PFC/RR-85-21

DOE/ET-51013-161
UC20b

APPROXIMATIONS

TO TOROIDAL HARMONICS

Patrick A. Pribyl

October, 1985

Plasma Fusion Center
Massachusetts Institute of Technology
Cambridge, MA 02139

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Approximations to Toroidal Harmonics

Abstract

Toroidal harmonics $P^1_{n-1/2}(\cosh\mu)$ and $Q^1_{n-1/2}(\cosh\mu)$ are useful in solutions to Maxwell's equations in toroidal coordinates. In order to speed their computation, a set of approximations has been developed that is valid over the range $0 < \mu < \infty$. The functional form used for these approximations is dictated by their behavior as $\mu \to 0$ and as $\mu \to \infty$, and is similar to that used by Hastings in his approximations to the elliptic integrals K and E. This report lists approximations of several mathematical forms with varying numbers of terms; approximations to the above Legendre functions are given for $n = 0$ through 6. Coefficients of each expansion have been adjusted to distribute the relative error in equi-amplitude peaks over some range, typically $.05 < \mu < 5$, and in the best cases these peaks are less than 10^{-10}. The simple method used to determine the approximations is described. Relative error curves are also presented, obtained by comparing approximations to the more accurate values computed by direct summation of the hypergeometric series.

Contents

1. Introduction

The vector potential in toroidal coordinates for a current distribution that is symmetric in ϕ is given by

$$\vec{A} = \hat{\phi}\sqrt{\cosh\mu - \cos\eta}\sum_{n=0}^{\infty}(a_n \cos\, n\eta + b_n \sin\, n\eta) \times$$
$$\times \left\{ c_n P^1_{n-1/2}(\cosh\mu) + d_n Q^1_{n-1/2}(\cosh\mu) \right\},$$

where $P^1_{n-1/2}$ and $Q^1_{n-1/2}$ are Legendre functions of the first and second kind. (The toroidal coordinate system, μ, η, ϕ, is described in Morse and Feshback[2].) The $P^1_{n-1/2}$s are singular as $\mu \to \infty$, and represent field contributions due to currents nearer the axis of the torus than is the region of interest, as measured by the coordinate μ. $Q^1_{n-1/2}$s are singular for $\mu \to 0$, and represent field contributions due to conductors external to the region of interest. In the neighborhood of a very thin torus ($\mu \to \infty$) these solutions reduce to those found in cylindrical coordinates, $\left(\frac{a}{r}\right)^n$ and $\left(\frac{r}{a}\right)^n$ respectively. For $n = 0$, the quantity $\sqrt{\cosh\mu}\, Q^1_{-1/2}(\cosh\mu)$ approaches a constant as $\mu \to \infty$, while $\sqrt{\cosh\mu}\, P^1_{-1/2}(\cosh\mu)$ approaches $\ln\frac{r}{a}$, thus reproducing the full expression for the vector potential in cylindrical coordinates.

2. Definitions and Asymptotic Behaviors

The Legendre functions are defined as follows:

$$P^1_{n-1/2}(\cosh\mu) = \frac{\Gamma(n+\frac{3}{2})\ \tanh\mu}{2\,\Gamma(n-\frac{1}{2})\ \cosh^{n+1/2}\mu} F\left(\frac{n+\frac{3}{2}}{2}, \frac{n+\frac{5}{2}}{2}; 2; \tanh^2\mu\right), \quad \text{and}$$

$$Q^1_{n-1/2}(\cosh\mu) = \frac{-\sqrt{\pi}\,\Gamma(n+\frac{3}{2})\ \tanh\mu}{\Gamma(n-\frac{1}{2})\ 2^{n+1/2}\ \cosh^{n+1/2}\mu} F\left(\frac{n-\frac{3}{2}}{2}, \frac{n-\frac{5}{2}}{2}; n+1; \text{sech}^2\mu\right),$$

where $F(a, b; c; z)$ is the hypergeometric function. For the approximation analysis, their asymptotic behavior for $\mu \to \infty$ is required, as well as their behavior near $\mu \to 0$. These are listed following:

Asymptotic behavior of $P^1_{n-1/2}(\cosh\mu)$ as $\mu \to \infty$ is given by

$$P^1_{n-1/2}(\cosh\mu) = \begin{cases} \dfrac{-\sqrt{2}}{\pi}\ (\ln 4 - 2 + \mu)\ \cosh^{-1/2}\mu, & \text{for } \mu \to \infty \text{ and } n = 0, \\ \dfrac{2^{n-1/2}\ (n-1)!}{\sqrt{\pi}\ \Gamma(n-1/2)}\ \cosh^{n-1/2}\mu, & \text{for } \mu \to \infty \text{ and } n \geq 1. \end{cases}$$

In the opposite limit of small μ,

$$P^1_{n-1/2}(\cosh\mu) = \left(n^2 - \tfrac{1}{4}\right)\frac{\mu}{2} + O\left(\mu^3\right), \qquad \text{for } \mu \to 0.$$

Asymptotic behavior of $Q^1_{n-1/2}(\cosh\mu)$ as $\mu \to \infty$ is given by

$$Q^1_{n-1/2}(\cosh\mu) = \frac{-\sqrt{\pi}\ \Gamma(n+3/2)}{2^{n+1/2}\ n!}\ \frac{1}{\cosh^{n+1/2}\mu}, \qquad \text{for } \mu \to \infty.$$

In the limit of small μ,

$$(\sinh\mu)\ Q^1_{n-1/2}(\cosh\mu) = -1 + \frac{\mu^2}{4}\left(n^2 - \frac{1}{4}\right)\left[1 + 2\alpha_n + 2\ln\frac{8}{\mu}\right] + O\left(\mu^4\right),$$

the coefficients α_n being given by

$$\alpha_n = \sum_{i=0}^{n} C_i^n \sum_{j=1}^{n-i} \frac{1}{1-2j},$$

where for $n = 0$: $\alpha_0 = 0$

and for $n > 0$, $i = 0$: $C_0^n = 2^{2n-1}$

$i = 1$: $C_1^n = -n\ 2^{2n-2}$

$i > 1$: $C_i^n = (-1)^i\ \dfrac{n}{i}\dbinom{2n-i-1}{i-1}2^{2(n-i)}$

3. Determination of Approximation Coefficients

The mathematical form of the approximations to the Legendre functions is constrained in several respects. In order for the error to decrease to zero as $\mu \rightarrow \infty$, the approximation must have the same asymptotic behavior as the Legendre function. Further, in order for the relative error to approach zero as $\mu \rightarrow 0$, the approximation must also have the same functional dependance on small μ. In the case of the $P^1_{n-1/2}$ s, which approach 0 as $\mu \rightarrow 0$, this requirement is a consequence of l'Hospital's rule. For $Q^1_{n-1/2}$ s, which are singular at the origin, this requirement arises because the asymptotic behaviors of the approximation and the function as $\frac{1}{\mu} \rightarrow \infty$ must again coincide in order for the error to approach zero.

Once the form of the function is established, the fact that the approximation satisfies the above constraints may be used to eliminate a number of coefficients. In principle, any of the coefficients could be solved for; however, for ease of coding, those multiplying the lowest order terms of the polynomials in the approximation were the ones eliminated. The remaining coefficients must then be determined in order to obtain the approximation.

A first guess to the coefficients is found by forming the best approximation to the Legendre function in a least-squares sense. Minimization of the sum of the squared relative errors, $E = \sum(\frac{\text{Approx}}{M^1_{n-1/2}} - 1)^2$, determines a set of N coefficients. At this point, the value of the relative error for the approximation using these coefficients oscillates about zero in a series of $N+1$ peaks with alternating signs and varying amplitudes. Typically their amplitudes differ by as much as a factor of ten, so the next goal is to distribute the error as evenly as possible.

After the initial guesses to the coefficients have been determined, an iterative procedure is used to relax them to a set that result in equi-amplitude peaks of the relative error about zero. For a given set of coefficients C_j, the amplitude of each peak in the relative error is given by

$$E_i = \frac{\sum_{j=1}^{N} T_j(\mu_i)C_j}{M^1_{n-1/2}(\cosh \mu_i)} - 1,$$

where $T_j(\mu)$ represents the functional dependence of each term of the approximation, μ_i is the value of μ at the i^{th} peak, and M represents either P or Q. (The approximation is given by $M^*(\mu) = \sum T_j(\mu)C_j$.) For the final result, a set of coefficients $\hat{C}_j$ is sought for which the sign of the error alternates between peaks of equal magnitude, each located at values of $\mu = \hat{\mu}_i$ that are not known *a priori*. That is, a solution is sought to the overdetermined system of $N+1$ equations given by $E_i = \pm d$.

The iterative scheme that has been developed is based on solving the following system of equations for a set of corrections to each of the coefficients, ΔC_j:

$$\Delta E_i = \frac{\sum_{j=1}^{N} T_j(\mu_i)\ \Delta C_j}{M^1_{n-1/2}(\cosh \mu_i)}.$$

The left-hand-side is determined by the difference of the i^{th} peak from the average magnitude of all the peaks, divided by a relaxation parameter R; specifically $\Delta E_i = \pm(|E_i| - <|E_i|>)/R$. The sign is such that $|E_i + \Delta E_i|$ is closer than $|E_i|$ to the average $<|E_i|>$. A new set of peaks in the relative error is then computed using the corrected coefficients, and the new values of ΔE_i and μ_i substituted into the above system of equations for ΔC_j. The system is solved for the ΔC_j s, and the iteration repeated. R greater than some minimum value seems to be required in order to assure convergence; $R = 5$ has been found to converge reliably and in a reasonable number of iterations. Typically 50-80 iterations result in all peaks being within 10^{-4} of their average magnitude.

Note: there is one more equation than unknown with this scheme. The equation with the smallest absolute value of ΔE_i is ignored on each iteration, and the remaining N equations solved for the N corrections ΔC_j.

4. Approximations to $P^1_{n-1/2}(\cosh\mu)$

Two different functional forms are presented for the approximations to the toroidal harmonics $P^1_{n-1/2}(\cosh\mu)$. The first was obtained by examination of the approximations to the elliptic integrals K and E given by Hastings[1]. The identity

$$P^1_{-1/2}(\cosh\mu) = -\frac{1}{\pi}\frac{1}{\sqrt{\cosh\mu+\sinh\mu}}\left[\left(\frac{2}{k^2}-1\right)K(k) - \frac{2}{k^2}E(k)\right],$$

$$\text{where}\quad k^2 = \frac{2\tanh\mu}{1+\tanh\mu},$$

may be used to derive an expression for $P^1_{-1/2}(\cosh\mu)$ involving the coefficients from Hastings' approximations, and the resulting approximation generalized for the higher order Legendre functions $P^1_{n-1/2}(\cosh\mu)$. Noting that $\cosh\mu+\sinh\mu = e^\mu$, and k^2 as defined above $= 1-e^{-2\mu}$, the general form of these approximations is given by

$$P_n* = N_n\frac{e^{(n-1/2)\mu}}{1-x}\Big[\ (a_0+a_1x+a_2x^2+a_3x^3+\ldots)+ + (b_0+b_1x+b_2x^2+b_3x^3+\ldots)\ln x\ \Big]$$

$$\text{for } x = e^{-2\mu}.$$

N_n represents a normalization factor for the asymptotic behavior as $\mu\to\infty$ and is given by

$$N_n = \frac{(n-1)!}{\sqrt{\pi}\,\Gamma(n-1/2)}.$$

The coefficients a_0 and b_0 are taken directly from the asymptotic behavior of $P^1_{n-1/2}(\cosh\mu)$ as $\mu\to\infty$; in particular, for $n=0$, $a_0 = \ln 4-2$ and $b_0 = -1/2$, while for $n>0$, $a_n = 1$ and $b_n = 0$. Three equations result from setting the behavior of the approximation for $\mu\to 0$ equal to that of the Legendre function, and these were used to eliminate three additional coefficients before solving for the approximation. (Section 2 lists the behavior of $P^1_{n-1/2}$ as $\mu\to\infty$ and as $\mu\to 0$.) The resulting approximations are very good for the first several harmonics, though for $n \gtrsim 4$ the accuracy rapidly deteriorates. A slight additional limitation arises from the factor

$(1-x)$ in the denominator of the leading order term; as $\mu \to 0$, x approaches unity, resulting in worsening errors for $\mu < .001$.

A second general form used for the approximations to $P^1_{n-1/2}(\cosh\mu)$ results in much better accuracy for the higher values of n, though is not as good for the lower ones. In this case the approximation is given by

$$P_n* = N_n \cosh^{n-1/2}\mu \left[\quad (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots) + \right.$$
$$\left. + (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots)\ln x \right]$$

$$\text{for } x = 1 - \tanh\mu, \qquad \text{and} \quad N_n = \frac{2^{n-1/2}\,(n-1)!}{\sqrt{\pi}\,\Gamma(n-1/2)}.$$

The coefficients a_0 and b_0 are determined by the same method as above, but in this case are equal to $\ln 32 - 4$ and -1 respectively. For $n > 0$, a_n and b_n are again equal to 1 and 0. For this approximation, however, only two coefficients were eliminated by setting the functional dependences equal for $\mu \to 0$.

Section 7 contains tables of coefficients for these approximations, and Section 8 contains plots of the error associated with each.

5. Approximations to $Q^1_{n-1/2}(\cosh\mu)$

Approximations to $Q^1_{n-1/2}(\cosh\mu)$ require only a single functional form for $0 \le n \le 6$ in order to achieve accuracy similar to that obtained for $P^1_{n-1/2}(\cosh\mu)$ with comparable numbers of terms. Here the approximation is given by

$$Q_n* = \frac{N_n}{\sinh\mu\ \cosh^{n-1/2}\mu}\left[\quad (a_0 + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots) + \right.$$
$$\left. + (b_2 x^2 + b_3 x^3 - b_4 x^4 + \ldots)\ln x \right]$$

$$\text{for } x = 1 - e^{-2\mu}, \qquad \text{and} \quad N_n = \frac{-\sqrt{\pi}\,\Gamma(n+3/2)}{2^{n+1/2}\,n!}$$

For these functions, $a_1 = b_0 = b_1 = 0$, so a given number of terms results in the order of the approximation being one higher than that for the same number of terms for P_n*. Three additional equations result from setting the behavior of the approximation for $\mu \to 0$ equal to that of the Legendre function, and these were used to eliminate three of the coefficients before solving for the approximation. The sum of coefficients a_i is constrained to equal 1, so that N_n again represents the multiplicative factor for the asymptotic behavior as $\mu \to \infty$. (Section 2 lists the behavior of $Q^1_{n-1/2}$ as $\mu \to \infty$ and as $\mu \to 0$.)

Section 7 contains tables of coefficients for these approximations, and Section 8 contains plots of the error associated with each.

6. Error Summary

Relative errors for each of the approximations presented above are summarized in Table 1. Entries in the table represent the amplitude of the peaks in the relative error plots of Section 8. The relative error is the difference in the value of the approximation and the "true" value of the Legendre function, divided by the true value; the latter was computed either by evaluation of the integral definition or by direct summation of the hypergeometric series, and was typically calculated to 17 decimal places.

The first approximation to $P^1_{n-1/2}(\cosh\mu)$ described above is extremely good for $n = 0$, and acceptable for $n = 1 - 3$, but deteriorates rapidly for $n \gtrsim 4$; it is also problematic for $\mu < .001$. On the other hand, the second approximation is best for $n \gtrsim 3$, and converges better for $\mu \to 0$, though for values of $\mu \ll 10^{-5}$ (10^{-4} for the Order=6 approximations, $n > 3$) the relative accuracy again worsens. In particular, since $P^1_{n-1/2}(\cosh\mu) = 0$ for $\mu = 0$, any nonzero sum of the coefficients in the approximation results in unbounded relative error.

Only one form of approximation to $Q^1_{n-1/2}(\cosh\mu)$ was necessary for similar accuracy to that obtained with a comparable number of terms for $P^1_{n-1/2}$. In addition, the approximations converged well for all values of μ, though in principle there should be precision errors as $\mu \to \infty$ similar to those for the P_n's as $\mu \to 0$.

Table 1a. Relative Errors in Approximations to $P^1_{n-1/2}(\cosh\mu)$
First Approximation, $10^{-3} \le \mu < \infty$

n, Order:	3	4	5
	$\times 10^{-5}$	$\times 10^{-7}$	$\times 10^{-9}$
0	.0255	.0291	.204
1	.119	.123	.166
2	.665	.457	.509
3	16.8	2.49	1.73
4	370.	77.2	9.95
5	1460.	2470.	386.
6	3430.	13300.	17000.

Table 1b. Relative Errors in Approximations to $P^1_{n-1/2}(\cosh\mu)$
Second Approximation, $10^{-4} \le \mu < \infty$

n, Order:	3	4	5	6
	$\times 10^{-4}$	$\times 10^{-6}$	$\times 10^{-8}$	$\times 10^{-9}$
0	.997	7.01	51.2	38.5
1	.273	1.29	7.20	4.45
2	.0303	.138	.721	.404
3	.0548	.119	.378	.148
4	.768	.103	.232	.0736
5	.416	.800	.490	.0915
6	2.99	6.60	1.48	.153

Table 1. Relative errors in approximations to the Legendre functions are summarized in Table 1. Entries in the table refer to the peak amplitude of the relative error, that is the peak deviation of the approximation from the true value of the function normalized to the true value. Relative errors for the first approximation to $P^1_{n-1/2}(\cosh\mu)$ described in Section 4 are presented in Table 1a, while relative errors for the second are shown in Table 1b. Table 1c (next page) summarizes relative errors for the approximations to $Q^1_{n-1/2}(\cosh\mu)$.

Table 1c. Relative Errors in Approximations to $Q^1_{n-1/2}(\cosh\mu)$				
n, Order:	4	5	6	7
	$\times 10^{-4}$	$\times 10^{-6}$	$\times 10^{-7}$	$\times 10^{-9}$
0	.173	.890	.547	3.77
1	.0491	.150	.0676	.378
2	.108	.143	.0207	.0343
3	.415	.659	.138	.331
4	.960	1.62	.350	.857
5	2.31	3.88	.846	2.19
6	6.65	7.33	1.73	4.76

Table 1, continued. Table 1c summarizes relative errors in approximations to the Legendre functions $Q^1_{n-1/2}(\cosh\mu)$ as described in Section 5. Entries in the table refer to the peak amplitude of the relative error, that is the peak deviation of the approximation from the true value of the function normalized to the true value.

7. Coefficient Tables

This section contains tables of the coefficients found for approximations to the Legendre functions $P^1_{n-1/2}(\cosh\mu)$ and $Q^1_{n-1/2}(\cosh\mu)$. Table 2 contains the coefficients for the first approximation to $P^1_{n-1/2}$, for n = (0-6) and the order of the polynomials = (3-5), and Table 3 contains those for the second approximation, for the same range in n but for the order = (3-6). Table 4 contains coefficients for the approximations to $Q^1_{n-1/2}$ for n = (0-6) and polynomial order = (4-7).

Table 2a. Coefficients for P_n*, Order=3 (First Approximation)

The first approximation to $P^1_{n-1/2}(\cosh\mu)$ for Order = 3 is given by

$$P_n* = N_n \frac{e^{(n-1/2)\mu}}{1-x}\Big[\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) + + \left(b_0 + b_1x + b_2x^2 + b_3x^3\right)\ln x \Big]$$

$$\text{with} \quad x = e^{-2\mu}, \quad \text{and} \quad N_n = \frac{(n-1)!}{\sqrt{\pi}\,\Gamma(n-1/2)}.$$

The coefficients for each value of $n \le 6$ are listed below. Error curves for each approximation are shown on pages 36 through 39.

$n = 0$; Peak Error = 0.255×10^{-6}

a_0	= -0.61370563888010937	b_0	= -0.50000000000000000
a_1	= 0.59678190215115219	b_1	= -0.12495244836240303
a_2	= 0.01505562976639772	b_2	= -0.00702839989954492
a_3	= 0.00186810696255946	b_3	= -0.00051663430967806

$n = 1$; Peak Error = 0.119×10^{-5}

a_0	= 1.00000000000000000	b_0	= 0.00000000000000000
a_1	= -1.32849199797948855	b_1	= 0.75020412730542663
a_2	= 0.31343560710103691	b_2	= -0.08972306956739813
a_3	= 0.01505639087845164	b_3	= -0.00402944659596868

$n = 2$; Peak Error = 0.665×10^{-5}

a_0	= 1.00000000000000000	b_0	= 0.00000000000000000
a_1	= -1.24360138040689322	b_1	= 0.00134436684400737
a_2	= -0.08130433703025459	b_2	= 0.49683478061285590
a_3	= 0.32490571743714780	b_3	= -0.06668624530090429

$n = 3$; Peak Error = 0.168×10^{-3}

a_0	= 1.00000000000000000	b_0	= 0.00000000000000000
a_1	= -0.71329261486799964	b_1	= 0.03407537259887128
a_2	= 0.48867677307564913	b_2	= 0.67251468795620560
a_3	= -0.77538415820764950	b_3	= 1.35550148278457300

$n = 4$; Peak Error $= 0.370 \times 10^{-2}$

$a_0 =$	1.00000000000000000	$b_0 =$	0.00000000000000000
$a_1 =$	2.47223427289702746	$b_1 =$	0.69269057515168786
$a_2 =$	15.28657750626309264	$b_2 =$	11.83674456315041001
$a_3 =$	-18.75881177916012010	$b_3 =$	10.70161091375504991

$n = 5$; Peak Error $= 0.146 \times 10^{-1}$

$a_0 =$	1.00000000000000000	$b_0 =$	0.00000000000000000
$a_1 =$	11.10194233272704345	$b_1 =$	2.57273282686029603
$a_2 =$	47.76268737042280677	$b_2 =$	40.12451926692116011
$a_3 =$	-59.86462970314985022	$b_3 =$	30.26931994209544019

$n = 6$; Peak Error $= 0.343 \times 10^{-1}$

$a_0 =$	1.00000000000000000	$b_0 =$	0.00000000000000000
$a_1 =$	25.63498626555872306	$b_1 =$	5.79086260747795478
$a_2 =$	96.86453178313747614	$b_2 =$	85.71069674869941046
$a_3 =$	-123.49951804869619920	$b_3 =$	59.63294495807755968

Table 2b. Coefficients for P_n*, Order=4 (First Approximation)

The first approximation to $P^1_{n-1/2}(\cosh\mu)$ for Order = 4 is given by

$$P_n* = N_n \frac{e^{(n-1/2)\mu}}{1-x}\Big[\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4\right) + $$
$$+ \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4\right) \ln x \Big]$$

$$\text{with} \quad x = e^{-2\mu}, \quad \text{and} \quad N_n = \frac{(n-1)!}{\sqrt{\pi}\,\Gamma(n-1/2)}.$$

The coefficients for each value of $n \leq 6$ are listed below. Error curves for each approximation are shown on pages 40 through 43.

$n = 0$; Peak Error = 0.291×10^{-8}

a_0	= -0.61370563888010937	b_0	= -0.50000000000000000
a_1	= 0.59658519488816578	b_1	= -0.12499785953630699
a_2	= 0.01410870772114893	b_2	= -0.00771237059592766
a_3	= 0.00262303189171999	b_3	= -0.00141402370455686
a_4	= 0.00038870437907468	b_4	= -0.00010226968513078

$n = 1$; Peak Error = 0.123×10^{-7}

a_0	= 1.00000000000000000	b_0	= 0.00000000000000000
a_1	= -1.32940346404233611	b_1	= 0.75000661003987845
a_2	= 0.30790128581388143	b_2	= -0.09335488334636968
a_3	= 0.01912510722944853	b_3	= -0.00930914892300754
a_4	= 0.00237707099900617	b_4	= -0.00062529104029817

$n = 2$; Peak Error = 0.457×10^{-7}

a_0	= 1.00000000000000000	b_0	= 0.00000000000000000
a_1	= -1.24983425049876898	b_1	= 0.00002822161323613
a_2	= -0.12257245977527165	b_2	= 0.47058659329283160
a_3	= 0.35532732408150240	b_3	= -0.10561479254983950
a_4	= 0.01707938619253820	b_4	= -0.00432036932157596

$n = 3$; Peak Error = 0.249×10^{-6}

a_0	= 1.00000000000000000	b_0	= 0.00000000000000000
a_1	= -0.87406257250952349	b_1	= 0.00015887079948609
a_2	= -0.51721080821847371	b_2	= 0.01048169824992055
a_3	= 0.08094734931219300	b_3	= 0.47424274938011670
a_4	= 0.31032603141580420	b_4	= -0.06054530308284824

$n = 4$; Peak Error $= 0.772 \times 10^{-5}$

a_0	= 1.0000000000000000	b_0	= 0.0000000000000000
a_1	= -0.72174153223564469	b_1	= 0.00481209197026763
a_2	= 0.51121599123450129	b_2	= 0.30374176828586330
a_3	= 0.74761258763212740	b_3	= 1.66084998327031999
a_4	= -1.53708704663098400	b_4	= 1.63541612986774501

$n = 5$; Peak Error $= 0.247 \times 10^{-3}$

a_0	= 1.0000000000000000	b_0	= 0.0000000000000000
a_1	= 0.15905403377342253	b_1	= 0.14566427936530424
a_2	= 22.37702276359091691	b_2	= 8.50635175108090991
a_3	= 18.32696286086377002	b_3	= 39.86833033931265025
a_4	= -41.86303965822810991	b_4	= 19.03782411960701015

$n = 6$; Peak Error $= 0.133 \times 10^{-2}$

a_0	= 1.0000000000000000	b_0	= 0.0000000000000000
a_1	= 3.68600305787949623	b_1	= 0.75216421514184617
a_2	= 107.48202154369647587	b_2	= 41.54005656855844020
a_3	= 61.01805038840713014	b_3	= 177.81365980048099829
a_4	= -173.18607498998310135	b_4	= 70.93422206525728946

Table 2c. Coefficients for P_n*, Order=5 (First Approximation)

The first approximation to $P^1_{n-1/2}(\cosh\mu)$ for Order = 5 is given by

$$P_n* = N_n \frac{e^{(n-1/2)\mu}}{1-x}\Big[\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5\right) + \\ + \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5\right)\ln x \Big]$$

$$\text{with} \quad x = e^{-2\mu}, \quad \text{and} \quad N_n = \frac{(n-1)!}{\sqrt{\pi}\,\Gamma(n-1/2)}.$$

The coefficients for each value of $n \leq 6$ are listed below. Error curves for each approximation are shown on pages 44 through 47.

$n = 0$; Peak Error = 0.204×10^{-9}

a_0	= -0.61370563888010937	b_0	= -0.50000000000000000
a_1	= 0.59657414033579320	b_1	= -0.12499991218689401
a_2	= 0.01388118292387826	b_2	= -0.00780297996425344
a_3	= 0.00237363256083113	b_3	= -0.00183089067584704
a_4	= 0.00077594725289831	b_4	= -0.00040597226678944
a_5	= 0.00010073580670847	b_5	= -0.00002511681739474

$n = 1$; Peak Error = 0.166×10^{-9}

a_0	= 1.00000000000000000	b_0	= 0.00000000000000000
a_1	= -1.32944011496172576	b_1	= 0.75000021523436217
a_2	= 0.30691746124960534	b_2	= -0.09371928508459080
a_3	= 0.01773339163710342	b_3	= -0.01125521599830550
a_4	= 0.00424910804701094	b_4	= -0.00218220300802118
a_5	= 0.00054015402800609	b_5	= -0.00013569592031398

$n = 2$: Peak Error = 0.509×10^{-9}

a_0	= 1.00000000000000000	b_0	= 0.00000000000000000
a_1	= -1.24999502118707079	b_1	= 0.00000073966455696
a_2	= -0.12734764755078450	b_2	= 0.46886499371714300
a_3	= 0.34794222170587740	b_3	= -0.11535534478788500
a_4	= 0.02661071556191640	b_4	= -0.01234358888648580
a_5	= 0.00278973147006147	b_5	= -0.00069466813429452

$n = 3;$ Peak Error $= 0.173 \times 10^{-8}$

a_0	= 1.0000000000000000	b_0	= 0.0000000000000000
a_1	= -0.8749823750670716O	b_1	= 0.0000026049358272 4
a_2	= -0.5453138681330870 2	b_2	= 0.0004167475055064 2
a_3	= 0.0377284747368799 9	b_3	= 0.4168707709832221 0
a_4	= 0.3673126813841619 0	b_4	= -0.1067480448437732 0
a_5	= 0.0152550870791167 3	b_5	= -0.0036435523904081 6

$n = 4;$ Peak Error $= 0.995 \times 10^{-8}$

a_0	= 1.0000000000000000	b_0	= 0.0000000000000000
a_1	= -0.74989797445248321	b_1	= 0.00001507503926984
a_2	= -0.31915131729314498	b_2	= 0.00240274502536429
a_3	= -0.33956158106391390	b_3	= 0.03779684817419871
a_4	= 0.11961350430808970	b_4	= 0.49682504953381330
a_5	= 0.28899736850145240	b_5	= -0.05359522528175207

$n = 5;$ Peak Error $= 0.386 \times 10^{-6}$

a_0	= 1.0000000000000000	b_0	= 0.0000000000000000
a_1	= -0.68366681449270034	b_1	= 0.00056883508355859
a_2	= 0.06441501851164178	b_2	= 0.08738761877868293
a_3	= 2.05481261489056399	b_3	= 1.28551731174868600
a_4	= 0.03100110413160455	b_4	= 3.29494259901631398
a_5	= -2.46656192304110999	b_5	= 1.93078776684961501

$n = 6;$ Peak Error $= 0.170 \times 10^{-4}$

a_0	= 1.0000000000000000	b_0	= 0.0000000000000000
a_1	= -0.48905364047194766	b_1	= 0.02403836948519533
a_2	= 12.50685709874961660	b_2	= 3.51051507767910798
a_3	= 79.31242478434097087	b_3	= 47.66207114844119008
a_4	= -12.93927962511278995	b_4	= 104.16198010983189981
a_5	= -79.39094861750585075	b_5	= 30.89132197249282008

Table 3a. Coefficients for P_n*, Order=3 (Second Approximation)

The second approximation to $P^1_{n-1/2}(\cosh\mu)$ for Order = 3 is given by

$$P_n* = N_n \cosh^{n-1/2}\mu \left[(a_0 + a_1 x + a_2 x^2 + a_3 x^3) + \right.$$
$$\left. + (b_0 + b_1 x + b_2 x^2 + b_3 x^3) \ln x \right]$$

$$\text{with} \quad x = 1 - \tanh\mu, \quad \text{and} \quad N_n = \frac{2^{n-1/2}\,(n-1)!}{\sqrt{\pi}\,\Gamma(n-1/2)}.$$

The coefficients for each value of $n \leq 6$ are listed below. Error curves for each approximation are shown on pages 48 through 51.

$n = 0$; Peak Error = 0.997×10^{-4}

a_0	= -0.534264097200273	b_0	= -1.000000000000000
a_1	= -0.416322778249672	b_1	= -0.921662913456429
a_2	= -0.441499264592456	b_2	= -1.394727510153524
a_3	= 1.392086140042402	b_3	= -0.671267423622260

$n = 1$; Peak Error = 0.273×10^{-4}

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -0.702875822954688	b_1	= 0.369144755798906
a_2	= -0.285654038649561	b_2	= 0.129902269298870
a_3	= -0.011470138395751	b_3	= -0.023493260561405

$n = 2$; Peak Error = 0.303×10^{-5}

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -0.873548857702661	b_1	= 0.000363029432287
a_2	= -0.066411050508437	b_2	= 0.115811915698895
a_3	= -0.060040091788902	b_3	= 0.029015600324192

$n = 3$; Peak Error = 0.548×10^{-5}

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -1.193635839709137	b_1	= -0.001259726569337
a_2	= 0.268987823384164	b_2	= -0.028231475358594
a_3	= -0.075351983675027	b_3	= 0.000069243341812

$n = 4$; Peak Error $= 0.768 \times 10^{-5}$

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -1.618863656415857	b_1	= 0.001342641185508
a_2	= 0.935374640256880	b_2	= 0.019036526999520
a_3	= -0.316510983841023	b_3	= -0.006085417673869

$n = 5$; Peak Error $= 0.416 \times 10^{-4}$

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -2.120219709406006	b_1	= -0.006040327882229
a_2	= 1.673324579432837	b_2	= -0.066798767009067
a_3	= -0.553104870026831	b_3	= 0.035918671383742

$n = 6$; Peak Error $= 0.299 \times 10^{-3}$

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -2.744614750664497	b_1	= -0.039476925869316
a_2	= 2.545235127232209	b_2	= -0.404037653856612
a_3	= -0.800620376567712	b_3	= 0.194146575945696

Table 3b. Coefficients for P_n*, Order=4 (Second Approximation)

The second approximation to $P^1_{n-1/2}(\cosh\mu)$ for Order = 4 is given by

$$P_n* = N_n \cosh^{n-1/2}\mu \Big[\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4\right) + \\ + \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4\right) \ln x \Big]$$

$$\text{with} \quad x = 1 - \tanh\mu, \quad \text{and} \quad N_n = \frac{2^{n-1/2}\,(n-1)!}{\sqrt{\pi}\,\Gamma(n-1/2)}.$$

The coefficients for each value of $n \le 6$ are listed below. Error curves for each approximation are shown on pages 52 through 55.

$n = 0$; Peak Error = 0.701×10^{-5}

a_0	= -0.534264097200273	b_0	= -1.000000000000000
a_1	= -0.287204999939953	b_1	= -0.887874653369265
a_2	= -1.347593031045908	b_2	= -1.416516344194158
a_3	= 0.033459265932411	b_3	= -2.636281934463109
a_4	= 2.135602862253724	b_4	= -0.830445987293418

$n = 1$; Peak Error = 0.129×10^{-5}

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -0.679108712765667	b_1	= 0.374235169533159
a_2	= -0.196104034340361	b_2	= 0.211370445581290
a_3	= -0.209676891599472	b_3	= -0.014176261328610
a_4	= 0.084889638705499	b_4	= -0.043681003267727

$n = 2$; Peak Error = 0.138×10^{-6}

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -0.874620775296809	b_1	= 0.000067761114539
a_2	= -0.044975160603561	b_2	= 0.119966943282330
a_3	= -0.036246011409509	b_3	= 0.071450693341156
a_4	= -0.044158052690121	b_4	= 0.017155255124050

$n = 3$; Peak Error = 0.119×10^{-6}

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -1.187952237133550	b_1	= -0.000076585175766
a_2	= 0.302305025325207	b_2	= -0.005019437895781
a_3	= -0.117151969019563	b_3	= 0.021529301307634
a_4	= 0.002799180827906	b_4	= -0.003970010557895

$n = 4;$ Peak Error $= 0.103 \times 10^{-6}$

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -1.624683322918256	b_1	= 0.000055363077883
a_2	= 0.925091521079852	b_2	= 0.002864812725363
a_3	= -0.264558021489821	b_3	= 0.009199212038030
a_4	= -0.035850176671775	b_4	= 0.016102087159833

$n = 5;$ Peak Error $= 0.800 \times 10^{-6}$

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -2.096555526189290	b_1	= -0.000480335034892
a_2	= 1.628304212413069	b_2	= -0.029495458841583
a_3	= -0.785958200525633	b_3	= -0.154775670363624
a_4	= 0.254209514301854	b_4	= -0.104070474155645

$n = 6;$ Peak Error $= 0.660 \times 10^{-5}$

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -2.596992669683981	b_1	= -0.003791673483050
a_2	= 2.125388786392214	b_2	= -0.224469004151843
a_3	= -2.172282180123772	b_3	= -1.130421825281543
a_4	= 1.643886063415539	b_4	= -0.659173743158298

Table 3c. Coefficients for P_n*, Order=5 (Second Approximation)

The second approximation to $P^1_{n-1/2}(\cosh\mu)$ for Order = 5 is given by

$$P_n* = N_n \cosh^{n-1/2}\mu \Big[\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5\right) + \\ + \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5\right) \ln x \Big]$$

$$\text{with} \quad x = 1 - \tanh\mu, \quad \text{and} \quad N_n = \frac{2^{n-1/2}\,(n-1)!}{\sqrt{\pi}\,\Gamma(n-1/2)}.$$

The coefficients for each value of $n \leq 6$ are listed below. Error curves for each approximation are shown on pages 56 through 59.

$n = 0$; Peak Error = 0.512×10^{-6}

a_0	= -0.534264097200273	b_0	= -1.000000000000000
a_1	= -0.233939773628904	b_1	= -0.877618113612578
a_2	= -0.994012369745059	b_2	= -1.151859729917774
a_3	= -3.960101642331980	b_3	= -3.705159046119528
a_4	= 2.473089405053885	b_4	= -5.328453318002163
a_5	= 3.249228477852331	b_5	= -1.083861096249782

$n = 1$; Peak Error = 0.720×10^{-7}

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -0.675293638402000	b_1	= 0.374903324454901
a_2	= -0.117985497733141	b_2	= 0.244258949858002
a_3	= -0.318616821353870	b_3	= 0.058514445111453
a_4	= -0.047057693549232	b_4	= -0.192152324759523
a_5	= 0.158953651038243	b_5	= -0.057987328633920

$n = 2$: Peak Error = 0.721×10^{-8}

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -0.874939389973730	b_1	= 0.000009204248706
a_2	= -0.046639236561549	b_2	= 0.118358182427859
a_3	= 0.005703714705223	b_3	= 0.085085034991842
a_4	= -0.047382878744061	b_4	= 0.067190328788835
a_5	= -0.036742209425882	b_5	= 0.012405841998708

$n = 3$; Peak Error $= 0.378 \times 10^{-8}$

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -1.187536700734583	b_1	= -0.000005462066188
a_2	= 0.313378705556708	b_2	= -0.000824469960609
a_3	= -0.113461936296202	b_3	= 0.039227147936564
a_4	= -0.019304882321331	b_4	= -0.002746489554452
a_5	= 0.006924813795408	b_5	= -0.003028270089267

$n = 4$; Peak Error $= 0.232 \times 10^{-8}$

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -1.624980632227915	b_1	= 0.000002941224528
a_2	= 0.919287204304596	b_2	= 0.000379177002391
a_3	= -0.255692148302061	b_3	= 0.004144894736949
a_4	= -0.026546779450641	b_4	= 0.027759610746449
a_5	= -0.012067644323979	b_5	= 0.004367807323043

$n = 5$; Peak Error $= 0.490 \times 10^{-8}$

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -2.093794116689689	b_1	= -0.000006628298482
a_2	= 1.705587674742494	b_2	= -0.000940839177930
a_3	= -0.709483882692027	b_3	= -0.012786087983951
a_4	= 0.099493295684059	b_4	= -0.030133605378040
a_5	= -0.001802971044838	b_5	= -0.003826335521243

$n = 6$; Peak Error $= 0.148 \times 10^{-7}$

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -2.574877144715508	b_1	= 0.000018661661637
a_2	= 2.740299352812521	b_2	= 0.002454947931291
a_3	= -1.473981735735666	b_3	= 0.030060937946438
a_4	= 0.426500741519116	b_4	= 0.055874474063492
a_5	= -0.117941213880463	b_5	= 0.006144098337009

Table 3d. Coefficients for P_n*, Order=6 (Second Approximation)

The second approximation to $P^1_{n-1/2}(\cosh\mu)$ for Order = 6 is given by

$$P_n* = N_n \cosh^{n-1/2}\mu \Big[\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6\right) + \\ + \left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\right)\ln x \Big]$$

$$\text{with} \quad x = 1 - \tanh\mu, \quad \text{and} \quad N_n = \frac{2^{n-1/2}\,(n-1)!}{\sqrt{\pi}\,\Gamma(n-1/2)}.$$

The coefficients for each value of $n \le 6$ are listed below. Error curves for each approximation are shown on pages 60 through 63.

$n = 0$; Peak Error = 0.385×10^{-7}

a_0	= -0.534264097200273	b_0	= -1.000000000000000
a_1	= -0.220629192934283	b_1	= -0.875449829295413
a_2	= -0.481780273571535	b_2	= -0.976249252004682
a_3	= -5.261736015097365	b_3	= -3.166901168898319
a_4	= -7.738525266565616	b_4	= -11.252830250851290
a_5	= 9.367861383611383	b_5	= -10.520151458002740
a_6	= 4.869073461757689	b_6	= -1.449387612458285

$n = 1$; Peak Error = 0.445×10^{-8}

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -0.674738359680759	b_1	= 0.374988089147577
a_2	= -0.083874523669485	b_2	= 0.254435214435597
a_3	= -0.255067913074569	b_3	= 0.142728224741304
a_4	= -0.503917490889110	b_4	= -0.291328878774636
a_5	= 0.288068134859095	b_5	= -0.436763849933655
a_6	= 0.229530152454828	b_6	= -0.071259829745445

$n = 2$; Peak Error = 0.404×10^{-9}

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -0.874992074922425	b_1	= 0.000001080530432
a_2	= -0.049376952465954	b_2	= 0.117489479517649
a_3	= 0.011091744000587	b_3	= 0.081370996941955
a_4	= 0.018836092795211	b_4	= 0.101705970704736
a_5	= -0.071477310004615	b_5	= 0.074787825713374
a_6	= -0.034081499402803	b_6	= 0.010345881072612

$n = 3$; Peak Error $= 0.148 \times 10^{-9}$

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -1.187503104816146	b_1	= -0.000000419884678
a_2	= 0.315853991974570	b_2	= -0.000126130349578
a_3	= -0.101784314011056	b_3	= 0.047428753521249
a_4	= -0.034427199413571	b_4	= 0.007169415166483
a_5	= 0.000451343153581	b_5	= -0.010993062630517
a_6	= 0.007409283112623	b_6	= -0.002472212264148

$n = 4$; Peak Error $= 0.736 \times 10^{-10}$

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -1.624998604205032	b_1	= 0.000000191069957
a_2	= 0.918190509370846	b_2	= 0.000051996220418
a_3	= -0.258012918013335	b_3	= 0.001367361074799
a_4	= -0.013148226277460	b_4	= 0.030034981965276
a_5	= -0.015416747718318	b_5	= 0.015152909788544
a_6	= -0.006614013156700	b_6	= 0.002056045111979

$n = 5$; Peak Error $= 0.915 \times 10^{-10}$

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -2.093751808109838	b_1	= -0.000000246230661
a_2	= 1.708678860001157	b_2	= -0.000070502745476
a_3	= -0.693005964851439	b_3	= -0.002053345877064
a_4	= 0.088835696025209	b_4	= -0.011759200463299
a_5	= -0.015168071897540	b_5	= -0.004875243615622
a_6	= 0.004411288832451	b_6	= -0.001605220142260

$n = 6$; Peak Error $= 0.153 \times 10^{-9}$

a_0	= 1.000000000000000	b_0	= 0.000000000000000
a_1	= -2.574997161564676	b_1	= 0.000000389357318
a_2	= 2.732091735765760	b_2	= 0.000105521366802
a_3	= -1.513696842391935	b_3	= 0.002873009271986
a_4	= 0.459783012358527	b_4	= 0.014645280664929
a_5	= -0.089787669416908	b_5	= 0.014417186196661
a_6	= -0.013393074750768	b_6	= 0.004653946822857

Table 4a. Coefficients for Q_n*, Order=4

The approximation to $Q^1_{n-1/2}(\cosh\mu)$ for Order = 4 is given by

$$Q_n* = \frac{N_n}{\sinh\mu\ \cosh^{n-1/2}\mu}\Big[\ (a_0 + a_2x^2 + a_3x^3 + a_4x^4) + \\ + (b_2x^2 + b_3x^3 + b_4x^4)\ln x\ \Big]$$

$$\text{with}\quad x = 1 - e^{-2\mu}, \quad \text{and}\quad N_n = \frac{-\sqrt{\pi}\,\Gamma(n+3/2)}{2^{n+1/2}\,n!}.$$

The coefficients for each value of $n \le 6$ are listed below. Error curves for each approximation are shown on pages 64 through 67.

$n = 0$; Peak Error = 0.173×10^{-4}

a_0 = 0.900316316157106			
a_2 = 0.035804137199687		b_2 = -0.028134884879910	
a_3 = 0.110664873431048		b_3 = 0.003378342935433	
a_4 = -0.046785326787842		b_4 = 0.057522636629864	

$n = 1$; Peak Error = 0.491×10^{-5}

a_0 = 1.200421754876141	
a_2 = -0.068190189118991	b_2 = 0.112539539519638
a_3 = -0.074881698262097	b_3 = 0.092916735120946
a_4 = -0.057349867495053	b_4 = 0.010185888879103

$n = 2$; Peak Error = 0.108×10^{-4}

a_0 = 1.920674807801826	
a_2 = -0.185394986489085	b_2 = 0.900316316157106
a_3 = -0.139480208204056	b_3 = 0.784578714838759
a_4 = -0.595799613108685	b_4 = 0.237982117372083

$n = 3$; Peak Error = 0.415×10^{-4}

a_0 = 3.292585384803131	
a_2 = 0.287352986794639	b_2 = 3.601265264628424
a_3 = -0.010749195917023	b_3 = 2.992274073031487
a_4 = -2.569189175680747	b_4 = 0.955282032889213

$n = 4$; Peak Error $= 0.960 \times 10^{-4}$

$a_0 =$ 5.853485128538899			
$a_2 =$ 3.480429301478592	$b_2 =$ 11.524048846810958
$a_3 =$ 1.316880077152199	$b_3 =$ 9.234373284623569
$a_4 =$ -9.650794507169690	$b_4 =$ 3.788443491953949

$n = 5$; Peak Error $= 0.231 \times 10^{-3}$

$a_0 =$ 10.642700233707090	
$a_2 =$ 15.930602599970358	$b_2 =$ 32.925853848031308
$a_3 =$ 5.052476995820293	$b_3 =$ 24.953467278551680
$a_4 =$ -30.625779829497740	$b_4 =$ 13.494338491722520

$n = 6$; Peak Error $= 0.665 \times 10^{-3}$

$a_0 =$ 19.648061969920780	
$a_2 =$ 55.989649537565295	$b_2 =$ 87.802276928083488
$a_3 =$ 4.132954390804304	$b_3 =$ 58.970411631414320
$a_4 =$ -78.770665898290380	$b_4 =$ 38.871021515160690

Table 4b. Coefficients for Q_n*, Order=5

The approximation to $Q^1_{n-1/2}(\cosh\mu)$ for Order = 5 is given by

$$Q_n* = \frac{N_n}{\sinh\mu\ \cosh^{n-1/2}\mu}\Big[\ (a_0 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5) + $$
$$+ (b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5)\ln x\ \Big]$$

$$\text{with}\quad x = 1 - e^{-2\mu},\quad \text{and}\quad N_n = \frac{-\sqrt{\pi}\,\Gamma(n+3/2)}{2^{n+1/2}\,n!}.$$

The coefficients for each value of $n \le 6$ are listed below. Error curves for each approximation are shown on pages 68 through 71.

$n = 0$; Peak Error = 0.890×10^{-6}

a_0 =	0.900316316157106		
a_2 =	0.035804137199687	b_2 =	-0.028134884879910
a_3 =	0.113309132451351	b_3 =	-0.004887204703456
a_4 =	0.222295573390245	b_4 =	0.207636272395787
a_5 =	-0.271725159198390	b_5 =	0.133227986594393

$n = 1$; Peak Error = 0.150×10^{-6}

a_0 =	1.200421754876141		
a_2 =	-0.068190189118991	b_2 =	0.112539539519638
a_3 =	-0.038032912613493	b_3 =	0.106143653107053
a_4 =	-0.108508394112938	b_4 =	0.033602108620482
a_5 =	0.014309740969281	b_5 =	-0.014310039551116

$n = 2$; Peak Error = 0.143×10^{-6}

a_0 =	1.920674807801826		
a_2 =	-0.185394986489085	b_2 =	0.900316316157106
a_3 =	0.174214108910371	b_3 =	0.879893594599449
a_4 =	-0.521546046465109	b_4 =	0.714998841698949
a_5 =	-0.387947883758003	b_5 =	0.128871920019061

$n = 3$; Peak Error = 0.659×10^{-6}

a_0 =	3.292585384803131		
a_2 =	0.287352986794639	b_2 =	3.601265264628424
a_3 =	1.542962725559433	b_3 =	3.476288829204327
a_4 =	-2.455790576739893	b_4 =	3.168307710193999
a_5 =	-1.667110520417310	b_5 =	0.521810670673672

$n = 4$; Peak Error $= 0.162 \times 10^{-5}$

a_0	= 5.853485128538899		
a_2	= 3.480429301478592	b_2	= 11.524048846810958
a_3	= 7.025679246193246	b_3	= 11.015390854243580
a_4	= -9.047123887182945	b_4	= 11.993296261365530
a_5	= -6.312469789027792	b_5	= 2.030350179604152

$n = 5$; Peak Error $= 0.388 \times 10^{-5}$

a_0	= 10.642700233707090		
a_2	= 15.930602599970358	b_2	= 32.925853848031308
a_3	= 24.093610701493343	b_3	= 31.007705219765920
a_4	= -30.531684437967850	b_4	= 39.665668019092220
a_5	= -19.135229097202940	b_5	= 5.936984919208991

$n = 6$; Peak Error $= 0.733 \times 10^{-5}$

a_0	= 19.648061969920780		
a_2	= 55.989649537565295	b_2	= 87.802276928083488
a_3	= 73.061008146601095	b_3	= 81.601520376007160
a_4	= -91.532248101070911	b_4	= 125.666406001157100
a_5	= -56.166471553016260	b_5	= 15.622427497055180

Table 4c. Coefficients for Q_n*, Order=6

The approximation to $Q^1_{n-1/2}(\cosh\mu)$ for Order = 6 is given by

$$Q_n* = \frac{N_n}{\sinh\mu\ \cosh^{n-1/2}\mu}\Big[\ \left(a_0 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6\right) + + \left(b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\right)\ln x\ \Big]$$

$$\text{with} \quad x = 1 - e^{-2\mu}, \quad \text{and} \quad N_n = \frac{-\sqrt{\pi}\,\Gamma(n+3/2)}{2^{n+1/2}\,n!}.$$

The coefficients for each value of $n \leq 6$ are listed below. Error curves for each approximation are shown on pages 72 through 75.

$n = 0$; Peak Error = 0.547×10^{-7}

a_0	= 0.900316316157106		
a_2	= 0.035804137199687	b_2	= -0.028134884879910
a_3	= 0.082449689918055	b_3	= -0.015426750804935
a_4	= 0.676616917069956	b_4	= 0.305372149560013
a_5	= -0.024569570366638	b_5	= 0.861719837426173
a_6	= -0.670617489978167	b_6	= 0.247591524353457

$n = 1$; Peak Error = 0.676×10^{-8}

a_0	= 1.200421754876141		
a_2	= -0.068190189118991	b_2	= 0.112539539519638
a_3	= -0.022576413033331	b_3	= 0.110328181869239
a_4	= -0.130904306713725	b_4	= 0.053941976022660
a_5	= -0.062690954516259	b_5	= -0.081101500260343
a_6	= 0.083940108506164	b_6	= -0.033166480403613

$n = 2$; Peak Error = 0.207×10^{-8}

a_0	= 1.920674807801826		
a_2	= -0.185394986489085	b_2	= 0.900316316157106
a_3	= 0.251234312289922	b_3	= 0.897719547819729
a_4	= -0.079841854405996	b_4	= 1.004085531559376
a_5	= -0.657024038290415	b_5	= 0.592809818201771
a_6	= -0.249648240906253	b_6	= 0.074533044801632

$n = 3$; Peak Error $= 0.138 \times 10^{-7}$

$a_0 =$ 3.292585384803131
$a_2 =$ 0.287352986794639 $b_2 =$ 3.601265264628424
$a_3 =$ 1.985557963465428 $b_3 =$ 3.581154544103700
$a_4 =$ -0.300809254067363 $b_4 =$ 4.695747517699175
$a_5 =$ -3.243380255421183 $b_5 =$ 2.666686530556666
$a_6 =$ -1.021306825574652 $b_6 =$ 0.284247091207232

$n = 4$; Peak Error $= 0.350 \times 10^{-7}$

$a_0 =$ 5.853485128538899
$a_2 =$ 3.480429301478592 $b_2 =$ 11.524048846810958
$a_3 =$ 8.799539396920812 $b_3 =$ 11.436970742640360
$a_4 =$ -0.442663285956600 $b_4 =$ 18.092228517308290
$a_5 =$ -12.491217680165180 $b_5 =$ 10.672092303192740
$a_6 =$ -4.199572860816523 $b_6 =$ 1.189365744170567

$n = 5$; Peak Error $= 0.846 \times 10^{-7}$

$a_0 =$ 10.642700233707090
$a_2 =$ 15.930602599970358 $b_2 =$ 32.925853848031308
$a_3 =$ 30.634437292160800 $b_3 =$ 32.577949710854710
$a_4 =$ -0.590800020967027 $b_4 =$ 61.487506309489400
$a_5 =$ -41.575529036790410 $b_5 =$ 35.685050391277940
$a_6 =$ -14.041411068080810 $b_6 =$ 3.923445497077095

$n = 6$; Peak Error $= 0.173 \times 10^{-6}$

$a_0 =$ 19.648061969920780
$a_2 =$ 55.989649537565295 $b_2 =$ 87.802276928083488
$a_3 =$ 93.761662013532309 $b_3 =$ 86.585523750896551
$a_4 =$ 3.090234388678536 $b_4 =$ 194.550038852518700
$a_5 =$ -125.743425294776699 $b_5 =$ 110.528166593172701
$a_6 =$ -45.746182614920220 $b_6 =$ 12.995875053266850

Table 4d. Coefficients for Q_n*, Order=7

The approximation to $Q^1_{n-1/2}(\cosh\mu)$ for Order = 7 is given by

$$Q_n* = \frac{N_n}{\sinh\mu\ \cosh^{n-1/2}\mu}\Big[\ \left(a_0 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7\right) + + \left(b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7\right)\ln x\ \Big]$$

$$\text{with} \quad x = 1 - e^{-2\mu}, \quad \text{and} \quad N_n = \frac{-\sqrt{\pi}\,\Gamma(n+3/2)}{2^{n+1/2}\,n!}.$$

The coefficients for each value of $n \le 6$ are listed below. Error curves for each approximation are shown on pages 76 through 79.

$n = 0$; Peak Error = 0.377×10^{-8}

a_0 =	0.900316316157106		
a_2 =	0.035804137199687	b_2 =	-0.028134884879910
a_3 =	0.050255378507067	b_3 =	-0.022913294941738
a_4 =	0.820395221471889	b_4 =	0.260739681119272
a_5 =	2.117086662758014	b_5 =	1.942140151062161
a_6 =	-1.603509450908733	b_6 =	2.456767042935584
a_7 =	-1.320348265185031	b_7 =	0.415506703650598

$n = 1$; Peak Error = 0.378×10^{-9}

a_0 =	1.200421754876141		
a_2 =	-0.068190189118991	b_2 =	0.112539539519638
a_3 =	-0.015780965100240	b_3 =	0.111833791556540
a_4 =	-0.112990349448017	b_4 =	0.076101017123960
a_5 =	-0.316597094327360	b_5 =	-0.161442351511524
a_6 =	0.151793227124054	b_6 =	-0.283874798082833
a_7 =	0.161343615994412	b_7 =	-0.051651673083095

$n = 2$; Peak Error = 0.343×10^{-10}

a_0 =	1.920674807801826		
a_2 =	-0.185394986489085	b_2 =	0.900316316157106
a_3 =	0.263511936467362	b_3 =	0.900106991823023
a_4 =	0.156896568694855	b_4 =	1.098519120082788
a_5 =	-0.314950261060373	b_5 =	1.058745716725335
a_6 =	-0.670077446632615	b_6 =	0.487290707726612
a_7 =	-0.170660618781970	b_7 =	0.047529346875610

$n = 3$; Peak Error $= 0.331 \times 10^{-9}$

a_0	= 3.292585384803131		
a_2	= 0.287352986794639	b_2	= 3.601265264628424
a_3	= 2.072358799321038	b_3	= 3.598521625866331
a_4	= 1.082277869683218	b_4	= 5.285528923151802
a_5	= -1.915736136867913	b_5	= 5.109492080569266
a_6	= -3.223132762440099	b_6	= 2.032538107168125
a_7	= -0.595706141294014	b_7	= 0.151680447429029

$n = 4$; Peak Error $= 0.857 \times 10^{-9}$

a_0	= 5.853485128538899		
a_2	= 3.480429301478592	b_2	= 11.524048846810958
a_3	= 9.172305728406928	b_3	= 11.511731255459780
a_4	= 5.462277023522988	b_4	= 20.612728309257640
a_5	= -6.687800211917144	b_5	= 21.149822313056400
a_6	= -13.595496984735910	b_6	= 8.836715812516606
a_7	= -2.685199985294353	b_7	= 0.696452584648511

$n = 5$; Peak Error $= 0.219 \times 10^{-8}$

a_0	= 10.642700233707090		
a_2	= 15.930602599970358	b_2	= 32.925853848031308
a_3	= 32.098623731360098	b_3	= 32.873833661344660
a_4	= 21.651681635461780	b_4	= 71.117734556739480
a_5	= -21.262330943372130	b_5	= 74.475372463445160
a_6	= -48.615889318261660	b_6	= 31.412729971099690
a_7	= -9.445387938865535	b_7	= 2.430379144205808

$n = 6$; Peak Error $= 0.476 \times 10^{-8}$

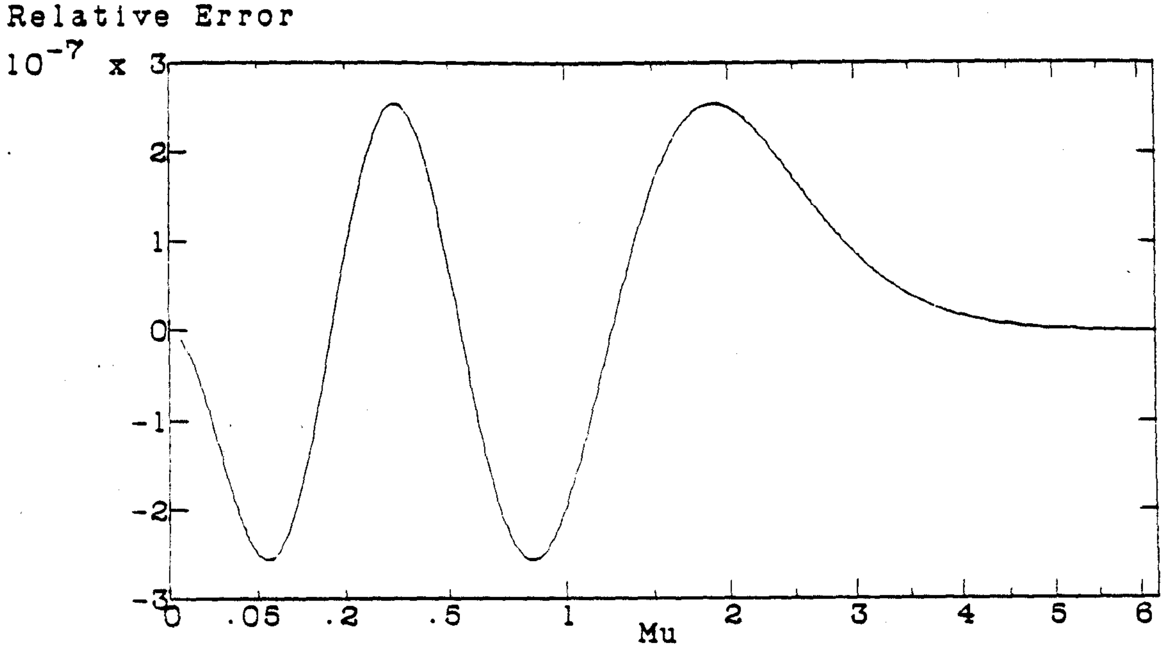
a_0	= 19.648061969920780		
a_2	= 55.989649537565295	b_2	= 87.802276928083488
a_3	= 98.799147171298294	b_3	= 87.609059938004631
a_4	= 77.635907192105920	b_4	= 227.100805837940001
a_5	= -59.709114868176460	b_5	= 239.583530676601399
a_6	= -160.147631976405300	b_6	= 103.762790747451000
a_7	= -31.216019026308530	b_7	= 8.057523927571967

8. Plots of Relative Errors

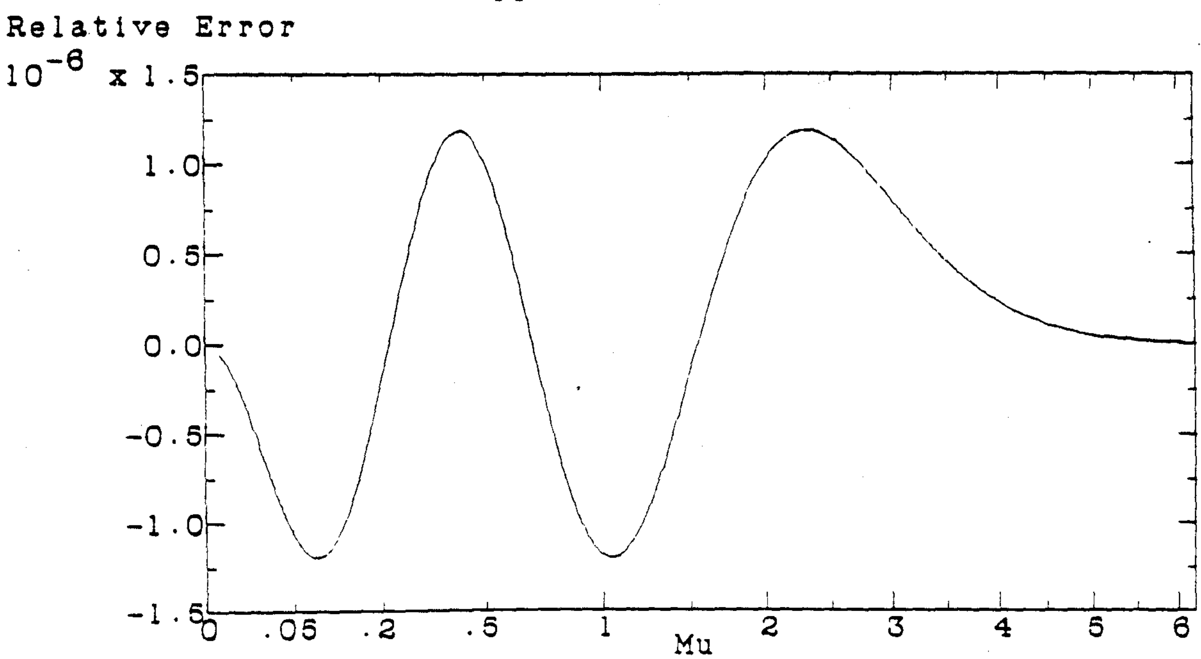
The relative error associated with each approximation listed in the previous section is plotted in the following pages. The relative error is the difference in the value of the approximation and the "true" value of the Legendre function, divided by the true value. The latter was obtained primarily by direct summation of the hypergeometric series, terminated when successive contributions to the sum left the digits in the first 17 decimal places unchanged. When convergence of the sum was too slow, as when $\mu < .02$ for $Q^1_{n-1/2}(\cosh\mu)$, the integral definition of the Legendre function was numerically evaluated. Trapezoidal integration with Romberg iterative improvement also made it possible to achieve 17 decimal places of accuracy.

Relative errors are plotted against μ. In order to spread the peaks in the error more evenly across the page, the horizontal axis is linear in $\sqrt{\mu}$. The "first" and "second" approximations to $P^1_{n-1/2}(\cosh\mu)$ are described in Section 4, while approximations to $Q^1_{n-1/2}(\cosh\mu)$ are given in Section 5.

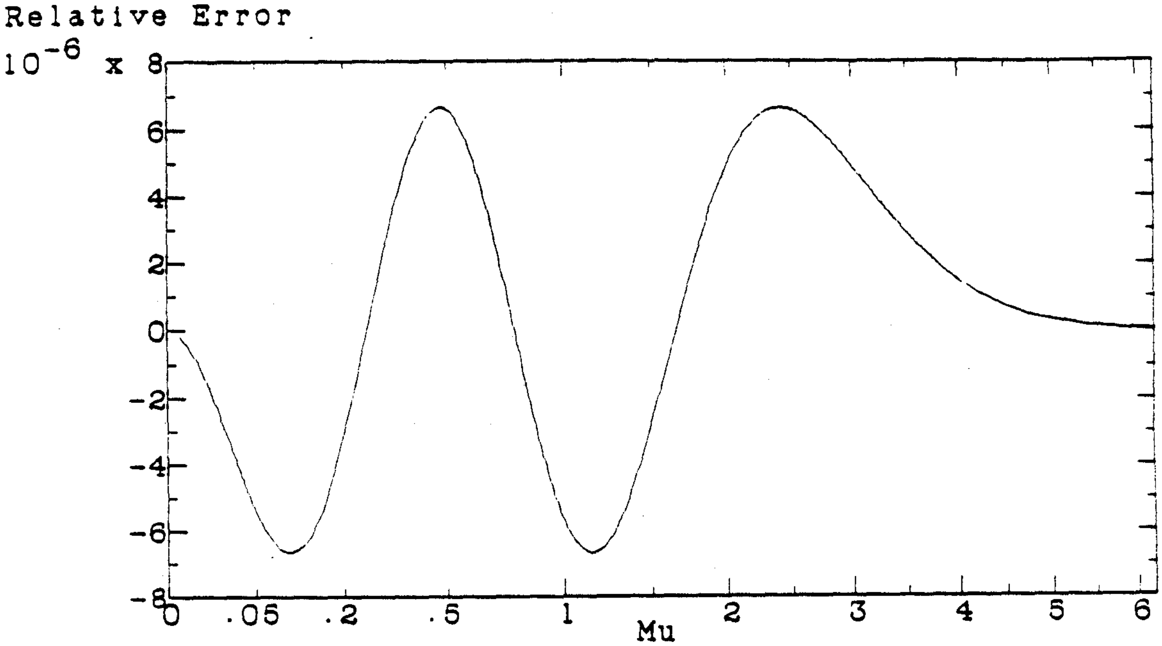
Relative Error for Pn*, n = 0, Order = 3
First Approximation (Mu › .001)



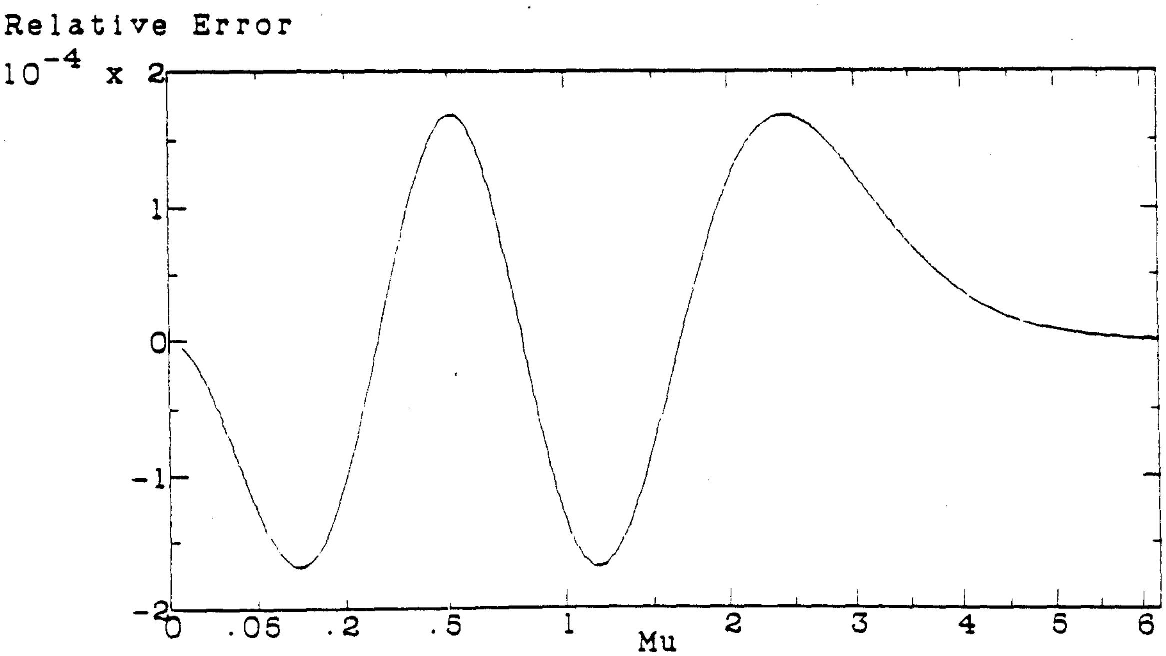
Relative Error for Pn*, n = 1, Order = 3
First Approximation (Mu › .001)



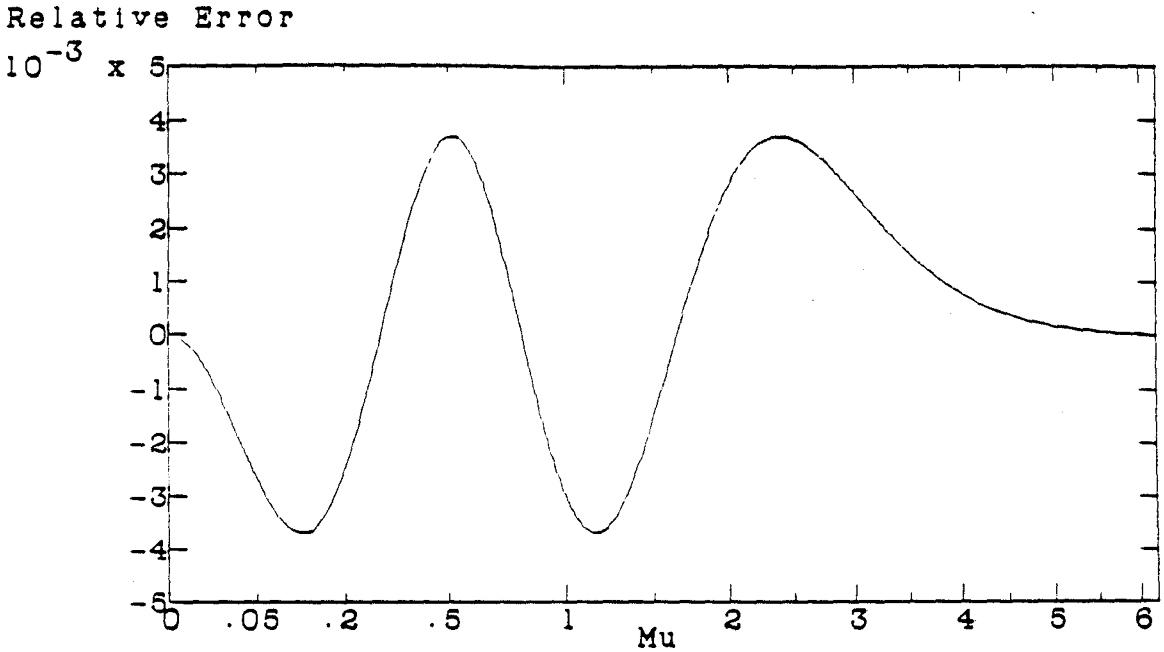
Relative Error for Pn*, n = 2. Order = 3
First Approximation (Mu › .001)



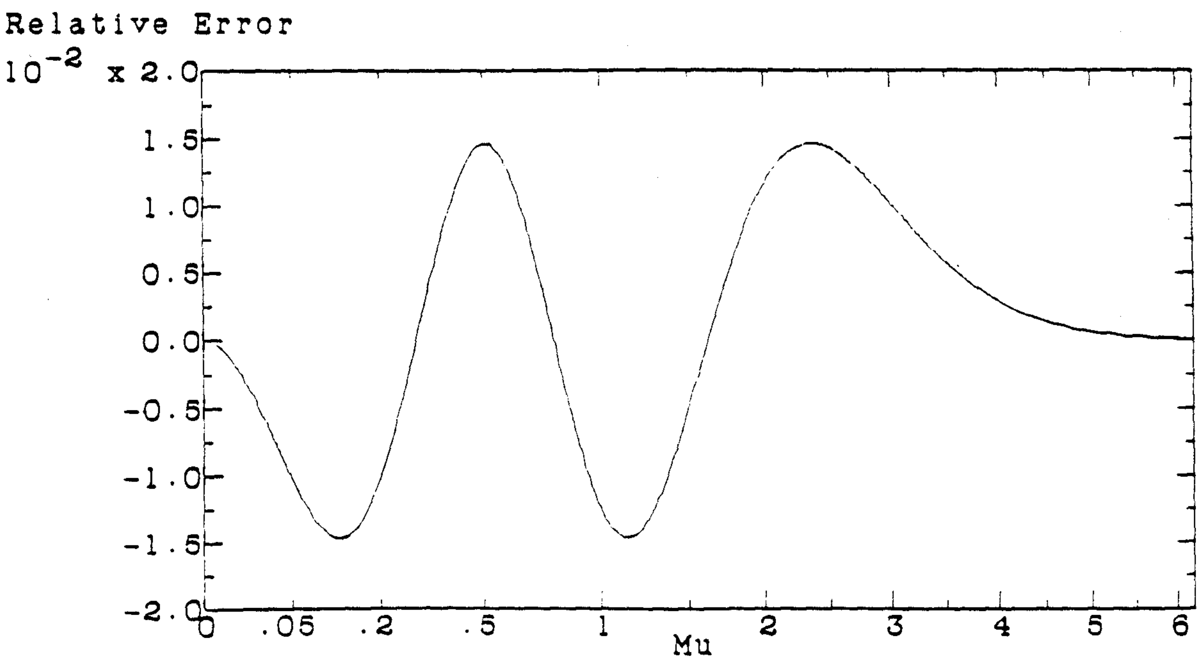
Relative Error for Pn*, n = 3, Order = 3
First Approximation (Mu › .001)



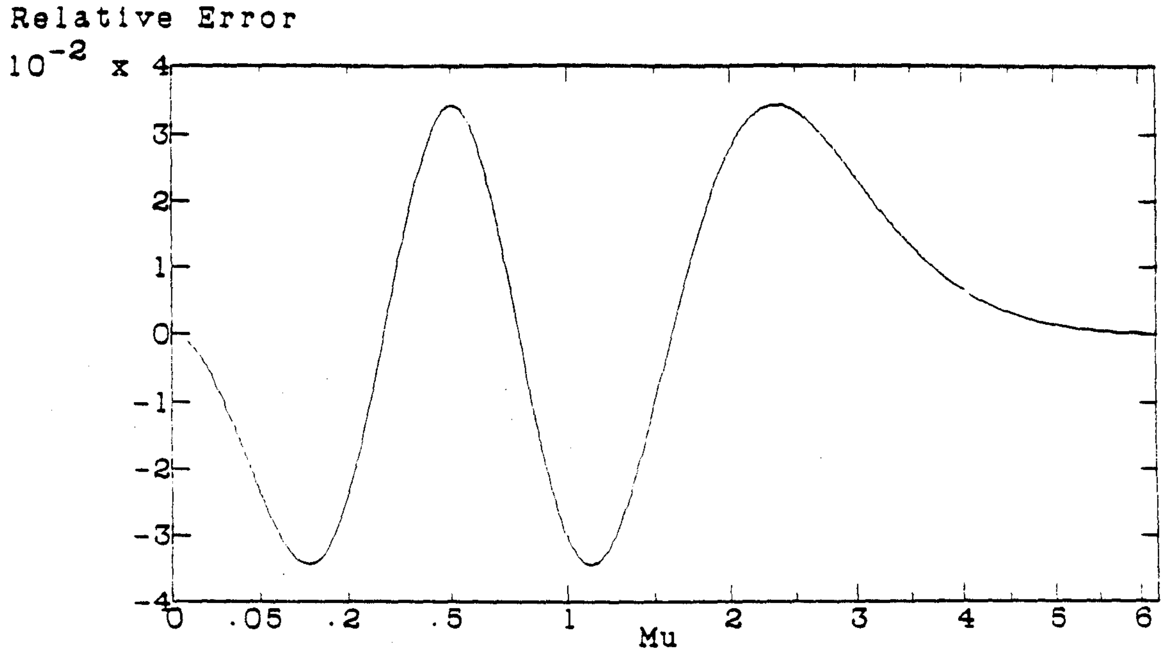
Relative Error for Pn*,   n = 4,   Order = 3
First Approximation   (Mu > .001)



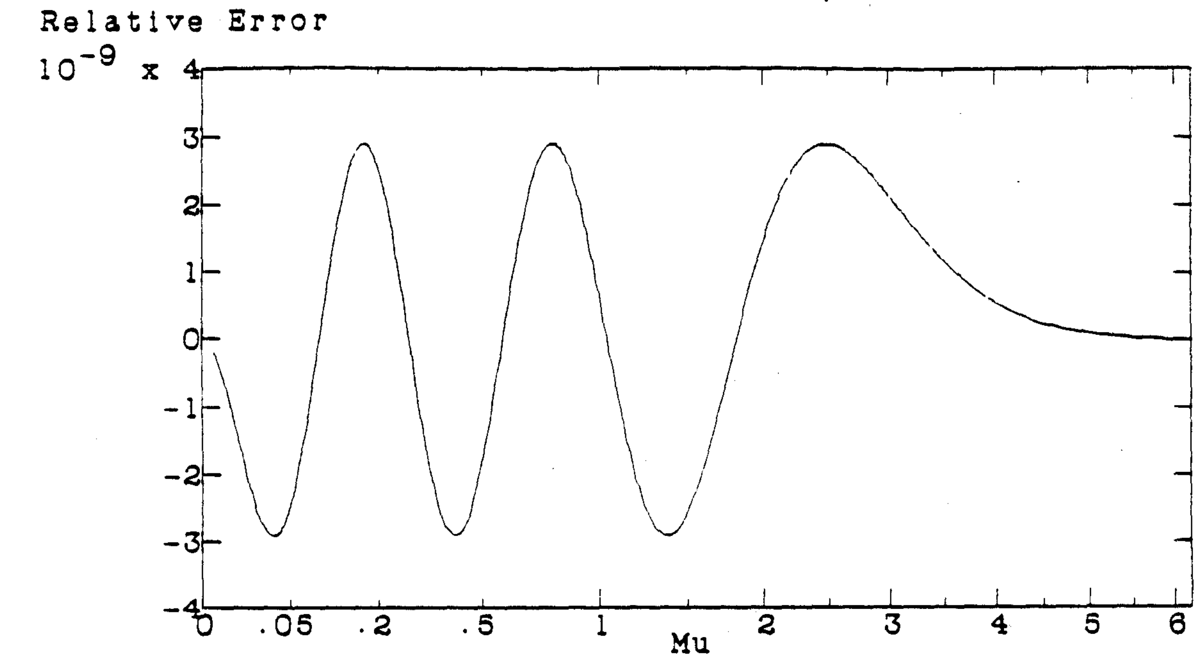
Relative Error for Pn*,   n = 5,   Order = 3
First Approximation   (Mu > .001)



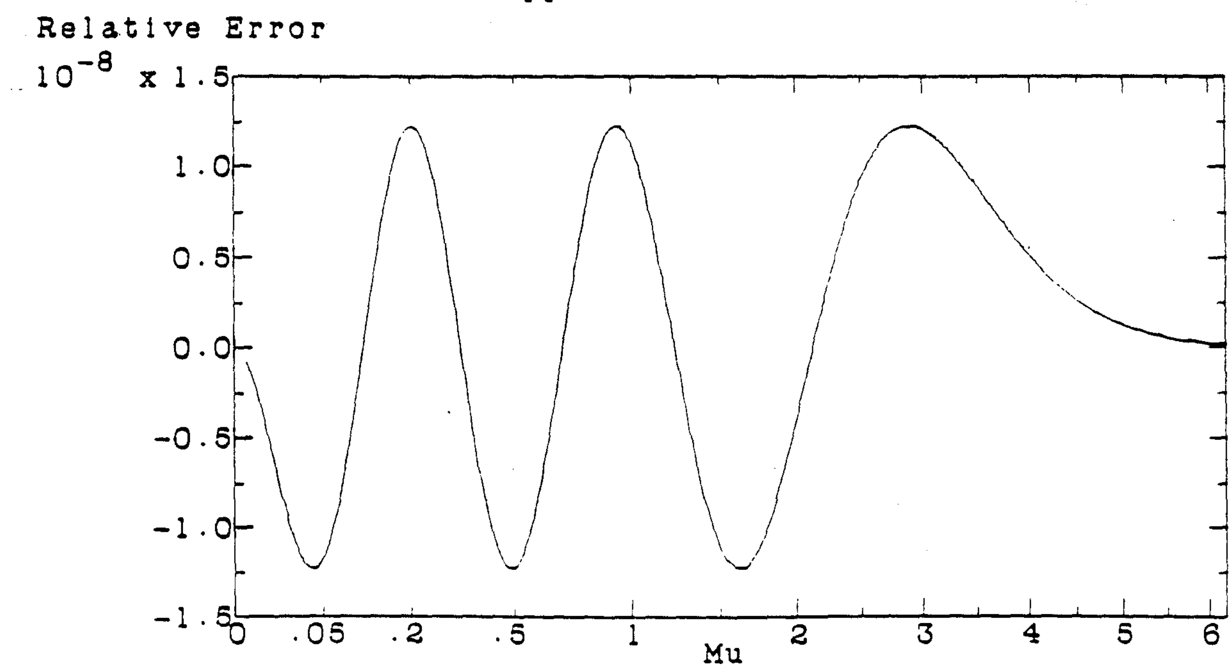
Relative Error for Pn*. n = 6. Order = 3
First Approximation (Mu > .001)



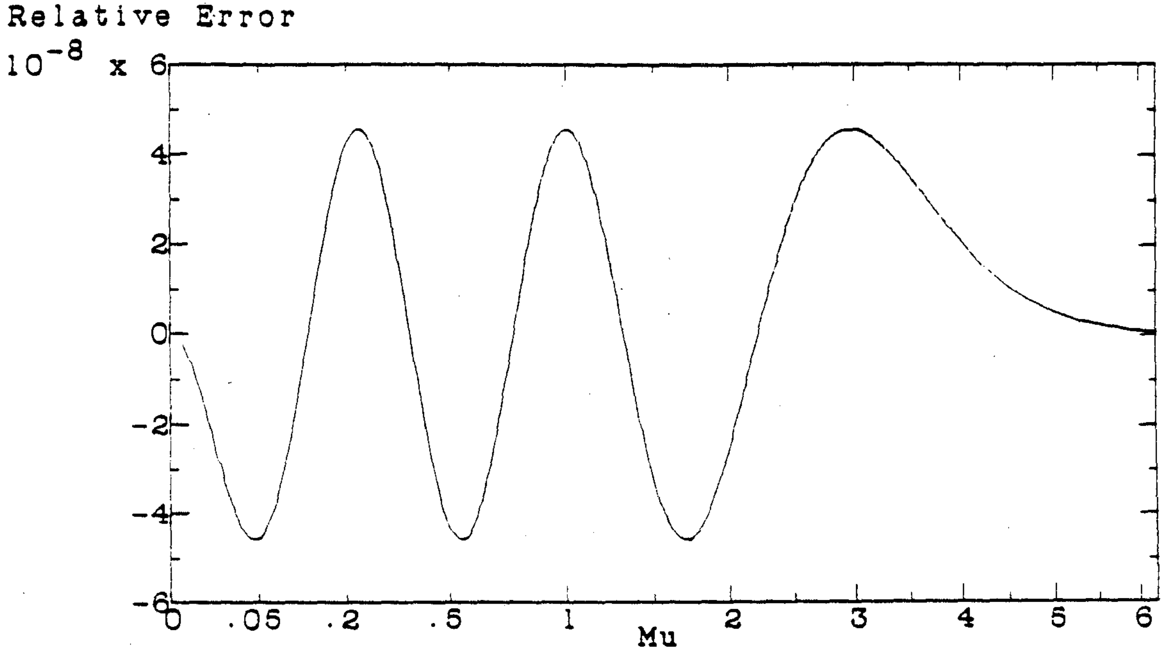
Relative Error for Pn*, n = 0, Order = 4
First Approximation (Mu > .001)



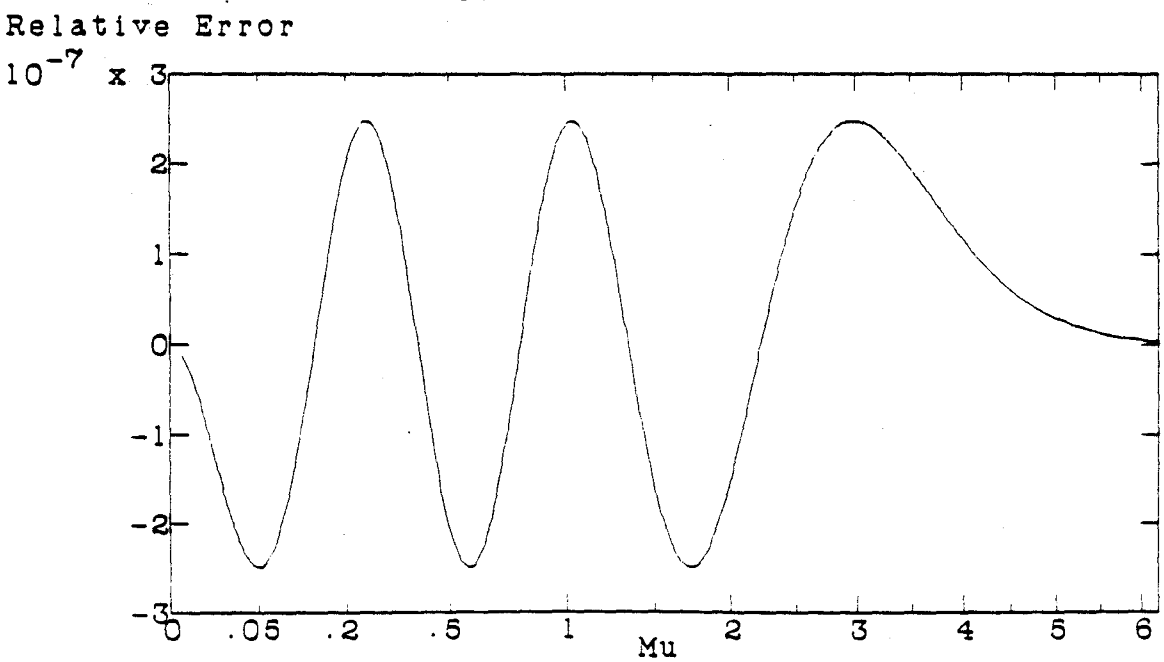
Relative Error for Pn*, n = 1, Order = 4
First Approximation (Mu > .001)



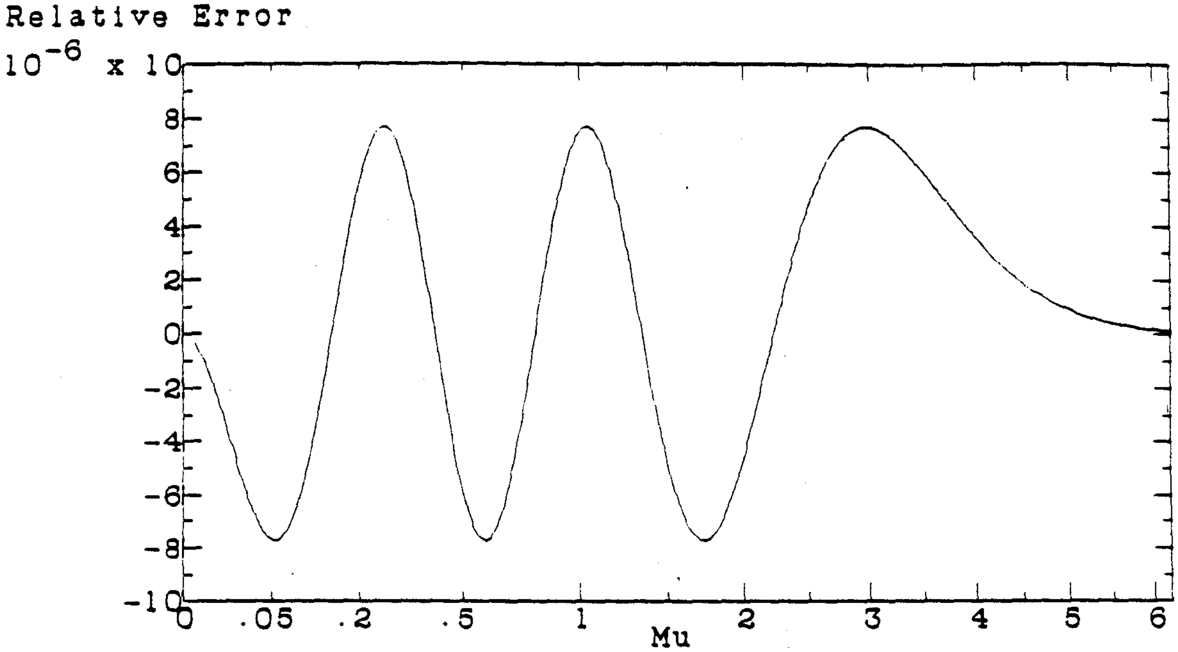
Relative Error for Pn*, n = 2, Order = 4
First Approximation (Mu › .001)



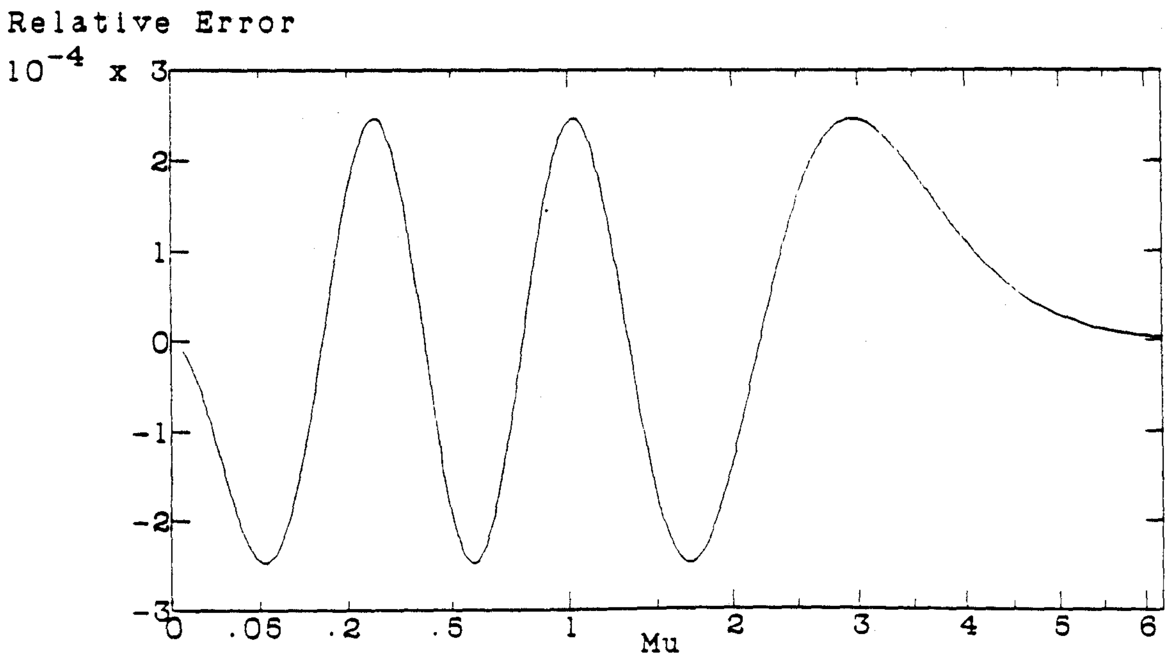
Relative Error for Pn*, n = 3, Order = 4
First Approximation (Mu › .001)



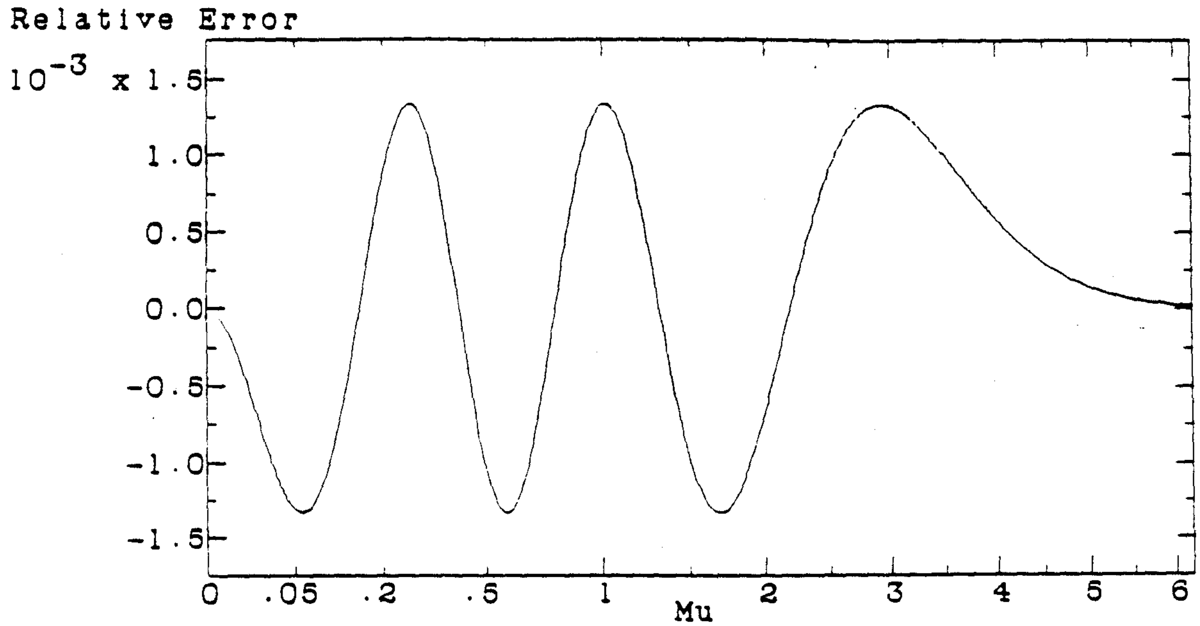
Relative Error for Pn*, n = 4, Order = 4
First Approximation (Mu > .001)

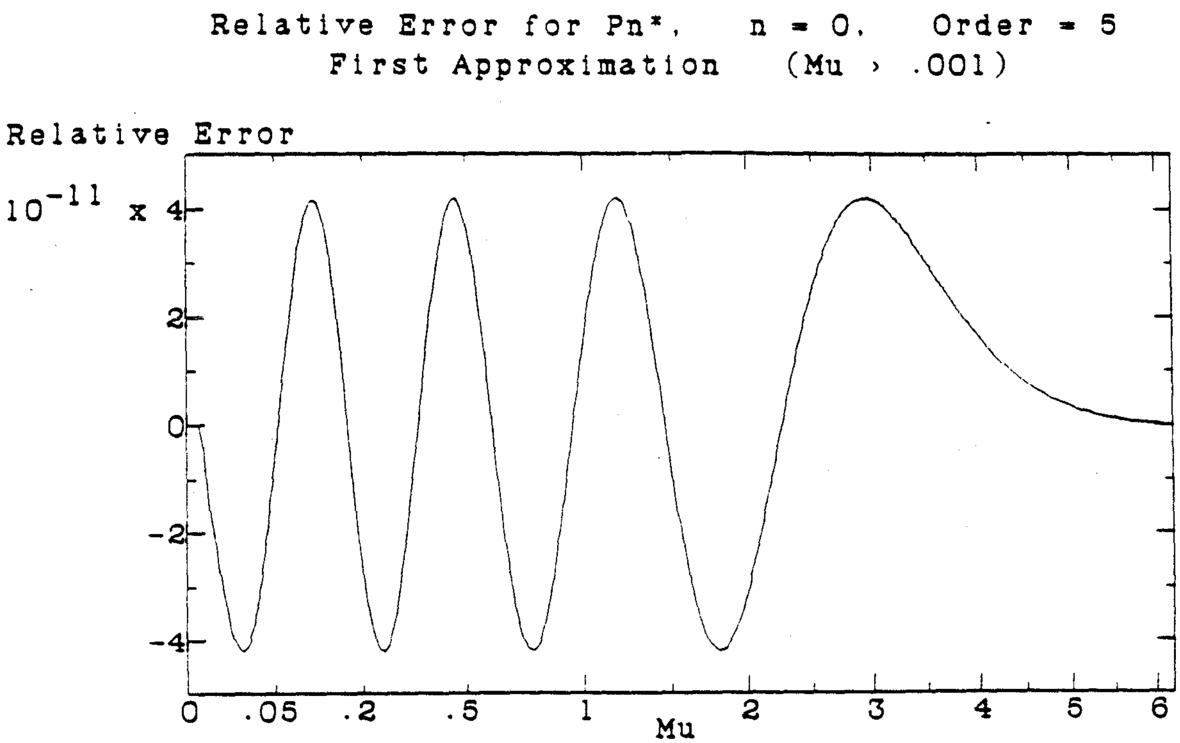


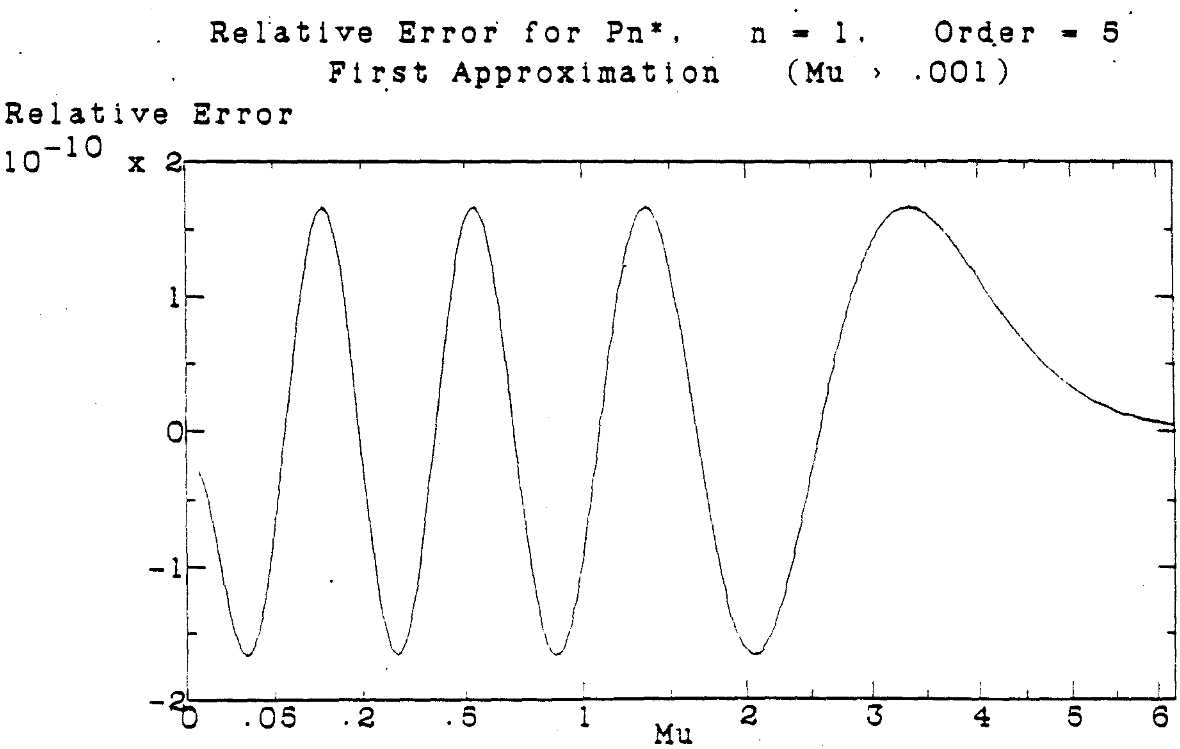
Relative Error for Pn*, n = 5, Order = 4
First Approximation (Mu > .001)



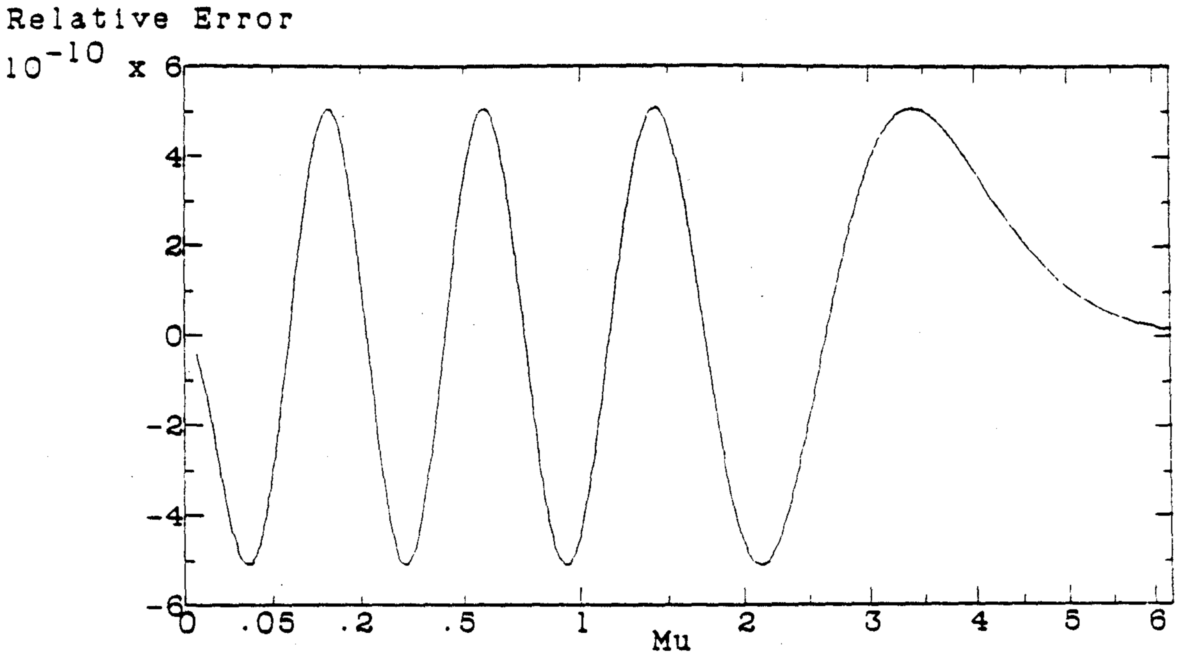
Relative Error for Pn*, n = 6, Order = 4
First Approximation (Mu › .001)



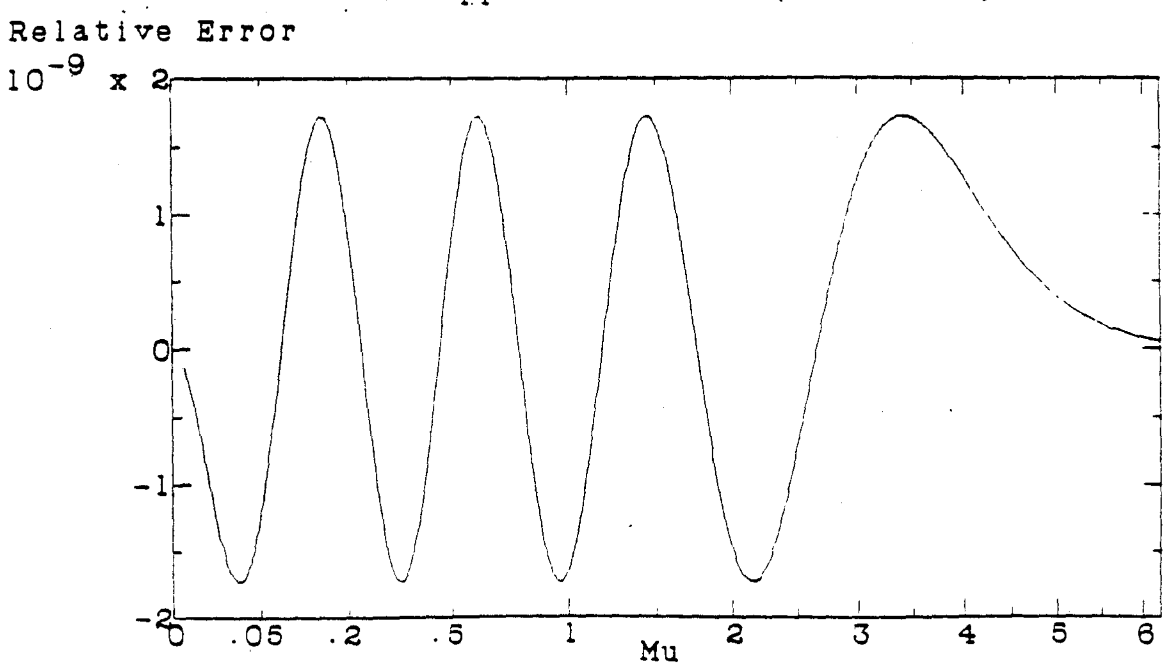

Relative Error for Pn*, n = 0, Order = 5
First Approximation (Mu › .001)
Relative Error
10^{-11} x
4
2
0
-2
-4
0
.05
.2
.5
1
2
3
4
5
6
Mu


Relative Error for Pn*, n = 1, Order = 5
First Approximation (Mu › .001)
Relative Error
10^{-10} x
2
1
0
-1
-2
0
.05
.2
.5
1
2
3
4
5
6
Mu

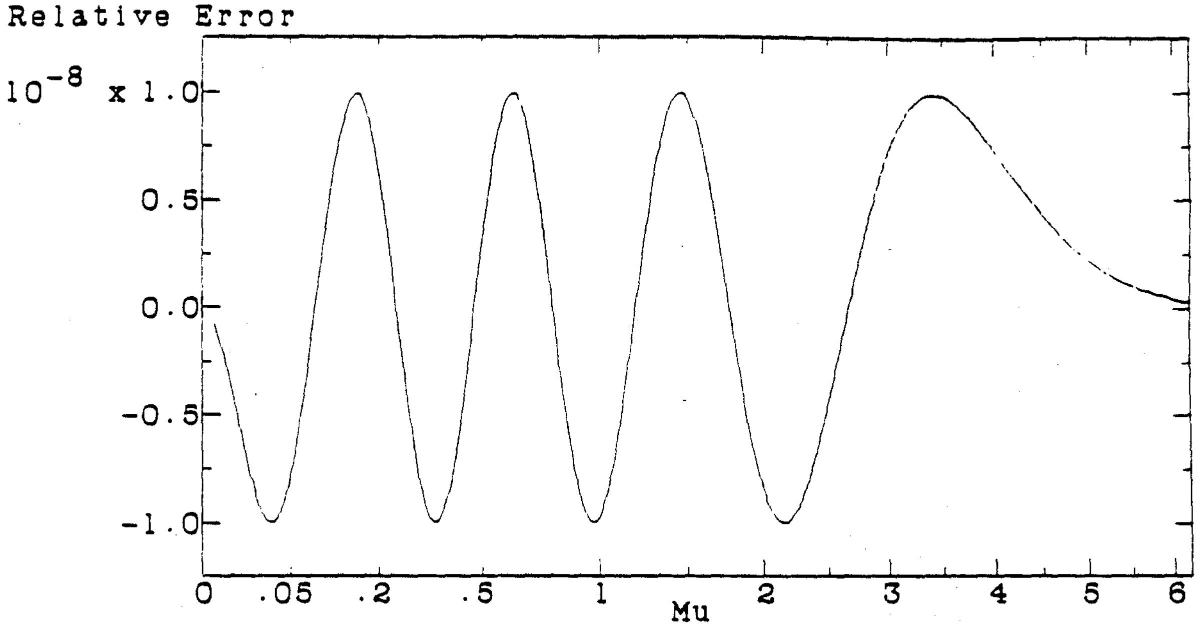
Relative Error for Pn*,   n = 2,   Order = 5
First Approximation   (Mu › .001)



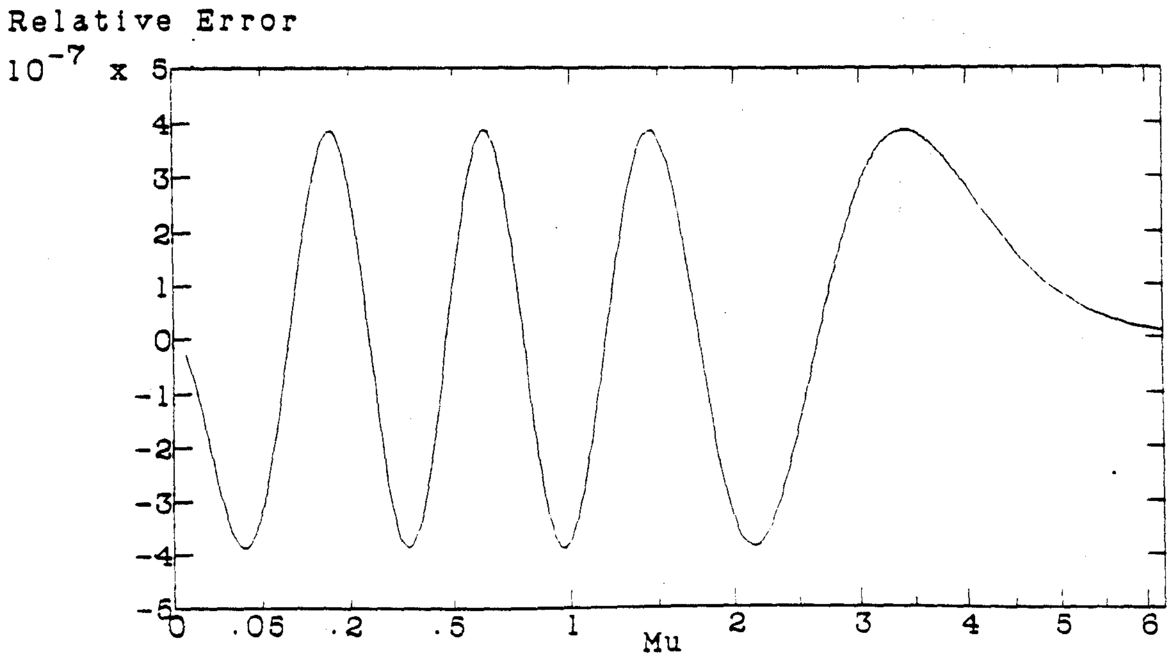
Relative Error for Pn*,   n = 3,   Order = 5
First Approximation   (Mu › .001)



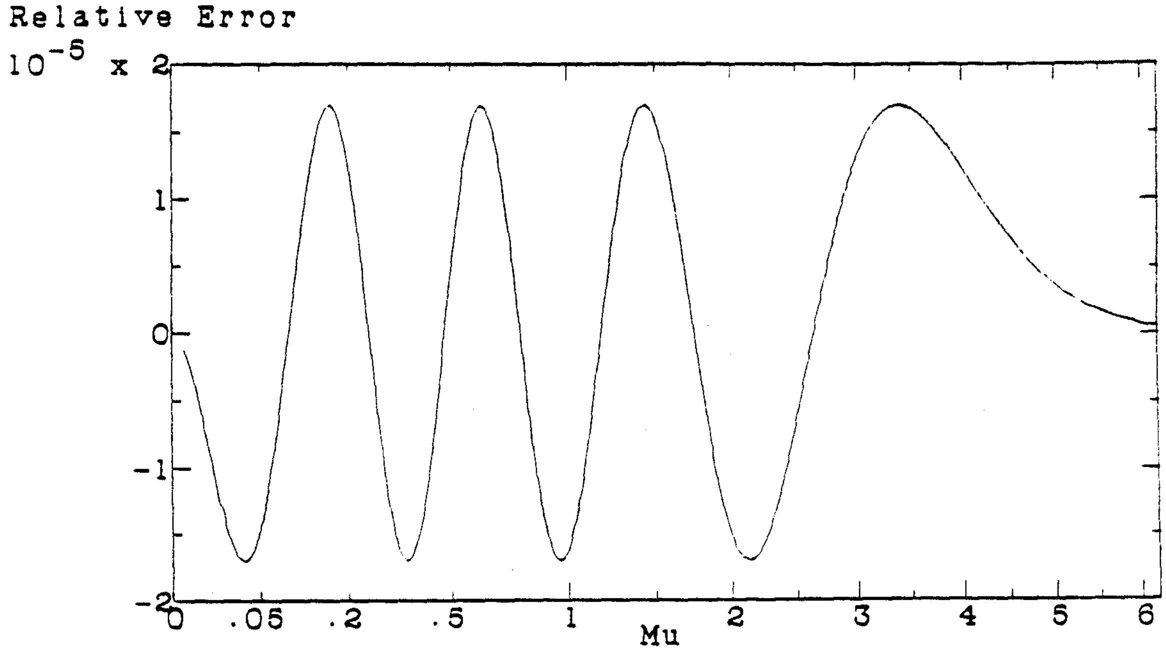
Relative Error for Pn*, n = 4, Order = 5
First Approximation (Mu > .001)



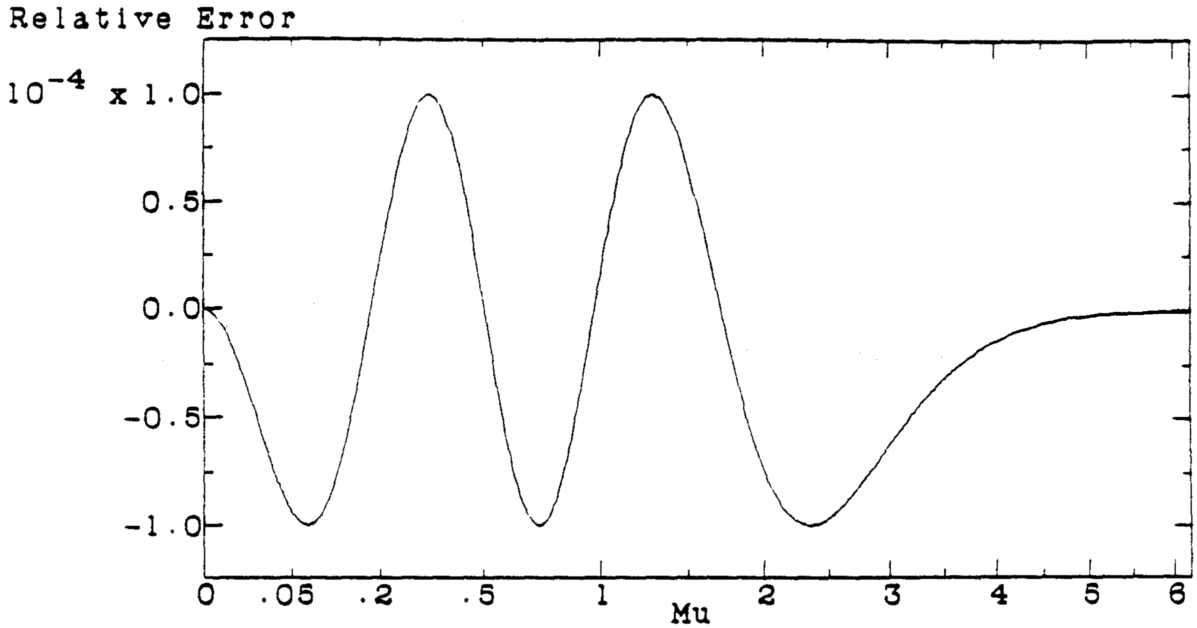
Relative Error for Pn*, n = 5, Order = 5
First Approximation (Mu > .001)



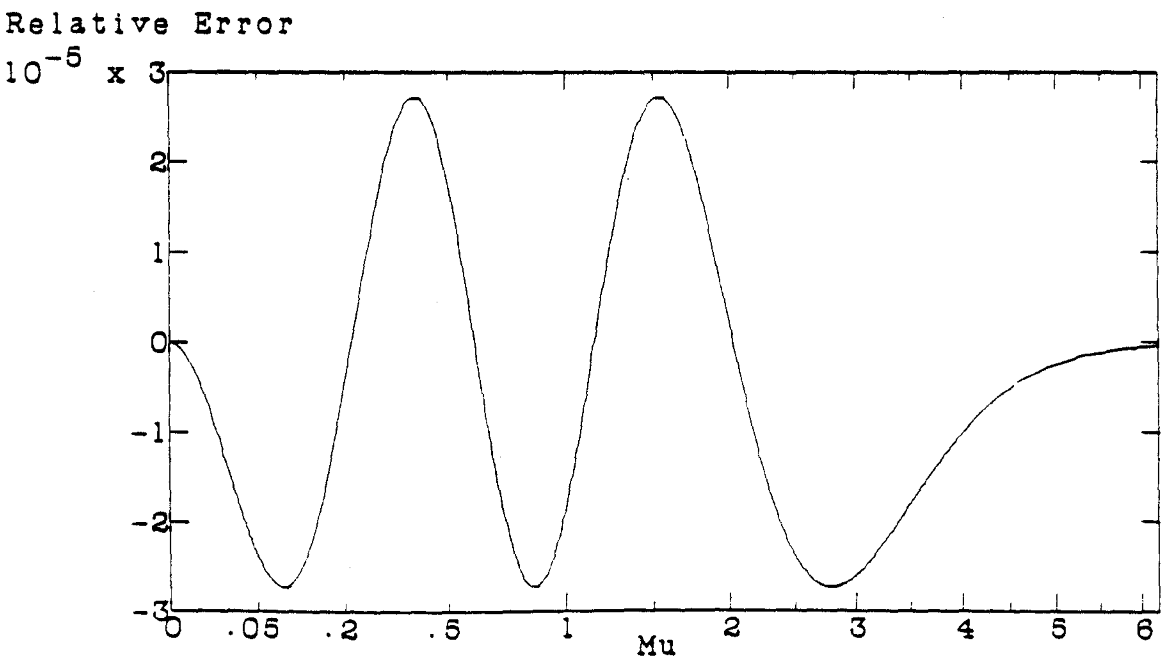
Relative Error for Pn*. n = 6. Order = 5
First Approximation (Mu > .001)



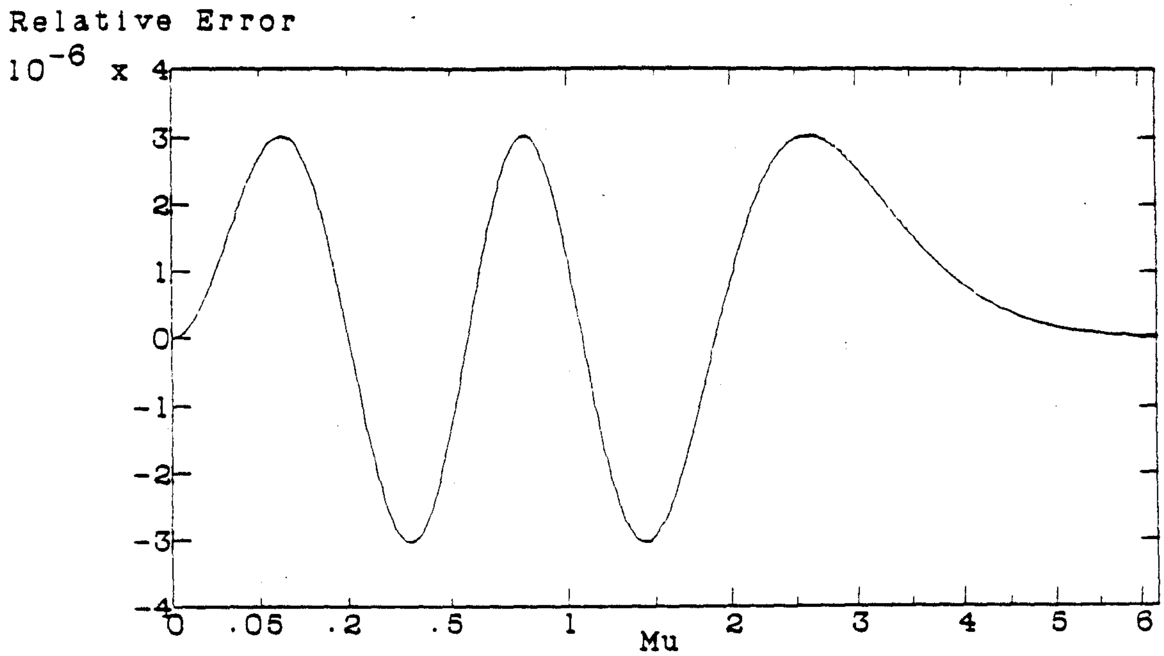
Relative Error for Pn*, n = 0, Order = 3
Second Approximation



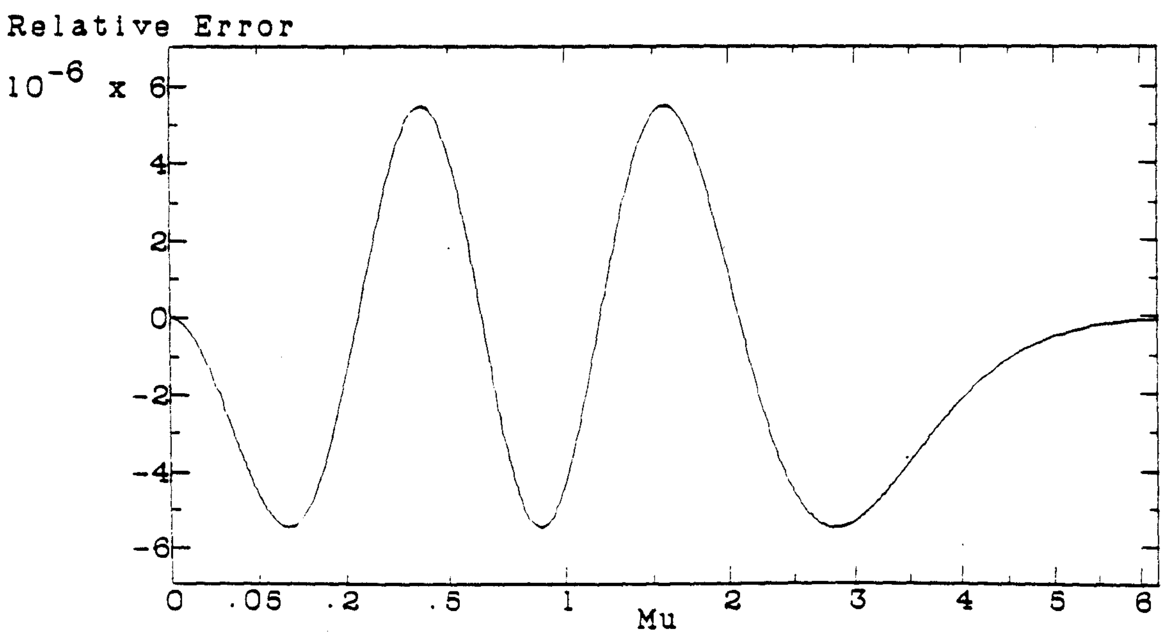
Relative Error for Pn*, n = 1, Order = 3
Second Approximation



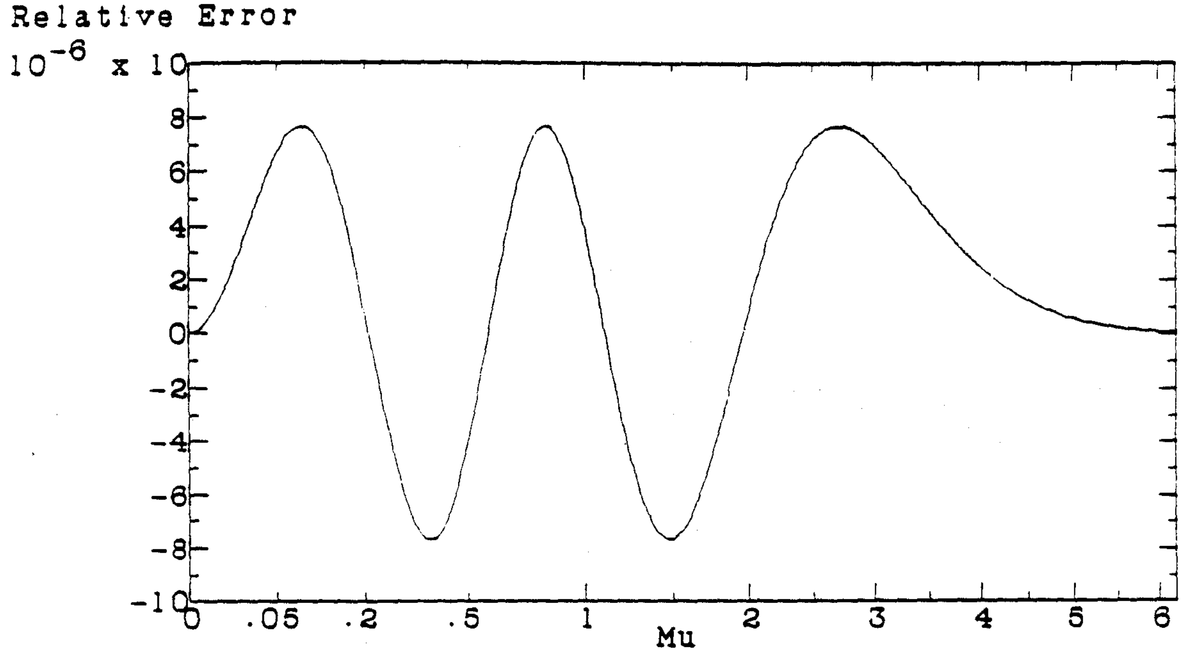
Relative Error for Pn*, n = 2, Order = 3
Second Approximation



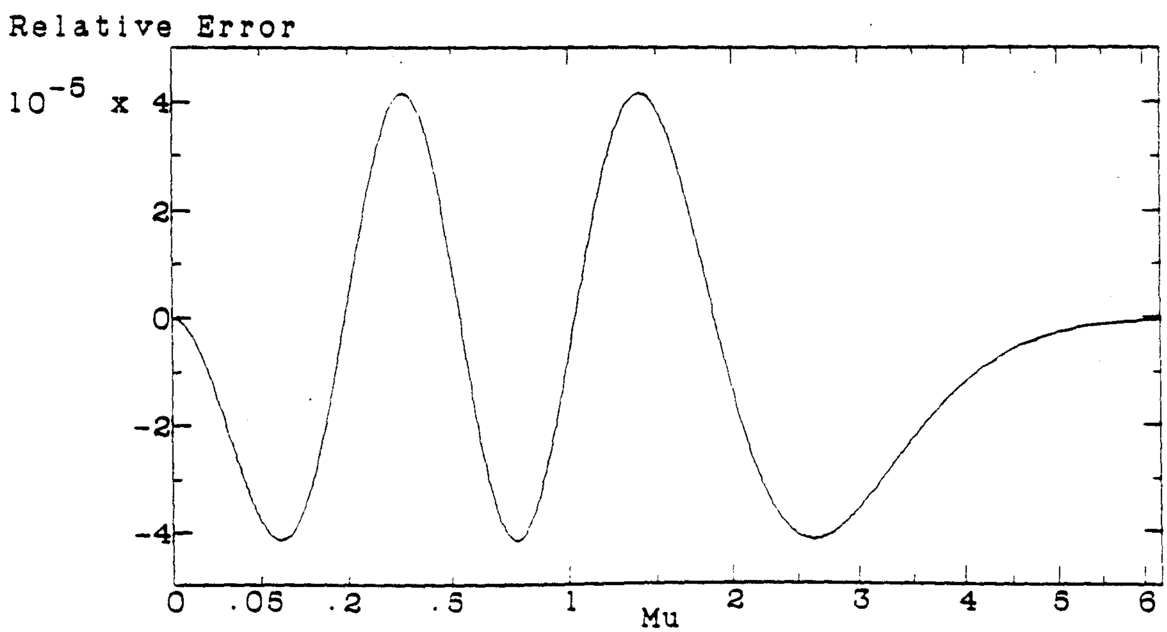
Relative Error for Pn*, n = 3, Order = 3
Second Approximation

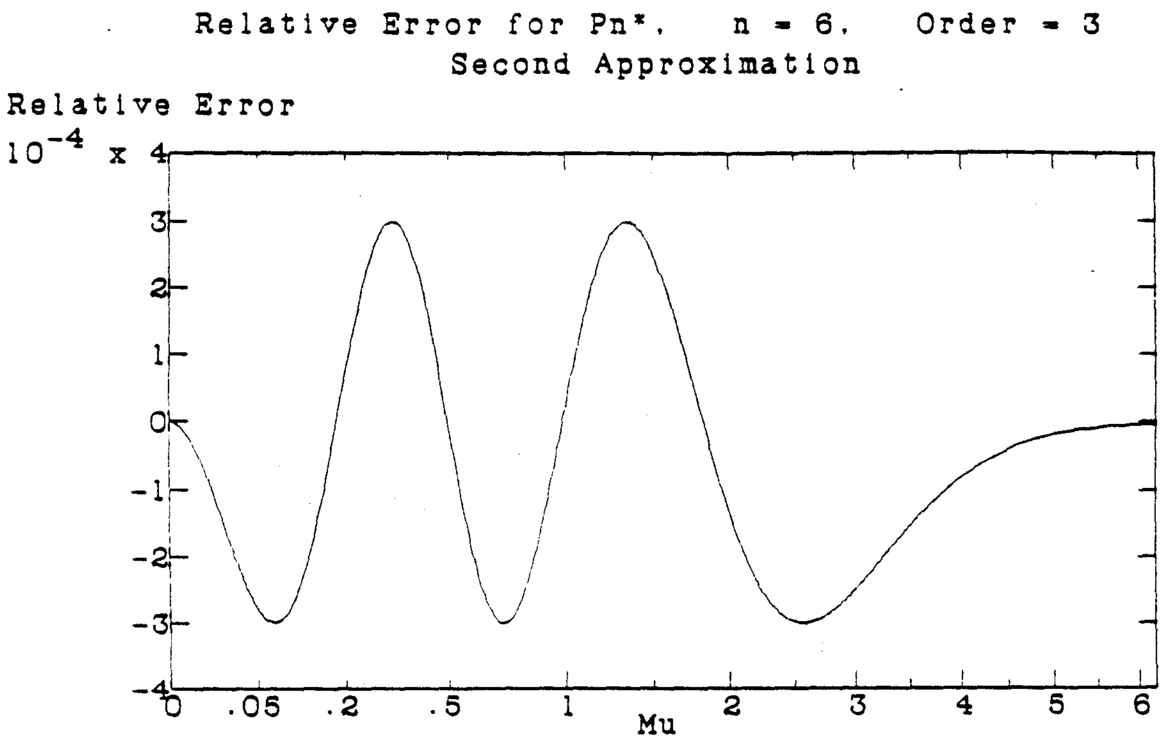


Relative Error for Pn*, n = 4, Order = 3
Second Approximation

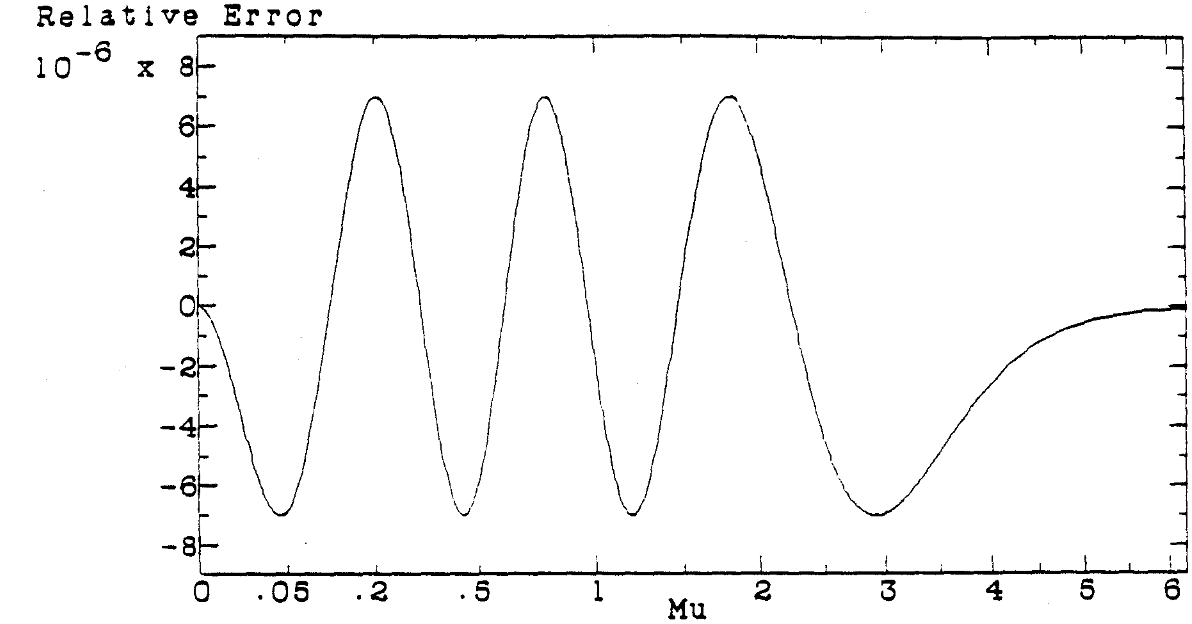


Relative Error for Pn*, n = 5, Order = 3
Second Approximation

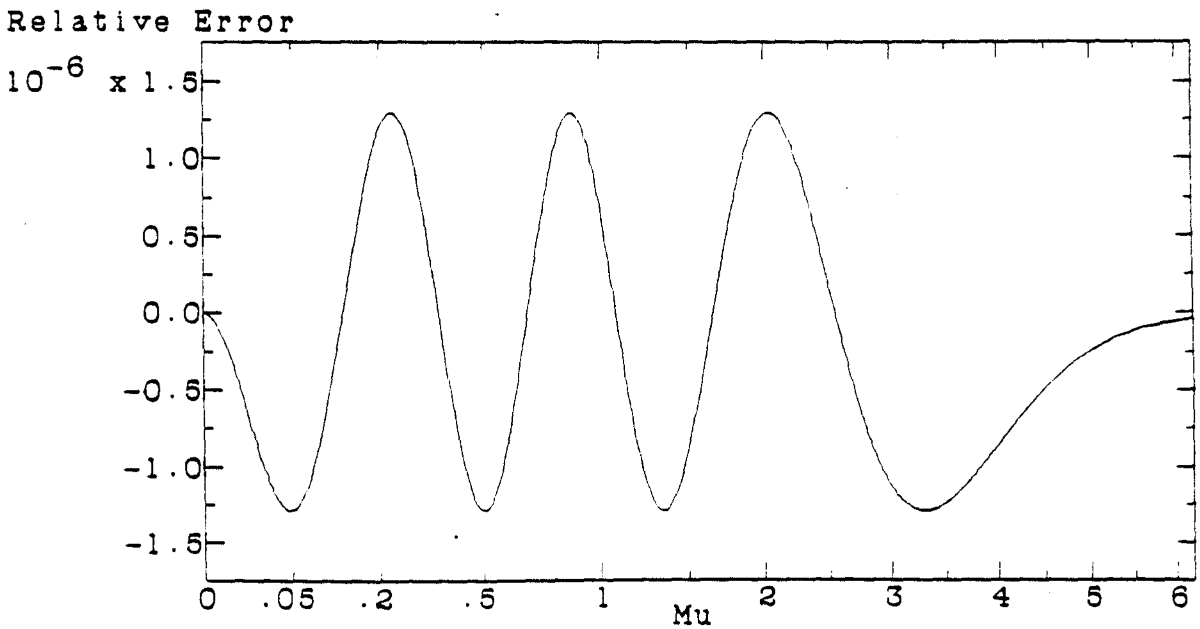



Relative Error for Pn*, n = 6, Order = 3
Second Approximation
Relative Error
10^{-4} x
4
3
2
1
0
-1
-2
-3
-4
0
.05
.2
.5
1
2
3
4
5
6
Mu

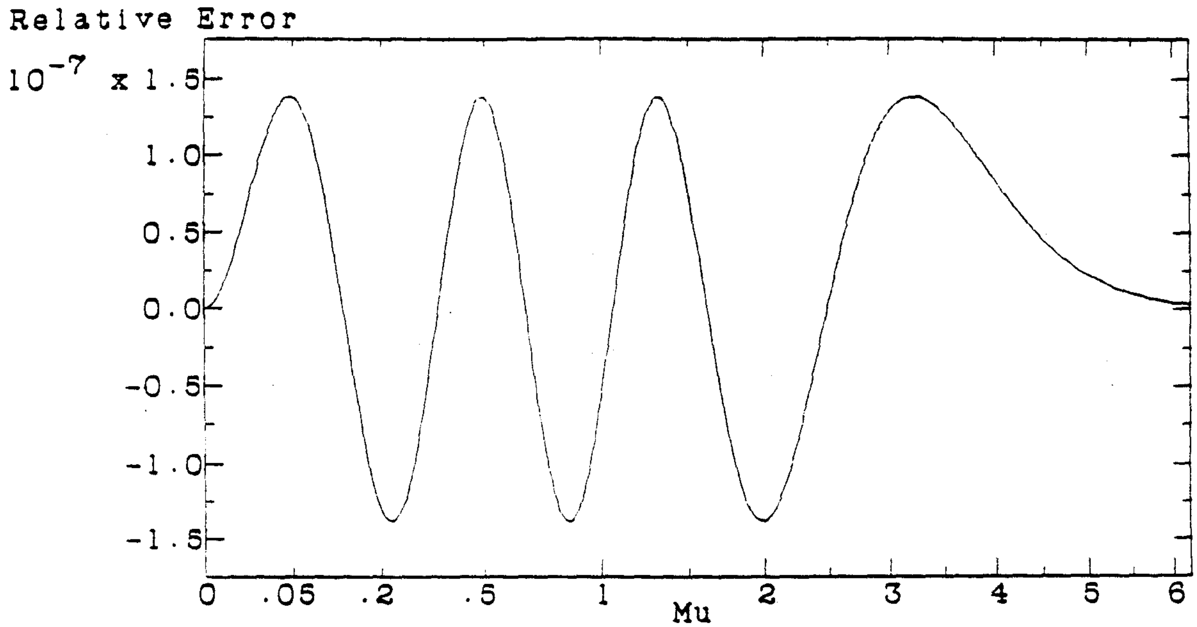
Relative Error for Pn*, n = 0, Order = 4
Second Approximation



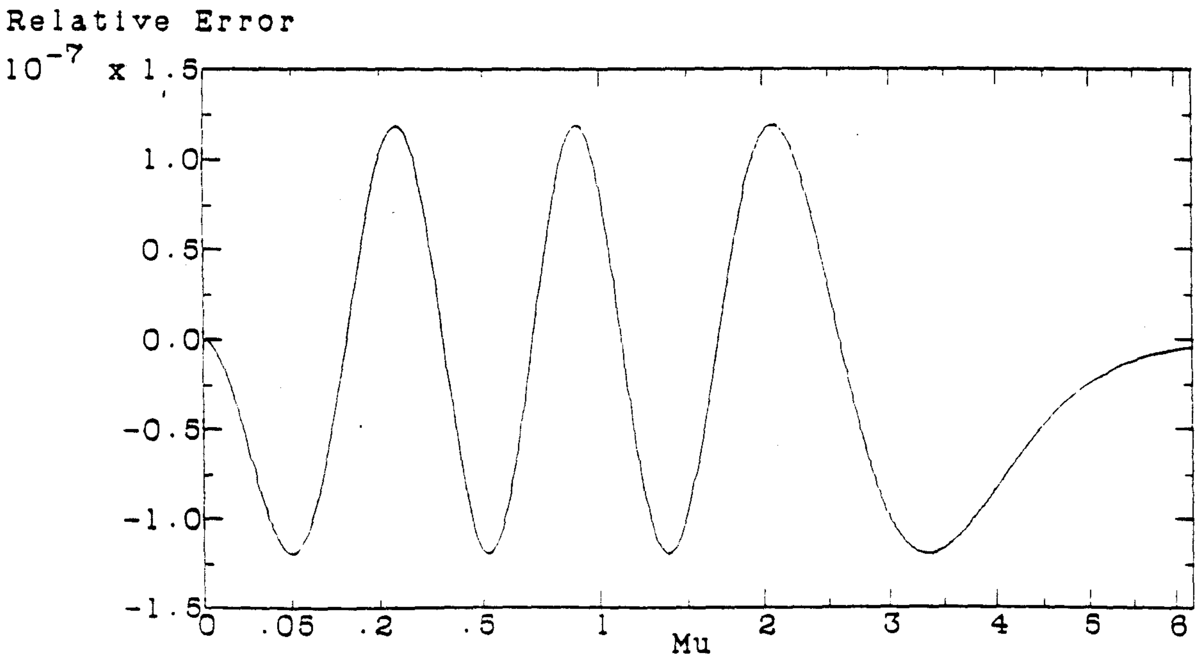
Relative Error for Pn*, n = 1, Order = 4
Second Approximation

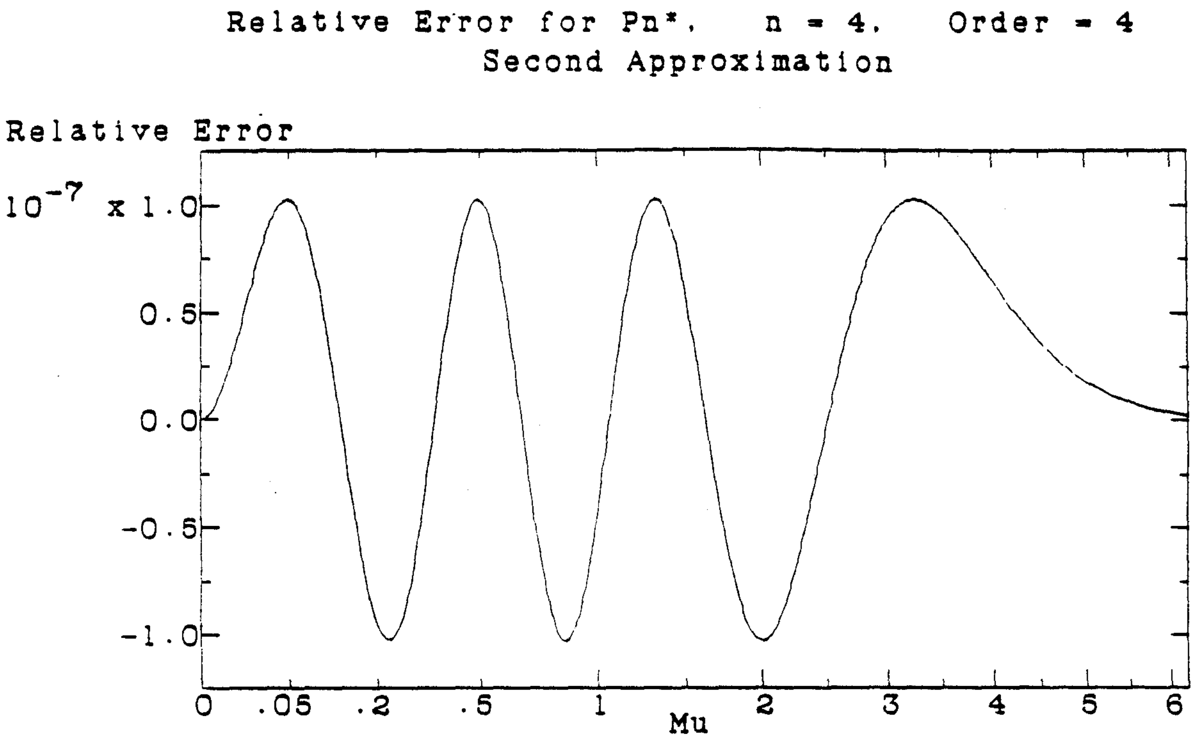


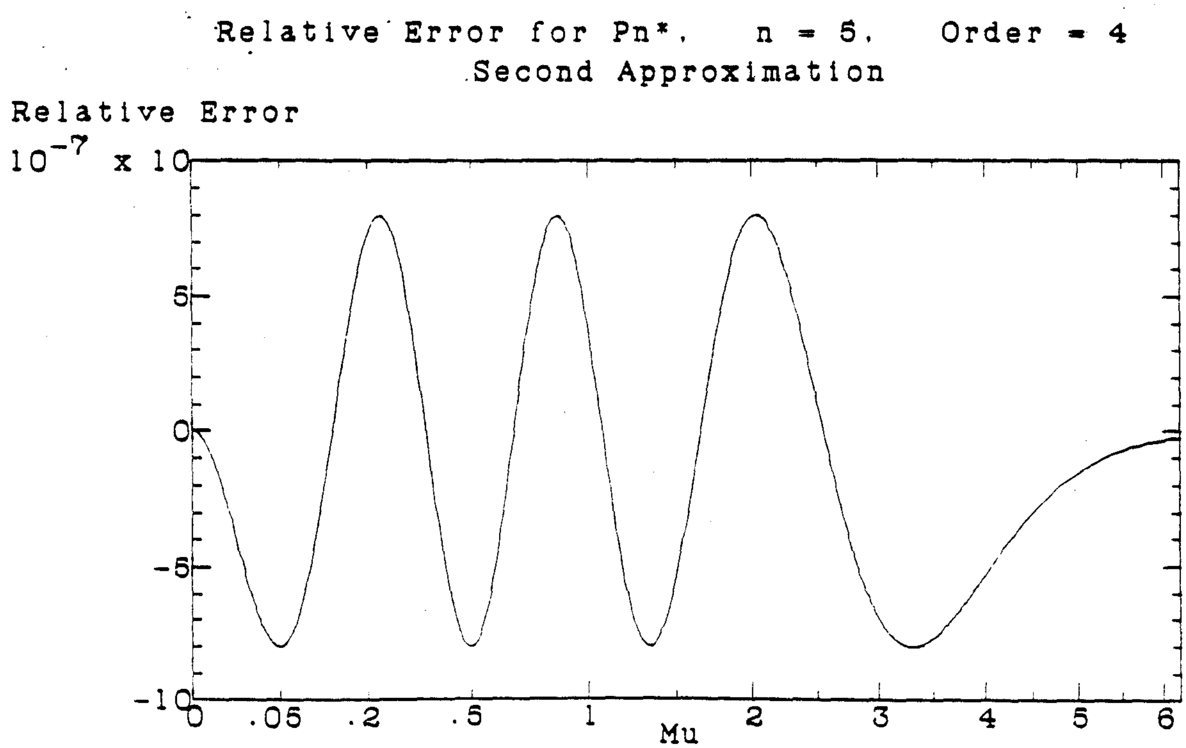
Relative Error for Pn*. n = 2. Order = 4
Second Approximation



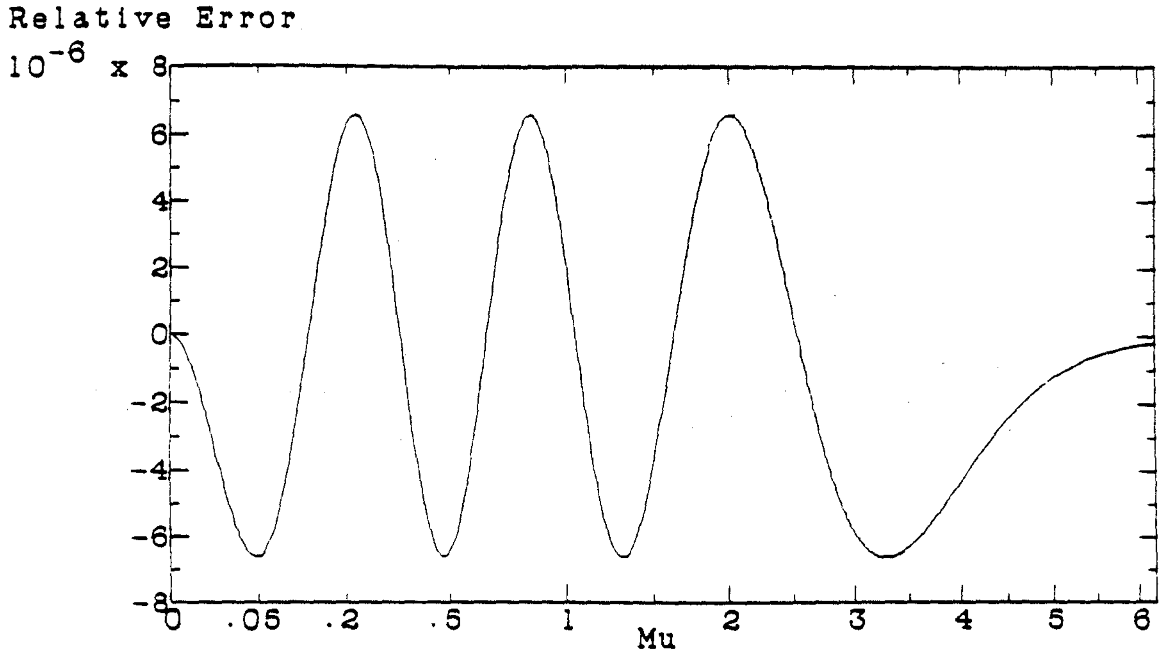
Relative Error for Pn*. n = 3. Order = 4
Second Approximation



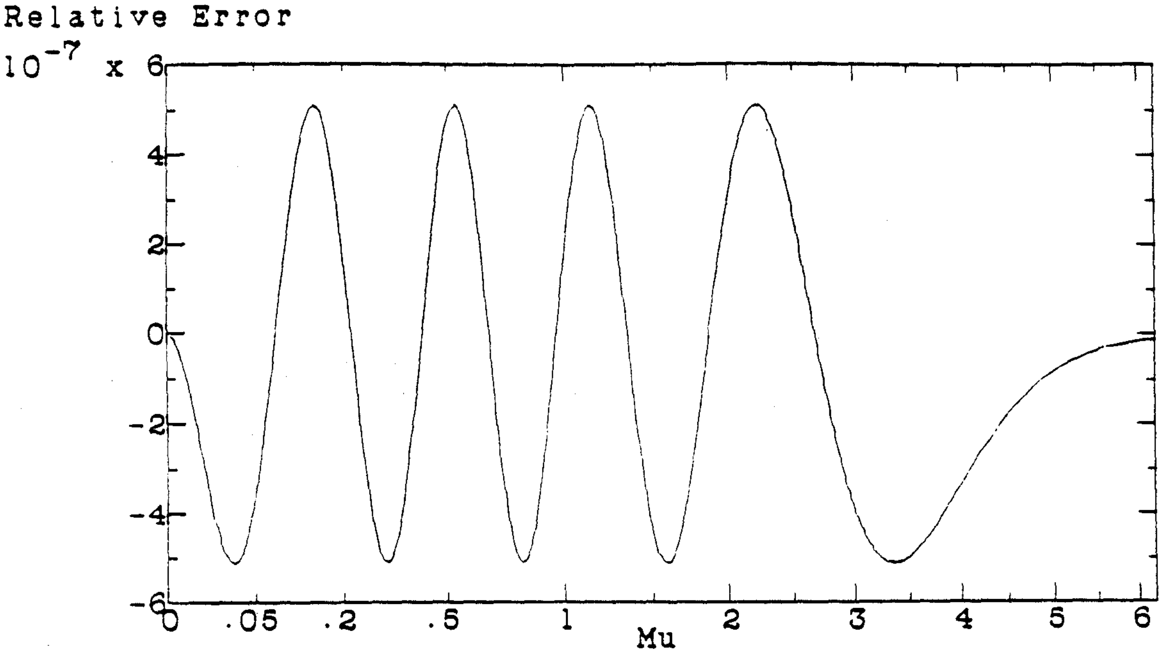

Relative Error for Pn*, n = 4, Order = 4
Second Approximation
Relative Error
10^-7 x 1.0
0.5
0.0
-0.5
-1.0
0
.05
.2
.5
1
2
3
4
5
6
Mu


Relative Error for Pn*, n = 5, Order = 4
Second Approximation
Relative Error
10^-7 x 10
5
0
-5
-10
0
.05
.2
.5
1
2
3
4
5
6
Mu

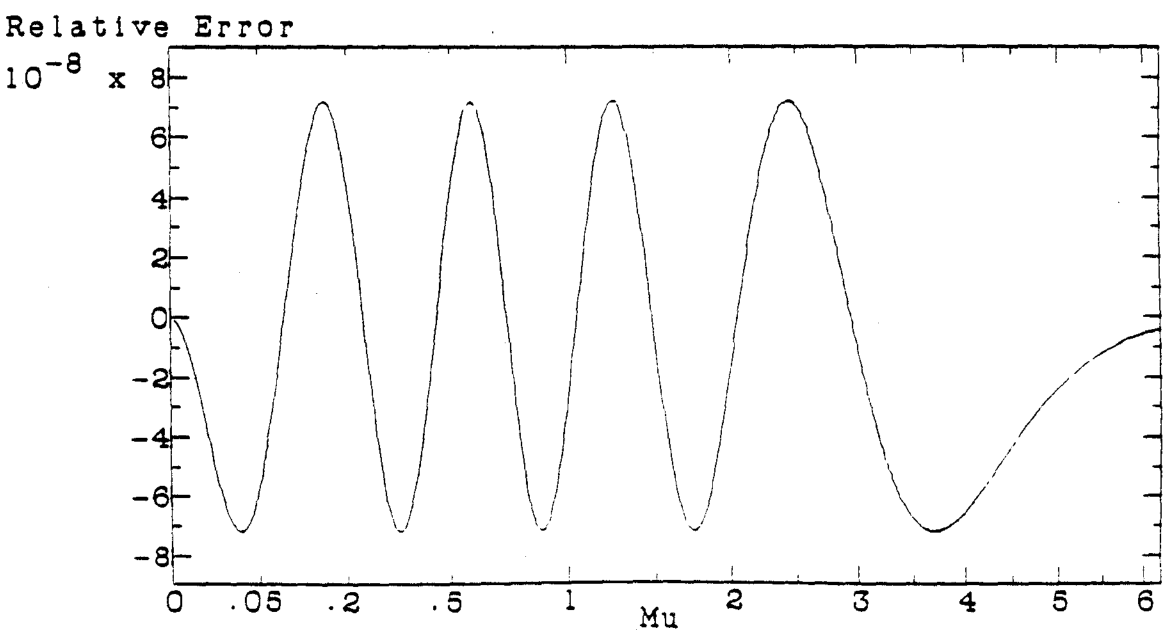
Relative Error for Pn*. n = 6. Order = 4
Second Approximation



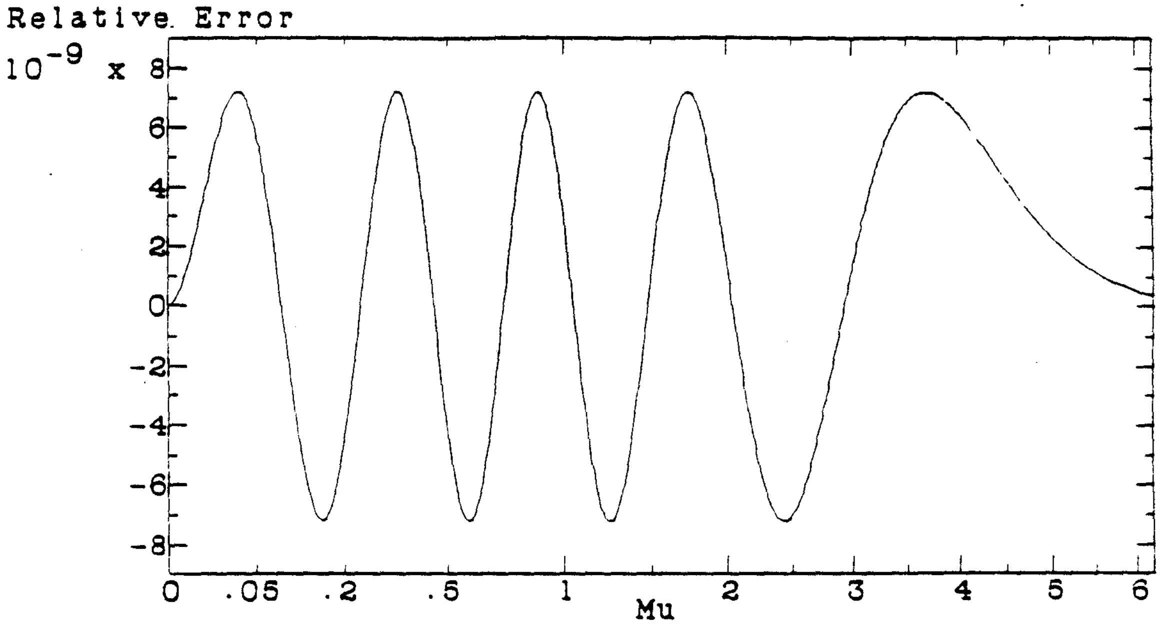
Relative Error for Pn*. n = 0. Order = 5
Second Approximation



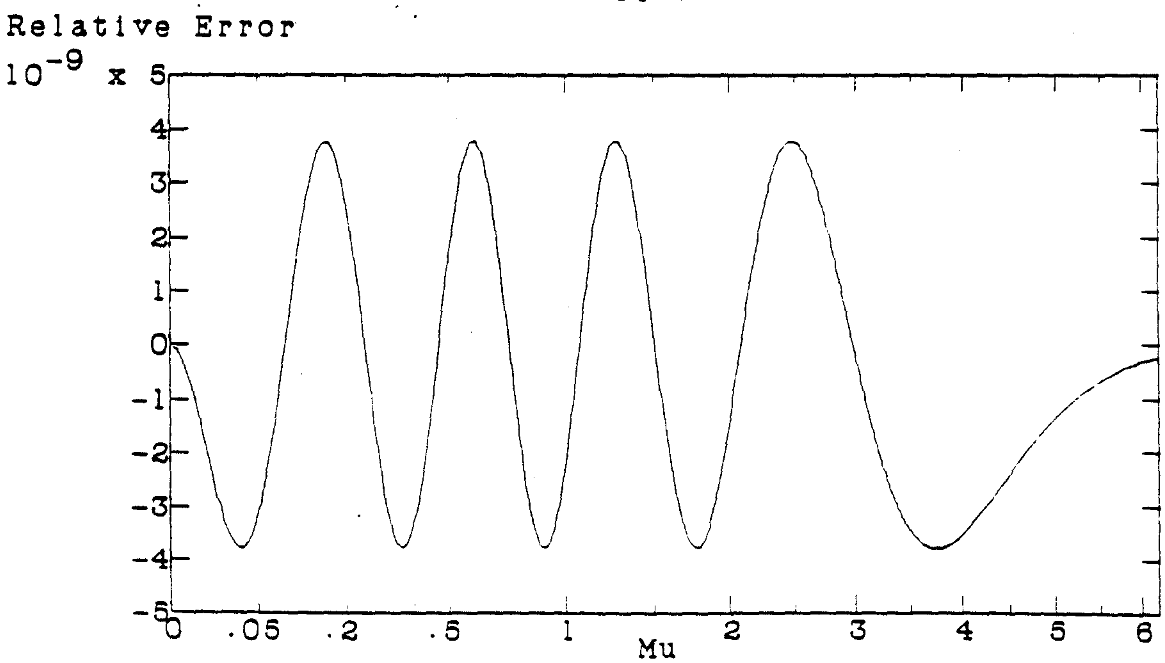
Relative Error for Pn*. n = 1. Order = 5
Second Approximation



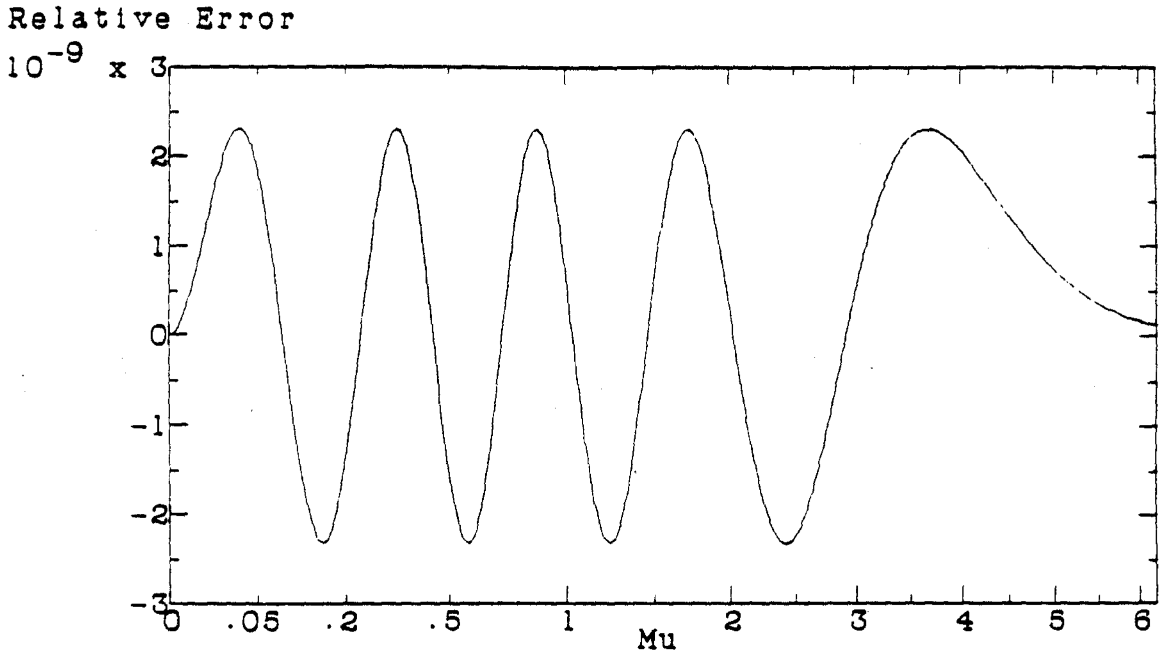
Relative Error for Pn*.   n = 2.   Order = 5
Second Approximation



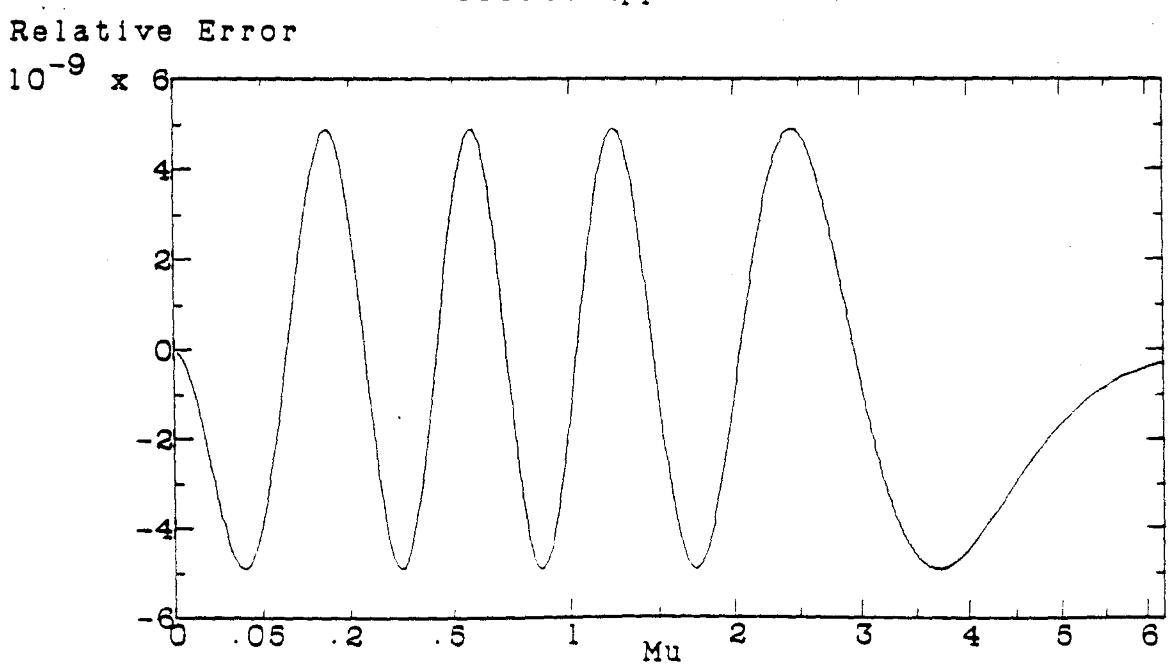
Relative Error for Pn*.   n = 3.   Order = 5
Second Approximation

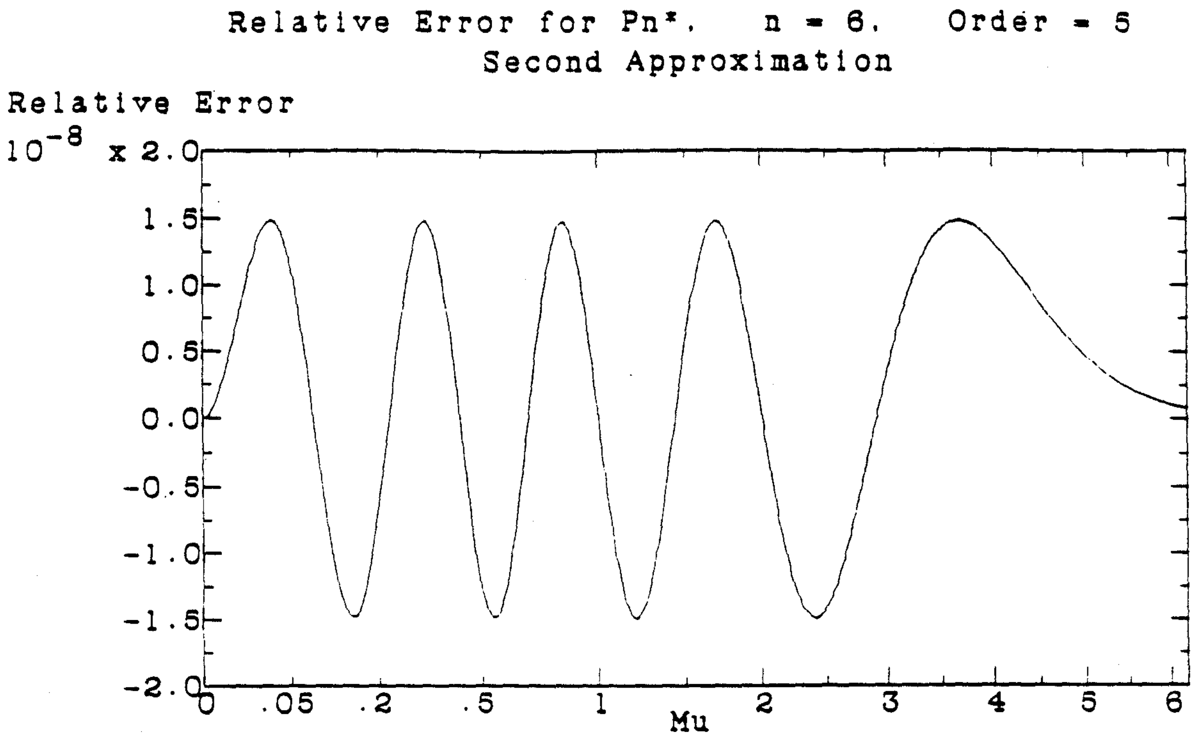


Relative Error for Pn*, n = 4, Order = 5
Second Approximation

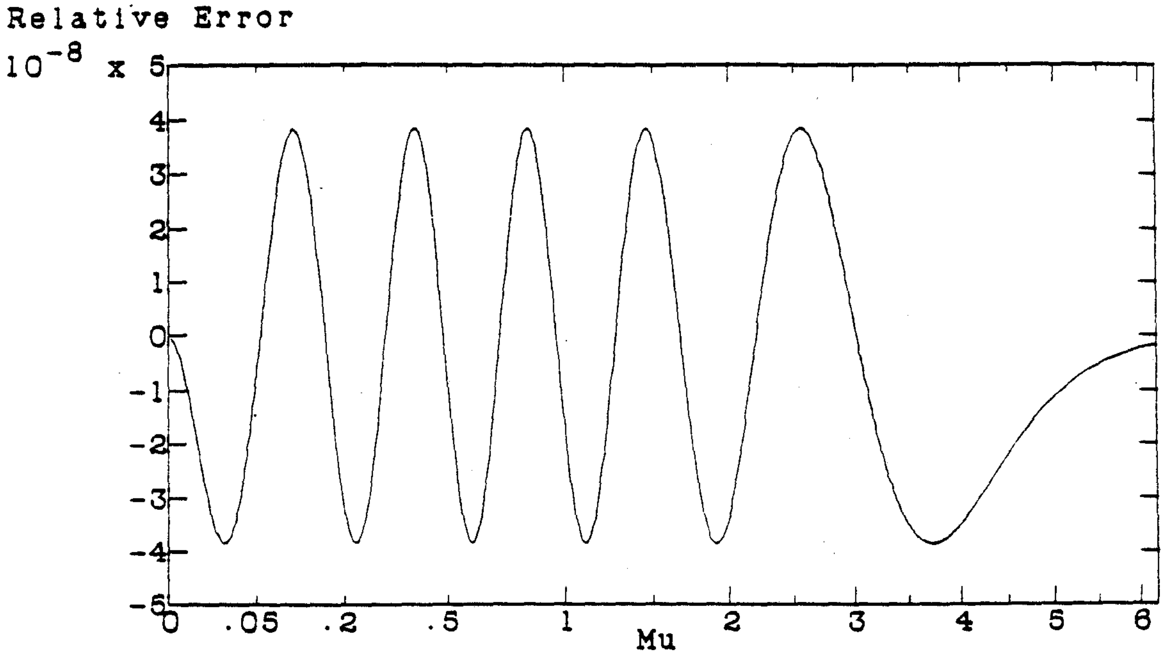


Relative Error for Pn*, n = 5, Order = 5
Second Approximation

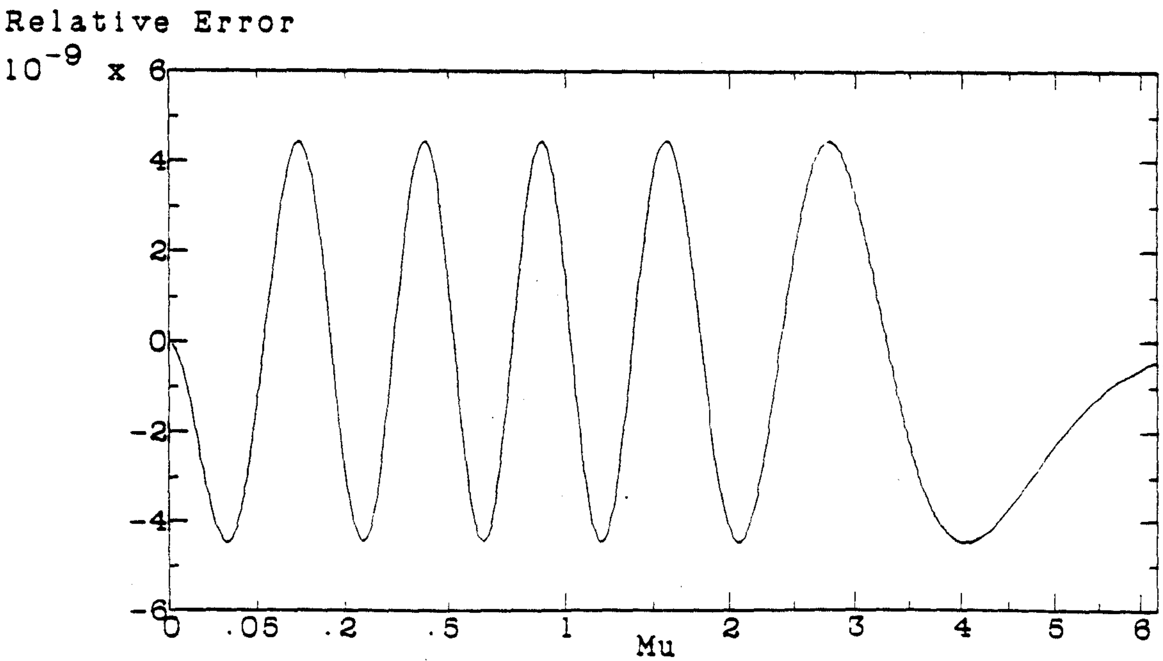


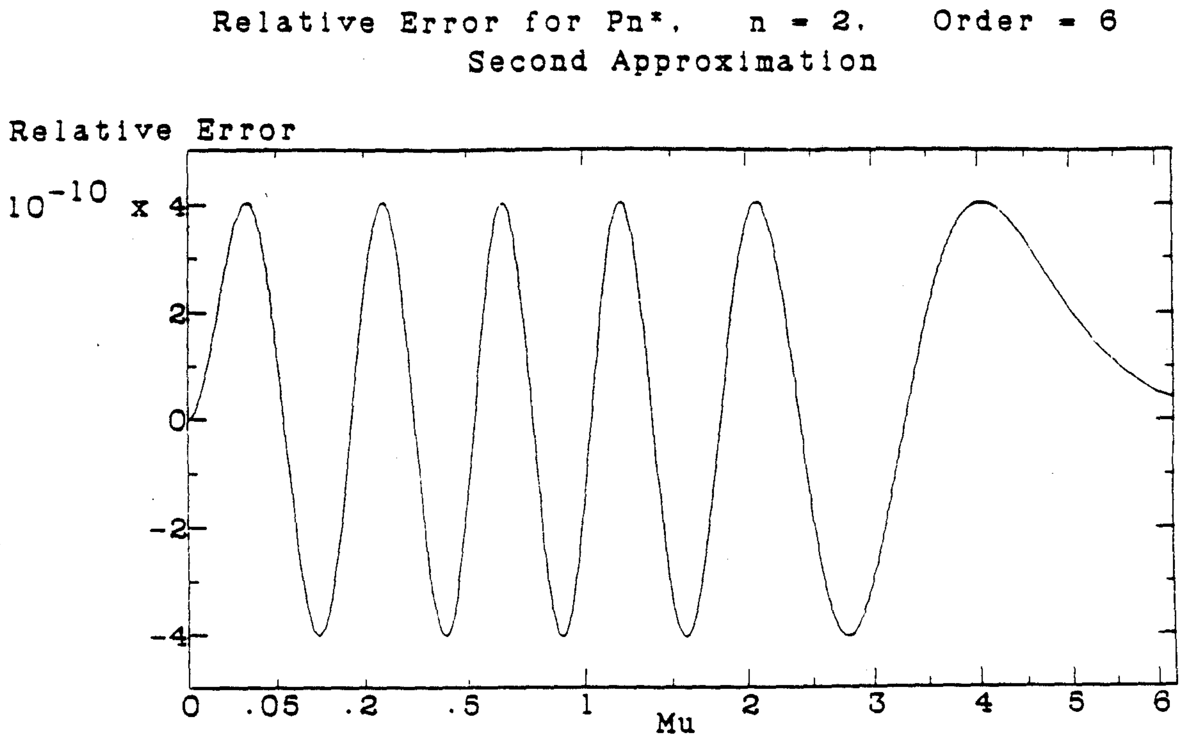

Relative Error for Pn*.   n = 6.   Order = 5
Second Approximation
Relative Error
10^{-8} x
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
0
.05
.2
.5
1
2
3
4
5
6
Mu

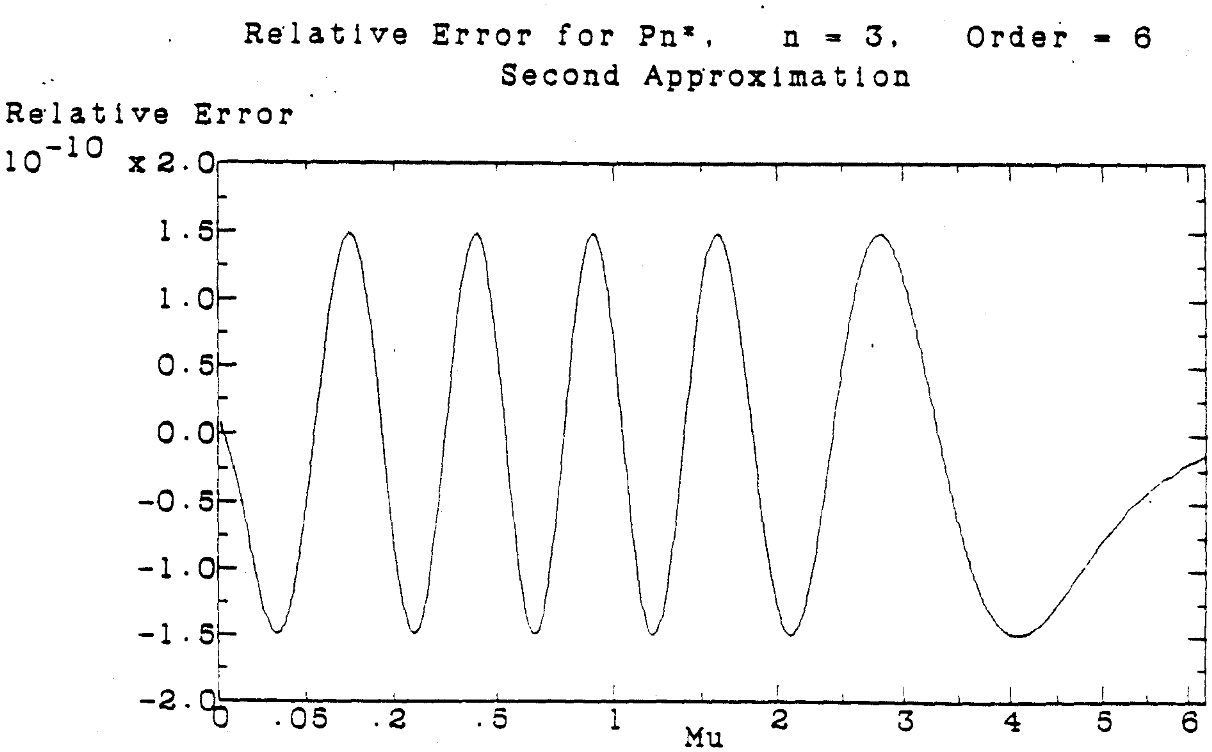
Relative Error for Pn*, n = 0, Order = 6
Second Approximation



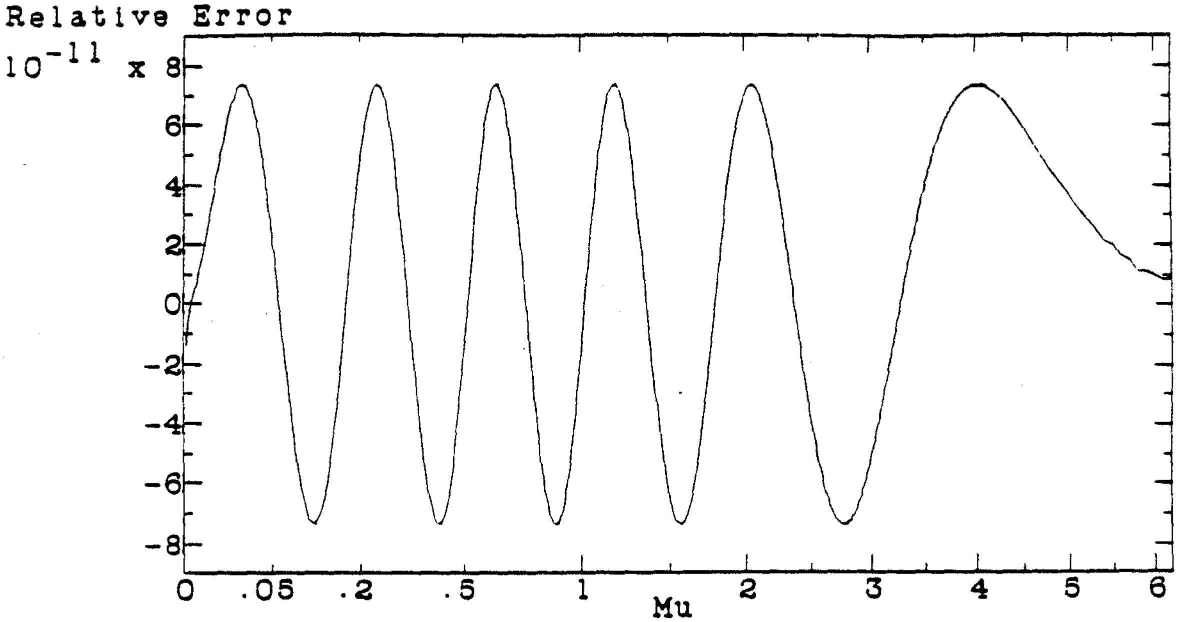
Relative Error for Pn*, n = 1, Order = 6
Second Approximation



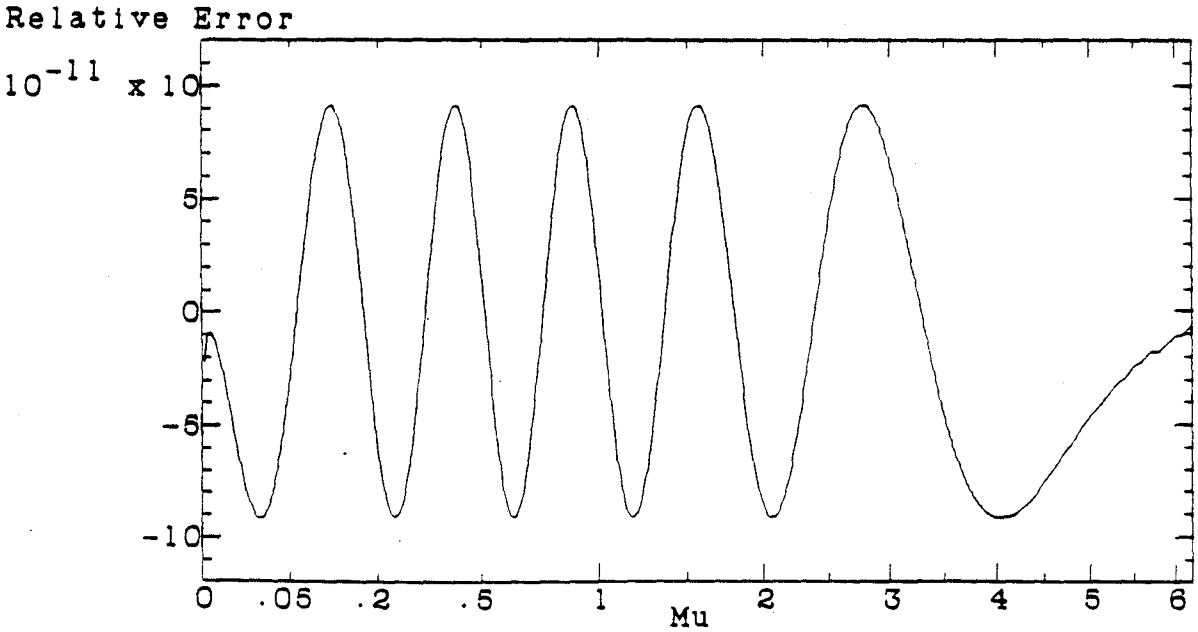

Relative Error for Pn*, n = 2, Order = 6
Second Approximation
Relative Error
10^-10 x
4
2
0
-2
-4
0
.05
.2
.5
1
2
3
4
5
6
Mu


Relative Error for Pn*, n = 3, Order = 6
Second Approximation
Relative Error
10^-10 x
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
0
.05
.2
.5
1
2
3
4
5
6
Mu

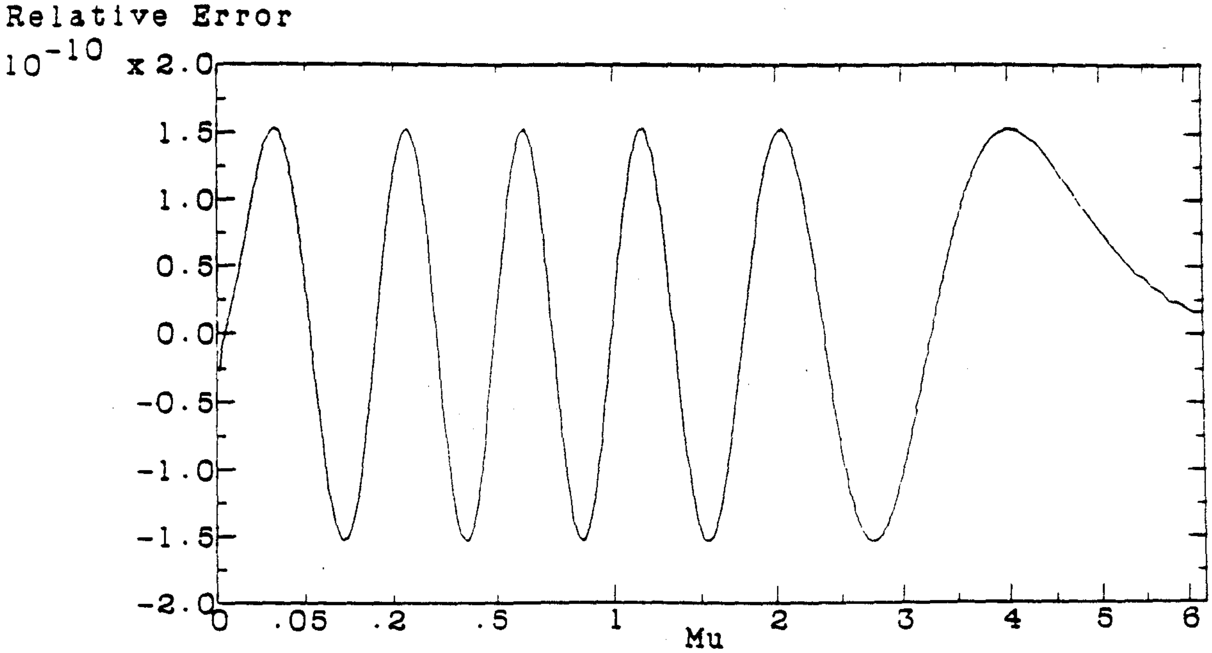
Relative Error for Pn*, n = 4, Order = 6
Second Approximation



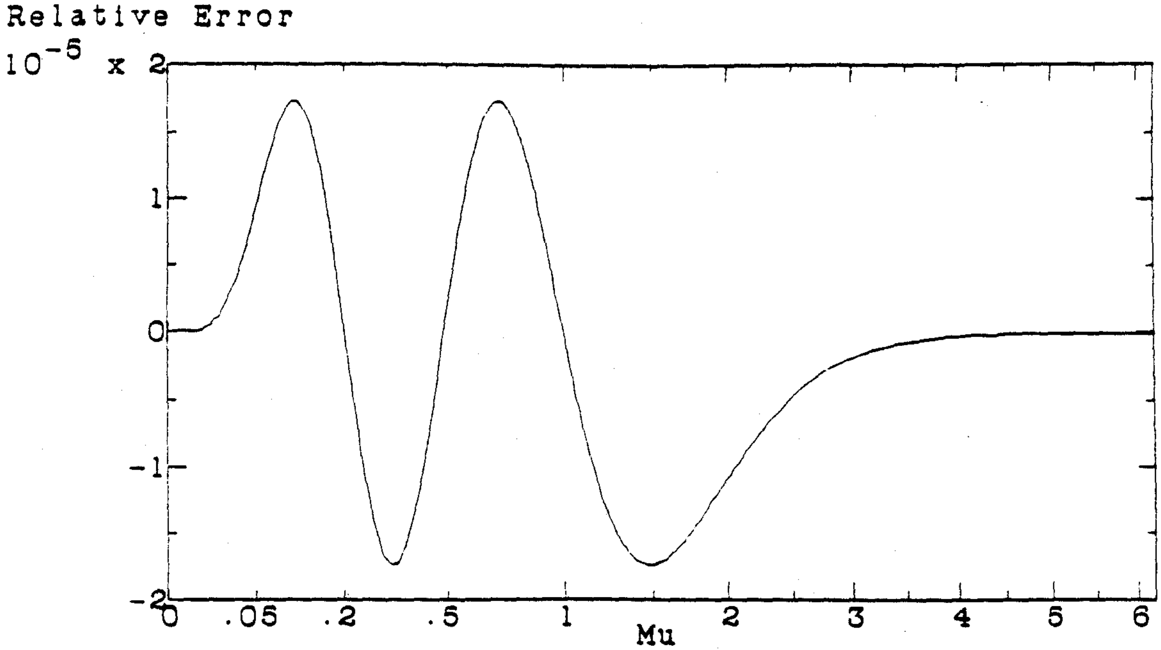
Relative Error for Pn*, n = 5, Order = 6
Second Approximation



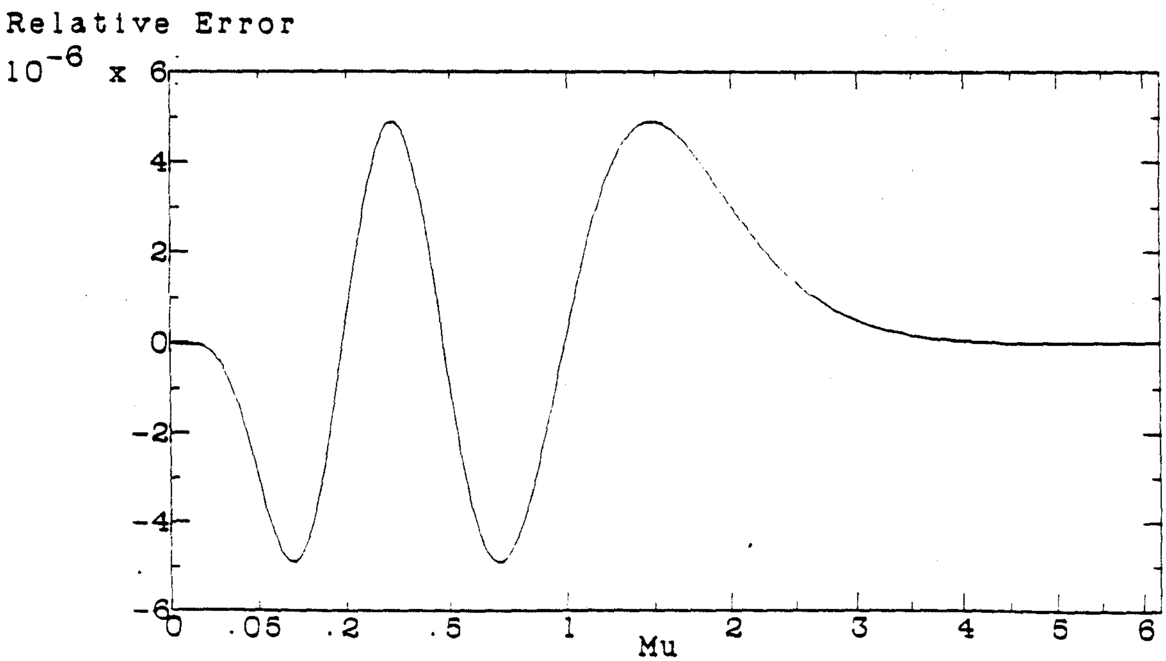
Relative Error for Pn*. n = 6. Order = 6
Second Approximation

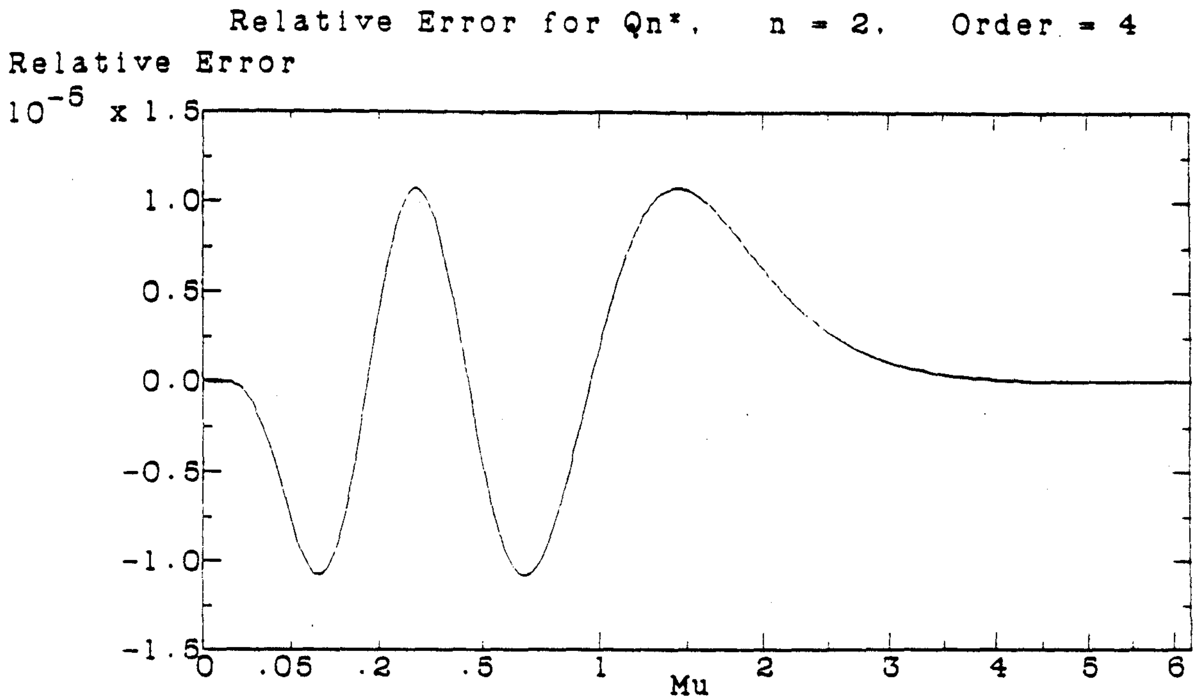


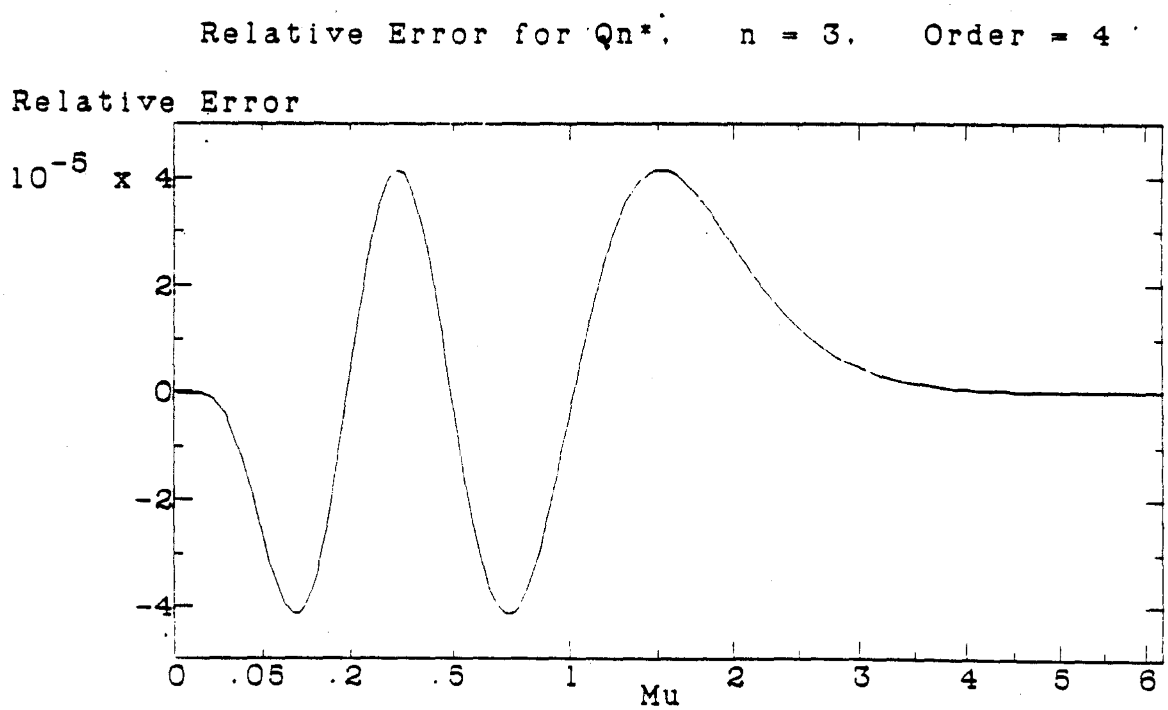
Relative Error for Qn*, n = 0, Order = 4



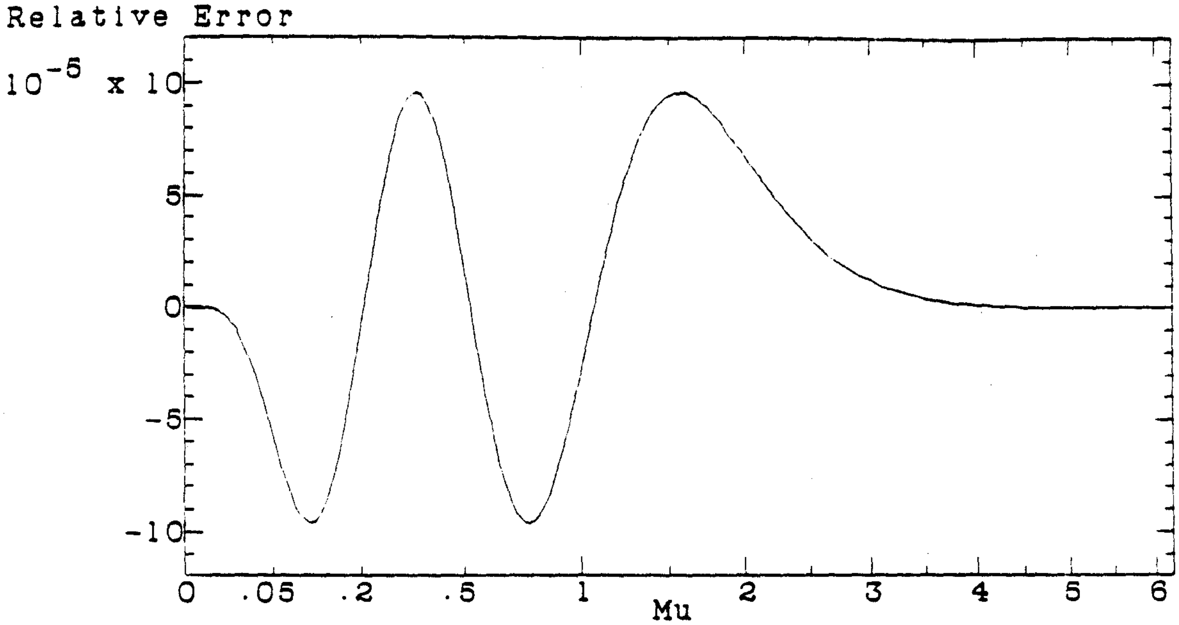
Relative Error for Qn*, n = 1, Order = 4



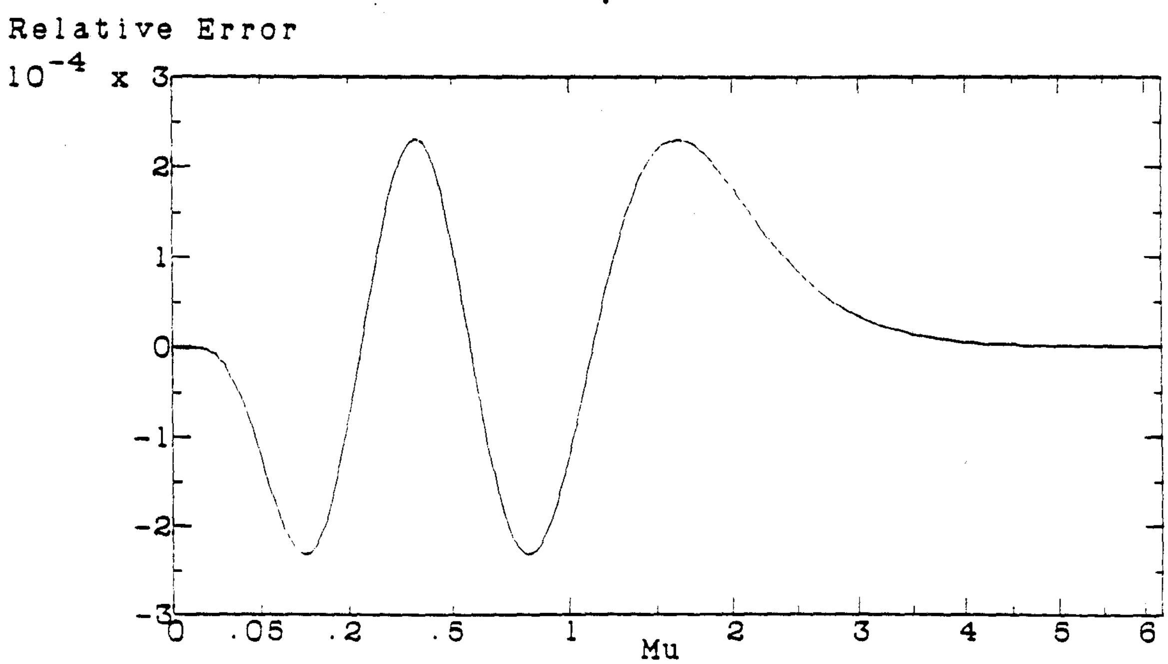

Relative Error for Qn*,   n = 2,   Order = 4
Relative Error
10^{-5} x 1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
0
.05
.2
.5
1
Mu
2
3
4
5
6

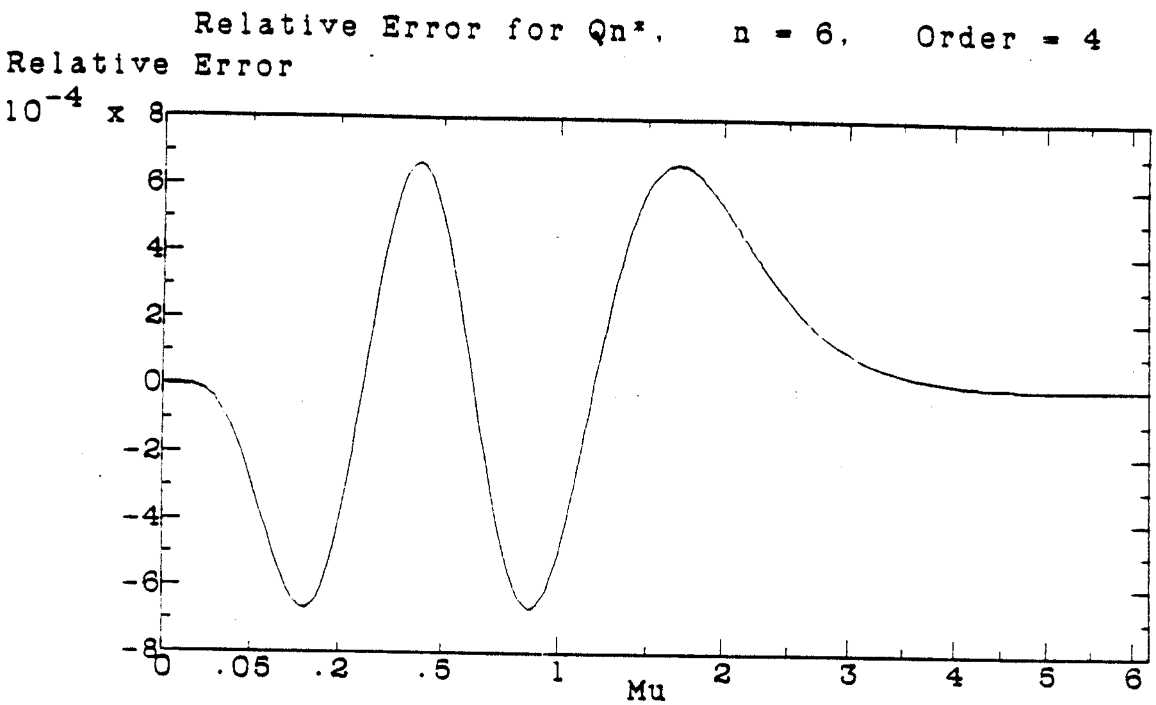

Relative Error for Qn*,   n = 3,   Order = 4
Relative Error
10^{-5} x 4
2
0
-2
-4
0
.05
.2
.5
1
Mu
2
3
4
5
6

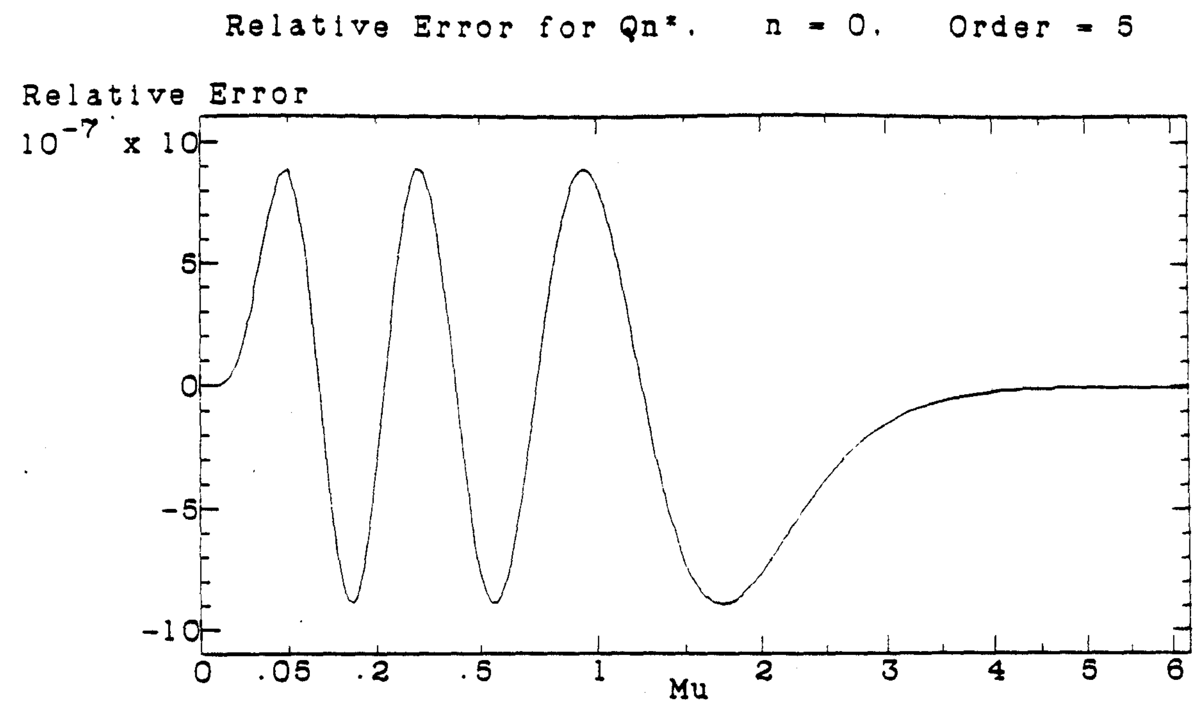
Relative Error for Qn*. n = 4. Order = 4

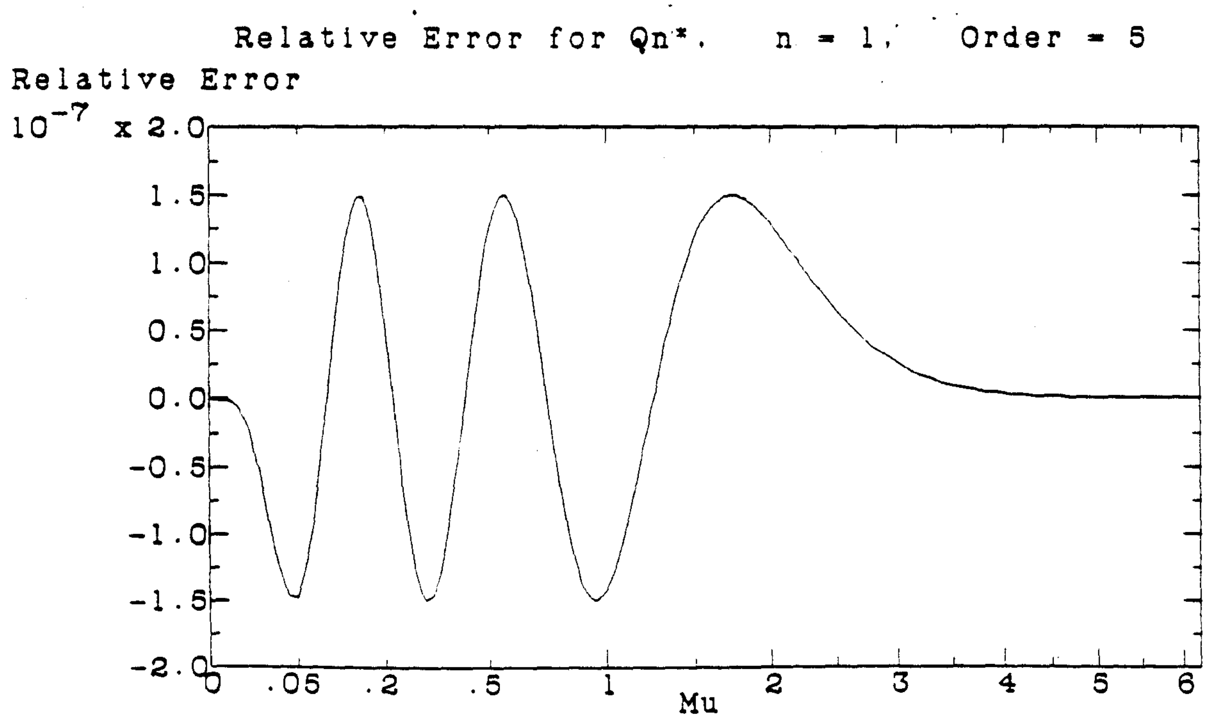


Relative Error for Qn*. n = 5. Order = 4

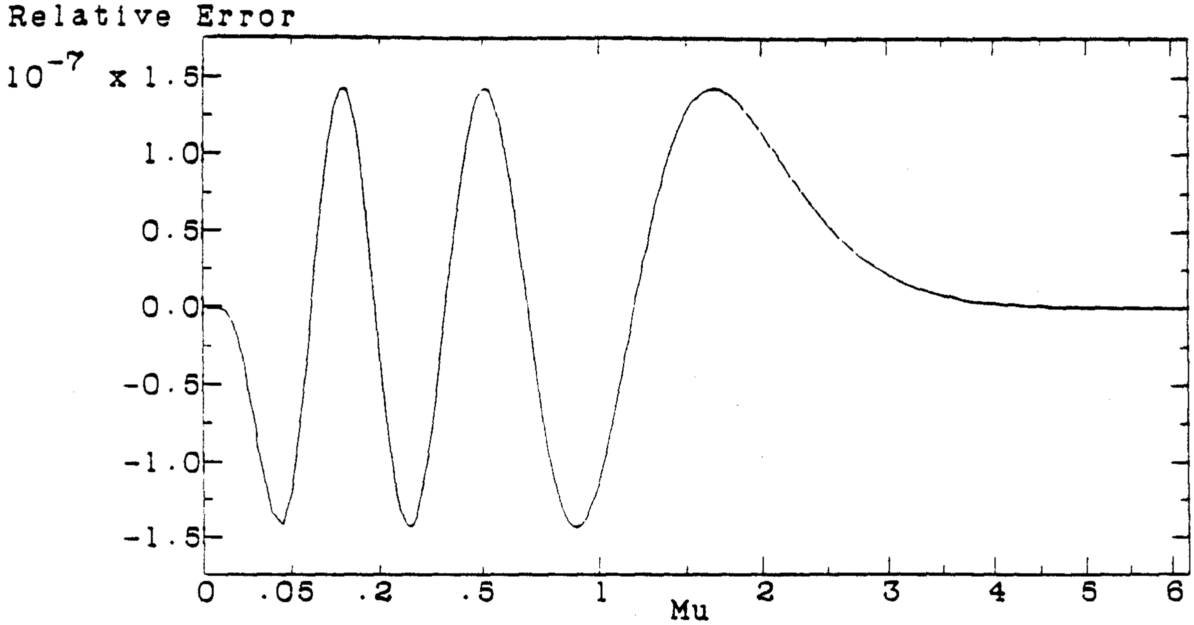



Relative Error for Qn*, n = 6, Order = 4
Relative Error
10^{-4} x
8
6
4
2
0
-2
-4
-6
-8
0
.05
.2
.5
1
2
3
4
5
6
Mu

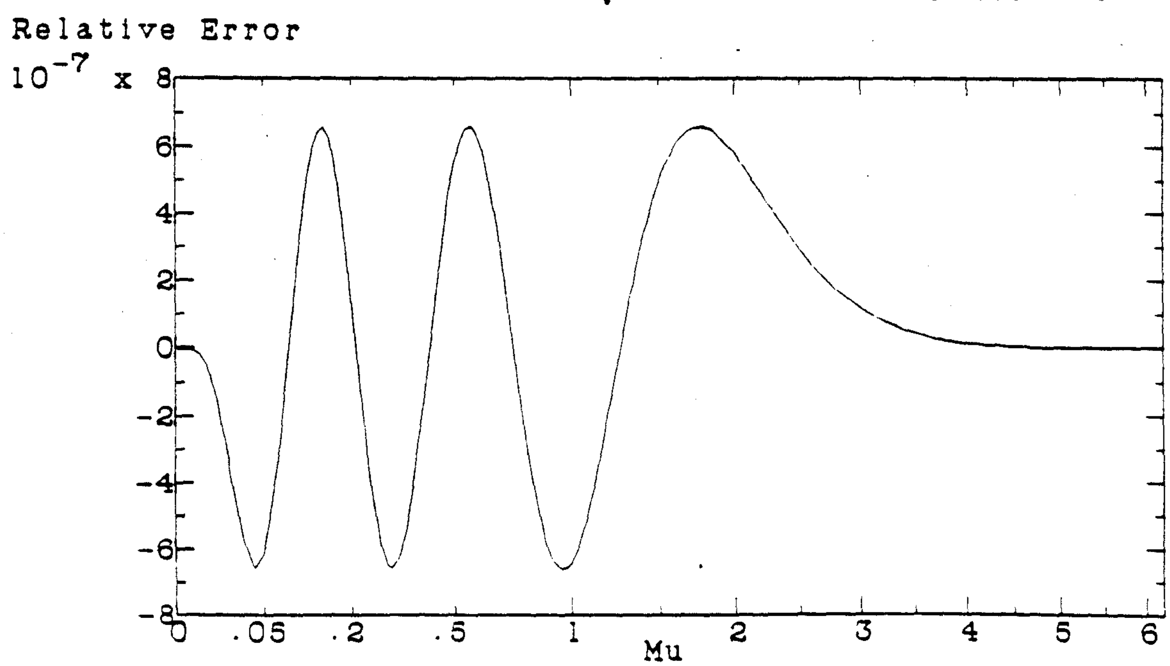

Relative Error for Qn*. n = 0. Order = 5
Relative Error
10^-7 x 10
5
0
-5
-10
0
.05
.2
.5
1
2
3
4
5
6
Mu


Relative Error for Qn*. n = 1. Order = 5
Relative Error
10^-7 x 2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
0
.05
.2
.5
1
2
3
4
5
6
Mu

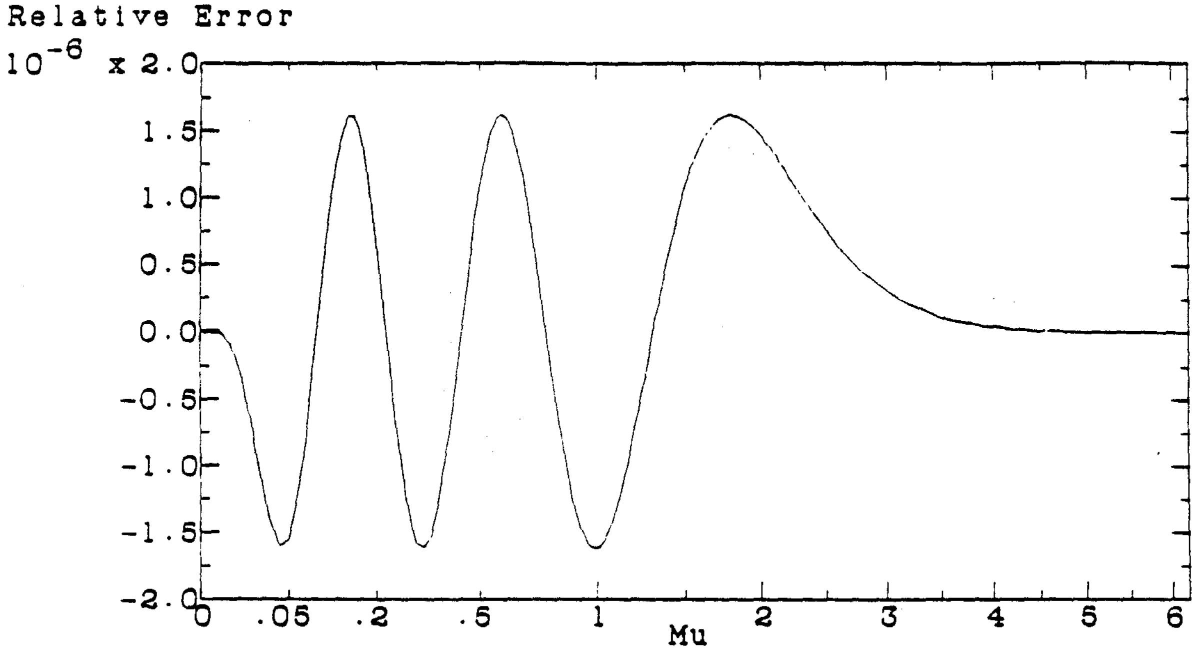
Relative Error for Qn*.   n = 2.   Order = 5



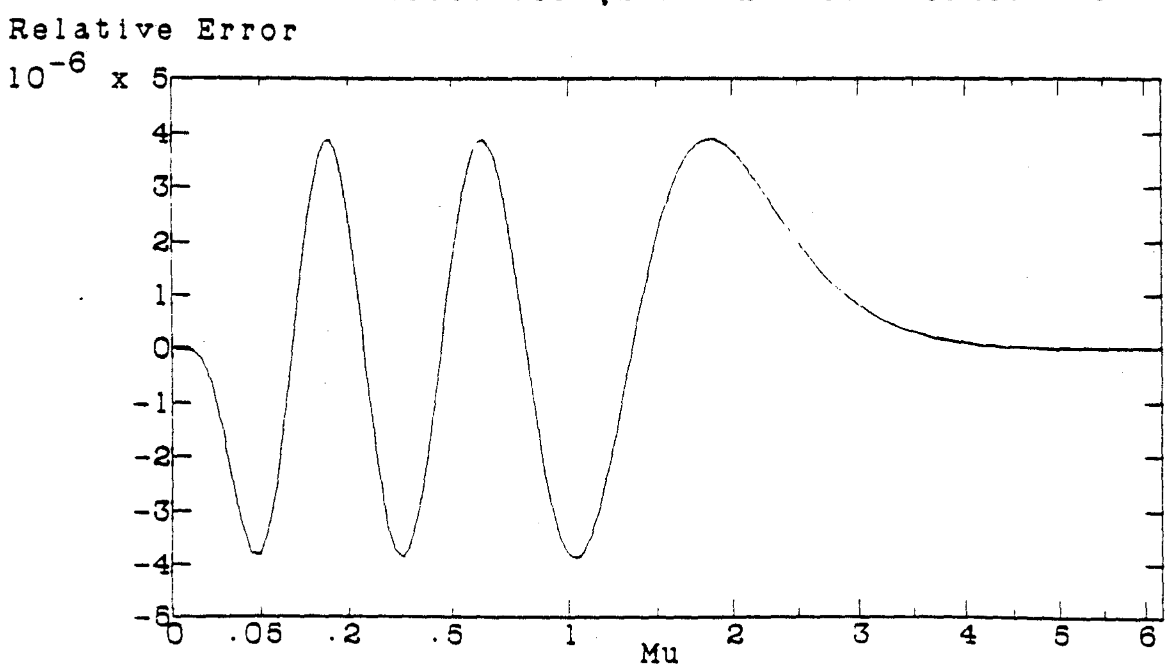
Relative Error for Qn*.   n = 3.   Order = 5

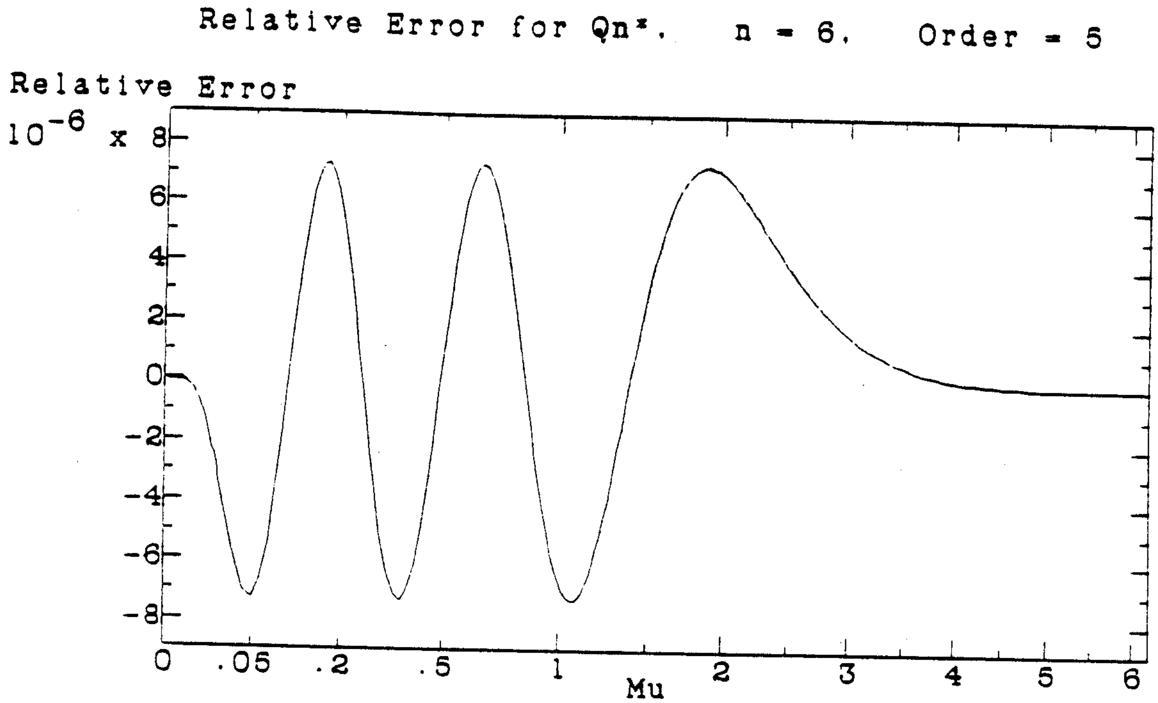


Relative Error for Qn*. n = 4. Order = 5

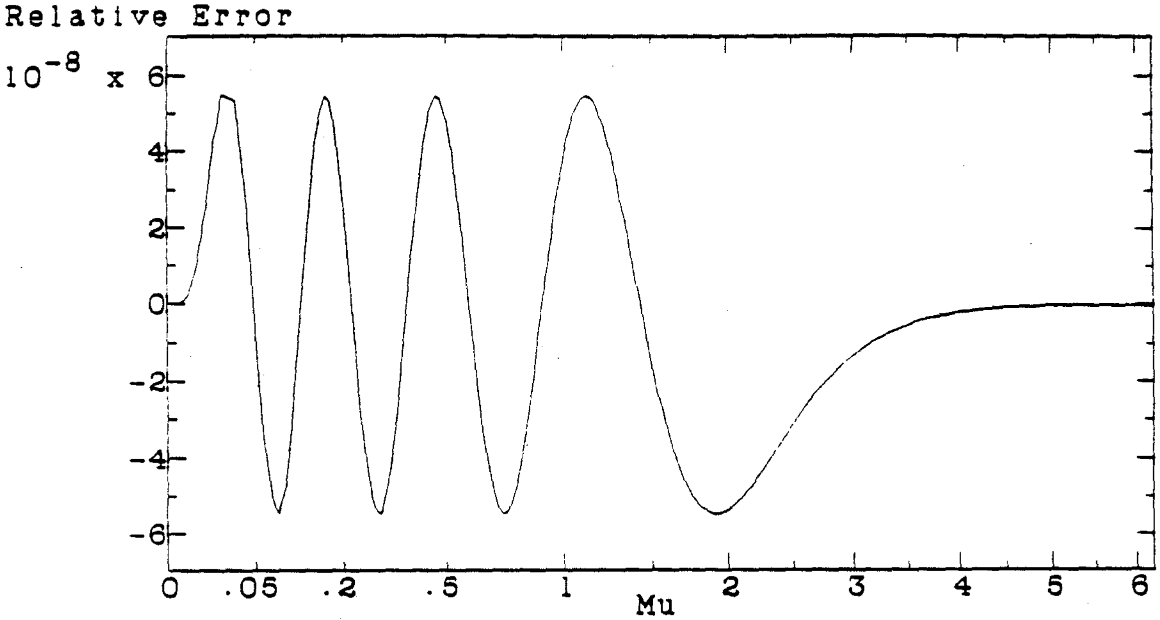


Relative Error for Qn*. n = 5. Order = 5

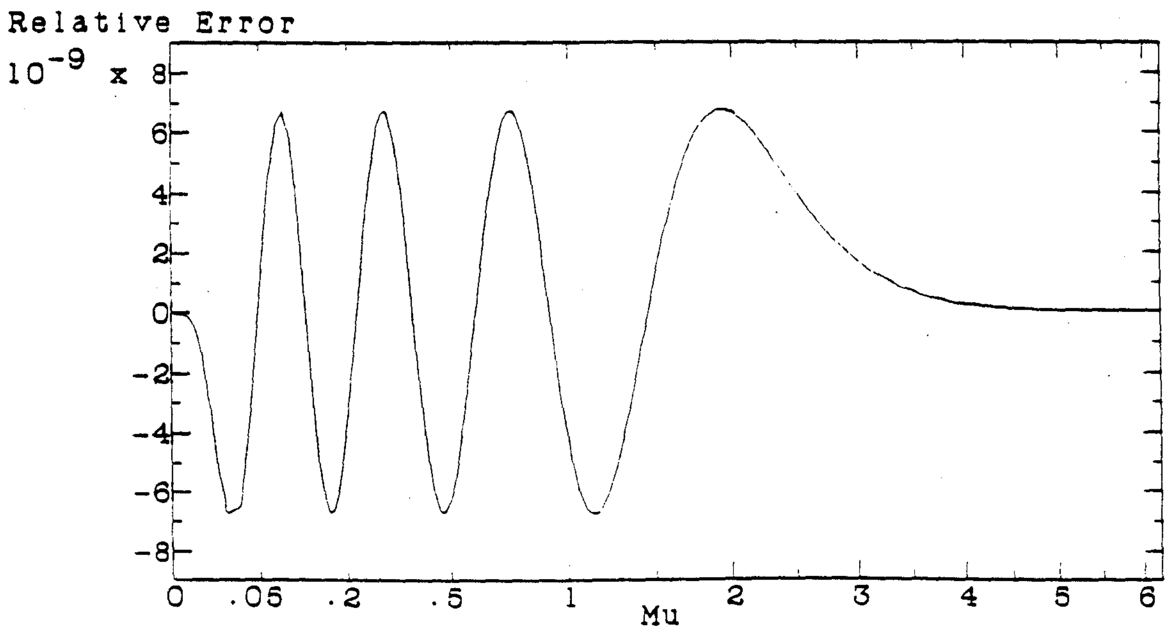



Relative Error for Qn*. n = 6. Order = 5
Relative Error
10^{-6} x
8
6
4
2
0
-2
-4
-6
-8
0
.05
.2
.5
1
2
3
4
5
6
Mu

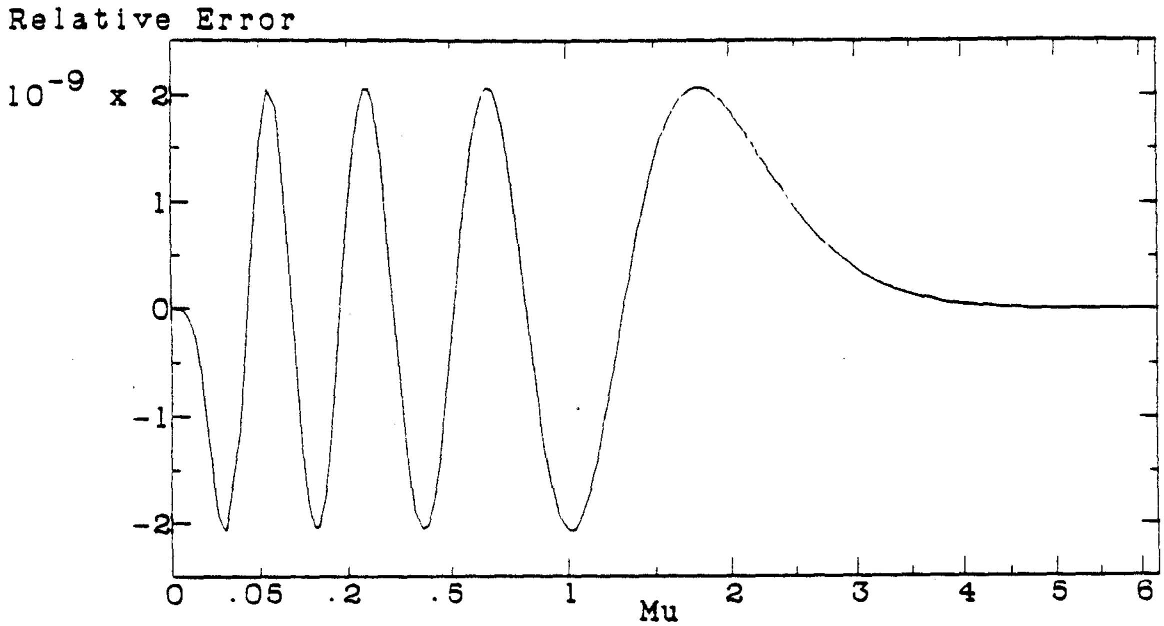
Relative Error for Qn*, n = 0, Order = 6



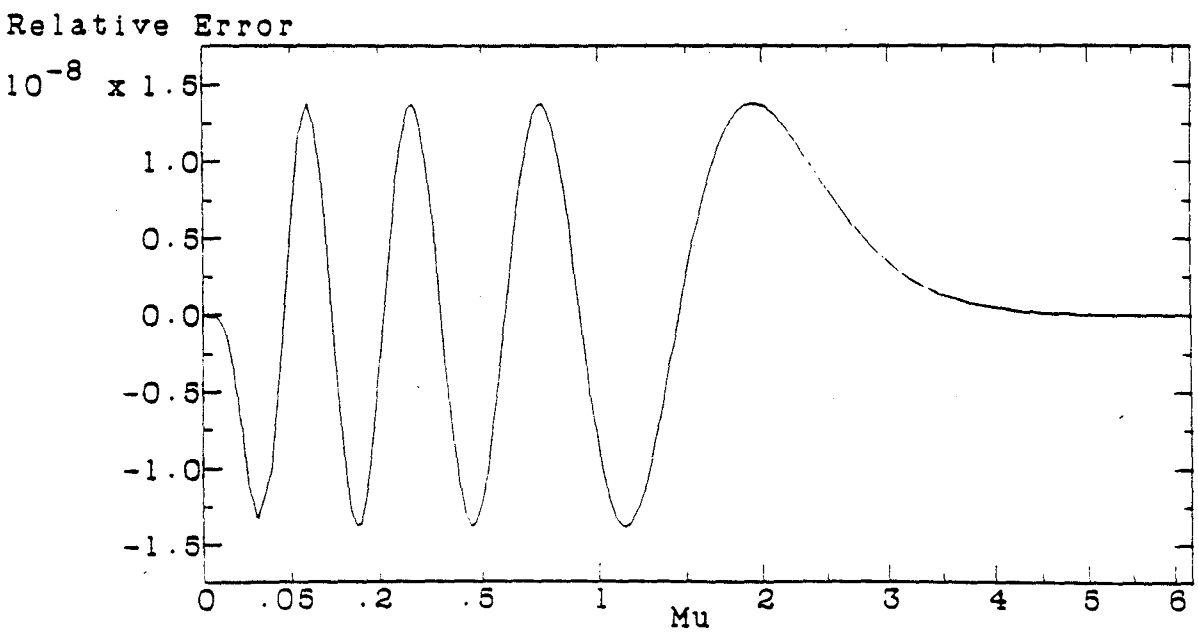
Relative Error for Qn*, n = 1, Order = 6



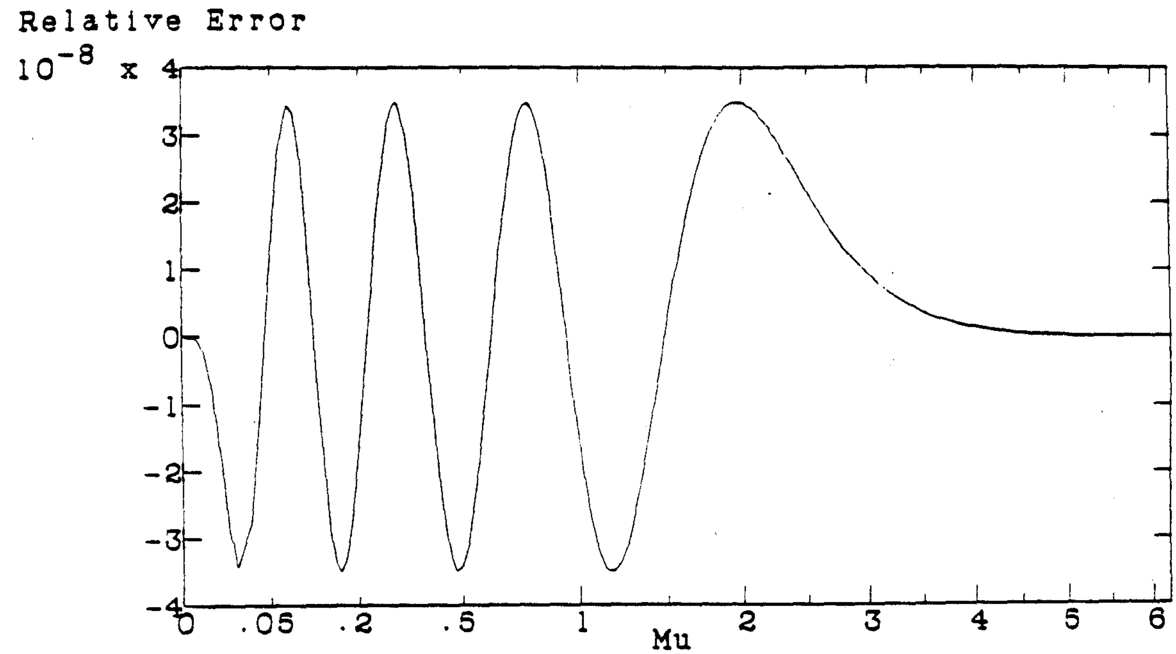
Relative Error for Qn*, n = 2, Order = 6



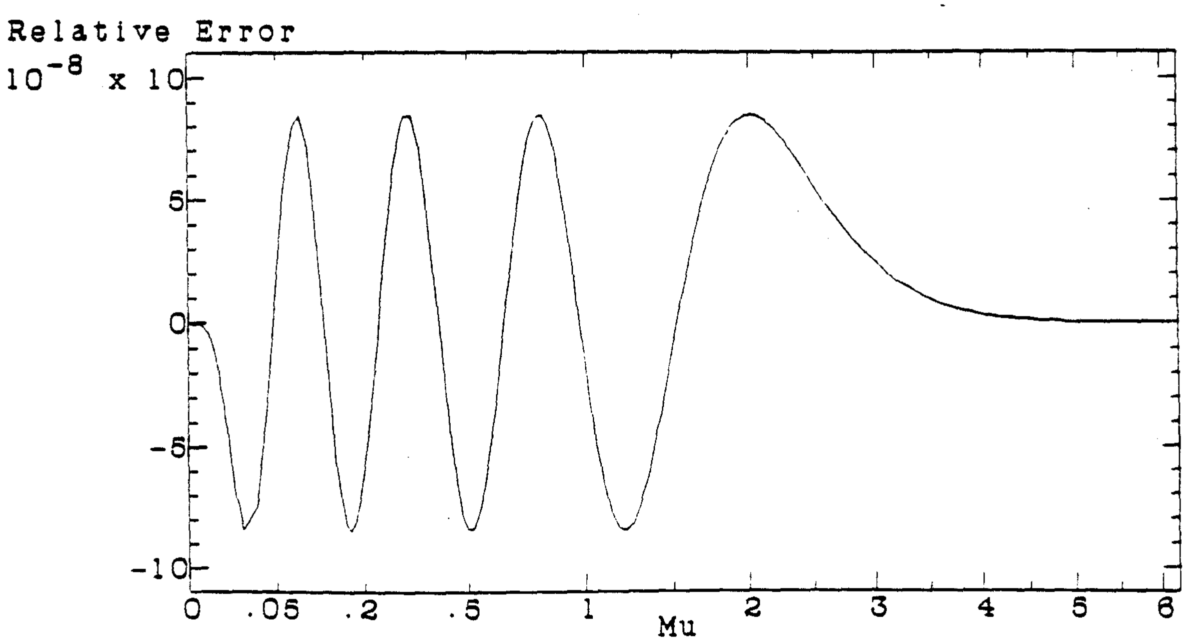
Relative Error for Qn*, n = 3, Order = 6

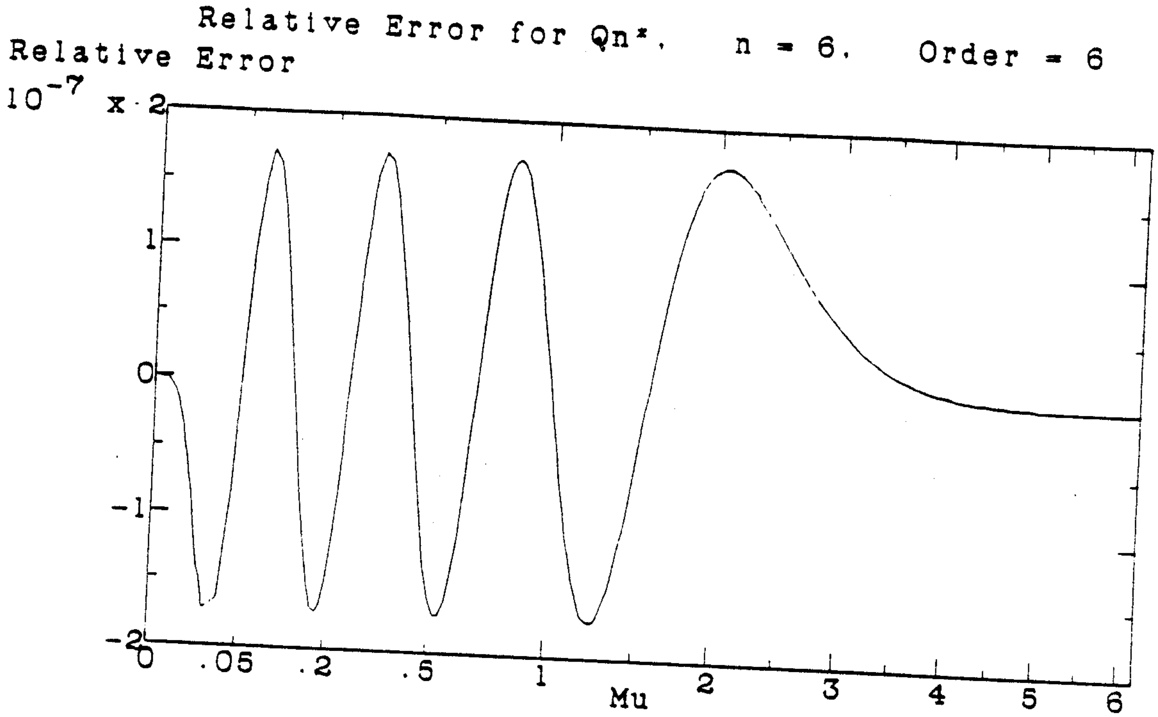


Relative Error for Qn*.   n = 4.   Order = 6

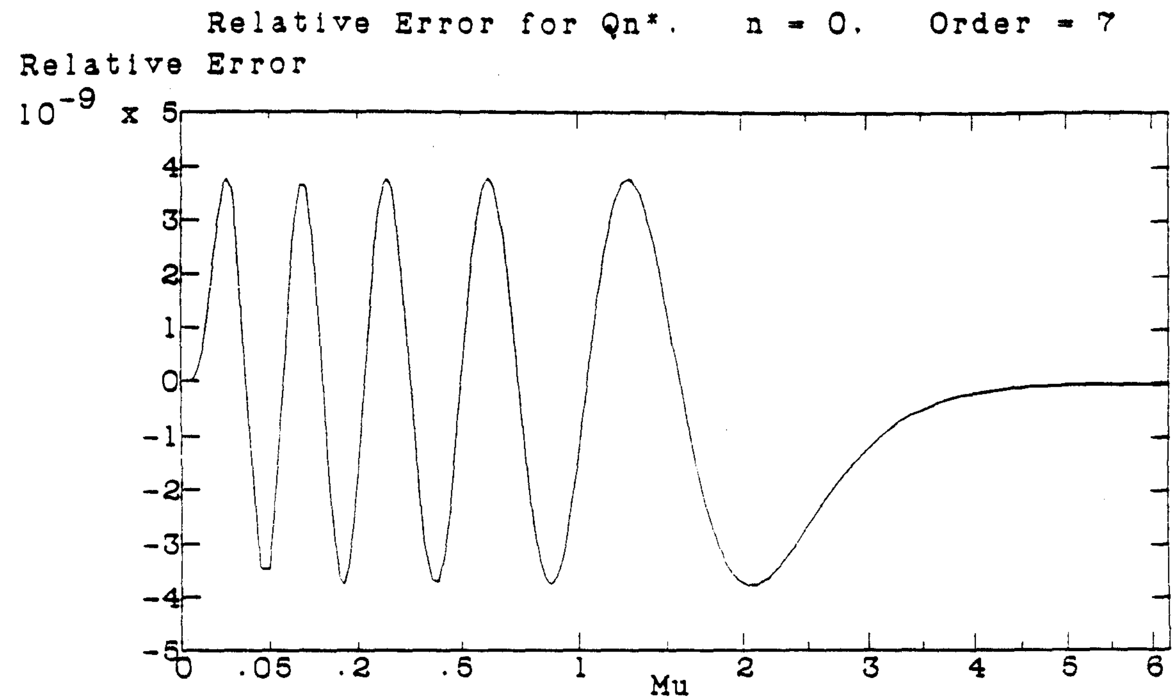


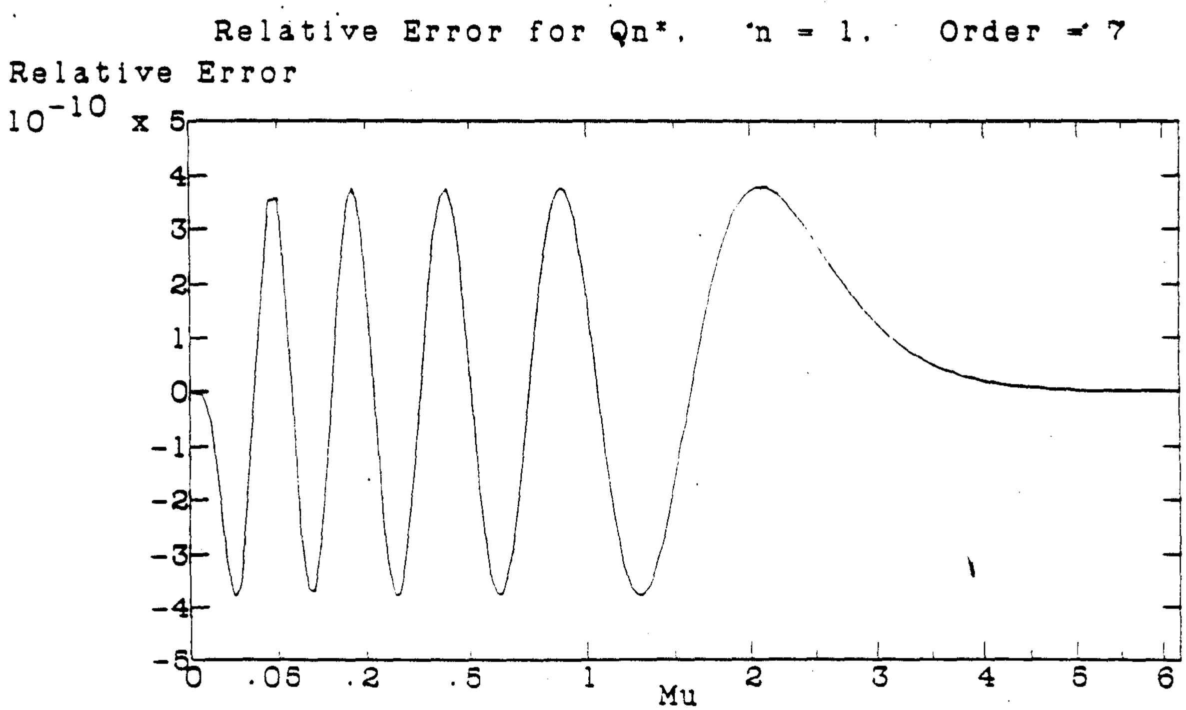
Relative Error for Qn*.   n = 5.   Order = 6

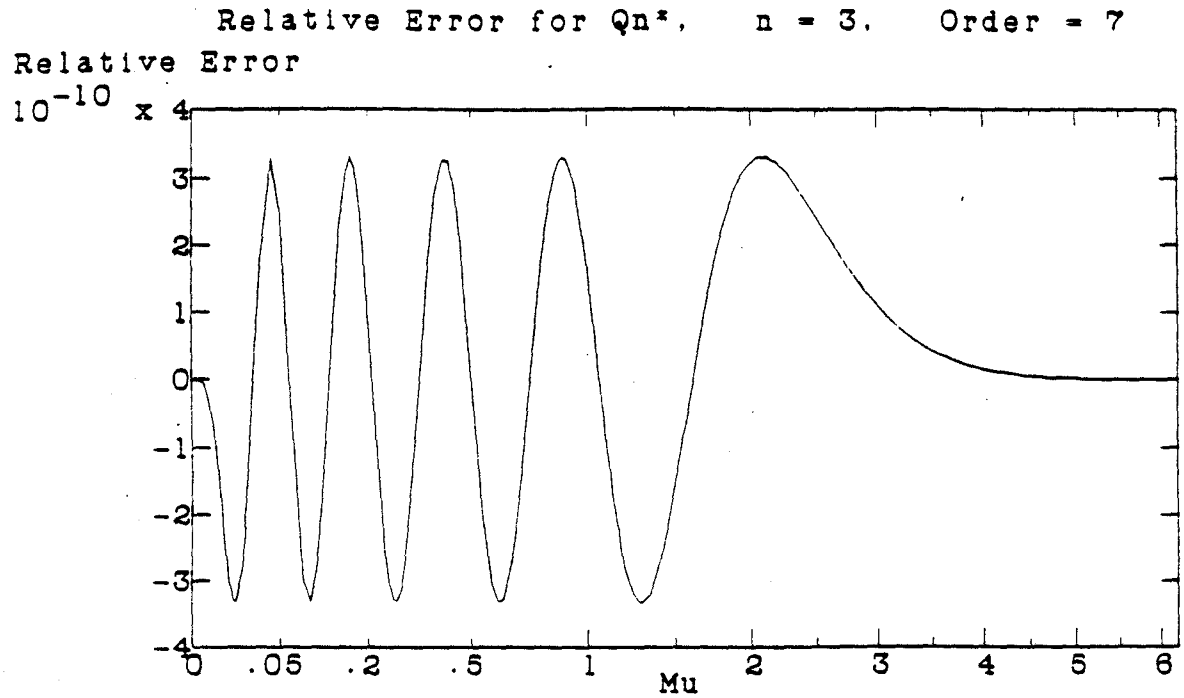


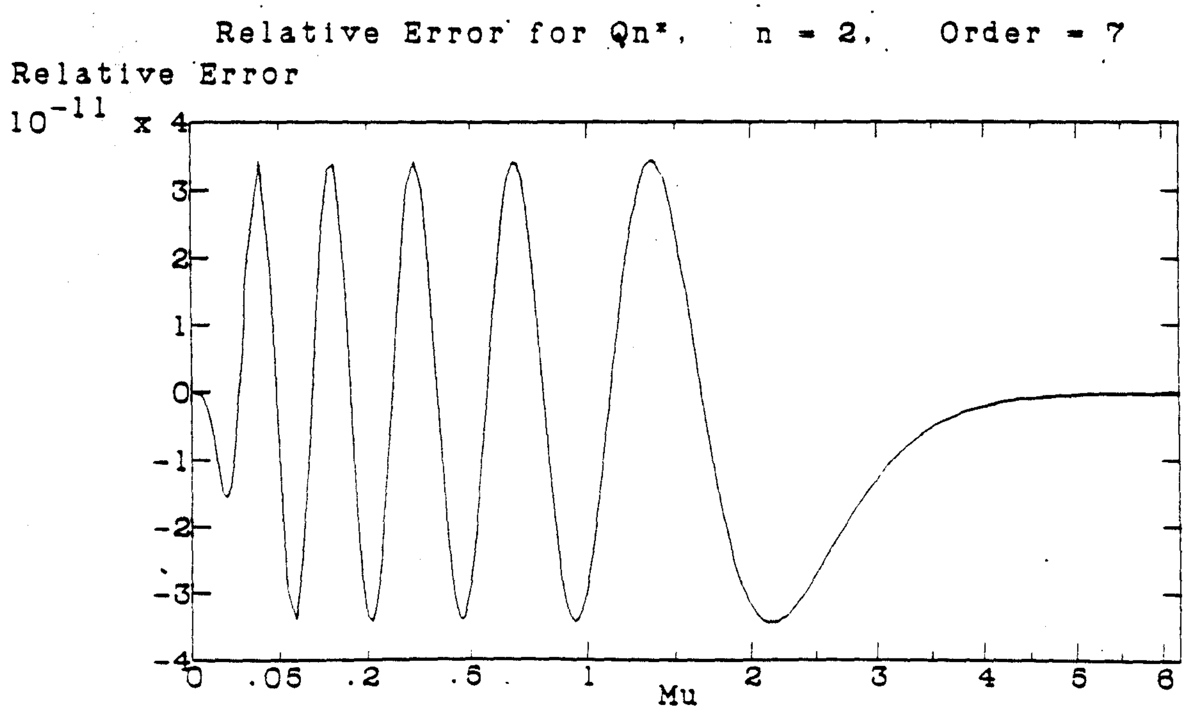


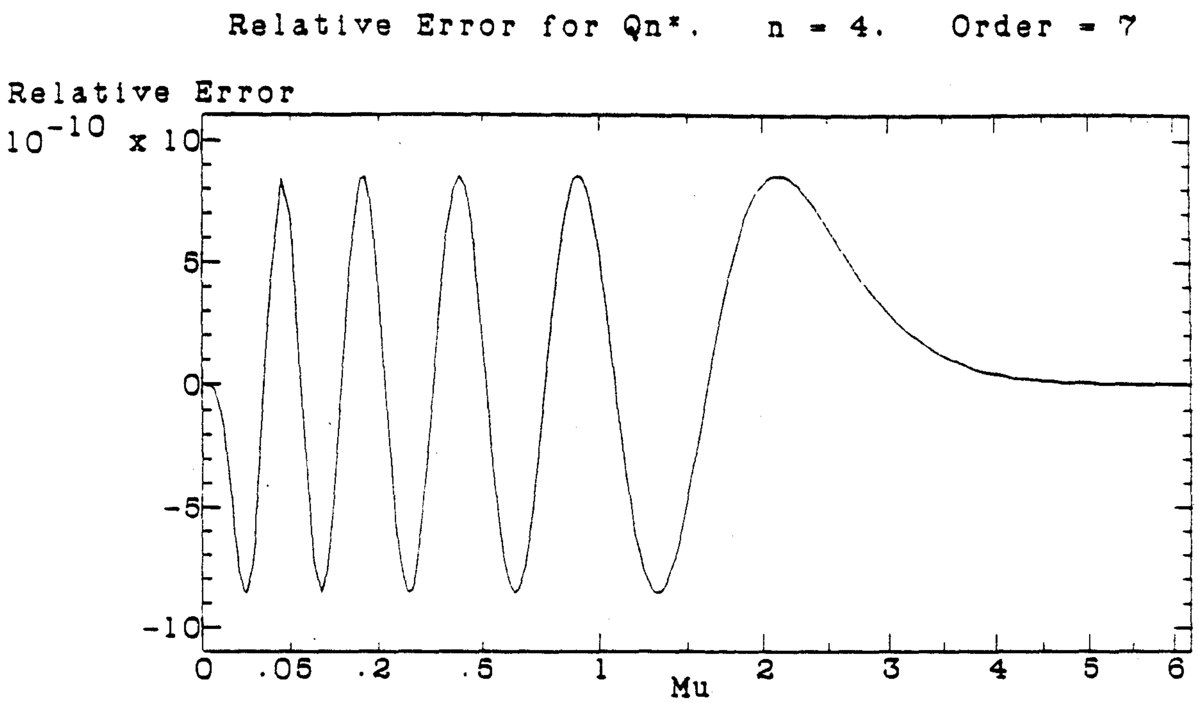
Relative Error for Qn*. n = 6. Order = 6
Relative Error
10^-7 x 2
1
0
-1
-2
0
.05
.2
.5
1
Mu
2
3
4
5
6

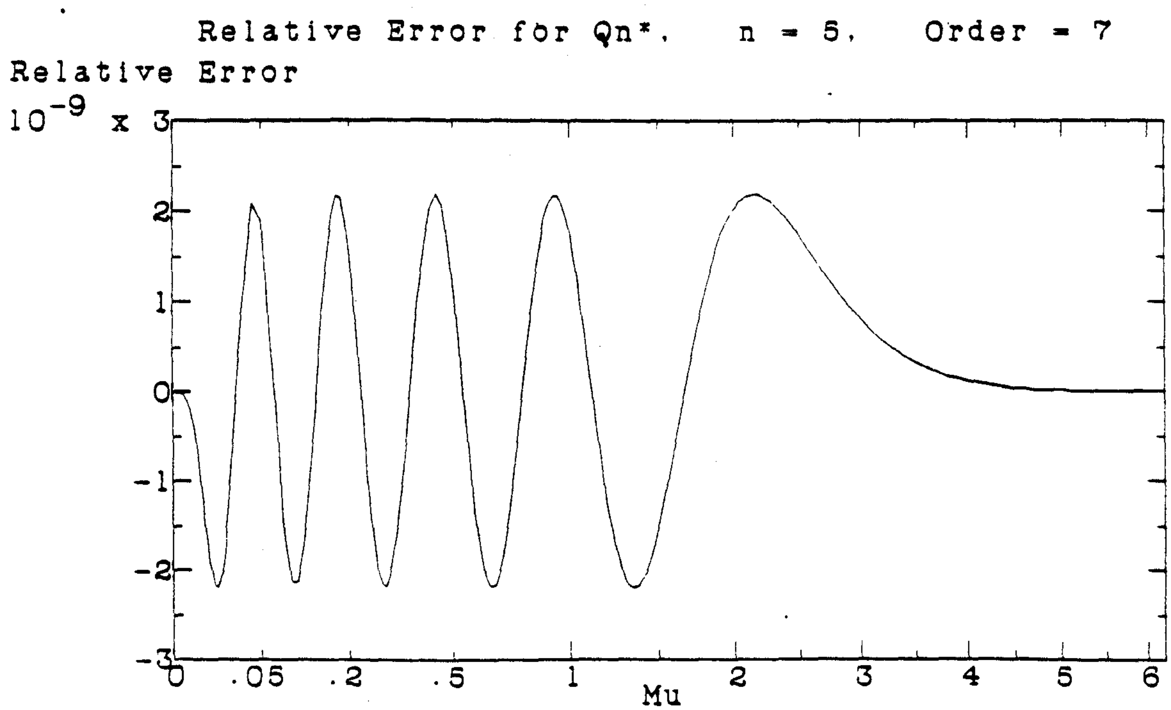

Relative Error for Qn*. n = 0. Order = 7
Relative Error
10^-9 x
5
4
3
2
1
0
-1
-2
-3
-4
-5
0
.05
.2
.5
1
2
3
4
5
6
Mu


Relative Error for Qn*. n = 1. Order = 7
Relative Error
10^-10 x
5
4
3
2
1
0
-1
-2
-3
-4
-5
0
.05
.2
.5
1
2
3
4
5
6
Mu

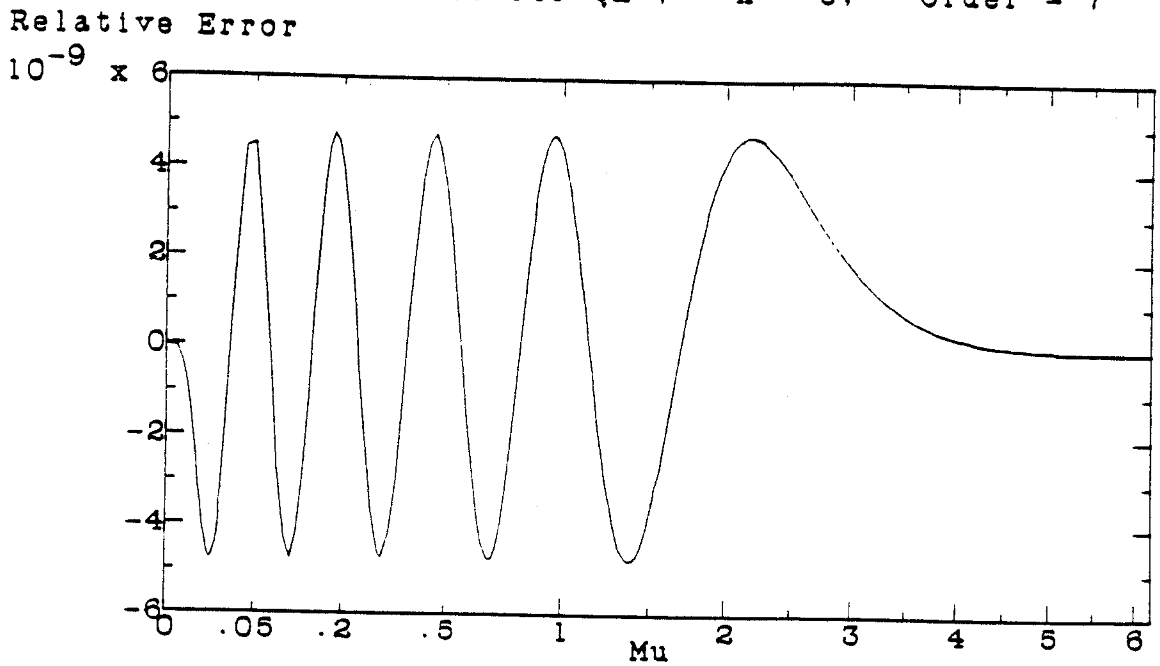

Relative Error for Qn*, n = 3, Order = 7
Relative Error
10^{-10} x
4
3
2
1
0
-1
-2
-3
-4
0
.05
.2
.5
1
2
3
4
5
6
Mu


Relative Error for Qn*, n = 2, Order = 7
Relative Error
10^{-11} x
4
3
2
1
0
-1
-2
-3
-4
0
.05
.2
.5
1
2
3
4
5
6
Mu


Relative Error for Qn*. n = 4. Order = 7
Relative Error
10^-10 x 10
5
0
-5
-10
0 .05 .2 .5 1 2 3 4 5 6
Mu


Relative Error for Qn*. n = 5. Order = 7
Relative Error
10^-9 x 3
2
1
0
-1
-2
-3
0 .05 .2 .5 1 2 3 4 5 6
Mu

Relative Error for Qn*,   n = 6,   Order = 7



References

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2. P. Morse, H. Feshback, **Methods of Theoretical Physics**, McGraw-Hill Book Company, New York, 1953.