SYMMETRIES IN DISSIPATION-FREE LINEAR MODE CONVERSION

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Symmetries in Dissipation-Free Linear Mode Conversion

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ABSTRACT

Linear mode conversions (MC) in loss-free (LF) regions of an inhomogeneous, Vlasov plasma in a magnetic field are shown to obey certain symmetries [1]. These are illustrated and interpreted for situations relevant to plasma heating and/or current drive.

INTRODUCTION

Consider a one-dimensional (in $x$) generic propagation and MC situation in an inhomogeneous plasma, with unperturbed (equilibrium) parameters (e.g., density, temperature, and magnetic field) that vary in $x$, as shown schematically in Figure 1. For homogeneity along $\vec{B}_0 = \hat{z}B_0(x)$, Landau and/or Doppler shifted cyclotron resonance absorption for any $k_z$ is assumed to occur outside the LF-MCR.

![Figure 1:](image)

PROOF OF SYMMETRIES

In the WKB regions to the right and left of the LF-MCR, let the complex field amplitudes of (e.g., forward) waves with energy flow into and out of the LF-MCR be, respectively,

$$a_p \sim \exp(i k_{px} x - i \omega t) \quad \text{and} \quad b_p \sim \exp(-i k_{px} x - i \omega t),$$

(1)

normalized so that: $|a_p|^2 = \text{wave energy flow density into the LF-MCR}; |b_p|^2 = \text{wave energy flow density out of the LF-MCR}. [\text{For backward waves, retaining the energy flow normalizations, the signs of the } k_{px}'s \text{ in (1) will change.}] \text{ In Figure 1, such modes on the left of the LF-MCR have } p = m (\text{there can be any number of such modes: } m_1, m_2, \ldots), \text{ and on}
the right of the LF-MCR, similarly, \( p = n \) designating any number of modes \((n_1, n_2, \ldots)\).

For a weakly dissipative mode, the total wave energy flow density (electromagnetic plus kinetic) is given, in general, by [2]

\[
\langle s_x \rangle_p = \left[ \frac{1}{2} \text{Re} \left( \vec{E} \times \vec{H}^* \right)_x - \frac{\varepsilon_0}{4} \omega \frac{\partial \chi_{H_\alpha\beta}}{\partial k_x} E_\alpha E_\beta^* \right]_p ,
\]

(2)

where \( \chi_{H_\alpha\beta} \) is the Hermitian part of the susceptibility tensor \( \chi_{H}(\vec{k}, \omega) \) with \( \vec{k} \) and \( \omega \) real and the star superscript denotes the complex conjugate. Since the full-wave equations describing the LF-MCR are linear (in general, linear integro-partial-differential equations) with appropriate boundary conditions, the complex field amplitudes \( a_p \) and \( b_p \) are related by a unique scattering matrix \( \vec{S} \)

\[
\vec{b} = \vec{S} \cdot \vec{a}
\]

(3)

where \( \vec{b} \) and \( \vec{a} \) are column vectors containing complex amplitudes of all \( b_p \) and all \( a_p \), respectively.

From energy flow conservation applied to the LF-MCR, we have \( \sum_p (|a_p|^2 - |b_p|^2) = 0 \), where the sum is over all \( m \)'s and \( n \)'s. Using (3), we can express this as \( \vec{a}^\dagger \cdot (\vec{I} - \vec{S}^\dagger \cdot \vec{S}) \cdot \vec{a} = 0 \), where the dagger superscript on \( \vec{S} \) denotes the complex-conjugate-transpose of \( \vec{S} \). Since this must hold true for arbitrary \( \vec{a} \), it follows that

\[
\vec{S}^\dagger = \vec{S}^{-1} .
\]

(4)

Next, consider wave energy flow under time reversibility. For the time reversed system, the direction of time-averaged energy flow density changes sign. In other words, the reversal of time changes time-averaged energy flow into the mode conversion region to time-averaged energy flow out of the mode conversion region, and vice versa. From (2), energy flow reversal is obtained by setting \( \vec{E} \rightarrow \vec{E}^* \), \( \vec{H} \rightarrow -\vec{H}^* \), \( \vec{k} \rightarrow -\vec{k} \) and, by (1), time reversal gives \( a_p \rightarrow b_p^* \) and \( b_p \rightarrow a_p^* \), where the star superscript denotes the complex conjugate. Referring to Figure 1, the effect of time reversal is to change \( a \) to \( b^* \) and \( b \) to \( a^* \), with arrows pointing in the same direction as indicated in the figure. Thus \( \vec{a}^* = \vec{S} \cdot \vec{b}^* \) or, taking the complex conjugate, \( \vec{a} = \vec{S}^* \cdot \vec{b} \). But from (3) \( \vec{a} = \vec{S}^{-1} \cdot \vec{b} \), hence

\[
\vec{S}^* = \vec{S}^{-1} .
\]

(5)

Combining (4) and (5), we finally obtain:

\[
\vec{S}^\dagger = \vec{S}^* \quad \text{or equivalently} \quad \vec{S}^T = \vec{S}^{-1} .
\]

(6)
where the $T$ superscript on $\mathbf{S}$ denotes the transpose of $\mathbf{S}$. Hence, the LF-MCR scattering matrix is symmetric. The symmetry of the LF-MC scattering matrix, $S_{ij} = S_{ji}$, entails important relationships for various power coefficients of the mode conversion process:

$$|S_{ij}|^2 = \left| \frac{b_i}{a_j} \right|^2 = \left| \frac{b_j}{a_i} \right|^2 = |S_{ji}|^2.$$  \hfill (7)

For MCs near the upper-hybrid resonance involving ordinary, extraordinary and electron Bernstein waves, the symmetries have been described in [3,4]. Here we illustrate the symmetries in two scenarios of MC near the ion-ion hybrid resonance (IIHR).

**MODE CONVERSIONS AT THE IIHR**

We assume conditions such that the individual ion-cyclotron resonances are outside the MCR containing the IIHR. MC is between fast Alfvén waves (FAW) and ion Bernstein waves (IBW).

1. **Cutoff on High-Field Side Following IIHR is Within MCR**

   The local dispersion relation in the LF-MCR for given $(\omega, k_z)$, and the WKB modes outside its boundaries, are illustrated in Figure 2. The associated scattering matrix is given by:

   $$\begin{pmatrix} b_B \\ b_F \end{pmatrix} = \begin{pmatrix} S_B & S_{FB} \\ S_{BF} & S_F \end{pmatrix} \begin{pmatrix} a_B \\ a_F \end{pmatrix}.$$  \hfill (8)

   From (7): $|S_{FB}|^2 = |S_{BF}|^2$ gives the symmetry in excitations by MCs between FAW and IBW. In addition, (5) gives a reflectivity symmetry $|S_B|^2 = |S_F|^2$.

![Figure 2](image1.png)

![Figure 3](image2.png)
2. No Cutoff on High-Field Side Following IIHR in MCR

The local dispersion relation in the LF-MCR for given \((\omega, k_z)\), and the boundaries of the LF-MCR with WKB mode fields outside of its boundaries are shown in Figure 3. The associated scattering matrix is given by:

\[
\begin{pmatrix}
    b_B \\
    b_H \\
    b_L
\end{pmatrix}
= 
\begin{pmatrix}
    S_B & S_{BH} & S_{BL} \\
    S_{HB} & S_H & S_{HL} \\
    S_{LB} & S_{LH} & S_L
\end{pmatrix}
\begin{pmatrix}
    a_B \\
    a_H \\
    a_L
\end{pmatrix}.
\tag{9}
\]

From (7): \(|S_{BH}|^2 = |S_{HB}|^2\) and \(|S_{BL}|^2 = |S_{LB}|^2\) give symmetries, respectively, in excitations by MCs between high-field side FAW and IBW, and low-field side FAW and IBW; \(|S_{HL}|^2 = |S_{LH}|^2\) gives the symmetry in transmissions of FAWs.

\[\text{Figure 4:}\]

\textbf{GENERALIZATION}

For 3-D propagation and mode conversion, the LF-MCR is identified by the breakdown of the eikonal description of modes. Outside the LF-MCR, where WKB eikonal descriptions are assumed to apply, and weakly dissipative modes are found to approach the LF-MCR by ray tracing, wave energy flow density is given by \[2\]

\[
\langle \vec{s}_p \rangle = \left[ \frac{1}{2} \text{Re} \left( \vec{E} \times \vec{H}^* \right) - \frac{\epsilon_0}{4} \omega \frac{\partial \chi^H_{\alpha \beta}}{\partial \vec{k}} \right]_p = \vec{v}_{gp} \langle w_p \rangle
\]

\tag{10}

where \(\vec{v}_{gp}\) and \(\langle w_p \rangle\) are, respectively, the group velocity and wave energy density of mode \(p\). Defining the mode amplitudes \((a_p, b_p)\) along \(\vec{v}_{gp}\) (see Figure 4), the symmetry of their scattering matrix \(\overline{S}\) is proven along lines identical to (3)–(6).
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