FLUID EFFECTS IN VIBRATING MICROMACHINED STRUCTURE

by

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Submitted to the Department of Mechanical Engineering
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ABSTRACT

This thesis presents the study of the fluid damping and surfboarding effects for
the tuning fork gyroscope. The quality factors in the drive and sense axes will be
evaluated and compared with the experimental results for a range of pressures. The
effects of the holes and the proof mass thickness (chimney) will be derived and
discussed, and a parametric study on several design parameters will be performed.
An analytical model based on the classic slider bearing with slip boundary will be derived
and numerical models will be developed to estimate the lift force from “surfboarding”,
and the numerical solution will be compared with the bias of the TFG from experiments
over a range of pressures.

Original contribution includes 1). Experimental work performed to obtain the in-
phase bias and quality factors in the drive and sense axes, 2). Data post-processing
technique developed to obtain the structural and fluid damping of the tuning fork
gyroscope, 3). Numerical simulations of the normalized Reynolds squeeze film equation
and normalized Reynolds slider bearing equation on nontrivial geometry, and 4).
Network model developed to solve for the pressure distribution from surfboarding with
the chimney effect.

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Peter Knook
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Chapter 1.
Introduction

1.1 - Background

Recent advancements in the microfabrication technology have enabled various designs of microelectromechanical systems (MEMS), such as microaccelerometers and micromechanical tuning fork gyroscopes. MEMS were first developed in silicon using techniques such as deposition, etching, doping and lithography. The small size scale of MEMS in general allows the devices to be very portable, and the inertia and gravitational effects are greatly reduced compared with conventional devices. In addition, the fabrication techniques allow a large number of identical devices to be processed on a single wafer at the same time. This greatly reduces the cost of manufacturing processes. As a result of these generic advantages, MEMS are widely developed and applied in various areas, including radiation sensors for space applications, inertial navigation for munitions guidance, and devices detecting strain, force, pressure, flow, acceleration, position, temperature, chemicals, etc.

A micromechanical tuning fork gyroscope (TFG) is a device that detects angular velocity (angular rate). It uses two parallel plates (proof masses) with square holes evenly distributed as electrodes for capacitive sensing. The proof masses are attached to the anchors with elastic beams and separated from the substrate by a small gap h. The three coordinate axes, x, y, z, are referred as the drive, sense and input axes, respectively. An illustration of the TFG is shown in Fig. 1.1.
Voltage supplied from the outer combs will cause the proof masses to oscillate out of phase in the drive axis. When an angular rate is applied to the input axis, the two parallel plates will have displacements in the sense axis caused by the Coriolis acceleration. Thus the tuning fork gyroscope detects angular rate by capturing the differential capacitance. An actual picture of a TFG is shown in Fig. 1.2.
Figure 1.2: A picture of a micromechanical tuning fork gyroscope (TFG).

The performance of the tuning fork gyroscope is evaluated based on the quality factor in the drive and sense axes, which are referred as the drive $Q$ and sense $Q$, respectively. The quality factor is inversely proportional to the total damping, which is the sum of the structural and fluid damping in the system. A high drive $Q$ system provides better mechanical gain, thus only low voltage is needed to excite the oscillation, and significant amount of power can be saved. A high sense $Q$ system will have higher sense displacement for a given input rate, thus amplifies the signal and provides better sensitivity. Therefore in order to improve the accuracy of the angular rate measurement, it is imperative to understand the detailed flow characteristics within the gyroscope unit.

In many designs the tuning fork gyroscope is placed inside a sealed package with a low ambient pressure in order to minimize the fluid damping caused by air viscosity. The gas molecules become further away from each other as the ambient pressure decreases, and eventually can be treated as discrete particles instead of a fluid continuum. This is referred as the rarefaction effect. The gas flow can be classified into four regimes: continuum flow, slip flow, transitional flow and molecular flow regimes. Each regime corresponds to the degree of rarefaction, where continuum is the most dense and molecular flow is the least dense. Holes through the proof masses are designed to relieve
the pressure buildup underneath when the proof mass is oscillating in the sense axis, and the combs are used to provide power as mentioned. This nontrivial geometry further complicates the airflow. An illustration of the design parameters, such as the gap height, hole spacing, combs overlap, etc., is shown in Fig. 1.3, also typical dimensions of TFG A and B are tabulated in Table 1.1.

Figure 1.3: An illustration of the design parameters.
<table>
<thead>
<tr>
<th>Parameters</th>
<th>TFG A</th>
<th>TFG B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proof mass length (L)</td>
<td>1000µm</td>
<td>450µm</td>
</tr>
<tr>
<td>Proof mass width (b)</td>
<td>1000µm</td>
<td>400µm</td>
</tr>
<tr>
<td>Width of hole (L_h)</td>
<td>5.5µm</td>
<td>4.5µm</td>
</tr>
<tr>
<td>Hole spacing (L_p)</td>
<td>12µm</td>
<td>10µm</td>
</tr>
<tr>
<td>Thickness (t)</td>
<td>20µm</td>
<td>10µm</td>
</tr>
<tr>
<td>Combs air gap (g_c)</td>
<td>2µm</td>
<td>3µm</td>
</tr>
<tr>
<td>Combs overlap length (L_c)</td>
<td>25µm</td>
<td>25µm</td>
</tr>
<tr>
<td>Gap height (h)</td>
<td>3µm</td>
<td>3µm</td>
</tr>
</tbody>
</table>

Table 1.1: Typical dimensions of the design parameters for TFG.

In addition to the fluid damping in the drive and sense axes, experiments had found that there exists a bias in the angular rate measurement that is in-phase with the drive velocity of the TFG. This in-phase bias is contributed by several sources, one of which is the “surfboarding” effect.

Ideally, the two proof masses are perfectly flat and parallel to the substrate. Hence the gap height is uniform across the proof mass. However, the gap between the proof masses and substrate is very often not uniform across when fabricated. The proof mass is tilted at an angle with respect to the substrate, causing the pressure to build up underneath the proof masses and generating unwanted lift force, known as the surfboarding effect. This lift force can contribute significant error to the angular rate measurement. An illustration is shown in Fig. 1.4:
Figure 1.4: Oscillation of a tilted proof mass and the pressure profile underneath the plate.

This thesis presents the study of the fluid damping and surfboarding effects for the tuning fork gyroscope. The quality factors in the drive and sense axes will be evaluated and compared with the experimental results for a range of pressures. The effects of the holes and the proof mass thickness (chimney) will be derived and discussed, and a parametric study on several design parameters will be performed.

Analytical model based on the classic slider bearing with slip boundary will be derived and numerical models will be developed to estimate the lift force from surfboarding. The numerical solution will be compared with the bias of the TFG, which can be interpreted as a Coriolis force, from experiments over a range of pressures.

Original contribution includes 1). Experimental work performed to obtain the in-phase bias and quality factors in the drive and sense axes, 2). Data post-processing technique developed to obtain the structural and fluid damping of the tuning fork gyroscope, 3). Numerical simulations of the normalized Reynolds squeeze film equation and normalized Reynolds slider bearing equation on nontrivial geometry, and 4).
Network model developed to solve for the pressure distribution from surfboarding with the chimney effect.

The modified Reynolds equation, which accounts for slight rarefaction had been derived in 1959 by Burgdorfer [5] using the slip flow conditions from Schaaf and Sherman [28]. As the size of the structure became smaller in the last decade, the degree of rarefaction became larger. As a result, models for highly rarefied flow were developed. Veijola et al. [34] approximated the effective viscosity based on the Boltzmann equation. Many other approximation methods were derived from experimental data fitting, such as Seidel et al. [29]. In this study, the governing Couette flow equation will be re-derived from the Navier-Stokes equation with slip flow conditions. Arkillic and Breuer [7] used the slip flow boundary condition to model the Couette-Poiseuille laminar flow in a small channel, which can be used to derive the damping calculation in the drive axis. Similar approach was performed by Cho et al. [8].

Blech [4] had solved the linearized Reynolds equation for squeeze film damping. At low squeeze number, the solution showed that the gas film is dominated by the viscous damping effect, while at high squeeze number, it is dominated by the spring force. His work will be modified using slip flow boundary conditions to include the rarefaction effect to calculate the gas damping in the sense axis.

The surfboarding analysis is based on lubrication theory by Osborne Reynolds in 1886 [12], which was developed and applied for slider bearing by W. J. Harrison in 1913 [12]. The Reynolds equation for a steady incompressible gas film will be modified to include the rarefaction effect.

1.2 – Thesis Outline

In Chapter 2, the analytical evaluation of the quality factor of drive axis, will be presented. The continuum flow, slip flow and molecular flow theories are used to analyze the experimental results. The experimental setup and procedure will be described.
In Chapter 3, the analytical evaluation of the quality factor of sense axis, will be presented. The Reynolds equation for squeeze film with rarefaction effect will be derived and linearized. This equation will be solved and compared with the experimental results.

In Chapter 4, the in-phase bias caused by the surfboarding effect will be discussed. The Reynolds equation for slider bearing will be derived and linearized with rarefaction effect. Several numerical approaches for solving the slider bearing equation are presented.

Finally, a summary of this work and conclusion will be presented in Chapter 5.
Chapter 2.

Fluid Damping in the Drive Axis

In this chapter, analytical methods used to determine the quality factor of the drive axis will be presented. The rarefaction effect will be discussed. The procedure for the ring-down experiment will be described and performed on various gyros, and the results will be compared with the analytical solution.

2.1 - Analytical Evaluation of the Quality Factor in the Drive Axis

The two proof masses of the tuning fork gyroscope oscillate in the drive axis under a power supply, as shown in Fig. 2.1.

![Diagram](image)

Figure 2.1: An illustration on the drive motion of the tuning fork gyroscope.

Since the gap-height to plate-length ratio is very small, typically about 0.003 to 0.006, the two parallel plates moving in relative tangential motion will experience shear stress acting on the surfaces. The velocity profile of the fluid in the gap, with the top surface (proof mass) moving horizontally, is shown in Fig. 2.2.

![Diagram](image)

Figure 2.2: The velocity profile of the fluid flow.
The fluid flow is governed by the Navier-Stokes equation

\[ \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \]  

\[ (2.1a) \]

\[ \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \]  

\[ (2.1b) \]

\[ \rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \]  

\[ (2.1c) \]

where \( p \) is the pressure, \( \rho \) is the density, \( \mu \) is the viscosity of the fluid, and \( u, v \) and \( w \) are the velocities of the fluid in the \( x, y \) and \( z \)-directions, respectively.

The flow is driven by the velocity in the \( x \)-direction only, hence (2.1b) and (2.1c) and all the terms in (2.1a) with \( v \) and \( w \) can be eliminated, i.e.,

\[ \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) . \]  

\[ (2.1d) \]

For the gap-height to plate-length ratio to be very small, the assumption of an infinite plate in the \( x \)-direction is applicable, the terms with \( \frac{\partial}{\partial x} \) become very small compared to \( \frac{\partial}{\partial z} \) and therefore can be eliminated. Also the variation in the \( y \)-direction is neglected for the two-dimensional model considered, resulting:

\[ \rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2} . \]  

\[ (2.1e) \]
The unsteadiness term, $\rho \frac{\partial u}{\partial t}$, can be estimated using order of magnitude analysis as $\rho \frac{U}{T}$, where $U$ is the plate velocity and $T$ is the period of the oscillation. The period $T$ can be represented as $\frac{1}{\omega}$ where $\omega$ is the frequency of the oscillation. The viscous term, $\mu \frac{\partial^2 u}{\partial z^2}$, can be estimated as $\mu \frac{U}{H^2}$, where $H$ is the gap height.

The unsteadiness can be eliminated if the Stokes number, defined as the ratio of the unsteadiness to the viscous term, $\frac{\rho \omega H^2}{\mu}$, is small. A typical driving frequency of the TFG is about $10^5$ rad/s, the density of air is close to 1 kg/m$^3$, the viscosity of air is about $10^{-5}$ N-s/m$^2$ at standard conditions and the gap height is $10^{-6}$ m.

The Stokes number is therefore: \[
\frac{(1)(10^5)(10^{-12})}{(10^{-5})} = 10^2,
\] which indicates that the unsteadiness contributes roughly 1% to the solution. Note that this calculation is based on atmospheric conditions, and the Stokes number will become smaller at lower pressures because the density is greatly reduced. Hence it can be eliminated and the equation becomes:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2}. \quad (2.2)$$

The pressure is constant along $x$ and thus the left-hand side of (2.2) is zero. This flow type is known as Couette flow. Solving for $u$ with the no-slip boundary conditions: $u(0) = u(h) = 0$, we have:

$$u(z) = \frac{U}{h} z \quad (2.3)$$

or
\[
\frac{u(z)}{U} = \frac{z}{h}. \tag{2.4}
\]

The shear stress exerted on the moving plate is the product of the velocity gradient and the fluid viscosity, i.e.,

\[
\tau = \mu \frac{\partial u}{\partial z} = \mu \frac{U}{h}, \tag{2.5}
\]

where \(h\) is the distance between the plates. The shear force is determined by integrating the shear stress over the area, which in this case can be approximated as the product of the effective (solid) area and the shear stress obtained above, i.e.,

\[
F = \tau A = \mu \frac{U}{h} A, \tag{2.6}
\]

where \(A\) is the effective area of the plate.

For viscously-damped free vibration [24], the damping coefficient, \(c\), is found by dividing the shear force by the velocity of the plate,

\[
c = \frac{F}{U} = \frac{\mu A}{h}. \tag{2.7}
\]

The critical damping coefficient, \(C_c\), is defined as

\[
C_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n, \tag{2.8}
\]

and the damping ratio, \(\zeta\), is determined by the ratio of the damping coefficient to the critical damping coefficient, i.e.,
\[ \zeta = \frac{c}{C_c} = \frac{\mu A}{h(2\mu \omega_n)}. \] (2.9)

The quality factor, Q, is defined as

\[ Q = \frac{1}{2\zeta} = \frac{m \omega_n h}{\mu A}. \] (2.10)

It can be concluded from (2.10) that the continuum flow solution is independent of pressure since the viscosity at standard atmospheric conditions has slight dependence on the temperature only. It is important to note that the quality factor calculated in (2.10) has neglected the structural (solid) damping. In general, the total damping is the sum of the fluid and solid damping, i.e., \( \zeta_{\text{total}} = \zeta_{\text{fluid}} + \zeta_{\text{solid}} \). Thus (2.10) is valid for \( \zeta_{\text{solid}} \ll \zeta_{\text{fluid}} \).

2.2 - Rarefaction Consideration

In a gas at sufficiently low pressure the length of the molecular mean free path – a measure of the probability to undergo interactions between the molecules – will become comparable to the characteristic length of the system [13], which is the gap height (3μm) in this case. The gas subjected to this condition does not behave entirely as a continuous fluid but rather exhibits characteristics as discrete molecules. This is referred as rarefied gas. The dimensionless ratio, \( \frac{\lambda}{h} \) (Knudsen number), is a measure of the degree of rarefaction. When this ratio is large, i.e., the mean free path, \( \lambda \), is much greater than the characteristic length, \( h \), the flow phenomena are mostly dictated by the molecular-surface interaction. Based on the Knudsen number, the flow regime can be divided into continuum flow, slip flow, transitional flow, and free molecular flow regimes, as shown in Table 2.1.
<table>
<thead>
<tr>
<th>Continuum Flow</th>
<th>Kn &lt; 0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slip Flow</td>
<td>0.001 &lt; Kn &lt; 0.1</td>
</tr>
<tr>
<td>Transitional Flow</td>
<td>0.1 &lt; Kn &lt; 10</td>
</tr>
<tr>
<td>Molecular Flow</td>
<td>Kn &gt; 10</td>
</tr>
</tbody>
</table>

Table 2.1. Flow regimes and the corresponding range of Knudsen numbers

The mean free path of the fluid can be evaluated by the total distance traveled by the molecules in time $\Delta t$ divided by the number of collisions in this time [14][18], i.e.,

$$\lambda = \frac{\bar{v} \Delta t}{\sqrt{2n} \pi D^2 \bar{v} \Delta t}.$$  \hspace{1cm} (2.11)

where $\bar{v}$ denotes the average velocity of the molecules, and the collision frequency is given as $\sqrt{2n} \pi D^2$, where $n$ is the number of molecules, and $D$ is the molecular diameter ($3.7 \times 10^{-10}$ m for air [9]). The number of molecules is governed by the pressure and temperature of the gas, i.e.,

$$n = \frac{N_a P}{RT},$$  \hspace{1cm} (2.12)

where $N_a$ is the Avogadro’s number ($6.02 \times 10^{23}$ molecules/gm-mole), $T$ is the absolute temperature (300 K) and $R$ is the universal gas constant (8.31 N-m/K-gm-mole). Substituting we obtain

$$\lambda = \frac{1}{\sqrt{2\pi} \frac{N_a P}{RT} D^2}.$$  \hspace{1cm} (2.13)

Knowing that the TFG operates typically in the pressure range from 1 to 50 mTorr (0.133 Pa to 6.68 Pa), the Knudsen number is on the order of $10^4$ with a 3$\mu$m gap. This
shows that the fluid flow is in the molecular regime. In this study, both the molecular flow model and the slip flow model will be used to characterize the discrepancy.

2.2a – Slip Flow Model

If the gap height is comparable to the mean free path of the fluid particles, slip flow will occur at the boundary. The velocity profile will change by the slip velocities at the boundaries as shown in Fig 2.3:

Figure 2.3: The slip velocity profile of the fluid flow.

The slip flow effect will reduce the shear force exerted on the plates and decrease the viscous damping.

The Couette flow with slip has been derived [1][27] from the Navier-Stokes equation (2.1a-2.1c). With the same order of magnitude analysis performed in the last section, the equation can be simplified as:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2}. \quad (2.14)$$

The slip boundary conditions are $u = \lambda \frac{\partial u}{\partial z}$ at $z = 0$ and $u = U - \lambda \frac{\partial u}{\partial z}$ at $z = h$.

Solving for $u,$
\[ u(z) = \frac{U}{h + 2\lambda}(z + \lambda) \]  \hspace{1cm} (2.15)

or

\[ \frac{u(z)}{U} = \frac{(z + \lambda)}{h} = \frac{z}{h} + \frac{Kn}{1 + 2Kn} \]  \hspace{1cm} (2.16)

As Kn goes to 0 (continuum regime), the velocity profile becomes

\[ \frac{u(z)}{U} = \frac{z}{h}, \]  \hspace{1cm} (2.17)

which is the same as the classic Couette flow solution.

The shear stress is found by the product of the velocity gradient and the dynamic viscosity, i.e.,

\[ \tau = \mu \frac{\partial u}{\partial y} = \mu \left( \frac{U}{h + 2\lambda} \right), \]  \hspace{1cm} (2.18)

and the shear force becomes:

\[ F = \frac{\mu UA}{(1 + 2Kn)h}. \]  \hspace{1cm} (2.19)

Following (2.7) to (2.10) above, we obtain

\[ Q = \frac{m\omega_n (1 + 2Kn)}{\mu A}. \]  \hspace{1cm} (2.20)
When the proof mass is oscillating in the drive axis, the combs on the both sides of the proof mass and the inner and outer combs are moving tangentially with each other, as shown in Fig. 2.4.

![Diagram of combs](image)

Figure 2.4: An illustration of the combs.

t is the thickness of the proof mass, \( g_c \) is the air gap between the combs and \( L_c \) is the nominal overlap length of the combs.

The oscillating motion induces Couette flow in between the combs, thus generating additional damping. The flow between the combs is modeled the same way as the proof mass and the substrate, with \( g_c \) being the characteristic length. For \( g_c \ll L_c \), the assumption of infinite plate and steady flow applies in the combs also. In addition, the squeezing of the fluid by the tip of each comb toward the "pocket" at the end will result
in three-dimensional flow phenomenon, thus it is neglected for simplicity. Solving for the simplified Navier-Stokes equation,

\[
\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}
\]  \hspace{1cm} (2.21)

with slip boundary conditions, \( u = \lambda \frac{\partial u}{\partial z} \) at \( y = 0 \) and \( u = U - \lambda \frac{\partial u}{\partial z} \) at \( y = g_e \), we obtain

\[
u(y) = \frac{U}{g_e + 2\lambda} (y + \lambda)
\]  \hspace{1cm} (2.22)

or

\[
\frac{\nu(y)}{U} = \frac{\frac{y + Kn}{g_e}}{\frac{g_e}{g_e + 2\lambda}} = \frac{g_e}{1 + 2Kn}.
\]  \hspace{1cm} (2.23)

The damping generated from the combs is therefore

\[
c_{\text{combs}} = \frac{2nA_{\text{combs}} \mu}{g_e + 2\lambda},
\]  \hspace{1cm} (2.24)

where \( n \) is the total number of combs on both sides of the proof mass.

Note that \( A_{\text{combs}} \), the overlapping area of each pair of combs, remains constant, i.e., \( t \times L_e \). Because as the proof mass moves to the right, the decrease in the overlapping area on the left will be equal to the increase of the overlapping area on the right. The total shearing area is therefore \( 2nA_{\text{combs}} \) for each comb has both the upper and lower surfaces that shear. Summing the damping from the proof mass and the combs, we have
\[ c = \frac{\mu A}{h + 2\\lambda} + \frac{\mu 2nA_{\text{comb}}}{g_c + 2\\lambda}, \]  \hspace{1cm} (2.25)

and the resulting \( Q \) is

\[ Q = \frac{m\omega_n}{\frac{\mu A}{h + 2\\lambda} + \frac{\mu 2nA_{\text{comb}}}{g_c + 2\\lambda}}. \]  \hspace{1cm} (2.26)

2.2b – Molecular Flow Model

For free molecular flow, the friction factor is given as [27]:

\[ C_f = \sqrt{\frac{2}{\pi \gamma}} \frac{1}{M}, \]  \hspace{1cm} (2.27)

where \( \gamma = 1.4 \) for air.

\( M \) is the Mach number, defined as

\[ M = \frac{U}{a}, \]  \hspace{1cm} (2.28)

where \( a \) is the speed of sound.

The shear stress \( \tau \) is then evaluated as follow:

\[ \tau = \frac{1}{2} \rho U^2 C_f = \frac{1}{2} \rho U \sqrt{\frac{2}{\pi \gamma} a}. \]  \hspace{1cm} (2.29)

Note that the shear stress is independent of the spacing between the surfaces. Therefore, it is important to realize that the shear force exerted on the bottom of the proof mass no
longer dominates at molecular flow regime. The shear stress should account for all the shearing area, including the top surface. An illustration is shown below:

![Illustration of shear stress at molecular regime](image)

Figure 2.5: An illustration of the shear stress at molecular regime.

The total shear force becomes:

$$ F = \tau A = \frac{1}{2} \rho U \sqrt{ \frac{2}{\pi \gamma} a A } , $$

where $A$ is the total effective area, including the top surface.

The damping from the proof mass is obtained by dividing the shear force by the velocity $U$, i.e.,

$$ c = \frac{1}{2} \rho A \sqrt{ \frac{2}{\pi \gamma} a } , $$

where $\rho$ is the density of the fluid.

The damping from the combs can be calculated by substituting the $A$ in (2.31) with $2nA_{\text{combs}}$, i.e.,

$$ c_{\text{combs}} = \frac{1}{2} \rho 2nA_{\text{combs}} \sqrt{ \frac{2}{\pi \gamma} a } , $$

and the total damping becomes
\[ c_{total} = \frac{1}{2} \rho a \sqrt{\frac{2}{\pi \gamma}} (A + 2nA_{cmbhs}). \] (2.33)

The quality factor is therefore

\[ Q = \frac{m\omega_n}{\rho a \sqrt{\frac{2}{\pi \gamma}} (A + 2nA_{cmbhs})}. \] (2.34)

The analytical solutions for the quality factor are therefore obtained for both slip flow and molecular flow. The solutions for TFG A are plotted in Fig. 2.6.

Figure 2.6: Quality Factor in the drive axis vs. Knudsen number (based on 3μm gap) for various models, with dimensions based on TFG A.
It can be seen from Fig. 2.6 that as the Knudsen number goes to 0, the slip flow model converges to the continuum model, which agrees with the physics. It is important to realize that the top-surface shear assumption should also be applied on the slip flow model when the mean free path is much greater than the gap height, i.e.,

\[ c = \frac{\mu A}{h + 2\lambda} + \frac{\mu^2 n A_{comb}}{g_s + 2\lambda} = \frac{\mu A}{2\lambda} + \frac{\mu^2 n A_{comb}}{2\lambda}, \]  

(2.35)

when \( \lambda \gg h \) and \( g_s \). Hence the slip flow model is also independent of the distance between the moving plates, and the assumption of top-surface shear should be applied. It was found that the solutions of the slip flow and molecular flow models are indeed very close at the higher Knudsen number region if the same effective area is used for both cases. This validates the slip flow model at highly rarefied regimes for simple geometry, despite the fact that the model should not apply. However to avoid discrete curves for illustration purpose, the area of the top surface was not included in the slip flow model and was included to the molecular model throughout the entire pressure range.

2.3 - Experimental Apparatus

The objective of the experiment was to determine the quality factor of the tuning fork gyroscope over a range of pressures when the proof masses were driven in the drive axis. A bell jar, a roughing pump and a turbo pump were used to achieve the range of pressures. A piezoelectric valve was used to govern the airflow into and out of the bell jar, thus controls the pressure in the bell jar dynamically. A digital capacitance manometer was used to measure the pressure inside the bell jar with a precision of \( \pm 1 \) mTorr. Resonant drive motion was stimulated by supplying the TFG with an AC voltage at a frequency very close to the natural drive frequency of the TFG. The response was captured by the oscilloscope as output voltage signal. The temperature of the TFG is monitored by a thermistor, and the output from the thermistor is displayed on the voltage
meter. A picture of the bell jar station and a pictorial representation are shown in Fig. 2.7 and 2.8.

Figure 2.7: A picture of the bell jar station.

Figure 2.8: Pictorial representation of the bell jar station.
The decay response of the TFG is captured by turning off the drive voltage, which triggers the oscilloscope. Fig. 2.9 illustrates a typical signal for TFG A.

![Graph showing decay response](image)

Figure 2.9: A decaying signal recorded in the oscilloscope at pressure = 100mTorr and temperature = 305K. (LCCC 701).

Knowing that the proof mass is decaying exponentially (free vibration with viscous damping), the damping ratio can be found by extrapolating the upper envelope of the output signal. The result after the extraction is shown below:
Figure 2.10: The extracted decay curve from the output signal in Fig. 2.9.

In each measurement, 450 data points were recorded. The time interval between each data point would be the total decay time divided by 450. Hence to obtain a better resolution, the data acquisition should be terminated once the output signal has reached a constant level. This was achieved by adjusting the time scale of the oscilloscope iteratively to determine the minimal time required for capturing the exponential decay. It was found that if the data were continually acquired, the output signal would be similar to the one illustrated in Fig. 2.11.
Figure 2.11: A decaying signal recorded in the oscilloscope at pressure = 150mTorr and temperature = 305K. (LCCC 701).

As shown in Fig. 2.11, the measurements from 2.5 to 4.5 seconds are not necessary. The constant output voltage observed at the end of the decaying signal was found to be contributed by the environmental noises and random vibrations. This noise floor in the output voltage was eliminated through post-processing of the acquired data.

The constant noise voltage, in this case 0.075V, was subtracted from the data in Fig. 2.10, and an exponential curve was fitted to the data. The post-processed result is illustrated in Fig. 2.12.
Figure 2.12: An exponential curve fit of the post-processed data from Fig. 2.10.

Excellent agreement was observed from the curve fit. The damping ratio of the TFG is equal to $b$ of the exponential equation, $y = ae^{-bx}$, determined from the curve fit. In this case, the damping ratio is 1.0213, and the quality factor can be calculated by (2.10).

The measurements and curve fitting procedure were completed for a range of pressures. The results are presented in the following.

2.4 – Results from the measurements

Two sets of gyros - 4 TFG A: LCCC 701, LCCC 702, LCCC 703, LCCC 704 and 4 TFG B: LCCC 760, LCCC 761, LCCC 762, and LCCC 763, were tested. LCCC stands for “Leadless Ceramic Chip Carrier”. The plot of quality factor vs. Kn, in log-log scale, is shown in Fig. 2.13.
Figure 2.13: Plot of Quality factor in the drive axis vs Kn for TFG A units.

It can be seen that the structural damping, which is independent of pressure, dominates the high Knudsen number (low pressure) range. Thus at lower pressure, the curve flattens to a constant. As mentioned in the previous section, the damping ratio from the curve fit consists of both the fluid damping and also the structural damping, i.e. $\zeta = \zeta_s + \zeta_f$. To isolate effects of the fluid damping from the total damping, we can see that

\[
Q = \frac{1}{2\zeta} = \frac{1}{2\zeta_f + 2\zeta_s},
\]

(2.36)

and

\[
Q_s = \frac{1}{2\zeta_s}.
\]

(2.37)
We use

$$
\frac{Q_s - Q}{Q_s} = \frac{1}{2\zeta_s} \left( 1 - \frac{1}{2\zeta_s + 2\zeta_f} \right) = \frac{2\zeta_f}{2\zeta_s + 2\zeta_f} = \frac{Q}{Q_f}
$$

(2.38)

and obtain

$$
Q_f = \frac{QQ_s}{Q_s - Q}.
$$

(2.39)

The $Q_s$ can be extracted by curve fitting the $Q$ vs. $Kn$ curve. A curve fitting function,

$$
Q = \frac{Q_s}{1 + \left( \frac{P}{\alpha} \right)^\beta},
$$

(2.40)

was used to curve fit the $Q$ vs. $Kn$ curve, where $P$ is the pressure, $Q_s$ is the structural damping, $\alpha$ and $\beta$ are constants. By iterating the constants $\alpha$, $\beta$ and $Q_s$ such that the function is fitting precisely on the measurements, $Q_s$ is obtained. An example is shown in Fig. 2.14 for LCCC 701.
Figure 2.14: Curve fitting the $Q$ measurement of LCCC 701 using (2.40) to extract the $Q_s$.

As shown in Fig. 2.14, excellent agreement is obtained from the curve fit, and the $Q_s$ was determined to be 135000, with $\alpha$ and $\beta$ equal to 65.0 and 0.87, respectively.

Once the structural damping is determined, we applied (2.39) and obtained the following plot of $Q_t$ for TFG A:
Figure 2.15: Plot of Quality factor with structural damping extracted vs Kn for TFG A units.

Fig. 2.15 clearly shows that the quality factor has a constant slope throughout the pressure range considered, with slight discrepancies at the low-pressure range (Kn > 5000, P < 3mTorr) that are due to drifts of the manometer. Similar plots are shown for TFG B in Fig. 2.16 and 2.17:
Figure 2.16: Plot of Quality factor in the drive axis vs Kn for TFG B units.

Figure 2.17: Plot of Quality factor with structural damping extracted vs Kn for TFG B units.
2.5 – Comparison of the theoretical solution and experimental results

The experimental results of the TFG A and TFG B compared with the continuum, slip flow and molecular flow model are shown below:

![Graph showing the comparison of quality factor and Knudsen number](image)

Figure 2.18: Comparison of the Quality Factor measurement (LCCC 701) with various analytical models for TFG A.
Figure 2.19: Comparison of the Quality Factor measurement (LCCC 760) with various analytical models for TFG B.

The measurements are shown to have similar trends with both the molecular flow model and the slip flow model. The discrepancy, defined as

\[
\text{Discrepancy (\%)} = \left| \frac{\text{Analytical Solution} - \text{Measurement}}{\text{Analytical Solution}} \right| \times 100\% , \tag{2.40}
\]

are calculated for each model. The results are tabulated in Table 2.2.
<table>
<thead>
<tr>
<th>Discrepancy in</th>
<th>TFG A</th>
<th>TFG B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slip Flow model</td>
<td>Min: 22.34%</td>
<td>Min: 47.37%</td>
</tr>
<tr>
<td></td>
<td>Max: 72.38%</td>
<td>Max: 77.59%</td>
</tr>
<tr>
<td></td>
<td>Average: 43.57%</td>
<td>Average: 57.96%</td>
</tr>
<tr>
<td>Molecular Flow model</td>
<td>Min: 0.16%</td>
<td>Min: 6.49%</td>
</tr>
<tr>
<td></td>
<td>Max: 54.27%</td>
<td>Max: 61.31%</td>
</tr>
<tr>
<td></td>
<td>Average: 21.31%</td>
<td>Average: 26.77%</td>
</tr>
</tbody>
</table>

Table 2.2: Discrepancies of the analytical models to the measurements for TFG A and B.

For the solutions spanned over a range of five orders of magnitude, the resulting discrepancies of roughly 25% and 50% for molecular flow and slip flow model, respectively, are considered to be good estimates. It is obvious that the rarefaction effect is important in this case, hence the continuum flow model has large errors compared with the measurements.

The effect of comb calculated by

\[
\text{Contribution from Combs (\%) = } \left| \frac{Q_{\text{total}} - Q_{\text{without combs}}}{Q_{\text{total}}} \right| \times 100\% , \quad (2.41)
\]

and it was also found that the contribution from combs to the quality factor for slip flow model and molecular flow model are about 42% and 21% for TFG A, and about 39% and 19% for TFG B, respectively. Thus it can be concluded that the combs are an important source of damping.

The discrepancy is contributed from two sources: analytical assumptions and experimental error. Analytical discrepancies include the assumption of infinite plate for both the proof mass and the combs, which neglects the end effect. Also neglecting the boundary conditions for the holes and simply multiplying the solid area to the shear stress to determine the shear force might be another source of error. Furthermore, the variation
in the y-axis and the unsteadiness are neglected for the models, thus contribute to the discrepancy. It is important to note that even though the experiments were performed in the free molecular flow regime, the slip flow model can still predict the fluid damping precisely.

It was found that the temperature variation within ±5 K does not contribute much of an error. The experimental errors are mainly contributed from the measurements of pressure, curve fitting of the decaying signal, and the extraction of the structural damping from the data. Exact measures on these errors can be difficult, a sample estimation of the percent error at low pressure (5 mTorr) is summarized below:

1). Error from manometer: ±1 mTorr = 20% error
2). Error from curve fit of decay signal: 5% from the noise floor adjustment
   1% from the curve fitting

![Graph](image)

Figure 2.20: An illustration of the curve fit of the decay signal, with noise floor = 0.05 volt subtracted.
Figure 2.21: An illustration of the curve fit of the decay signal, with noise floor = 0.06 volt subtracted.

It can be seen from Fig. 2.20 and 2.21 that an acceptable curve fit can be obtained with slight variation on the noise floor adjustment. The two curve fits yield about 5% difference in the damping ratio, and the curve fit itself is fitting up to 99% accuracy, with 1% error.

3). Error from structural damping extraction: 20% error
Figure 2.22: An illustration of the curve fit of the Q measurement to extract $Q_s$.

From Fig. 2.22 we can see that acceptable curve fits can be obtained with slight variation on the $Q_s$, the $Q_f$ is 1824623 and 1410260 for $Q_s = 135000$ and 138000, respectively, which results about 20% difference in the $Q_f$.

This concludes the analysis on the drive Q.
Chapter 3.
Fluid Damping in the Sense Axis

In this chapter, analytical methods used to determine the quality factor of the sense axis will be presented. The linearized Reynolds squeeze film equation solved analytically by Blech[4] will be modified with slip boundary condition. The resulting equation will be solved numerically. The effect of the proof mass thickness (chimney) and the sensitivity of various design parameters will also be discussed. The experimental setup will be described and the results on various gyros will be compared with the numerical solutions.

3.1 - Analytical Evaluation of the Quality Factor in the Sense Axis

When the excitation frequency matches the sense frequency of the tuning fork gyro (TFG), the proof masses will oscillate out of phase in the z-direction with a power supplied from the outer combs. The front view of the proof mass oscillation is shown in Fig. 3.1:

![Diagram of oscillation](image)

Figure 3.1: An illustration on the sense motion of the tuning fork gyro.

For very small gap-height to plate-length ratio, the two parallel plates moving in relative normal direction motion will experience a backward pressure force acting on the
surfaces of the plates. The velocity profile of the fluid with the top plate squeezing down is shown in Fig. 3.2.

![Figure 3.2: The velocity profile of the fluid flow.](image)

The fluid flow is governed by the Navier-Stokes equations

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{3.1a}
\]

\[
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \tag{3.1b}
\]

\[
\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right), \tag{3.1c}
\]

where \( p \) is the pressure, \( \rho \) is the density, \( \mu \) is the viscosity of the fluid, and \( u, v \) and \( w \) are the velocities of the fluid in the \( x, y \) and \( z \)-directions, respectively.

The main flow is driven by the pressure gradient in the \( x \)-direction, hence all the terms with \( v \) and \( w \) are comparatively small and can be eliminated, i.e.,

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right). \tag{3.1d}
\]

The assumption of an infinite plate in the \( x \)-direction is applicable, and the unsteadiness term, \( \rho \frac{\partial u}{\partial t} \), can also be eliminated using order of magnitude analysis as described in Chapter 2. Hence the equation becomes:
\[
\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2}.
\]  

(3.2)

Unlike Couette flow, the pressure gradient on the left-hand side is not zero. This flow type is known as Poiseuille flow. Notice that the symmetry of the flow allows the coordinate system to be placed at the center. With no-slip boundary conditions and zero ambient pressure, the pressure gradient along the plate is:

\[
\frac{\partial p}{\partial x} = \frac{12\mu Q}{h^3 b}
\]  

(3.3)

where \( b \) is the width of the plate, and \( Q \) is the flow rate of the fluid, defined as

\[
Q = VA = \text{velocity} \times \text{cross sectional area of the flow for incompressible flow.}
\]

Substituting \( W \) for velocity and \( bx \) for cross sectional area, the pressure distribution becomes:

\[
p(x) = \frac{6\mu Wb}{h^3} x^2,
\]  

(3.4)

where \( h \) is the distance between the plates, and \( b \) is the width of the plate. The lift force is determined by integrating the pressure over the area, i.e.,

\[
F = \int_0^L p \, dx = \frac{2\mu WbL^3}{h^3},
\]  

(3.5)

where \( L \) is the length of the plate.

Following (2.7) to (2.10), the quality factor for sense mode, \( Q \), becomes:
\[ Q = \frac{1}{2\zeta} = \frac{m\omega_h h^3}{\mu b L^3}. \] (3.6)

It can be concluded from above that the continuum flow solution is independent of pressure.

### 3.2 - Rarefaction Consideration

If the gap height is comparable to the mean free path of the fluid particles, slip velocity will occur at the boundary, as described in Chapter 2.2a. The slip flow velocity profile is shown in Fig. 3.3.

![Figure 3.3: The slip velocity profile of the fluid flow.](image)

The slip flow effect will reduce the shear force exerted on the plates and decrease the viscous damping. In this study, the slip flow model will be derived and compared to the experimental results to measure the discrepancy.

The Poiseuille flow with slip is derived from the Navier-Stokes equation [1], which can be simplified using the order of magnitude analysis discussed in the last section. The simplified Navier-Stokes equation is:

\[ \frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2}. \] (3.7)
with boundary conditions \( u = \lambda \frac{\partial u}{\partial z} \) at \( y = 0 \) and \( u = -\lambda \frac{\partial u}{\partial z} \) at \( y = h \).

Solving for \( u \),

\[
u(z) = \frac{1}{\mu} \frac{\partial p}{\partial x} \left( \frac{x^2}{2} \frac{h}{z} \frac{h}{2} \lambda \right).
\] (3.8)

Integrating the continuity equation from 0 to \( h \):

\[
\int_0^h \left( \frac{\partial (\rho w)}{\partial x} + \frac{\partial (\rho w)}{\partial z} + \frac{\partial \rho}{\partial t} \right) dz = 0.
\] (3.9)

Recognizing that

\[
\int_0^h \frac{\partial (\rho w)}{\partial z} dz = \rho \int_0^h \frac{\partial w}{\partial z} dz + w \int_0^h \frac{\partial \rho}{\partial z} dz,
\] (3.10)

for \( \frac{\partial \rho}{\partial z} \) is zero,

\[
\rho \int_0^h \frac{\partial w}{\partial z} dz = \rho W - \rho(0) = \rho W = \rho \frac{\partial h}{\partial t}.
\] (3.11)

Combining \( \rho \frac{\partial h}{\partial t} \) and \( h \frac{\partial \rho}{\partial t} \), and substitute \( u(z) \) into the equation:

\[
\frac{\partial}{\partial x} \left( \frac{\rho \frac{\partial p}{\partial x}}{\mu} (h^3 + 6h^2 \lambda) \right) = 12 \frac{\partial (\rho h)}{\partial t}
\] (3.12)

For an isothermal process, the density is directly proportional to the pressure. Therefore substituting \( p \) for \( \rho \):
\[
\frac{\partial}{\partial x} \left( \frac{p}{\mu} \frac{\partial p}{\partial x} (h^3 + 6h^2 \lambda) \right) = 12 \frac{\partial (ph)}{\partial t}.
\]  
(3.13)

(3.13) is the Reynolds squeeze film equation with rarefaction effect.

We proceed by defining several non-dimensional parameters for normalization:

Non-dimensional pressure: \( \Psi = \frac{p}{P_a} \);  
(3.14)

Non-dimensional gap height: \( H = \frac{h}{h_o} \);  
(3.15)

Non-dimensional coordinate: \( X = \frac{x}{L} \);  
(3.16)

Non-dimensional time: \( T = \omega t \).  
(3.17)

In addition, we define \( Kn_o \) as the Knudsen number with the mean free path at the initial condition, i.e.,

\[
Kn_o = \frac{\lambda_o}{h_o}. 
\]  
(3.18)

Substituting them into the Reynolds equation we have

\[
\frac{\partial}{\partial X} \left( \Psi H^3 \frac{\partial \Psi}{\partial X} \right) + \frac{\partial}{\partial X} \left( 6Kn_o H^2 \frac{\partial \Psi}{\partial X} \right) = \sigma \frac{\partial}{\partial T} (\Psi H),
\]  
(3.19)

where \( \sigma \) is the squeeze number, defined as \( \frac{12 \mu \omega L^2}{P_a h_o^2} \).

Applying perturbation method by substituting

\[
\Psi = 1 + \varepsilon \Psi'
\]  
(3.20)
H = 1 + \varepsilon \cos T \quad (3.21)

into (3.19), where \( \varepsilon = \frac{\delta}{h_o} \), which is the ratio of the sense displacement, \( \delta \), to the nominal gap height. Neglecting higher order terms of \( \varepsilon \), (3.19) becomes

\[
\varepsilon \left[ \frac{\partial^2 \Psi'}{\partial X^2} (1 + 6Kn_o) - \sigma \frac{\partial \Psi'}{\partial T} = -\sigma \sin T \right] 
\quad (3.22)
\]

Assuming the solution is in the form \( \Psi' = \Psi_0 \sin T + \Psi_1 \cos T \) [16], substituting them into (3.22) we obtain:

\[
\frac{\partial^2 \Psi_0}{\partial X^2} + \frac{\sigma}{(1 + 6Kn_o)} \Psi_1 + \frac{\sigma}{(1 + 6Kn_o)} \Psi_0 = 0 \quad (3.23)
\]

\[
\frac{\partial^2 \Psi_1}{\partial X^2} - \frac{\sigma}{(1 + 6Kn_o)} \Psi_0 = 0 \quad (3.24)
\]

(3.23) and (3.24) are the coupled normalized equations we will apply to solve the rarefied squeeze film problem. The non-dimensional damping and spring forces can be calculated from the solution:

\[
f_o = \frac{1}{A} \int_A \Psi_0 dA \quad (3.25)
\]

\[
f_i = \frac{1}{A} \int_A \Psi_1 dA \quad (3.26)
\]

Note that the damping force evaluated is on the order of \( \varepsilon \), which cancelled with the plate velocity. Thus to evaluate the damping coefficient, \( \nu \), the non-dimensional damping force is multiplied by \( P_o A \), where \( P_o \) is the ambient pressure and \( A \) is the total area where the pressure is acted on. The quality factor can be determined as described from (2.7) to (2.10).
3.3 - Numerical Approach

Blech [4] has solved the linearized squeeze film equation for Kn = 0 analytically for rectangular plate, i.e.,

\[ \frac{\partial^2 \Psi_0}{\partial X^2} + \sigma \Psi_1 + \sigma = 0 \quad (3.27) \]
\[ \frac{\partial^2 \Psi_1}{\partial X^2} - \sigma \Psi_0 = 0 \quad (3.28) \]

The closed form solution for the non-dimensional damping and spring forces are shown below:

\[ f_0 = \frac{64\sigma \varepsilon}{\pi^6} \sum_{m,n}^{m \neq n, n \text{ odd}} \frac{m^2 + (n/\beta)^2}{(mn)^2 \left\{ [(m^2 + (n/\beta)^2)^2 + \sigma^2/\pi^4] \right\} } \quad (3.29) \]
\[ f_1 = \frac{64\sigma^2 \varepsilon}{\pi^8} \sum_{m,n}^{m \neq n, n \text{ odd}} \frac{1}{(mn)^2 \left\{ [(m^2 + (n/\beta)^2)^2 + \sigma^2/\pi^4] \right\} } \quad (3.30) \]

where \( \beta \) is the rectangular plate's length/width ratio. Since the TFG has holes distributed across the proof mass, closed form analytical solution becomes impossible. Therefore, numerical approach is used to solve (3.23) and (3.24). A finite element program, PDEase, which can solve field equation in arbitrary two-dimensional domain, is used to solve the coupled equations on the proof mass. It is noticed that the difference between (3.23 & 3.24) and (3.27 & 3.28) is the modified squeeze number to account for the rarefaction effect, thus the program can be validated through the simulation of a square plate (\( \beta = 1 \)) and compare the solution with Blech's solution above for a wide range of squeeze numbers. The mesh and contours from PDEase simulations are shown in Fig. 3.4, 3.5 and 3.6, and the numerical solution is plotted against the Blech's solution in Fig. 3.7.
Figure 3.4: The mesh used for the rectangular plate simulation, with $\beta = 1$;

(3.27 & 3.28) are the governing equations.

Figure 3.5: The contour of $\Psi_0$ at $\sigma = 10$. 
Figure 3.6: The contour of $\Psi_1$ at $\sigma = 10$.

Figure 3.7: Comparison of the numerical solution with Blech’s solution.

The PDEase program is validated by the excellent agreement obtained from the simulated and analytical results.
3.4 – Chimney Boundary Condition

As the plate is squeezed down, the fluid will flow through the hole and out of the proof mass. Due to geometric symmetry, the fluid under the solid area will be bounded by the symmetry lines, and will only flow through the hole as shown below:

Figure 3.8: An illustration of the flow through the hole in the proof mass.

Hence only one “cell” is needed for the simulation, the domain and boundary conditions are specified as follows:

Figure 3.9: An illustration of the domain and boundary conditions used in the simulations.
No pressure gradient exist at the outer boundary; thus both $\partial \Psi_0$ and $\partial \Psi_1$ are zero as the outer boundary condition. To define the inner boundary condition for the hole, we recall that the fluid is squeezed from the bottom to the top surface of the proof mass, where the pressure is ambient. The flow along the "chimney" can be represented by pipe flow, with the length being the thickness of the proof mass, $t$, as shown in Fig. 3.10.

![Diagram showing the chimney model used to determine the inner boundary condition.](image)

Figure 3.10: The chimney model used to determine the inner boundary condition; $W$ is the downward velocity of the proof mass in the $z$ direction.

The fluid flow along the chimney is also rarefied, thus we will model the flow with slip with the assumption of circular cross section for simplicity. The Reynolds number in the chimney is $\frac{\rho U D_h}{\mu}$, where $D_h$ is the hydraulic diameter for rectangular cross section,

defined as $\frac{2ab}{a+b} = \frac{2L_h L_h}{L_h + L_h} = L_h$. The Reynolds number is in the order of

$$\left(\frac{1 \text{ kg/m}^3}{\text{m}^3} \right) \left(\frac{1 \text{ m}}{\text{s}} \right) \left(10^{-4} \text{ m}\right) \left(10^{-5} \frac{\text{Ns}}{\text{m}^2}\right) = 1 \times 10^{-1}. $$

The entrance length, which is estimated by

$$\frac{L_e}{D_h} = \frac{0.6}{1 + 0.035 \text{Re}} + 0.056 \text{Re}[30] \approx 0.6 \text{ D}_h = 0.6 \times 5.5 \mu\text{m} = 3.3 \mu\text{m} \text{ for TFG A, which is}

about 16.5\% \text{ of the chimney length (t = 20\mu m). For TFG B, the entrance length is about}
0.6 \text{ D}_h = 0.6*4.5\mu m = 2.7\mu m$, which is about 27% of the chimney length ($t = 10\mu m$). We will neglect the entrance length effect for first approximation. Also the arguments presented in 3.1 still hold, and the governing equation of the fluid becomes:

\[
\frac{\partial p}{\partial z} = \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) \right),
\] (3.31)

with boundary conditions: \( u(R) = -\frac{\lambda}{\mu} \frac{\partial u}{\partial r} \) and \( \frac{\partial u}{\partial r} = 0 \) at \( r = 0 \)

Solving for \( u(r) \), we obtain:

\[
\frac{u(r)}{4\mu} = \frac{1}{4\mu} \frac{\partial p}{\partial z} \left( r^2 - R^2 - 2R\lambda \right)
\] (3.32)

The flow rate, \( Q \), is obtained by integrating \( u(r) \) over the cross section. The average flow velocity is determined by dividing the flow rate by the cross sectional area, \( \pi R^2 \), i.e.,

\[
\frac{V_{chimney}}{A_{chimney}} = \frac{Q}{\pi R^2} = \frac{1}{2\mu} \frac{\partial p}{\partial z} \left( -\frac{R^2}{4} - R\lambda \right)
\] (3.33)

From energy conservation [23], the pressure drop along the chimney is balanced by the friction losses from the flow, i.e.,

\[
P - P_a = f \frac{1}{D_h} \frac{\rho V_{chimney}^2}{2}
\] (3.34)

where \( f \) is the friction factor, defined as

\[
f = \frac{C}{R_e}
\] (3.35)
where the shape constant, C, is 64 for a circular pipe. Other cross sectional shape such as rectangular and triangular, the shape constants are obtained by simulations [23].

Substituting \( \text{Re} = \frac{\rho V L_h}{\mu} \), and \( V = V_{\text{chimney}} \) into \( C = f \text{Re} \), recognizing that \( \frac{P - P_a}{t} = \frac{\partial P}{\partial z} \), we obtain

\[
C = \frac{64}{1 + 8 \frac{\lambda}{L_h}} \tag{3.36}
\]

Substitute (3.36) into (3.34),

\[
P - P_a = \frac{64 \mu V}{\left(1 + 8 \frac{\lambda}{L_h}\right) 2 L_h^2} \tag{3.37}
\]

Normalized the pressure by the ambient pressure, \( P_a \), we have

\[
\Psi = \frac{32 \mu V}{\left(1 + 8 \frac{\lambda}{L_h}\right) L_h^2 P_a} + 1 \tag{3.38}
\]

Applying perturbation using (3.20), we obtain

\[
\varepsilon^{\Psi}_{\text{hole boundary}} = \frac{32 \mu V}{\left(1 + 8 \frac{\lambda}{L_h}\right) L_h^2 P_a} \tag{3.39}
\]

For the squeeze number (\( \sigma \)) is low, the flow can be assumed to be incompressible, thus conservation of mass (volume) can be applied:

\[
WA_p = VA_{\text{hole}} \tag{3.40}
\]
where \( W = \omega \delta \)

\[
A_p = L_p^2 - L_h^2
\]

\[
A_{hole} = L_h^2
\]

and \( \delta \) is the plate displacement.

The chimney velocity becomes

\[
V = \frac{\omega \delta (L_p^2 - L_h^2)}{L_h^2}
\]  

(3.41)

Substituting \( V \) in (3.39),

\[
\varepsilon \Psi' = \frac{32 \mu \omega \delta (L_p^2 - L_h^2)}{(1 + 8 \frac{\lambda}{L_h}) L_h^4 P_a}
\]

(3.42)

Rearranging the terms,

\[
\varepsilon \Psi' = \frac{12 \mu L_p^2 \omega}{h_o^2 P_a} \frac{\delta}{h_o} \left( \frac{1 - \frac{L_h^2}{L_p^2}}{h_o^3} \right) \left( 1 + 8 \frac{\lambda}{L_h} \right) \frac{1}{12 L_h^4}
\]

(3.43)

note that the squeeze number \( \sigma = \frac{12 \mu \omega L_p^2}{P_a h_o^2} \) and \( \varepsilon = \frac{\delta}{h_o} \) can be extracted, thus we can express the chimney boundary condition as:
\[
\psi_{\text{hole boundary}} = \left[ \frac{32}{L_p^2} \frac{h_t^2}{L_p^2} \left( 1 - \frac{L_h^2}{L_p^2} \right) \right] \left[ \frac{1}{1 + 8 \frac{\lambda}{L_h}} \right] \sigma
\]

\[= K_{\text{geometry}} \times K_{\text{rarefaction}} \times \sigma \]  

(3.44)

where \( K_{\text{geometry}} \) is a function of \( t/L_p \), \( h/L_p \) and \( L_h/L_p \), and \( K_{\text{rarefaction}} \) is a function of the Knudsen number.

Because the parameter \( t/L_p \) affects the constant \( K_{\text{geometry}} \) linearly, \( K_{\text{geometry}} \) can be minimized by varying \( h/L_p \) and \( L_h/L_p \). A contour plot is shown below for \( K_{\text{geometry}} \).

![Contour plot](image)

**Figure 3.11**: A contour of \( \log_{10}(K_{\text{geometry}}) \) with various \( L_h/L_p \) and \( h/L_p \) ratios; X depicts the design of TFG A.
To minimize chimney resistance, it is desirable to have $K_{\text{geometry}}$ as low as possible. The above shows the contour of $\log_{10}(K_{\text{geometry}})$ and the cross shows where the design TFG A is.

The problem is therefore completely defined; numerical simulation of (3.23 & 3.24) is performed on a square cell with a square hole at the center. The outer boundary condition have zero non-dimensional pressure gradient, while the inner boundary condition is $\Psi^*_\text{hole boundary} = K_{\text{geometry}} \times K_{\text{rarefaction}} \times \sigma$, as shown below:

$$\frac{\partial^2 \Psi^*}{\partial \xi^2} + \frac{\sigma}{(1 + 6 \xi n_a)} \Psi^*_1 + \frac{\sigma}{(1 + 6 \xi n_a)} = 0$$

$$\frac{\partial^2 \Psi^*}{\partial \eta^2} - \frac{\sigma}{(1 + 6 \xi n_a)} \Psi^*_0 = 0$$

$\nabla \Psi^* = 0$

Mathematical expression:

$$\Psi^*_\text{hole boundary} = \left[ \frac{32}{L_p} \left( \frac{\eta^2}{L_p} \right) \left( 1 - \frac{L_p}{L_p} \right) \left( \frac{\eta^2}{L_p} \right) \right] \left[ \frac{1}{1 + 8 \frac{\lambda}{L_p}} \right] \sigma$$

Figure 3.12: The complete problem statement for squeeze film damping with rarefaction effect.

With the chimney boundary condition determined, the simulation of a single cell was carried out. As stated, only the non-dimensional damping pressure is needed to evaluate the damping force. The mesh used and a typical simulated results are shown in Fig. 3.13 and 3.14.
Figure 3.13: The mesh used for the cell simulation.

Figure 3.14: The solution contour of the non-dimensional damping pressure at ambient pressure of 5mTorr (0.668 Pa) – TFG A.

The damping coefficient, $c$, is calculated by $f_0P_aA$ as mentioned in the previous section. The quality factor can then be determined following (2.7) to (2.10).

Similar to oscillation in the drive axis, the combs will also generate shear stress with oscillation in the sense axis, as shown below:
Figure 3.15: An illustration of the comb-shearing in sense axis.

There will be variation of the shearing area, but the average shearing area will be estimated as $A_{\text{combs}} = L_c t$ and used for the comb shear calculation. The model developed in Chapter 2.2 applies directly to the situation, thus the additional damping is

$$c_{\text{combs}} = \frac{2nA_{\text{combs}} \mu}{g_c + 2\lambda}, \quad (3.45)$$

and the quality factor can be calculated by (2.7) to (2.10).

3.5 – Sensitivity of the Quality Factor in the Sense Axis to Geometry

In the last section the $K_{\text{geometry}}$ was evaluated, it is desirable to investigate how sensitive the quality factor is to the change in geometry. We proceed by simulating the solution with various parameter changes. For instance, the sensitivity of the hole spacing ($L_p$) is obtained by varying the hole spacing while keeping the hole size ($L_h$), gap height ($h$) and the thickness ($t$) constant at their nominal dimensions in Table 1.1. The sensitivity of the gap height is obtained similarly by varying the gap height while keeping
the other three parameters constant. Note that the effect of the geometry changes to the mass, oscillating frequency, and number of holes, are not included. The results from the sensitivity analysis for pressure = 5 mTorr, are presented in the following plots:

![Graph showing the relationship between Quality Factor in Sense Axis and Lp (m) for 10.5um < Lp < 13.5um.](image)

Figure 3.16: Plot of the Quality Factor in the sense axis vs. Lp, based on TFG A.
Figure 3.17: Plot of the Quality Factor in the sense axis vs. $L_h$, based on TFG A.

Figure 3.18: Plot of the Quality Factor in the sense axis vs. $h$, based on TFG A.
Figure 3.19: Plot of the Quality Factor in the sense axis vs. $t$, based on TFG A.

The four plots above are plotted on the same scale in Fig. 3.20 below:
Figure 3.20: Plot of the Quality Factor in the sense axis vs. Geometry changes, based on TFG A. Multiple x-axes are used to illustrate the range considered for each parameter.

It can be seen from Fig. 3.20 that the hole size and the hole spacing are the most sensitive parameters while the gap height and the proof mass thickness are less sensitive to the sense Q over the range of reasonable deviations. As described in the last section, the constant $K_{\text{geometry}}$ governs the sensitivity of sense Q by the four geometric parameters. Since $K_{\text{geometry}}$ only has a linear dependence on the thickness of the proof mass, $t$, it has less effect than the fourth-order dependency on both the hole size ($L_h$) and hole spacing ($L_p$). The sensitivity of the gap height, $h$, which has a third-order dependency, lies in between as shown.
3.6 - Experimental Apparatus

The experimental setup for sense Q measurement was very similar to the one for drive Q, described in Chapter 2.3. The excitation frequency is adjusted to match the natural sense frequency of the TFG. The proof mass plate and the substrate are used as capacitors. As a result, any physical mismatches, such as variation of gap height, effective area, etc. between the left and right proof masses, will induce error in the signal read at the sense pre-amplifier. This is called the feedthrough of the system. It was not an issue for drives Q measurement because the AC voltage across the motor capacitor is held at virtual ground via an Op-Amp. To overcome the error from feedthrough, a carrier is introduced to separate the frequency of the excitation signal from the frequency of the real motion. The signal of interest, which is proportional to sense axis motion is then separated after demodulating the output signal by the carrier. Hence the output of the sense motion is obtained. A pictorial representation of the experimental setup is shown below:

![Diagram of experimental setup]

Figure 3.21: The experimental setup for sense Q measurement.

The signal will be demodulated and captured by the oscilloscope. A typical decaying signal is presented below:
Figure 3.22: An illustration of a demodulated signal from the oscilloscope at pressure =200mTorr, temperature = 305 K. (LCCC 704).

Using the same procedure for noise floor cancelation described in Chapter 2.3, the quality factor for sense axis were measured over a range of pressures.

3.7 – Results from the measurements

Two sets of gyros - 5 TFG A, LCCC 569, LCCC 584, LCCC 702, LCCC 704, LCCC 705 and 5 TFG B, LCCC 759, LCCC 760, LCCC 761, LCCC 762, and LCCC 763, were tested. The plot of the quality factor in the sense axis vs. Kn, in log-log scale, is shown in Fig. 3.23.
Figure 3.23: Plot of Quality Factor in the sense axis vs Kn for TFG A units.

The quality factor contribution from fluid damping, $Q_f$, can be extracted using the same method described in Chapter 2.3. Plots of $Q_f$ vs. Kn for TFG A is shown in Fig. 3.24.
Figure 3.24: Quality Factor vs Kn with structural damping extracted (TFG A).

It can be seen that the $Q_f$ data agree well over the entire range of pressures, despite the difference in the structural damping as shown in Fig. 3.23. Similar plots are shown for TFG B in Fig. 3.25 and 3.26:
Figure 3.25: Plot of Quality Factor in the sense axis vs Kn for TFG B units.

Figure 3.26: Quality Factor vs Kn with structural damping extracted (TFG B).
The $Q_f$ data for TFG B also reveals good agreement between the units.

3.8 – Comparison of the numerical solution and experimental results

The experimental results of the TFG B and TFG A compared with the continuum, slip flow model are shown below:

![Graph showing comparison of measurement (LCCC 569) with continuum and slip flow models for TFG A.](image)

Figure 3.27: Comparison of the measurement (LCCC 569) with continuum and slip flow models for TFG A.
Figure 3.28: Comparison of the measurement (LCCC 759) with continuum and slip flow models for TFG B.

The measurements are shown to have similar trends with the slip flow model. The discrepancy, defined as

\[
\text{Discrepancy (\%) = } \left| \frac{\text{Analytical Solution} - \text{Measurement}}{\text{Analytical Solution}} \right| \times 100\% ,
\]  

(3.46)

are calculated for each model. The results are tabulated in Table 3.1.
<table>
<thead>
<tr>
<th>Discrepancy in</th>
<th>TFG B</th>
<th>TFG A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slip Flow model</td>
<td>Min: 2.92%</td>
<td>Min: 37.04%</td>
</tr>
<tr>
<td></td>
<td>Max: 34.47%</td>
<td>Max: 50.14%</td>
</tr>
<tr>
<td></td>
<td>Average: 28.53%</td>
<td>Average: 48.61%</td>
</tr>
</tbody>
</table>

Table 3.1: Discrepancies of the analytical models to the measurements for TFG A and B.

For the solutions spanned over a range of five orders of magnitude, the resulting discrepancies of roughly 30% and 50% for TFG A and B, respectively, are considered to be good estimates. It is obvious that the rarefaction effect is important in this case, hence the continuum flow model has large errors compared with the measurements. The contribution from the combs is calculated by

\[
\text{Contribution from Combs} = \left( \frac{Q_{\text{total}} - Q_{\text{without combs}}}{Q_{\text{total}}} \right) \times 100\% , \tag{3.47}
\]

and it was found that the combs contribute about 0.9% to the total quality factor at high Knudsen number regime for TFG A, and about 0.7% for TFG B. Thus it can be concluded that the combs can be neglected for damping calculation.

The contribution from the chimney is calculated by

\[
\text{Contribution from Chimney} = \left( \frac{Q_{\text{total}} - Q_{\text{without chimney}}}{Q_{\text{total}}} \right) \times 100\% , \tag{3.48}
\]

it was found that the difference of \(Q\) is about 3400% at atmospheric pressures (760 Torr) to 5100% at low pressures (1 mTorr) for TFG A. Thus the chimney is very significant in sense axis damping.

The discrepancy is contributed from two sources: analytical assumptions and experimental error. Analytical discrepancies include the assumption of infinite plate for both the proof mass and the combs, neglecting the end effects. The boundary conditions
for the holes is based on a circular pipe instead of a square duct, which might be another source of error. Furthermore, the variation in the y-axis and the unsteadiness are neglected for the models, thus contribute to the discrepancy. It is important to note that even though the experiments were performed in the free molecular flow regime, the slip flow model can still predict the fluid damping precisely.

It was also found that for sense axis damping, the temperature variation within ±5 K does not contribute much of an error. The experimental errors are mainly contributed from the measurements of pressure, curve fitting of the decaying signal, and the extraction of the structural damping from the data. A sample calculation of the experimental error from Chapter 2 can be applied for sense axis damping also.

This concludes the analysis for quality factor in the sense axis.
Chapter 4.

Bias from Surfboarding

In this chapter, analytical methods used to determine the bias caused by surfboarding will be presented. The surfboarding effect will be modeled based on lubrication theory developed by Osborne Reynolds in 1886 [12]. The lubrication theory was later developed and applied for slider bearing by W. J. Harrison in 1913 [12]. The slider bearing equation will be modified and presented with the rarefaction effect. The experimental procedure will be described and performed on the gyro's. The results will be presented and compared to the numerical solution.

4.1 – Analytical Evaluation of the bias from surfboarding with rarefaction effect

As stated in Chapter 2, the TFG consists of two proof masses that oscillate out of phase in the x-direction under a voltage from the outer combs.

![Diagram of tuning fork gyro oscillating in the drive axis](image)

Figure 4.1: An illustration of the tuning fork gyro oscillating in the drive axis.

Ideally, the two proof masses are perfectly flat with respect to the substrate thus the gap height is uniform across the proof mass. However, the gap between the proof masses and substrate is very often not uniform across, but tilted at an angle with respect to the substrate when fabricated. This causes the pressure to build up underneath the proof masses and generates unwanted lift force, which can contribute significant error to the angular rate measurement. This in-phase bias is often referred as the surfboarding effect. An illustration is shown in Fig. 4.2:
Figure 4.2: Oscillation of a tilted proof mass and the pressure profile underneath the plate.

The flow can be modeled by a stationary tilted plate with a substrate moving horizontally [31], as shown in Fig. 4.3.

Figure 4.3: Two-dimensional model of the fluid flow problem; y-direction into the page.

The fluid flow is governed by the Navier-Stokes equations

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)
\]  (4.1a)
\[ \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \]  
\hfill (4.1b) 

\[ \rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right), \]  
\hfill (4.1c)

where \( p \) is the pressure, \( \rho \) is the density, \( \mu \) is the viscosity of the fluid, and \( u, v \) and \( w \) are the velocities of the fluid in the \( x, y \) and \( z \)-directions, respectively.

The main flow is driven by the plate velocity in the \( x \)-direction. However, since the pressure gradient along \( x \) and \( y \)-directions will allow flow in both directions, all the terms with \( u \) and \( v \) remain and all \( w \) terms, which are comparatively small, can be eliminated, i.e.,

\[ \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \]  
\hfill (4.1d) 

\[ \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right). \]  
\hfill (4.1e)

For the gap-height to plate-length and plate-width ratio to be very small, the terms with \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \) become very small compared to \( \frac{\partial}{\partial z} \) thus can be eliminated.

\[ \rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2} \]  
\hfill (4.1f) 

\[ \rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial z^2}. \]  
\hfill (4.1g)

Since there are no oscillations in the \( y \)-direction, the term \( \rho \frac{\partial v}{\partial t} \) can be eliminated.

The unsteadiness terms, \( \rho \frac{\partial u}{\partial t} \), can be eliminated based on the discussion in Chapter 2 and the equation becomes:
\[ \frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2} \]  
\[ (4.2) \]

\[ \frac{\partial p}{\partial y} = \mu \frac{\partial^2 v}{\partial z^2}. \]  
\[ (4.3) \]

We will proceed using the slip flow theory, and the boundary conditions are \( u(0) = U + \lambda \frac{\partial u}{\partial z} \) and \( u(h) = -\lambda \frac{\partial u}{\partial z} \); \( v(0) = \lambda \frac{\partial v}{\partial z} \) and \( v(h) = -\lambda \frac{\partial v}{\partial z} \) [5]

Solving for \( u \) and \( v \), we have

\[ u(z) = \frac{1}{\mu} \frac{\partial P}{\partial x} \frac{z^2}{2} - \frac{1}{\mu} \frac{\partial P}{\partial x} \frac{h}{2} z - \frac{U}{h + 2\lambda} z - \frac{1}{\mu} \frac{\partial P}{\partial x} \frac{h}{2} \lambda - \frac{U}{h + 2\lambda} \lambda + U \]  
\[ (4.4) \]

\[ v(z) = \frac{1}{\mu} \frac{\partial P}{\partial y} \frac{z^2}{2} - \frac{1}{\mu} \frac{\partial P}{\partial y} \frac{h}{2} z - \frac{1}{\mu} \frac{\partial P}{\partial y} \frac{h}{2} \lambda. \]  
\[ (4.5) \]

Substituting (4.3 & 4.4) into the continuity equation:

\[ \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} + \frac{\partial \rho}{\partial t} = 0, \]  
\[ (4.6) \]

\( w \) is negligible compared with \( u \) and \( v \), and the unsteady term can be eliminated. Integrating the equation from 0 to \( h \) we have:

\[ \int_0^h \frac{\partial (\rho u)}{\partial x} dz + \int_0^h \frac{\partial (\rho v)}{\partial y} dz = 0. \]  
\[ (4.7) \]

Interchanging the order of integration and differentiation and substituting \( u(z) \) and \( v(z) \) into the equation:
\[
\frac{\partial}{\partial x} \left( \frac{\rho \partial P}{\mu \partial x} (h^3 + 6h^2 \lambda) \right) + \frac{\partial}{\partial y} \left( \frac{\rho \partial P}{\mu \partial y} (h^3 + 6h^2 \lambda) \right) = \frac{\partial}{\partial x} (6 \rho U h) 
\] (4.8)

This is the modified Reynolds equation for slider bearing with rarefaction effect [6].

We proceed by defining several non-dimensional parameters for normalization:

Non-dimensional pressure: \( \Psi = \frac{P}{P_a} \) \hspace{1cm} (4.9)

Non-dimensional gap height: \( H = \frac{h}{h_o} \) \hspace{1cm} (4.10)

Non-dimensional coordinate: \( X = \frac{x}{L} \) \hspace{1cm} (4.11)

Non-dimensional time: \( T = \omega t \) \hspace{1cm} (4.12)

In addition, we define \( Kn_1 \) as the Knudsen number of the mean free path at the higher end of the proof mass, i.e.,

\[
Kn_1 = \frac{\lambda_i}{h_i} 
\] (4.13)

Substituting the non-dimensional parameters in (4.8), and note that for isothermal process, \( \rho \) is linearly proportional to the pressure thus substitute \( P \) for \( \rho \). We obtain:

\[
\frac{\partial}{\partial X} \left( \Psi H^3 \frac{\partial \Psi}{\partial X} + 6H^2 Kn_1 \frac{\partial \Psi}{\partial X} \right) + \frac{\partial}{\partial Y} \left( \Psi H^3 \frac{\partial \Psi}{\partial Y} + 6H^2 Kn_1 \frac{\partial \Psi}{\partial Y} \right) = \Lambda \left( \frac{\partial}{\partial X} (\Psi H) \right) 
\] (4.14)

where \( \Lambda \) is the bearing number, defined as

\[
\Lambda = \frac{6\mu LU}{P_a h_i^2} 
\] (4.15)

This is the normalized slider bearing equation with slip flow effect.
For the gap height, \( h = h_1 - \frac{h_1 - h_2}{L} x \)

We define \( H \) as

\[
H = \frac{h}{h_1} = 1 - \frac{h_1 - h_2}{h_1} X = 1 - \varepsilon X
\]

where \( \frac{h_1 - h_2}{h_1} = \varepsilon \) \hspace{1cm} (4.16)

Applying the perturbation method by substituting \( \Psi = 1 + \varepsilon \Psi' \), and neglecting higher order terms, we have:

\[
\varepsilon \left[ \left( \frac{\partial^2 \Psi'}{\partial X^2} + \frac{\partial^2 \Psi'}{\partial Y^2} \right) \right] = \frac{\Lambda}{(1 + 6 Kn_i)} \left( \frac{\partial \Psi'}{\partial X} - 1 \right)
\]

\hspace{1cm} (4.17)

The boundary condition is the following:

\( X = 0, \, P = \text{Pa}: \, \Psi = 1 \) and thus \( \Psi' = 0 \)

\( X = 1, \, P = \text{Pa}: \, \Psi = 1 \) and thus \( \Psi' = 0 \)

\( Y = 0, \, P = \text{Pa}: \, \Psi = 1 \) and thus \( \Psi' = 0 \)

\( Y = 1, \, P = \text{Pa}: \, \Psi = 1 \) and thus \( \Psi' = 0 \)

Equation (4.17) will be used for the surfboading analysis.
4.2 – Numerical Approach

Equation (4.17) governs the slip flow for a slider bearing. A numerical approach is necessary for the nontrivial geometry of the proof mass. A PDEase program is used to solve (4.17) for the normalized perturbed pressure, $\Psi'$. The PDEase code is verified by the numerical results published in [12].

A square plate of length 1 was used to implement equation (4.17) with the boundary conditions described above. Bearing numbers 1, 5 and 10 will be simulated. The results, $\Psi'$, are equal to $\frac{\Psi' - 1}{\varepsilon} = \frac{P - P_o}{P_o \varepsilon}$. Integrating the solution $\varepsilon \Psi'$ and normalizing the lift force (bearing load) by the area and the ambient pressure, i.e., $\frac{\varepsilon \int \Psi' \, dA}{P_o A}$, the normalized bearing load is obtained.

The bearing load is then compared with the simulated results published in [12]. An illustration of the mesh used and the results are shown in Fig. 4.4 & 4.5.

![Figure 4.4: The mesh used in the PDEase program for slider bearing analysis.](image_url)
Figure 4.5: The solution contour of $\Psi'$ with $\Lambda = 5$, moving in the $x$-direction.

The simulated results are plotted against the published data [12] in Fig. 4.6:

Figure 4.6: Comparison of the PDEase simulations to the published data for slider bearing equation.

Good agreement between numerical solutions and published data exists for low bearing numbers and small tilts. Hence the code is verified for the problem considered.

From the last chapter, it was learned that the chimney resistance is an important source of pressure buildup. However, the chimney boundary condition requires the flow rate of the fluid to
be solved simultaneously. Thus without the knowledge of the exact air flow rate, the PDEase simulation can only be used with the following scenarios: 1). A flat plate, i.e., no-hole/infinite chimney resistance assumption; or 2). A proof mass with perfectly vented holes, i.e., zero chimney resistance. Since the fluid flow is three-dimensional in the proof mass, massive effort will be necessary to determine the exact flow rate. Thus we will approach with the upper and lower bound of the problem.

The proof mass of TFG B is not a square (450μm in x-direction and 400μm in y-direction), therefore the simulation is very similar to the one in Fig. 4.4, with slight change of dimension as shown:

![Figure 4.7: The mesh used for upper-bound approximation to the slider bearing equation.](image)

The bearing number is calculated as \( \frac{6\mu LU}{(1 + 6Kn_1)P_o h_i^2} \), and it is approximately 0.62 with \( L = 450 \mu m \). The solution contour is shown in Fig. 4.8.
Figure 4.8: The solution contour of $\Psi^*$ for the flat plate with $\Lambda = 0.62$, moving in the x-direction.

Proceeding with lower bound calculations, i.e., solving (4.17) with totally vented holes. Since no flow symmetry can be assumed, the 1800 hole of the TFG B have to be implemented. However, due to computational limitations, the maximum number of holes implemented in the model is 81. The mesh and resulting contour of $\Psi^*$ of a 49-hole mesh are shown in Fig. 4.9 and 4.10, with $\Lambda = 0.62$ for all cases.
Figure 4.9: The mesh with 49 holes used in slider bearing analysis.

Figure 4.10: The solution contour at the upper-left corner of the 49-hole domain.

The solution was noticed to be repetitive for each hole, it was verified by plotting the solution value along the centerlines in both $x$ and $y$-directions in Fig. 4.9, as shown below:

Figure 4.11: The value of the solution along the $(X)(X)$ centerline in Fig. 4.9.
Figure 4.12: The value of the solution along the (Y)(Y) centerline in Fig. 4.9.

It can be seen from Fig. 4.11 and 4.12 that the solution can be represented by multiple single cell simulations. We proceed by simulating two rows of holes and comparing the solution to the full square plate, the mesh used is shown below:

Figure 4.13: The mesh used with infinite plate assumption in the y-direction (Two-Column).

By multiplying the solution obtained from the 32-hole mesh by 8, the solution for 16 by 16 (256) holes was obtained. Plotting the solutions vs. number of holes we have:
Figure 4.14: The normalized force, \( \int \Psi' dA \), for various number of holes.

It can be concluded from Fig. 4.14 that the end effect quickly disappear as the curve follows a straight line as the number of holes increases. Based on this conclusion we can simulate one cell and multiply the solution throughout the plate as for the squeeze film problem. The mesh used for the cell is the same as the one for squeeze film problem, as shown below:
\[
\begin{aligned}
\frac{\partial^2 \Psi' + \partial^2 \Psi'}{\partial x^2 + \partial y^2} &= \frac{\Lambda}{(1+6\kappa_1)} \left( \frac{\partial \Psi'}{\partial x} - 1 \right)
\end{aligned}
\]

Figure 4.15: The mesh, governing equation and boundary conditions used for the cell simulation \((\Lambda = 0.01376 \text{ based on } L = 10 \mu m)\).

The maximum tolerance for deviation in gap height is 0.5\(\mu\)m, which yields an \(\varepsilon\) of \(\frac{(h_1-h_2)}{h_1} = 0.5\mu m/3.25\mu m = 0.154\). Plotting the upper and lower bound solutions for the lift force generated over the range of pressures considered, the results in Fig. 4.17 is obtained.
Figure 4.17: Plot of the upper and lower bound solutions of lift force generated, with the tilt $\varepsilon = \frac{(h_1 - h_2)}{h_1} = 0.154$.

The two bounds are 2 orders of magnitudes apart, thus the chimney resistance is an important contribution to the pressure buildup. Hence it is necessary to solve the flow rate and evaluate the chimney resistance. A flow network lumped parameter model is developed using Matlab, as shown below:
Figure 4.18: The one-dimensional network model used to solve the lift force generated from surfboarding.

Looking down, the flow is coming in the x-direction:

Figure 4.19: The one-dimensional network model in x-y plane.

Each column of holes can be represented by a long slot, and the variation in y-direction will be accounted for afterward.

For n holes, there are 2n+1 pressure nodes, n chimney resistance ($R_c$), 2n+2 plate resistance ($R$), each "block" can be as follows:
The plate resistance is obtained from Kirchoff’s law:

\[
\left( \frac{1}{R_1} + \frac{1}{R_2} \right) P = (Q_1 - Q_2),
\]

(4.18)

with \( P_a \) set to zero.

The driving flow rates, \( Q_1 \) and \( Q_2 \), are:

\[
Q_1 = \frac{bh_1 U}{2}
\]

(4.19)

\[
Q_2 = \frac{bh_2 U}{2}.
\]

(4.20)

Solving for \( R_1 \):

\[
R_1 = \frac{12 \mu l}{bh_1^3}
\]

(4.21)

\[
R_2 = \frac{12 \mu l}{bh_2^3}
\]

(4.22)

Thus the plate resistance can be determined by

\[
R_i = \frac{12 \mu l}{bh_i^3}
\]

(4.23)
The rarefaction effect can be included by modifying the viscosity by the factor \( \frac{1}{1 + 6Kn} \) [5], i.e.,

\[
\mu_{\text{eff}} = \frac{\mu}{1 + 6Kn} \tag{4.24}
\]

The Chimney resistance is determined the same way as described in Chapter 3.

\[
R_i = \frac{C\mu t}{\left(1 + 8\frac{\lambda}{L_n}\right)2L_n^4} \tag{4.25}
\]

A set of equations can therefore be written based on conservation of mass, i.e.,

\( P_n \): for \( n \) is odd (pressure node under the plate):

\[
\frac{P_{n-1} - P_n}{R_n} + Q_n = \frac{P_n - P_{n+1}}{R_{n+1}} + Q_{n+1} \tag{4.26}
\]

\( P_n \): for \( n \) is even (pressure node under the hole):

\[
\frac{P_{n-1} - P_n}{R_n} + Q_n = \frac{P_n - P_{n+1}}{R_{n+1}} + Q_{n+1} + \frac{P_n - P_0}{R_c} \tag{4.27}
\]

Rearranging:

\[
P_i \left( \frac{1}{R_i} + \frac{1}{R_{i+1}} \right) - \frac{P_{i-1}}{R_i} - \frac{P_{i+1}}{R_{i+1}} = Q_i - Q_{i+1} \tag{4.28}
\]

for \( i = \) odd:

\[
P_i \left( \frac{1}{R_i} + \frac{1}{R_{i+1}} + \frac{1}{R_c} \right) - \frac{P_{i-1}}{R_i} - \frac{P_{i+1}}{R_{i+1}} = Q_i - Q_{i+1} \tag{4.29}
\]
for \( i = \text{even}. \)

The \( 2n+1 \) equations can be rearranged in matrix form, i.e.,

\[
\begin{pmatrix}
\frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} \\
-\frac{1}{R_2} & \frac{1}{R_3} + \frac{1}{R_r} & -\frac{1}{R_3} \\
& -\frac{1}{R_3} & \frac{1}{R_4} & -\frac{1}{R_4} \\
& & & \ddots & \ddots & \ddots \\
& & & & & \frac{1}{R_{2n}} + \frac{1}{R_{2n+1}} & -\frac{1}{R_{2n+1}} \\
& & & & & -\frac{1}{R_{2n+1}} & \frac{1}{R_{2n+2}} & -\frac{1}{R_{2n+2}}
\end{pmatrix}
\begin{pmatrix}
P_1 \\
P_2 \\
P_3 \\
\vdots \\
P_{2n+1}
\end{pmatrix}
= \begin{pmatrix}
Q_1 - Q_2 \\
Q_2 - Q_3 \\
Q_3 - Q_4 \\
\vdots \\
Q_{2n+1} - Q_{2n+2}
\end{pmatrix}
\]

(4.30)

Solving for the pressure nodes simultaneously, we obtained the pressure distribution, and by integrating the pressure over the area of the proof mass, the lift force is determined.

To verify the network program, it is compared against the analytical solution for one-dimensional slider bearing [31]:

\[
P = \frac{6\mu U}{\alpha(h_1 + h_2)} \left[ \frac{(h_1 - h)(h - h_2)}{h^2} \right]
\]

(4.31)

where \( \alpha \) is \((h_1 - h_2)/L\).

To verify the validity of the network model, the chimney resistance is adjusted to a very high number by letting \( L_n = 0.1 \mu \text{m} \) and \( t = 1^{25} \text{m} \). Hence the plate can be viewed as if it has no through hole, and the result from a 45-hole model (91 nodes total) is shown in Fig. 4.21.
Figure 4.21: Comparison of the network solution with “infinite” chimney resistance to the one-dimensional flat plate solution. $L = 459.9 \, \mu m$, $\mu_{eff} = 1.74108e^{-7} \, N \cdot s/m^2$, corresponding to Knudsen number of 17.5. The tilt, $\varepsilon = (h_1 - h_2)/h_1$, is 0.0645.

Excellent agreement is obtained thus the network model is validated. Plotted below are the solutions of pressure distributions along the length of proof mass from a 45-hole model (91 nodes total) with various proof mass thicknesses (various chimney resistances).
Figure 4.22: The network solution of the pressure profile with “zero” chimney resistance. $L = 455 \mu m$, $\mu_{eff} = 1.74108e-7$ N-s/m$^2$, corresponding to Knudsen number of 17.5.

The tilt, $e = (h_1-h_2)/h_1$, is 0.0645.
Figure 4.23: Comparison of the flat plate solution and network model with $t = 8\mu m$.

$L = 455\ \mu m$, $\mu_{\text{eff}} = 1.74108e^{-7}\ N\cdot s/m^2$, corresponding to Knudsen number of 17.5.

The tilt, $\varepsilon = (h_{2}-h_{1})/h_{1}$, is 0.0645.

As shown in Fig. 4.22, the pressure distribution fluctuates across the proof mass as the chimney length, $t$, goes to zero ($1^{-20}\ m$). This is similar to the PDEase simulation with perfectly vented holes. As the chimney resistance increases, the pressure distribution becomes the solution of a flat plate as illustrated in Fig. 4.21. It is noted that the pressure profile in Fig. 4.22 is increasing with $x$ while the PDEase solution has a constant peak (Fig. 4.11). It is because the PDEase solution was simulated with a bearing number based on an average gap height ($h = 3\mu m$), thus the variation of the bearing number was not included. The plate resistances of the network model have accounted for the variation in gap height, thus the pressure profile is more accurately captured.

For the network model, the variation in the $y$-direction can be approximated as parabolic profile based on Fig. 4.8. The parabolic profile is approximated as $4\frac{y}{b} \left( 1 - \frac{y}{b} \right)$, with the one-
dimensional solution assumed to be 1 as shown below. Integrating the parabola we found that the lift force from the one-dimensional model was overestimated by $1/3$, i.e., $F_{1,D} = (3/2)F_{2,D}$.

![Pressure vs. y](image)

Figure 4.24: An illustration of the variation of pressure in $y$-direction.

Integrating the pressure profiles from the network model with $L_n = 5\mu m$ and compare with the lift forces obtained from PDEase simulations, we obtained the following:
Figure 4.25: Comparison of the lift force evaluated from the network and PDEase models to the flat plate solution.

The upper bound solutions from PDEase simulation and the network model agree well with the one-dimensional flat plate solution. The difference between the flat plate solution and the PDEase simulation can be explained by over-estimating the $y$-variation in the flat plate solution; and difference between the flat plate solution and the network model exists because the chimney resistance cannot represent a flat plate with $L_n = 5\mu m$, regardless how high the thickness ($t$) was used. However they both represent an acceptable approximation of the upper bound solution.

The lower bound solutions from PDEase simulation and the network model do not agree well. It is due to neglecting some solid area during the transformation from two-dimensional domain to a one-dimensional network, as shown below.
More rigorous investigations have to be performed to account for the neglected area, such as development of two-dimensional network model. We will therefore use the upper and lower bounds from the PDEase simulations and proceed with the analysis.

4.3 – Experimental Apparatus

The experimental setup for bias measurement is essentially combining the setup for both drive Q and sense Q measurements. The TFG unit is placed inside the bell jar with the desired pressure and supplied with an AC voltage at the resonant drive frequency to stimulate the drive mode at a fixed and stable resonant frequency. At the same time the output signal of the sense axis is monitored as described in the sense Q measurement section. A pictorial representation is shown in Fig. 4.27.
Figure 4.27: A pictorial representation of the experimental setup for bias measurements.

Because the TFG is not subjected to rotation, any output signal from the sense axis is the bias of the unit.

It is important to understand that there are many possible combinations of the surfboarding effect. Both of the proof masses can be tilted at a positive angle, or one positive and one negative. There can also be only one proof mass tilted. Since the output signal is the sum of the two proof masses, it will be very difficult to analyze the TFG with both proof masses oscillating. Therefore to separate the combining effect, only one proof mass is excited each time.

A TFG with the following dimensions has been tested:
<table>
<thead>
<tr>
<th>Parameters</th>
<th>TFG C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proof mass length (L)</td>
<td>450µm</td>
</tr>
<tr>
<td>Proof mass width (b)</td>
<td>400µm</td>
</tr>
<tr>
<td>Width of hole (Lₘ)</td>
<td>4.5µm</td>
</tr>
<tr>
<td>Hole spacing (Lₚ)</td>
<td>10µm</td>
</tr>
<tr>
<td>Thickness (t)</td>
<td>8µm</td>
</tr>
<tr>
<td>Combs air gap (gₖ)</td>
<td>3µm</td>
</tr>
<tr>
<td>Combs overlap length (Lₖ)</td>
<td>25µm</td>
</tr>
<tr>
<td>Gap height (h)</td>
<td>3µm</td>
</tr>
</tbody>
</table>

Table 4.1: Typical dimensions of the design parameters for TFG C.

The in-phase bias, in degree/second, for the left and right proof mass are plotted in the following:

Figure 4.28: The bias, in deg/s, generated from TFG C (LCCC 775).
The results from Fig. 4.28 show that the two proof masses of the TFG are tilted oppositely at similar angles. We will focus on the left proof mass and determine its approximate tilt with the network model. The bias, interpreted as Coriolis acceleration, is converted into lift force by:

\[ F = 2m\Omega x\omega_d, \tag{4.32} \]

where \( m \) is the mass, \( \Omega \) is the bias in deg/sec, \( x \) is the amplitude of the oscillation, and \( \omega_d \) is the drive frequency of the TFG. Fig. 4.29 shows the lift force evaluated from the in-phase bias vs. Knudsen number for the left proof mass.

![Graph showing lift force vs. Knudsen number](image)

Figure 4.29: The lift force evaluated from the in-phase bias vs. Knudsen number.
4.4 – Comparison of the numerical solutions with the experimental results

The network model was simulated over a range of pressures, and the results are plotted against the three numerical methods used in Fig. 4.30.

![Graph](image)

Figure 4.30: Plot of various approximations vs. the measurement, the tilt, $h_1-h_2/h_1$, is 0.0645.

By fitting the solution curve from the network model to the measurements, the tilt can be determined. The result indicates that the left proof mass has a gap height difference of approximately 0.2μm from one end to the other, i.e., $h_1 = 3.1\mu m$ and $h_2 = 2.9\mu m$, which is within the tolerance for the fabrication. However, there are some other pressure sensitive factors contributing to the in-phase bias of the TFG in addition to the surfboarding. As a result of the uncertainties, the network model for surfboarding effect can be concluded to provide a reasonable order of magnitude calculation of the lift force.

This concludes the analysis for surfboarding.
Chapter 5.

Conclusion

In this thesis, the effects of rarefied gas damping on the micromechanical tuning fork gyroscope were investigated. The quality factors in the drive and sense axes, with the rarefaction effect included, were derived. Closed-form solutions based on slip flow and molecular flow theories were obtained for the quality factor in the drive axis. By comparing the solutions for slip and molecular flow to the continuum flow solution, it was found that the rarefaction effect contributed 1 to 5 orders of magnitude increase in quality factor over the entire range of pressures considered (1 to 10³ mTorr).

The slip flow solution yielded the same result as the molecular flow solution for the simple geometry used (flat plate with effective area). Any difference in solution (Fig. 2.6) is due to the different effective areas used. Thus the same result will be obtained if same effective area is used in both models. This validates the validity of the slip flow model in high Knudsen number regime, despite the fact that slip flow assumption should be invalid at free molecular level.

The molecular flow solution had an average discrepancy of 21% for TFG A and 27% for TFG B. These solutions are considered to be good approximations for the range of quality factors obtained (5 orders of magnitude). It was found that the shearing between the combs contributes significantly (about 20%) to the quality factor calculations for both TFG A and B.

For quality factor in the sense axis, the Reynolds equation for isothermal squeeze film damping with slip flow boundary conditions was solved numerically to calculate the fluid damping. The rarefaction effect contributed 1 to 5 orders of magnitude increase of quality factor over the entire range of pressures considered (1 to 10³ mTorr).

The slip flow solution had an average discrepancy of 50% and 30% for TFG A and B, respectively, which is an acceptable accuracy at high Knudsen number regime. It was found that the comb contribution is insignificant, only about 1% to the solution. However, the chimney resistance from the proof mass provide an important source of
damping. The quality factor determined without chimney resistance is greater than the quality factor with chimney resistance by a factor of 3.4 at atmospheric pressure (760 Torr) and 5.1 at low pressure (1 mTorr).

For surfboard analysis, the Reynolds equation for isothermal slider bearing was solved numerically with the slip flow boundary conditions. The upper and lower bounds, which correspond to the no hole and perfectly vented holes situation, respectively, were obtained from the PDEase model. The two bounds were used as references for the development of the one-dimensional network model, which simultaneously solved the pressure underneath the proof mass. Reasonable estimation of the tilt of the proof mass was obtained and the model can be used for parametric study of the design parameters. If the tilt can be experimentally determined, the resulting lift force from the network model can be used to measure the discrepancy.

The discrepancy for the quality factors in the drive and sense axes is contributed from two sources: analytical assumptions and experimental error. Analytical discrepancies include 1). Slip flow assumption used at highly rarefied regime; 2). Neglecting end/edge effects; 3). Perturbation method used to simplify the normalized equations; 4). Circular cross section used to determine the chimney resistance; 5). Perfectly flat proof mass surface assumed; 6). Perfectly parallel plates assumed for drive and sense Q calculations; 7). Isothermal process assumed; 8). All holes are assumed to be uniform in size and spacing. 9). Neglecting unsteadiness; 10). Neglecting variation in y-direction. These assumptions and simplifications can be the sources of error.

The experiments were performed at room temperature. It was found that a temperature variation within ±5 K did not alter the quality factors measurably. Main source of experimental discrepancy include the measurements of pressure, curve fitting of the decaying signal, and the extraction of the structural damping from the data.

The accuracy of the quality factor evaluation can be improved if the structural damping can be theoretically determined, because the fluid damping extraction depends heavily on the value of the structural damping, especially at low pressures. Furthermore, fluid models that are developed for high Knudsen number regime can be used to better capture the rarefaction effect.
For surfboarding, it is recommended that a more sophisticated two-dimensional network model, which includes more nodes underneath the plate and hole, should be developed. Thus the variation in the y-direction and the solid area between the holes can be better captured.

To achieve better performance, it is recommended that the design should focus on the hole size and hole spacing, because these two parameters are the most sensitive parameters for sense damping. The chimney resistance can be reduced if the through holes on the proof mass are circular instead of rectangular.

In summary, the quality factor in drive and sense axes have been successfully determined with an acceptable accuracy. The surfboarding effect is also predicted by the one-dimensional network flow model, which can assist in the development of more advanced tuning fork gyroscopes.
References


Appendix A

PDEase Code for Isothermal Squeeze Film Problem

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Title
"Isothermal Squeeze Film Solution for Unit Cell"

Select
epsilon = 1e-7

Variables
psi0  psi1

Definitions
a = 1
b = 1
po = 0.04334
sigma = 0.025356
c = 0.27085
d = 0.72915

Equations
div(grad(psi1)) - sigma*psi0 = 0
div(grad(psi0)) + sigma*psi1 = -sigma
**Boundaries**

region 1

start (0,0)
natural(psi0)= po
natural(psi1)= po
line to (a,0) to (a,b) to (0,b)
natural(psi0)= po
natural(psi1)= po
line to finish

exclude 2

start(c, c)
value(psi0)= po
value(psi1)= po
line to (c,d) to (d,d) to (d,c)
value(psi0)= po
value(psi1)= po
line to finish

**Plots**

contour(psi0)

**End**
Appendix B

PDEase Code for Isothermal Slider Bearing Problem

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Title
"Isothermal Slider Bearing Solution for Rectangular Plate"

Select
errlim= 5e-4

Variables
p

Definitions
n = 1
a = 1
b = 1
xa = 1/(4*n)
ya = 1/(4*n)
l = 1/(2*n)
po = 0
bearing = .01376
Equations

\text{div(grad(p))} = \text{bearing*[dx(p) -1]}

Boundaries

region 1

start (0,0)
natural(p)= po
line to (a,0) to (a,b) to (0,b)
natural(p)= po
line to finish
exclude 2

start(xa, ya)
value(p)= po
line to (xa, ya+l) to (xa+l, ya+l) to (xa+l, ya)
value(p)= po
line to finish

Plots

contour(p)

End
Appendix C

Matlab Code for Surfboarding Network Flow Problem

```matlab
clc;
n = 45;  %number of holes%
H1 = 3.25e-6;  %edge height (largest)%
H2 = 2.75e-6;  %edge height (smallest)%
Lp = 10e-6;  %length of hole spacing%
Lh = 2.5e-6;  %hole size%
l = Lp - Lh;  %length of unit cell%
lc = 8e-6;  %length of chimney%
ueff = 1.73011e-7;  % effective viscosity %
b = 400e-6;  %width of proof mass cell%
C = 64;  %geometric constant for chimney%
L = (n+1)*l+n*Lh;  %length of proof mass%
e = (H1-H2)/L;  %amount of tilt%
U = 1.5949;  %proof mass velocity%

%------------------ Defining the gap height at various chimney points ---------%

xh(1) = l+Lh/2;
hhole(1) = H1 - e*xh(1);
for i=2:1:n
    xh(i) = i*(l+Lh)-Lh/2;
    hhole(i) = H1 - e*xh(i);
end

%------------------ Defining the gap height at various nodal points ---------%
```
xp(1) = l/4;  \% first set \%
hp(1) = H1 - e*xp(1);

xp(2*(n+1)) = L-(l/4);  \% final set \%
hp(2*(n+1)) = H1 - e*xp(2*(n+1));

for j=1:1:n  \% chimney counter \%
    xp(2*j) = (j*(l+Lh)-(Lh/2))-(l/4 + Lh/2);  \%chimney location - (l/4+Lh/2)\%
    hp(2*j) = H1 - e*xp(2*j);
    xp(2*j+1) = (j*(l+Lh)-(Lh/2))+(l/4 + Lh/2);  \%chimney location + (l/4+Lh/2)\%
    hp(2*j+1) = H1 - e*xp(2*j+1);
end

\%------------------- Defining the plate and chimney resistance -------------------\%

for i=1:1:2*(n+1)
    rp(i) = 12*ueff*(l/2)/(b*hp(i)*hp(i)*hp(i));
end

rc = C*ueff*lc/(2*Lh*Lh*Lh*Lh);

\%------------------- Defining the R matrix (odd and even rows) -------------------\%

for i=1:2:2*n-1
    R(i,i) = 1/rp(i) + 1/rp(i+1);
    R(i,i+1) = -1/rp(i+1);
    if i>1
        R(i,i-1) = -1/rp(i);
    end
end
R(2*n+1,2*n) = -1/rp(2*n+1);
R(2*n+1,2*n+1) = 1/rp(2*n+1) + 1/rp(2*n+2);

for i=2:2:2*n-2
    R(i,i) = 1/rp(i) + 1/rp(i+1) + 1/rc;
    R(i,i-1) = -1/rp(i);
    R(i,i+1) = -1/rp(i+1);
end
R(2*n,2*n) = 1/rp(2*n) + 1/rp(2*n+1) + 1/rc;
R(2*n,2*n-1) = -1/rp(2*n);
R(2*n,2*n+1) = -1/rp(2*n+1);

%--------------------- Defining the Flow vector (Q's) ---------------------%

for i=1:1:2*n+1
    q(i) = U*b*(hp(i)-hp(i+1))/2;
end

%--------------------- Calculating the Pressure (P's) ---------------------%

Q = q(:);
P = inv(R)*Q;
j = 1;

%------ Separate the P vector into pressure-under-plate/dP-along-chimney ------%

for i=1:2:2*n+1
    pres(j) = P(i);
    j = j+1;
end

j = 1;
for i=2:2:2*n
    presc(j) = P(i);
    j = j+1;
end

pressure = pres(:);
dpchimney = presc(:);

%%%------- Combining the two position vectors (xh's and xp's) into X's -------%

k = 0;
for i=1:2:2*n+1
    x(i) = l/2 + k*(l + Lh);
    k = k + 1;
end

k = 1;
for i=2:2:2*n
    x(i) = xh(k);
    k = k + 1;
end
x(2*n+2) = L;

P(2*n+2) = 0;
X = x(:);

%%%-------------------------------- End of Program -------------------------%
THESIS PROCESSING SLIP

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