Self-consistent radial electric field
in collisional screw-pinches and magnetic
dipoles in the absence of fluctuations

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Abstract

Collisional plasma confined by magnetic fields of a screw-pinch and a magnetic dipole (or any other axisymmetric and up-down symmetric closed magnetic field line configuration) is considered, and equations governing evolution of the self-consistent radial electric field are derived for each case, provided that effects of plasma fluctuations are negligible.
I. INTRODUCTION

It has long been realized that in magnetic confinement devices a profile of electric field, $E$, develops, which is consistent with profiles of confining magnetic field, $B$, plasma density, $n$, and electron and ion temperatures, $T$ and $T_e$, respectively. In confined plasmas in general, the radial electric field is determined by the condition that radial fluxes of charged particles be ambipolar. In axisymmetric magnetic confinement devices, such as a tokamak or a magnetic dipole, the particle flux ambipolarity condition is equivalent to conservation of toroidal angular momentum. However, there is an important difference between the two types of devices: unlike in a tokamak, charged particles in a closed magnetic field line dipole do not exhibit neoclassical behavior since the departures from field lines is only due to gyromotion. For an axially uniform cylindrical screw-pinch the departure from field lines is again only due to gyromotion, so no neoclassical behavior is possible. In this case, the radial electric field is determined by solving the total momentum conservation equation for the perpendicular current density and then demanding that its radial component vanish.

Resulting radial electric field equations depend parametrically on the parallel component of the plasma flow, which has to be evaluated from the parallel momentum conservation equation. Such parallel flow in a dipole depends on its up-down asymmetry and can usually be neglected for fairly up-down symmetric magnetic flux surfaces. The parallel flow evolution equation in a screw-pinch, which usually has to be retained together with the radial electric field evolution equation, can be thought of as the classical (as opposed to neoclassical) equivalent of the poloidal flow damping calculation in a tokamak.\(^1\)

In the absence of plasma fluctuations, the self-consistent radial electric field in a
tokamak can be evaluated by employing neoclassical theory. This has been done in regimes of both low (banana)\(^2,3\) and high (Pfirsch-Schlüter)\(^4,5\) plasma collisionality. In geometries with only gyroradius departures from flux surfaces, the radial electric field evaluation has been performed by using Braginskii’s MHD-ordered closure\(^6\) for collisional plasmas (see for example Ref. 7 for a screw-pinch calculation). However, recent work\(^4,8\) indicates that the drift-ordered collisional closure,\(^9\) which includes all the Braginskii’s terms and more, is often more appropriate. We are not aware of a calculation for the “classical” radial electric field in configurations like a screw-pinch or a dipole that employs such a drift-ordered closure. Consequently, in this article we use these more general results to obtain the self-consistent equations governing the radial electric field and parallel flow in plasmas confined by screw pinches and axisymmetric closed field line devices, such as magnetic dipoles.

The article is organized in the following way. Section II summarizes orderings and assumptions used in the calculation. Section III describes some necessary steps for evaluating parallel plasma flows and heat fluxes, which are common for the two magnetic configurations discussed herein, and gives general expressions for the ion viscosity. Sections IV and V build on this formalism to derive the radial electric field evolution equations in a screw-pinch and a dipole, respectively. The parallel flow equation for a screw-pinch is also derived. Finally, Sec. VI summarizes the results.

**II. ORDERINGS AND ASSUMPTIONS**

To derive the evolution equation for the radial electric field we adopt the standard collisional transport orderings for magnetized plasmas (see, for example, Ref. 10). The primary expansion parameters are based on smallness of the ion gyroradius,
\[ \rho = \frac{v_{Ti}}{\Omega}, \text{and the ion mean-free path,} \quad \lambda = \frac{v_{Ti}}{\nu}, \text{as compared with the characteristic length scale,} \quad L: \]

\[
\delta \equiv \frac{\rho}{L} \ll 1, \\
\Delta \equiv \frac{\lambda}{L} \ll 1.
\]

Here, \( v_{Ti} = \sqrt{2T/M} \) is the ion thermal speed, with \( M \) the ion mass and \( T \) the ion temperature, \( \Omega = (eB/Mc) \) is the ion gyrofrequency, with \( e \) the unit electric charge, \( B = |B| \) the magnitude of the magnetic field, and \( c \) the speed of light (for simplicity, plasma consisting of electrons and singly-charged ions is considered), and \( \nu = (4\sqrt{\pi}/3)(\ln \Lambda n e^4/M^{1/2}T^{3/2}) \) is the ion collision frequency, with \( n \) the plasma density and \( \ln \Lambda \) the Coulomb logarithm. To evaluate the radial electric field we will use the short mean-free path expressions for the ion viscosity\(^9\) (given in Sec. III), which were obtained by assuming \( \nu/\Omega \ll 1 \). For the single length scale ordering considered here, we must assume

\[
\frac{\delta}{\Delta} = \frac{\nu}{\Omega} \ll 1. \tag{2}
\]

We also assume that, in leading order, the plasma density, \( n \), the electron and the ion temperatures, \( T_e \) and \( T \), and the electrostatic potential, \( \varphi \), are functions of only the local radial coordinate. In fact, these quantities are functions of only this coordinate to all orders for the screw-pinch, but not for the dipole, where we assume that the dominant parallel (to the magnetic field) variation is due to the parallel variation of the magnetic field by taking

\[
\frac{B \cdot \nabla \ln T}{B \cdot \nabla \ln B} \ll 1. \tag{3}
\]

It can be shown \textit{a posteriori} that the left-hand side of Eq. (3) is of order \((\delta/\Delta)^2 \ll 1\) for a magnetic dipole. As a result, the ion temperature will be treated as a flux function.
The time scale of interest is assumed to be that associated with the collisional ion radial heat transport, namely

$$\frac{\partial}{\partial t} \sim \frac{\chi}{L^2} \sim \nu \delta^2,$$  

(4)

where $\chi \sim \nu \rho^2$ is the ion thermal diffusivity. This time scale is assumed to be much shorter than the characteristic time scale for the variation of the vector potential, $A$, which is determined by the resistive diffusion of the magnetic field, so that $\beta \sqrt{M/m} \gg 1$, with $\beta \equiv 8\pi n(T + T_e)/B^2$ and $m$ the electron mass. As a result, the electric field, $E = -\nabla \varphi - c^{-1}\partial A/\partial t$, is electrostatic to the order we require,

$$c^{-1}|\partial A/\partial t| \ll \frac{\delta}{\Delta} \ll 1,$$  

(5)

where we estimate $A \sim BL$ and $e\varphi \sim T_e \sim T$.

### III. PRELIMINARY REMARKS

We now present the background needed to proceed with the radial electric field and the parallel flow calculations. To evaluate the necessary components of the ion gyroviscous stress tensor to the same order as those of the ion perpendicular viscosity, we have to know the ion particle and heat flows to order $(\delta^2/\Delta) \ll 1$. As usual, to the order required, the ion flow velocity is given by the sum of the parallel, $E \times B$, and diamagnetic velocities,

$$V = V_\parallel + V_\perp = V_\parallel \hat{b} + c \frac{\hat{b} \times \nabla \varphi}{B} + \frac{\hat{b} \times \nabla p}{Mn\Omega},$$  

(6)

where $p$ is the ion pressure and $\hat{b} \equiv B/B$ is the unit vector along $B$. The lowest order continuity equation minus its flux-surface average,

$$\nabla \cdot (nV) = \langle \nabla \cdot (nV_\perp) \rangle_\theta,$$  

(7)
places a constraint on $V_∥$. Here, the flux surface average is defined as $\langle \cdot \cdot \cdot \rangle_\theta \equiv (V')^{-1} \oint \langle \cdot \cdot \cdot \rangle d\theta / (B \cdot \nabla \theta)$, with $V' \equiv \oint d\theta / (B \cdot \nabla \theta)$. It follows from the symmetry properties of the two magnetic configurations and from the lowest order ion parallel momentum equation (which only matters for a dipole),

$$B \cdot (\nabla p + en \nabla \phi) = 0,$$

that $\nabla \cdot (n V_\perp) = 0$. Therefore, Eq. (7) requires

$$B \cdot \nabla \left( \frac{n V_∥}{B} \right) = 0.$$  \hspace{1cm} (9)

In the following sections we will use Eq. (9) along with the flux-surface average of the parallel component of the plasma momentum equation (with electron inertial and viscous effects neglected),

$$Mn \left( \frac{\partial V}{\partial t} + V \cdot \nabla V \right) + \nabla (p + p_e) + \nabla \cdot \pi = \frac{1}{c} J \times B,$$

with $p_e = n T_e$ the electron pressure and

$$J = \frac{c}{4\pi} \nabla \times B$$  \hspace{1cm} (11)

the plasma current, to evaluate $V_∥$ separately for each magnetic configuration.

To evaluate the parallel heat flux we employ the general short mean-free path expression,

$$q = q_∥ + q_\perp = q_∥ \hat{b} + \frac{5cp}{2eB} \hat{b} \times \nabla T + q_e,$$

with $q_∥ = -\kappa_∥ \nabla_∥ T$, $\kappa_∥ = (125p/32M\nu)$, and $q_e = -\kappa_\perp \nabla_\perp T$, $\kappa_\perp = (2p\nu/M\Omega^2)$. To lowest order the ion temperature evolution equation minus its flux-surface average must be satisfied,

$$\nabla \cdot \left[ q + \left( e\varphi + \frac{5}{2} T \right) n V \right] = \langle \nabla \cdot q_\perp \rangle_\theta.$$  \hspace{1cm} (13)
It is clear that evaluation of $q\parallel$ is equivalent to evaluation of the parallel ion temperature gradient. It follows from the symmetry properties of the two magnetic configurations that $\nabla \cdot q_\perp = \nabla \cdot q_c$, which, along with Eq. (8) and the total pressure and electron temperature being flux functions to the order required, gives

$$B \cdot \nabla \left[ \frac{1}{B} \left( q_\parallel + \frac{5}{2} p V_\parallel \right) \right] = \langle \nabla \cdot q_c \rangle_o - \nabla \cdot q_c + n V_\parallel \left( \frac{T_e}{T+T_e} \right) \nabla || T. \quad (14)$$

We will use Eq. (14) to evaluate $q\parallel$ separately for each magnetic configuration in the following sections.

In the calculations that follow we require the short mean-free path expression for the ion viscosity $\pi = \pi_\parallel + \pi_g + \pi_\perp$, where $\pi_\parallel$, $\pi_g$, and $\pi_\perp$ are the parallel, gyro-, and perpendicular viscosities, respectively, as derived in Ref. 9. The parallel viscosity is given by

$$\pi_\parallel = \left( \hat{b} \hat{b} - \frac{1}{3} I \right) \pi_\parallel, \quad (15)$$

with

$$\pi_\parallel = -\eta (3\hat{b} \hat{b} - I) : (\tilde{\alpha} - \xi \tilde{\gamma}) + \zeta, \quad \tilde{\alpha} = \nabla V + \frac{2}{5p} \nabla q, \quad \tilde{\gamma} = \frac{2}{5p} \left[ q \nabla \ln p - \left( 2q + \frac{4}{15} q_\parallel \right) \nabla \ln T - \nabla q + \frac{4}{15} \nabla q_\parallel \right],$$

$$\eta = 0.96 \frac{p}{\nu}, \quad \xi = 0.61, \quad \zeta = \frac{3M}{4pT} (0.115 q_\parallel^2 - 0.085 q_\perp^2),$$

and $I$ the unit dyad. The gyroviscosity is

$$\pi_g = \frac{p}{4\Omega} \left\{ \hat{b} \times [\tilde{\alpha} + \tilde{\alpha}^T] \cdot (3\hat{b} \hat{b} + I) - (3\hat{b} \hat{b} + I) \cdot [\tilde{\alpha} + \tilde{\alpha}^T] \times \hat{b} \right\}, \quad (16)$$

with $\tilde{\alpha}^T$ denoting a transpose of $\tilde{\alpha}$. And the perpendicular viscosity is

$$\pi_\perp = -\frac{3\nu}{100\Omega^2} \left[ \tilde{W} + 3\hat{b} \cdot \tilde{W} + 3 \tilde{W} \cdot \hat{b} \hat{b} \right] - \frac{9M\nu}{200pT\Omega} \left[ \hat{b} \times q \left( q + \frac{31}{15} q_\parallel \right) + \left( q + \frac{31}{15} q_\parallel \right) \hat{b} \times q \right]. \quad (17)$$
with

\[
W = \frac{3}{10} \left[ -\nabla q - \frac{1}{10} \nabla q_\parallel + \left( q - \frac{1}{6} q_\parallel \right) \nabla \ln p - \left( \frac{3}{4} q - \frac{13}{120} q_\parallel \right) \nabla \ln T \right] \\
+ p \left[ \vec{\alpha} - \frac{1}{3} (\vec{I} : \vec{\alpha}) \vec{I} \right] + \text{Transpose.}
\]

Notice, that only the diamagnetic contribution to \( q_\perp \) has to be retained in \( \vec{\pi}_\parallel \) and \( \vec{\pi}_\perp \), but both the diamagnetic and the classical collisional contribution \( q_c \) to \( q_\perp \) are required in \( \vec{\pi}_g \).

IV. RADIAL ELECTRIC FIELD IN A SCREW-PINCH

This section derives an equation for the radial electric field in a screw-pinch, i.e. in a cylinder with a circular cross-section having both the axial and the azimuthal magnetic fields.

Before proceeding, we introduce a convenient coordinate system by employing the radial coordinate, \( r \), the azimuthal angle, \( \theta \), and the axial coordinate, \( z \). These coordinates form the right-hand set \((r, \theta, z)\). Symmetry of the screw-pinch requires

\[
\partial(\text{any scalar})/\partial \theta = \partial(\text{any scalar})/\partial z = 0.
\]

The confining magnetic field is conveniently written as

\[
B = B_\theta(r) \hat{\theta} + B_z(r) \hat{z},
\]

where \( \hat{\theta} \) and \( \hat{z} \) are unit vectors in \( \theta \) and \( z \) directions, respectively, with \( \hat{\theta} \cdot \hat{z} = 0 \), giving \( B_\theta^2 + B_z^2 = B^2 \).

Since both \( n \) and \( B \) are functions of \( r \) and \( t \) only, Eq. (9) gives the obvious answer

\[
V_\parallel = V_\parallel(r, t),
\]
so that, in accordance with Eq. (6),

\[
\mathbf{V} = -\frac{c}{B_\theta} \left( \frac{\partial \varphi}{\partial r} + \frac{1}{en} \frac{\partial p}{\partial r} \right) \hat{z} + \left[ \frac{V_\parallel}{B} + \frac{cB_z}{B_\theta B^2} \left( \frac{\partial \varphi}{\partial r} + \frac{1}{en} \frac{\partial p}{\partial r} \right) \right] \mathbf{B}
\]

(20)

\[
\equiv \omega(r, t) \hat{z} + u(r, t) \mathbf{B}.
\]

To determine \( V_\parallel \) we have to employ the parallel component of Eq. (10). Dotting Eq. (10) by \( \mathbf{B} \), employing expression (20) for \( \mathbf{V} \), and noticing that \( \nabla \hat{z} = 0 \) and \( \nabla \hat{\theta} = -\hat{\theta} \hat{r}/r \), with \( \hat{r} \) the unit vectors in \( r \) direction, we obtain

\[
M n \left[ B_z \frac{\partial \omega}{\partial t} + B \frac{\partial (uB)}{\partial t} \right] + \frac{B_\theta}{r^2} \frac{\partial}{\partial r} \left[ r^2 (\hat{r} \cdot \vec{\pi} \cdot \hat{\theta}) \right] + \frac{B_z}{r} \frac{\partial}{\partial r} \left[ r (\hat{r} \cdot \vec{\pi} \cdot \hat{z}) \right] = 0.
\]

Neglecting magnetic field diffusion, the first term simplifies and we arrive at the evolution equation for \( V_\parallel \):

\[
M n B \frac{\partial V_\parallel}{\partial t} + \frac{B_\theta}{r^2} \frac{\partial}{\partial r} \left[ r^2 (\hat{r} \cdot \vec{\pi} \cdot \hat{\theta}) \right] + \frac{B_z}{r} \frac{\partial}{\partial r} \left[ r (\hat{r} \cdot \vec{\pi} \cdot \hat{z}) \right] = 0.
\]

(21)

To obtain an equation for the radial electric field we notice that, in accordance with Eq. (11), \( \mathbf{J} \cdot \hat{r} = 0 \). Crossing the momentum equation (10) by \( \mathbf{B} \), dotting by \( \hat{r} \) to evaluate \( \mathbf{J} \cdot \hat{r} \), and setting the result to zero gives

\[
(B_\theta \hat{z} - B_z \hat{\theta}) \cdot \left[ M n \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) + \nabla \cdot \vec{\pi} \right] = 0.
\]

(22)

Employing expression (20) for \( \mathbf{V} \) we can rewrite this equation as

\[
M n \left[ B_\theta \frac{\partial \omega}{\partial t} + u \left( B_\theta \frac{\partial B_z}{\partial t} - B_z \frac{\partial B_\theta}{\partial t} \right) \right] + \frac{B_\theta}{r} \frac{\partial}{\partial r} \left[ r(\hat{r} \cdot \vec{\pi} \cdot \hat{z}) \right] - \frac{B_z}{r^2} \frac{\partial}{\partial r} \left[ r^2 (\hat{r} \cdot \vec{\pi} \cdot \hat{\theta}) \right] = 0.
\]

(23)

Continuing to neglect magnetic field diffusion, the first term simplifies, and we arrive at the final equation

\[
M n B_\theta \frac{\partial \omega}{\partial t} + \frac{B_\theta}{r} \frac{\partial}{\partial r} \left[ r(\hat{r} \cdot \vec{\pi} \cdot \hat{z}) \right] - \frac{B_z}{r^2} \frac{\partial}{\partial r} \left[ r^2 (\hat{r} \cdot \vec{\pi} \cdot \hat{\theta}) \right] = 0.
\]

(24)
Once the quantities $\hat{r} \cdot \vec{\pi} \cdot \hat{z}$ and $\hat{r} \cdot \vec{\pi} \cdot \hat{\theta}$ are known, Eqs. (21) and (24) form a closed system of two equations for the two unknowns, $\omega(r, t)$ [or equivalently $E_r = -\partial \varphi(r, t)/\partial r$] and $V_{\parallel}(r, t)$. Notice, that only the $\vec{\pi}_g$ and $\vec{\pi}_\bot$ portions of the ion viscosity $\vec{\pi}$ contribute to $\hat{r} \cdot \vec{\pi} \cdot \hat{z}$ and $\hat{r} \cdot \vec{\pi} \cdot \hat{\theta}$. By combining Eqs. (21) and (24) we obtain the equation advancing $u$ to be

$$MnB_\theta \frac{\partial u}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 (\hat{r} \cdot \vec{\pi} \hat{\theta}) \right] = 0,$$

(25)

which can be used in place of either Eq. (21) or Eq. (24).

To evaluate the quantities $\hat{r} \cdot \vec{\pi} \cdot \hat{z}$ and $\hat{r} \cdot \vec{\pi} \cdot \hat{\theta}$ we require an expression for $q$ that is accurate to order $(\delta^2/\Delta)$. Since $\nabla_{\parallel} T = (B_\theta/B) \partial T/\partial \theta + (B_z/B) \partial T/\partial z = 0$, it immediately follows from the short mean-free path expression for the parallel heat flux that $q_{\parallel} = 0$ to all orders, and Eq. (12) can be rewritten as

$$q = \frac{5}{2} \left( -\frac{c_p}{eB_\theta} \frac{\partial T}{\partial r} \hat{\theta} + \frac{c_p}{eB^2 B_\theta} \frac{\partial T}{\partial r} B \right) - \kappa_{\bot} \frac{\partial T}{\partial r} \hat{r} \tag{26}$$

$$\equiv \frac{5}{2} [s(r, t) \hat{z} + g(r, t) B] - \kappa_{\bot} \frac{\partial T}{\partial r} \hat{r}.$$

Notice, that constraint (14) is satisfied identically in this case, and that $gB^2 + sB_z = 0$ to satisfy $q_{\parallel} = 0$.

Employing expressions (20) and (26) for $V$ and $q$, respectively, and noticing that $\nabla \hat{r} = \hat{\theta} \hat{r}/r$, $\nabla \hat{\theta} = -\hat{\theta} \hat{r}/r$, we obtain

$$p \hat{\alpha} = \left[ \frac{\partial (\omega + uB_z)}{\partial r} + \frac{\partial (s + gB_z)}{\partial r} \right] \hat{r} \hat{z} + \left[ \frac{\partial (uB_\theta)}{\partial r} + \frac{\partial (gB_\theta)}{\partial r} \right] \hat{r} \hat{\theta}$$

$$\frac{B_\theta}{r} (pu + g) \hat{\theta} \hat{r} - \frac{2}{5} \frac{\partial}{\partial r} \left( \frac{\kappa_{\bot} \partial T}{\partial r} \right) \hat{r} \hat{r} - \frac{2}{5} \frac{\kappa_{\bot} \partial T}{\partial r} \hat{\theta} \hat{\theta}.$$
Then, Eq. (16) gives

\[
\pi_g = \frac{1}{4\Omega} \left\{ \frac{\partial (\omega + uB_z)}{\partial r} + \frac{\partial (s + gB_z)}{\partial r} \right\}
\]

\[
\times \left\{ \frac{B_z}{B} \left[ 1 + \frac{3(B_z^2 - B_\theta^2)}{B^2} \right] \left( \hat{\theta} \hat{z} + \hat{z} \hat{\theta} \right) + \frac{2B_\theta}{B} \left[ \hat{z} \hat{r} - \hat{r} \hat{z} + \frac{3B_\theta^2}{B^2} (\hat{\theta} \hat{\theta} - \hat{z} \hat{z}) \right] \right\}
\]

\[
+ \frac{1}{4\Omega} \left\{ \frac{\partial (uB_\theta)}{\partial r} + \frac{\partial (gB_\theta)}{\partial r} - (pu + g) \frac{B_\theta}{r} \right\}
\]

\[
\times \left\{ \frac{B_\theta}{B} \left[ -1 + \frac{3(B_z^2 - B_\theta^2)}{B^2} \right] \left( \hat{\theta} \hat{z} + \hat{z} \hat{\theta} \right) + \frac{2B_z}{B} \left[ \hat{\theta} \hat{r} - \hat{r} \hat{\theta} + \frac{3B_\theta^2}{B^2} (\hat{\theta} \hat{\theta} - \hat{z} \hat{z}) \right] \right\}
\]

\[
- \frac{1}{5\Omega} \frac{\partial}{\partial r} \left( \kappa_{\perp} \frac{\partial T}{\partial r} \right) \left[ \frac{B_z}{B} \left( \hat{\rho} \hat{r} + \hat{r} \hat{\rho} \right) - \frac{B_\theta}{B} \left( \hat{z} \hat{r} + \hat{r} \hat{z} \right) \right]
\]

\[
+ \frac{1}{5\Omega} \frac{\partial T}{\partial r} \left[ \frac{B_z}{B} \left( \hat{\rho} \hat{r} + \hat{r} \hat{\rho} \right) + \frac{3B_\theta^2}{B^2} \left( \hat{\theta} \hat{\theta} - \hat{z} \hat{z} \right) \right],
\]

so that

\[
\hat{r} \cdot \pi_g \cdot \hat{\rho} = - \frac{B_z}{5\Omega B} \left[ \frac{\partial}{\partial r} \left( \kappa_{\perp} \frac{\partial T}{\partial r} \right) - \left( 1 + \frac{3B_\theta^2}{B^2} \right) \frac{\kappa_{\perp} \frac{\partial T}{\partial r}}{r} \right] \quad (28)
\]

and

\[
\hat{r} \cdot \pi_g \cdot \hat{z} = \frac{B_\theta}{5\Omega B} \left[ \frac{\partial}{\partial r} \left( \kappa_{\perp} \frac{\partial T}{\partial r} \right) + \frac{3B_\theta^2 \kappa_{\perp} \frac{\partial T}{\partial r}}{B^2 r} \right]. \quad (29)
\]

To evaluate the perpendicular viscosity contribution we first employ expression (27) for \(p \hat{\rho} \) and expression (26) for \(q \) (both with the \(\kappa_{\perp} \) terms neglected) to obtain

\[
\vec{W} = \xi_1 (\hat{r} \hat{z} + \hat{z} \hat{r}) + \xi_2 \left( \hat{r} \hat{\theta} + \hat{\theta} \hat{r} \right), \quad (30)
\]

with

\[
\xi_1 \equiv p \frac{\partial (\omega + uB_z)}{\partial r} + \frac{1}{4} \frac{\partial (s + gB_z)}{\partial r} + \frac{3}{4} (s + gB_z) \left( \frac{\partial \ln p}{\partial r} - \frac{3}{4} \frac{\partial \ln T}{\partial r} \right),
\]

\[
\xi_2 \equiv pr \frac{\partial}{\partial r} \left( \frac{uB_\theta}{r} \right) + r \frac{\partial}{\partial r} \left( \frac{gB_\theta}{r} \right) + \frac{3}{4} gB_\theta \left( \frac{\partial \ln p}{\partial r} - \frac{3}{4} \frac{\partial \ln T}{\partial r} \right),
\]

and

\[
\hat{b} \times q \cdot q + q \hat{b} \times q = \frac{25B_\theta}{4B} \left[ (s + gB_z)(\hat{r} \hat{z} + \hat{z} \hat{r}) + gB_\theta (\hat{r} \hat{\theta} + \hat{\theta} \hat{r}) \right]. \quad (31)
\]
We then use results (30) and (31) to write the perpendicular viscosity as

\[
\pi_\perp = -\frac{3\nu}{10\Omega^2} \xi_1 \left[ \left( 1 + \frac{3B_z^2}{B^2} \right) (\hat{r} \hat{z} + \hat{z} \hat{r}) + \frac{3B_\theta B_z}{B^2} (\hat{r} \hat{\theta} + \hat{\theta} \hat{r}) \right] \\
- \frac{3\nu}{10\Omega^2} \xi_2 \left[ \left( 1 + \frac{3B_z^2}{B^2} \right) (\hat{r} \hat{\theta} + \hat{\theta} \hat{r}) + \frac{3B_\theta B_z}{B^2} (\hat{r} \hat{z} + \hat{z} \hat{r}) \right] \\
- \frac{9M\nu}{32pT\Omega} \frac{sB_\theta}{B} \left[ (s + gB_z)(\hat{r} \hat{z} + \hat{z} \hat{r}) + gB_\theta (\hat{r} \hat{\theta} + \hat{\theta} \hat{r}) \right],
\]

so that

\[
\hat{r} \cdot \pi_\perp \cdot \hat{\theta} = -\frac{3\nu}{10\Omega^2} \left[ \frac{3B_\theta B_z}{B^2} \xi_1 + \left( 1 + \frac{3B_z^2}{B^2} \right) \xi_2 \right] - \frac{9M\nu}{32pT\Omega} \frac{sgB_\theta^2}{B} 
\]

and

\[
\hat{r} \cdot \pi_\perp \cdot \hat{z} = -\frac{3\nu}{10\Omega^2} \left[ \left( 1 + \frac{3B_z^2}{B^2} \right) \xi_1 + \frac{3B_\theta B_z}{B^2} \xi_2 \right] - \frac{9M\nu}{32pT\Omega} \frac{s(s + gB_z)B_\theta}{B}.
\]

Equations (21) and (24) with (\hat{r} \cdot \pi \cdot \hat{\theta}) = (\hat{r} \cdot \pi_g \cdot \hat{\theta}) + (\hat{r} \cdot \pi_\perp \cdot \hat{\theta}) and (\hat{r} \cdot \pi \cdot \hat{z}) = (\hat{r} \cdot \pi_g \cdot \hat{z}) + (\hat{r} \cdot \pi_\perp \cdot \hat{z}) given by Eqs. (28), (29), (32), and (33) fully describe \(V_\parallel(r, t)\) and \(\omega(r, t)\) provided the magnetic field, plasma density, and ion temperature profiles are known.

It is instructive to study the limiting cases of \(B_z = 0\), \(B_\theta = 0\), and \(\partial/\partial t = 0\). We first consider a Z-pinch \((B_\theta \neq 0)\) by taking \(B_z = 0\). Then, Eqs. (21) and (24) decouple to become

\[
Mn \frac{\partial V_\parallel}{\partial t} - \frac{1}{r^2} \frac{\partial}{\partial r} \left[ \frac{6\nu pr^3}{5\Omega^2} \frac{\partial}{\partial r} \left( \frac{V_\parallel}{r} \right) \right] = 0,
\]

\[
Mn \frac{\partial \omega}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left\{ \frac{r}{5\Omega} \frac{\partial}{\partial r} \left( \kappa_\perp \frac{\partial T}{\partial r} \right) - \frac{9M\nu s^2 r}{32pT\Omega} \right\} - \frac{3\nu r}{10\Omega^2} \left[ \frac{\partial \omega}{\partial r} + \frac{1}{4} \frac{\partial s}{\partial r} + \frac{3s}{4} \left( \frac{\partial \ln p}{\partial r} - \frac{3\partial \ln T}{\partial r} \right) \right] = 0.
\]

In this case, \(V_\parallel = uB\) satisfies a homogeneous equation allowing rigid azimuthal steady-state rotation \((V_\parallel/r = \text{constant})\) that is completely decoupled from the axial \(\omega\)
flow driven by radial gradients of $n$ and $T$. When the time variation of $\omega$ is negligible and in the absence of momentum sources and sinks Eq. (34) gives

$$c \left( \frac{\partial \varphi}{\partial r} + \frac{1}{en} \frac{\partial p}{\partial r} \right) = B_\theta \int^r \frac{1}{p} \left[ \frac{3}{8} v q_c \left( \frac{\partial \ln p}{\partial r} - \frac{2}{r} \frac{\partial \ln T}{\partial r} \right) + \frac{1}{8} \frac{\partial (v q_c)}{\partial r} + \frac{2}{3} v \frac{\partial q_c}{\partial r} \right],$$

where $v \equiv (\Omega/\nu)$, $q_c \equiv -\kappa_\perp (\partial T/\partial r)$, and the right-hand side vanishes for $\partial T/\partial r = 0$ to give a Maxwell-Boltzmann response, as required.

A similar situation occurs for a $\theta$-pinch ($B_z \neq 0$) for which $B_\theta = 0$. In this case, $\omega$, $u$, $s$ and $g$ are proportional to $1/B_\theta$ as $B_\theta \to 0$, so it is convenient to define $\tilde{\omega} \equiv \omega B_\theta/B$ and $\tilde{s} = s B_\theta/B$. Then, Eqs. (21) and (24) become

$$M \frac{\partial V}{\partial t} - \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{6 \nu p r}{5 \Omega^2} \frac{\partial V}{\partial r} \right] = 0,$$

$$M \frac{\partial \tilde{\omega}}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left[ \frac{r^3}{5 \Omega} \frac{\partial}{\partial r} \left( \frac{\kappa_\perp T}{r} \frac{\partial T}{\partial r} \right) - \frac{9 M \nu s^2 r^2}{32 p T \Omega} \right]$$

$$- \frac{3 \nu r^2}{10 \Omega^2} \left[ p r \frac{\partial}{\partial r} \left( \frac{\tilde{\omega}}{r} \right) + r \frac{\partial}{\partial r} \left( \frac{\tilde{s}}{r} \right) + \frac{3 \tilde{s}}{4} \left( \frac{\partial \ln p}{\partial r} - \frac{3}{4} \frac{\partial \ln T}{\partial r} \right) \right] = 0,$$

Again, $V_\parallel$ satisfies a homogeneous equation decoupled from that for $\tilde{\omega}$, which allows a solution $V_\parallel =$constant. Notice that in this case $\mathbf{V} = V_\parallel \hat{z}$, as can be seen from Eq. (20), but the radial electric field is determined by the equation for $\tilde{\omega}$. When the time variation of $\tilde{\omega}$ is negligible and in the absence of momentum sources and sinks Eq. (36) gives

$$c \left( \frac{\partial \varphi}{\partial r} + \frac{1}{en} \frac{\partial p}{\partial r} \right) = B_z r \int^r \frac{1}{r p} \left[ \frac{3}{8} v q_c \left( \frac{\partial \ln p}{\partial r} - \frac{2}{r} \frac{\partial \ln T}{\partial r} \right) + \frac{r}{8} \frac{\partial (v q_c)}{\partial r} + \frac{2}{3} r v \frac{\partial q_c}{\partial r} \right].$$

Once again the right-hand side of the equation for $\tilde{\omega}$ vanishes to give a generalized Maxwell-Boltzmann response if $\partial T/\partial r = 0$. 

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Finally, if both \( B_z \neq 0 \) and \( B_\theta \neq 0 \), and the time variation of \( V_\parallel \) and \( \omega \) is negligible, Eqs. (21) and (24) result in

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r^2 \left( \hat{r} \cdot \hat{\pi} \cdot \hat{\theta} \right) \right] = 0 = \frac{\partial}{\partial r} \left[ r \left( \hat{r} \cdot \hat{\pi} \cdot \hat{z} \right) \right],
\]

which means, in the absence of momentum sources and sinks, that

\[
\hat{r} \cdot \hat{\pi} \cdot \hat{\theta} = \hat{r} \cdot \hat{\pi} \cdot \hat{z} = 0.
\]

Equation (39) can be solved in a straightforward (although somewhat tedious) way to give

\[
V_\parallel = -\frac{B_z}{B} I_1 + \frac{B_\theta}{B} I_2,
\]

\[
c \left( \frac{\partial \varphi}{\partial r} + \frac{1}{en} \frac{\partial p}{\partial r} \right) = B_\theta I_1 + B_z I_2,
\]

with

\[
I_1 = \int_0^r dr \left[ \frac{3}{8wp} \left( \frac{\partial \ln p}{\partial r} - 2 \frac{\partial \ln T}{\partial r} \right) + \frac{1}{8} \frac{\partial (wp)}{\partial r} + \frac{2}{3} \frac{\partial q_c}{\partial r} \right],
\]

\[
I_2 = \int_0^r r dr \left[ \frac{3}{8zq_c} \left( \frac{\partial \ln p}{\partial r} - 2 \frac{\partial \ln T}{\partial r} \right) + \frac{r}{8} \frac{\partial (zq_c)}{\partial r} + \frac{2}{3} \frac{r^2 q_c}{r} \right].
\]

Here, \( w \equiv (\Omega/\nu)(B_\theta/B) \) and \( z \equiv (\Omega/\nu)(B_z/B) \). Quantities \( I_1 \) and \( I_2 \) characterize flows in the \( \hat{z} \) and \( \hat{\theta} \) directions. Indeed, we can easily see from Eqs. (20), (40), and (41) that

\[
\mathbf{V} = -I_1 \hat{z} + I_2 \hat{\theta}.
\]

Eqs. (40) and (41) predict that both \( V_\parallel \) and \( (\partial \varphi/\partial r + e^{-1}n^{-1}\partial p/\partial r) \) are generally finite when \( B_z \neq 0 \) and \( B_\theta \neq 0 \). If \( \partial T/\partial r = 0 \) then the general expressions simplify since \( I_1 \rightarrow \) constant and \( I_2/r \rightarrow \) constant and we obtain a rigidly azimuthally rotating homogeneous axial flow and a “generalized” Maxwell-Boltzmann response. Of course, Eq. (41) reproduces results of Eqs. (35) and (37) when \( B_z \) and \( B_\theta \), respectively, are set to zero.
V. RADIAL ELECTRIC FIELD IN A MAGNETIC DIPOLE

This section derives an equation for the radial electric field in a magnetic dipole configuration, i.e. in an axisymmetric doughnut-like configuration with closed poloidal magnetic field lines. We assume for simplicity that the dipole is up-down symmetric. This assumption is not crucial and can easily be relaxed to obtain somewhat more complicated expressions that typically give very small corrections.

For a dipole, it is convenient to use as coordinates the poloidal magnetic flux, $\psi$, the poloidal angle, $\theta$, and the toroidal angle, $\zeta$. The symmetry requires in this case that $\partial (\text{any scalar})/\partial \zeta = 0$. The dipolar magnetic field can be written as

$$ B = \nabla \zeta \times \nabla \psi. $$

(42)

First, we evaluate the parallel flow velocity for the dipole configuration. From $\nabla \cdot J = 0$ and the poloidal component of Ampere’s law we find $J_\parallel = 0$. Consequently, we anticipate that $V_\parallel = 0$ since the system is up-down symmetric. More systematically, we use Eq. (9) to write

$$ V_\parallel = K(\psi) B_n. $$

The unknown flux function $K(\psi)$ can be evaluated from the flux-surface averaged parallel momentum equation. In this equation the standard Pfirsch-Schlüter tokamak treatment only retains the ion parallel viscosity, since it is $\Delta/\delta$ larger than the ion gyroviscosity, to find to lowest order that $K(\psi)$ is proportional to the toroidal magnetic field$^{8,11}$ and so vanishes in a dipole.

If higher order corrections (including those from the ion gyroviscosity) are retained then we find that $K(\psi) \propto \langle R^2 \nabla_\parallel B \rangle_\theta$, where the quantity $\langle R^2 \nabla_\parallel B \rangle_\theta$ is equal to zero for an up-down symmetric configuration. Therefore, up-down asymmetry results in finite parallel plasma flows. However, in addition to being proportional to
the asymmetry factor $\langle R^2 \nabla_B \rangle_\theta$, such parallel flows are already small compared with the toroidal flows because the toroidal magnetic field vanishes. As a result, typically, $V_\parallel / V_\perp \ll \delta / \Delta$. Therefore, the approximation $V_\parallel = 0$ is accurate enough for our purposes not only for up-down symmetric, but also for moderately up-down asymmetric configurations. Taking this fact into account and employing Eq. (8) we can express the full plasma flow (6) in a dipole as

$$V = -c \left( \frac{\partial \varphi}{\partial \psi} + \frac{1}{en} \frac{\partial p}{\partial \psi} \right) R^2 \nabla \zeta \equiv \omega(\psi, \theta, t) R^2 \nabla \zeta. \quad (43)$$

Since $n, p, \varphi$ are flux function to leading order, so is $\omega$.

Next, we evaluate the parallel ion heat flux $q_\parallel$. Employing Eq. (14) with $V_\parallel = 0$ we obtain

$$q_\parallel = L(\psi) B + B \int \frac{d\theta}{B \cdot \nabla_\theta} \left( \langle \nabla \cdot q_c \rangle_\theta - \nabla \cdot q_c \right), \quad (44)$$

where the unknown flux function $L(\psi)$ can be determined from the constraint $^6 \langle B q_\parallel \rangle_\theta = 0$. Evaluating $L(\psi)$ and plugging the result back into Eq. (44) we obtain the final expression for $q_\parallel$:

$$q_\parallel = B \int \frac{d\theta}{B \cdot \nabla_\theta} \left( \langle \nabla \cdot q_c \rangle_\theta - \nabla \cdot q_c \right) - \frac{B}{\langle B^2 \rangle_\theta} \left( B^2 \int \frac{d\theta}{B \cdot \nabla_\theta} \left( \langle \nabla \cdot q_c \rangle_\theta - \nabla \cdot q_c \right) \right)_\theta. \quad (45)$$

Notice that according to Eq. (45) $q_\parallel \sim p v_{Ti} (\delta^2 / \Delta)$, so that $|\nabla_\parallel T|/|\nabla_\perp T| \sim (\delta / \Delta)^2$. Poloidal variations of $n$ and $\varphi$ are comparable with that of $T$. This result is different from that of the standard tokamak Pfirsch-Schlüter theory,$^{10}$ which predicts that $q_\parallel \sim p v_{Ti} \delta$ and $|\nabla_\parallel T|/|\nabla_\perp T| \sim (\nu / \Omega) \sim (\delta / \Delta)$, because of the absence of a toroidal magnetic field. Taking this estimate for $q_\parallel$ into account we can write the lowest order heat flux in a dipole as

$$q = -\frac{5cp}{2e} \frac{\partial T}{\partial \psi} R^2 \nabla \zeta \equiv \frac{5}{2} s(\psi, \theta, t) R^2 \nabla \zeta. \quad (46)$$
Having evaluated $V_\parallel$ and $q_\parallel$ we can now proceed to obtaining the equation governing self-consistent radial electric field. Dotting momentum equation (10) by $R^2 \nabla \zeta$, employing expression (43) for $V$, using Ampere’s law (11), and flux-surface averaging, we obtain

$$\frac{\partial}{\partial t} \left( M n \langle \omega R^2 \rangle_\theta \right) + \frac{1}{V' \partial \psi} \left( V' \left( R^2 \nabla \zeta \cdot \overrightarrow{\pi} \cdot \nabla \psi \right)_\theta \right) = 0,$$

(47)

where $\omega$ is defined by Eq. (43). Since $\nabla \zeta \cdot \nabla \psi = \hat{b} \cdot \nabla \psi = 0$ the parallel viscosity $\overrightarrow{\pi}_\parallel$ does not contribute to the second term on the left-hand side. Therefore, we only need $R^2 \nabla \zeta \cdot \overrightarrow{\pi}_g \cdot \nabla \psi$ and $R^2 \nabla \zeta \cdot \overrightarrow{\pi}_\perp \cdot \nabla \psi$.

To evaluate the gyroviscous contribution we use expression (16) for the ion gyroviscosity and vector identities $\hat{b} \times \nabla \psi = -BR^2 \nabla \zeta$, $\hat{b} \times R^2 \nabla \zeta = \nabla \psi / B$, and $\nabla \psi \nabla \psi = R^2 B^2 \left( \overrightarrow{I} - \hat{b} \hat{b} - R^2 \nabla \zeta \nabla \zeta \right)$ to write

$$R^2 \nabla \zeta \cdot \overrightarrow{\pi}_g \cdot \nabla \psi = \frac{pB R^2}{2 \Omega} \left[ 2R^2 \left( \nabla \zeta \cdot \overrightarrow{\alpha} \cdot \nabla \zeta \right) + \left( \hat{b} \cdot \overrightarrow{\alpha} \cdot \hat{b} \right) - \left( \overrightarrow{I} \cdot \overrightarrow{\alpha} \right) \right].$$

(48)

Noticing that

$$2pR^4 \left( \nabla \zeta \cdot \overrightarrow{\alpha} \cdot \nabla \zeta \right) = \left( pV + \frac{2}{5} q \right) \cdot \nabla R^2$$

$$= \nabla \cdot \left[ \left( pV + \frac{2}{5} q \right) R^2 \right] - R^2 \nabla \cdot \left( pV + \frac{2}{5} q \right),$$

employing the ion temperature evolution equation (13), which reads to the order required as

$$\nabla \cdot \left( pV + \frac{2}{5} q \right) = \frac{2}{5} \langle \nabla \cdot q_c \rangle_\theta,$$

(49)

and noticing that $V \cdot \nabla \psi = 0$ to the order required, we obtain

$$\left\langle 2pR^4 \left( \nabla \zeta \cdot \overrightarrow{\alpha} \cdot \nabla \zeta \right) \right\rangle_\theta = \frac{2}{5} \left\langle \nabla \cdot \left( q_c R^2 \right) \right\rangle_\theta - \frac{2}{5} \langle R^2 \rangle_\theta \langle \nabla \cdot q_c \rangle_\theta.$$

(50)
Rewriting
\[ p \hat{b} \cdot \hat{\alpha} \cdot \hat{b} = p \nabla_{\parallel} V_{\parallel} + \frac{2}{5} \nabla_{\parallel} q_{\parallel} - \kappa \cdot \left( p \nabla + \frac{2}{5} q \right) = \frac{2}{5} \nabla_{\parallel} q_{\parallel} - \frac{2}{5} \kappa \cdot q_c, \quad (51) \]

with \( \kappa \equiv \hat{b} \cdot \nabla \hat{b} \) the magnetic field line curvature, employing expression (45) for \( q_{\parallel} \), and noticing that \( \kappa \cdot \nabla \psi = -R^2 B \nabla \cdot (\nabla \psi / R^2 B) \), so that
\[
\kappa \cdot q_c \approx -\frac{1}{B} \nabla \cdot \left( \frac{\nabla \psi}{R^2 B} \right) \left( q_c \cdot \nabla \psi \right)
\]
(where we neglected poloidal temperature variation), we arrive at
\[
\left\langle R^2 p \left( \hat{b} \cdot \hat{\alpha} \cdot \hat{b} \right) \right\rangle_\theta = \frac{2}{5} \left\langle R^2 \right\rangle_\theta \left\langle \nabla \cdot q_c \right\rangle_\theta - \frac{2}{5} \left\langle R^2 \left( \nabla \cdot q_c \right) \right\rangle_\theta \quad (52)
\]
\[
+ \frac{2}{5} \left\langle \frac{R^2}{B} \nabla \cdot \left( \frac{\nabla \psi}{R^2 B} \right) \left( q_c \cdot \nabla \psi \right) \right\rangle_\theta + \frac{2}{5} \left\langle R^2 \nabla_{\parallel} B \left[ \int \frac{d\theta}{B \cdot \nabla \theta} \left( \left\langle \nabla \cdot q_c \right\rangle_\theta - \nabla \cdot q_c \right) \right] \right\rangle_\theta.
\]

Finally, noticing that
\[
p \hat{\pi} \cdot \hat{\alpha} = p \nabla \cdot \mathbf{V} + \frac{2}{5} \nabla \cdot q \quad (53)
\]
and employing Eq. (49) with \( \mathbf{V} \cdot \nabla p \) neglected as small, we find
\[
\left\langle R^2 p \hat{\pi} \cdot \hat{\alpha} \right\rangle_\theta = \frac{2}{5} \left\langle R^2 \right\rangle_\theta \left\langle \nabla \cdot q_c \right\rangle_\theta \cdot (54)
\]

Putting results (50), (52), and (54) together we can write the required gyroviscous contribution as
\[
\left\langle R^2 \nabla \zeta \cdot \hat{\pi}_g \cdot \nabla \psi \right\rangle_\theta = \frac{B}{5 \Omega} \left\{ \left\langle q_c \cdot \nabla R^2 \right\rangle_\theta - \left\langle R^2 \right\rangle_\theta \left\langle \nabla \cdot q_c \right\rangle_\theta \right\} (55)
\]
\[
+ \left\langle \frac{R^2}{B} \nabla \cdot \left( \frac{\nabla \psi}{R^2 B} \right) \left( q_c \cdot \nabla \psi \right) \right\rangle_\theta + \left\langle R^2 \nabla_{\parallel} B \left[ \int \frac{d\theta}{B \cdot \nabla \theta} \left( \left\langle \nabla \cdot q_c \right\rangle_\theta - \nabla \cdot q_c \right) \right] \right\rangle_\theta \right\}.
\]

To evaluate the contribution from the perpendicular viscosity we use Eq. (17) to write to the order required
\[
R^2 \nabla \zeta \cdot \hat{\pi}_p \cdot \nabla \psi = -\frac{3\nu}{10 \Omega^2} \left( R^2 \nabla \zeta \cdot \hat{W} \cdot \nabla \psi \right) - \frac{9M \nu B}{200 pT \Omega} \left( R^2 \nabla \zeta \cdot q \right)^2. \quad (56)
\]
Employing the lowest order expressions (43) and (46) for \(V\) and \(q\), respectively, we find

\[
R^2 \nabla \zeta \cdot \vec{\omega} \cdot \nabla \psi = R^4 B^2 \left[ \frac{\partial \omega}{\partial \psi} + \frac{1}{4} \frac{\partial s}{\partial \psi} + \frac{3}{4} s \left( \frac{\partial \ln p}{\partial \psi} - \frac{3}{4} \frac{\partial \ln T}{\partial \psi} \right) \right]
\]

and \(R^2 \nabla \zeta \cdot q = 5sR^2/2\), so that to the order required

\[
\langle R^2 \nabla \zeta \cdot \vec{\omega} \cdot \nabla \psi \rangle = - \frac{3B^2}{10\Omega^2} \langle \nu \rangle \langle R^4 \rangle \left[ \langle p \rangle \left( \frac{\partial \omega}{\partial \psi} + \frac{1}{4} \frac{\partial s}{\partial \psi} + \frac{3}{4} s \left( \frac{\partial \ln p}{\partial \psi} - 2 \frac{\partial \ln T}{\partial \psi} \right) \right) \right].
\]

Equation (47) with \(\langle R^2 \nabla \zeta \cdot \vec{\omega} \cdot \nabla \psi \rangle\) given by the sum of right-hand sides of Eqs. (55) and (57) describes evolution of radial electric field in an up-down symmetric or even moderately up-down asymmetric magnetic dipole (or any other closed field line) configuration. Since \(\omega\) is a lowest order flux function, \(\omega \approx \langle \omega \rangle = -c(\partial \langle \varphi \rangle / \partial \psi + e^{-1} \langle n \rangle^{-1} \partial \langle p \rangle / \partial \psi\), we can use \(\langle \omega R^2 \rangle \approx \langle \omega \rangle \langle R^2 \rangle\) to write the full evolution equation for \(\langle \omega \rangle\) as

\[
M \langle R^2 \rangle \frac{\partial}{\partial t} \langle n \rangle \langle \omega \rangle + \frac{B}{5\Omega V} \frac{\partial}{\partial \psi} \left\{ V' \left( \langle q \cdot \nabla R^2 \rangle - \langle R^2 \rangle \langle \nabla \cdot q \rangle \right) \right\} + V' \left( \frac{R^2}{B} \nabla \cdot \left( \frac{\nabla \psi}{R^2 B} \right) \langle q \cdot \nabla \psi \rangle \right) - \frac{3B^2}{10\Omega^2 V} \frac{\partial}{\partial \psi} \left\{ V' \langle \nu \rangle \langle R^4 \rangle \left[ \langle p \rangle \left( \frac{\partial \omega}{\partial \psi} + \frac{1}{4} \frac{\partial s}{\partial \psi} + \frac{3}{4} s \left( \frac{\partial \ln p}{\partial \psi} - 2 \frac{\partial \ln T}{\partial \psi} \right) \right) \right] \right\}
\]

Here, to the order required \(\langle s \rangle = -ce^{-1} \langle p \rangle \partial \langle T \rangle / \partial \psi\). Of course, only classical transport effects enter this equation for \(\langle \omega \rangle\), and if \(\partial T / \partial \psi = 0\) we obtain the simple result

\[
M \langle R^2 \rangle \frac{\partial}{\partial t} \langle n \rangle \langle \omega \rangle - \frac{3B^2}{10\Omega^2 V^2} \frac{\partial}{\partial \psi} \left( V' \langle p \rangle \langle \nu \rangle \langle R^4 \rangle \frac{\partial \langle \omega \rangle}{\partial \psi} \right) = 0,
\]

giving the Maxwell-Boltzmann behavior \(\langle \omega \rangle = \text{constant}\) in the steady state.
VI. CONCLUSIONS

In the preceding sections we consider collisional plasma confined by the magnetic field of a screw-pinch and a dipole magnetic configuration (or any axisymmetric configuration with closed poloidal field lines). We derive the equations describing the evolution of the self-consistent radial electric field and parallel plasma flow for situations when the effects of plasma turbulence are negligible.

For a general screw-pinch configuration, when both axial and azimuthal magnetic fields are present, the radial electric field evolution equation, Eq. (24), is coupled with the parallel flow evolution equation, Eq. (21), and both must be solved simultaneously. The off-diagonal ion viscosity components \((\mathbf{r} \cdot \mathbf{\pi} \cdot \mathbf{\hat{\theta}}) = (\mathbf{r} \cdot \mathbf{\pi}_g \cdot \mathbf{\hat{\theta}}) + (\mathbf{r} \cdot \mathbf{\pi}_\perp \cdot \mathbf{\hat{\theta}})\) and \((\mathbf{r} \cdot \mathbf{\pi} \cdot \mathbf{\hat{z}}) = (\mathbf{r} \cdot \mathbf{\pi}_g \cdot \mathbf{\hat{z}}) + (\mathbf{r} \cdot \mathbf{\pi}_\perp \cdot \mathbf{\hat{z}})\) occurring in these equations are given by Eqs. (28), (29), (32), and (33). For steady state situations and in the absence of momentum sources and sinks Eqs. (21) and (24) can be solved analytically. The solution is given by Eqs. (40) and (41). When one of the magnetic field components is equal to zero, i.e. the screw-pinch degenerates into a \(\theta\)-pinch or a Z-pinch, the parallel flow and the radial electric field equations decouple. The parallel flow equation admits a simple solution \(V_{\parallel}(r, t) = \text{constant}\) for a \(\theta\)-pinch and \(V_{\parallel}(r, t)/r = \text{constant}\) for a Z-pinch, whereas the radial electric field equation can again be solved analytically, provided a steady state situation without momentum sources and sinks is assumed. The solutions are given by Eqs. (35) for a Z-pinch \((B_z = 0)\) and (37) for a \(\theta\)-pinch \((B_\theta = 0)\). They correspond to particular cases of the general solution given by Eq. (41).

For a plasma confined by a dipole magnetic field, we considered for simplicity the case of an up-down symmetric (or moderately up-down asymmetric) configuration. The parallel plasma flow is equal to zero (or negligibly small) in this case and
The results obtained herein along with the more familiar number and energy conservation equations\textsuperscript{10} allow the electrostatic potential, the plasma density, and the ion and electron temperatures to be evaluated in screw-pinch and dipole configurations.

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