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Comment on: *"On higher order corrections to gyrokinetic Vlasov–Poisson equations in the long wavelength limit"* [Ref. W.W. Lee et al., Phys. Plasmas 16, 044506 (2009)]

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Comment on Ref. [W. W. Lee *et al.*, Phys. Plasmas 16, 044506 (2009)]

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Abstract

A recent publication [F. I. Parra *et al.*, Plasma Phys. Control. Fusion **50**, 065014 (2008)] warned against the use of the lower order gyrokinetic Poisson equation at long wavelengths because the long wavelength, radial electric must remain undetermined to the order the equation is obtained. Another reference [W. W. Lee *et al.*, Phys. Plasmas **16**, 044506 (2009)] criticizes these results by arguing that the higher order terms neglected in the most common gyrokinetic Poisson equation are formally smaller than the terms that are left. This argument is naive and ignores that the lower order terms, although formally larger, vanish without determining the long wavelength, radial electric field. The reason for this cancellation is discussed. In addition, the origin of a nonlinear term present in the gyrokinetic Poisson equation of Ref. [F. I. Parra *et al.*, Plasma Phys. Control. Fusion **50**, 065014 (2008)] is explained.

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This Comment on the article by Lee and Kolesnikov [1] clarifies several misunderstandings about our work [2]. In Ref. 1, the usual electrostatic gyrokinetic model, composed of a gyrokinetic Vlasov equation and a gyrokinetic Poisson equation, is analyzed following the method developed in Ref. 3. Two main conclusions are drawn that, in opinion of Lee and Kolesnikov, invalidate our results in Ref. 2. On the one hand, the terms that are traditionally ignored in the lower order gyrokinetic Poisson equation are found to be smaller than the linear polarization density. On the other hand, a nonlinear term that appears in Eq. (55) of Ref. 2 is not recovered by the method in Ref. 3. Here we explain that the fact that higher order terms are formally smaller does not invalidate our conclusions because we find that the formally larger terms vanish at long wavelengths. We also explain how the nonlinear term in the Poisson equation (55) of Ref. 2 is obtained.

First we address the issue of the higher order terms in the gyrokinetic Poisson equation, leaving aside the nonlinear term in Eq. (55) of Ref. 2. The result found in Ref. 1 is not at all surprising; it is obvious that higher order terms are formally smaller than lower order terms! However, the higher order terms are crucial for the final result because the lower order polarization density exactly cancels with other contributions without determining the long wavelength radial electric field. Our second order calculation for a non-turbulent θ-pinch in Ref. 2 shows that the lowest order polarization density and the solution of the second order gyrokinetic equation cannot determine the axisymmetric long wavelength electrostatic potential. This example illustrates a problem that exists even in the more complex tokamak geometry and in the presence of turbulence. The long wavelength, axisymmetric piece of the electrostatic potential must remain undetermined unless the ion distribution function is determined to higher order than second in an expansion on the small ratio $\delta_i = \rho_i/L \ll 1$ between the ion gyroradius ρ_i and the characteristic length L. In full f gyrokinetic simulations [4–7] that advance the distribution function in time and obtain the electrostatic potential from the gyrokinetic Poisson equation, the fact that the long wavelength, axisymmetric piece of the potential is undetermined means that, in the best case, the long wavelength piece depends exclusively on the initial condition. In the worst (and most probable) scenario, the calculated long wavelength radial electric field is erroneous unless the ion distribution function and the polarization density are calculated to the necessary order.

In Ref. 8, we studied the gyrokinetic Poisson equation in the presence of steady-state turbulence by taking its time derivative. We obtain that to calculate the self-consistent

long wavelength radial electric field in a turbulent tokamak, the gyrokinetic Fokker-Planck equation, $df_i/dt = C\{f_i\}$, must be correct at least up to order $\delta_i^3 f_{Mi} v_i/L$ because the long wavelength flux surface averaged charge density is bounded by $\partial\langle e(Zn_i - n_e)\rangle_\psi/\partial t \lesssim \delta_i^3 e n_e v_i/L$, that is, in statistical steady-state the time derivative of quasineutrality vanishes to a very high order without determining the long wavelength radial electric field. Here n_i and n_e are the ion and electron total densities (including both polarization and gyrocenter densities), $v_i = \sqrt{2T_i/M}$ is the ion thermal velocity, f_{Mi} is the lowest order ion distribution function, assumed to be Maxwellian, $\langle \ldots \rangle_\psi$ is the usual flux surface average in tokamaks, and Ze and e are the ion charge and the electron charge magnitude. Importantly, in Ref. 8 we also argued that $\partial\langle e(Zn_i - n_e)\rangle_\psi/\partial t \lesssim \delta_i^3 e n_e v_i/L$ is only an upper bound that may overestimate the real size of the long wavelength contribution to $\partial\langle e(Zn_i - n_e)\rangle_\psi/\partial t$ that we believe is

$$\frac{\partial}{\partial t}\langle e(Zn_i - n_e)\rangle_\psi \sim \delta_i^4 e n_e \frac{v_i}{L}. \tag{1}$$

This order of magnitude estimate is derived in detail in the paragraphs after equation (37) of Ref. 8. Its origin is the relation between the long wavelength, radial electric field and the toroidal rotation. Using charge conservation,

$$\frac{\partial}{\partial t}\langle e(Zn_i - n_e)\rangle_\psi = -\frac{1}{V'}\frac{\partial}{\partial \psi} V' \langle \mathbf{J}\cdot\nabla\psi\rangle_\psi, \tag{2}$$

where $\mathbf{J}$ is the current density, ψ is the poloidal magnetic flux, and $V' \equiv dV/d\psi$ is the flux surface volume. To determine the total radial current $\langle \mathbf{J}\cdot\nabla\psi\rangle_\psi$, we use the total momentum conservation [see Eq. (19) of Ref. 8]. In particular, the conservation of toroidal angular momentum gives

$$\begin{aligned}\frac{1}{c}\langle \mathbf{J}\cdot\nabla\psi\rangle_\psi = &\frac{\partial}{\partial t}\langle R n_i M \mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}}\rangle_\psi \\ &+\frac{1}{V'}\frac{\partial}{\partial\psi}V'\langle R\hat{\boldsymbol{\zeta}}\cdot\overleftrightarrow{\boldsymbol{\pi}}_i\cdot\nabla\psi\rangle_\psi,\end{aligned} \tag{3}$$

where $\mathbf{V}_i$ is the ion average velocity, R is the major radius, $\hat{\boldsymbol{\zeta}}$ is the unit vector in the toroidal direction, and $\overleftrightarrow{\boldsymbol{\pi}}_i$ are the off-diagonal terms of the stress tensor, including Reynolds stress, gyroviscosity and perpendicular viscosity. In Eq. (3), the Lorentz force in the toroidal direction gives rise to the term $\langle \mathbf{J}\cdot\nabla\psi\rangle_\psi$ since $R(\mathbf{J}\times\mathbf{B})\cdot\hat{\boldsymbol{\zeta}} = \mathbf{J}\cdot\nabla\psi$. Eq. (3) makes explicit the relation between quasineutrality and radial transport of toroidal angular momentum, be this transport turbulent or neoclassical. On the one hand, Eq. (2) implies that $\langle \mathbf{J}\cdot\nabla\psi\rangle_\psi = 0$,

i.e., the Lorentz force cannot spin the plasma because of quasineutrality. The radial electric field is then given by the transport of toroidal angular momentum that determines the toroidal rotation profile and hence the long wavelength axisymmetric piece of the potential through the relation between the toroidal velocity and $\partial\phi/\partial\psi$ [see Eq. (17) of Ref. 8 and the discussion in the paragraphs above and below]. But Eq. (3) also gives the maximum deviation of the radial current from zero, given by the radial transport of toroidal angular momentum, $\langle R\hat{\boldsymbol{\zeta}}\cdot\overleftrightarrow{\boldsymbol{\pi}}_i\cdot\nabla\psi\rangle_\psi$. Assuming that the transport is at the gyroBohm level, $\langle R\hat{\boldsymbol{\zeta}}\cdot\overleftrightarrow{\boldsymbol{\pi}}_i\cdot\nabla\psi\rangle_\psi \sim |\nabla\psi|D_{gB}\nabla(Rn_iMV_i) \sim |\nabla\psi|R\delta_i^3 p_i$, where $D_{gB}\sim\delta_i\rho_i v_i$ is the gyroBohm transport coefficient, and $V_i\sim\delta_i v_i$ is the average ion velocity, of the order of the drifts. With this estimate, Eq. (3) gives that the long wavelength contribution to $\langle\mathbf{J}\cdot\nabla\psi\rangle_\psi$ is of the order of $\delta_i^4 en_e v_i|\nabla\psi|$, and lower order contributions cancel exactly at long wavelengths. Employing this estimate and Eq (2), the order of magnitude estimate of Eq. (1) is readily found.

The order of magnitude estimate in Eq. (1) implies that the lower order polarization density must cancel with other lower order terms without determining the long wavelength radial electric field. Notice that usual gyrokinetic formulations only contain the lowest order drifts, being then of order $\mathbf{v}_d\cdot\nabla_{\mathbf{R}}f_i\sim\delta_i v_i f_{Mi}/L$ because $\mathbf{v}_d\sim\delta_i v_i$ and $\nabla_{\mathbf{R}}f_i\sim k_\perp\delta_i f_{Mi}$, with $k_\perp\rho_i\sim 1$. If the long wavelength flux surface averaged charge density $\langle e(Zn_i-n_e)\rangle_\psi$, formally of order $\delta_i en_e$, is zero initially, it will remain so for times of order L/v_i without determining the radial electric field because Eq. (1) gives $\partial\langle e(Zn_i-n_e)\rangle_\psi/\partial t \ll \delta_i en_e v_i/L$. The radial electric field is then given by the initial condition (see Ref. 8 for more details on its relation with toroidal rotation). Notice that this corresponds to saying that the lower order terms like the linear polarization density combine together to give zero without self-consistently determining the radial electric field, invalidating the claim of Ref. 1. Times longer than L/v_i are not consistent with usual gyrokinetic simulations because the physical cancellations in $\partial\langle e(Zn_i-n_e)\rangle_\psi/\partial t$ to orders $\delta_i^2 en_e v_i/L$ and $\delta_i^3 en_e v_i/L$ may not occur due to lack of next order corrections to the drifts. Thus, in the best case, $\langle e(Zn_i-n_e)\rangle_\psi=0$ without self-consistently determining the radial electric field and just giving a radial electric field that depends on the initial condition. In the worst case, higher order terms that have not cancelled give the incorrect result $\partial\langle e(Zn_i-n_e)\rangle_\psi/\partial t \gg \delta_i^4 en_e v_i/L$, and imposing $\langle e(Zn_i-n_e)\rangle_\psi=0$ leads to an unphysical radial electric field. In Ref. 9 we study the particular case of slab geometry, showing that the formalism in Ref. 3 leads to the incorrect long wavelength electric field because it introduces an unphysical momentum source. This

reference presents the problem from yet another point of view. The radial electric field can be obtained from two different equations, namely the gyrokinetic Poisson equation and the transport of momentum. The transport of momentum determines the velocity profile and the velocity profile is uniquely related to the long wavelength electric field (in the slab geometry of Ref. 9 this relation is simply the $\mathbf{E}\times\mathbf{B}$ drift). Both approaches to obtain the electric field will give conflicting results at long wavelengths unless Eq. (3) is satisfied, that is, the radial current $\langle \mathbf{J}\cdot\nabla\psi\rangle_\psi$ is of the order of the right side, requiring then as high order as $\langle \mathbf{J}\cdot\nabla\psi\rangle_\psi \sim \delta_i^4 e n_e v_i |\nabla\psi|$.

We now comment on the nonlinear term of Eq. (55) in Ref. 2. Lee and Kolesnikov [1] seem to ignore that there is freedom in the choice of gyrokinetic variables. Following our method [2], any gyrophase independent function can be added to the definition of the gyrokinetic variables, except for the magnetic moment that is chosen to be an adiabatic invariant order by order. In the work by Dubin *et al.* [3], the definition of the gyrokinetic variables is much more constrained because the transformation between guiding center and gyrokinetic variables is chosen to be area-conserving or symplectic; in the more general approach of Ref. 10, different gyrokinetic variables are possible by choosing different gyrophase independent symplectic components of the phase-space Lagrangian. Due to the freedom in the definition of the gyrokinetic variables, our derivation of the gyrokinetic equation [2] and the original work by Dubin *et al.* [3] differ slightly. The differences are explained in detail in Ref. 11. The nonlinear term $(ZMc^2 n_i/2T_i B^2)|\nabla_\perp\phi|^2$ in Eq. (55) of Ref. 2 is due to the following definition of our gyrokinetic kinetic energy [see Eq. (30) of Ref. 11]

$$E=\frac{1}{2}\overline{v}_{||}^2+\mu B+\frac{Ze}{2M}\left[\frac{Ze}{MB}\frac{\partial}{\partial\mu}\langle\tilde{\phi}^2\rangle+\frac{c}{2B\Omega_i}\langle(\nabla_{\mathbf{R}}\tilde{\Phi}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}}\tilde{\phi}\rangle\right], \tag{4}$$

where $\overline{v}_{||}$, $\tilde{\phi}$ and $\tilde{\Phi}$ are defined as in Ref. 3. Notice that the extra term is gyrophase independent and thus preserves the main property of gyrokinetic transformations, i.e., the equations of motion do not depend on the gyrophase. The rest of the gyrokinetic variables in Refs. 2 and 3 coincide or differ by trivial factors, i.e., $\mathbf{R}$, μ and φ from Ref. 2 are $\overline{\mathbf{X}}$, $(Ze/Mc)\overline{\mu}$ and $-\overline{\theta}-\pi/2$ in Ref. 3. The relation between the distribution function $f_i(\mathbf{R},E,\mu,t)$ obtained with our gyrokinetic variables and the distribution function $f_{iD}(\overline{\mathbf{X}},\overline{v}_{||},\overline{\mu},t)$ obtained with Dubin's is calculated by Taylor expanding the dependence of $f_i(\mathbf{R},E,\mu,t)$ on E about

$(1/2)\overline{v}_{||}^2 + \mu B$, giving

$$f_i(\mathbf{R}, E, \mu, t) \simeq f_{iD}(\overline{\mathbf{X}}, \overline{v}_{||}, \overline{\mu}, t) + \frac{Ze}{2M}\left[\frac{Ze}{MB}\frac{\partial}{\partial \mu}\langle\tilde{\phi}^2\rangle + \frac{c}{2B\Omega_i}\langle(\nabla_{\mathbf{R}}\tilde{\Phi}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}}\tilde{\phi}\rangle\right]\frac{\partial f_i}{\partial E}. \quad (5)$$

Since the last term is of order $\delta_i^2 f_{Mi}$, only the lowest order Maxwellian distribution function $f_i \simeq f_{Mi}$ is needed to determine the ion density to order $\delta_i^2 n_e$. Thus, $\partial f_i/\partial E \simeq (-M/T_i) f_{Mi}$. In addition, since Eq. (55) of Ref. 2 is a long wavelength approximation, we can use the long wavelength results $\partial\langle\tilde{\phi}^2\rangle/\partial\mu \simeq (M^2c^2/Z^2e^2B)|\nabla_\perp\phi|^2$ and $\langle(\nabla_{\mathbf{R}}\tilde{\Phi}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}}\tilde{\phi}\rangle \simeq 0$. Then, the contribution of the difference in the definition of the kinetic energy leads to the difference in the polarization density

$$\frac{Ze}{2M}\int d^3v\left[\frac{Ze}{MB}\frac{\partial}{\partial\mu}\langle\tilde{\phi}^2\rangle + \frac{c}{2B\Omega_i}\langle(\nabla_{\mathbf{R}}\tilde{\Phi}\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}}\tilde{\phi}\rangle\right]\frac{\partial f_i}{\partial E} \simeq -\frac{Mc^2 n_i}{2T_i B^2}\int d^3v\,|\nabla_\perp\phi|^2. \quad (6)$$

This is exactly the missing nonlinear term.

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