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Abstract

A comprehensive analysis of the finite-Larmor-radius (FLR) fluid moment equations for collisionless magnetized plasmas is presented. It is based on perturbative but otherwise general solutions for the second and third rank fluid moments (the stress and stress flux tensors) with closure conditions on the fourth rank moment. The single expansion parameter is the ratio between the gyroradius of the plasma species under consideration and any other characteristic length, which is assumed to be small but finite in a magnetized medium. This formalism allows a complete account of the gyroviscous stress, the pressure anisotropy and the anisotropic heat fluxes, and is valid for arbitrary magnetic geometry, arbitrary plasma pressure and fully electromagnetic nonlinear dynamics. As the result, very general yet notably compact perturbative systems of FLR collisionless fluid equations, applicable to either fast (magnetohydrodynamic) or slow (diamagnetic) motions, are obtained.

I. Introduction.

Fluid models play a central role in plasma research because their reduced dimensionality makes it more feasible to analyze realistic configurations, with broad ranges of plasma parameters, in three-dimensional space geometry. The more standard fluid models are derived for regimes of high collisionality[1,2], but a majority of plasmas of interest in space and in magnetic fusion experiments are collisionless or weakly collisional. For these, a fluid description can still make sense under strong magnetization conditions, at least as far as the dynamics perpendicular to the magnetic field is concerned. Fluid systems of equations for collisionless magnetized plasmas are then derived by means of perturbative expansions in powers of the ratio between the gyroradius of each species and any other characteristic length, $\delta \sim \rho/L \ll 1$. In addition to a small value of δ, a meaningful fluid description of a collisionless plasma is limited to low-frequency phenomena whose characteristic rate of temporal variation is also small compared to the gyrofrequency of the species under consideration. Here, two different ordering assumptions can be made. In the first one, to be called "fast dynamics" and also sometimes referred to as "magnetohydrodynamic ordering", the time derivative is assumed to be first-order in δ relative to the gyrofrequency, $\partial/\partial t \sim \delta\Omega_c$, and the flow velocity is assumed to be comparable to the thermal speed $u \sim v_{th}$. In the second one, to be called "slow dynamics" and also sometimes referred to as "drift ordering", the time derivative and the flow velocity are assumed to be respectively second-order and first-order, $\partial/\partial t \sim \delta^2\Omega_c$ and $u \sim \delta v_{th}$.

The fluid theory of collisionless plasmas was pioneered in the classic work of Chew, Goldberger and Low (CGL)[3]. The CGL analysis is restricted to the lowest order or zero-Larmor-radius limit, which is consistent only with the fast dynamics ordering. Besides, the CGL analysis leaves as unspecified closure variables the lowest order parallel heat fluxes. These are set equal to zero in the double adiabatic model, but this is recognized to be a poor approximation at low collisionality. Moreover, in order to take into account important diamagnetic and other multi-fluid effects, it is necessary to go to higher orders in the gyroradius expansion. With the fast dynamics ordering, A. Macmahon derived the general first-order finite-Larmor-radius (FLR) equations for the full stress tensor and the perpendicular heat fluxes[4]. Macmahon's results have remained the state of art in fast dynamics FLR collisionless fluid theory, but they do not include the parallel heat flux equations either. Dynamic evolution equations for the parallel heat fluxes, with closure conditions on fourth rank fluid moments (energy-weighed stress tensors), have been obtained more recently, only in the zero-Larmor-radius limit[5,6]. Other outstanding issues concern the slow dynamics ordering, and are related to the fact that this ordering does not lead in general to a strictly consistent asymptotic expansion of the fluid moment equations. This was already pointed out in Ref.[4] where, like in the early slow dynamics papers [7,8], the slow dynamics ordering was applied only under the extreme assumptions of constant magnetic field, no parallel flow, and low ratio β between plasma and magnetic pressures. To this day there is no universal agreement on slow dynamics subsidiary orderings, and the derivation of suitable FLR fluid systems in low collisionality regimes is still the subject of active investigation and debate[9–15]. Since the main advantage of the fluid description (compared to the more accurate but higher dimensionality kinetic description) is the better ability to analyze complex configurations, it should be desirable that fluid models not be thwarted by restrictive assumptions on variables such as β, the magnetic geometry, or the degrees of inhomogeneity and anisotropy.

This article presents a general derivation of first-significant-order FLR systems of fluid moment equations for collisionless magnetized plasmas. The third rank moment (the stress flux tensor) is solved for on the same footing and to the same degree of accuracy in the perturbative expansion in δ as the second rank moment (the stress tensor), and this provides the sought after evolution equations for the parallel heat fluxes. The momentum conservation equation, which evolves the first rank moment (the particle flux), is assumed to be satisfied exactly for whatever expression of the stress tensor is provided. This guarantees the existence of an exact energy conservation law, and allows an exact algebraic elimination of the electric field. The closure condition is imposed by specifying some explicit representation of the fourth rank fluid moment. The more traditional approach based on solving perturbatively for the fluid moments of the Vlasov equation[4,6,9,13] is followed, as opposed to taking moments of the gyrophase-averaged drift-kinetic[5] or gyrokinetic[10–12,14,15] equations. This approach has the advantage of readily yielding unambiguous results without the recourse to any further assumptions. Hence, the results presented here are valid for inhomogeneous and anisotropic plasmas of arbitrary β in any magnetic geometry. Also, this general formalism can be equally applied to the fast and slow dynamics ordering schemes. As mentioned earlier, the slow dynamics ordering is plagued by a consistency problem and this will be discussed openly. The adopted course is to retain maximum generality and to show clearly what the slow dynamics ordering, without other subsidiary assumptions, can and cannot do. The first part of the paper, through Section IV, presents a number of general relations for the collisionless fluid moments, which establish the framework of our analysis. The first-significant-order FLR perturbative systems are derived in Section V for the fast dynamics ordering, and in Sections VI and VII for the slow dynamics ordering. The paper ends with a note on the energy conservation law in Section VIII, and some concluding remarks.

II. Fluid variables and collisionless fluid moment equations.

The distribution function of a collisionless plasma species, $f(\mathbf{v}, \mathbf{x}, t)$, obeys the Vlasov equation,

$$\frac{\partial f}{\partial t} \; + \; v_i \frac{\partial f}{\partial x_i} \; + \; \frac{e}{m}\Big(E_i + \epsilon_{ijk} v_j B_k\Big)\frac{\partial f}{\partial v_i} \; = \; 0, \tag{1}$$

where $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ are the electric and magnetic fields, and m and e are the species mass and electric charge. All the results in this paper apply to each species independently, so the species index is dropped throughout. The velocity moments of the distribution function define the fluid variables we shall be concerned with. These are the particle density:

$$n(\mathbf{x}, t) \; = \; \int d^3\mathbf{v} \; f(\mathbf{v}, \mathbf{x}, t), \tag{2}$$

the particle flux:

$$n(\mathbf{x}, t) \; u_i(\mathbf{x}, t) \; = \; \int d^3\mathbf{v} \; v_i \; f(\mathbf{v}, \mathbf{x}, t), \tag{3}$$

where $\mathbf{u}(\mathbf{x}, t)$ represents the macroscopic fluid velocity, the second rank stress tensor:

$$P_{ij}(\mathbf{x}, t) \; = \; m \int d^3\mathbf{v} \; v_i v_j \; f(\mathbf{v}, \mathbf{x}, t), \tag{4}$$

the third rank stress flux tensor:

$$M_{ijk}(\mathbf{x}, t) \; = \; m \int d^3\mathbf{v} \; v_i v_j v_k \; f(\mathbf{v}, \mathbf{x}, t), \tag{5}$$

and the fourth rank tensor:

$$N_{ijkl}(\mathbf{x},t) \;=\; m\int d^3\mathbf{v}\; v_i v_j v_k v_l\; f(\mathbf{v},\mathbf{x},t). \tag{6}$$

Notice that the stress tensors have been defined in terms of the laboratory frame velocities, so that P_{ij} includes the Reynolds stress. By integrating the appropriately weighed Vlasov equation over velocity space (see e.g. Ref.[16] whose notation is largely followed here), one obtains the system of collisionless fluid moment equations:

$$\frac{\partial n}{\partial t} \;+\; \frac{\partial (nu_i)}{\partial x_i} \;=\; 0, \tag{7}$$

$$m\frac{\partial (nu_i)}{\partial t} \;+\; \frac{\partial P_{ij}}{\partial x_j} \;=\; en\Big(E_i \;+\; \epsilon_{ijk}u_jB_k\Big), \tag{8}$$

$$\frac{\partial P_{ij}}{\partial t} \;+\; \frac{\partial M_{ijk}}{\partial x_k} \;=\; enE_{[i}u_{j]} \;+\; \frac{e}{m}\epsilon_{[ikl}P_{kj]}B_l, \tag{9}$$

$$\frac{\partial M_{ijk}}{\partial t} \;+\; \frac{\partial N_{ijkl}}{\partial x_l} \;=\; \frac{e}{m}\Big(E_{[i}P_{jk]} \;+\; \epsilon_{[ilm}M_{ljk]}B_m\Big). \tag{10}$$

In our notation, the square brackets around indices represent the minimal sum over permutations of uncontracted indices needed to yield completely symmetric tensors. Thus, $E_{[i}u_{j]} \equiv E_iu_j + E_ju_i$, $E_{[i}P_{jk]} \equiv E_iP_{jk} + E_jP_{ki} + E_kP_{ij}$ and so on.

The continuity (7) and momentum conservation (8) equations will be assumed to be solved exactly for the particle density and the fluid velocity, once an expression of the stress tensor P_{ij} is provided. Then, eliminating the electric field between Eq.(8) and Eqs.(9,10), one obtains the following pair of equations for the stress and stress flux tensors:

$$\epsilon_{[ikl}P_{kj]}B_l \;=\; mn\epsilon_{[ikl}u_{j]}u_kB_l \;+\; \frac{m}{e}\left[\frac{\partial P_{ij}}{\partial t} \;+\; \frac{\partial M_{ijk}}{\partial x_k} \;-\; m\frac{\partial (nu_{[i})}{\partial t}u_{j]} \;-\; \frac{\partial P_{[ik}}{\partial x_k}u_{j]}\right], \tag{11}$$

$$\epsilon_{[ilm}M_{ljk]}B_m \;=\; \epsilon_{[ilm}P_{jk]}u_lB_m \;+\; \frac{m}{e}\left[\frac{\partial M_{ijk}}{\partial t} \;+\; \frac{\partial N_{ijkl}}{\partial x_l} \;-\; \frac{1}{n}\frac{\partial (nu_{[i})}{\partial t}P_{jk]} \;-\; \frac{1}{mn}\frac{\partial P_{[il}}{\partial x_l}P_{jk]}\right]. \tag{12}$$

The collisionless stress tensor can be represented as the sum of three terms,

$$P_{ij} \;=\; mnu_iu_j \;+\; P^{CGL}_{ij} \;+\; \hat{P}_{ij}, \tag{13}$$

where the first term is the Reynolds stress and

$$P_{ij}^{CGL} + \hat{P}_{ij} \;=\; m \int d^3\mathbf{v}\;(v_i - u_i)(v_j - u_j)\; f(\mathbf{v}, \mathbf{x}, t). \tag{14}$$

The second term in (13) is the CGL tensor, diagonal in a reference frame aligned with the magnetic field,

$$P_{ij}^{CGL} \;=\; p_\perp \delta_{ij} \;+\; (p_\parallel - p_\perp) b_i b_j, \tag{15}$$

where $p_\perp$ and $p_\parallel$ are the perpendicular and parallel pressures, and $\mathbf{b} \equiv \mathbf{B}/B$ is the magnetic unit vector. It is also useful to define the mean scalar pressure $p \equiv (2p_\perp + p_\parallel)/3$. The last term, $\hat{P}_{ij}$, is the gyroviscous stress that satisfies

$$\hat{P}_{ii} \;=\; \hat{P}_{ij} b_i b_j \;=\; 0. \tag{16}$$

Taking this representation to Eq.(11), one gets

$$\epsilon_{[ikl}\hat{P}_{kj]} b_l \;=\; \frac{m}{eB}\left[\frac{\partial P_{ij}}{\partial t} \;+\; \frac{\partial M_{ijk}}{\partial x_k} \;-\; m\frac{\partial (n u_{[i})}{\partial t} u_{j]} \;-\; \frac{\partial P_{[ik}}{\partial x_k} u_{j]}\right], \tag{17}$$

which will be written in shorthand form as

$$\epsilon_{[ikl}\hat{P}_{kj]} b_l \;=\; K_{ij}, \tag{18}$$

where K_{ij} stands identically for the right hand side of (17).

Similarly, the collisionless stress flux tensor can be represented as

$$M_{ijk} \;=\; -2mn u_i u_j u_k \;+\; P_{[ij} u_{k]} \;+\; M_{ijk}^{CGL} \;+\; \hat{M}_{ijk}, \tag{19}$$

where

$$M_{ijk}^{CGL} + \hat{M}_{ijk} \;=\; m \int d^3\mathbf{v}\;(v_i - u_i)(v_j - u_j)(v_k - u_k)\; f(\mathbf{v}, \mathbf{x}, t), \tag{20}$$

$$M_{ijk}^{CGL} \;=\; q_{T\parallel}\delta_{[ij} b_{k]} \;+\; (2q_{B\parallel} - 3q_{T\parallel}) b_i b_j b_k, \tag{21}$$

and

$$\hat{M}_{ijj} b_i \;=\; \hat{M}_{ijk} b_i b_j b_k \;=\; 0. \tag{22}$$

The variables $q_{T\|}$ and $q_{B\|}$ are the parallel fluxes of perpendicular heat and parallel heat respectively. The perpendicular flux of perpendicular heat is the vector with components $q_{T\perp i} \equiv \hat{M}_{ijk}(\delta_{jk} - b_j b_k)/2$, and the perpendicular flux of parallel heat is the vector with components $q_{B\perp i} \equiv \hat{M}_{ijk} b_j b_k/2$. The total parallel and perpendicular heat fluxes are respectively $q_\| \equiv q_{T\|} + q_{B\|}$ and $\mathbf{q}_\perp \equiv \mathbf{q}_{T\perp} + \mathbf{q}_{B\perp}$. Taking this representation of the stress flux tensor to Eq.(12), one gets

$$\epsilon_{[ilm}\hat{M}_{ljk]}b_m \;=\; \frac{m}{eB}\left[\frac{\partial M_{ijk}}{\partial t} \;+\; \frac{\partial N_{ijkl}}{\partial x_l} \;-\; \frac{1}{n}\frac{\partial(nu_{[i})}{\partial t}P_{jk]} \;-\; \frac{1}{mn}\frac{\partial P_{[il}}{\partial x_l}P_{jk]}\right] \;-\; u_{[i}K_{jk]}, \tag{23}$$

which will be written in shorthand form as

$$\epsilon_{[ilm}\hat{M}_{ljk]}b_m \;=\; G_{ijk}. \tag{24}$$

The right hand sides of Eqs.(17) and (23) are proportional to the inverse of the gyrofrequency, $\Omega_c = eB/m$. Therefore the tensors $\hat{P}_{ij}$ and $\hat{M}_{ijk}$ can be ordered at least as $O(\delta)$ quantities. By an algebraic iterative procedure, Eqs.(17) and (23) will provide explicit perturbative solutions for $\hat{P}_{ij}$ and $\hat{M}_{ijk}$ in powers of δ.

III. Fourth rank fluid moment and closure conditions.

For the above system of fluid moment equations to be closed, there remains to specify the fourth rank moment N_{ijkl}. We choose to represent the latter as

$$N_{ijkl} \;=\; 3mnu_iu_ju_ku_l \;-\; P_{[ij}u_ku_{l]} \;+\; M_{[ijk}u_{l]} \;+\; N^{2PC}_{ijkl} \;+\; \tilde{N}_{ijkl}, \tag{25}$$

where

$$N^{2PC}_{ijkl} + \tilde{N}_{ijkl} \;=\; m\int d^3\mathbf{v}\;(v_i-u_i)(v_j-u_j)(v_k-u_k)(v_l-u_l)\;f(\mathbf{v},\mathbf{x},t), \tag{26}$$

and

$$N^{2PC}_{ijkl} \;=\; \frac{m}{n}\left[\int d^3\mathbf{v}\;(v_{[i}-u_{[i})(v_j-u_j)\;f(\mathbf{v},\mathbf{x},t)\right]\left[\int d^3\mathbf{v}\;(v_k-u_k)(v_{l]}-u_{l]})\;f(\mathbf{v},\mathbf{x},t)\right] \tag{27}$$

or, equivalently,

$$N^{2PC}_{ijkl} \;=\; \frac{1}{mn}\left[p_\perp\delta_{[ij} + (p_\| - p_\perp)b_{[i}b_j + \hat{P}_{[ij}\right]\left[p_\perp\delta_{kl]} + (p_\| - p_\perp)b_kb_{l]} + \hat{P}_{kl]}\right]. \tag{28}$$

The first three terms in Eq.(25) which account for the convective part of N_{ijkl}, and the term N_{ijkl}^{2PC} which is the part of the fluid-rest-frame fourth rank moment that can be expressed as a symmetric sum of products of two-point correlations, are known in terms of previously defined variables. The remainder, $\tilde{N}_{ijkl}$, is the term that we cannot determine using fluid arguments alone and will be considered to be the closure variable in our formulation.

Our simplest fluid truncation scheme, a twenty moment generalization of Grad's thirteen moment closure for isotropic neutral gases[17], would therefore be to set $\tilde{N}_{ijkl} = 0$. This yields a non-dissipative model that would include all "purely fluid" effects (convective and diamagnetic), but would not include "purely kinetic" effects such as wave-particle resonances. The expressions for the different components of the stress tensor (in terms of the fluid velocity and heat fluxes) would be exact, but the heat fluxes themselves would not: the expressions for the heat fluxes would include correctly all the terms involving the anisotropic temperature gradient drives, but would miss additional contributions arising from the aforementioned "purely kinetic" effects.

A better, if still not completely rigorous approach would be to use for $\tilde{N}_{ijkl}$ only its zeroth-order form in the small gyroradius expansion. This leaves an expression with only three yet to be determined scalars, which we choose to write as

$$\tilde{N}_{ijkl}^{(0)} \;=\; \frac{1}{2}\,\Big[(\tilde{r}_{\perp}^{(0)} - \tilde{r}_{B\perp}^{(0)})\delta_{[ij}\delta_{kl]} \;+\; (5\tilde{r}_{B\perp}^{(0)} - \tilde{r}_{\perp}^{(0)})\delta_{[ij}b_k b_{l]} \;+\; (4\tilde{r}_{\parallel}^{(0)} + 3\tilde{r}_{\perp}^{(0)} - 35\tilde{r}_{B\perp}^{(0)})b_i b_j b_k b_l\Big]. \quad (29)$$

These three scalars, $\tilde{r}_{\perp}^{(0)}$, $\tilde{r}_{\parallel}^{(0)}$ and $\tilde{r}_{B\perp}^{(0)}$, are the components of the energy-weighed and parallel-energy weighed stress tensors in the fluid-rest-frame, evaluated on the difference between the actual zeroth-order distribution function and a two-temperature-Maxwellian:

$$\tilde{r}_{\perp}^{(0)}\delta_{ij} \;+\; (\tilde{r}_{\parallel}^{(0)} - \tilde{r}_{\perp}^{(0)})b_i b_j \;=\; \frac{m}{2}\int d^3\mathbf{v}\;|\mathbf{v}-\mathbf{u}|^2\;(v_i-u_i)(v_j-u_j)\;(f^{(0)}-f_M), \quad (30)$$

$$\tilde{r}_{B\perp}^{(0)}\delta_{ij} \;+\; (\tilde{r}_{\parallel}^{(0)} - 3\tilde{r}_{B\perp}^{(0)})b_i b_j \;=\; \frac{m}{2}\int d^3\mathbf{v}\;[(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]^2\;(v_i-u_i)(v_j-u_j)\;(f^{(0)}-f_M), \quad (31)$$

or, equivalently,

$$\tilde{r}_{\perp}^{(0)} = \frac{m}{4}\int d^3\mathbf{v}\ |\mathbf{v}-\mathbf{u}|^2\ \left(|\mathbf{v}-\mathbf{u}|^2 - [(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]^2\right)\ (f^{(0)} - f_M), \tag{32}$$

$$\tilde{r}_{\parallel}^{(0)} = \frac{m}{2}\int d^3\mathbf{v}\ |\mathbf{v}-\mathbf{u}|^2\ [(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]^2\ (f^{(0)} - f_M), \tag{33}$$

$$\tilde{r}_{B\perp}^{(0)} = \frac{m}{4}\int d^3\mathbf{v}\ [(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]^2\ \left(|\mathbf{v}-\mathbf{u}|^2 - [(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]^2\right)\ (f^{(0)} - f_M). \tag{34}$$

Here, $f^{(0)} = f^{(0)}(m|\mathbf{v}-\mathbf{u}|^2/2, \lambda, \mathbf{x}, t)$ is the zero-Larmor-radius distribution function which depends on the velocity space coordinates through the fluid-rest-frame energy, $m|\mathbf{v}-\mathbf{u}|^2/2$, and the pitch angle, $\sin\lambda = (\mathbf{v}-\mathbf{u})\cdot\mathbf{b}/|\mathbf{v}-\mathbf{u}|$, but is independent of the gyrophase. The two-temperature-Maxwellian is

$$f_M(m|\mathbf{v}-\mathbf{u}|^2/2, \lambda, \mathbf{x}, t) = \left(\frac{m}{2\pi}\right)^{3/2} \frac{n^{5/2}}{p_\perp\ p_\parallel^{1/2}}\ \exp\left[-\frac{m\ n\ |\mathbf{v}-\mathbf{u}|^2}{2}\left(\frac{\cos^2\lambda}{p_\perp} + \frac{\sin^2\lambda}{p_\parallel}\right)\right]. \tag{35}$$

Actually, since the part of $f^{(0)}$ that is odd in λ does not contribute to the integrals in Eqs.(32-34), only the even part of $f^{(0)} - f_M$ needs to be known here. Various expressions of $f^{(0)} - f_M$ have been derived as approximate solutions of the drift-kinetic equation[5,9,18,19]. Their moments (32-34) allow for non-local models of the Landau damping and other phase-mixing dissipative effects to be incorporated into the fluid formalism.

In any case, for the purposes of the present work, the specific choice of the closure condition on $\tilde{N}_{ijkl}$ will be left open. The different closure variables that stem from the $\tilde{N}_{ijkl}$ term will always be retained and will be denoted by a *tilde*.

IV. Formal solution for the second and third rank fluid moments.

The evolution equations for the stress and stress flux tensors (17,23), also written in shorthand form as (18,24), can be manipulated algebraically to obtain equivalent expressions that make more transparent the asymptotic expansion procedures to be carried out later. Considering Eq.(18) as an algebraic linear inhomogeneous equation for $\hat{P}_{ij}$, its right hand side must satisfy two solubility conditions:

$$K_{ii} = K_{ij}b_ib_j = 0. \tag{36}$$

Then, Eq.(18) can be inverted to yield the formal solution[20,21]

$$\hat{P}_{ij} \;=\; \frac{1}{4}\epsilon_{[ikl}b_k K_{lj]} \;+\; \frac{3}{4}\epsilon_{[ikl}b_{j]}b_k b_m K_{lm} \;, \tag{37}$$

while the solubility conditions (36) provide evolution equations for the two independent components of P_{ij}^{CGL}, $p \equiv (2p_\perp + p_\parallel)/3$ and $p_\parallel$. The condition $K_{ii} = 0$ yields

$$\frac{3}{2}\frac{dp}{dt} \;+\; \frac{5}{2}p\frac{\partial u_i}{\partial x_i} \;+\; (p_\parallel - p_\perp)\left(b_i b_j \frac{\partial u_i}{\partial x_j} - \frac{1}{3}\frac{\partial u_i}{\partial x_i}\right) \;+\; \frac{\partial(q_\parallel b_i + q_{\perp i})}{\partial x_i} \;+\; \hat{P}_{ij}\frac{\partial u_i}{\partial x_j} \;=\; 0, \tag{38}$$

where $d/dt \equiv \partial/\partial t + u_i \partial/\partial x_i$ is the convective time derivative. This, combined with the component of the momentum equation (8) in the direction of $\mathbf{u}$, is equivalent to the energy conservation equation

$$\frac{\partial}{\partial t}\Big(\frac{1}{2}mnu^2 + \frac{3}{2}p\Big) \;+\; \frac{\partial Q_i}{\partial x_i} \;-\; enE_i u_i \;=\; 0, \tag{39}$$

where $Q_i \equiv M_{ijj}/2$ is the total energy flux for the plasma species under consideration. The condition $K_{ij}b_i b_j = 0$ yields

$$\frac{1}{2}\frac{dp_\parallel}{dt} \;+\; \frac{1}{2}p_\parallel\frac{\partial u_i}{\partial x_i} \;+\; p_\parallel b_i b_j \frac{\partial u_i}{\partial x_j} \;+\; \frac{\partial(q_{B\parallel}b_i + q_{B\perp i})}{\partial x_i} \;+\; \frac{q_{T\parallel}}{B}b_i\frac{\partial B}{\partial x_i} \;-$$

$$-\; \hat{P}_{ij}b_i\left(\frac{\partial b_j}{\partial t} + u_k\frac{\partial b_j}{\partial x_k} - b_k\frac{\partial u_k}{\partial x_j}\right) \;-\; \hat{M}_{ijk}b_i\frac{\partial b_j}{\partial x_k} \;=\; 0. \tag{40}$$

Now we can expand the right hand side of (17) taking into account Eqs.(38,40), to obtain the general expression for K_{ij}:

$$K_{ij} \;=\; \frac{m}{eB}\Bigg\{\lambda_2\delta_{ij} \;+\mu_2 b_i b_j \;+\; p_\perp\frac{\partial u_{[i}}{\partial x_{j]}} \;+\; (p_\parallel - p_\perp)\left[\frac{\partial(b_i b_j)}{\partial t} + u_k\frac{\partial(b_i b_j)}{\partial x_k} + b_{[i}b_k\frac{\partial u_{j]}}{\partial x_k}\right] \;+$$

$$+\;\frac{\partial(q_{T\parallel}b_{[i})}{\partial x_{j]}} \;+\; (2q_{B\parallel} - 3q_{T\parallel})b_k\frac{\partial(b_i b_j)}{\partial x_k} \;+\; \frac{\partial\hat{P}_{ij}}{\partial t} \;+\; u_k\frac{\partial\hat{P}_{ij}}{\partial x_k} \;+\; \hat{P}_{ij}\frac{\partial u_k}{\partial x_k} \;+\; \hat{P}_{[ik}\frac{\partial u_{j]}}{\partial x_k} \;+\; \frac{\partial\hat{M}_{ijk}}{\partial x_k}\Bigg\}, \tag{41}$$

where λ_2 and μ_2 are two scalar functions that need not be of concern any longer because they do not contribute to $\hat{P}_{ij}$ when inserted in Eq.(37).

In completely analogous fashion, we consider Eq.(24) as an algebraic linear inhomogeneous equation for $\hat{M}_{ijk}$, which also requires its right hand side to satisfy two solubility conditions:

$$G_{ijj}b_i \;=\; G_{ijk}b_ib_jb_k \;=\; 0. \tag{42}$$

Then, Eq.(24) can be inverted to yield the formal solution

$$\hat{M}_{ijk} \;=\; \frac{1}{3}\epsilon_{[ilm}b_lG_{mjk]} \;-\; \frac{1}{12}\epsilon_{[ilm}b_jb_lb_nG_{mnk]} \;+$$

$$+\; \frac{2}{9}\epsilon_{[ilm}\epsilon_{jnp}\epsilon_{kqr]}b_lb_nb_qG_{mpr} \;+\; \frac{5}{6}\epsilon_{[ilm}b_jb_{k]}b_lb_nb_pG_{mnp}, \tag{43}$$

and the solubility conditions (42) provide evolution equations for the two independent components of M^{CGL}_{ijk}, $q_\parallel \equiv q_{T\parallel} + q_{B\parallel}$ and $q_{B\parallel}$. The condition $G_{ijj}b_i = 0$ yields

$$\frac{dq_\parallel}{dt} \;+\; \left(2q_\parallel - q_{B\parallel}\right)\frac{\partial u_i}{\partial x_i} \;+\; 3q_{B\parallel}b_ib_j\frac{\partial u_i}{\partial x_j} \;+\; \frac{p_\parallel}{m}b_i\frac{\partial}{\partial x_i}\left(\frac{2p_\perp + 3p_\parallel}{2n}\right) \;-\; \frac{p_\perp(p_\parallel - p_\perp)}{mnB}b_i\frac{\partial B}{\partial x_i} \;+$$

$$+\; \frac{1}{m}\hat{P}_{ij}\left[b_i\frac{\partial}{\partial x_j}\left(\frac{2p_\perp + 3p_\parallel}{2n}\right) + \left(\frac{p_\parallel - 2p_\perp}{n}\right)\frac{\partial b_i}{\partial x_j} - 2\left(\frac{p_\parallel - p_\perp}{n}\right)b_ib_k\frac{\partial b_j}{\partial x_k}\right] \;+\; \frac{p_\perp}{m}\frac{\partial}{\partial x_j}\left(\frac{1}{n}b_i\hat{P}_{ij}\right) \;+$$

$$+\; \frac{1}{m}\hat{P}_{ij}b_k\frac{\partial}{\partial x_j}\left(\frac{1}{n}\hat{P}_{ik}\right) \;-\; \frac{1}{2}\hat{M}_{ijj}\left(\frac{\partial b_i}{\partial t} + u_k\frac{\partial b_i}{\partial x_k} - b_k\frac{\partial u_k}{\partial x_i}\right) \;+\; \hat{M}_{ijk}b_i\frac{\partial u_j}{\partial x_k} \;+\; \frac{1}{2}b_i\frac{\partial \tilde{N}_{ijkk}}{\partial x_j} \;=\; 0, \tag{44}$$

and the condition $G_{ijk}b_ib_jb_k = 0$ yields

$$\frac{dq_{B\parallel}}{dt} \;+\; q_{B\parallel}\frac{\partial u_i}{\partial x_i} \;+\; 3q_{B\parallel}b_ib_j\frac{\partial u_i}{\partial x_j} \;+\; \frac{3p_\parallel}{2m}b_i\frac{\partial}{\partial x_i}\left(\frac{p_\parallel}{n}\right) \;+\; \frac{3}{2m}\hat{P}_{ij}\left[b_i\frac{\partial}{\partial x_j}\left(\frac{p_\parallel}{n}\right) - 2\frac{p_\parallel}{n}b_ib_k\frac{\partial b_j}{\partial x_k}\right] \;+$$

$$+\; \frac{3}{2m}\hat{P}_{ij}b_ib_kb_l\frac{\partial}{\partial x_j}\left(\frac{1}{n}\hat{P}_{kl}\right) \;-\; \frac{3}{2}\hat{M}_{ijk}b_ib_j\left(\frac{\partial b_k}{\partial t} + u_l\frac{\partial b_k}{\partial x_l} - b_l\frac{\partial u_l}{\partial x_k}\right) \;+\; \frac{1}{2}b_ib_jb_k\frac{\partial \tilde{N}_{ijkl}}{\partial x_l} \;=\; 0. \tag{45}$$

Finally, we expand the right hand side of (23) taking into account Eqs.(44,45), to obtain the general expression for G_{ijk}:

$$
\begin{aligned}
G_{ijk} \;=\; & \frac{m}{eB}\Bigg[\lambda_3\delta_{[ij}b_{k]} \;+\mu_3 b_i b_j b_k \;+\; q_{T\parallel}\delta_{[ij}\left(\frac{\partial b_{k]}}{\partial t} + u_l\frac{\partial b_{k]}}{\partial x_l} + b_l\frac{\partial u_{k]}}{\partial x_l}\right) \;+ \\
& +\;(2q_{B\parallel} - 3q_{T\parallel})b_{[i}b_j\left(\frac{\partial b_{k]}}{\partial t} + u_l\frac{\partial b_{k]}}{\partial x_l} + b_l\frac{\partial u_{k]}}{\partial x_l}\right) \;+\; q_{T\parallel}b_{[i}\frac{\partial u_j}{\partial x_{k]}} \;+\; \frac{p_\perp}{m}\delta_{[ij}\frac{\partial}{\partial x_{k]}}\left(\frac{p_\perp}{n}\right) \;+ \\
& +\;\frac{p_\perp}{m}b_{[i}b_j\frac{\partial}{\partial x_{k]}}\left(\frac{p_\parallel - p_\perp}{n}\right) \;+\; \frac{p_\perp(p_\parallel - p_\perp)}{mn}\frac{\partial(b_{[i}b_j)}{\partial x_{k]}} \;+\; \frac{(p_\parallel - p_\perp)^2}{mn}b_{[i}b_l\frac{\partial(b_j b_{k]})}{\partial x_l} \;+ \\
& +\;\frac{1}{m}\hat{P}_{[il}\frac{\partial}{\partial x_l}\left(\frac{1}{n}\hat{P}_{jk]}\right) \;+\; \frac{1}{m}\delta_{[ij}\frac{\partial}{\partial x_l}\left(\frac{p_\perp}{n}\right)\hat{P}_{lk]} \;+\; \frac{1}{m}b_{[i}b_j\frac{\partial}{\partial x_l}\left(\frac{p_\parallel - p_\perp}{n}\right)\hat{P}_{lk]} \;+\; \frac{p_\perp}{m}\frac{\partial}{\partial x_{[i}}\left(\frac{1}{n}\hat{P}_{jk]}\right) \;+ \\
& +\;\left(\frac{p_\parallel - p_\perp}{m}\right)b_{[i}b_l\frac{\partial}{\partial x_l}\left(\frac{1}{n}\hat{P}_{jk]}\right) \;+\; \frac{\partial \hat{M}_{ijk}}{\partial t} \;+\; u_l\frac{\partial \hat{M}_{ijk}}{\partial x_l} \;+\; \hat{M}_{ijk}\frac{\partial u_l}{\partial x_l} \;+\; \hat{M}_{[ijl}\frac{\partial u_{k]}}{\partial x_l} \;+\; \frac{\partial \tilde{N}_{ijkl}}{\partial x_l}\Bigg], \quad (46)
\end{aligned}
$$

where, again, λ_3 and μ_3 are two scalar functions that do not contribute to $\hat{M}_{ijk}$ when Eq.(46) is taken to (43).

Equations (37,41), (38,40), (43,46) and (44,45) constitute our "formal solutions" for the $\hat{P}_{ij}$, P_{ij}^{CGL}, $\hat{M}_{ijk}$ and M_{ijk}^{CGL} tensors respectively. These equations are exact and nothing more than an algebraic rearrangement of the original system (7-10). Their advantage is that they are cast in a convenient form that makes it straightforward to carry out a systematic expansion in powers of δ. This will yield the sought after explicit systems of FLR reduced equations, as shown in the next Sections.

V. Perturbative FLR system in the fast dynamics ordering.

Perturbative systems of collisionless fluid equations are based on asymptotic expansions in powers of the ratio $\delta \sim \rho/L \ll 1$ between the gyroradius of the species under consideration and any (i.e. the shortest) characteristic length other than the gyroradii. The fast dynamics ordering assumes the time derivative to be first-order in δ relative to the gyrofrequency, $\partial/\partial t \sim \delta\Omega_c$, and the flow velocity to be of the order of the thermal speed, $u \sim v_{th} \equiv \sqrt{2p/(mn)}$. In this Section, we shall carry out the asymptotic expansion of the collisionless fluid equations under the fast dynamics ordering. This ordering implies that the perpendicular heat flux is a first-order variable, $q_{\perp j} \sim \delta p v_{th}$, but the parallel heat flux is zeroth-order, $q_{\|} \sim p v_{th}$. Also, the gyroviscous stress is first-order relative to the mean scalar pressure, $\hat{P}_{ij} \sim \delta p$, but the pressure anisotropy is zeroth-order, $p_{\|} - p_{\perp} \sim p$. The first significant FLR terms are obtained in the first order of the δ asymptotic expansion. Accordingly, we need to evaluate the first-order gyroviscous stress tensor, $\hat{P}^{(1)}_{ij}$, and the first-order perpendicular stress flux tensor, $\hat{M}^{(1)}_{ijk}$.

Keeping only first-order accuracy and dropping the inconsequential terms proportional to δ_{ij} and $b_i b_j$, Eq.(41) becomes

$$K^{(1)}_{ij} = \frac{m}{eB}\Bigg\{ p_{\perp} \frac{\partial u_{[i}}{\partial x_{j]}} + (p_{\|} - p_{\perp}) \left[\frac{1}{B} b_{[i} \left(\frac{\partial B_{j]}}{\partial t} \right)^{(0)} + u_k \frac{\partial (b_i b_j)}{\partial x_k} + b_{[i} b_k \frac{\partial u_{j]}}{\partial x_k} \right] +$$

$$+ \frac{\partial (q_{T\|} b_{[i})}{\partial x_{j]}} + (2 q_{B\|} - 3 q_{T\|}) b_k \frac{\partial (b_i b_j)}{\partial x_k} \Bigg\}. \tag{47}$$

Here, only the-zeroth order time derivative of the magnetic field is needed, as given by Faraday's law with the electric field derived from the momentum equation (8) in zeroth-order:

$$\left(\frac{\partial \mathbf{B}}{\partial t} \right)^{(0)} = \nabla \times (\mathbf{u} \times \mathbf{B}). \tag{48}$$

Taking this to Eq.(37), one gets the first-order gyroviscous stress tensor

$$\hat{P}^{(1)}_{ij} = b_{[i} h^{(1)}_{\perp j]} + \epsilon_{[ikl} b_k (\delta_{mj]} - b_m b_{j]}) S^{(1)}_{lm}, \tag{49}$$

where the vector with components $h^{(1)}_{\perp j} \equiv b_i \hat{P}^{(1)}_{ij}$ is

$$\mathbf{h}^{(1)}_{\perp} = \frac{m}{eB} \mathbf{b} \times \left[2 p_{\|} (\mathbf{b} \cdot \nabla) \mathbf{u} + p_{\perp} \mathbf{b} \times \omega + \nabla q_{T\|} + 2 (q_{B\|} - q_{T\|}) \kappa \right], \tag{50}$$

$\omega \equiv \nabla \times \mathbf{u}$ is the vorticity, $\kappa \equiv (\mathbf{b} \cdot \nabla) \mathbf{b}$ is the magnetic curvature, and the second rank tensor $S^{(1)}_{ij}$ is

$$S^{(1)}_{ij} = \frac{m}{4eB} \left[p_{\perp} \frac{\partial u_{[i}}{\partial x_{j]}} + q_{T\|} \frac{\partial b_{[i}}{\partial x_{j]}} \right]. \tag{51}$$

Similarly, keeping only first-order accuracy and dropping the terms proportional to $\delta_{[ij} b_{k]}$ and $b_i b_j b_k$, Eq.(46) becomes

$$G^{(1)}_{ijk} \;=\; \frac{m}{eB}\left\{ q_{T\|}\delta_{[ij}\left[\frac{1}{B}\left(\frac{\partial B_{k]}}{\partial t}\right)^{(0)} + u_l\frac{\partial b_{k]}}{\partial x_l} + b_l\frac{\partial u_{k]}}{\partial x_l}\right] + \right.$$

$$+\;(2q_{B\|}-3q_{T\|})b_{[i}b_j\left(\frac{\partial b_{k]}}{\partial t} + u_l\frac{\partial b_{k]}}{\partial x_l} + b_l\frac{\partial u_{k]}}{\partial x_l}\right) \;+\; q_{T\|}b_{[i}\frac{\partial u_j}{\partial x_{k]}} \;+\; \frac{p_\perp}{m}\delta_{[ij}\frac{\partial}{\partial x_{k]}}\left(\frac{p_\perp}{n}\right) +$$

$$\left. +\;\frac{p_\perp}{m}b_{[i}b_j\frac{\partial}{\partial x_{k]}}\left(\frac{p_\|-p_\perp}{n}\right) \;+\; \frac{p_\perp(p_\|-p_\perp)}{mn}\frac{\partial(b_{[i}b_j)}{\partial x_{k]}} \;+\; \frac{(p_\|-p_\perp)^2}{mn}b_{[i}b_l\frac{\partial(b_jb_{k]})}{\partial x_l} \;+\; \frac{\partial\tilde{N}^{(0)}_{ijkl}}{\partial x_l}\right\}. \tag{52}$$

Taking this to Eq.(43), one gets

$$\hat{M}^{(1)}_{ijk} \;=\; 2b_{[i}b_jq^{(1)}_{B\perp k]} \;+\; \frac{1}{2}(\delta_{[ij}-b_{[i}b_j)q^{(1)}_{T\perp k]} \;+\; \epsilon_{[ilm}b_jb_l(\delta_{nk]}-b_nb_{k]})T^{(1)}_{mn}, \tag{53}$$

where the first-order perpendicular heat flux vectors are

$$\mathbf{q}^{(1)}_{B\perp} \;=\; \frac{m}{eB}\mathbf{b}\times\left[\frac{p_\perp}{2m}\nabla\left(\frac{p_\|}{n}\right) + \frac{p_\|(p_\|-p_\perp)}{mn}\kappa + 2q_{B\|}(\mathbf{b}\cdot\nabla)\mathbf{u} + q_{T\|}\mathbf{b}\times\omega\right] \;+\; \tilde{\mathbf{q}}^{(1)}_{B\perp}, \tag{54}$$

$$\mathbf{q}^{(1)}_{T\perp} \;=\; \frac{m}{eB}\mathbf{b}\times\left[\frac{2p_\perp}{m}\nabla\left(\frac{p_\perp}{n}\right) + 4q_{T\|}(\mathbf{b}\cdot\nabla)\mathbf{u}\right] \;+\; \tilde{\mathbf{q}}^{(1)}_{T\perp}, \tag{55}$$

and the second rank tensor $T^{(1)}_{ij}$ is

$$T^{(1)}_{ij} \;=\; \frac{m}{4eB}\left[q_{T\|}\frac{\partial u_{[i}}{\partial x_{j]}} \;+\; \frac{p_\perp(p_\|-p_\perp)}{mn}\frac{\partial b_{[i}}{\partial x_{j]}}\right] \;+\; \tilde{T}^{(1)}_{ij}. \tag{56}$$

These expressions include the closure terms

$$\tilde{\mathbf{q}}^{(1)}_{B\perp} \;=\; \frac{m}{eB}\mathbf{b}\times\left[\nabla\tilde{r}^{(0)}_{B\perp} + (\tilde{r}^{(0)}_\| - 5\tilde{r}^{(0)}_{B\perp})\kappa\right], \tag{57}$$

$$\tilde{\mathbf{q}}^{(1)}_{T\perp} \;=\; \frac{m}{eB}\mathbf{b}\times\left[\nabla(\tilde{r}^{(0)}_\perp - \tilde{r}^{(0)}_{B\perp}) + (5\tilde{r}^{(0)}_{B\perp} - \tilde{r}^{(0)}_\perp)\kappa\right], \tag{58}$$

and

$$\tilde{T}^{(1)}_{ij} \;=\; \frac{m}{2eB}(5\tilde{r}^{(0)}_{B\perp} - \tilde{r}^{(0)}_\perp)\frac{\partial b_{[i}}{\partial x_{j]}}. \tag{59}$$

Shown here in a form that singles out as closure variables the three independent components (32-34) of the $\tilde{N}^{(0)}_{ijkl}$ tensor, the first-order results (53-59) for $\hat{M}^{(1)}_{ijk}$ and (49-51) for $\hat{P}^{(1)}_{ij}$ can be verified to be equivalent to those given in Ref.[4].

Next we consider the pressure evolution equations (38,40) which, keeping first-order accuracy in order to retain the first significant FLR terms, read

$$\frac{3}{2}\frac{dp}{dt} \ + \ \frac{5}{2}p\frac{\partial u_i}{\partial x_i} \ + \ (p_\parallel - p_\perp)\left(b_i b_j \frac{\partial u_i}{\partial x_j} - \frac{1}{3}\frac{\partial u_i}{\partial x_i} \right) \ + \ \frac{\partial (q_\parallel b_i + q^{(1)}_{\perp i})}{\partial x_i} \ + \ \hat{P}^{(1)}_{ij}\frac{\partial u_i}{\partial x_j} \ = \ 0, \tag{60}$$

and

$$\frac{1}{2}\frac{dp_\parallel}{dt} \ + \ \frac{1}{2}p_\parallel\frac{\partial u_i}{\partial x_i} \ + \ p_\parallel b_i b_j \frac{\partial u_i}{\partial x_j} \ + \ \frac{\partial (q_{B\parallel} b_i + q^{(1)}_{B\perp i})}{\partial x_i} \ + \ \frac{q_{T\parallel}}{B} b_i \frac{\partial B}{\partial x_i} \ -$$

$$- \ \hat{P}^{(1)}_{ij} b_i \left[\frac{1}{B}\left(\frac{\partial B_j}{\partial t}\right)^{(0)} + u_k \frac{\partial b_j}{\partial x_k} - b_k \frac{\partial u_k}{\partial x_j} \right] \ - \ \hat{M}^{(1)}_{ijk} b_i \frac{\partial b_j}{\partial x_k} \ = \ 0. \tag{61}$$

Substituting the previous results for the first order stress (49-51) and stress flux (53-59) tensors, one gets the two first-order FLR pressure evolution equations:

$$\frac{3}{2}\frac{dp}{dt} \ + \ \frac{5}{2}p\nabla\cdot\mathbf{u} \ + \ (p_\parallel - p_\perp)\Big\{\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] - \frac{1}{3}\nabla\cdot\mathbf{u}\Big\} \ + \ \nabla\cdot(q_\parallel\mathbf{b} + \mathbf{q}^{(1)}_\perp) \ +$$

$$+ \ \mathbf{h}^{(1)}_\perp \cdot [2(\mathbf{b}\cdot\nabla)\mathbf{u} + \mathbf{b}\times\omega] \ + \ q_{T\parallel}\sigma^{(1)} \ = \ 0, \tag{62}$$

and

$$\frac{1}{2}\frac{dp_\parallel}{dt} \ + \ \frac{1}{2}p_\parallel\nabla\cdot\mathbf{u} \ + \ p_\parallel\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] \ + \ \nabla\cdot(q_{B\parallel}\mathbf{b} + \mathbf{q}^{(1)}_{B\perp}) \ + \ q_{T\parallel}\mathbf{b}\cdot\nabla(\ln\ B) \ +$$

$$+ \ \mathbf{h}^{(1)}_\perp\cdot(\mathbf{b}\times\omega) \ - \ 2\mathbf{q}^{(1)}_{B\perp}\cdot\kappa \ + \ q_{T\parallel}\sigma^{(1)} \ = \ 0, \tag{63}$$

where

$$\sigma^{(1)} \ = \ \frac{m}{4eB}\epsilon_{ijk} b_i \frac{\partial b_{[j}}{\partial x_{l]}}(\delta_{lm} - b_l b_m)\frac{\partial u_{[k}}{\partial x_{m]}}. \tag{64}$$

In Eqs.(62,63), the zeroth-order terms, i.e. those without the $^{(1)}$ superscript, reproduce the CGL equations derived in Ref.[3]. The terms involving the variables $\mathbf{h}^{(1)}_\perp$, $\mathbf{q}^{(1)}_{B\perp}$, $\mathbf{q}^{(1)}_\perp \equiv \mathbf{q}^{(1)}_{B\perp} + \mathbf{q}^{(1)}_{T\perp}$ and

$\sigma^{(1)}$ given by Eqs.(50,54,55,57,58,64), can be shown to be equivalent to the first-order terms derived by Macmahon[4], although they appear here in a more compact form. They provide the most general first-order FLR corrections to the pressure evolution equations of a collisionless plasma species, under the fast dynamics ordering.

The last step, which was not carried out in Ref.[4], is to obtain the first-order equations for the parallel heat fluxes. Keeping first-order accuracy in Eqs.(44,45), we have

$$\frac{dq_\parallel}{dt} \;+\; \left(2q_\parallel - q_{B\parallel}\right)\frac{\partial u_i}{\partial x_i} \;+\; 3q_{B\parallel} b_i b_j \frac{\partial u_i}{\partial x_j} \;+\; \frac{p_\parallel}{m} b_i \frac{\partial}{\partial x_i}\left(\frac{2p_\perp + 3p_\parallel}{2n}\right) \;-\; \frac{p_\perp(p_\parallel - p_\perp)}{mnB} b_i \frac{\partial B}{\partial x_i} \;+$$

$$+\; \frac{1}{m}\hat{P}^{(1)}_{ij}\left[b_i \frac{\partial}{\partial x_j}\left(\frac{2p_\perp + 3p_\parallel}{2n}\right) + \left(\frac{p_\parallel - 2p_\perp}{n}\right)\frac{\partial b_i}{\partial x_j} - 2\left(\frac{p_\parallel - p_\perp}{n}\right) b_i b_k \frac{\partial b_j}{\partial x_k}\right] \;+\; \frac{p_\perp}{m}\frac{\partial}{\partial x_j}\left(\frac{1}{n} b_i \hat{P}^{(1)}_{ij}\right) \;-$$

$$-\; \frac{1}{2}\hat{M}^{(1)}_{ijj}\left[\frac{1}{B}\left(\frac{\partial B_i}{\partial t}\right)^{(0)} + u_k \frac{\partial b_i}{\partial x_k} - b_k \frac{\partial u_k}{\partial x_i}\right] \;+\; \hat{M}^{(1)}_{ijk} b_i \frac{\partial u_j}{\partial x_k} \;+\; \frac{1}{2} b_i \frac{\partial(\tilde{N}^{(0)}_{ijkk} + \tilde{N}^{(1)}_{ijkk})}{\partial x_j} \;=\; 0 \qquad (65)$$

and

$$\frac{dq_{B\parallel}}{dt} \;+\; q_{B\parallel}\frac{\partial u_i}{\partial x_i} \;+\; 3q_{B\parallel} b_i b_j \frac{\partial u_i}{\partial x_j} \;+\; \frac{3p_\parallel}{2m} b_i \frac{\partial}{\partial x_i}\left(\frac{p_\parallel}{n}\right) \;+\; \frac{3}{2m}\hat{P}^{(1)}_{ij}\left[b_i \frac{\partial}{\partial x_j}\left(\frac{p_\parallel}{n}\right) - 2\frac{p_\parallel}{n} b_i b_k \frac{\partial b_j}{\partial x_k}\right] \;-$$

$$-\; \frac{3}{2}\hat{M}^{(1)}_{ijk} b_i b_j \left[\frac{1}{B}\left(\frac{\partial B_k}{\partial t}\right)^{(0)} + u_l \frac{\partial b_k}{\partial x_l} - b_l \frac{\partial u_l}{\partial x_k}\right] \;+\; \frac{1}{2} b_i b_j b_k \frac{\partial(\tilde{N}^{(0)}_{ijkl} + \tilde{N}^{(1)}_{ijkl})}{\partial x_l} \;=\; 0. \qquad (66)$$

Now, substituting the expressions (49-51) for $\hat{P}^{(1)}_{ij}$, (53-59) for $\hat{M}^{(1)}_{ijk}$ and (29) for $\tilde{N}^{(0)}_{ijkl}$, we get the two FLR evolution equations for the parallel heat fluxes:

$$\frac{dq_\parallel}{dt} \;+\; 2q_\parallel \nabla\cdot\mathbf{u} \;+\; q_{B\parallel}\{3\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] - \nabla\cdot\mathbf{u}\} + \frac{p_\parallel}{m}\mathbf{b}\cdot\nabla\left(\frac{2p_\perp + 3p_\parallel}{2n}\right) \;-\; \frac{p_\perp(p_\parallel - p_\perp)}{mn}\mathbf{b}\cdot\nabla(\ln\ B) \;+$$

$$+\; \frac{1}{m}\mathbf{h}^{(1)}_\perp \cdot \left[\nabla\left(\frac{2p_\perp + 3p_\parallel}{2n}\right) - \frac{p_\parallel}{n}\kappa\right] \;+\; \frac{p_\perp}{m}\nabla\cdot\left(\frac{1}{n}\mathbf{h}^{(1)}_\perp\right) \;+\; \mathbf{q}^{(1)}_\perp \cdot (\mathbf{b}\times\omega) \;+\; 2\mathbf{q}^{(1)}_{B\perp}\cdot[2(\mathbf{b}\cdot\nabla)\mathbf{u} + \mathbf{b}\times\omega] \;+$$

$$+\left[\frac{p_\perp^2}{mn} + 2(5\tilde{r}^{(0)}_{B\perp} - \tilde{r}^{(0)}_\perp)\right]\sigma^{(1)} \;+\; \mathbf{b}\cdot\nabla\tilde{r}^{(0)}_\parallel \;+\; (\tilde{r}^{(0)}_\perp - \tilde{r}^{(0)}_\parallel)\mathbf{b}\cdot\nabla(\ln\ B) \;+\; \frac{1}{2}\tilde{n}^{(1)} \;=\; 0 \qquad (67)$$

and

$$\frac{dq_{B\parallel}}{dt} \;+\; q_{B\parallel}\nabla\cdot\mathbf{u} \;+\; 3q_{B\parallel}\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] \;+\; \frac{3p_\parallel}{2m}\mathbf{b}\cdot\nabla\left(\frac{p_\parallel}{n}\right) \;+\; \frac{1}{m}\mathbf{h}^{(1)}_\perp\cdot\left[\nabla\left(\frac{3p_\parallel}{2n}\right) - \frac{3p_\parallel}{n}\kappa\right] +$$

$$+\; 3\mathbf{q}^{(1)}_{B\perp}\cdot(\mathbf{b}\times\omega) \;+\; \mathbf{b}\cdot\nabla(\tilde{r}^{(0)}_\parallel - 2\tilde{r}^{(0)}_{B\perp}) \;+\; (5\tilde{r}^{(0)}_{B\perp} - \tilde{r}^{(0)}_\parallel)\mathbf{b}\cdot\nabla(\ln\ B) \;+\; \frac{1}{2}\tilde{n}^{(1)}_B \;=\; 0, \qquad (68)$$

where the additional first-order closure terms are $\tilde{n}^{(1)} \equiv b_i\partial\tilde{N}^{(1)}_{ijkk}/\partial x_j$ and $\tilde{n}^{(1)}_B \equiv b_ib_jb_k\partial\tilde{N}^{(1)}_{ijkl}/\partial x_l$. The zero-Larmor-radius limit of these parallel heat flux equations, i.e. the terms without the $^{(1)}$ superscript, was derived by different methods in Refs.[5] and [6] (there is a discrepancy with one term of Ref.[5] whose origin has not yet been clarified). Here, the first significant FLR corrections have also been obtained. Notice the symmetry between the structure of these parallel heat flux evolution equations and the pressure evolution equations (62,63). This set (62,63,67,68), along with the first-order explicit formulas for the gyroviscous stress (49-51) and the perpendicular stress flux (53-59), complete the general, fast-dynamics-ordered FLR system.

VI. Perturbative FLR system in the slow dynamics ordering.

This Section will deal with the perturbative expansion of the collisionless fluid system, assuming the slow dynamics ordering. Under this ordering scheme, the time derivative is assumed to be second-order relative to the gyrofrequency, $\partial/\partial t \sim \delta^2\Omega_c$, and the flow velocity is taken as first-order relative to the thermal speed, $u \sim \delta v_{th}$. The parallel and perpendicular heat fluxes are assumed to be comparable, and all the components of the fluid-rest-frame stress flux tensor are first-order quantities, $M^{CGL}_{ijk} \sim \hat{M}_{ijk} \sim \delta p v_{th}$. In the stress tensor, the gyroviscous term is comparable to the Reynolds term and second-order relative to the CGL term, $\hat{P}_{ij} \sim mnu_iu_j \sim \delta^2 P^{CGL}_{ij} \sim \delta^2 p$. The perturbative expansion proceeds by incremental powers of δ^2, and the first significant FLR terms in the system of fluid equations are second-order. The crucial observation is that, unlike the fast dynamics ordering, the the slow dynamics ordering does not lead in general to a strictly consistent asymptotic expansion of the fluid equations. However, since important physical phenomena such as diamagnetic flows and drift waves do take place on the slow dynamics time scale and less than rigorous expansions based on the slow dynamics ordering are widely used, we shall proceed anyway for the sake of completeness. The goal is to go as far as can be justified without invoking additional assumptions (which still will yield a number of useful results) and to draw attention to the unresolved issues.

The fundamental difference in the slow dynamics expansion, compared with the fast dynamics one, is that the zeroth-order terms of the parallel component of the momentum conservation equation and the two parallel heat flux evolution equations, do not involve the dynamical variables that are advanced in time according to these equations. The time derivatives of these variables (the parallel component of the fluid velocity and the two parallel heat fluxes) appear only among the second-order terms, and the zeroth-order terms yield some non-trivial quasi-static constraints that must be satisfied in lowest order by the pressures. Specifically, retaining only the zeroth-order terms, the component of the momentum equation (8) parallel to the magnetic field yields

$$\mathbf{b}\cdot\nabla p_\parallel \;-\; (p_\parallel - p_\perp)\mathbf{b}\cdot\nabla(\ln B) \;-\; en\mathbf{b}\cdot\mathbf{E} \;\simeq\; 0, \tag{69}$$

and the parallel heat flux equations (44,45) yield

$$\frac{p_\parallel}{m}\mathbf{b}\cdot\nabla\left(\frac{2p_\perp + 3p_\parallel}{2n}\right) \;-\; \frac{p_\perp(p_\parallel - p_\perp)}{mn}\mathbf{b}\cdot\nabla(\ln\ B) \;+\; \mathbf{b}\cdot\nabla\tilde{r}_\parallel^{(0)} \;+\; (\tilde{r}_\perp^{(0)} - \tilde{r}_\parallel^{(0)})\mathbf{b}\cdot\nabla(\ln\ B) \simeq 0, \tag{70}$$

$$\frac{3p_\parallel}{2m}\mathbf{b}\cdot\nabla\left(\frac{p_\parallel}{n}\right) \;+\; \mathbf{b}\cdot\nabla(\tilde{r}_\parallel^{(0)} - 2\tilde{r}_{B\perp}^{(0)}) \;+\; (5\tilde{r}_{B\perp}^{(0)} - \tilde{r}_\parallel^{(0)})\mathbf{b}\cdot\nabla(\ln\ B) \;\simeq\; 0, \tag{71}$$

where "approximately equal to zero" means that the left hand sides of Eqs.(69-71) must actually be second-order quantities, comparable to the next-higher-order terms in their respective complete equations. The compatibility of these quasi-static constraints with the independent dynamic evolution equations for the pressures, is a necessary condition for the validity of the slow dynamics asymptotic expansion. Even if we can assume that the above constraints are satisfied (one might think that they specify the closure variables $\tilde{r}_\parallel^{(0)},\ \tilde{r}_\perp^{(0)},\ \tilde{r}_{B\perp}^{(0)}$), we are faced with the fact that to get the leading, first-order solutions for the parallel flow velocity and the parallel heat fluxes, we must consider the FLR, second-order terms of their respective dynamic evolution equations (hence the slow-dynamics-ordered models are intrinsically finite-Larmor-radius). These FLR terms involve the second-order corrections to the stress tensor, i.e. $P_{ij} = O(p) + O(\delta^2 p)$, but only yield first-order-accurate solutions for the parallel flow velocity and the parallel heat fluxes, i.e. $u_\parallel = O(\delta v_{th})$ and $q_\parallel \sim q_{B\parallel} = O(\delta p v_{th})$. However, to obtain the CGL part of the stress tensor accurate to $O(\delta^2 p)$, one must solve the FLR, second-significant-order pressure evolution equations, which require knowledge of the fluid velocity and the heat fluxes to third-order accuracy, i.e. $\mathbf{u} = O(\delta v_{th}) + O(\delta^3 v_{th})$ and $\mathbf{q} = O(\delta p v_{th}) + O(\delta^3 p v_{th})$. Thus the required accuracy in the parallel component of the fluid velocity cannot be achieved consistently. The required accuracy in the parallel heat fluxes cannot be achieved either, and in the case of the perpendicular heat fluxes it is not practical. These difficulties, which do not arise in the fast dynamics ordering scheme, are most often glosssed over in the slow dynamics or "drift ordering" based literature.

With the above cautions in mind, let us proceed formally with the slow dynamics expansion. First, we evaluate the first-order stress flux tensor $\hat{M}^{(1)}_{ijk}$. The result is the one obtained under the fast dynamics scheme, without the terms involving products of the parallel heat fluxes and the gradients of the fluid velocity which are now two orders higher in δ. Thus $\hat{M}^{(1)}_{ijk}$ is given by an expression identical to Eq.(53), where now we have

$$\mathbf{q}^{(1)}_{B\perp} = \frac{m}{eB}\mathbf{b}\times\left[\frac{p_\perp}{2m}\nabla\left(\frac{p_\parallel}{n}\right) + \frac{p_\parallel(p_\parallel - p_\perp)}{mn}\kappa + \nabla\tilde{r}^{(0)}_{B\perp} + (\tilde{r}^{(0)}_\parallel - 5\tilde{r}^{(0)}_{B\perp})\kappa\right], \tag{72}$$

$$\mathbf{q}^{(1)}_{T\perp} = \frac{m}{eB}\mathbf{b}\times\left[\frac{2p_\perp}{m}\nabla\left(\frac{p_\perp}{n}\right) + \nabla(\tilde{r}^{(0)}_\perp - \tilde{r}^{(0)}_{B\perp}) + (5\tilde{r}^{(0)}_{B\perp} - \tilde{r}^{(0)}_\perp)\kappa\right], \tag{73}$$

and

$$T^{(1)}_{ij} = \frac{m}{4eB}\left[\frac{p_\perp(p_\parallel - p_\perp)}{mn} + 2(5\tilde{r}^{(0)}_{B\perp} - \tilde{r}^{(0)}_\perp)\right]\frac{\partial b_{[i}}{\partial x_{j]}}. \tag{74}$$

The second step is to evaluate the gyroviscous stress tensor, whose perturbative expansion begins now in second order. Keeping second-order accuracy and dropping terms proportional to δ_{ij} and $b_i b_j$, Eq.(41) gives

$$K^{(2)}_{ij} = \frac{m}{eB}\left\{p_\perp\frac{\partial u_{[i}}{\partial x_{j]}} + (p_\parallel - p_\perp)\left[\frac{1}{B}b_{[i}\left(\frac{\partial B_{j]}}{\partial t}\right)^{(1)} + u_k\frac{\partial(b_i b_j)}{\partial x_k} + b_{[i}b_k\frac{\partial u_{j]}}{\partial x_k}\right] + \right.$$

$$\left. + \frac{\partial(q_{T\parallel}b_{[i})}{\partial x_{j]}} + (2q_{B\parallel} - 3q_{T\parallel})b_k\frac{\partial(b_i b_j)}{\partial x_k} + \frac{\partial\hat{M}^{(1)}_{ijk}}{\partial x_k}\right\}. \tag{75}$$

Here, a first-order time derivative of the magnetic field is needed. This is provided by Faraday's law with a first-order electric field derived from the slow-dynamics-ordered momentum equation:

$$\left(\frac{\partial \mathbf{B}}{\partial t}\right)^{(1)} \;=\; \nabla\times\left\{\mathbf{u}\times\mathbf{B} \;-\; \frac{1}{en}\left[\nabla p_{\perp} \;+\; (\mathbf{B}\cdot\nabla)\left(\frac{p_{\parallel}-p_{\perp}}{B^2}\;\mathbf{B}\right)\right]\right\}. \tag{76}$$

The algebra will not be carried any further, but all the terms in $K_{ij}^{(2)}$ are now explicitly known. The second-order gyroviscosity is

$$\hat{P}_{ij}^{(2)} \;=\; \frac{1}{4}\epsilon_{[ikl}b_k(\delta_{mj]}+3b_m b_{j]})K_{lm}^{(2)}. \tag{77}$$

Next we turn to the pressure evolution equations. Under the slow dynamics ordering, the lowest-order terms in Eqs.(38,40) are already of order $\delta p v_{th}/L$, which we consider as first-order. Therefore, in order to retain the first significant FLR terms, it is necessary to expand them keeping third-order accuracy:

$$\frac{3}{2}\frac{dp}{dt} \;+\; \frac{5}{2}p\frac{\partial u_i}{\partial x_i} \;+\; (p_{\parallel}-p_{\perp})\left(b_i b_j\frac{\partial u_i}{\partial x_j}-\frac{1}{3}\frac{\partial u_i}{\partial x_i}\right) \;+\; \frac{\partial(q_{\parallel}b_i+q_{\perp i}^{(1)}+q_{\perp i}^{(3)})}{\partial x_i} \;+\; \hat{P}_{ij}^{(2)}\frac{\partial u_i}{\partial x_j} \;=\; 0, \tag{78}$$

and

$$\frac{1}{2}\frac{dp_{\parallel}}{dt} \;+\; \frac{1}{2}p_{\parallel}\frac{\partial u_i}{\partial x_i} \;+\; p_{\parallel}b_i b_j\frac{\partial u_i}{\partial x_j} \;+\; \frac{\partial(q_{B\parallel}b_i+q_{B\perp i}^{(1)}+q_{B\perp i}^{(3)})}{\partial x_i} \;+\; \frac{q_{T\parallel}}{B}b_i\frac{\partial B}{\partial x_i} \;-$$

$$-\;\hat{P}_{ij}^{(2)}b_i\left[\frac{1}{B}\left(\frac{\partial B_j}{\partial t}\right)^{(1)}+u_k\frac{\partial b_j}{\partial x_k}-b_k\frac{\partial u_k}{\partial x_j}\right] \;-\; (\hat{M}_{ijk}^{(1)}+\hat{M}_{ijk}^{(3)})b_i\frac{\partial b_j}{\partial x_k} \;=\; 0. \tag{79}$$

These are the slow dynamics FLR pressure equations and they involve the third-order perpendicular stress flux tensor, $\hat{M}_{ijk}^{(3)}$, which remains to be evaluated (recall that the third-order perpendicular heat flux vectors are $q_{\perp i}^{(3)}=\hat{M}_{ijj}^{(3)}/2$ and $q_{B\perp i}^{(3)}=\hat{M}_{ijk}^{(3)}b_j b_k/2$). In the present formulation, the pressures, the parallel heat fluxes and the flow velocity are not expanded as explicit series in powers of δ. Instead, they are considered to be exact solutions of their respective dynamic evolution differential equations for whatever approximate coefficient functions are available. The accuracy of these solutions is obviously only as good as that of the coefficient functions. Accordingly, the solution for the parallel heat fluxes $q_{\parallel}$ and $q_{B\parallel}$ to be used in Eqs.(78,79) should also be accurate to $O(\delta^3 p v_{th})$, and the solution for the flow velocity should be accurate to $O(\delta^3 v_{th})$. As mentioned earlier, this loop cannot be closed consistently because the second-order-accurate pressure solutions derived from Eqs.(78,79) only guarantee a first-order-accurate solution for the parallel flow velocity when taken to the slow-dynamics-ordered parallel component of the momentum conservation equation:

$$mn\, b_i\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) \; + \; b_i\frac{\partial p_\parallel}{\partial x_i} \; - \; (p_\parallel - p_\perp)b_i\frac{\partial(\ln B)}{\partial x_i} \; + \; b_i\frac{\partial \hat{P}^{(2)}_{ij}}{\partial x_j} \; - \; enb_iE_i \; = \; 0, \qquad (80)$$

where the quasi-static constraint (69) is assumed to be satisfied.

Finally, let us examine the parallel heat flux equations. Assuming that the quasi-static constraints (70,71) are satisfied, the first surviving terms in Eqs.(44,45) are of order $\delta^2 p v_{th}^2/L$, which we consider as second-order. To get the next correction that would yield parallel heat flux solutions accurate to $O(\delta^3 p v_{th})$, Eqs.(44,45) would have to be expanded keeping fourth-order accuracy. This would require knowledge of solutions for $u_\parallel$ accurate to $O(\delta^3 v_{th})$ and for P^{CGL}_{ij} accurate to $O(\delta^4 p)$, which are not available. Also, the fourth-order parallel heat flux equations would involve the previously encountered third-order $\hat{M}^{(3)}_{ijk}$ plus the fourth-order $\hat{P}^{(4)}_{ij}$ and $\tilde{N}^{(4)}_{ijkl}$ and a double product of the second-order $\hat{P}^{(2)}_{ij}$, whose evaluation is an impractical task. The best course of action is to consider only the second-order parallel heat flux equations, which yield solutions accurate to $O(\delta p v_{th})$, and treat the third-order corrections to the parallel heat fluxes as additional unspecified terms. These can be lumped together with the still undetermined third-order corrections to the perpendicular heat fluxes, $\mathbf{q}^{(3)}_\perp$, $\mathbf{q}^{(3)}_{B\perp}$, in the pressure equations (78,79). Retaining second-order accuracy with the slow dynamics ordering, the parallel heat flux equations (44,45) become

$$\frac{dq_\parallel}{dt} \; + \; \Big(2q_\parallel - q_{B\parallel}\Big)\frac{\partial u_i}{\partial x_i} \; + \; 3q_{B\parallel}b_ib_j\frac{\partial u_i}{\partial x_j} \; + \; \frac{p_\parallel}{m}b_i\frac{\partial}{\partial x_i}\left(\frac{2p_\perp + 3p_\parallel}{2n}\right) \; - \; \frac{p_\perp(p_\parallel - p_\perp)}{mnB}b_i\frac{\partial B}{\partial x_i} \; +$$

$$+ \; \frac{1}{m}\hat{P}^{(2)}_{ij}\left[b_i\frac{\partial}{\partial x_j}\left(\frac{2p_\perp + 3p_\parallel}{2n}\right) + \left(\frac{p_\parallel - 2p_\perp}{n}\right)\frac{\partial b_i}{\partial x_j} - 2\left(\frac{p_\parallel - p_\perp}{n}\right)b_ib_k\frac{\partial b_j}{\partial x_k}\right] \; + \; \frac{p_\perp}{m}\frac{\partial}{\partial x_j}\left(\frac{1}{n}b_i\hat{P}^{(2)}_{ij}\right) \; -$$

$$- \; \frac{1}{2}\hat{M}^{(1)}_{ijj}\left[\frac{1}{B}\left(\frac{\partial B_i}{\partial t}\right)^{(1)} + u_k\frac{\partial b_i}{\partial x_k} - b_k\frac{\partial u_k}{\partial x_i}\right] \; + \; \hat{M}^{(1)}_{ijk}b_i\frac{\partial u_j}{\partial x_k} \; + \; \frac{1}{2}b_i\frac{\partial(\tilde{N}^{(0)}_{ijkk} + \tilde{N}^{(2)}_{ijkk})}{\partial x_j} = O(\delta^4 p v_{th}^2/L) \, (81)$$

and

$$\frac{dq_{B\parallel}}{dt} \;+\; q_{B\parallel}\frac{\partial u_i}{\partial x_i} \;+\; 3q_{B\parallel}b_ib_j\frac{\partial u_i}{\partial x_j} \;+\; \frac{3p_\parallel}{2m}b_i\frac{\partial}{\partial x_i}\left(\frac{p_\parallel}{n}\right) \;+\; \frac{3}{2m}\hat{P}^{(2)}_{ij}\left[b_i\frac{\partial}{\partial x_j}\left(\frac{p_\parallel}{n}\right) - 2\frac{p_\parallel}{n}b_ib_k\frac{\partial b_j}{\partial x_k}\right] -$$

$$-\;\frac{3}{2}\hat{M}^{(1)}_{ijk}b_ib_j\left[\frac{1}{B}\left(\frac{\partial B_k}{\partial t}\right)^{(1)} + u_l\frac{\partial b_k}{\partial x_l} - b_l\frac{\partial u_l}{\partial x_k}\right] \;+\; \frac{1}{2}b_ib_jb_k\frac{\partial(\tilde{N}^{(0)}_{ijkl} + \tilde{N}^{(2)}_{ijkl})}{\partial x_l} \;=\; O(\delta^4 p v_{th}^2/L). \qquad (82)$$

Here we have kept a reminder of the terms of order $\delta^4 p v_{th}^2/L$ that would be necessary to obtain the desirable third-order-accurate parallel heat flux solution, but cannot be evaluated with the slow dynamics ordering scheme. Analogous to the case of the parallel momentum equation (80), the parallel heat flux equations (81,82) require the subsidiary quasistatic constraints (70,71) to be satisfied and yield only first-order-accurate solutions for $q_\parallel$ and $q_{B\parallel}$.

VII. Slow dynamics equations with weak anisotropy.

The slow dynamics analysis of the previous Section assumed a strong anisotropy, $p_\parallel - p_\perp \sim p$, as should be appropriate in the absence of collisions. However, most slow dynamics studies rely also on the weak anisotropy ordering, $p_\parallel - p_\perp \sim \delta^2 p$, such that the anisotropic part of the CGL stress (also sometimes referred to as "parallel viscosity") is comparable to the gyroviscous stress. This is the natural ordering in high collisionality regimes, but it cannot be justified in principle at low collisionality, except for some special situations such as axisymmetric equilibria with closed magnetic surfaces. The slow-dynamics-ordered fluid equations become much simpler in the case of weak anisotropy. So, it is worthwhile to investigate the conditions under which a weakly anisotropic limit of our slow dynamics collisionless equations could be established.

In our collisionless formulation, the weak anisotropy limit corresponds to assuming the orderings $p_\parallel - p_\perp \sim \delta^2 p$, $2q_{B\parallel} - 3q_{T\parallel} \sim \delta^2 q_\parallel$, $\tilde{r}^{(0)}_\parallel - \tilde{r}^{(0)}_\perp \sim \delta^2 \tilde{r}^{(0)}$ and $5\tilde{r}^{(0)}_{B\perp} - \tilde{r}^{(0)}_\perp \sim \delta^2 \tilde{r}^{(0)}$, where $\tilde{r}^{(0)} \equiv (2\tilde{r}^{(0)}_\perp + \tilde{r}^{(0)}_\parallel)/3$. With these orderings, the first quasi-static constraint (69) reduces to

$$\mathbf{b}\cdot\nabla p \;-\; en\mathbf{b}\cdot\mathbf{E} \;=\; O(\delta^2 p/L), \qquad (83)$$

and the other two (70,71) are fulfilled simultaneously with

$$\frac{5p}{2m}\mathbf{b}\cdot\nabla\left(\frac{p}{n}\right) \;+\; \mathbf{b}\cdot\nabla\tilde{r}^{(0)} \;=\; O(\delta^2 p v_{th}^2/L). \tag{84}$$

To obtain the first-order perpendicular stress flux tensor, we bring the weak anisotropy orderings to Eqs.(72-74). Retaining only first-order accuracy, we get

$$5\mathbf{q}_{B\perp}^{(1)} \;=\; \frac{5}{4}\mathbf{q}_{T\perp}^{(1)} \;=\; \mathbf{q}_{\perp}^{(1)} \;=\; \frac{m}{eB}\mathbf{b}\times\left[\frac{5p}{2m}\nabla\left(\frac{p}{n}\right)+\nabla\tilde{r}^{(0)}\right], \tag{85}$$

and

$$T_{ij}^{(1)} \;=\; 0. \tag{86}$$

Thus, the full first-order perpendicular stress flux tensor (53) reduces to

$$\hat{M}_{ijk}^{(1)} \;=\; \frac{2}{5}\delta_{[ij}q_{\perp k]}^{(1)}. \tag{87}$$

Now we can evaluate the second-order gyroviscous stress tensor. Bringing the expression (87) for $\hat{M}_{ijk}^{(1)}$ as well as the weak anisotropy orderings to Eq.(75), and keeping second-order accuracy, we get

$$K_{ij}^{(2)} \;=\; \frac{m}{eB}\left[p\frac{\partial u_{[i}}{\partial x_{j]}} \;+\; \frac{2}{5}\frac{\partial(q_{\parallel}b_{[i}+q_{\perp[i)}^{(1)})}{\partial x_{j]}}\right], \tag{88}$$

hence

$$\hat{P}_{ij}^{(2)} \;=\; \frac{1}{4}\epsilon_{[ikl}b_k(\delta_{mj]}+3b_m b_{j]})K_{lm}^{(2)} \;=\; b_{[i}h_{\perp j]}^{(2)} \;+\; \epsilon_{[ikl}b_k(\delta_{mj]}-b_m b_{j]})S_{lm}^{(2)}, \tag{89}$$

where

$$\mathbf{h}_{\perp}^{(2)} \;=\; \frac{m}{eB}\mathbf{b}\times\left\{p\Big[2(\mathbf{b}\cdot\nabla)\mathbf{u}+\mathbf{b}\times\omega\Big]+\frac{2}{5}(\nabla q_{\parallel}+q_{\parallel}\kappa)+\frac{2}{5}\Big[2(\mathbf{b}\cdot\nabla)\mathbf{q}_{\perp}^{(1)}+\mathbf{b}\times(\nabla\times\mathbf{q}_{\perp}^{(1)})\Big]\right\} \tag{90}$$

and

$$S_{ij}^{(2)} \;=\; \frac{m}{4eB}\left[p\frac{\partial u_{[i}}{\partial x_{j]}} \;+\; \frac{2}{5}q_{\parallel}\frac{\partial b_{[i}}{\partial x_{j]}}+\frac{2}{5}\frac{\partial q_{\perp[i}^{(1)}}{\partial x_{j]}}\right]. \tag{91}$$

This expression (88,89) for the second order gyroviscous stress tensor has the same form as the one derived in high collisionality theories under the slow dynamics ordering[2,16,20,22]. However, there are important differences in the way the heat flux vectors are determined. First, instead of being given by a collisional expression, the parallel heat flux is determined by its own dynamic evolution equation in our collisionless case. Second, our collisionless perpendicular heat flux (85) has the additional term proportional to $\mathbf{b}\times\nabla\tilde{r}^{(0)}$ besides the conventional diamagnetic term proportional to $\mathbf{b}\times\nabla(p/n)$. This additional closure term accounts for "strictly kinetic" effects such as the Landau damping.

Finally, we consider the evolution equations for the pressures and the parallel heat fluxes. It is now convenient to use p, $(p_\parallel - p_\perp)$, $q_\parallel$ and $(2q_{B\parallel} - 3q_{T\parallel})$ as the four independent CGL variables. Also, it is convenient to use $\tilde{r}^{(0)}$, $(\tilde{r}^{(0)}_\parallel - \tilde{r}^{(0)}_\perp)$ and $(5\tilde{r}^{(0)}_{B\perp} - \tilde{r}^{(0)}_\parallel)$ as the three independent zeroth-order closure variables. With the weak anisotropy orderings, and taking into account the above weak anisotropy results, the slow dynamics parallel heat flux equations (81,82) become

$$\frac{dq_\parallel}{dt} + q_\parallel\left(\frac{7}{5}\frac{\partial u_i}{\partial x_i} + \frac{9}{5}b_i b_j\frac{\partial u_i}{\partial x_j}\right) + \left[\frac{5p}{2m} + \frac{5(p_\parallel - p_\perp)}{3m}\right] b_i\frac{\partial}{\partial x_i}\left(\frac{p}{n}\right) + \frac{2p}{3m}b_i\frac{\partial}{\partial x_i}\left(\frac{p_\parallel - p_\perp}{n}\right) -$$

$$- \frac{p(p_\parallel - p_\perp)}{mnB}b_i\frac{\partial B}{\partial x_i} + \frac{1}{m}\hat{P}^{(2)}_{ij}\left[b_i\frac{\partial}{\partial x_j}\left(\frac{5p}{2n}\right) - \frac{p}{n}\frac{\partial b_i}{\partial x_j}\right] + \frac{p}{m}\frac{\partial}{\partial x_i}\left(\frac{1}{n}b_i\hat{P}^{(2)}_{ij}\right) -$$

$$- q^{(1)}_{\perp i}\left[\frac{1}{B}\left(\frac{\partial B_i}{\partial t}\right)^{(1)} + u_j\frac{\partial b_i}{\partial x_j} - \frac{7}{5}b_j\frac{\partial u_j}{\partial x_i} - \frac{2}{5}b_j\frac{\partial u_i}{\partial x_j}\right] +$$

$$+ b_i\frac{\partial}{\partial x_i}\left[\tilde{r}^{(0)} + \frac{2}{3}(\tilde{r}^{(0)}_\parallel - \tilde{r}^{(0)}_\perp)\right] - \frac{(\tilde{r}^{(0)}_\parallel - \tilde{r}^{(0)}_\perp)}{B}b_i\frac{\partial B}{\partial x_i} + \frac{1}{2}\tilde{n}^{(2)} = O(\delta^4 p v_{th}^2/L) \qquad (92)$$

and

$$\frac{dq_\parallel}{dt} + q_\parallel\left(\frac{\partial u_i}{\partial x_i} + 3b_i b_j\frac{\partial u_i}{\partial x_j}\right) + \left[\frac{5p}{2m} + \frac{5(p_\parallel - p_\perp)}{3m}\right] b_i\frac{\partial}{\partial x_i}\left(\frac{p}{n}\right) + \frac{5p}{3m}b_i\frac{\partial}{\partial x_i}\left(\frac{p_\parallel - p_\perp}{n}\right) +$$

$$+ \frac{1}{m}\hat{P}^{(2)}_{ij}\left[b_i\frac{\partial}{\partial x_j}\left(\frac{5p}{2n}\right) - \frac{5p}{n}b_i b_k\frac{\partial b_j}{\partial x_k}\right] - q^{(1)}_{\perp i}\left[\frac{1}{B}\left(\frac{\partial B_i}{\partial t}\right)^{(1)} + u_j\frac{\partial b_i}{\partial x_j} - b_j\frac{\partial u_j}{\partial x_i}\right] +$$

$$+ b_i\frac{\partial}{\partial x_i}\left[\tilde{r}^{(0)} + \frac{2}{3}(\tilde{r}^{(0)}_\parallel - \tilde{r}^{(0)}_\perp) - \frac{2}{3}(5\tilde{r}^{(0)}_{B\perp} - \tilde{r}^{(0)}_\parallel)\right] + \frac{5(5\tilde{r}^{(0)}_{B\perp} - \tilde{r}^{(0)}_\parallel)}{3B}b_i\frac{\partial B}{\partial x_i} + \frac{5}{6}\tilde{n}^{(2)}_B = O(\delta^4 p v_{th}^2/L), \quad (93)$$

where the second-order closure terms are $\tilde{n}^{(2)} \equiv b_i\partial\tilde{N}^{(2)}_{ijkk}/\partial x_j$ and $\tilde{n}^{(2)}_B \equiv b_i b_j b_k\partial\tilde{N}^{(2)}_{ijkl}/\partial x_l$. As anticipated, the zeroth-order limit of these two equations is compatible with the single quasi-static constraint given in Eq.(84). These second-order parallel heat flux equations do not involve the third-order variable $(2q_{B\|} - 3q_{T\|})$. Instead, Eqs.(92,93) provide one dynamic evolution equation for $q_\|$, and one additional quasi-static constraint that results from their difference:

$$\frac{2}{5}q_\| \left(3b_i b_j \frac{\partial u_i}{\partial x_j} - \frac{\partial u_i}{\partial x_i}\right) \; + \; \frac{p}{m} b_i \frac{\partial}{\partial x_i}\left(\frac{p_\| - p_\perp}{n}\right) \; + \; \frac{p(p_\| - p_\perp)}{mnB} b_i \frac{\partial B}{\partial x_i} \; +$$

$$+ \; \frac{p}{mn}\hat{P}^{(2)}_{ij}\left(\frac{\partial b_i}{\partial x_j} - 5 b_i b_k \frac{\partial b_j}{\partial x_k}\right) \; - \; \frac{p}{m}\frac{\partial}{\partial x_i}\left(\frac{1}{n} b_i \hat{P}^{(2)}_{ij}\right) \; - \; \frac{2}{5} q^{(1)}_{\perp i} b_j \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \; -$$

$$- \; \frac{2}{3} b_i \frac{\partial (5\tilde{r}^{(0)}_{B\perp} - \tilde{r}^{(0)}_\|)}{\partial x_i} \; + \; \frac{3(\tilde{r}^{(0)}_\| - \tilde{r}^{(0)}_\perp) + 5(5\tilde{r}^{(0)}_{B\perp} - \tilde{r}^{(0)}_\|)}{3B} b_i \frac{\partial B}{\partial x_i} \; + \; \frac{5}{6}\tilde{n}^{(2)}_B \; - \; \frac{1}{2}\tilde{n}^{(2)} = O(\delta^4 p v_{th}^2 / L). \quad (94)$$

Of the two slow dynamics pressure equations (78,79), the first one is already written in its most convenient form. As the second independent one, it is now useful to take the linear combination that produces the time derivative of $(p_\| - p_\perp)$:

$$\frac{d(p_\| - p_\perp)}{dt} \; + \; (p_\| - p_\perp)\left(\frac{4}{3}\frac{\partial u_i}{\partial x_i} + b_i b_j \frac{\partial u_i}{\partial x_j}\right) \; + \; p\left(3 b_i b_j \frac{\partial u_i}{\partial x_j} - \frac{\partial u_i}{\partial x_i}\right) \; +$$

$$+ \; \frac{1}{5} \; \frac{\partial\left[4q_\| b_i + 3(2q_{B\|} - 3q_{T\|})b_i - 2q^{(1)}_{\perp i} - 5q^{(3)}_{\perp i} + 15 q^{(3)}_{B\perp i}\right]}{\partial x_i} \; + \; \frac{6q_\| - 3(2q_{B\|} - 3q_{T\|})}{5B} b_i \frac{\partial B}{\partial x_i} \; -$$

$$- \; \hat{P}^{(2)}_{ij}\left\{3b_i\left[\frac{1}{B}\left(\frac{\partial B_j}{\partial t}\right)^{(1)} + u_k \frac{\partial b_j}{\partial x_k} - b_k \frac{\partial u_k}{\partial x_j}\right] + \frac{\partial u_i}{\partial x_j}\right\} \; - \; \frac{6}{5} q^{(1)}_{\perp i} b_j \frac{\partial b_i}{\partial x_j} \; - \; 3\hat{M}^{(3)}_{ijk} b_i \frac{\partial b_j}{\partial x_k} \; = \; 0. \quad (95)$$

This is the only equation in the weak-anisotropy-ordered system that involves the third order variable $(2q_{B\|} - 3q_{T\|})$. Therefore we may consider it to be decoupled from the rest, and assume $(p_\| - p_\perp)$ to be determined by Eq.(94). However, the leading terms, of order $\delta p v_{th}/L$, in Eq.(95) involve neither $(2q_{B\|} - 3q_{T\|})$ nor $(p_\| - p_\perp)$. The near cancellation of these terms imposes another quasi-static constraint, necessary for the consistency of the weak anisotropy assumption:

$$p\Big\{3\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] - \nabla\cdot\mathbf{u}\Big\} + \frac{2}{5}\nabla\cdot\left(2q_\|\mathbf{b} - \mathbf{q}^{(1)}_\perp\right) + \frac{6}{5}\left[q_\|\mathbf{b}\cdot\nabla(\ln\, B) - \mathbf{q}^{(1)}_\perp\cdot\kappa\right] = O(\delta^3 p v_{th}/L). \quad (96)$$

Summarizing, with the weak anisotropy and slow dynamics assumptions, compact expressions have been obtained for the perpendicular heat fluxes (85) and the gyroviscous stress tensor (89-91). Only the three CGL variables p, $(p_\parallel - p_\perp)$ and $q_\parallel$ are involved in the coupled system of equations. The mean pressure p and the total parallel heat flux $q_\parallel$ are determined by their dynamic evolution equations (78) and (93), whereas the variation of the pressure anisotropy $(p_\parallel - p_\perp)$ along the magnetic field is determined by the quasi-static equation (94). In addition, three quasi-static consistency equations must be satisfied independently, namely Eq.(96) for the validity of the weak anisotropy ordering and Eqs.(83,84) for the validity of the slow dynamics ordering. The fulfillment of these consistency constraints must be verified on a case by case basis. Bringing the weak anisotropy form of the gyroviscosity (89-91) to the pressure equation (78), the latter becomes:

$$\frac{3}{2}\frac{dp}{dt} \ + \ \frac{5}{2}p\nabla\cdot\mathbf{u} \ + \ (p_\parallel - p_\perp)\Big\{\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] - \frac{1}{3}\nabla\cdot\mathbf{u}\Big\} \ + \ \nabla\cdot(q_\parallel\mathbf{b} + \mathbf{q}_\perp^{(1)} + \mathbf{q}_\perp^{(3)}) \ +$$

$$+ \ \mathbf{h}_\perp^{(2)}\cdot[2(\mathbf{b}\cdot\nabla)\mathbf{u} + \mathbf{b}\times\omega] \ + \ \frac{2}{5}q_\parallel\sigma^{(2)} \ + \ \tau_u^{(3)} \ = \ 0, \tag{97}$$

where the scalar $\sigma^{(2)}$ is the same defined in Eq.(64) as $\sigma^{(1)}$ (only now being labeled second-order because $\mathbf{u}$ is first-order) and the third-order scalar $\tau_u^{(3)}$ is

$$\tau_u^{(3)} \ = \ \frac{m}{10eB}\epsilon_{ijk}b_i\frac{\partial q_{\perp[j}^{(1)}}{\partial x_{l]}}(\delta_{lm} - b_l b_m)\frac{\partial u_{[k}}{\partial x_{m]}}. \tag{98}$$

Similarly, the parallel heat flux equation (93) becomes:

$$\frac{dq_\parallel}{dt} \ + \ q_\parallel\Big\{\nabla\cdot\mathbf{u} + 3\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}]\Big\} \ + \ \left[\frac{5p}{2m} + \frac{5(p_\parallel - p_\perp)}{3m}\right]\mathbf{b}\cdot\nabla\left(\frac{p}{n}\right) \ + \ \frac{5p}{3m}\mathbf{b}\cdot\nabla\left(\frac{p_\parallel - p_\perp}{n}\right) \ +$$

$$+ \ \frac{1}{m}\mathbf{h}_\perp^{(2)}\cdot\left[\nabla\left(\frac{5p}{2n}\right) - \frac{5p}{n}\kappa\right] \ + \ \mathbf{q}_\perp^{(1)}\cdot\left(\mathbf{b}\times\omega - \frac{1}{eBn^2}\nabla n\times\nabla p\right) \ + \ \frac{5}{6}\tilde{n}_B^{(2)} \ +$$

$$+ \ \mathbf{b}\cdot\nabla\Big[\tilde{r}^{(0)} + \frac{2}{3}(\tilde{r}_\parallel^{(0)} - \tilde{r}_\perp^{(0)}) - \frac{2}{3}(5\tilde{r}_{B\perp}^{(0)} - \tilde{r}_\parallel^{(0)})\Big] + \frac{5}{3}(5\tilde{r}_{B\perp}^{(0)} - \tilde{r}_\parallel^{(0)})\mathbf{b}\cdot\nabla(\ln\ B) = O(\delta^4 p v_{th}^2/L), \tag{99}$$

and the quasi-static pressure anisotropy equation (94) becomes:

$$\frac{p}{mB}\mathbf{b}\cdot\nabla\left[\frac{B(p_\| - p_\perp)}{n}\right] \;+\; \frac{2}{5}q_\|\Big\{3\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] - \nabla\cdot\mathbf{u}\Big\} \;-\; \frac{4p}{mn}\mathbf{h}_\perp^{(2)}\cdot\kappa\; -$$

$$-\;\frac{p}{m}\nabla\cdot\left(\frac{1}{n}\mathbf{h}_\perp^{(2)}\right) \;-\; \frac{2}{5}\mathbf{q}_\perp^{(1)}\cdot[2(\mathbf{b}\cdot\nabla)\mathbf{u} + \mathbf{b}\times\omega] \;-\; \frac{p^2}{mn}\sigma^{(2)} \;+\; \frac{p}{mn}\tau_b^{(2)}\; -$$

$$-\;\frac{2}{3}\mathbf{b}\cdot\nabla(5\tilde{r}_{B\perp}^{(0)} - \tilde{r}_\|^{(0)}) + \Big[(\tilde{r}_\|^{(0)} - \tilde{r}_\perp^{(0)}) + \frac{5}{3}(5\tilde{r}_{B\perp}^{(0)} - \tilde{r}_\|^{(0)})\Big]\mathbf{b}\cdot\nabla(\ln\;B) \;+\; \frac{5}{6}\tilde{n}_B^{(2)} - \frac{1}{2}\tilde{n}^{(2)} = 0, \quad (100)$$

where the second-order scalar $\tau_b^{(2)}$ is

$$\tau_b^{(2)} \;=\; \frac{m}{10eB}\epsilon_{ijk}b_i\frac{\partial q^{(1)}_{\perp[j}}{\partial x_{l]}}(\delta_{lm} - b_l b_m)\frac{\partial b_{[k}}{\partial x_{m]}}. \quad (101)$$

With first-order-accurate solutions for the fluid velocity and the heat fluxes, Eq.(100) is sufficient to provide the required accuracy in the pressure anisotropy, $(p_\| - p_\perp) \sim \delta^2 p$. However, Eq.(100) specifies only the variation of $B(p_\| - p_\perp)/n$ along the magnetic field, and obtaining a global solution for $(p_\| - p_\perp)$ would require consideration of the consistency constraints should the magnetic field lines form closed magnetic surfaces.

VIII. Energy conservation law.

In all the ordering schemes discussed in the previous three Sections, the mean pressure evolution equations, i.e. Eqs.(60,62,78,97), have the form:

$$\frac{3}{2}\frac{dp}{dt} \;+\; \frac{5}{2}p\frac{\partial u_i}{\partial x_i} \;+\; (p_\| - p_\perp)\left(b_i b_j\frac{\partial u_i}{\partial x_j} - \frac{1}{3}\frac{\partial u_i}{\partial x_i}\right) \;+\; \frac{\partial q_i^{(*)}}{\partial x_i} \;+\; \hat{P}_{ij}^{(*)}\frac{\partial u_i}{\partial x_j} \;=\; 0, \quad (102)$$

where $q_i^{(*)}$ is some approximation for the heat flux and $\hat{P}_{ij}^{(*)}$ is some approximation for the gyroviscous stress. In addition, throughout our analysis, the fluid velocity is assumed to be an exact solution of the momentum conservation equation (8), where the available approximation $\hat{P}_{ij}^{(*)}$ is used in the gyroviscous part of the stress tensor. So, taking the component of the momentum equation in the direction of $\mathbf{u}$ and using the continuity equation, we have:

$$\frac{1}{2}mn\frac{du^2}{dt} \; + \; u_i\frac{\partial p}{\partial x_i} \; + \; u_i\frac{\partial}{\partial x_j}\Big[(p_\parallel - p_\perp)\Big(b_ib_j - \frac{1}{3}\delta_{ij}\Big) + \hat{P}^{(*)}_{ij}\Big] \; - \; enu_iE_i \; = \; 0. \tag{103}$$

Now, combining Eqs.(102) and (103), integrating by parts and using again the continuity equation, we get:

$$\frac{\partial}{\partial t}\Big(\frac{1}{2}mnu^2 + \frac{3}{2}p\Big) \; + \; \nabla\cdot\mathbf{Q}^{(*)} \; - \; en\;\mathbf{u}\cdot\mathbf{E} \; = \; 0, \tag{104}$$

where

$$Q^{(*)}_i \; = \; \Big(\frac{1}{2}mnu^2 + \frac{5}{2}p\Big)u_i \; + \; \Big[(p_\parallel - p_\perp)\Big(b_ib_j - \frac{1}{3}\delta_{ij}\Big) + \hat{P}^{(*)}_{ij}\Big]u_j \; + \; q^{(*)}_i \tag{105}$$

is an approximation for the total energy flux of the plasma species under consideration. Therefore, summing over all the species and using the definition of the current density and Faraday's and Ampere's laws (where the displacement current is neglected),

$$\sum_{species} en\;\mathbf{u}\cdot\mathbf{E} \; = \; \mathbf{j}\cdot\mathbf{E} \; = \; -\frac{1}{2}\frac{\partial B^2}{\partial t} \; - \; \nabla\cdot(\mathbf{E}\times\mathbf{B}), \tag{106}$$

we obtain the total energy conservation law:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}B^2 + \sum_{species}\Big(\frac{1}{2}mnu^2 + \frac{3}{2}p\Big)\right] \; + \; \nabla\cdot\Big[\mathbf{E}\times\mathbf{B} + \sum_{species}\mathbf{Q}^{(*)}\Big] \; = \; 0. \tag{107}$$

This energy conservation is exact even though the available heat flux vectors and gyroviscosity tensors are only approximate. An exact energy conservation law is usually lost when the momentum conservation equation is solved approximately, with the fluid velocity split into a parallel component and a series of perpendicular "drifts". The higher moment analysis described in this paper could be carried out in a coherent fashion by treating the whole fluid velocity vector as the exact solution of a momentum conservation equation. The fact that this guarantees an exact energy conservation is another welcome consequence.

IX. Concluding remarks.

The guiding principle behind this work has been to obtain general and rigorous results that can be used as a firm basis for a wide variety of more specialized applications. Thus, besides the collisionless idealization and the small-δ perturbative expansion to the first significant FLR order, no other simplifications have been introduced. Every relevant term has been kept in our equations, including those whose evaluation is beyond the possibilities of the fluid theory in general or the slow dynamics ordering in particular. Accordingly, the use of multiple expansion parameters and subsidiary orderings has been keenly avoided. It is left for the case of each specific application to choose the appropriate model of the closure terms, and to possibly carry out further reductions by taking advantage of other applicable small parameters. In particular, except for the Larmor radius, no separation of length scales has been assumed. In situations where several disparate length scales other than the gyroradii are physically relevant, the shortest of them is to be taken when defining the ratio $\delta \sim \rho/L$. Small ratios of two such additional characteristic lengths, for example those perpendicular and parallel to the magnetic field $L_\perp/L_\|$, can be used afterwards to derive more specific reduced systems applicable to those situations. However, regarding the use of $L_\perp/L_\|$ as a subsidiary expansion parameter, it is worth pointing out that the smallness of this ratio has a different meaning depending on whether it refers to the equilibrium or the perturbations. Small $L_\perp/L_\|$ orderings entail a distinction between equilibrium and perturbations, that the general results shown in this article do not make.

The present analysis has also avoided deliberately to make explicit use of the "gyroviscous cancellation" [7,8,23]. This is a partial cancellation between terms in the divergences of the gyroviscous stress and the Reynolds stress, that is apparent when the perpendicular flow velocity is expanded as a sum of $\mathbf{E} \times \mathbf{B}$, diamagnetic and polarization drifts. However, even for the simplest form of the gyroviscosity in the weakly anisotropic slow dynamics, the clutter originating from the numerous remaining terms and from having to use the expanded form of the flow velocity, far outweighs the benefits of the cancellation. Moreover, as mentioned in the previous Section, this usually causes the violation of the exact energy conservation law. Therefore, even though the "gyroviscous cancellation" is implicit in our equations, it is deemed advantageous not to be concerned about it.

The most serious limitation of this work is likely to be the complete neglect of collisions. However low, a non-zero collisionality rate is needed physically in most cases. A realistic analysis at low but finite collisionality cannot take advantage of short-mean-free-path asymptotic expansions. Hence it modifies the present formulation only by adding the terms arising from the velocity moments of the collision operator part of the kinetic equation, but leaving everything else unchanged. The best approach towards an account of these low but finite collisionality effects, is probably to evaluate the collision integrals with trial distribution functions that yield identically the known fluid moments, i.e. a Chapman-Enskog-like approach[17,24,25]. This will be the subject of future investigations.

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