed states. The 54Hz pick-up is on. The model of intrinsic noise is non-uniform with $\alpha^2 = [4, 4, 4, 64, 64, 4]$. 
One important issue is how state reconstruction depends on the accuracy of the model of the plant. In order to answer this question, we tried two different simulations with measurement noise only:\footnote{Model reduction guarantees the identity of the input-output response, however there is a certain variability in the sign of the coefficients of the input matrix $b_r$, and, in a consistent way, the signs of the coefficients of the output matrix $C_r$. This variability does not affect the input-output response, but does affect the states. It is important to make sure that the sign convention is consistent when comparing states of reduced models of different systems.}

1. the Kalman filter and the pseudo-inverse observer are built from an equilibrium with growth rate $\lambda = 314\text{rad/s}$ (the corresponding matrices are extracted from simulations of shot 1080430028 at 1.1s), about 15% smaller than the unstable eigenvalue of the plant;

2. the Kalman filter and the pseudo-inverse observer are built from an equilibrium with growth rate $\lambda = 282\text{rad/s}$ (the corresponding matrices are extracted from simulations of shot 1080430028 at 0.9s), about 24% smaller than the unstable eigenvalue of the plant.

In both cases the numbers are similar to those in table 5.2, proving a robust behavior of the filter with respect to (small) errors in the model of the plant.

### 5.4 Conclusions

Our linear simulations prove that a Kalman filter performs better than a simple pseudo-inverse observer in reconstructing the states of a system from available observations, provided adequate models of the system, the inputs and the intrinsic and measurement noise are available.

If the states of the system are known in real-time, it is possible to place the poles of the closed loop arbitrarily with a feedback law of the form $u = -Gx$, provided the system is observable: the loop response can then be shaped to behave optimally in the
presence of step-wise or sinusoidal perturbations. It is not our intent to cover the topic here, however we should point at one difficulty with full-state feedback control based on the model \( \{ \mathbf{A}_r, \mathbf{b}_r, \mathbf{C}_r \} \) which is the poor controllability of the slow modes of \( \mathbf{A}_r \) with a single actuator consisting of the chopper and the EFC coil. This is immediately evident from the magnitude of the coefficients of \( \mathbf{b}_r \) corresponding to the slow modes. Therefore pole placement can be effectively implemented only for a sub-system excluding these modes or in a full MIMO framework.
Chapter 6

Model-based Filter for the Vertical Observer Based on a Kalman Filter

At limit elongations two concurrent elements affect plasma control: firstly, the phase margin is small and the linear stability of the feedback loop is marginal; secondly, the maximum controllable displacement becomes comparable with the broadband noise in C-Mod, $\delta z_{\text{exp}}/\sigma_z \sim 1$, and plasma control may be compromised by the system noise. An optimized filter might help alleviate the second effect.

A model-based filter uses a-priori knowledge of the plant so it could in principle have an advantage over low-pass filters. In this chapter we discuss a model-based filter for the vertical observer based on a Single-Input Single-Output (SISO) implementation of the Kalman filter. We describe the analytical derivation of the filter’s equations and transfer function and report on the experimental test of the filter in the vertical stabilization loop of C-Mod. The filter performance is evaluated in comparison to an optimized low-pass filter.

6.1 SISO Filter Equations

Let’s consider the single-input single-output problem:
\[
\frac{d(\delta I_{cv})}{dt} = A \delta I_{cv} + bV_{vs}
\]  
(6.1)

\[
ZCUR = c_z \delta I_{cv}
\]  
(6.2)

where \(ZCUR\) is the vertical position times the plasma current. The goal here is to filter noise from the vertical observer, based on the predictions of the model in equations 6.1, 6.2.

The output matrix \(c_z\) is obtained by multiplying the SIMO output \(C\) times the vertical observer \(A_{\text{out\_3}}\):

\[
c_z = A_{\text{out\_3}} \ast C
\]  
(6.3)

The procedure for model reduction, compounding and digitization are similar to those discussed for the SIMO system, with final result:

\[
x(k) = A_d x(k - 1) + b_d u(k - 1) + w(k)
\]  
(6.4)

\[
ZCUR(k) = c_z d x(k) + r(k)
\]  
(6.5)

In the SISO case \(R \equiv \sigma_R^2 = \sigma_{ZCUR}^2 \sim (1mm \ast MA)^2\) is readily available from the analysis of noise. Equation 5.16 becomes:

\[
\tilde{x}(k) = A_d \tilde{x}(k - 1) + B_d u(k - 1) + kZCUR_{\text{exp}}(k)
\]  
(6.6)

where the Kalman gain \(k\) is evaluated by solving equations 5.12, 5.13, 5.15 until convergence, and the output of the filter is given by:

\[
ZCUR(k) = c_z d \tilde{x}(k)
\]  
(6.7)
The filter frequency response is obtained by calculating the z-transform$^1$ of equations 6.6 and 6.7:

$$\hat{X}(z) = A_d \frac{\hat{X}(z)}{z} + \frac{b_d'}{z} U(z) + kZCUR_{\exp}(z) \quad (6.8)$$

$$ZCUR(z) = c_{zd} \left( zI - A_d' \right)^{-1} b_d' U(z) + z c_{zd} \left( zI - A_d' \right)^{-1} k ZCUR_{\exp}(z) \quad (6.9)$$

$$\frac{ZCUR(z)}{ZCUR_{\exp}(z)} = c_{zd} \left( zI - A_d' \right)^{-1} b_d' \frac{U(z)}{ZCUR_{\exp}(z)} + z c_{zd} \left( zI - A_d' \right)^{-1} k \quad (6.10)$$

$$F_{KF} = F_{SYS} \ast F_{CTRL} + z \ast F_K \quad (6.11)$$

where $F_{KF} \equiv \frac{ZCUR(z)}{ZCUR_{\exp}(z)}$, $F_{SYS} \equiv c_{zd} \left( zI - A_d' \right)^{-1} b_d'$, $F_{CTRL} \equiv \frac{U(z)}{ZCUR_{\exp}(z)}$, $F_K \equiv c_{zd} \left( zI - A_d' \right)^{-1} k$.

$F_{CTRL}$ is the transfer function of the PD controller in the vertical stabilization loop:

$$F_{CTRL} = m \ast \left( \frac{Dz - 1}{Tz} + P \right) \quad (6.12)$$

where $T$ is the sampling time and $m$ is a scalar gain. The filter behavior is therefore affected by internal parameters, i.e. the noise ratio $\sigma_R^2/\sigma_w^2$, which can be expressed again in terms of $\alpha^2$, $\sigma_R^2/\sigma_w^2 = \|c_{zd}\|^2 \sigma_{ZCUR/\text{intrinsic}}^2 = \|c_{zd}\|^2 \alpha^2$, and by external parameters, i.e. the controller gains $P$ and $D$. By applying the transformation from $z$ to $s$-domain, $z = \frac{2 + sT}{2 - sT}$, it is possible to draw the Bode plots for different parameter values.

Figure 6-1 shows the Bode plots for different orders of the reduced model. When the

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$^1$A brief introduction of the z-transform is given in appendix F.
order of the model is $N_{\text{red}} \geq 3$ the slow modes are included and the frequency response has a unitary DC gain, For $N_{\text{red}} \geq 6$ the frequency response is practically unchanged.

![Frequency Response Graph](image)

**Figure 6-1**: Frequency response of the filter for different orders of the reduced model. In this example $D = 6$, $P = 4$, $\alpha^2 = 4$.

Because only the measurement of the vertical position is used to correct the prediction of multiple states, the problem is under-constrained and the internal states of the filter differ from the physical states of the plant.

The broad minimum $\sim 60Hz$ and the broad maximum at $\sim 500Hz$ are caused by the combination of the terms in equation 6.11, i.e. the low-pass behavior of $z \ast F_K$, the high pass behavior of $F_{CTRL}$ and the cut-off of the chopper at 800Hz. The location and width of these features change with the value of the derivative and proportional gains. However, these gains are chosen in order to stabilize the feedback loop. Therefore only the noise ratio is available for filter tuning. Figures 6-2, 6-3, 6-4, 6-5 show the Bode plots.
of $F_{KF}$, $F_{SYS}$, $F_{CTRL}$ and $z \ast F_K$ with $N_{red} = 6$, $D = 6$, $P = 4$, for different values of the noise ratio.

![Frequency response of the filter](image)

Figure 6-2: Frequency response of the filter for different values of the noise ratio $\alpha^2$. Solid lines show the analytic transfer function, diamonds are the frequency response calculated from the time traces of Alcasim simulations.

The filter does not perform as well as an optimized low-pass filter in terms of pass-band distortion and stop-band attenuation. By comparison, the 5-th order FIR Equiripple discussed in section 5.3 achieves $-40dB$ stop-band attenuation with phase lag $< 10$ deg at $200Hz$ and negligible amplitude distortion.
Figure 6-3: Frequency response of the system term $F_{SYS}$ for different values of the noise ratio $\alpha^2$. 
Figure 6-4: Frequency response of the PD controller $F_{CTRL}$. 
Figure 6-5: Frequency response of the Kalman term \( z \ast F_K \) for different values of the noise ratio \( \alpha^2 \).
6.2 Experimental Results

The SISO Kalman filter was programmed in IDL and incorporated in the C-Mod Plasma Control System (PCS) as a real-time procedure on the vertical observer. It was initially tested with Alcasim simulations and off-line on the machine. Figure 6-6 shows the comparison of analytic and experimental transfer functions in shot 1080430020. In this experiment the filter was outputting its results on a channel not used for feedback control.

![Comparison of analytic and experimental transfer functions](image)

Figure 6-6: Comparison of analytic and experimental transfer functions. Diamonds are experimental results from shot 1080430020, where the filter was operating off-line. The gain of the chopper used in the linear model of the filter was 200 and the noise ratio was set to a large value $\alpha^2 = 16$. 

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At the vertical signal frequency of approximately $120\, Hz$, the filter introduces a phase lead that actually improves the stability of the closed loop and an attenuation of approximately $-4\, dB$ that changes the dynamic behavior of the loop (the filter attenuation is equivalent to changing the control gains), but did not destabilize plasmas up to $\kappa = 1.84$. Figure 6-7 shows results from a shot where the filter was operating in the vertical stabilization loop (1080430028).

### 6.3 Conclusions

We discussed a model-based filter for the vertical observer based on a SISO implementation of the Kalman filter. Simulations and experiments show that the filter rejects high-frequency noise without destabilizing high-elongation plasmas.

We only implemented a simple model with uniform intrinsic noise by which the filter does not perform as well as an optimized low-pass filter. However, a more general implementation with different intrinsic-noise levels might change this conclusion.
Figure 6-7: Comparison of two plasma discharges, with the Kalman filter (1080430028) and without (1080430027). The left-hand panels show the plasma elongation, the vertical observer, the chopper demand and the chopper current without the filter, while the right-hand panels show the same quantities with the filter applied after 0.8s. The elongation is ramped during the discharges. The ripple starting at 1s, when the elongation peaks, is real plasma motion, with amplitude 5mm in physical units.
Chapter 7

Current Saturation Avoidance with Real-time Control using DPCS

Tokamak ohmic-transformer and equilibrium-field coils need to be able to operate near their maximum current capabilities. However if they reach their limits during high-performance discharges or in the presence of strong off-normal events, shape control is compromised, and instability, even plasma disruptions can result.

In this chapter we discuss the design and experimental test of an anti-saturation routine, which detects the impending saturation of ohmic and equilibrium field currents and interpolates to a neighboring safe equilibrium in real-time. This approach becomes necessary when poor redundancy of control coils, and therefore absence of a suitable null space, may require mid-shot pulse rescheduling [30], [31], as opposed to an adaptation in control [32].

The routine was implemented with a multi-processor, multi-time-scale control scheme, which is based on a master process and multiple asynchronous slave processes. The scheme is general and can be used for any computationally intensive algorithm.
7.1 Anti-saturation Adaptive Control Routine

The ohmic and equilibrium field coils in C-Mod are illustrated in figure 1-1. The EF3 coils are connected in series to generate primarily a vertical field for radial force balance and position control. The EFC coils are connected in anti-series to generate primarily a radial field for fast vertical stabilization. Table 7.1 summarizes the characteristics of the power supplies connected to the control coils.

<table>
<thead>
<tr>
<th>Coils</th>
<th>Rating</th>
<th>Limits</th>
<th>Power Supply</th>
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<th>Configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td>OH1</td>
<td>±50kA</td>
<td>±35kA</td>
<td>12-pulse thyristor</td>
<td>930V/500V</td>
<td>2kV Interrupter Lockout Dual</td>
</tr>
<tr>
<td>OH2U/L</td>
<td>±50kA</td>
<td>±35kA</td>
<td>12-pulse thyristor</td>
<td>240V/100V</td>
<td>2kV Interrupter Circulating Dual</td>
</tr>
<tr>
<td>EF1U/L</td>
<td>±15kA</td>
<td>[+7.5, −5] kA</td>
<td>12-pulse thyristor</td>
<td>420V/200V</td>
<td>2kV Interrupter Circulating Dual</td>
</tr>
<tr>
<td>EF2U/L</td>
<td>+7kA</td>
<td>+5.5kA</td>
<td>2-quadrant TMX</td>
<td>680V/560V</td>
<td>35mΩ ballast</td>
</tr>
<tr>
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<td>3645V/2400V</td>
<td>Series Coils</td>
</tr>
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<td>GA Chopper/TMX</td>
<td>1000V</td>
<td>Anti-series Coils 80mΩ ballast</td>
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</tbody>
</table>

Table 7.1: C-Mod poloidal field system.

Different control currents are more effective for changing different shape and position parameters: for example, OH2U and OH2L in anti-phase for the vertical position of the plasma; EF1U and EF1L for the vertical position of the upper and lower X-point (i.e. the elongation); EF2U and EF2L for the radial position of the upper and lower X-point (i.e. the top and bottom triangularity); EF3U and EF3L, connected in series, for the
radial position of the plasma; etc. However it is really a combination of all the currents that produces the magnetic moment necessary to obtain the desired shape. Therefore the control of the plasma shape and position is a Multiple-Input Multiple-Output (MIMO) problem. The error signal between the real-time value and the pre-programmed target of a quantity under feedback control produces a weighted demand to all of the power supplies, to generate a magnetic moment which does not affect the other controlled quantities. A comprehensive description of the synthesis of the controllers used in C-Mod is given in [29].

When a current reaches saturation, the controllers that include that current are unable to produce the intended magnetic moments and the magnetic topology is distorted. There are cases when current saturation produces only shape degradation, without loss of vertical control; however, when only one of symmetric pairs of control coils saturates, a radial field may arise and cause a vertical displacement event.

If the number of control coils is larger than the number of controlled parameters, it is possible to deploy some of the extra coils to help those that are saturating. In this case the system is said to have a null space. Another way of expressing the concept is to say that the control coils are redundant. Evidently, large redundancy implies large capital cost, therefore fusion reactors usually have small null space. This is also the case in C-Mod, where redundancy is minimal with the EF4 coils potentially doubling symmetric currents in the EF2 and EF3 coils, however with much longer penetration time.

In the presence of small or zero null space, the solution is to intervene on the cause that is pushing the currents to their rails. Changing the control gains alone would at most change the time horizon of a saturation event and might work for transient saturations but not when the pre-programmed target is too aggressive and ultimately inaccessible. In this case the targets need to be modified in real-time, a technique known as pulse rescheduling. Pulse rescheduling may also involve the modification of the control gains or a more general modification of the control scheme. It is immediately evident that a problem with this approach is to guarantee compatibility and smooth transition between
The anti-saturation routine implemented in C-Mod linearly interpolates to a new equilibrium where the original targets are relaxed. The routine constantly monitors the ohmic and equilibrium field currents, which are available as DPCS inputs. The measured values are averaged with a window filter in order to reject noise. The average time derivatives of the currents are also calculated. The ratio of the distance of the currents from their rails and the time derivatives gives the time horizon for a saturation event. When this time becomes smaller than a user-defined threshold, a flag is raised and the target and feed-forward waveforms are linearly interpolated to those of the safe equilibrium\(^1\). Figure 7-1 shows the flow-chart of the routine.

The minimum time horizon has to be small enough to avoid false positives, but large enough to avoid the occurrence of current saturation during the transition to the safe equilibrium. Similarly, the interpolation time has to be long enough to avoid transients in the currents and the plasma and short enough to save the discharge by quickly moving to the safe equilibrium. Obviously, time horizon and interpolation time are related to one another: a rule of thumb is that the interpolation time should be about twice as large as the time horizon and about the same order of magnitude as the time constant of the currents whose saturation is to be avoided. It might also be possible to derive an analytical relationship between time horizon, interpolation time and the current time constants, however the problem is significantly complicated by the high degree of coupling of all currents, therefore a heuristic approach with optimization through simulations is probably more effective.

The design of compatible safe equilibria represents an important issue of this application. A safe equilibrium can be completely different from the original target and even realize special cases such as early ramp-down and discharge termination, however it ought to be possible to move to this equilibrium from the original while avoiding current rails

\(^1\)Non-linear methods might be more suitable for interpolating between different equilibria, however we did not investigate this issue.
Figure 7-1: Flow chart of the anti-saturation routine.
and large transients. The simplest safe equilibrium is designed by relaxing the original high performance target and by verifying with simulations that the currents are indeed far from their rails. For example, the high-triangularity shape illustrated in figure 7-2 may cause the EF2L current to reach its upper rail: the corresponding safe equilibria were designed by reducing the bottom triangularity of the original target and were verified with Alcasim simulations.

All relevant parameters are input to the anti-saturation routine from the PCS interface: the safe equilibrium to which the routine interpolates in case of a saturation event; the depth of the window filter; the time horizon at which a saturation event is triggered; the time duration of the interpolation; the option of reading the current rails directly from the engineering tree.

### 7.2 Multi-processor Multi-timescale Control Scheme

The anti-saturation routine requires significant computation resources and its calculation in the main real-time process is marginal at the nominal cycle speed 100µs. Therefore, we developed a more general multi-processor multi-timescale architecture for which the routine was a convenient test-bed.

Figure 7-3 illustrates the real-time operation of the anti-saturation routine, with a dual-processor dual-timescale control scheme.

The master process is synchronous with 100µs cycles. The anti-saturation routine is incorporated via an observer procedure and a controller procedure and is active only during a time window which is controlled from the PCS interface.

After reading new input data and calculating the observers and the "normal" target waveforms, the master tries to write the current data to Shared Memory I. It does so by interrogating the status of a semaphore, a software instrument used to manage conflicts in shared memory access\(^2\). Subsequently, the master interrogates the status of a flag,

\(^2\)A semaphore is a flag that signals the use of a block of shared memory by a certain process. If the
Figure 7-2: The red solid line shows the high-triangularity equilibrium developed for use in similarity studies with JFT-2M. The blue dashed line is the original JFT-2M shape, the green solid line is a standard C-Mod shape. From [33].
Figure 7-3: Dual-processor dual-timescale control scheme used for the anti-saturation routine. The orange blocks highlight the operations of semaphoring and data exchange from master process to slave process. The yellow blocks highlight the reverse operations. Data exchange happens through blocks of shared memory.
which signals if a saturation event has occurred or not. In the former case, and in the case that Shared Memory II is accessible for reading, the normal target waveforms are overwritten with the values from Shared Memory II, which stores the results of the slave’s computations. Otherwise, the normal targets are either overwritten with previous values. In the case that a saturation event hasn’t occurred, the normal targets are left unchanged.

The master process calculates the error signals, the PID controllers and the feed-forward waveforms of the original target equilibrium, then the controller portion of the anti-saturation routine overwrites these waveforms or not according to similar conditions as described before.

The slave process is asynchronous: the duration of a cycle depends on the context, which is determined through conditional statements. The slave process starts in a tight loop that checks for the time index of the data in Shared Memory I. This is done without trying to lock the memory through semaphored access: a problem with an earlier version of the routine was that the slave would own Shared Memory I for most of the time, given its short cycle time, and therefore cause a large number of conflicts with the master. With the new solution, the slave tries to lock the memory only if there is a good chance that new data is available (i.e. the time index has changed).

The next important conditional statement in the slave’s flow is the status of the saturation flag. If a saturation event hasn’t occurred yet, the slave continues to compute the time horizon of the various currents and to check for the threshold. Otherwise, the slave starts to compute the interpolated values of the target and feed-forward waveforms for the time step and writes the results to Shared Memory II.

Since the master’s process can’t be delayed or broken by conflicts in shared memory access, this case is handled by using the values of the target and feed-forward waveforms at the previous time step. This is a legitimate operation because of the slow evolution of poloidal equilibrium. The slave aborts the current cycle in case of conflicts.

flag is high, other processes can’t access the block of memory. The flag is set high by the first process that tries to read or write and is released by the same process at the end.
The init portion of the anti-saturation routine initializes the block of shared memory (with the IDL instructions SHMMAP and SHMVAR), creates the semaphore objects (with the IDL instruction SEM_CREATE) and spawns the slave process (with the IDL method IDL_BRIDGE). The init portion of the slave process initializes a local pointer to the shared memory (SHMMAP and SHMVAR) and creates the semaphore objects (SEM_CREATE). After the real-time phase is concluded, in the clean-up portion of the anti-saturation routine, the master process frees the block of shared memory (SHMUNMAP), kills the slave process (OBJ_DESTROY) and destroys the semaphore objects (SEM_DELETE). If the interrupts are disabled during the real-time phase, it is necessary to issue an exit command to the slave process before the OBJ_DESTROY, in order to terminate it. The reason why this extra operation is needed is still unknown.

The scheme can be easily generalized to other applications and to a larger number of processes/processors.

### 7.3 Experimental Results

The anti-saturation routine was successfully tested with the new control system DPCS3, comprising a dual-core quad-processor machine pcdaqdpcs3, two DTACQ Acq196 digitizers with 96 inputs and 16 analog outputs each and one INCAA DI02 decoder/timing module. The PCI bus of the host is connected to the cPCI crate containing the digitizers by a SBS bus extender module. This system is by default configured to duplicate the functionality of the standard DPCS control system.

The high performance target was the high-triangularity equilibrium developed for use in similarity studies with JFT2-M, where the EF2L current may reach its upper rail and the plasma disrupt. In this specific example the identity of the saturating current is known, however the anti-saturation routine is more general because it always checks for saturation of all currents. The anti-saturation routine and the new control computer were extensively tested in Alcasim simulations: firstly, the safe equilibrium was designed in the
PCS interface by reducing the bottom triangularity of the original target and Alcasim was used to verify that all currents would stay far from their rails; secondly, Alcasim was used to find a suitable setting for interpolation time and time horizon and to test the multi-processor operation of DPCS3.

The anti-saturation routine was then successfully tested in experiments, as shown in figure 7-4.

![Adaptive Interpolation to Safe Equilibrium](image)

Figure 7-4: The anti-saturation routine detects a saturation event at 0.66s in shots 1080521013 (blue trace) and 1080521016 (red trace) and interpolates to the safe equilibria. The safe equilibrium is designed by simply relaxing the triangularity of the original target plasma (black trace). As a consequence, the EF2L current is further away from its rail.

In these experiments the time horizon for saturation was 10\(ms\) and the interpolation time was 100\(ms\). The equilibrium field and ohmic currents and the plasma parameters did not experience large transients during the transition to the new equilibria. However, in a previous shot 1080521011, the fast interpolation time 20\(ms\) caused a large perturbation of the vertical position and the plasma disrupted, as it is shown in figure 7-5. The
interpolation time is a critical parameter and should be carefully chosen.

Figure 7-5: The short interpolation time 20\(ms\) causes a large perturbation of the vertical position in shot 1080521011 and the plasma disrupts. The same experiment with interpolation time 100\(ms\) succeeds in shot 1080521013.

### 7.4 Conclusions

We demonstrated an application of pulse rescheduling in the specific case of the impending saturation of the EF2L current during a high-triangularity discharge. A high-performance multi-processor multi-timescale architecture was specifically developed and successfully tested.

The application is quite general, at least in an operation scenario where the targets are progressively more ambitious (elongation, triangularity, plasma current, etc.) until the machine limits and the safe equilibrium is set to be a previously run, less demanding
discharge. An advantage of continuing the discharge, instead of terminating it, might be the possibility of dynamically scheduling alternative experiments. However, a simple pair of scalar parameters time horizon and interpolation time might not be sufficient in a more general application, for example the time horizon might need to be different for different coils and maybe for different saturation scenarios (i.e. real-time time-horizon calculation), at least for a most efficient use of resources.

More generally, the anti-saturation routine should be able to handle cases where an arbitrary number of currents is saturating and to decide in real-time how to interpolate and to which alternative scenario: for example, an early termination might be a preferable solution in the case of volt-second over-consumption. Finally, certain pulse-rescheduling applications may require the interpolation of control gains: it is then important to verify that the dynamic behavior of the loop is not significantly perturbed by such transitions and the plasma remains stable.
Chapter 8

Conclusions

Next-generation tokamak fusion reactors will need to operate near their magneto-hydrodynamic limits at high elongation and growth rate of the vertical instability. Because of the constraints of machine cost, and therefore minimum hardware overhead, it is evident that a large responsibility falls on the control design. The problem of vertical stabilization can be broken into several parts:

- Analyze the dynamics of the vertical position, in particular the growth rate of the unstable mode and the location of its currents;

- Design actuators that can effectively counteract the unstable mode;

- Identify the variables that affect the vertical growth rate and select the most unstable equilibria for control design;

- Design realistic scenarios of noise and disturbances;

- Design advanced controllers to stabilize all equilibria robustly in the presence of noise and disturbances;

- Iterate through hardware upgrades if sufficient margins are not obtainable with current machine design.
The design of the ITER vertical stabilization system went through a similar experience. After the discovery that the plasma internal inductance could surpass the design specifications at high-$q_{95}$ phases during the pulse (current rise, ohmic operation, L-mode operation, etc.), a design upgrade was proposed with a secondary vertical stabilization circuit VS2 and higher voltage rating for the original circuit VS1, as illustrated in figure 8-1.

![Figure 8-1: Vertical stabilization system on ITER with the new VS2 circuit. On one hand, the new VS2 circuit can effectively counteract the currents on the inner wall and allow the machine to operate safely at lower stability margin (higher elongation). On the other hand, the higher voltage rating allows to increase the maximum controllable displacement, meaning the system should be able to recover from larger disturbances. Credits D.A. Humphreys.](image)
C-Mod is well positioned to contribute to the study of potential issues and the design of optimal control strategies for next generation tokamaks, for several reasons, among which plasma parameters; machine design, in particular small redundancy of control coils; and fast digital control architecture. At C-Mod, we made progress on several of the previously discussed points:

- Calibrated Alcasim simulations allow precise calculation of the growth rate of C-Mod equilibria and the visualization of the topology of stable and unstable modes.

- A large database analysis of C-Mod equilibria confirms the analytical calculation of an upper bound of the internal inductance as a function of the elongation and the safety factor; calibrated Alcasim simulations prove that the stability margin of C-Mod plasmas is a decreasing function of the internal inductance and that the safe operational region of Alcator C-Mod is described by $m_s \geq 0.3$, $m_\varphi \geq 10$ deg, $\delta z_{max}/a \geq 5\%$, $\delta z_{max}/\sigma_z \geq 10$.

- Broadband noise originating from the plasma and the magnetic diagnostics limits the resolution of the vertical observer at $\sim 1mm*MA$ and the velocity resolution at $\sim 1m/s*MA$, which may affect plasma controllability at limit elongations $\kappa \sim 1.85$. Large pick-ups on some of the power supplies drive large vertical oscillations of high-elongation plasmas with potentially destabilizing effects.

- A state observer based on a Kalman filter is a promising candidate for state reconstruction for full-state vertical stabilization, provided adequate models of the system, the inputs, the intrinsic and measurement noise and an adequate set of diagnostic measurements are available. A model-based filter for the vertical observer based on a SISO implementation of the Kalman filter with uniform intrinsic noise rejects high frequency noise without destabilizing high-elongation plasmas, however it does not perform as well as an optimized low-pass filter.

- Pulse rescheduling is effective at avoiding power supply saturation in a proof-of-
principle application where a high-triangularity equilibrium is interpolated to a more conservative target. The rate of interpolation is critical to avoid transients and potentially loss of plasma control.

There is potential for future work on each one of these topics, in particular: the possibility of extending the safe operational region of C-Mod by adding fast actuators on the inboard of the vacuum vessel, similar to the VS2 circuit in ITER; the possibility of increasing controllability of plasmas at limit elongations by eliminating the large pick-ups of the power supplies and by reducing noise in the vertical observer; the possibility of increasing resiliency to noise and disturbances by full-state feedback control; the demonstration of a more general implementation of the anti-saturation routine; all these results being transferable to ITER and next-generation tokamak machines.
Chapter 9

Appendix A. Inductance of a Uniform-current Ellipse

The poloidal magnetic field is calculated by differentiating the flux in equations 2.29 and 2.30:

\[ B_\psi^2 = \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 = 4\psi_a^2 \left( \frac{x^2}{a^4} + \frac{y^2}{b^4} \right) \]  \hspace{1cm} (9.1)

\[ \psi_a = \frac{\mu_0 I_p}{2\pi} \frac{ab}{a + b} \] \hspace{1cm} (9.2)

where \( a \) and \( b \) are the minor and major axes, \( x \) and \( y \) are the Cartesian coordinates relative to the center of the plasma and \( I_p \) is the total plasma current.

The volume average poloidal field energy density is calculated by integrating equation 9.1 over the cross-section:

\[ \langle B_\psi^2 \rangle = \frac{2\psi_a^2}{\mu_0 \pi ab} \int_{-a}^{a} \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} \left( \frac{x^2}{a^4} + \frac{y^2}{b^4} \right) \, dy \, dx \] \hspace{1cm} (9.3)
\[
\frac{\langle B^2_\varphi \rangle}{2\mu_0} = \frac{4\psi_a^2}{\mu_0\pi ab} \int_{-a}^{a} ab \sqrt{1-x^2/a^2} \left( \frac{x^2}{a^4} + \frac{1 - x^2/a^2}{3b^2} \right) \frac{dx}{a} \quad (9.4)
\]

\[
\frac{\langle B^2_\vartheta \rangle}{2\mu_0} = \frac{4\psi_a^2}{\mu_0\pi} \int_{-1}^{1} \sqrt{1-t^2} t^2 \left( \frac{1}{a^2} - \frac{1}{3b^2} \right) + \sqrt{1-t^2} \frac{1}{3b^2} dt \quad (9.5)
\]

Using:

\[
\int_{0}^{1} \sqrt{1-t^2} dt = \int_{0}^{\pi/2} \cos \vartheta d\vartheta = \frac{\pi}{4} \quad (9.6)
\]

\[
\int_{0}^{1} \sqrt{1-t^2} t^2 dt = \int_{0}^{\pi/2} \cos \vartheta \sin^2 \vartheta d\vartheta = \int_{0}^{\pi/2} \left( \frac{\sin 2\vartheta}{2} \right)^2 d\vartheta = \frac{\pi}{16} \quad (9.7)
\]

the volume average density is obtained:

\[
\frac{\langle B^2_\vartheta \rangle}{2\mu_0} = \frac{\psi_a^2}{2\mu_0} \frac{a^2 + b^2}{a^2 b^2} = \frac{\mu_0 I_p^2}{8\pi^2} \frac{1}{a^2 + b^2} \quad (9.8)
\]

The poloidal field energy density at the edge of the plasma is calculated using the ITER definition of \( l_i = l_i(3) \):

\[
\frac{B^2_\varphi}{2\mu_0} \equiv \frac{\mu_0 I_p^2}{4V_p/R} = \frac{\mu_0 I_p^2}{8\pi^2 a b} \quad (9.9)
\]

Therefore:

\[
l_i = \frac{\langle B^2_\varphi \rangle}{B^2_\varphi} = \frac{1}{2} \frac{2ab}{a^2 + b^2} = \frac{1}{2} \frac{2\kappa}{1 + \kappa^2} \quad (9.10)
\]
Chapter 10

Appendix B. Chopper Power Supply

A simplified diagram of the analog circuitry of the chopper is shown in figure 10-1.

Figure 10-1: Simplified schematic of the analog circuitry of the chopper.

The analysis of a chopper cycle starts with assuming that the thyristors $Th1$, $Th2$ and $Th3$ are off. There is a residual current flowing in the EFC load in the direction of the arrow. The voltage across the load is clamped to the voltage across the varistor $V_{var}$, approximately $-1500V$. The on-pulse is fired and $Th1$ becomes conducting, the
current is pushed by the power supply assembly $PS1$ and the current path commutates from $V1-D3$ to $F1-D1-Th3$. After a short delay ($6\mu s$ nominal), a reset pulse is sent to $Th1$ and the capacitor $C2$ is reverse-charged through $L1$. The reset pulse is generated from the on-pulse on the reset gate drive board. The resonant reverse-charge of $C2$ ultimately turns off $Th1$. Also $Th3$ is turned off at this point and the off-pulse is fired. The instantaneous voltage across the load jumps to the sum of the voltages across $C1$ and $C2$. This spike is seen on the monitor scope $V\, dynamic$ at the onset of the off-pulse. Capacitors $C1$ and $C2$ discharge on one another and on the load, until $Th2$ is turned off. Simultaneously, the current is commutated to the varistor path $V1-D3$ and the voltage across the load clamps back to its negative minimum $-V_{var}$. The dynamic voltage is bipolar, but the current keeps flowing in the same direction.

The control circuitry of chopper is shown in figure 10-2.

![Figure 10-2: Schematic of the control circuitry of the chopper.](image)

The pre-conditioning stages include clamping circuits to limit the input voltage to either positive or negative values ($U1$ and $U11$) and additional buffer stages ($U2$ and $U12$. $U12$ is also inverting). The voltage to frequency conversion is done on each branch with one $AD650$ converter in unipolar configuration. The loading resistors and capacitors are set in order to have a maximum pulsing frequency of $\sim 650kHz$ on either branch.
This is larger than on the DIII-D chopper power supplies\textsuperscript{1}. The pulses generated by the AD650 chips go through two stages of frequency division, nominally \( /9.9 \) (chips \( U_4 \) and \( U_{14} \)) and \( /9 \) (chips \( U_5 \) and \( U_{15} \)). The output of the frequency dividers/pulse counters is fed back through a logic network whose purpose is to avoid that the on and off-counters work simultaneously. In other words, assuming that a forward polarity voltage is being applied to the coils, the on-counters are working while the off-counters are disabled. At zero input voltage the output frequency is \( 650kHz/180 \approx 3.6kHz \). The on-pulse and the off-pulse are sent to the drive boards of the chopper through optical links.

### 10.1 Simulation of the Chopper

The Simulink model of the chopper is discussed in detail in an internal Alcator report. Figure 10-3 shows the highly non-linear DC characteristic.

Table 10.1 summarizes the chopper voltage gain as a function of the input frequency and amplitude:

\[
\begin{array}{cccccc}
V_{\text{Gain}} & Freq [Hz] & 50 & 100 & 150 & 200 \\
V_{\text{in}} [V] & & & & & \\
0.5 & & 150 & 150 & 150 & 120 \\
1 & & 150 & 150 & 150 & 120 \\
1.5 & & 160 & 160 & 150 & 140 \\
2 & & 160 & 160 & 150 & 140 \\
2.5 & & 200 & 200 & 160 & 150 \\
\end{array}
\]

Table 10.1: Non-linear gain of the chopper.

Figure 10-3: DC characteristic of the chopper.
The chopper typically operates at full input amplitude and $100\,Hz \div 250\,Hz$ input frequency, corresponding to the vertical instability of C-Mod plasmas. Our simulations show that, in these conditions, the gain is approximately 200. This value is significantly larger than the value 80 used in the linear models of the vertical stabilization loop (which was calibrated in order to reproduce the condition of marginal stability of high elongation plasmas): one obvious reason is that the load used in simulations in table 9 is the LR series of the EFC coil and is insufficient to represent the real dynamic load which includes also the mutual coupling of the EFC coil with the tokamak structures and also plasma mediated effects.
Chapter 11

Appendix C. DFT Algorithm

Given a series of \( N \) time samples \( x(n) = x(nT) \), the Discrete Fourier Transform (DFT) is calculated as:

\[
X(k) \equiv \sum_{n=0}^{N-1} x(n)w(n)e^{\frac{2\pi}{N}i kn}
\]  

(11.1)

where \( w(n) \) are the weights from a Hanning window:

\[
w(n) = \frac{1}{2} \left[ 1 - \cos \left( \frac{2\pi n}{N} \right) \right]
\]  

(11.2)

Then the Power Spectral Density (PSD) is normalized in the following fashion:

\[
P(0) = \frac{1}{W} |X(0)|^2
\]  

(11.3)

\[
P\left( f_k = k/(NT) \right) = \frac{2}{W} |X(k)|^2, \ k = 1, 2, ..., \frac{N}{2} - 1
\]  

(11.4)

\[
P(f_{Nyq}) = \frac{1}{W} |X(N/2)|^2
\]  

(11.5)
\[ W = N \sum_{n=0}^{N-1} w(n)^2 \]  

(11.6)

The frequency resolution is given by \( \Delta f = 1/(NT) \).
Chapter 12

Appendix D. Schur Balanced Truncation

Schur balanced truncation is a method of model reduction based on Hankel Singular Values, i.e. on the input-output frequency response of the system \{A, B, C\}. Given the controllability grammian \(P\):

\[
P \equiv \int_0^\infty \exp(At)BB^H \exp(A^Ht)dt
\]

and the observability grammian \(Q\):

\[
Q \equiv \int_0^\infty \exp(A^Ht)C^H C \exp(At)dt
\]

the Hankel Singular Values (HSV's) are defined as the square root of the eigenvalues of the product of the grammians:

\[
\sigma_i \equiv \sqrt{\lambda_i(PQ)}
\]

Because \(P\) and \(Q\) are real symmetric matrices, there exists a real matrix \(R\) such that \(Q = R^HR\) and \(RPR^H = U^H \Sigma^2 U\), where \(U\) is a unitary matrix and \(\Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_N)\). We are looking for a state-space balanced transformation of the
where the reduced model \( \{ A_r, B_r, C_r \} \) is obtained by partitioning the transformed matrices and contains the \( N_{\text{red}} \) largest HSV’s. It has been shown \cite{34} that such transformation is given by \( T_{BAL} = -\Sigma^{-1/2} U^H R \). With this transformation, the new grammians are \( \tilde{P} = \tilde{Q} = \Sigma \).

Glover \cite{35} showed that the transfer functions of the full model and the reduced model differ by an error which is at most equal to twice the sum of the neglected singular values:

\[
\| G(i\omega) - G_r(i\omega) \|_\infty \leq 2 \sum_{i=N_{\text{red}}+1}^{N} \sigma_i \quad (12.7)
\]

where the infinity norm signifies the largest singular value of a matrix.

The transformation matrix \( T_{BAL} \) is not necessarily orthogonal and the balancing transformation can be badly conditioned when the system is nearly unobservable or uncontrollable (i.e. \( P \) or \( Q \) are close to singular). Safonov and Chiang \cite{36} proposed a set of transformations which works also for systems that are nearly unobservable or uncontrollable and produces the same reduced-model transfer function. Given a real matrix with real eigenvalues such as \( PQ \), it is possible to find a real orthogonal matrix \( V \) such that \( V^T PQ V \) is an upper triangular matrix with the eigenvalues of \( PQ \) on the main diagonal. This matrix is known as the Schur form of \( PQ \). Two orthogonal real transformations \( V_A \) and \( V_D \) can be found by which the eigenvalues of \( PQ \) appear in ascending or descending order. These transformations can be partitioned to isolate the
$N_{red}$ largest singular values $V_A = \begin{bmatrix} V_{A2} & V_{Ar} \end{bmatrix}$, $V_D = \begin{bmatrix} V_{Dr} & V_{D2} \end{bmatrix}$. Next, a new matrix $E$ is formed and decomposed according to its singular values:

$$E \equiv V_{Dr} V_{Dr} = U_E \Sigma_E V_E^T$$

The truncation matrices are then given by:

$$T_l = \Sigma_E^{-1/2} U_E V_{Ar}$$

$$T_r = V_{Dr} V_E \Sigma_E^{-1/2}$$

The following identities are satisfied:

$$A_r = T'_l \cdot A \cdot T_r$$

$$B_r = T'_l \cdot B$$

$$C_r = C \cdot T_r$$

The Matlab code for our custom implementation of Schur balanced truncation is copied below:

```matlab
[V,L]=eig(A);
L=diag(L);
[dummy3,IX]=sort(real(L),’ascend’);
L=L(IX);
V=V(:,IX);
A=diag(L);
B=inv(V)*B;
```
b = B(:,2);
C = C*V;

[GRED, redinfo, Tl, Tr, se] = ...
my_schurmr(ss(A(1:end-1,1:end-1),b(1:end-1),C(:,1:end-1),zeros(96,1)),5);
% my_schurmr is equivalent to schurmr.m available as a standard routine
% in the Matlab Robust Control Toolbox, but it has been modified to return
% extra arguments, namely the truncation matrices "Tl", "Tr" and the
% normalization matrix "se"

Ar = blkdiag(GRED.a, A(end,end));
br = [GRED.b; b(end)];
Cr = [GRED.c, C(:,end)];
Tl = blkdiag(Tl, 1);
Tr = blkdiag(Tr, 1);
[Vr, Lr] = eig(Ar);
Lr = diag(Lr);
[dummy3, IX] = sort(real(Lr), 'ascend');
Lr = Lr(IX);
Vr = Vr(:, IX);
Ar = diag(Lr);
br = inv(Vr) * br;
Cr = Cr * Vr;
Tl = Tl * (inv(Vr))';
Tr = Tr * Vr;

GRED is the reduced model and Tl and Tr are the left and right truncation matrices.
It is easy to show that the equations of model truncation are satisfied:

\[
Ar = \text{blkdiag}(\text{GRED.a}, A(\text{end}, \text{end})) = Tl' \cdot A \cdot Tr
\]  

(12.14)
\[ br = [GRED.b; b(\text{end})] = Tl' \cdot b \]  

(12.15)

\[ Cr = [GRED.c, C(:, \text{end})] = C \cdot Tr \]  

(12.16)
Appendix E. Linear Model of the Plant

The linear model of the chopper consists of a 4-th order Butterworth filter and a constant gain. The matrices of the filter are:

\[
A_{ps} = 10^3 \begin{bmatrix}
-9.2879 & -5.026500 & 0 & 0 \\
5.0265 & 0 & 0 & 0 \\
0 & 5.0265 & -3.8472 & -5.0265 \\
0 & 0 & 5.0265 & 0
\end{bmatrix} \text{[rad/s]} \quad (13.1)
\]

\[
b_{ps} = 10^3 \begin{bmatrix}
5.0265 \\
0 \\
0 \\
0
\end{bmatrix} \text{[rad/Vs]} \quad (13.2)
\]

\[
c_{ps} = \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} \text{[V]} \quad (13.3)
\]

A high-elongation equilibrium with growth rate \( \lambda = 370 \text{rad/s} \) is the reference for numerical studies and simulations of the Kalman filter. The corresponding matrices are
extracted from calibrated Alcasim simulations of shot 1080430028 at 1.5s during the pulse\textsuperscript{1}:

\[
A_r = 10^{3} \times \begin{bmatrix}
-2.603 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.8389 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.2124 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.0028 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.0014 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.3700
\end{bmatrix} \text{[rad/s]} \quad (13.4)
\]

\[
b_r = 10^{4} \times \begin{bmatrix}
-0.8570 \\
2.709 \\
1.095 \\
0.0018 \\
-0.0007 \\
-0.6825
\end{bmatrix} \text{[A/V * rad/s]} \quad (13.5)
\]

\textsuperscript{1}Alcasim simulation settings: } Z_\sigma = 0.7, j/j_0 = 0.42, l_i calculated from q_0.
\[ C_r = 10^3 [\begin{array}{cccccc}
0.0002 & -0.0004 & -0.0021 & 0.0043 & -0.0053 & -0.0044 \\
-0.0006 & -0.0010 & 0.0015 & 0.0037 & 0.0026 & -0.0006 \\
-0.0005 & -0.0007 & 0.0033 & 0.0042 & 0.0042 & 0.0027 \\
0.0003 & -0.0001 & 0.0023 & 0.0058 & -0.0022 & 0.0029 \\
0.0006 & 0.0003 & 0.0009 & 0.0048 & -0.0096 & 0.0018 \\
0.0004 & 0.0003 & 0.0019 & -0.0060 & -0.0076 & 0.0033 \\
-0.0000 & 0.0000 & 0.0011 & -0.0052 & -0.0099 & 0.0014 \\
0.0009 & 0.0004 & 0.0007 & -0.0056 & 0.0113 & 0.0016 \\
-0.0040 & 0.0015 & -0.0012 & -0.0129 & 0.0185 & 0.0060 \\
0.0022 & 0.0014 & 0.0002 & -0.0154 & 0.0067 & 0.0030 \\
0.0016 & 0.0004 & 0.0004 & -0.0123 & 0.0005 & 0.0006 \\
-0.0018 & -0.0007 & -0.0009 & 0.0119 & -0.0096 & -0.0022 \\
-0.0026 & -0.0015 & 0.0001 & 0.0142 & -0.0109 & -0.0028 \\
0.0036 & -0.0012 & 0.0034 & 0.0091 & -0.0139 & -0.0017 \\
-0.0009 & -0.0002 & 0.0002 & 0.0027 & -0.0126 & 0.0005 \\
-0.0001 & 0.0003 & -0.0001 & 0.0027 & 0.0113 & 0.0007 \\
-0.0006 & 0.0000 & -0.0005 & 0.0030 & 0.0110 & -0.0002 \\
-0.0007 & -0.0001 & -0.0000 & -0.0059 & 0.0139 & 0.0002 \\
-0.0003 & 0.0003 & -0.0015 & -0.0064 & 0.0069 & -0.0010 \\
0.0005 & 0.0007 & -0.0035 & -0.0030 & -0.0027 & -0.0031 \\
0.0008 & 0.0005 & -0.0042 & 0.0020 & -0.0085 & -0.0050 \\
0.0005 & 0.0000 & -0.0002 & -0.0002 & 0.0038 & -0.0005 \\
-0.0003 & -0.0004 & -0.0008 & 0.0043 & -0.0043 & -0.0021 \\
-0.0003 & -0.0002 & -0.0003 & 0.0029 & -0.0027 & -0.0009
\end{array}] [\text{V/A}] \ (13.6)\]
\[
\begin{bmatrix}
-0.0007 & 0.0001 & 0.0006 & 0.0021 & 0.0012 & 0.0015 \\
0.0004 & 0.0000 & -0.0001 & -0.0002 & 0.0053 & -0.0003 \\
-0.0006 & -0.0001 & -0.0002 & 0.0042 & 0.0000 & -0.0001 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-0.0019 & -0.0016 & 0.0053 & -0.0015 & 0.0170 & 0.0038 \\
-0.0013 & -0.0056 & -0.0051 & 0.0294 & -0.0210 & -0.0242 \\
0.0013 & -0.0045 & -0.0108 & 0.0505 & -0.0449 & -0.0305 \\
0.0029 & -0.0016 & -0.0088 & 0.0386 & -0.0379 & -0.0199 \\
0.0008 & -0.0013 & -0.0041 & -0.0076 & -0.0445 & -0.0107 \\
0.0003 & -0.0009 & -0.0019 & -0.0028 & -0.0477 & -0.0057 \\
-0.0011 & 0.0010 & 0.0010 & -0.0065 & 0.0007 & 0.0054 \\
-0.0000 & 0.0010 & 0.0000 & -0.0065 & 0.0052 & 0.0034 \\
-0.0054 & -0.0024 & 0.0069 & -0.0043 & -0.0102 & 0.0073 \\
0.0016 & 0.0010 & 0.0094 & 0.0000 & 0.0141 & 0.0133 \\
-0.0020 & 0.0002 & 0.0008 & 0.0076 & 0.0082 & 0.0032 \\
0.0003 & 0.0006 & 0.0008 & -0.0052 & 0.0043 & 0.0030 \\
0.0000 & 0.0001 & 0.0000 & -0.0006 & 0.0008 & 0.0003 \\
-0.0001 & 0.0004 & 0.0014 & -0.0038 & 0.0041 & 0.0034 \\
0.0004 & 0.0007 & 0.0005 & -0.0050 & 0.0025 & 0.0024 \\
-0.0020 & 0.0004 & 0.0016 & 0.0037 & 0.0057 & 0.0050 \\
0.0013 & 0.0007 & 0.0092 & 0.0012 & 0.0118 & 0.0122 \\
-0.0055 & -0.0027 & 0.0057 & -0.0021 & -0.0171 & 0.0047 \\
0.0002 & 0.0009 & -0.0010 & -0.0048 & 0.0020 & 0.0013 \\
\end{bmatrix}
\]

... = 10^3[V/A] (13.7)
$\begin{bmatrix}
-0.0009 & 0.0007 & -0.0002 & -0.0038 & -0.0029 & 0.0029 \\
0.0001 & -0.0007 & -0.0009 & -0.0031 & -0.0419 & -0.0035 \\
0.0005 & -0.0011 & -0.0024 & -0.0089 & -0.0359 & -0.0071 \\
0.0026 & -0.0009 & -0.0057 & 0.0300 & -0.0329 & -0.0129 \\
0.0007 & -0.0037 & -0.0067 & 0.0402 & -0.0361 & -0.0219 \\
-0.0012 & -0.0059 & -0.0068 & 0.0322 & -0.0295 & -0.0275 \\
-0.0008 & -0.0048 & -0.0084 & 0.0279 & -0.0305 & -0.0266 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0.0001 & 0.0001 & -0.0003 & 0.0001 & -0.0026 & -0.0002 \\
0.0021 & -0.0016 & 0.0093 & -0.0572 & 0.0364 & 0.0064 \\
-0.0024 & 0.0015 & -0.0092 & 0.0531 & -0.0403 & -0.0064 \\
0.0010 & -0.0001 & 0.0016 & 0.0130 & 0.0373 & 0.0014 \\
-0.0012 & 0.0000 & -0.0016 & -0.0102 & -0.0331 & -0.0013 \\
0.0019 & 0.0009 & -0.0023 & -0.0002 & -0.0147 & 0.0017 \\
-0.0024 & -0.0011 & 0.0026 & 0.0013 & 0.0182 & -0.0016 \\
\end{bmatrix}$

... $= 10^3$
\[
\begin{bmatrix}
-0.0001 & -0.0001 & 0.0001 & 0.0009 & 0.0021 & 0.0001 \\
-0.0001 & -0.0001 & 0.0001 & 0.0009 & 0.0021 & 0.0001 \\
-0.0004 & -0.0002 & 0.0003 & 0.0004 & 0.0034 & 0.0001 \\
0.0001 & 0.0000 & -0.0001 & 0.0047 & 0.0041 & 0.0000 \\
-0.1425 & 0.1174 & 0.1806 & 0.0140 & 0.0093 & 0.0579 \\
0.1425 & -0.1174 & -0.1806 & -0.0140 & -0.0093 & -0.0579 \\
0.1174 & -0.0967 & -0.1487 & -0.0115 & -0.0077 & -0.0477 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[c_z = 10^3 * \begin{bmatrix}
-0.0062 & 0.0175 & 0.0862 & -0.1656 & 0.2722 & 0.1773
\end{bmatrix} [V/A] \quad (13.10)\]
Chapter 14

Appendix F. Z-Transform

Given a discrete-time signal $x(n) = x(nT)$, the bilateral z-transform is defined as:

$$X(z) = Z \{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (14.1)$$

where $z$ is a complex number. For causal signals $x(n) \equiv 0 \ \forall n < 0$, the unilateral z-transform is defined as:

$$X(z) = Z \{x(n)\} = \sum_{n=0}^{\infty} x(n)z^{-n} \quad (14.2)$$

The inverse transform is given by:

$$x(n) = Z^{-1} \{X(z)\} = \frac{1}{2\pi i} \oint_{C} X(z)z^{n-1}dz \quad (14.3)$$

where $i$ is the imaginary unit and $C$ is a counter-clockwise closed path encircling the origin and entirely in the region of convergence of $X(z)$. The contour or path must encircle all of the poles of $X(z)$. One special case is when $C$ is the unitary circle, $C = e^{i\omega}$. In this case 14.3 becomes the expression of the inverse Discrete Time Fourier Transform (DTFT, not to be confused with the Discrete Fourier Transform, DFT), therefore:
\[ X(i\omega) = DTFT\{x(n)\} \quad (14.4) \]

The unilateral z-transform is a generalization of the Discrete Time Fourier Transform alike the Laplace Transform is a generalization of the Fourier Transform for causal continuous-time signals. The unilateral z-transform is the Laplace transform of the ideal sampled signal:

\[ X(z) = X(e^{sT}) = L\left\{ \sum_{n=0}^{\infty} x(t)\delta(t - nT) \right\} \quad (14.5) \]

with the substitution \( z = e^{sT} \). This also implies that the left, stable semi-plane in the Laplace domain corresponds to the region inside the unitary circle in the z domain. An immediate corollary is: a discrete-time system is stable if and only if all its poles fall in the unitary circle in the z domain.

Two useful approximations allow to move from Laplace to z-domain and vice versa:

\[ s = \frac{2}{T} \frac{z - 1}{z + 1} \quad (14.6) \]

\[ z = \frac{2 + sT}{2 - sT} \quad (14.7) \]

The variable transformation 14.7 is especially useful for analyzing the frequency response of discrete-time filters.
Bibliography


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