Pfirsch-Schlüter Electric Field in a Tokamak

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This work was supported by the U.S. Department of Energy, Grant No. DE-FG02-91ER54109. Reproduction, translation, publication, use and disposal, in whole or in part, by or for the United States government is permitted.

Submitted for publication in the Physics of Plasma (March 2009)
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Abstract

A concise and complete differential equation determining the Pfirsch-Schlüter radial electric field in an up-down symmetric tokamak is presented in the limit of weak poloidal magnetic field.

The differential equation for the Pfirsch-Schlüter electric field in a tokamak has been evaluated for large aspect ratio circular flux surfaces [1], for general cross sections [2,3], and for tokamaks having the poloidal magnetic field $B_p$ small compared to the toroidal magnetic field $B_t$ with a weak ion diamagnetic drift [4]. Recent measurements in Alcator C-Mod [5] and DIII-D [6] for subsonic ion flows indicate that the ion diamagnetic flow must be retained since it tends to cancel with the $E \times B$ drift velocity, where $\vec{B} = I(\psi) \nabla \zeta + \nabla \zeta \times \nabla \psi$ is the axisymmetric tokamak magnetic field with $\psi$ the flux function, $\zeta$ is the toroidal angle variable, $I = RB_t$, $R$ is the major radius, and $\vec{E} = -(\partial \Phi / \partial \psi) \nabla \psi$ is the lowest order electric field with the electrostatic potential $\Phi$ a flux function to lowest order. Consequently, it is convenient to use the results of [2-4] to generalize the $B_p << B_t$ expression of [4] so that it retains the ion diamagnetic drift. To do so is very straightforward since the result of [4] is consistent with the $B_p << B_t$ limit of [2,3] within the ion diamagnetic drift term that now has to be retained. Defining the ion density, temperature and pressure as $n$, $T$ and $p$, and taking $Z = 1$ for the ion charge number, the lowest order ion flow velocity $\vec{V}$ on a flux surface is the usual Pfirsch-Schlüter result of Hazeltine [7]:

$$\vec{V} = \omega R^2 \nabla \zeta + u(\psi) \vec{B}$$

with

$$\omega = -c[(\partial \Phi / \partial \psi) + (en)^{-1}(\partial p / \partial \psi)],$$

and
\[ u = -\frac{c}{e} \left[ 1.78 \frac{1}{\langle B^2 \rangle} + 0.057 \frac{\langle (\nabla_n/\langle nB \rangle)^2 \rangle}{\langle (\nabla_n B)^2 \rangle} \right] \frac{\partial T}{\partial \psi}, \]  

where \( B = B_{\text{II}}, \nabla_n = B^{-1} \mathbf{B} \cdot \nabla \), and \( \langle \ldots \rangle \) denotes a flux surface average. The departure of \( \partial \Phi / \partial \psi \) from a generalized radial Maxwell-Boltzmann relation (\( \omega = \text{constant} \)) for an up-down symmetric tokamak is then found from the radial flux of toroidal angular momentum by employing

\[
-\frac{5}{6 \nu} \langle R^2 \nabla \zeta \cdot \nabla \psi \rangle = \left\{ \frac{R^4 B^2}{\Omega^2} \frac{\partial \omega}{\partial \psi} + 0.19 \frac{c I \Omega}{e T (T + T_e) \Omega^2} \left[ R^2 \left\{ 1 - \frac{B^2}{\langle B^2 \rangle} \right\} \right] \right. 

-0.53 \frac{c T^{1/2}}{e n^2} \frac{\partial}{\partial \psi} \left\{ \frac{1}{R^2} \frac{\partial T}{\partial \psi} \left[ \frac{1}{T} - \frac{B}{\langle B^2 \rangle} \right] \right\} \left\{ - \frac{R^2}{B^2} \left[ 1 - \right\{ \frac{R^2}{B^2} \right\} \right\} \right. 

\]

where \( T_e \) is the electron temperature, \( \nu = (4\pi^{1/2} e^4 n / n \Lambda) / 3M^{1/2} T^{3/2} \) is the ion-ion collision frequency, and \( \mathbf{\pi} \) is the ion stress tensor. Setting

\[
\langle R^2 \nabla \zeta \cdot \nabla \psi \rangle = 0
\]

gives the differential equation determining the steady state Pfirsch-Schlüter radial electric field once a boundary condition is specified. Except for the retention of the \( \partial p / \partial \psi \) term in the \( \omega \) appearing in (4), Eqs. (2) and (4) are the same as Eqs. (68) and (75) of Ref. [4] for an up-down symmetric tokamak. Moreover, Eqs. (2) and (4) correspond precisely to the \( B_p \ll B \), limit of Eq. (60) of Ref. [2], and are consistent with the \( T_e = T \) large aspect ratio result given in Ref. [1] for which the terms in the second line of Eq. (4) may be neglected as smaller than those in the first line by an inverse aspect ratio squared. The up-down asymmetric term of Refs. [2,4] may be added into (4) as necessary.

The correct isothermal limit is now recovered [8] by (5), while for the more typical non-isothermal cases the \( \partial p / \partial \psi \) term in \( \omega \) is comparable to the small numerical coefficient second term on the right when \( T \partial [n \partial p / \partial \psi] \sim 0.2 q^2 (\partial T / \partial \psi)^2 \), with \( q \) the safety factor. In fact, the ion pressure gradient term is required to understand the Alcator C-Mod measurements in collisional to semi-collisional pedestals [5] because it nearly cancels the \( \partial \Phi / \partial \psi \) term in \( \omega \). Moreover, based on the entropy production argument of [9], in a banana regime pedestal of poloidal ion gyroradius width, \( T \partial p / \partial \psi >> p \partial T / \partial \psi \) is required, as has been observed in DIII-D [6].
The temporal evolution of the electrostatic potential is found from the flux surface average of conservation of toroidal angular momentum:

\[
\frac{\partial}{\partial t} \left[ M_{n}\langle \omega (R^{2} \rangle + Iu \rangle \right] + \frac{1}{\nabla^{2}} \frac{\partial}{\partial \psi} \left[ V' \langle R^{2} \nabla \zeta \cdot \nabla \psi \rangle \right] = 0 ,
\]

where \( V' = \langle ... \rangle \).

In summary, we have presented a concise and complete expression for the differential equation determining the Pfirsch-Schlüter radial electric field in an up-down symmetric tokamak in the limit of weak poloidal magnetic field.

ACKNOWLEDGEMENTS
The authors are indebted to Kai Wong and Vincent Chan for a number of important conversations regarding the neoclassical electric field in tokamaks and appreciate their insightful comments during the preparation of this brief communication. Research supported by U.S. Department of Energy grants DE-FG02-91ER-54109 at MIT, and DE-AC52-06NA-25396 at LANL.

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