PSFC/JA-08-38

QUASILINEAR THEORY OF ELECTRON TRANSPORT BY RADIO FREQUENCY WAVES AND NON-AXISYMMETRIC PERTURBATIONS IN TOROIDAL PLASMAS

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November 2008

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This work was supported by the U.S. Department of Energy Contracts No. DE-FG02-91ER-54109 and DE-FG02-99ER-54521.

To appear in *Physics of Plasmas.*

Quasilinear theory of electron transport by radio frequency waves and non-axisymmetric perturbations in toroidal plasmas

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(Dated: November 26, 2008)

Abstract

The use of radio frequency (RF) waves to generate plasma current and to modify the current profile in magnetically confined fusion devices is well documented. The current is generated by the interaction of electrons with an appropriately tailored spectrum of externally launched RF waves. In theoretical and computational studies, the interaction of RF waves with electrons is represented by a quasilinear diffusion operator. The balance, in steady state, between the quasilinear operator and the collision operator gives the modified electron distribution from which the generated current can be calculated. In this paper the relativistic operator for momentum and spatial diffusion of electrons due to RF waves and non-axisymmetric magnetic field perturbations is derived. Relativistic treatment is necessary for the interaction of electrons with waves in the electron cyclotron (EC) range of frequencies. The spatial profile of the RF waves is treated in general so that diffusion due to localized beams is included. The non-axisymmetry magnetic field perturbations can be due to magnetic islands as in neoclassical tearing modes. The plasma equilibrium is expressed in terms of the magnetic flux coordinates of an axisymmetric toroidal plasma. The electron motion is described by guiding center coordinates using the action-angle variables of motion in an axisymmetric toroidal equilibrium. The Lie perturbation technique is used to derive a diffusion operator which is non-singular and time dependent. The resulting action diffusion equation describes resonant and non-resonant momentum and spatial diffusion. Momentum space diffusion leads to current generation in the plasma and spatial diffusion describes the effect of RF waves and magnetic perturbations on spatial evolution of the current profile. Depending on the symmetry of the equilibrium and the corresponding relation of the action variables to the configuration space variables, additionally to diffusion along the radial direction, poloidal and toroidal electron diffusion is also described. In deriving the diffusion operator, no statistical assumption, such as the Markovian assumption, for the underlying electron dynamics, is imposed. Consequently, the operator is time dependent and valid for a dynamical phase space that is a mix of correlated regular orbits and decorrelated chaotic orbits. The diffusion operator is expressed in a form suitable for implementation in a numerical code.

I. INTRODUCTION

The steady state operation of a tokamak fusion device will require some externally generated plasma current. Radio frequency waves are a desirable option as it is possible to control the spatial location in a plasma where they can drive current. Among the various radio frequency waves electron cyclotron (EC) waves have been extensively used to generate plasma currents and to modify the current profile. In DIII-D EC waves were to used to generate plasma current to control the growth of neoclassical tearing modes [1]. In TCV EC current drive (CD) was not only used for controlling the neoclassical tearing mode [2] but also to provide the total confining current [3].

A theoretical description of the interaction of radio frequency (RF) waves with electrons in tokamaks requires an accounting of the toroidal magnetic field geometry. Furthermore, for EC waves, the description has to be relativistic so that the damping of the waves and their interaction with electrons are described correctly [4–7]. In this paper, for an axisymmetric toroidal equilibrium, we derive a relativistic diffusion operator for the interaction of RF waves with electrons in the presence of non-axisymmetric magnetic field perturbations. We use magnetic flux coordinates to describe the equilibrium magnetic field and the electron motion is expressed in terms of the canonical guiding center variables [8]. The Lie transform perturbation theory [9] is used to determine effects of RF waves and non-axisymmetric magnetic perturbations on the electron motion. The ordering parameter, assumed to be small, in the perturbation expansion is taken to be the ratio of both the strength of the RF fields and of the non-axisymmetric magnetic field perturbations to the confining magnetic field. In deriving the diffusion equation for the electron distribution function we show that the Lie perturbation expansion needs to be carried out to first order in order to obtain a diffusion equation which is accurate to second order in the ordering parameter [10].

The non-axisymmetric magnetic perturbations could be due to the formation of magnetic islands, e.g., the neoclassical tearing modes, in a plasma. Even though a magnetic island may evolve in time, we assume that, on the time scales relevant to the interaction of RF waves with electrons, the island is essentially stationary. This is a reasonable approximation. In experiments, an external control system is used to guide the EC waves to the location of the island [1]. The current profile is actively modified in the island region indicating that the movement of the island is slow compared to the time it takes for the EC waves to interact with electrons.

There have been a number of studies on quasilinear diffusion due to plasma turbulence and plasma waves [11–16]. In a broad sense, there are essentially two approaches to the derivation of the diffusion equation. One follows the approach of Kennel and Engelmann [11] for a uniform plasma in a spatially uniform magnetic field. In order to obtain the quasilinear diffusion equation the initial, zero-order, particle distribution function is assumed to be a random distribution of the phase of particle gyro motion. The uniformity assumption brings about peculiar limits in the evaluation of the diffusion coefficient which cannot be justifiably extended to an inhomogeneous plasma. The Kennel-Engelmann form is also not suitable for addressing experiments in tokamaks in which the launched RF wave has a fixed frequency. The non-relativistic Kennel-Engelmann approach has been extended to relativistic plasmas [12]. The second approach is due to Kaufman [13] and applies to axisymmetric toroidal plasmas. In this approach the non-relativistic electron motion is described in terms of the guiding center variables – an approach that is not necessary in the uniform plasma description of Kennel-Engelmann. The quasilinear evolution equation is obtained from the continuity equation for the complete electron distribution function, and the diffusion operator is expressed in terms of the action variables which are invariants of an axisymmetric toroid.

Our formulation of the diffusion equation and the diffusion operator follows some aspects of Kaufman's approach. The dynamical variables describing the electron motion are the three canonical actions related to poloidal flux (radial coordinate), momentum parallel to magnetic field, and magnetic moment, and their corresponding canonical angles. The actions are constants of the motion when the magnetic perturbations and the RF wave fields are ignored. In the presence of perturbations, the canonical Lie transform theory is used to determine, perturbatively, the evolution, over a finite time interval, [17] of the dynamical variables and any arbitrary function of these variables. A special case of such a function is the electron distribution. We show that the evolution of canonical angle-averaged distribution function can be evaluated to second order in the perturbation parameter by solving for the electron dynamics to first order in this parameter [10]. The evolution equation for the distribution function is a diffusion equation in action space. Depending on the symmetries of the magnetic field, the action variables depend, additionally to the parallel momentum, magnetic moment and radial coordinate, also on the poloidal and the toroidal coordinates.

Therefore, the action diffusion equation describes momentum and particle transport along the respective dimensions. Elements of the diffusion tensor are non-singular functions of the actions and time, and include both resonant and non-resonant diffusion. The time dependence of the diffusion operator is a consequence of the finite time interval used in calculating changes in the dynamical variables. Consequently, singular Dirac delta functions, which appear in the Kennel-Engelmann and Kaufman approaches and are commonly treated by including collisonal effects resulting to phase decorelation and resonance broadening [14], are not present in our diffusion operator.

The Kennel-Engelmann and Kaufman approaches invoke the Markovian assumption in order to obtain a diffusion equation. In the Kennel-Engelmann approach it is assumed that the turbulence affects all particles in such a way that the distribution function is independent of gyro phase of the particles. In the Kaufman approach the Markovian assumption is made to justify evaluation of the distribution function. It is also applied to the diffusion operator where the upper limit of the time integral is extended to infinity. In both approaches, due to the Markovian assumption, terms in the diffusion tensor contain a delta function that is a function of the wave-particle resonance condition. So the diffusion tensor is non-zero for a discrete set of action surfaces which satisfy the exact resonant condition. This leads to mathematical and numerical difficulties. A way out of this dilemma is to invoke small nonlinearities that broaden the delta function and lead to a continuous diffusion tensor [13]. The delta function singularities reflect an underlying dynamical phase space in which the particle motion is chaotic. This leads to a loss of memory and phase mixing – the basic assumptions for a Markovian process. For such a process the motion is assumed to be chaotic over any time scale so that the upper limit of the time integral in the diffusion tensor can be extended to infinity. This leads to a delta function singularity. However, in many cases of interest, the Markovian assumption does not hold. The underlying phase space of the particle motion contains not only chaotic regions but also islands pertaining to regular or quasiperiodic motion. In this more general case, the change in dynamics has to evaluated over finite time intervals so that the diffusion tensor is a smooth function of time and actions localized around the linear wave-particle resonances.

This paper is organized as follows. In Section II we set up the toroidal coordinates for an axisymmetric magnetic equilibrium. The action-angle variables for electron motion in this geometry are defined. In Sections III and IV the perturbations due to a non-axisymmetric magnetic field and the RF wave fields, respectively, are included in the Hamiltonian. Parts of the Lie transform perturbation theory that are relevant to our studies are outlined in Section V. The Lie perturbation theory is used in Section VI to evaluate the leading order effects on the electron orbits due to the magnetic perturbations and the RF waves. In section VII, as a response to these perturbations, we derive, in action space, the diffusion equation for the electron distribution function. The final results are summarized and discussed in Section VIII.

II. CANONICAL GUIDING CENTER HAMILTONIAN IN ACTION-ANGLE VARIABLES

In a general magnetic configuration, consisting of nested toroidal magnetic surfaces, the covariant representation of the magnetic field is [8]

$$\mathbf{B} = g(\psi_p)\nabla\zeta + I(\psi_p)\nabla\theta + \delta(\psi_p, \theta)\nabla\psi_p \tag{1}$$

where ψ_p, ζ, and θ are, respectively, the poloidal flux, the toroidal angle, and the poloidal angle. The functions g and I are related to the poloidal and toroidal currents, respectively, and [8]

$$\delta(\psi, \theta, \zeta) = \frac{-\left(I\nabla\theta\cdot\nabla\psi + g\nabla\zeta\cdot\nabla\psi\right)}{|\nabla\psi|^2} \tag{2}$$

is related to the degree of non-orthogonality of the coordinate system. The magnetic field lines are straight lines in the (ζ, θ) plane. The guiding center Hamiltonian is obtained from the guiding center Lagrangian [18]

$$L_{gc} = \frac{e}{c}\mathbf{A}^{\star}\cdot\mathbf{v} + \frac{mc}{e}\mu\dot{\xi} - H_{gc} \tag{3}$$

where $\mathbf{v}$ is the guiding center velocity, c is the speed of light, e is the electron charge, m is the electron mass, $\mathbf{A}^{\star} = \mathbf{A} + (mc/e)u_{\parallel}\hat{\mathbf{b}}$, $\mathbf{A}$ is the vector potential, $\mathbf{u} = \gamma\mathbf{v}$, $\gamma = (1 - v^2/c^2)^{-1/2}$, $v_{\parallel}$ is the component of $\mathbf{v}$ along $\mathbf{B}$, $\hat{\mathbf{b}} = \mathbf{B}/B$, $\mu = mu_{\perp}^2/2B$ is the magnetic moment, and ξ is the gyrophase. The dot represents a derivative with respect to time. The corresponding Hamiltonian is

$$H_{gc} = \left(m^2c^4 + m^2c^2u_{\parallel}^2 + 2mc^2\mu B\right)^{1/2} + e\Phi \tag{4}$$

where Φ is the electrostatic potential.

A canonical variable set can be obtained using the the formalism of [18, 19]. Following, the derivation of [18] we multiply the Lagrangian (3) by the constant factor c/e so that we either have to use a Hamiltonian which is c/e times the energy [18] or measure time in c/e units, which is the case we consider in the following. The rest of the derivation follows the standard procedure utilized in [19]. In Eq. (3) we replace $\mathbf{v}$ by $\mathbf{v}+\mathbf{w}$, where $\mathbf{v}$ describes the guiding center motion and $\mathbf{w}$ is given by $\mathbf{A}^{\star}\cdot\mathbf{w} = -\delta\rho_{\|}\dot{\psi}_p$ with $\rho_{\|} = mcu_{\|}/eB$. Then the two sets of canonically conjugate variables are (P_θ, θ) and (P_ζ, ζ) where

$$P_\theta = \psi + \rho_{\|} I \tag{5}$$

$$P_\zeta = \rho_{\|} g - \psi_p \tag{6}$$

ψ, the toroidal flux, is given by $d\psi/d\psi_p = q(\psi_p)$ with $q(\psi_p)$ being the safety factor. ψ_p and $\rho_{\|}$ are functions of P_θ and P_ζ only, and

$$\begin{aligned} \frac{\partial\psi_p}{\partial P_\theta} &= \frac{g}{D}, \qquad \frac{\partial\psi_p}{\partial P_\zeta} = \frac{-I}{D} \\ \frac{\partial\rho_{\|}}{\partial P_\theta} &= \frac{1-\rho_{\|} g'}{D}, \qquad \frac{\partial\rho_{\|}}{\partial P_\zeta} = \frac{q+\rho_{\|} I'}{D} \end{aligned} \tag{7}$$

where $D = gq + I + \rho_{\|}(gI' - Ig')$ with the prime indicating differentiation with respect to ψ_p. The third set of canonically conjugate variables is (μ, ξ). Since the gyrophase ξ is a cyclic coordinate, μ is a constant of the motion. For the toroidally symmetric configuration, ζ is also a cyclic coordinate so that P_ζ is conserved. Since the Hamiltonian H is time independent, it is also a constant of the motion

$$H_{gc}(P_\theta, \theta; P_\zeta, \mu) = W = \text{const.} \tag{8}$$

Thus, the three-degree of freedom system (4) has three independent conserved quantities (μ, P_ζ, W) and the particle motion is completely integrable. The Hamiltonian describes magnetically trapped particles moving in banana orbits, and passing particles circulating in the toroidal direction.

A canonical action-angle transformation can be used to eliminate θ from the Hamiltonian. A new action $\hat{P}_\theta$ where

$$\hat{P}_\theta = \oint P_\theta(\theta; \mu, P_\zeta, W) d\theta \tag{9}$$

along with the canonical transformation obtained from the generating function

$$S(\xi, \zeta, \theta; \hat{\mu}, \hat{P}_\zeta, \hat{P}_\theta) = \xi\hat{\mu} + \zeta\hat{P}_\zeta + \int_0^\theta P_\theta(\theta'; \hat{\mu}, \hat{P}_\zeta, \hat{P}_\theta) d\theta' \tag{10}$$

eliminates the θ dependence in (8) while preserving μ and P_ζ. In the transformation given above, the hatted variables are the new action-angle variables and $\hat{\mu} = \mu$ and $\hat{P}_\zeta = P_\zeta$. We will use the new action-angle variables and drop, without leading to any confusion, the hat over this variable set.

III. HAMILTONIAN WITH NON-AXISYMMETRIC, STATIC, MAGNETIC FIELD PERTURBATIONS

Our aim in this paper is to construct a model for the diffusion of electrons due to the combined effects of RF waves and non-axisymmetric magnetic field perturbations. For the time scale of interest we can assume that any magnetic perturbations to the magnetic equilibrium discussed in the previous section are static. These magnetic perturbations, for example those due to neoclassical tearing modes [20], are assumed to evolve on a time scale that is long compared to the time it takes for the RF waves to modify the local electron distribution function.

A general perturbation is given by a vector potential

$$\tilde{\mathbf{A}} = a_\zeta \nabla\zeta + a_\theta \nabla\theta + a_{\psi_p} \nabla\psi_p \tag{11}$$

where $a_\zeta, a_\theta, a_{\psi_p}$ are functions of position. Following [19], the canonical variables are modified as follows

$$P'_\theta = P_\theta + a_\theta(\psi_p, \theta, \zeta) \tag{12}$$

$$P'_\zeta = P_\zeta + a_\zeta(\psi_p, \theta, \zeta) \tag{13}$$

where $\mathbf{w}$ is given by $\mathbf{A}^\star \cdot \mathbf{w} = -(\delta\rho_\parallel + a_{\psi_p})\dot{\psi}_p$. For most applications a perturbed field of the restricted form

$$\tilde{\mathbf{A}} = a\mathbf{B} \tag{14}$$

with

$$a(\psi_p, \theta, \zeta) = \sum_{m_1, m_2} a_{m_1,m_2}(\psi_p) e^{i(m_1\theta + m_2\zeta)} \tag{15}$$

can be used [21]. This form, while not completely general, is sufficient to exactly represent the $\nabla\psi$ component of any magnetic perturbation. The other components are not important as they contribute only to the nonresonant perturbations of the equilibrium. Perturbations of the form (14) modify the parallel canonical momentum

$$\rho_c = \rho_\parallel + a \tag{16}$$

so that

$$H_{gc} = \left[m^2c^4 + e^2(\rho_c - a)^2 B^2 + 2mc^2\mu B\right]^{1/2} + e\Phi \tag{17}$$

IV. HAMILTONIAN INCLUDING RF WAVE FIELDS

The scalar and vector potentials corresponding to RF wave fields are represented in an eikonal form [22]

$$\begin{aligned} \Phi_{rf}(\mathbf{x},t) &= \tilde{\Phi}_{rf}(\mathbf{x})e^{i\Psi(\mathbf{x},t)} \\ \mathbf{A}_{rf}(\mathbf{x},t) &= \tilde{A}_{rf}(\mathbf{x})\mathbf{P}_{rf}e^{i\Psi(\mathbf{x},t)} \end{aligned} \tag{18}$$

where $\tilde{\Phi}_{rf}$ and $\tilde{A}_{rf}$ are amplitudes of the scalar and vector potentials, respectively, Ψ is the phase, and $\mathbf{P}_{rf}$ is the wave polarization vector. The local wave vector $\mathbf{k}$ and the angular frequency ω of the wave fields are given by

$$\begin{aligned} \mathbf{k}(\mathbf{x},t) &= \nabla\Psi(\mathbf{x},t) \\ \omega(\mathbf{x},t) &= -\frac{\partial\Psi(\mathbf{x},t)}{\partial t} \end{aligned} \tag{19}$$

The Lagrangian of a particle in a static inhomogeneous magnetic field interacting with RF waves is

$$L = [m\mathbf{u} + (e/c)(\mathbf{A} + \mathbf{A}_{rf})] \cdot \dot{\mathbf{x}} - H \tag{20}$$

where

$$H = (m^2c^4 + c^2u^2)^{1/2} + e\Phi + e\Phi_{rf} \tag{21}$$

and the potentials Φ, $\mathbf{A}$ correspond to the static inhomogeneous magnetic field discussed in the previous section.

In order to make use of the guiding center magnetic coordinates (ψ_p, θ, ζ), we define a transformed velocity [23, 24]

$$m\mathbf{u}_0 = m\mathbf{u} + \frac{e}{c}\mathbf{A}_{rf} \tag{22}$$

Then

$$L = \left[m\mathbf{u}_0 + \frac{e}{c}\mathbf{A} \right] \cdot \dot{\mathbf{x}} - H \tag{23}$$

with

$$H = \left[m^2c^4 + c^2 \left(m\mathbf{u}_0 - (e/c)\mathbf{A}_{rf} \right)^2 \right]^{1/2} + e\left(\Phi + \Phi_{rf} \right) \tag{24}$$

We will assume that $|\mathbf{A}| \gg |\mathbf{A_{rf}}|$ and $|\Phi| \gg |\Phi_{rf}|$ so that the particle orbits are perturbed by the RF fields. Using a formal perturbation parameter ϵ that multiplies $\mathbf{A_{rf}}$ and Φ_{rf}, we obtain, to second order in ϵ,

$$\begin{aligned} H &= mc^2\gamma_0 + e\Phi \\ &\quad + \epsilon e \left[-(1/\gamma_0 c)\mathbf{u}_0 \cdot \mathbf{A}_{rf} + \Phi_{rf} \right] \\ &\quad + \epsilon^2 (e^2/2mc^2\gamma_0) \left[A_{rf}^2 - (1/c^2\gamma_0^2)(\mathbf{u}_0 \cdot \mathbf{A}_{rf})^2 \right] \end{aligned} \tag{25}$$

where $\gamma_0 = (1 + u_0^2/c^2)^{1/2}$. Eventually, ϵ will be set to one.

The ϵ^0-term is the guiding center Hamiltonian given in Eq. (17). The higher order terms in ϵ need to be expressed in terms of the action-angle variables of the unperturbed Hamiltonian (8). We define the following transformation

$$\mathbf{x} = \mathbf{X} + \rho\hat{a} \tag{26}$$

$$\mathbf{u}_0 = u_{0\|}\hat{b} + u_{0\perp}\hat{c} \tag{27}$$

where $\mathbf{X}$ is the position of the center of the gyration and ρ is the Larmor radius of the particle. The unit vector $\hat{b}$ is along the axisymmetric magnetic field. The unit vectors $\hat{a}$ and $\hat{c}$ are perpendicular to $\hat{b}$, $\hat{a} = \hat{b} \times \hat{c}$, and gyrating with the particle. In terms of the fixed coordinate system,

$$\begin{aligned} \hat{a} &= \cos\xi\hat{\tau}_1 - \sin\xi\hat{\tau}_2 \\ \hat{c} &= -\sin\xi\hat{\tau}_1 - \cos\xi\hat{\tau}_2 \end{aligned} \tag{28}$$

where $\hat{\tau}_1$ and $\hat{\tau}_2$ are fixed unit vectors with $\hat{\tau}_1 \times \hat{\tau}_2 = \hat{b}$.

When the spatial variation of the eikonal phase in Eq. (18) is assumed to be small compared to the Larmor radius of the particle, the transformations in Eqs. (26, 27) give

$$\exp\left[i\Psi(\mathbf{x}, t) \right] \simeq \exp\left[i\Psi(\mathbf{X}, t) + i\mathbf{k} \cdot \rho\hat{a} \right] \tag{29}$$

If ϕ is the angle between $\mathbf{k}_\perp$ and $\hat{\tau}_1$ then

$$\mathbf{k} = k_\| \hat{b} + k_\perp (\cos\phi\,\hat{\tau}_1 + \sin\phi\,\hat{\tau}_2) \tag{30}$$

and

$$e^{i\mathbf{k}\cdot\rho\hat{a}} = \sum_{l=-\infty}^{+\infty} J_l(k_\perp \rho) e^{il(\xi+\phi+\pi/2)} \tag{31}$$

where J_l is the $l-$th order Bessel function. All quantities on the right hand side of Eqs. (29), (30), (31) are evaluated in the guiding center coordinates. If the polarization vector is expressed in terms of the right hand and left hand circular polarizations (P_{rf}^+, P_{rf}^-) and the parallel component $P_{rf}^{\|}$, we obtain

$$\begin{aligned}
\mathbf{u}_0 \cdot \mathbf{A}_{rf} &= \left[\Omega_c(\rho_c - a)P_{rf}^{\|} + u_\perp \left(P_{rf}^+ e^{i\xi} + P_{rf}^- e^{-i\xi}\right)\right] \tilde{A}_{rf}(\mathbf{X} + \rho\hat{a}) \\
&\quad e^{i\Psi(\mathbf{X},t)} \sum_l J_l(k_\perp\rho) e^{il(\xi+\phi+\pi/2)} \\
&= \tilde{A}_{rf}(\mathbf{X}+\rho\hat{a}) e^{i\Psi(\mathbf{X},t)} \\
&\quad \sum_l \left[\Omega_c(\rho_c - a)P_{rf}^{\|} J_l + u_\perp \left(P_{rf}^+ J_{l-1} + P_{rf}^- J_{l+1}\right)\right] e^{il(\xi+\phi+\pi/2)}
\end{aligned} \tag{32}$$

where $\Omega_c = eB/mc$ is the electron gyrofrequency. So far we have not made any assumptions about the ratio of particle Larmor radius to scale length over which the wave field amplitudes vary. For the case where this ratio is small, we can simply replace $\mathbf{x}$ by $\mathbf{X}$.

V. LIE TRANSFORM CANONICAL PERTURBATION THEORY

Here we will summarize some basic aspects of Lie transform perturbation theory [9] which will be used in the subsequent sections.

Consider a Hamiltonian $H(\mathbf{z})$ which is a function of the set of phase space variables $\mathbf{z}(t)$. The time evolution of $\mathbf{z}(t;t_0)$ from some initial time t_0 to t is governed by Hamilton's equations of motion with the initial condition $\mathbf{z}(t_0;t_0) = \mathbf{z}_0$. The time evolution of any function $f(\mathbf{z},t)$ of $\mathbf{z}(t)$ and time t from time t_0 to time t is given by

$$f\left(\mathbf{z}(t;t_0), t\right) = S_H(t;t_0) f\left(\mathbf{z}_0, t_0\right) \tag{33}$$

where $S_H(t;t_0)$ is the time evolution operator. The derivation of $S_H(t;t_0)$ is equivalent to solving the equations of motion. This may not be possible for the variables in which the problem is originally posed. In this case one generally tries to transform to a new set of variables $\mathbf{z}'$ using an operator $T(\mathbf{z},t)$

$$\mathbf{z}' = T(\mathbf{z},t)\mathbf{z} \tag{34}$$

The Hamiltonian $K(\mathbf{z}')$ generated by this transformation is such that the corresponding time evolution operator $S_K(t;t_0)$ can be more easily evaluated. This is the case, for example, when one transforms to action-angle variables $\mathbf{z}' = (\mathbf{J}', \theta')$ and K depends only on the actions $\mathbf{J}'$. This case is relevant to our studies and we will further pursue this line of thought.

For $K = K(\mathbf{J}')$, the actions are constants of the motion so that the operator $S_K(t;t_0)$ evolves the angles $\boldsymbol{\theta}'$ only

$$f\left(\mathbf{z}'(t;t_0),t\right) = S_K(t;t_0) f\left(\mathbf{z}'_0,t_0\right) = f(\mathbf{J}'_0, \boldsymbol{\theta}'_0 + \boldsymbol{\theta}') \tag{35}$$

where

$$\boldsymbol{\theta}' = \int_{t_0}^{t} \boldsymbol{\omega}_K(\mathbf{J}'_0, s)ds, \qquad \boldsymbol{\omega}_K(\mathbf{J}'_0, t) = \frac{\partial K(\mathbf{J}'_0, t)}{\partial \mathbf{J}'_0} \tag{36}$$

According to Lie transform theory, the operator T is

$$T = e^{-L} \tag{37}$$

where $Lf = [w, f]$ with $[a,b] = \nabla_{\boldsymbol{\theta}} a \cdot \nabla_{\mathbf{J}} b - \nabla_{\mathbf{J}} a \cdot \nabla_{\boldsymbol{\theta}} b$ denoting the Poisson bracket. The function $w(\mathbf{z})$ is defined as the Lie generator. The inverse transformation is $T^{-1} = e^L$. The Lie transform operator is important in that it generates canonical transformations and commutes with any function of the phase space variables. The latter property implies that the evolution of $f(\mathbf{z},t)$ can be evaluated by transforming to the new variable set $\mathbf{z}'$, applying the time evolution operator $S_K(t;t_0)$ to the transformed function, and then transforming back to the original variables $\mathbf{z}$, according to

$$f\left(\mathbf{z}(t;t_0),t\right) = T(\mathbf{z}_0,t_0) S_K(t;t_0) T^{-1}(\mathbf{z}_0,t_0) f\left(\mathbf{z}_0,t_0\right) \tag{38}$$

This procedure applies to an integrable Hamiltonian. However, it is even more useful in generating a perturbation scheme for a nearly integrable Hamiltonian system. If such a system has a small non-integrable part of order ϵ, a canonical transform T can be constructed as a power series in ϵ, by following the scheme developed by Deprit [25]. According to this scheme, the old Hamiltonian H, the new Hamiltonian K, the transformation operator T, and the Lie generator w are expanded in power series of ϵ:

$$H(\mathbf{z}, t, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n H_n(\mathbf{z}, t) \tag{39a}$$

$$K(\mathbf{z}, t, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n K_n(\mathbf{z}, t) \tag{39b}$$

$$T(\mathbf{z}, t, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n T_n(\mathbf{z}, t) \tag{39c}$$

$$w(\mathbf{z}, t, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n w_{n+1}(\mathbf{z}, t) \tag{39d}$$

where w_0 is chosen so that $T_0 = I$ is the identity transformation. Through second order the transformations T and T^{-1} are

$$T_0 = I \tag{40a}$$

$$T_1 = -L_1 \tag{40b}$$

$$T_2 = -\frac{1}{2}L_2 + \frac{1}{2}L_1^2 \tag{40c}$$

and

$$T_0^{-1} = I \tag{41a}$$

$$T_1^{-1} = L_1 \tag{41b}$$

$$T_2^{-1} = \frac{1}{2}L_2 + \frac{1}{2}L_1^2 \tag{41c}$$

respectively. T_0 has been chosen to be the identity operator. To second order, the Lie generator w and the new Hamiltonian K are

$$K_0 = H_0 \tag{42}$$

$$\frac{\partial w_1}{\partial t} + [w_1, H_0] = K_1 - H_1 \tag{43}$$

$$\frac{\partial w_2}{\partial t} + [w_2, H_0] = 2(K_2 - H_2) - L_1(K_1 + H_1) \tag{44}$$

The left hand side of Eqs. (43)-(44) are the total time derivatives of w_1 and w_2 along the unperturbed orbits given by H_0. Thus, the solutions are provided by integrating the right hand side along these known unperturbed orbits. The choice of K_n's is arbitrary and depends on the physical situation. For example, in the study on the effect of ponderomotive force on the distribution function [26], K_n's are chosen so that only the slowly varying terms appear in the new Hamiltonian. The resulting system is, in general, non-integrable. In our case, it is convenient to choose K_n's so as to eliminate the $\boldsymbol{\theta}$ dependence in the new Hamiltonian. Then the transformed system is integrable, and we can explicitly calculate the evolution of the distribution function.

VI. CANONICAL PERTURBATION THEORY FOR THE PERTURBED HAMILTONIAN

From Eqs. (17) and (25)

$$\begin{aligned} H &= mc^2\Gamma + e\Phi \\ &+\epsilon e\left(-\frac{1}{\Gamma c}\mathbf{u}_0\cdot\mathbf{A}_{rf} + \Phi_{rf}\right) \\ &+\epsilon^2\frac{e^2}{2mc^2\Gamma}\left[A_{rf}^2 - \frac{1}{c^2\Gamma^2}(\mathbf{u}_0\cdot\mathbf{A}_{rf})^2\right] \end{aligned} \tag{45}$$

where

$$\Gamma = \left[1 + e^2(\rho_c - a)^2B^2/m^2c^4 + 2\mu B/mc^2\right]^{1/2} \tag{46}$$

The Hamiltonian with the RF wave fields is a function of the canonically conjugate (action-angle) variables (P_ζ, ζ), (P_θ, θ) and (μ, ξ). In the absence of static magnetic field perturbations, $a = 0$, the order ϵ^0 terms form the unperturbed system which is an integrable Hamiltonian. The static magnetic field perturbations for $a \neq 0$ are assumed to be small, of the same order ϵ with the wave fields, compared to the unperturbed part of the full Hamiltonian. Then to second order in the ordering parameter ϵ

$$H = H_0 + \epsilon H_1 + \epsilon^2 H_2 \tag{47}$$

where

$$H_0 = mc^2\Gamma_0 + e\Phi \tag{48}$$

$$H_1 = -\frac{e}{\Gamma_0 c}\left(\Omega_c\rho_c\hat{b} + u_\perp\hat{c}\right)\cdot\mathbf{A}_{rf} + e\Phi_{rf} - \frac{m\Omega_c^2}{\Gamma_0}\rho_c a \tag{49}$$

$$\begin{aligned} H_2 &= \frac{e^2}{2mc^2\Gamma_0}\left[A_{rf}^2 - \frac{1}{c^2\Gamma_0^2}\left\{\left(\Omega_c\rho_c\hat{b} + u_\perp\hat{c}\right)\cdot\mathbf{A}_{rf}\right\}^2\right] \\ &+\frac{m\Omega_c^2}{2\Gamma_0^3}\left(\Gamma_0^2 - \frac{\Omega_c^2}{c^2}\rho_c^2\right)a^2 + \frac{e\Omega_c}{c\Gamma_0^3}\left(\Gamma_0^2 - \frac{\Omega_c^2}{c^2}\rho_c^2\right)(\hat{b}\cdot\mathbf{A}_{rf})a \\ &-\frac{e\Omega_c^2}{c^3\Gamma_0^3}u_\perp(\hat{c}\cdot\mathbf{A}_{rf})\rho_c a \end{aligned} \tag{50}$$

and

$$\Gamma_0 = \left[1 + \Omega_c^2\rho_c^2/c^2 + u_\perp^2/c^2\right]^{1/2} \tag{51}$$

Following the discussion in the previous section on Lie transform perturbation theory, the first order Lie generator, obtained from Eq. (43) by seting $K_1 = 0$, is

$$w_1 = \int_{t_0}^{t} \left[\frac{e}{\Gamma_0 c} \quad \tilde{A}_{rf}(\mathbf{x})e^{i\Psi(\mathbf{x},s)} \left(\frac{eB}{mc}\rho_c\hat{b} + \left(\frac{2\mu B}{m}\right)^{1/2}\hat{c}\right)\cdot\mathbf{P}_{rf}\right. \tag{52}$$

$$\left. -e\tilde{\Phi}_{rf}(\mathbf{x})e^{i\Psi(\mathbf{x},s)} - \frac{e^2B^2}{\Gamma_0 mc^2}\rho_c a\right] ds \tag{53}$$

where the integration is along the unperturbed orbits obtained from H_0 in Eq. (48). Note that the RF wave fields are a function of $\mathbf{x} = \mathbf{X} + \rho\hat{a}$, where $\mathbf{X} = (\psi_p(P_\theta, P_\zeta, \mu), \theta, \zeta)$, while the other terms depend only on $\mathbf{X}$.

If we assume that the RF field is a slowly varying wavepacket so that the spatial scale over which its phase and amplitude vary is much longer than the Larmor radius, then Eq. (32) yields

$$w_1 = \int_{t_0}^{t} \left\{e^{i\Psi(\mathbf{X},s)}\sum_l \left[\frac{e}{\Gamma_0 c}\tilde{A}_{rf}(\mathbf{X})\left(\frac{eB}{mc}\rho_c P_{rf}^{\parallel}J_l + \left(\frac{2\mu B}{m}\right)^{1/2}\left(P_{rf}^{+}J_{l-1} + P_{rf}^{-}J_{l+1}\right)\right)\right.\right.$$

$$\left.\left. -e\tilde{\Phi}_{rf}(\mathbf{X})\right]e^{il(\xi+\phi+\pi/2)} - \frac{e^2B^2}{\Gamma_0 mc^2}\rho_c\sum_{m_1,m_2} a_{m_1,m_2}(\psi_p)e^{i(m_1\theta+m_2\zeta)}\right\}ds \tag{54}$$

where Fourier representation of the static magnetic field perturbations (15) has been used. The integrand is a function of the action-angle variables and the integration is along the unperturbed orbits. Since $\mathbf{X}$ is periodic in θ and ζ

$$\left[\frac{e}{\Gamma_0 c}\tilde{A}_{rf}(\mathbf{X}) \quad \left\{\frac{eB}{mc}\rho_c P_{rf}^{\parallel}J_l + \left(\frac{2\mu B}{m}\right)^{1/2}\left(P_{rf}^{+}J_{l-1} + P_{rf}^{-}J_{l+1}\right)\right\}\right.$$

$$\left. -e\tilde{\Phi}_{rf}(\mathbf{X})\right]e^{ik_{\psi_p}\psi_p} = \sum_{n_1,n_2} G_{n_1,n_2}(\mathbf{J})e^{i(n_1\theta+n_2\zeta)} \tag{55}$$

and

$$-\frac{e^2B^2}{\Gamma_0 mc^2}\rho_c = \sum_{n_1} F_{n_1}(\mathbf{J})e^{in_1\theta} \tag{56}$$

where the coefficients of the Fourier series are functions of $\mathbf{J} = (P_\theta, P_\zeta, \mu)$. The phase function in the eikonal of Eq. (54) is $\Psi(\mathbf{X}, t) = k_{\psi_p}\psi_p + k_\theta\theta + k_\zeta\zeta - \omega t$, where we have neglected the constant phase term $il(\phi + \pi/2)$. The Fourier expansions include spatial inhomogeneity of the equilibrium magnetic field, and perturbations due to RF wave fields and static magnetic fields. We can rewrite w_1 as

$$
\begin{aligned}
w_1 &= \int_{t_0}^{t} \sum_{n_1,n_2,l} G_{n_1,n_2,l}(\mathbf{J}) e^{i\left[(n_1+k_\theta)\theta+(n_2+k_\zeta)\zeta+l\xi-\omega s\right]} ds \\
&+ \int_{t_0}^{t} \sum_{n_1,m_1,m_2} F_{n_1}(\mathbf{J}) a_{m_1,m_2}(\mathbf{J}) e^{i[(n_1+m_1)\theta+m_2\zeta]} ds
\end{aligned}
\tag{57}
$$

Since the actions are constants of H_0 in Eq. (48), the integrals in Eq. (57) involve only the angles $\boldsymbol{\theta} = (\theta, \zeta, \xi)$. Using the unperturbed orbits

$$
\begin{aligned}
\mathbf{J}(s) &= \text{const.} \\
\boldsymbol{\theta}(s) &= \boldsymbol{\theta}(t) + \boldsymbol{\omega}_{\boldsymbol{\theta}}(s-t)
\end{aligned}
\tag{58}
$$

where $\boldsymbol{\omega}_{\boldsymbol{\theta}} = \partial H_0(\mathbf{J})/\partial \mathbf{J}$ is the (constant) frequency vector of the unperturbed system H_0,

$$
\begin{aligned}
w_1 &= \sum_{n_1,n_2,l} G_{n_1,n_2,l}(\mathbf{J}) \int_{t_0}^{t} e^{i\left[(n_1+k_\theta)\theta+(n_2+k_\zeta)\zeta+l\xi-\omega s\right]} ds \\
&+ \sum_{n_1,m_1,m_2} F_{n_1}(\mathbf{J}) a_{m_1,m_2}(\mathbf{J}) \int_{t_0}^{t} e^{i[(n_1+m_1)\theta+m_2\zeta]} ds
\end{aligned}
\tag{59}
$$

The time integration over the angles yields

$$
\begin{aligned}
w_1 &= \sum_{n_1,n_2,l} G_{n_1,n_2,l}(\mathbf{J}) e^{i\mathbf{N}_{n_1,n_2,l}\cdot(\boldsymbol{\theta}-\boldsymbol{\omega}_{\boldsymbol{\theta}} t)} \frac{e^{i\left(\mathbf{N}_{n_1,n_2,l}\cdot\boldsymbol{\omega}_{\boldsymbol{\theta}}-\omega\right)t} - e^{i\left(\mathbf{N}_{n_1,n_2,l}\cdot\boldsymbol{\omega}_{\boldsymbol{\theta}}-\omega\right)t_0}}{i(\mathbf{N}_{n_1,n_2,l}\cdot\boldsymbol{\omega}_{\boldsymbol{\theta}}-\omega)} \\
&+ \sum_{n_1,m_1,m_2} F_{n_1}(\mathbf{J}) a_{m_1,m_2}(\mathbf{J}) e^{i\mathbf{M}_{n_1,m_1,m_2}\cdot(\boldsymbol{\theta}-\boldsymbol{\omega}_{\boldsymbol{\theta}} t)} \frac{e^{i\mathbf{M}_{n_1,m_1,m_2}\cdot\boldsymbol{\omega}_{\boldsymbol{\theta}} t} - e^{i\mathbf{M}_{n_1,m_1,m_2}\cdot\boldsymbol{\omega}_{\boldsymbol{\theta}} t_0}}{i(\mathbf{M}_{n_1,m_1,m_2}\cdot\boldsymbol{\omega}_{\boldsymbol{\theta}})}
\end{aligned}
\tag{60}
$$

where $\mathbf{N}_{n_1,n_2,l} = (n_1+k_\theta, n_2+k_\zeta, l)$ and $\mathbf{M}_{n_1,m_1,m_2} = (n_1+m_1, m_2, 0)$. Both sums in the above expression include a functional dependence of the form

$$
\mathcal{R}(\Omega; t, t_0) = \frac{e^{i\Omega t} - e^{i\Omega t_0}}{i\Omega} = \int_{t_0}^{t} e^{i\Omega s} ds \tag{61}
$$

This function is smooth and localized around $\Omega = 0$ and indicates a resonance between the particle motion and the perturbations. The first sum in (60) includes resonance between RF waves and the particles and depends on the three angles. The second sum in (60) includes resonance between magnetic perturbations and the particles and depends on the two angles θ and ζ. For long times

$$
\lim_{t\to\infty} \mathcal{R}(\Omega; t, -t) = 2\pi\delta(\Omega) \tag{62}
$$

where $\delta(\Omega)$ is the Dirac delta function. This delta function appears in the conventional quasilinear theories [11–13].

We can similarly obtain the second order generating function w_2. However, this is not necessary for a diffusion equation that is accurate to second order in the perturbation parameters (ϵ, λ). As we will show, an evolution equation, accurate to second order in the perturbation parameters, for the action dependent distribution function depends only on results from a first order canonical perturbation analysis [10].

VII. EVOLUTION OF THE ANGLE-AVERAGED DISTRIBUTION FUNCTION

The evolution, over an infinitesimal time interval $[t_0, t_0 + \Delta t]$, of any function $f(\boldsymbol{\theta}, \mathbf{J}, t)$ of the phase space variables and time is given by Eq. (38). From Eq. (60), $w_1(\mathbf{z}_0, t_0) = 0$, where $\mathbf{z}_0 = (\boldsymbol{\theta}_0, \mathbf{J}_0)$ is the value of the canonical variables at the initial time t_0. Then $T(\mathbf{z}_0, t_0) = I$, and, since we have chosen $K_n = 0$ for $n = 1, 2$, it follows that the time evolution of S_K is given by the H_0

$$S_K = S_{K_0} = S_{H_0} \tag{63}$$

Consequently

$$f(\mathbf{z}_{t+\Delta t}, t + \Delta t) - f(\mathbf{z}_t, t) = \left[T^{-1} - I\right] (\mathbf{z}_t + \Delta\mathbf{z}, t + \Delta t) f(\mathbf{z}_t, t) \tag{64}$$

where $f(\mathbf{z}_t, t) = f(\mathbf{z}(t), t)$. The variation $\Delta\mathbf{z}$ is obtained from H_0 by integrating over unperturbed orbits. Upon dividing Eq. (64) by Δt and taking the limit $\Delta t \to 0$ we obtain

$$\frac{\partial f(\mathbf{z}, t)}{\partial t} = \frac{\partial \left[T^{-1} - I\right] (\mathbf{z}, t)}{\partial t} f(\mathbf{z}, t) \tag{65}$$

If $f(\mathbf{z}, t)$ is taken to be the particle distribution function, Eq. (65) is an approximation, to the same order as T^{-1}, of the original Vlasov (Liouville) equation.

Consider a function $F(\mathbf{J}, t)$ which is an average of $f(\boldsymbol{\theta}, \mathbf{J}, t)$ over the canonical angles $\boldsymbol{\theta}$,i.e.,

$$F(\mathbf{J}, t) = \langle f(\boldsymbol{\theta}, \mathbf{J}, t) \rangle_{\boldsymbol{\theta}} \tag{66}$$

Then,

$$\frac{\partial F(\mathbf{J}, t)}{\partial t} = \frac{\partial \left\langle [T^{-1} - I](\mathbf{z}, t) \right\rangle_{\boldsymbol{\theta}}}{\partial t} F(\mathbf{J}, t). \tag{67}$$

From Eq. (41c)

$$T^{-1} - I = L_1 + (1/2)L_2 + (1/2)L_1^2 \tag{68}$$

with

$$L_n F(\mathbf{J}, t) = [w_n(\boldsymbol{\theta}, \mathbf{J}, t), F(\mathbf{J}, t)] = \nabla_{\boldsymbol{\theta}} w_n \cdot \nabla_{\mathbf{J}} F, \qquad \text{for } n = 1, 2 \tag{69}$$

and

$$\begin{aligned} L_1^2 F(\mathbf{J}, t) &= [w_1(\boldsymbol{\theta}, \mathbf{J}, t), [w_1(\boldsymbol{\theta}, \mathbf{J}, t), F(\mathbf{J}, t)]] \\ &= \nabla_{\boldsymbol{\theta}} w_1 \cdot \nabla_{\mathbf{J}} \left(\nabla_{\boldsymbol{\theta}} w_n \cdot \nabla_{\mathbf{J}} F\right) - \nabla_{\mathbf{J}} w_1 \cdot \nabla_{\boldsymbol{\theta}} \left(\nabla_{\boldsymbol{\theta}} w_n \cdot \nabla_{\mathbf{J}} F\right) \end{aligned} \tag{70}$$

On integrating by parts and using the fact that the dependence on all the angles is periodic, we find that

$$\langle L_n F(\mathbf{J}, t)\rangle_{\boldsymbol{\theta}} = 0, \qquad \text{for } n = 1, 2 \tag{71}$$

and

$$\left\langle L_1^2 F(\mathbf{J}, t)\right\rangle_{\boldsymbol{\theta}} = \nabla_{\mathbf{J}} \cdot \left[\langle \nabla_{\boldsymbol{\theta}} w_1 \nabla_{\boldsymbol{\theta}} w_1\rangle_{\boldsymbol{\theta}} \cdot \nabla_{\mathbf{J}} F(\mathbf{J}, t)\right] \tag{72}$$

Since this equation depends only on w_1, an important point emerges from this calculation. The angle-averaged operators of Eq. (68), needed in the evolution equation (67), can be evaluated up to second order in the perturbation parameter using results from first order perturbation theory [10, 27]. An analogous result has been obtained for the distribution function, averaged over the fast time scale, in the presence of a ponderomotive force [26]. However, a "fake diffusion" contribution also appears in the equation for the distribution function.

The evolution equation (67) takes on the form

$$\frac{\partial F(\mathbf{J}, t)}{\partial t} = \nabla_{\mathbf{J}} \cdot [\mathbf{D}(\mathbf{J}, t) \cdot \nabla_{\mathbf{J}} F(\mathbf{J}, t)] \tag{73}$$

where

$$\mathbf{D}(\mathbf{J}, t) = \frac{1}{2} \frac{\partial \left\langle \nabla_{\boldsymbol{\theta}} w_1(\boldsymbol{\theta}, \mathbf{J}, t) \nabla_{\boldsymbol{\theta}} w_1(\boldsymbol{\theta}, \mathbf{J}, t)\right\rangle_{\boldsymbol{\theta}}}{\partial t} \tag{74}$$

is the generalized quasilinear tensor. If $f = \mathbf{J}$ in Eq. (64), then we obtain the first order momentum variation

$$\langle \Delta\mathbf{J}\Delta\mathbf{J}\rangle_{\boldsymbol{\theta}} = \langle \nabla_{\boldsymbol{\theta}} w_1 \nabla_{\boldsymbol{\theta}} w_1\rangle_{\boldsymbol{\theta}} \tag{75}$$

so that

$$D(\mathbf{J}, t) = \lim_{\Delta t \to 0} \frac{\langle \Delta\mathbf{J}\Delta\mathbf{J}\rangle_{\boldsymbol{\theta}}}{2\Delta t} \tag{76}$$

This is the common definition of the quasilinear diffusion tensor.

Using Eq. (60) in Eq. (74) we obtain

$$\mathbf{D}(\mathbf{J},t) = \sum_{n_1,n_2,l} \mathbf{N}_{n_1,n_2,l}\mathbf{N}_{n_1,n_2,l}\left|G_{n_1,n_2,l}(\mathbf{J})\right|^2 \frac{\sin\left[\left(\mathbf{N}_{n_1,n_2,l}\cdot\boldsymbol{\omega_\theta}-\omega\right)t\right]}{\mathbf{N}_{n_1,n_2,l}\cdot\boldsymbol{\omega_\theta}-\omega} + \sum_{n_1,m_1,m_2} \mathbf{M}_{n_1,m_1,m_2}\mathbf{M}_{n_1,m_1,m_2}\left|F_{n_1}(\mathbf{J})\right|^2\left|a_{m_1,m_2}(\mathbf{J})\right|^2 \frac{\sin\left[\left(\mathbf{M}_{n_1,m_1,m_2}\cdot\boldsymbol{\omega_\theta}\right)t\right]}{\mathbf{M}_{n_1,m_1,m_2}\cdot\boldsymbol{\omega_\theta}} \quad (77)$$

. This diffusion tensor depends on time and is non-singular. The time dependence is a result of carrying out the perturbation theory over finite time intervals. As a consequence the diffusion tensor is non-singular. The tensor depends on the resonance conditions $\Omega = 0$ through the continuous smooth functions $\mathcal{R}(\Omega;t,t_0)$ in Eq. (61). This diffusion tensor is in contrast to the commonly used singular quasilinear tensors which depend on Dirac's delta function [11, 13]. The delta function occurs due to the Markovian assumption that has been made in previous derivations of the quasilinear diffusion tensor. The Markovian assumption is applied to electron dynamics. It is assumed that the applied RF perturbations phase mix the electron motion and the orbits are completely decorrelated [13, 28–30]. In this case, the upper limit of the time integral in w_1 can be extended to infinity, or, equivalently, the time interval Δt in the definition (76) of the diffusion tensor can be taken as infinite. Then the functions $\mathcal{R}(\Omega;t,t_0)$ tend to Dirac's delta functions. The Markovian assumption for the decorrelation of particle orbits is closely related to an underlying phase space. It is assumed that resonance overlap occurs over an extended region of phase space resulting in a complete chaos [31]. This is a very strong assumption. Although large chaotic phase space regions may exist for certain ranges of parameters, it is quite common to have phase space islands comprised of of regular quasiperiodic motion. For quasiperiodic motion, the particle orbits are strongly correlated. This inhomogeneous structure of phase space does not allow for a global Markovian assumption. Since the diffusion tensor is for the entire range of actions, we need to incorporate finite time intervals in the evaluation of the tensor. This lead to a time dependent diffusion tensor consisting of continuous smooth functions which are localized, in action space, around the resonances.

The evolution equation (73) can be transformed from the canonical action variables to the physical space configuration variables. If the Jacobian $\mathcal{J}$ transforms actions $\mathbf{J}$ to physical space variables $\mathbf{P}$, then Eq. (73) becomes

$$\frac{\partial F}{\partial t} = (\mathcal{J}\cdot\nabla_{\mathbf{P}})\cdot\left[\mathbf{D}(\mathbf{P},t)\cdot(\mathcal{J}\cdot\nabla_{\mathbf{P}})F\right] \quad (78)$$

Let us consider the case when there are no magnetic field perturbations, i.e.,$\alpha = 0$. The canonical momenta P_θ, P_ζ depend only on the radial coordinate ψ_p and the parallel momentum $\rho_\parallel$. The third canonical momentum μ depends on the perpendicular momentum. If we consider a cylindrically symmetric equilibrium which does not depend on the poloidal angle θ, the unperturbed guiding center Hamiltonian (8) is independent of θ, so that P_θ is a conserved action. Then the diffusion equation describes momentum diffusion and spatial diffusion in the radial direction. The former leads to heating and current drive, while the latter leads to radial particle transport. The Jacobian $\mathcal{J}$ is directly obtained from Eqs. (7).

For an axisymmetric toroidal equilibrium, the unperturbed guiding center Hamiltonian (8) also depends on θ, and an additional canonical transformation (9)-(10) is needed to describe the system in action-angle variables. The third action $\hat{P}_\theta$ depends on the other actions and also on θ. Then the action diffusion equation also includes spatial diffusion along the poloidal direction. The corresponding Jacobian is obtained from Eqs. (7) and (10). If nonaxisymmetric magnetic field perturbations are also included, the respective modification of the definition of the canonical variables (16) also include θ and ζ. Then the action diffusion equation includes diffusion in all the spatial direction.

For the cylindrical and axisymmetric toroidal equilibria, even when non-axisymmetric perturbations are included, the derivation procedure as well as the form of the action diffusion equation are identical. It is the topology of the magnetic field that determines the relationship between the action variables and the physical configuration space variables through the canonical transformations. As the number of degrees of symmetry is increased, configuration space diffusion occurs in fewer dimensions.

The quasilinear tensor (74) is determined from the first order Lie generating function w_1 (60). Thus, the collective particle behavior, represented by the distribution function, is obtained from the single particle dynamics. This is a consequence of the fact that Lie operators commute with any function of the phase space variables. This property allows for the unification of the test particle approach with the kinetic approach [30]. The Lie generating functions used in the quasilinear tensor are also related to approximate invariants of the motion. The solution to Eq. (43) results in the approximate invariants of the motion $\bar{\mathbf{J}}$

$$\bar{\mathbf{J}} = \mathbf{J} + \frac{\partial w_1(\mathbf{J}, \boldsymbol{\theta}, t)}{\partial \boldsymbol{\theta}} = \text{const.} \tag{79}$$

These first order approximate invariants of the motion contain essential information about the resonant structure of the phase space [32]. The inhomogeneity of the phase space, due to the coexistence of resonant islands and chaotic space, is contained in the quasilinear tensor through w_1. Thus, the kinetic equation takes into account the entire topology of phase space.

VIII. SUMMARY

In Fokker-Planck equations used for studying heating and current drive by RF waves, the wave-particle interaction is generally represented by a quasilinear diffusion operator. Using the powerful Lie transform perturbation technique we have derived a diffusion operator that includes the interaction of RF waves with electrons and also the effect of non-axisymmetric magnetic field perturbations on the motion of electrons. Our formalism is fully relativistic and uses the magnetic field geometry of an axisymmetric tokamak. The diffusion operator can be implemented in a numerical code using the following steps:

- The magnetic field $\mathbf{B}$ of an axisymmetric toroidal plasma can be obtained from an equilibrium code that solves the Grad-Shafranov equation. The spatial dependence is given by $\mathbf{X} = (\psi_p, \theta, \zeta)$.
- The RF fields $\mathbf{A}_{rf}, \mathbf{\Phi}_{rf}$ (18) can be provided by a ray tracing or a full wave code and expressed in terms of $\mathbf{X}$.
- Then we express $\mathbf{B}$ in terms of the canonical variables $(P_\theta, P_\zeta, \theta)$ using the transformation $\psi_p = \psi_p(P_\theta, P_\zeta)$ given by Eqs. (5), (6), and (16).
- Next we transform to action-angle variables using Eqs. (9) and (10). The spatial dependence of the perturbations a, $\mathbf{A}_{rf}$, and $\mathbf{\Phi}_{rf}$, and of $\rho_c = \rho_c(P_\theta, P_\zeta)$ is also transformed to action-angle variables.
- Then the coefficients of the Fourier expansions (55) and (56) can be readily obtained.
- The quasilinear tensor $\mathbf{D}$ is then provided directly by Eq. (77).
- The results can be readily transformed back to the physical variable set by using the inverse transformations from action variables to physical variables.

The quasilinear operator contains momentum and configuration space diffusion due to RF waves and static magnetic field perturbations. The momentum space diffusion leads to current generation by electrons and the configuration space diffusion leads to spatial modifications of the current profile. The relativistic formalism is suitable for electron cyclotron drive – the primary RF scheme for stabilization of neoclassical tearing modes in ITER.

In our derivation, the respective action diffusion tensor is nonsingular, as a result of calculating the change in the actions of the electrons in finite times. The latter is related to the fact that no statistical assumption, such as the Markovian assumption, related to strongly chaotic electron dynamics, is imposed. Therefore, we account for both chaotic and quasi-periodic motion of the electrons in determining the diffusion operator. The Markovian assumption completely eliminates any quasi-periodic motion from being included in the quasilinear description and leads to a diffusion operator that is singular. Our quasilinear diffusion operator, obtained using the Lie perturbation technique in finite time intervals, is a time dependent and continuous smooth function of the action variables.

IX. ACKNOWLEDGEMENTS

This work has been supported in part by the European Fusion Programme (EURATOM) and the Greek General Secretariat of Research and Technology, and in part by the DoE grants DE-FG02-99ER-54521 and DE-FG02-91ER-54109.

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