Numerical Study of the Nonlinear Dynamics of the Acoustic Drops and Bubbles

by

Yu-Hsuan Su

B.S., Civil Engineering
National Chung Hsing University, Taiwan (1987)

S.M., Power Mechanical Engineering
National Tsing Hua University, Taiwan (1989)

S.M., Civil Engineering
Massachusetts Institute of Technology (1994)

Submitted to the Department of Mechanical Engineering in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1999

© Massachusetts Institute of Technology 1999. All rights reserved.

Author

Department of Mechanical Engineering
June, 1999

Certified by

Zaichun Feng
Associate Professor
Thesis Supervisor

Accepted by

Amit A. Sonin
Chairman, Department Committee on Graduate Students
Numerical Study of the Nonlinear Dynamics of the Acoustic Drops and Bubbles

by

Yu-Hsuan Su

Submitted to the Department of Mechanical Engineering
on May 7, 1999, in partial fulfillment of the
requirements for the degree of
DOCTOR OF PHILOSOPHY

Abstract

The dynamics of liquid drops and bubbles held together by surface tension and perturbed by small disturbances is of great interest to many researchers. Its essential physical nature is characterized by a nonlinear moving-boundary problem complicated by the interfacial stress interaction between two domains, each governed by their own dynamical systems respectively.

In this thesis, the dynamics of an acoustically levitated drop is investigated. A low dimensional phase plane approach is used to interpret the nonlinear dynamics of the drop motions. It is found that the stability of shape oscillations imposes an upper limit on the acoustic bond number that can be used, while the lower limit is set by the stability of translational motion. The static equilibrium shapes can be obtained by incorporating the artificial damping into the system. The static equilibrium shapes thus found agree very well with the experimental data.

In addition, the two-to-one internal resonance of a single bubble between the volume mode and one of the shape modes is carefully examined. Instability wedges for unstable volume oscillations on the plane of volume oscillation amplitude versus frequency are identified numerically. Furthermore, the dynamical behaviors of the bubbles with parameters within the instability wedges can be divided into stable bubble oscillations and transient bubble oscillations. Attention is focused on the transient bubble oscillations. Numerical simulation shows that liquid jets form at the two poles of the transient bubble and lead to the breakup of the bubble. A possible mechanism resulting in the formation of the liquid jets is proposed and demonstrated with numerical simulation examples.

Bjerknes forces between two bubbles are also investigated. It is found that the Bjerknes forces between two attracting bubbles can be predicted with a formula derived by Crum with amazing accuracy. However, numerical simulations indicate that a multiplication factor is needed for the cases of two repelling bubbles within short distance. The effect of shape oscillations on the translational motions of two bubbles is also examined. Interestingly, the shape oscillation has little effect on attracting bubbles, while significant effect on the translational motion of two repelling bubbles within short distance is observed.

Thesis Supervisor: Zaichun Feng
Title: Associate Professor
Acknowledgments

I wish to express my sincere gratitude to Professor Zaichun Feng for his guidance, support, and understanding, in ways too numerous to list, during the past six years. It has truly been a pleasure and indeed, a privilege to have had him as my doctoral thesis advisor. His kindness and warmth are beyond words.

I am deeply grateful to Professors Triantaphyllos R. Akylas and Michael P. Brenner for their insightful advice and invaluable suggestions while serving as my thesis committee members.

I am indebted to a number of colleagues who share their good humor and brilliant intellect with me, in particular, Srikanth Vedantam, Vitaly J. Napadow, Alfred M. Pettinger and Jin Qiu for the quality time we shared. The administrative support from Ms. Debra R. Blanchard and Leslie Regan is gratefully acknowledged.

Special thanks go to Tyson R. Browning, my Ashdown roommate, for teaching me English and so many things about the American culture.

I also would like to take this opportunity to thank the many friends who have helped at various stages, and in many different ways over the past years. Tesan Liao and Chiaming Liu offer me their cozy friendships more than I can ever think of. Actually, we are more like brothers than just friends. Kuo-Shen Chen and Jec-Kong Gone, two alumni from NTHU, were always there when needed. So many delicious meals and enlightening chats offered by Shih-En Shih and Su-Chuan Cheng are most appreciated. The ten-day trip to California with Ilin Wang and Emerson's comrades is the most memorable traveling experience I had during this period. The friendships from Long-Sheng Kuo, Wen-Tang Kuo, Chihhao Ho, and many others from ROCSA made my MIT life so much easier to endure.

Last but not least, I am particularly appreciative of my parents, younger brother Yu-Nung, and younger sister Yu-Chen for their unconditional support and ceaseless love. This thesis is dedicated to them.
# Contents

1 Introduction .......................................................... 21
   1-1 Overview ......................................................... 21
   1-2 Literature review .............................................. 22
   1-3 Organization .................................................... 24
References ............................................................... 25

2 Mathematical Formulations ........................................... 27
   2-1 Introduction ..................................................... 27
   2-2 General considerations ........................................ 27
      2-2.1 Viscosity: viscous damping ............................... 28
      2-2.2 Compressibility: radiation damping ..................... 28
      2-2.3 Thermal conduction: thermal damping ................... 30
      2-2.4 Contributions of primary damping effects ............. 31
   2-3 Equations of motion: Drop .................................... 31
   2-4 Conservation laws: Drop ....................................... 33
   2-5 Equations of motion: Bubble ................................. 34
   2-6 Conservation laws: Bubble ..................................... 37
   2-7 Concept of mode shapes ....................................... 38
      2-7.1 Zeroth mode: radial motion (Rayleigh-Plesset equation) 40
      2-7.2 First mode: translational motion ....................... 42
      2-7.3 Shape modes: shape oscillation ........................ 42
   2-8 Non-dimensionalization ........................................ 44
References ............................................................... 45

3 Acoustic Radiation .................................................... 47
   3-1 Introduction ..................................................... 47
   3-2 Acoustic radiation pressure in compressible fluid .......... 49
   3-3 Steady state and the Helmholtz equation .................... 51
   3-4 General solutions of the Helmholtz equation ............... 52
   3-5 Scattering of a planar wave by a compressible sphere ....... 54
   3-6 Time-averaged radiation pressure ............................ 57
References ............................................................... 60
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-1</td>
<td>Comparison of bubble damping effects.</td>
<td>30</td>
</tr>
<tr>
<td>2-2</td>
<td>Geometry description of the drop configuration</td>
<td>31</td>
</tr>
<tr>
<td>2-3</td>
<td>Geometry description of the bubble configuration</td>
<td>35</td>
</tr>
<tr>
<td>2-4</td>
<td>The multivalueness of radii at certain angles makes the Legendre decomposition impossible.</td>
<td>40</td>
</tr>
<tr>
<td>2-5</td>
<td>Shapes obtained from the superposition upon a unit sphere a pure $n$-th shape mode of amplitude $\pm 0.3$ ($n = 2, 3, 4, 5, 6, 7$).</td>
<td>43</td>
</tr>
<tr>
<td>3-1</td>
<td>Geometric configuration of scattering problem</td>
<td>54</td>
</tr>
<tr>
<td>3-2</td>
<td>$</td>
<td>\Phi_s</td>
</tr>
<tr>
<td>3-3</td>
<td>$</td>
<td>\Phi_s</td>
</tr>
<tr>
<td>3-4</td>
<td>Radiation pressure $\hat{p}$ vs. polar angle for a rigid sphere with $ka = 2.0$ and $kh = 0.0$ in a planar standing wave field. The radial length deviating from the unit circle is proportional to the value of $\hat{p}$ on the surface in that direction. Over regions of increased pressure, the radial length is less than 1, indicating a net inward force on the sphere. Over regions of decreased pressure, the radial length is greater than 1, indicating a net outward force on the sphere.</td>
<td>59</td>
</tr>
<tr>
<td>3-5</td>
<td>Radiation pressure $\hat{p}$ vs. polar angle for a rigid sphere with $ka = 2.0$ and $kh = 0.0$ in a planar progressive wave field.</td>
<td>59</td>
</tr>
<tr>
<td>4-1</td>
<td>Phase diagram for the $n$-th Legendre mode perturbation imposed on the volume mode oscillation with frequency detuning ratios $\kappa = 0.8, -0.8, 5.0, -5.0$, respectively. Note that the stability of the spherical volume mode oscillation is represented by the stability of the origin. (a),(b) show that the origin is a center and thus stable for $\kappa_0$ small. (c),(d) reveals that the origin is a saddle point and thus unstable when $\kappa_0$ is large.</td>
<td>75</td>
</tr>
<tr>
<td>5-1</td>
<td>Geometric configuration for interior problem.</td>
<td>82</td>
</tr>
<tr>
<td>5-2</td>
<td>Geometric configuration for exterior problem.</td>
<td>82</td>
</tr>
</tbody>
</table>
5–3 Second mode oscillations with initial conditions. $r = 1, \phi = 0.6p_2, N = 41, \Delta t = 0.005$. (a) Amplitude of the tip motion, $z(1) - 1.0$. (b) Energy tracing. The upper oscillatory curve is the kinetic energy, the lower oscillatory curve is potential energy, and the nearly straight line is the total energy. 

5–4 Free oscillation of an ideal gas bubble with initial conditions. $R = 1.5a, \phi = 0, p_\infty = 2.0, \gamma = 1.4, N = 41, \Delta t = 0.0001$. (a) Time history of bubble radius $R(t)$. (b) Energy tracing. The oscillatory curves, from the thickest to the thinnest, are the kinetic energy, surface energy, potential energy at infinity, and internal energy respectively and the straight line is the total energy.

5–5 Comparison of Rayleigh’s frequencies for a bubble between theoretical and simulation results.

6–1 The geometric configuration of the scattering problem.

6–2 Levitation coefficients $Y_s$ for a rigid sphere located at $kh = \pi/4$ in a standing wave. The levitation coefficients $Y_s$ obtained from BEM are represented by circles $(\circ)$ and those obtained from Hasegawa’s method are represented by the solid line. Note the point at $ka = \pi$, which is the eigenfrequency of the BEM formulation, deviates from Hasegawa’s value. But BEM still gives very good results for the neighboring points $ka = 3.14, 3.16$.

6–3 Levitation force for a rigid sphere located in a standing wave at various position. (a) Velocity potential of the standing wave. (b) Pressure field of the standing wave. (c) Levitation coefficients for $Y_s > 0$. In this case, the pressure node is a stable equilibrium position, if without gravity. (d) Levitation coefficients for $Y_s < 0$. In this case, the pressure anti-node is a stable equilibrium position, if without gravity.

6–4 Conservative lower limits on the acoustic bond numbers $B_0$, for bond number $B = 0.1, 0.2, 0.3$, based on the spherical shape approximation.

6–5 Translational frequencies under various acoustic bond numbers. The translational frequencies based on spherical shape approximation are represented by the solid lines. The frequencies obtained from numerical simulations are represented by the cross signs $(\times)$. (a) $B = 0.0$, (b) $B = 0.1$, (c) $B = 0.2$. 

6–6 Orbits in the phase plane $(Z, \dot{Z})$. The pressure field (broken line) and the time-averaged levitation force (solid line) are shown schematically in (a). (b) $B = 0$, (c) $B = 0.136$. 


6–7 Equilibrium centroidal positions of a drop in an acoustic field. The unstable and stable equilibrium positions are represented by the crosses (×) and the circles (○) respectively. To depict the size of the trapping zone, the maximum z-coordinates of the homoclinic orbits are also shown in this Figure and are represented by the stars (*). (a)ka = 0.575, B = 0.136, (b)ka = 0.2, B = 0.136. .......................... 112

6–8 Shape oscillation frequencies under various acoustic bond numbers. Note that the frequencies have been normalized with respect to Rayleigh’s free oscillation frequency. (a)B = 0.0(‘+’:ka = 0.2, ‘o’:ka = 0.4, ‘*’:ka = 0.6), (b)B = 0.1(‘+’:ka = 0.2, ‘o’:ka = 0.4, ‘*’:ka = 0.6), (c)B = 0.2(‘+’:ka = 0.4, ‘o’:ka = 0.6, ‘*’:ka = 0.8). ........................... 114

6–9 Shape oscillation frequencies for drops with different initial conditions. Note that the frequencies have been normalized with respect to Rayleigh’s free oscillation frequency. The data points for drop released from spherical shape are represented by circles (o) and the data points for drop released from nearly steady static equilibrium shape are represented by the cross sign (×). .............................. 115

6–10 Large-amplitude shape oscillation of a drop with ka = 0.2, B_a = 2.56, B = 0. The initial aspect ratio is 1.71(prolate). ............................. 116

6–11 Phase plot of the large-amplitude shape oscillation of a drop with ka = 0.575, B = 0, B_a = 2.15. Six orbits (A-F) with different initial equatorial radii, R = 0.9, 0.977, 0.978, 1.0, 1.1, 1.2, are shown. ...... 117

6–12 Time evolution of the equatorial radii for drops with ka = 0.575, B = 0.0, and B_a = 2.15 starting with different initial equatorial radii R = 0.9, 0.977, 0.978, 1.0, 1.1, 1.2. ................................. 117

6–13 Stable equilibrium shape for a drop with ka = 0.575, B = 0.0, B_a = 2.15. ................................................................. 118

6–14 Time evolution of the drop shape for ka = 0.575, B = 0.0, B_a = 2.15, R = 0.978. ............................................................. 119

6–15 Equilibrium radii vs. acoustic bond number for a drop with ka = 0.575, and B = 0.0. The equatorial radius of a stable equilibrium shape is represented by a circle (o) and that of an unstable equilibrium shape by a cross (×). A saddle-node bifurcation is observed. The saddle-node bifurcation defines an upper threshold, B_a,cr, for the acoustic bond number. Currently, B_a,cr = 2.482 and the maximum stable equilibrium radius is 1.443. ................................. 120
6–16 Equilibrium radii vs. acoustic bond number for a drop with $ka = 0.2$, and $B = 0.0$. The equatorial radius of a stable equilibrium shape is represented by a circle (o) and that of an unstable equilibrium shape is represented by a cross (x). A saddle-node bifurcation is also observed. The upper threshold, $B_{ac}$, in this case is 3.126 and the maximum stable equilibrium radius is 1.5.

6–17 Unbounded growth of the equatorial radii for a drop with $ka = 0.575$, $B = 0$, $B_a = 2.15$, $R = 0.977$. Only the drop shapes at the first 4 time units are shown.

6–18 Deformed drop shapes under various acoustic bond numbers for a drop with $ka = 0.4$ and $B = 0.0$. The drop shapes calculated from Marston’s formula are represented by the dash lines. The drop shapes obtained from numerical simulations are represented by the stars(*).

6–19 Aspect ratios under various acoustic bond numbers for a drop with $ka = 0.4$ and $B = 0.0$.

6–20 Deformed drop shapes under various acoustic bond numbers for a drop with $ka = 0.4$ and $B = 0.1$.

6–21 Aspect ratios under various acoustic bond numbers for a drop with $ka = 0.4$ and $B = 0.1$.

6–22 Aspect ratios under various sound intensities. A comparison of the results given in Tian, Holt & Apfel (1993) (solid line) and those from BEM numerical simulations (represented by “o”). Note Apfel’s definition of the aspect ratio is different from ours.

6–23 Comparison of the results given in Tian, Holt & Apfel (1993) (p.3100, Figure 2.) (represented by “x”), the experimental data given in Trinh & Hsu (1986) (p.1337, Figure 4.) (represented by “+”), and the results from the BEM numerical simulations (represented by “o”).

6–24 Time evolution of the centroidal position(a) and equatorial radius(b) of an initially spherical drop with $ka = 0.575$, $B = 0.136$, and $B_a = 1.071$ released at different positions, $Z_0 = -2.332, -1.00, -0.378$, in the acoustic field.

6–25 Time evolution of the centroidal positions and the equatorial radii of two drops both with $ka = 0.575$, $B = 0.136$, $B_a = 1.07$ but one with initial equatorial radius $R = 1.037$ (thick line) and the other one with $R = 1.040$ (thin line).

7–1 Time evolution of the radius of spherical bubble oscillation with $p_\infty = 2.0$, $R_0 = 3.0$. For the purpose of comparison, the result obtained from numerical integration of the Rayleigh-Plesset equation is also shown. Kinetic energy, Surface energy, internal energy, energy flowing to infinity, and total energy are shown below.
7-2 Spherical bubble oscillation is unstable to small shape perturbation. 
\( p_\infty = 180, \ P_2 = 0.02 \) .............................................. 136

7-3 Two-to-one resonance between \( P_0 \) and \( P_2 \) modes, Initial perturbations: \( P_0 = 0.05, \ P_2 = 0.001 \) .................................................. 138

7-4 Two-to-one resonance between \( P_0 \) and \( P_2 \) modes, Initial perturbations: \( P_0 = 0.08, \ P_2 = 0.001 \) .................................................. 139

7-5 Legendre decomposition of a transient bubble with \( p_\infty = 9.0 \) and 
initial perturbations \( P_0 = -0.2, \ P_2 = 0.01 \) .................. 140

7-6 Bubble shape at \( t = 5.0 \). Note: The multivalueness of radii at certain angles is the pathological cause for the failure of Legendre mode 
decomposition. .............................................................. 140

7-7 Energy tracing of a transient bubble with \( p_\infty = 9.0 \) and initial perturbations \( P_0 = -0.2, \ P_2 = 0.01 \) .................. 141

7-8 Tip velocity at the north pole. \( (p_\infty = 9.0, \ P_0 = -0.2, \ P_2 = 0.01) \) ... 141

7-9 Transient bubble shapes at \( t = 4.4, 4.6, 4.8, 5.0 \). \( (p_\infty = 9.0, \ P_0 = -0.2, \ P_2 = 0.01) \) .............................................. 142

7-10 Instability wedges for detuned two-to-one resonance between volume 
mode and shape modes. ..................................................... 142

7-11 Legendre mode decomposition for the dynamics of a bubble with 
initial shape perturbations \( P_0 = 0.01 \) and velocity perturbation in 
5-th mode subject to pressure at infinity \( p_\infty = 180 \). Note that \( P_5 \) 
mode grows but not exponentially. And the volume oscillation is 
stable. .......................................................... 143

7-12 Two-to-one resonance between \( P_0 \) mode and \( P_6 \) mode with different 
detuning ratio. .............................................................. 144

7-13 Two-to-one resonance between \( P_0 \) mode and \( P_5 \) mode with different 
detuning ratio. .............................................................. 145

7-14 Exact two-to-one resonance between volume mode and \( P_6 \) mode. 
\( P_0 = 0.01, \ P_6 = 0.001 \) and \( p_\infty = 265.14 \). ......................... 146

7-15 Legendre mode decomposition for the dynamics of a bubble with 
initial shape perturbations \( P_0 = 0.028, \ P_6 = 0.01 \) subject to pressure 
at infinity \( p_\infty = 210 \). The computation stops at \( t = 3.2 \). The abrupt 
changes near the end of computation are caused by the failure of 
Legendre mode decomposition due to the multivalueness of radial 
length in certain angle \( \theta \). .............................................................. 147

7-16 Tip motions just a few moments before the computation stops. Note 
that the tip oscillates up and down several times and finally shoots 
into the bubble.............................................................. 148
7-17 Energy tracing of the bubble. The heavy dashed line represents the total energy, the heavy solid line is the kinetic energy, the light solid line is the surface energy, and the dot-dashed line stands for the internal energy. For the purpose of clarity, the energy associated with the pressure at infinity is not shown.

7-18 Boundary separating the stable bubble oscillations from the transient bubble oscillations in the instability wedge of mode 6.)

7-19 Geometric amplification for a contracting bubble. Note that the tip acceleration grows when the surface area decrease.

7-20 Geometric amplification for a expanding bubble. Note that the tip acceleration decays when the surface area grows.

7-21 Local geometric amplification for a bubble with $P_0 = 0.1$ and pressure at infinity $P_\infty = 210.0$. A local disturbance is imposed by moving the north pole by 0.001 along the symmetric axis. Note that $P_{10}$ mode is excited due to the near two-to-one resonance with $P_0$ mode.

7-22 Tip velocity of the north pole corresponding to the same simulation with $P_\infty = 210.0, P_0 = 0.028$, and $P_0 = 0.01$

7-23 Direct correlation between the growth of tip acceleration and the local surface area decrease caused by the collapsing $P_0$ mode are clearly seen in this figure.

8-1 Geometric configuration for two bubble problem

8-2 Geometric configuration for Crum's model.

8-3 Translational motions induced by the pulsation of two identical bubbles with various amplitudes. $P_\infty = 1333.33, \tilde{R}_1 = \tilde{R}_2 = 1.0, D = 3.0, \delta_1 = \delta_2 = 0.01, 0.02, 0.03, 0.1$.

8-4 The shapes of the two bubbles just before touching each other for the cases with parameters: $P_\infty = 1333.33, D_1 = D_2 = 1.5, \delta_1 = \delta_2 = 0.01, 0.02, 0.03, 0.1$, and equilibrium radii $\tilde{R}_1 = \tilde{R}_2 = 1.0$ The shape in (d), which is smooth on one side and severely deformed on the other, is referred as the spherical cap shape.

8-5 Translational motions induced by the pulsation of two in-phase bubbles with various far field pressures. $\tilde{R}_1 = \tilde{R}_2 = 1.0, D = 3.0, \delta_1 = \delta_2 = 0.01, P_\infty = 50, 400, 1333.33$.

8-6 Volume mode oscillation for one of the two bubbles oscillating in phase with $P_\infty = 400, D_1 = D_2 = 1.5, \delta_1 = \delta_2 = 0.01$, and $\tilde{R}_1 = \tilde{R}_2 = 1.0$.

8-7 Large amplitude volume mode oscillation for one of the two bubbles oscillating in phase with $P_\infty = 2, 20, 100, D_1 = D_2 = 3.0, \delta_1 = \delta_2 = 0.3$, and $\tilde{R}_1 = \tilde{R}_2 = 1.0$.
8–8 Translational motions induced by the pulsation of two anti-phase bubbles with various amplitudes. \( p_\infty = 400, \bar{R}_1 = \bar{R}_2 = 1.0, \ D = 3.0, \ \delta_1 = \delta_2 = 0.01, 0.02, 0.03. \) ........................................... 169

8–9 Translational motions induced by the pulsation of two anti-phase bubbles with various far field pressures. \( \bar{R}_1 = \bar{R}_2 = 1.0, \ D = 3.0, \ \delta_1 = \delta_2 = 0.01, \ p_\infty = 300, 400, 500. \) ........................................... 170

8–10 Translational motion (repulsion) induced by two alternating-phase bubbles. The far field pressure \( p_\infty \) is set to 400.0. Initial perturbation (\( P_0 = 0.01 \)) is imposed on bubble 1, while bubble 2 is stationary. .. 171

8–11 In-phase shape oscillations result in the attraction of the two bubbles. The initial perturbations \( P_0 = 0.1 \) are imposed on both bubbles. Due to the symmetry, only the data of bubble 1 are shown. The far field pressure is set to \( p_\infty = 2.0 \). The distance between the two centroids is 2.5. .......................................................... 171

8–12 Anti-phase shape oscillation leads to the repulsion of the two bubbles. 173

8–13 Alternating-phase shape oscillations (a)\( D = 4.0 \), (b)\( D = 3.0 \), (c)\( D = 2.8 \), (d)\( D = 2.6 \), and (e)\( D = 2.4 \). Subscripts are used to specify the bubble under discussion. ................................. 174

8–14 Bubble shapes at the moment of maximum deformation in the second bubble. (a)\( D = 4.0 \), (b)\( D = 3.0 \), (c)\( D = 2.8 \), (d)\( D = 2.6 \), and (e)\( D = 2.4 \). Subscripts are used to specify the bubble under discussion. .... 175

8–15 No significant effect of shape oscillation on the translational motion of two attracting bubbles is observed. .................................................. 175

8–16 Effect of the shape oscillation on the translational motion of two repelling bubbles is quite significant. ............................................. 176

8–17 Two-to-one resonance is suppressed by the two bubble configuration. 177

8–18 Study of the effect of two-to-one resonance on the translational motions. ............................................................. 178

8–19 The dynamics of dominant modes for the case with parameters given in Table 8.2 (a). ................................................................. 179

8–20 The dynamics of dominant modes for the case with parameters given in Table 8.2 (b). ................................................................. 179

8–21 The dynamics of dominant modes for the case with parameters given in Table 8.2 (c). ................................................................. 180

8–22 The dynamics of dominant modes for the case with parameters given in Table 8.2 (d). ................................................................. 180

8–23 Bubble shapes for the case with parameters given in Table 8.2(d) .. 181
8–24 Translational motions of the centroids of two bubbles of different sizes. Six different values of the smaller radii (bubble 2) are used; \( \bar{R}_2 = 0.25, 0.5, 0.6, 0.75, 0.8, 0.9 \). The far field pressure \( p_\infty \) is set to 50.0, the distance between the two centroids \( D = 3.0 \), and initial perturbation \( P_0 = 0.02 \) is imposed on the larger bubble (bubble 1) for all cases. ............................................................... 182

8–25 Shape modes in the small bubbles are excited via direct harmonical resonance with the volume modes of the large bubbles. .................. 182

8–26 Verification of the possibility of direct resonance between the volume mode of large bubble and \( P_2 \) mode of the small bubble. ............... 183

8–27 Verification of the possibility of direct resonance between the volume mode of large bubble and \( P_3 \) mode of the small bubble. ............... 183

8–28 The two bubbles repel each other when \( \bar{R}_2 \geq 0.86 \) while they attract each other when \( \bar{R}_2 \leq 0.84 \) .................................................. 184

8–29 Comparison between the simulation results produced from the code developed in the present work and those given in Pelekasis 1993B, Figure 5. ............................................................... 185

8–30 The fourth mode is excited in the second bubble due to the harmonic resonance with the volume mode of the first bubble. .................. 187

8–31 Several bubble shapes up to the breakdown of computations are shown here. ............................................................... 187

8–32 The volume of the second bubble increases significantly due to the harmonic resonance with forcing, which in turn enhances the translational motion involved. .................................................. 188

8–33 Several bubble shapes up to the breakdown of computations are show here. ............................................................... 188

8–34 It appears that the forcing is in subharmonic resonance with the fourth mode of the second bubble in the present case. .................. 189

8–35 Moving away from the blurry boundary, the two bubbles attract each other again when \( \omega_f = 30 \) .................................................. 190

8–36 The translational motions of the two bubbles switch from attraction to repulsion with the increase of the far field pressure. .................. 191
List of Tables

6.1 Comparison of the levitation coefficients $Y_s(Y_p)$ obtained from BEM (40 quadratic elements) and those obtained from Hasegawa's method [2] for a rigid immovable sphere in a standing (progressive) wave field. ......................................................... 105

6.2 Levitation coefficients $Y_s$ for spheroids obtained from BEM with 40 quadratic elements. ......................................................... 105

8.1 Parameters for studying the effect of shape oscillations on two repelling bubbles. ................................................................. 176

8.2 Parameters for studying two-to-one resonance. .......................... 177

8.3 The natural frequencies of the corresponding eigenmodes. .......... 186

B.1 Coordinates of integration points and weighting factors for Standard Gaussian Quadrature ......................................................... 202

B.2 Coordinates of integration points and weighting factors for Logarithmic Gaussian Quadrature ......................................................... 203
Chapter 1

Introduction

1-1 Overview

With numerous applications prevailing in science and engineering, the dynamics of liquid drops and bubbles held together by surface tension and perturbed by small disturbances is of great interest to many researchers. Its essential physical nature is characterized by a nonlinear moving-boundary problem complicated by interfacial stress interaction between two domains governed by their own dynamical systems respectively. Classical theories, based on the small-amplitude assumption, can be categorized into two different approaches: linearized analysis and perturbation analysis. Perhaps, Rayleigh's frequency governing the constant-volume free oscillation around the equilibrium shape is the most preeminent discovery in the linearized theory. Perturbation theory, suffered from the moving boundary, gives its major contributions to the understanding of a more restricted yet nonlinear case, the Rayleigh-Plesset equation. Rayleigh-Plesset equation dominates the radial spherical motion of a single bubble in an incompressible viscous fluid. After decades of effort, variations of Rayleigh-Plesset equation taking into account of the effects of compressibility, mass diffusion, and thermal convection are studied. Significant contributions are made to the fields of cavitation, sonochemistry, sonoluminesence and so on.

Though fruitful, Rayleigh-Plesset equation is challenged by the stability issue. Unlike the linear equation, nonlinear equations might possess many solutions or none at all. The value of the solution to a nonlinear equation is determined by its stability. Only the stable solutions can be realized in the physical world. Since the Rayleigh-Plesset equation relies on the spherical assumption, shape oscillation is the most detrimental concern to this assumption. More explicitly, is the spherical bubble oscillation stable subjected to the small shape perturbations? Nonlinear theory enriches the dynamics of drops and bubbles and many interesting phenomena that are not possible in the linear theory can occur. Perhaps, the secondary
internal resonance between two linearly uncoupled eigenmodes is one of the most prominent features of the nonlinear system. Through the secondary internal resonance, two eigenmodes may resonate and exchange energy with each other either subharmonically or superharmonically. This is especially true for nonlinear systems with commensurate natural frequencies. Due to the quadratic nonlinearity involved, the two-to-one subharmonical resonance between the volume mode and one of the shape modes (the natural frequency for the volume mode is nearly twice that of the shape mode coupled.) is the most significant. Perturbation analysis of this two-to-one subharmonic resonance shows that the initially small shape perturbation may grow exponentially due to the strong nonlinear interaction between between. The significant growth of the initially small shape perturbation thus induced will jeopardize the validity of perturbation analysis, therefore prediction of perturbation analysis might not be accurate. Though without the two-to-one resonance between the volume mode and shape modes in drop dynamics, the same dilemma occurs for large amplitude shape oscillation of liquid drops driven by the acoustic forcing. Furthermore, the dynamics of breakup of drops and bubbles is a long standing yet interesting topic that deserves special attention. It is generally believed that the breakups of drops and bubbles are caused by a Rayleigh-Taylor like instability on the interfaces that separate the fluids of different densities. The use of perturbation analysis is certainly quite limited over this area.

This leaves us only one option – numerical simulation. Numerical simulation of the drop dynamics and bubble dynamics is made feasible by the recent developments in computer technologies over the past decades. However, it is no panacea for the current nonlinear system under consideration. Restricted by the stiffness of the nonlinear system, node clustering effect, and stability of the numerical integration, numerical simulation is only feasible for a moderate range of parameters. The objective of this thesis is to explore the nonlinear dynamical behaviors of drops and bubbles numerically within this moderate range of parameters.

1-2 Literature review

Acoustically levitated drops

Liquid drops could be levitated and manipulated acoustically under microgravity environment. This provides an attractive solution to the containerless material processing, in which a sample is suspended and manipulated without touching contaminating containers. Two key issues involved are the calculation of the radiation pressure exerted on the drops and how the drops will respond to the radiation pressure imposed on the surface? King's work (1934) on the radiation pressure on a rigid sphere initiates the concept of acoustic levitation. The study of acoustic levitation
consists of two essential parts: the understanding of the acoustic forcing and the dynamic behavior of the drop. Because of the significant difference in time scales, the acoustic forcing problem can be reduced to a steady-state scattering problem governed by the Helmholtz equation. Analytic solutions are only available for problems with nearly spherical boundaries. By expanding the planar incident wave velocity potential in terms of spherical harmonics and matching the boundary conditions on the surface, King obtained the total velocity potential for the acoustic field scattered by a rigid sphere. This method was further modified by Hasegawa & Yosioka (1969) to account for the elasticity of the sphere. Yosioka & Kawasima (1955) derived the acoustic radiation force on an elastic sphere in a plane quasistationary wave field. Yosioka, Kawasima & Hirano (1964) also calculated the radiation force on a rigid sphere in a plane quasistationary wave field. By using basic properties of spherical Bessel functions of the first kind and second kind, Hasegawa (1977) simplified the results of the previous work.

Studies on the dynamic response of the drop by Marston (1980; 1981), and Tian, Holt & Apfel (1993) are mainly focused on the static deformations and the shape oscillations of the drops. Both of their methods are based on the same spherical harmonics expansion method. Experiments conducted by Trinh & Wang (1982) and Trinh & Hsu (1986) demonstrate consistent results with the theoretical predictions.

Single bubble

Cavitational phenomena were first documented in the literature by Reynolds (1894). The theoretical analysis on the problem of the collapse of an empty cavity in an incompressible liquid medium given by Rayleigh (1917) initiated the scientific study of cavitation and bubble dynamics. Based on Rayleigh’s equation, Plesset (1949) considered a gas-filled bubble with the effect of viscosity as well as surface tension and derived the so-called Rayleigh-Plesset equation. This equation describes the ‘inertially-dominated’ bubble dynamics in the absence of thermal consideration. More sophisticated models considering the effect of slight compressibility (Prosperetti & Lezzi 1986) of liquid, mass-diffusion (Hsieh & Plesset 1961, Eller & Flynn 1965), and thermal conduction (Scriven 1959, Flynn 1975) are now available. A comprehensive review of researches based on Rayleigh-Plesset equation prior to 1976 was given by Plesset & Prosperetti (1977). Intensive emphasis was placed on the radial motion of spherical bubbles. Recent review of studies on bubble dynamics taking the nonlinear dynamical systems approach was done by Feng & Leal (1997).

Two bubbles

Bjerknes (1906) found that the forces exerting on the two pulsating bodies in liquid are inversely proportional to the square of the distance between them. Based upon the law of kinematic buoyancy, which is analogous to the Archimedean law, Bjerknes
further proposed that "the kinematic buoyancy exerted on a body immersed in a fluid is equal to the mass of water displaced by the body times the acceleration of the body". Perhaps Hicks (1879; 1880) is the first one who derived such analytic expressions for the Bjerknes forces between two pulsating spheres. A brief literature review of the Bjerknes forces and related researches was given by Pelekasis & Tsamopoulos (1993 A).

1-3 Organization

This thesis contains nine chapters.

Chapter 1 outlines the objectives of current research and gives a brief literature review.

Chapter 2 develops a rigorous mathematical formulations for studying the nonlinear dynamics of a single drop and bubble. The most fundamental conservation laws such as mass conservation and energy conservation are derived for the drop and bubble cases, respectively. Concept of the eigenmodes for the corresponding linearized equation is introduced. The famous Rayleigh-Plesset equation is briefly derived and discussed.

Chapter 3 gives a detailed derivation of King's formula for calculating the acoustic radiation pressure due to a rigid spherical scatterer in a compressible fluid. The steady state wave equation, Helmholtz equation, is discussed. The rigid sphere scattering problem is solved analytically. The results are used to check the accuracy of numerical computations in later chapters.

Chapter 4 generalizes the asymptotic analysis to consider a degenerate case. Theoretic treatment of the two-to-one resonance between volume mode and one of the shape modes is also introduced.

Chapter 5 presents the Boundary Element Method (BEM) implementation of solving the Laplace equation and Helmholtz equation given in Chapter 2 and Chapter 3 to every detail. Stability issue of the time integration is addressed. Verifications of the code is done by comparing with the direct integration of Rayleigh-Plesset equation and the mass and energy conservation. Numerical simulation is performed with the same parameters given in Lundgren & Mansour (1988) and the two results appear to be the same.

Chapter 6 studies the dynamics of an acoustically levitated drop comprehensively by projecting the complex dynamics onto two uncoupled 2D phase plane. One for the shape oscillation with equatorial radius $R$ and its time derivative $\dot{R}$ as phase variables and the other for the translational motion with the centroidal position $Z$ and $\dot{Z}$ as its phase variables. Several key concepts such as minimum trapping pressure, and trapping region are grasped via this projection. Static equilibrium shapes are found by incorporating the artificial damping to get rid of transients. Finally, the possible coupling between the translation motion and shape oscillations
are examined.

Chapter 7 explores the subharmonic two-to-one resonance between the volume mode and one of the shape modes. Instability wedges for the unstable shape modes are identified numerically. A qualitative boundary separating the stable bubble oscillations from transient bubble oscillations within the instability wedge is sketched. The jet formation on the transient bubble is carefully verified. A Rayleigh-Taylor type instability like mechanism held responsible for the formation of liquid jet is proposed.

Chapter 8 adapts the BEM code developed in Chapter 5 to the case of two bubbles. Bubble motions driven by the secondary Bjerknes forces, either attraction or repulsion, are simulated and results are compared with those obtained from the Crum's simplified model of pulsating bubbles. Various new interesting phenomena and resonant interactions that are not encompassed by Crum's model are investigated. Bubbles of different sizes exhibiting certain peculiar translation motions are examined.

Chapter 9 summarizes the major contributions of the present thesis and the possible directions of the future work are addressed.

REFERENCES


MARSTON, P. 1981 Quadruple projection of the radiation pressure on a compressible sphere. 


RAYLEIGH, LORD. 1917 On the pressure developed in a liquid during the collapse of a 

REYNOLDS, O. 1894 Experiments showing the boiling of water in an open tube at ordinary 


TIAN, Y., HOLT, G. & APPFEL, E. 1993 Deformation and location of an acoustically 

TRINH, E. & WANG, T. G. 1982 Large amplitude free and driven drop-shape oscillations: 


YOSIOKA, K. KAWASIMA, Y. & HIRANO, H. 1964 On the absolute measurement of ultrasonic 
Chapter 2

Mathematical Formulations

2-1 Introduction

The dynamical responses of a drop/bubble to the external excitations which are either impulsive or time-varying is of intriguing interest to many fields in science and engineering. The purposes of this chapter are to develop a basic framework of mathematical modeling for the drop/bubble and to build up the necessary mathematical formulations that describe the motion of a drop/bubble immersed in its surrounding medium. Equations of motion are introduced. Conservation laws are presented.

2-2 General considerations

The aim of the present work is to simulate the nonlinear dynamics of a very complicated physical system with a simple mathematical model. It is more of arts than science to define the word ‘simple’. Perhaps the appropriate question to ask is what kinds of physical mechanisms should be included in order to best capture the essence of the dynamics. In other words, how to trade off between the accuracy of simulation and the simplicity of model? In this work, the liquid is assumed to be inviscid, incompressible, and irrotational as usual, e.g. Hall & Seminara (1980). And the gas undergoes adiabatic process while the drop/bubble is set in motion. All macroscopic properties of the gas are uniform. No mass exchange and thermal conduction between the two phases are allowed in the model. In the absence of any damping effect in the model, the energy of the system is conserved. In the following subsections, no rigorous argument is intended to verify the validity of these assumptions. Only heuristic models or relevant analogies are incorporated to uphold those assumptions instead!
2-2.1 Viscosity: viscous damping

The Navier-Stokes equation can be satisfied by the inviscid flow, except that the boundary condition cannot be matched on the interface which leads to the generation of vorticity on the interface. As is well known that vorticity may enter an initially irrotational flow only by means of the diffusion of vorticity across the boundary. Once vorticity has entered the flow, it is con~cuted by the velocity field while diffusing and intensifies or attenuates due to the action of vortex stretching. Based on Stokes' boundary layer analysis (the semi-infinite flow above a plane wall that is oscillating in its plane with an angular frequency \( \omega \)), the amplitude of the velocity decays exponentially along the normal direction and becomes vanishingly small at a typical distance \( (2\nu/\omega)^{1/2} \) away from the plane. The influence of viscosity is presumably confined within this so-called Stokes boundary layer. Analogously, we expect that vorticity associated with the drop/bubble oscillation will be confined within a thin layer at the interface. If the penetration depth of vorticity is small compared with the drop/bubble size \( a \) (radius of the drop/bubble), then the bulk of the fluid flow remains nearly irrotational. Thus the inviscid and irrotational assumption would be appropriate if \( (2\nu/\omega)^{1/2} \ll a \) can be justified.

2-2.2 Compressibility: radiation damping

The slight compressibility of the liquid leads to the radiation of energy from the bubble toward infinity via the pressure wave propagating outward. In order to grasp the idea of radiation damping and assess the effect on the bubble dynamics, a linearized model of the spherical bubble is cited here, Nayfeh & Mook (1979).

Consider a spherical bubble with equilibrium radius \( a \) pulsating in a quiescent fluid with density \( \rho_0 \) and pressure \( p_g^e \) in its equilibrium state. Let us express the radius of the pulsating bubble as

\[
R = a [1 + \epsilon \eta(t)].
\]  
(2-1)

Then the linearized boundary conditions at the bubble surface is

\[ p = p_g \text{ and } u = \epsilon \alpha \eta \text{ at } r = a. \]

Accordingly, the boundary conditions at infinity are

\[ u \to 0 \text{ and } p \to p_g^e \text{ as } r \to \infty. \]

The linearized equations governing the motion of the liquid are Conservation of mass

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 u \right) + \frac{1}{\rho_0} \frac{\partial \rho}{\partial t} = 0,
\]  
(2-2)

Euler’s equation of mot. n

\[
\frac{\partial u}{\partial t} = - \frac{1}{\rho_0} \frac{\partial p}{\partial r},
\]  
(2-3)
Constitutive equation

\[ dp = c^2 d\rho, \quad (2-4) \]

where \( u, \rho, p, \) and \( c \) are the radial velocity, the density, the pressure, and the speed of sound in liquid, respectively. Combining Eq.(2-2), (2-3), and (2-4), we obtain

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial p}{\partial r} \right) = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}. \quad (2-5) \]

The outgoing wave solution is

\[ p = p_g^* + \frac{f(r - ct)}{r}. \quad (2-6) \]

Hence

\[ \frac{\partial}{\partial t} (rp) = -c \frac{\partial}{\partial r} (rp) . \quad (2-7) \]

Integrating Eq.(2-3) from \( r = a \) to \( r = \infty \), one obtains

\[ \int_a^\infty \frac{\partial u}{\partial t} dr + \frac{1}{\rho_0} \left( p_g + \frac{2\sigma}{a} - p_g^* \right) = 0. \quad (2-8) \]

Integrate the unsteady term by parts as follows:

\[ \int_a^\infty \frac{\partial u}{\partial t} dr = \int_a^\infty \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 u \right) dr = -r \frac{\partial u}{\partial t} \bigg|_a^\infty + \int_a^\infty \frac{1}{r} \frac{\partial^2}{\partial r \partial t} \left( r^2 u \right) dr. \quad (2-9) \]

Making use of Eq.(2-2) and (2-4), one can write

\[ \frac{1}{r} \frac{\partial^2}{\partial r \partial t} \left( r^2 u \right) = -r \frac{\partial^2 p}{\partial t^2} = -\frac{1}{\rho_0 c^2} \frac{\partial^2}{\partial t^2} (rp). \quad (2-10) \]

Using Eq.(2-7), one can express the above equation as

\[ \frac{1}{r} \frac{\partial^2}{\partial r \partial t} \left( r^2 u \right) = \frac{1}{\rho_0 c^2} \frac{\partial}{\partial t} \frac{\partial p}{\partial r} = \frac{1}{\rho_0 c^2} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial t} \right). \quad (2-11) \]

Substituting Eq.(2-11) into (2-9), we obtain

\[ \int_a^\infty \frac{\partial u}{\partial t} dr = -r \frac{\partial u}{\partial t} \bigg|_a^\infty + \frac{1}{\rho_0 c} \frac{\partial p}{\partial t} \bigg|_a^\infty. \quad (2-12) \]

Applying the boundary conditions, we obtain

\[ -\epsilon \alpha^2 \eta + \frac{a}{\rho_0 c} \frac{\partial p_g}{\partial t} + \frac{1}{\rho_0} \left( p_g + \frac{2\sigma}{a} - p_g^* \right) = 0. \quad (2-13) \]

For adiabatic process, we have

\[ \frac{p_g}{p_g^*} = \left( \frac{a}{R} \right)^{3\gamma} = (1 + \epsilon \eta)^{-3\gamma} \sim 1 - 3\gamma \epsilon \eta. \quad (2-14) \]

Substitution of Eq.(2-14) into (2-13) yields

\[ \dot{\eta} + \frac{3\gamma p_g^*}{\rho_0 c a^2} \dot{\eta} + \frac{1}{\rho_0 a^2} \left( 3\gamma p_g^* - \frac{2\sigma}{a} \right) \eta = 0. \quad (2-15) \]

The second term is a damping term which is due to the radiation of energy from the bubble outward via the liquid. Note that the damping coefficient is inversely proportional to the equilibrium radius. A legitimate question to ask is under what
condition we can neglect the effect of compressibility. A balance of the order of magnitudes of the above equation shows that the compressibility can be neglected if \((\omega/c)a = ka \ll 1\), where \(\omega\) is the frequency of bubble pulsation.

It is believed that radiation damping can influence important characteristics of nonlinear forced oscillatory motion such as period doubling and transition to chaos, see Lauterborn (1985). Further analysis of the nonlinear effect of compressibility can be found in the literature. Interested readers are referred to Keller & Kolodner (1956), Epstein & Keller (1971), Keller & Miksis (1980), and Prosperetti & Lezzi (1986).

### 2-2.3 Thermal conduction: thermal damping

It is generally believed that the thermal effects are negligible for most of the collapsing stage, and they play an important role only in the final stage of collapse when the gas is highly compressed by the surrounding fluid. Due to nonlinearity, the elapsed times are so short that it seems reasonable to assume the process is adiabatic. However, the heat transfer between the two phases could be quite significant owing to the extremely high temperature gradient at the final stage of the collapse. This mechanism provides an alternative damping effect to the dynamics.
2-2.4 Contributions of primary damping effects

Chapman & Plesset (1971) presented a useful comparison of the relative importances of the three primary damping effects mentioned above in Figure 2-1. Obviously, viscosity dominates for very small bubbles. Radiation damping dominates for bubbles larger than 1cm, and thermal effects dominates for a broad range of bubble size.

2-3 Equations of motion: Drop

Consider an inviscid and incompressible fluid drop with density \( \rho \) and equilibrium radius \( a \) \((V = \frac{4}{3}\pi a^3, V: \) volume of the drop) oscillating in an unbounded quiescent fluid with negligible density \( \rho_f (\ll \rho) \). We denote the drop region by \( V \), the surrounding fluid region by \( V^c \), and the interface by \( S \), as illustrated in Figure 2-2. The surface tension coefficient \( \sigma \) on the interface is presumed to be a constant.

\[ P_e - P_\infty = \frac{2\sigma}{a} \text{ on } S, \]  

Figure 2-2 Geometry description of the drop configuration

Basic state

Let the undisturbed basic state be a spherical drop sitting in the surrounding fluid with uniform ambient pressure \( P_\infty \). Then the balance of normal stress on the interface gives

\[ P_e - P_\infty = \frac{2\sigma}{a} \text{ on } S, \]  

where \( P_e \) is the equilibrium pressure of the drop at basic state.

Irrotational flow

The condition of incompressibility gives

\[ \nabla \cdot U = 0 \text{ in } V, \]  

where \( U \) is the velocity field.
Let the shape of the drop in motion at time $t$ be described by $F(R, t) = 0$. Then the kinematic boundary condition can be expressed as
\[
\frac{DF}{Dt} = \frac{\partial F}{\partial t} + (U \cdot \nabla) F = 0 \quad \text{on } S, \tag{2-18}
\]
where $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (U \cdot \nabla)$ stands for the material derivative operator.

The Navier-Stokes equation for an inviscid fluid in the absence of body forces other than gravitational force is
\[
\rho \left[ \frac{\partial U}{\partial t} + (U \cdot \nabla) U \right] = -\nabla P - \rho g k \quad \text{in } V, \tag{2-19}
\]
where $P$ is the pressure field, and $g$ the gravitational constant.

The dynamical boundary condition, governed by the Laplace-Young equation, can be expressed as
\[
P - P_o = \sigma \nabla \cdot n \quad \text{on } S, \tag{2-20}
\]
where $P_o$ is the pressure of the surrounding medium immediately outside the interface and $n$ the outward unit normal on the interface $S$. The term $\nabla \cdot n$ represents the total curvature of a generic point on the interface.

If the flow is irrotational, the velocity field $U$ can be represented by the gradient of a scalar velocity potential $\Phi$:
\[
U = \nabla \Phi. \tag{2-21}
\]

The condition of incompressibility expressed in terms of velocity potential can then be written as
\[
\nabla^2 \Phi = 0 \quad \text{in } V. \tag{2-22}
\]

The kinematic boundary condition can be rewritten as
\[
\frac{DF}{Dt} = \frac{\partial F}{\partial t} + (\nabla \Phi \cdot \nabla) F = 0 \quad \text{on } S. \tag{2-23}
\]

Lagrangian description provides one alternative to trace the interface without referring to $F(R, t)$ explicitly as follows
\[
\frac{LR}{Dt} = \nabla \Phi, \tag{2-24}
\]
where $R$ is the position vector of a generic material point on the interface.

Substituting $U = \nabla \Phi$ into Eq.(2-19) and integrating the resulting equation with respect to the spatial variables yields the unsteady Bernoulli's equation
\[
\rho \frac{\partial \Phi}{\partial t} + \frac{1}{2} \rho \nabla \Phi \cdot \nabla \Phi + P + \rho g z = h(t) \quad \text{in } V, \tag{2-25}
\]
where $h(t)$ is an arbitrary function of time only\(^1\).

---

\(^1\) The following vector identity is used in the above derivation.
\[
(U \cdot \nabla) U = \frac{1}{2} \nabla (U \cdot U) - U \times (\nabla \times U).
\]
Combining Eq.(2–20) and (2–25) gives
\[ \rho \frac{\partial \Phi}{\partial t} + \frac{1}{2} \rho \nabla \Phi \cdot \nabla \Phi + P_0 + \sigma \nabla \cdot n + \rho g z = h(t) \quad \text{on } S, \] (2–26)

Rewriting the above equation in terms of material derivative, we have
\[ \rho \frac{D \Phi}{Dt} = \frac{1}{2} \rho \nabla \Phi \cdot \nabla \Phi - P_0 - \sigma \nabla \cdot n + \rho g z + h(t) \quad \text{on } S. \] (2–27)

Since we have the liberty to choose \( h(t) \) without changing the flow field, we will choose \( h(t) = 0 \) for convenience.

The governing equation Eq.(2–22) together with the boundary conditions Eq.(2–23) and (2–27) form a complete system of equations that characterizes the drop motion. For clarity, we put them together here:

\[
\begin{align*}
\nabla^2 \Phi &= 0 & \text{in } V, \quad (2–28) \\
\frac{D F}{Dt} &= 0 & \text{on } S, \quad (2–29) \\
\frac{D \Phi}{Dt} &= \frac{1}{2} \rho \nabla \Phi \cdot \nabla \Phi - P_0 - \sigma \nabla \cdot n + \rho g z & \text{on } S. \quad (2–30)
\end{align*}
\]

2-4 Conservation laws: Drop

Mass conservation and energy conservation are two non-negotiable laws to obey for all physical systems.

Mass Conservation

Mass conservation for a drop can be expressed as:
\[ \int_S \rho U \cdot n dA = 0. \] (2–31)

Substituting \( U = \nabla \Phi \) into Eq.(2–31) and applying the Gauss' divergence theorem, we have
\[ \rho \int_S \frac{\partial \Phi}{\partial n} dA = \rho \int_S \nabla^2 \Phi dA = 0 \] (2–32)

for an incompressible and irrotational flow.

Energy Conservation

In general, there are three kinds of energy changes involved in the drop oscillation: (1) change of the kinetic energy, \( \Delta K \), (2) change of the surface energy, \( \Delta S \), and (3) the work done by the external pressure, \( \Delta W \).
Change of the kinetic energy

The kinetic energy associated with the fluid flow can be expressed as:

\[ \Delta K = \frac{1}{2} \int_V \rho (U \cdot U) dV. \quad (2-33) \]

Substituting \( U = \nabla \Phi \) into Eq.(2-33), we obtain

\[ \Delta K = \frac{1}{2} \int_V \rho (\nabla \Phi \cdot \nabla \Phi) dV. \quad (2-34) \]

From vector identity, we have

\[ \nabla \cdot (\Phi \nabla \Phi) = \nabla \Phi \cdot \nabla \Phi - \Phi \nabla^2 \Phi. \quad (2-35) \]

Substituting Eq.(2-35) into Eq.(2-34) and applying the divergence theorem, we obtain

\[ \Delta K = \frac{\rho}{2} \int_S \Phi \frac{\partial \Phi}{\partial n} dA. \quad (2-36) \]

As mentioned earlier, \( \Phi \) is determined to a function of time only. This does not change the value of \( \Delta K \) due to the incompressibility condition given in Eq.(2-32).

Change of surface energy, \( \Delta S \)

The change of surface energy of the interface due to the surface tension can be written as

\[ \Delta S = \sigma (A - A_0), \quad (2-37) \]

where \( A_0 \) is the surface area of the drop at equilibrium state.

Work done by the external pressure

The work done by the external pressure is

\[ \Delta W = \int_S P_0 dV. \quad (2-38) \]

If the external pressure is constant, then the work done by the external pressure is 0 due to the incompressibility condition given in Eq.(2-32).

Since there is no energy dissipation mechanism in the current model, the total energy should conserve all the time. Thus we have

\[ \Delta K + \Delta S + \Delta W = 0. \quad (2-39) \]

2-5 Equations of motion: Bubble

We consider an ideal gas bubble with equilibrium radius \( a \) \( (V_e = \frac{4}{3} \pi a^3, V_e: \) the initial volume of the bubble) oscillating in an unbounded quiescent fluid with density \( \rho \). The fluid is assumed to be inviscid and incompressible and the surface tension coefficient \( \sigma \) is constant. Let the interface (denoted by \( S \) as shown in Figure 2–3) of
the bubble in motion at time $t$ be described by $F(R, t) = 0$. We denote the cavity region by $V$ and the fluid region by $V^c$. We will neglect the dynamics of the gas inside the bubble and assume that the gas pressure inside the bubble, $P_g$, is uniform. Because the density of the gas is different from that of liquid by several orders, i.e. $\rho_g \ll \rho$, the negligence of the dynamics of the gas is justified. The assumption that the pressure is uniform in the bubble is justified if the size of the bubble is small compared to the acoustic wavelength in the gas. Thermal coupling between the gas bubble and the surrounding fluid will not be considered here.

![Figure 2-3 Geometry description of the bubble configuration](image)

**Basic state**

Let the undisturbed basic state be an equilibrium bubble sitting in the unbounded fluid with ambient pressure $P_{\infty}$. As a consequence of the existence of surface tension, the gas pressure at this equilibrium state, $P_g^e$, is governed by the Laplace-Young equation:

$$ P_g^e - P_{\infty} = \frac{2\sigma}{a}. \quad (2-40) $$

Though thermal coupling is not considered, we would like to point out that Eq.(2-40) specifies the amount of gas inside the bubble implicitly.

**Irrotational flow**

As usual, we introduce the velocity potential $\Phi$ to express the velocity field $U$ in the fluid ($V^c$) as follows:

$$ U = \nabla \Phi \quad \text{in } V^c. \quad (2-41) $$

The incompressibility of the fluid gives

$$ \nabla \cdot U = \nabla^2 \Phi = 0 \quad \text{in } V^c. \quad (2-42) $$
Note that the region $V^c$ extends to infinity. The kinematic boundary condition on the free surface can be written as

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \nabla \Phi \cdot \nabla F = 0 \quad \text{on } S. \quad (2-43)$$

Eq.(2-43) is used to trace the evolution of the interface.

The dynamic boundary condition (Laplace-Young equation) can be expressed as

$$P_g - P_o = \sigma \nabla \cdot n \quad \text{on } S, \quad (2-44)$$

where $P_o$ is the liquid pressure on the free surface and $n$ the outward unit normal on the free surface pointing into the liquid. The term $\nabla \cdot n$ represents the total curvature of a generic point on the free surface.

The unsteady Bernoulli’s equation of this incompressible, inviscid, irrotational flow can be expressed as follows:

$$\rho \frac{\partial \Phi}{\partial t} + \frac{1}{2} \rho \nabla \Phi \cdot \nabla \Phi + P = h(t) \quad \text{in } V^c, \quad (2-45)$$

where $P$ is the fluid pressure and $h(t)$ is an arbitrary function of time that plays no role in the determination of the fluid flow. Applying Eq.(2-45) at the fluid particle on the interface and using Eq.(2-44), we obtain the following equation

$$\rho \frac{\partial \Phi}{\partial t} + \frac{1}{2} \rho \nabla \Phi \cdot \nabla \Phi + P_g - \sigma \nabla \cdot n = h(t) \quad \text{on } S. \quad (2-46)$$

A complete description of the present boundary value problem includes the boundary conditions at infinity. The physical boundary conditions are no fluid motion at infinity and constant pressure, $P_{\infty}$, at infinity. It follows that

$$\rho \frac{\partial \Phi}{\partial t} \bigg|_{\infty} + P_{\infty} = h(t). \quad (2-47)$$

Since we have the luxury of choosing $h(t)$ arbitrarily, we choose $h(t) = P_{\infty}$ so that $\frac{\partial \Phi}{\partial t} \bigg|_{\infty} = 0$. With this choice of $h(t)$ we have $\Phi|_{\infty} \equiv 0$, if $\Phi|_{\infty} = 0$ initially. Although the choice of $h(t)$ is immaterial to the determination of the fluid flow, it does play an important role in simplifying the formulation of BEM. Substituting $h(t) = P_{\infty}$ into Eq.(2-46), we then obtain

$$\rho \frac{\partial \Phi}{\partial t} + \frac{1}{2} \rho \nabla \Phi \cdot \nabla \Phi + P_g - \sigma \nabla \cdot n = P_{\infty} \quad \text{on } S. \quad (2-48)$$

Rewriting the above equation in terms of the material derivative, we have

$$\rho \frac{D \Phi}{Dt} = \frac{1}{2} \rho \nabla \Phi \cdot \nabla \Phi - P_g + \sigma \nabla \cdot n + P_{\infty} \quad \text{on } S. \quad (2-49)$$

The governing equation Eq.(2-42) together with the boundary conditions Eq.(2-43) and (2-49) form a complete system of equations that characterizes the bubble motion. For clarity, we put them together here:

[BUBBLE]

$$\nabla^2 \Phi = 0 \quad \text{in } V^c, \quad (2-50)$$
\begin{align*}
\frac{DF}{Dt} &= 0 \quad \text{on } S, \quad (2-51) \\
\rho \frac{D\Phi}{Dt} &= \frac{1}{2} \rho \nabla \Phi \cdot \nabla \Phi - P_g + \sigma \nabla \cdot n + P_\infty \quad \text{on } S. \quad (3-52)
\end{align*}

2-6 Conservation laws: Bubble

Unlike the drop case, the volume conservation cannot be easily verified as in the bubble case. However, the energy conservation can be used as another measure of verifying the accuracy of the numerical computation. Four kinds of energy changes are involved in the system: (1) change of the internal energy of the ideal gas inside the bubble, \( \Delta E \), (2) change of the surface energy of the interface, \( \Delta S \), (3) change of the kinetic energy of the fluid flow, \( \Delta K \), and (4) change of the potential energy stored at infinity, \( \Delta P \).

Change of internal energy, \( \Delta E \)

For convenience, adiabatic process is assumed. Hence the pressure \( P_g \) inside the gas bubble is governed by

\[ P_g = P_g^e (V_e/V)^\gamma, \quad (2-53) \]

where \( V \) is the volume of the bubble, and \( \gamma \) the ratio of the two specific heats. (Isothermal process can be used also and the only thing needed to be changed is the constitutive law for the process.) For ideal gas undergoing adiabatic process, we have

\[ P_g V^\gamma = P_g^e V_e^\gamma. \quad (2-54) \]

From the first law of thermodynamics, the change of internal energy of the ideal gas can be written as

\[ \Delta E = \Delta Q - \int_{V_e}^{V} PdV = \frac{P_g^e V_e}{\gamma - 1} \left[ \left( \frac{V}{V_e} \right)^{\gamma-1} - 1 \right], \quad (2-55) \]

where \( \Delta Q = 0 \) due to the adiabatic assumption.

Change of surface energy, \( \Delta S \)

The change of surface energy of the interface due to the surface tension can be written as

\[ \Delta S = \sigma (A - A_0). \quad (2-56) \]

Change of kinetic energy, \( \Delta K \)

The kinetic energy associated with the fluid flow can be expressed as

\[ \Delta K = \frac{1}{2} \int_{V_e} \rho (\nabla \Phi \cdot \nabla \Phi) dV. \quad (2-57) \]
Substituting Eq.(2–35) into Eq.(2–57) and applying the divergence theorem, we obtain

$$\Delta K = \frac{\rho}{2} \left( \int_S \Phi \frac{\partial \Phi}{\partial n} dA + \int_{\Sigma} \Phi \frac{\partial \Phi}{\partial n} dA \right)$$  \hspace{1cm} (2–58)

As mentioned in the previous section, with the choice of $h(t) = P_\infty$, $\Phi \equiv 0$ at infinity and the second term on the right hand side of the above equation is zero. Thus we have

$$\Delta K = \frac{\rho}{2} \int_S \Phi \frac{\partial \Phi}{\partial n} dA$$  \hspace{1cm} (2–59)

**Change of work done by fluid at infinity, $\Delta W$**

The work, $\Delta W$, done by the fluid at infinity during the bubble expansion (collapse) due to the pressure at infinity, $P_\infty$, can be expressed as

$$\Delta W = \int P_\infty dV.$$  \hspace{1cm} (2–60)

If the far field pressure $P_\infty$ is constant, then the work done can be expressed as

$$\Delta W = P_\infty (V - V_e).$$  \hspace{1cm} (2–61)

Since the fluid motion is isentropic and the ideal gas inside the bubble undergoes adiabatic process, no energy dissipation mechanism exists in the system. The total energy should conserve at all times. Thus we have

$$\Delta E + \Delta S + \Delta K + \Delta W = 0$$  \hspace{1cm} (2–62)

### 2-7 Concept of mode shapes

In order to investigate the bubble dynamics, we need a way to describe how the bubble interface changes as time elapses. Ideally, we can trace the interface evolution by deploying the nodes on the interface and following these nodes pointwisely. This can be easily done from the view point of numerics. However, two major obstacles prevent us from doing so. First, we would need as many degrees of freedom as the number of nodes to resolve the interface. Second, the dynamics of each node provides only the local information. It is hard to draw a general picture of the global bubble motion from the nodal data. Fortunately, a mode decomposition concept coming from the conventional analysis of Laplace equation provides a powerful way of capturing the dynamics of the bubble interface roughly with only a few terms. This technique is now standard and termed the *Legendre mode decomposition*.

For both drops and bubbles, the velocity potential is governed by the Laplace equation

$$\nabla^2 \Phi = 0.$$  \hspace{1cm} (2–63)

\(^2\) An interesting discussion of different flows governing by the Laplace equation can be found in Dowling & Fowcs Williams (1983)
As noted in Lamb (1932), the Laplacian operator is homogeneous with respect to \(x, y,\) and \(z.\) Therefore, the homogeneous terms of the same algebraic degree must satisfy the Laplace equation separately. Any such homogeneous solution is called a 'spherical solid harmonic' of the algebraic degree.

Let \(\Phi_n\) be a spherical solid harmonic of degree \(n,\) then if we write

\[ \Phi_n = r^n S_n, \]

\(S_n\) will be a function of \(\theta\) and \(\psi\) only. It is therefore called a 'spherical surface harmonic' of degree \(n.\)

In the spherical coordinate system \((r, \theta, \psi),\) Laplace equation can be written as

\[ \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \Phi}{\partial \psi^2} = 0. \]  

Substituting Eq.(2–64) into Eq.(2–65) and applying the method of separation of variables, we have

\[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial S_n}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 S_n}{\partial \psi^2} = n(n+1)S_n = 0. \]  

Since the product \(n(n + 1)\) is unchanged in value when substituting \(- (n + 1)\) for \(n,\) we have two spherical solid harmonics \(r^n S_n\) and \(r^{-(n+1)} S_n\) corresponding to one spherical surface harmonic \(S_n.\) Further separation of variables gives

\[ S_n = A_0 P_n(\eta) + \sum_{m=1}^{m=n} (A_m \cos m\psi + B_m \sin m\psi) P^m_n(\eta), \]

where \(P_n\) is the Legendre polynomial of order \(n,\) \(P^m_n\) the associated Legendre function, and \(\eta = \cos \theta.\)

For axisymmetric cases, we simply have

\[ \Phi = \sum_{n=0}^{\infty} \left[ A_n(t) r^n + B_n(t) r^{-(n+1)} \right] P_n(\eta), \]

and the shape function can be expressed as

\[ F(R, t) \equiv r - f(\theta, t) = 0. \]

The series expansion in Eq.(2–68) satisfies the Laplace equation automatically and the coefficients \(A_n(t)\) and \(B_n(t)\) are to be determined from the boundary conditions. For exterior problems such as bubbles, \(A_n = 0\) is required so that \(\Phi\) is finite at infinity. While for interior problems such as drops, \(B_n = 0\) is required so that \(\Phi\) is finite when \(r = 0.\) Under the present settings, it is natural to express the oscillatory shape changes in the interface in terms of Legendre polynomials as follows:

\[ f(\theta, t) = a \left[ 1 + \sum_{n=0}^{\infty} a_n(t) P_n(\cos \theta) \right], \]

where \(a_n(t)\) is the amplitude of the \(n\)-th mode.
Although the modal decomposition of the interface based on Legendre polynomials is very helpful in understanding the relevant dynamics, it is not always applicable. There is a serious constraint on this method. Eq.(2–69) assumes that $r$ is a single-valued function of $\theta$. However, this is not always the case, see Figure 2–4, and the modal decomposition method does not apply under this situation. This is particularly true for a non-convex interface containing higher modes with large amplitudes.

### 2-7.1 Zeroth mode: radial motion (Rayleigh-Plesset equation)

The zeroth mode does not change the spherical shape of the interface, instead it changes the volume enclosed by the interface. Hence it is sometimes called the *volume mode* or the *breathing mode*. Since the volume of an incompressible drop is constant, the amplitude of the volume mode is always zero. However, the volume of a bubble may vary with time and this leads to the expanding or collapsing of a spherical bubble.

The volume mode oscillation for a spherical bubble immersed in a surrounding fluid with viscosity $\mu$ is governed by the *Rayleigh-Plesset equation*. It has been extensively explored in the past due to its own importance in scientific research and industrial applications. For completeness, a brief discussion of the Rayleigh-Plesset equation is given below!

Consider a spherical gas bubble with radius $R(t)$ undergoing a time-dependent change of volume within an infinite region of incompressible fluid. Conservation of mass can be expressed as

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \Phi}{\partial r}) = 0.$$  \hspace{1cm} (2–71)

Thus we have

$$\frac{\partial \Phi}{\partial r} = \frac{F(t)}{r^2},$$  \hspace{1cm} (2–72)

where $F(t)$ is determined by the kinematic boundary condition.

For a pure gas bubble, the kinematic boundary condition gives

$$F(t) = R(t)^2 \dot{R}(t).$$  \hspace{1cm} (2–73)
Substitution of Eq.(2-72) into the radial component of the Navier-Stokes equation gives
\[
\frac{1}{r^2} \frac{\partial F}{\partial t} - \frac{2F^2}{r^5} + \frac{1}{\rho} \frac{\partial P}{\partial r} = 0
\]  
(2-74)

Integrating with respect to \( r \), one obtains
\[
\frac{1}{r} \frac{\partial F}{\partial t} - \frac{F^2}{2r^4} + \frac{1}{\rho} [P(r, t) - P_\infty(t)] = 0,
\]  
(2-75)

where \( P_\infty(t) \) is the pressure at infinity. Since the first term decays much slower than the second term, the gauge pressure \( P(r, t) - P_\infty(t) \) behaves like \( r^{-1} \) as \( r \to \infty \).

Evaluating Eq.(2-75) at the bubble surface \( r = R(t) \), one has
\[
R \ddot{R} + \frac{3}{2} \dot{R}^2 + \frac{1}{\rho} [P_\infty(t) - P(R, t)] = 0,
\]  
(2-76)

where \( P(R, t) \) is the fluid pressure at the bubble surface.

The normal stress balance on the bubble surface is governed by the Laplace-Young equation as follows:
\[
P_g(t) = P(R, t) + \frac{2\sigma}{R} - 2\mu \frac{\partial u_r}{\partial r} \bigg|_{r=R},
\]  
(2-77)

where \( P_g(t) \) is the pressure inside the gas bubble.

\[
P_g(t) = P_0 \left[ \frac{V_e}{V(t)} \right] \gamma
\]  
(2-78)

Combining Eq.(2-76),(2-77), and (2-78), we obtain
\[
R \ddot{R} + \frac{3}{2} \dot{R}^2 + \frac{1}{\rho} \left\{ P_\infty(t) - P_0 \left[ \frac{V_e}{V(t)} \right] \gamma + \frac{2\sigma}{R} + \frac{4\mu R}{R} \right\} = 0.
\]  
(2-79)

Usually, the Rayleigh-Plesset equation can not be integrated analytically and a numerical integration is required.

Consider the case that \( P_\infty \) is a constant, and \( \mu = 0 \).
\[
\ddot{R} + \frac{3}{2} \dot{R}^2 + \frac{1}{\rho R} \left\{ P_\infty - P_0 \left[ \frac{V_e}{V(t)} \right] \gamma + \frac{2\sigma}{R} \right\} = 0.
\]  
(2-80)

The equilibrium radius \( a \) is given by
\[
P_\infty - P_0 \left[ \frac{V_e}{V(t)} \right] \gamma + \frac{2\sigma}{R} = 0.
\]  
(2-81)

Linearization of the above equation about its equilibrium radius \( a \) gives:
\[
\dot{r} + \Omega_0^2 r = 0,
\]  
(2-82)

where \( R \equiv a + r \) and
\[
\Omega_0^2 \equiv \left[ 3\gamma \left( \frac{P_\infty}{\sigma/a} + 2 \right) - 2 \right] \frac{\sigma}{\rho a^3}.
\]  
(2-83)

From Eq.(2-40), we can rewrite the above equation as
\[
\Omega_0^2 = \frac{1}{\rho a^2} \left[ 3\gamma \frac{P_g^e}{a} - \frac{2\sigma}{a} \right].
\]  
(2-84)
Obviously, the equilibrium radius \( a \) is stable if
\[
P_{g}^{\ast} > \frac{2\sigma}{3\gamma a},
\]
and unstable if
\[
P_{g}^{\ast} < \frac{2\sigma}{3\gamma a}.
\]
For a given mass \( m \) of gas bubble at temperature \( T \), there exists a critical radius, \( a_{cr} \), where
\[
P_{g}^{\ast} = \frac{mR_{G}T}{\frac{4\pi}{3}a_{cr}^{3}} = \frac{2\sigma}{3\gamma a_{cr}}.
\]
\( R_{G} \) is the ideal gas constant. Consequently, we obtain
\[
a_{cr} = \sqrt{\frac{9\gamma mR_{G}T}{8\pi\sigma}}.
\]
Thus the bubble is stable if \( a < a_{cr} \) and unstable if \( a > a_{cr} \). This critical radius was first identified by Blake (1949) and is referred as the Blake critical radius.

2-7.2 First mode: translational motion

The first mode is related to the translational motion of a rigid sphere. If we take the origin at the center of the sphere, and the \( z \) axis the direction of the motion, the normal velocity at the interface is \( U\cos\theta \), where \( U \) is the velocity of the sphere. The velocity potential corresponding to this motion can be easily determined as follows:

For a drop,
\[
\frac{\partial \Phi}{\partial r} = U\cos\theta.
\]

Hence we obtain
\[
\Phi = U\frac{r}{a}\cos\theta.
\]

For a bubble,
\[
-\frac{\partial \Phi}{\partial r} = U\cos\theta.
\]

Hence we have
\[
\Phi = \frac{1}{2}U\frac{a^{3}}{r^{2}}\cos\theta.
\]

In both cases, the translational motion of a rigid sphere is described by the velocity potential of the first mode.

2-7.3 Shape modes: shape oscillation

All other modes \((n \geq 2)\) are termed the shape modes. The shape modes drive the interface into non-spherical shape oscillation without changing the volume enclosed. Figure 2-5 demonstrates the mode shapes obtained from the superposition upon a unit sphere of a single \( n \)-th shape mode of amplitude \( \pm 0.3 \) \((n = 2, 3, 4, 5, 6, 7)\).
Linear analysis of constant volume shape oscillation

Consider an incompressible liquid drop with density \( \rho \) oscillating in an infinite region of another incompressible fluid with density \( \rho' \) under 0g environment. A linearized formulation can be written down as follows:

Inside the drop we have

\[
\nabla^2 \Phi = 0 \quad \text{in } V, \\
\frac{\partial f}{\partial t} = \frac{\partial \Phi}{\partial r} \quad \text{at } r = a, \\
\rho \frac{\partial \Phi}{\partial t} = P_i^e - P_i \quad \text{at } r = a,
\]

and in the surrounding fluid we have

\[
\nabla^2 \Phi' = 0 \quad \text{in } V', \\
\frac{\partial f}{\partial t} = \frac{\partial \Phi'}{\partial r} \quad \text{at } r = a, \\
\rho' \frac{\partial \Phi'}{\partial t} = P_\infty - P_o \quad \text{at } r = a,
\]

where \( \rho \) is the density of the drop, \( \rho' \) the density of the surrounding fluid, \( \Phi \) the velocity potential of the drop, and \( \Phi' \) the velocity potential of the surrounding fluid.

On the interface, the normal stress balance gives

\[
P_i - P_o = \sigma \nabla \cdot n.
\]

While at equilibrium state, we have

\[
P_i^e - P_\infty = \frac{2\sigma}{a},
\]

where \( P_i^e \) is the equilibrium pressure of the drop.
Let us consider the pure \( n \)-th mode shape oscillation \((n \geq 2)\). Accordingly, we assume

\[
\Phi = A_n(t) r^n P_n(\cos \theta), \quad (2-101)
\]
\[
\Phi' = B_n(t) r^{-(n+1)} P_n(\cos \theta), \quad (2-102)
\]
\[
f = a [1 + a_n(t) P_n(\cos \theta)]. \quad (2-103)
\]

After some simple algebra, we obtain\(^3\)

\[
\nabla \cdot n \sim \frac{2}{a} + \frac{(n-1)(n+2)}{a} a_n P_n(\cos \theta). \quad (2-104)
\]

and

\[
\ddot{a}_n + \Omega_n^2 a_n = 0, \quad (2-105)
\]

where

\[
\Omega_n^2 = \frac{n(n-1)(n+1)(n+2) \sigma}{[(n + 1)\rho + n\rho'] \frac{1}{a^3}} \quad n \geq 2. \quad (2-106)
\]

In order to obtain the \( n \)-th mode shape oscillation frequency for a liquid drop in vacuum, we set \( \rho' = 0 \) and get the \textit{Rayleigh's frequency} for a drop:

\[
\Omega_n^2 = n(n-1)(n+2) \frac{\sigma}{\rho a^3} \quad n \geq 2. \quad (2-107)
\]

And the constant volume \( n \)-th mode shape oscillation frequency for a bubble is

\[
\Omega_n^2 = (n-1)(n+1)(n+2) \frac{\sigma}{\rho a^3} \quad n \geq 2. \quad (2-108)
\]

### 2-8 Non-dimensionalization

Physical laws do not depend on arbitrarily chosen basic units of measurement. Dimensional analysis reveals the essential nature of the physical problem under consideration, which would otherwise be very disillusive. In order to simplify the problem, we choose \( a \) as the characteristic length, \( \sigma / a \) as the characteristic pressure, and \( \sqrt{\rho a^3 / \sigma} \) as the characteristic time.

The continuity equation in the dimensionless form can be written as

\[
\nabla^2 \phi = 0. \quad (2-109)
\]

Kinematic boundary condition on the interface is

\[
\frac{\partial f}{\partial t} + \nabla \phi \cdot \nabla f = 0 \quad \text{on } r = f(\theta, t). \quad (2-110)
\]

The dynamical boundary condition for a drop can be expressed as

\[
\frac{D\phi}{Dt} = \frac{1}{2} \nabla \phi \cdot \nabla \phi - p_0 - \nabla \cdot n + Bz \quad \text{on } r = f(\theta, t), \quad (2-111)
\]

---

\(^3\) See the derivation of Eq.(4-26).
where $B \equiv \frac{\rho g a^2}{\sigma}$ is the bond number indicating the relative importance of the gravity with respect to surface tension effect. The dimensionless linear angular frequency for the $n$-th mode for a drop is given by

$$\omega_n^2 = n(n-1)(n+2) \quad n \geq 2. \quad (2-112)$$

The dynamical boundary condition for a bubble is

$$\frac{D\phi}{Dt} = \frac{1}{2} \nabla \phi \cdot \nabla \phi - p_g + \nabla \cdot n + p_\infty \quad \text{on} \ r = f(\theta, t). \quad (2-113)$$

The dimensionless linear angular frequency for the volume mode for a bubble is given by

$$\omega_0^2 = 3\gamma(p_\infty + 2) - 2, \quad (2-114)$$

and that for the $n$-th mode is given by

$$\omega_n^2 = (n-1)(n+1)(n+2) \quad n \geq 2. \quad (2-115)$$

Interestingly, with the chosen characteristic quantities, the dimensionless angular frequencies for the shape modes are functions of the mode number, $n$, only. While the dimensionless angular frequency for the volume mode depends on the dimensionless far field pressure, $p_\infty$.

REFERENCES


Chapter 3

Acoustic Radiation

3-1 Introduction

One of the major interest of the current thesis is to study the dynamics of acoustically levitated drops. In this chapter, the mathematical model for the acoustic radiation and the calculation of the radiation pressure will be introduced with great details. As is well known that an object placed in a sonically irradiated medium is subjected to a steady force resulting from the scattering of the acoustic wave. Acoustic standing waves have long been used to position and levitate small drops under both microgravity and 1g environments. Small specimen up to centimeters in size is levitated in either 'interference' type\(^1\) or 'resonance' type\(^2\) levitators with frequency ranging from 1 to 100kHz, see Barmatz (1982). The sound pressure levels used are so high (155-175 dB re: \(2 \times 10^{-4}\) dyne/cm\(^2\)) that nonlinear acoustic phenomena operate. The advantage of acoustic levitation over the other methods is that any material, conductor or insulator, magnetic or non-magnetic, may be levitated.

To date, acoustic levitation plays important roles in several industrial applications such as:

**Containerless material processing** Harmful contamination associated with containers is lethal to certain manufacturing processes of materials such as glass and ceramics. Avoidance of a container is thus essential to these sorts of material processing.

**Surface tension measurement** Measurement of the surface tension of high temperature molten melts poises serious difficulty in suspending the melts at precise position without direct physical contact. Acoustic levitation serves as an alternative.

---

\(^1\) Object is trapped at the distance about \(\lambda/4\) below the reflector of the levitator.

\(^2\) The lateral dimensions of the levitator are of the same order of the acoustic wavelength.
Meteorology In order to study the formation of clouds, the initiation of precipitation, and the formation of ice, acoustic levitator is a must-have in the laboratory, Lupi (1990).

Acoustic radiation is the most essential part of acoustic levitation. In the audible range of frequencies, it is possible to measure the radiation pressure amplitude in sound field physically. At ultrasonic frequencies, however, physical measurement of the radiation pressure is very difficult and almost impossible. Especially when the dimensions of resonators become comparable with the wavelength, diffraction problem makes physical measurement extremely difficult. Thus a systematic way of calculating the radiation pressure based on the hydrodynamics is substantial to the acoustic levitation. In order to calculate the radiation pressure acting on the surface of a scatterer, analysis of the scattering problem is inevitable. Study of scattering problem requires some knowledge in the wave mechanics. Therefore, in this chapter we will derive the pertinent wave equation based on a hydrodynamic model and solve the wave equation subjected to certain simple geometric boundary conditions analytically.

King (1935) first derived a formula for calculating the radiation pressure on a rigid sphere based on the hydrodynamic equation taking account of the second order effect of the sound pressure. Yosioka & Kawasima (1955) further modified the method by including the compressibility of the sphere. Hasegawa & Yosioka (1969), Hasegawa (1977), Hasegawa (1979) extended the same method by considering the elastic vibration of the solid sphere. Westervelt (1957) approached the rigid sphere problem from the linear momentum principle and obtained a simpler formula for the radiation pressure. Later, Maidanik & Westervelt (1957) applied the same approach to calculate the force acting on the fluid sphere. Alekseev (1983) derived a more general formula to calculate the radiation pressure on a sphere subjected to the incident wave field of arbitrary form.

The structure of this chapter is as follows: Section 3-2 introduces the hydrodynamic model proposed by King (1935), the wave equations, and the calculation of radiation pressure based on the velocity potential approach. Section 3-3 derives the reduced wave equation in the steady state, i.e. Helmholtz equation. Section 3-4 focuses on the analytic solutions of Helmholtz equation in the spherical polar coordinates. Section 3-5 presents the analysis of the scattering of a planar velocity potential incident on a compressible sphere. Section 3-6 gives the concept of time-averaged radiation pressure that is particularly useful in the acoustic levitation model.
3-2 Acoustic radiation pressure in compressible fluid

In this section, a brief theoretical derivation of the radiation pressure in an inviscid compressible fluid based on the excellent paper by King (1935) is introduced.

In general, Euler's equations of motion governing the *inviscid compressible* fluid flow in the absence of body force is

$$
\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x}, \quad \rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y}, \quad \rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z},
$$

(3-1)

where $\frac{D}{Dt}$ stands for the material derivative and can be expressed as

$$
\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}.
$$

(3-2)

Let's assume the fluid to be *barotropic*, i.e.

$$
p = f(\rho),
$$

(3-3)

and introduce $\varphi$ defined by

$$
\varphi(p) = \int \frac{dp}{\rho}.
$$

(3-4)

By chain rule, we have

$$
\frac{\partial \varphi}{\partial x} = \frac{1}{\rho} \frac{\partial \rho}{\partial x}, \quad \frac{\partial \varphi}{\partial y} = \frac{1}{\rho} \frac{\partial \rho}{\partial y}, \quad \frac{\partial \varphi}{\partial z} = \frac{1}{\rho} \frac{\partial \rho}{\partial z}.
$$

(3-5)

Substitute Eq.(3-5) into (3-1) and we obtain

$$
\frac{Du}{Dt} = -\frac{\partial \varphi}{\partial x}, \quad \frac{Dv}{Dt} = -\frac{\partial \varphi}{\partial y}, \quad \frac{Dw}{Dt} = -\frac{\partial \varphi}{\partial z}.
$$

(3-6)

If the flow is *irrotational*, then the velocity field $U = (u, v, w)$ can be determined by a scalar velocity potential $\Phi$ as follows

$$
u = \frac{\partial \Phi}{\partial x}, \quad v = \frac{\partial \Phi}{\partial y}, \quad w = \frac{\partial \Phi}{\partial z}.
$$

(3-7)

Then by exchanging the order of differentiation, we have

$$
\frac{Du}{Dt} = \frac{\partial}{\partial x} \left[ \frac{\partial \Phi}{\partial t} + \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial \Phi}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial \Phi}{\partial z} \right)^2 \right],
$$

$$
\frac{Dv}{Dt} = \frac{\partial}{\partial y} \left[ \frac{\partial \Phi}{\partial t} + \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial \Phi}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial \Phi}{\partial z} \right)^2 \right],
$$

(3-8)

$$
\frac{Dw}{Dt} = \frac{\partial}{\partial z} \left[ \frac{\partial \Phi}{\partial t} + \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial \Phi}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial \Phi}{\partial z} \right)^2 \right].
$$

Substituting Eq.(3-8) into Eq.(3-6), we have

$$
\varphi = \int \frac{dp}{\rho} = -\left[ \frac{\partial \Phi}{\partial t} + \frac{1}{2} (u^2 + v^2 + w^2) \right] + C_1 = -\left[ \frac{\partial \Phi}{\partial t} + \frac{1}{2} q^2 \right] + C_1,
$$

(3-9)

---

3 There is a sign difference in defining the velocity potential though. The notation used in this thesis is $U = \nabla \Phi$, instead of $U = -\nabla \Phi$. 

where \( q \equiv (u^2 + v^2 + w^2) \) and \( C_1 \) is an integration constant.

Continuity equation gives

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) = 0. \tag{3-10}
\]

\[
\frac{1}{\rho} \frac{D\rho}{Dt} = \frac{D}{Dt} \ln \rho = -\nabla^2 \Phi. \tag{3-11}
\]

For a fluid in which \( dp/d\rho = c^2 \), a constant, the quantity \( \varphi \) can be rewritten as

\[
\varphi = \int \frac{c^2 d\rho}{\rho} = c^2 \ln \rho + C_2, \tag{3-12}
\]

where \( C_2 \) is an integration constant.

Combining Eq.(3-9), (3-11), and (3-12), we get

\[
\frac{1}{c^2} \frac{D^2 \Phi}{Dt^2} - \frac{1}{2c^2} \frac{D}{Dt} \left[ \left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial y} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 \right] = \nabla^2 \Phi. \tag{3-13}
\]

Eq.(3-13) is the exact equation governing the velocity potential for the irrotational flow of an inviscid compressible fluid with constant sound speed. In acoustic problem, the term \( q^2/c^2 \ll 1 \) is usually neglected when calculating the pressure (This will be explained later!) and the approximate equation can be reduced to the wave equation as follows:

\[
\nabla^2 \Phi = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} \tag{3-14}
\]

Since for compressible fluid pressure is directly related to density and density variation is usually very small for liquid, i.e. the condensation \( s = (\rho - \rho_0)/\rho_0 \ll 1 \), we can approximate the pressure by the Taylor expansion of Eq.(3-3) as follows:

\[
p = f(\rho_0 + s\rho_0) = f(\rho_0) + f'(\rho_0)s\rho_0 + \frac{1}{2}f''(\rho_0)s^2\rho_0^2 + \cdots. \tag{3-15}
\]

Hence

\[
dp = \rho_0(f' + f''s\rho_0 + \cdots)ds,
\]

and by definition of \( s \) we have

\[
\frac{1}{\rho} = \frac{1}{\rho_0}(1 - s + s^2 - \cdots). \tag{3-17}
\]

Thus,

\[
\varphi = \int \frac{dp}{\rho} = f's + \frac{1}{2}(\rho_0f'' - f')s^2 + \cdots. \tag{3-18}
\]

Solving Eq.(3-18) for \( s \) in terms of \( \varphi \) we obtain

\[
s \sim \frac{\varphi}{f'} - \frac{1}{2} \frac{(\rho_0f'' - f')}{f'} \varphi^2 + \cdots. \tag{3-19}
\]

Substituting the above equation into Eq.(3-15) we have

\[
p - p_0 = \rho_0f' \left\{ \frac{\varphi}{f'} - \frac{1}{2} \frac{(\rho_0f'' - f')}{f'} \varphi^2 \right\} + \frac{1}{2} \rho_0^2 \frac{f''}{f'^2} \varphi^2 + \cdots, \tag{3-20}
\]
or,

\[ p - p_0 = \rho_0 \varphi + \frac{1}{2} \frac{\rho_0}{c^2} \varphi^2 + \cdots, \]  

(3-21)

where we have used the fact that \( f'(\rho_0) = c^2 \). Substitution of Eq.(3-9) into the above equation leads to

\[ p - p_0 = -\rho_0 \frac{\partial \Phi}{\partial t} + \frac{1}{2} \frac{\rho_0}{c^2} \left( \frac{\partial \Phi}{\partial t} \right)^2 - \frac{1}{2} \rho_0 \varphi^2. \]  

(3-22)

Note that Eq.(3-22) is correct to the terms of the order \( q^2 (\gg q^2/c^2) \), this justifies the validity of using the approximate equation Eq.(3-14) to calculate the radiation pressure.

### 3-3 Steady state and the Helmholtz equation

In general the transient behavior of a system subject to a time-harmonic input dies out very quickly (usually within several cycles), and the steady state response of the system synchronizes with the input frequency but with a certain phase angle. Steady state response is much more desired than the transient response since it characterizes the physical nature of the system yet avoids the mathematical complexity introduced by the time dependence. This approach of analyzing the steady state response subject to time-harmonic input is called “frequency domain analysis”.

In frequency domain analysis it is conventional and convenient to express a time-harmonic quantity \( \Phi(t) \) in the following form:

\[ \Phi(t) = a \cos(\omega t + \alpha) = \text{Re}\{Ae^{i\omega t}\}, \]  

(3-23)

where \( a \) is the real amplitude of \( \Phi(t) \), \( \omega \) the angular frequency, \( \alpha \) the phase angle, and \( A(t) \) the complex amplitude of \( \Phi(t) \). Complex amplitude \( A \) contains the information of both amplitude and phase angle. Obviously,

\[ \text{Re}\{A\} = a \cos \alpha \quad \text{Im}\{A\} = a \sin \alpha, \]  

(3-24)

and

\[ a = |A|. \]  

(3-25)

Here \( \text{Re}\{\cdot\} \) and \( \text{Im}\{\cdot\} \) stands for taking the real and imaginary parts of a complex number respectively, and \( |\cdot| \) denotes taking the modulus of a complex number.

In order to seek the steady state solution of the wave equation Eq.(3-14), \( \Phi \) is assumed to be

\[ \Phi(r, t) = \text{Re}\{\Psi(r)e^{i\omega t}\}, \]  

(3-26)

where \( \Psi(r) \) is the spatial factor of \( \Phi(r, t) \). After substituting Eq.(3-26) into Eq.(3-14), the wave equation is converted into the Helmholtz equation:

\[ \nabla^2 \Psi + k^2 \Psi = 0, \]  

(3-27)
where
\[ k = \frac{\omega}{c} \quad (3-28) \]
is the wave number of the acoustic wave corresponding to the frequency \( \omega \).

Note the Helmholtz equation is an elliptic type PDE instead of a hyperbolic type PDE. Thus the problem to be solved is a boundary value problem instead of an initial-boundary value problem, which is a great simplification to the original problem.

### 3-4 General solutions of the Helmholtz equation

In spherical polar coordinates, the Helmholtz equation can be expressed as
\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Psi}{\partial \psi^2} + k^2 \Psi = 0. \quad (3-29) \]

By the method of separation of variables, we try
\[ \Psi(r, \theta, \psi) = R(r) \Theta(\theta) G(\psi). \quad (3-30) \]

Substituting Eq.(3-30) into Eq.(3-29) and dividing by \( R \Theta G \), we have
\[ \frac{d^2 G}{d \psi^2} + \beta G = 0, \quad (3-31) \]
\[ \frac{1}{\sin \theta} \frac{d}{d \theta} \left( \sin \theta \frac{d \Theta}{d \theta} \right) + \left( \alpha - \frac{\beta}{\sin^2 \theta} \right) \Theta = 0, \quad (3-32) \]
\[ r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} + (k^2 r^2 - \alpha) R = 0. \quad (3-33) \]

From Eq.(3-31), we have \( G(\psi) = e^{\pm i \sqrt{\beta} \psi} \). Since \( G(\psi) \) is periodic with period \( 2\pi \) physically, \( \beta = m^2 \) and \( m \) is an integer.

Eq.(3-32) can be expressed as
\[ \frac{1}{\sin \theta} \frac{d}{d \theta} \left( \sin \theta \frac{d \Theta}{d \theta} \right) + \left( \alpha - \frac{m^2}{\sin^2 \theta} \right) \Theta = 0. \quad (3-34) \]

It can be easily shown that for the solutions of the above equation to converge, \( \alpha = n(n + 1) \), where \( n \) is an integer. This leads to the associated Legendre equation and its solutions are called the associated Legendre functions, \( P_n^m(\cos \theta) \).

Eq.(3-33) can be written as
\[ r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} + [k^2 r^2 - n(n + 1)] R = 0. \quad (3-35) \]

Eq.(3-35) is not a Bessel's equation. But if we substitute
\[ R(kr) = \frac{Z(kr)}{(kr)^{1/2}}, \quad (3-36) \]
Eq. (3-35) becomes
\[ r^2 \frac{d^2 Z}{dr^2} + r \frac{dZ}{dr} + \left[ k^2 r^2 - \left( n + \frac{1}{2} \right)^2 \right] Z = 0, \] (3-37)

which is Bessel's equation of order \( n + \frac{1}{2} \). The solutions for \( Z(kr) \) is a linear combination of Bessel's functions of order \( n + \frac{1}{2} \), \( J_{n+1/2}(kr) \) and \( N_{n+1/2}(kr) \):
\[ Z(kr) = a_n J_{n+1/2}(kr) + b_n N_{n+1/2}(kr), \] (3-38)

or it can be written as a linear combination of the Hankel functions\(^4\) as follows
\[ Z(kr) = c_n H^{(1)}_{n+1/2}(kr) + d_n H^{(2)}_{n+1/2}(kr). \] (3-39)

For convenience, the spherical Bessel functions are introduced as follows:
\[ j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x), \] (3-40)
\[ n_n(x) = \sqrt{\frac{\pi}{2x}} N_{n+1/2}(x), \] (3-41)
\[ h^{(1)}_n(x) = \sqrt{\frac{\pi}{2x}} H^{(1)}_{n+1/2}(x) = j_n(x) + in_n(x), \] (3-42)
\[ h^{(2)}_n(x) = \sqrt{\frac{\pi}{2x}} H^{(2)}_{n+1/2}(x) = j_n(x) - in_n(x), \] (3-43)

which have asymptotic properties:
\[ x \to 0 \quad \begin{cases} j_n(x) \sim [(2n+1)!!]^{-1} x^n, \\ n_n(x) \sim -[(2n-1)!!] x^{-(n+1)}, \end{cases} \] (3-44)

and
\[ x \to \infty \quad \begin{cases} j_n(x) \sim x^{-1} \sin \left( x - \frac{n \pi}{2} \right), \\ n_n(x) \sim -x^{-1} \cos \left( x - \frac{n \pi}{2} \right), \\ h^{(1)}_n(x) \sim -ix^{-1} e^{i(x-n \pi/2)}, \\ h^{(2)}_n(x) \sim ix^{-1} e^{-i(x-n \pi/2)}, \end{cases} \] (3-45)

where
\[ (2n+1)!! \equiv 1 \cdot 3 \cdot 5 \cdots (2n+1), \quad (2n)!! \equiv 2 \cdot 4 \cdot 6 \cdots (2n). \]

Thus the most general solution can be written as
\[ \Psi(r, \theta, \psi) = \sum_{n=0}^{\infty} \left[ a_n j_n(kr) + b_n n_n(kr) \right] \sum_{m=-n}^{m=n} P^m_n(\cos \theta)e^{im\psi}, \] (3-46)

\(^4\) The Hankel functions are defined as
\[ H^{(1)}_\nu(x) = J_\nu(x) + iN_\nu(x), \]
\[ H^{(2)}_\nu(x) = J_\nu(x) - iN_\nu(x). \]
\[ \Psi(r, \theta, \psi) = \sum_{n=0}^{\infty} \left[ c_n h_n^{(1)}(kr) + d_n h_n^{(2)}(kr) \right] \sum_{m=-n}^{m=n} P_n^m(\cos \theta)e^{im\psi}. \quad (3-47) \]

From the asymptotic behavior we realize that \( j_n(x) \) and \( n_n(x) \) are appropriate for the description of standing spherical waves; \( h_n^{(1)}(x) \) and \( h_n^{(2)}(x) \) are suitable for that of traveling waves. One important difference between \( j_n(x) \) and \( n_n(x) \) is \( j_n(x) \) is regular for all \( x \), while \( n_n(x) \) is singular at \( x = 0 \). Hence if there is no source point at \( x = 0 \), the coefficient of \( n_n(x) \) must vanish. With the choice of time dependence \( e^{i\omega t} \), \( h_n^{(1)}(x) \) gives an incoming traveling spherical wave while \( h_n^{(2)}(x) \) an outgoing wave.

For axisymmetric case, the most general solution reduces to

\[ \Psi(r, \theta, \psi) = \sum_{n=0}^{\infty} [a_n j_n(kr) + b_n n_n(kr)] P_n(\cos \theta), \quad (3-48) \]

or

\[ \Psi(r, \theta, \psi) = \sum_{n=0}^{\infty} \left[ c_n h_n^{(1)}(kr) + d_n h_n^{(2)}(kr) \right] P_n(\cos \theta). \quad (3-49) \]

### 3-5 Scattering of a planar wave by a compressible sphere

Consider a compressible sphere of radius \( a \) whose center is at distance \( h \) above the velocity potential anti-node of a planar standing (progressive) wave. Let \( z \)-axis be in the direction of the wave propagation of the standing (progressive) wave. See Figure 3-1.

![Figure 3-1 Geometric configuration of scattering problem](image-url)
Then the incident velocity potential can be written as

\[ \Phi_i = \begin{cases} \frac{1}{2} A e^{i\omega t} \left[ e^{-ik(z+h)} + e^{ik(z+h)} \right] & \text{for standing wave}, \\ Re \left\{ A e^{i\omega t} e^{-ik(z+h)} \right\} & \text{for progressive wave}, \end{cases} \]  

(3.50)

where \( k \) is the wavenumber of the incident wave outside the scatterer. It has been shown that

\[ e^{-ikz} = e^{-ikr \cos \theta} = \sum_{n=0}^{\infty} (2n+1)(-i)^n j_n(kr) P_n(\cos \theta). \]  

(3.51)

Therefore, Eq.(3.50) becomes

\[ \Phi_i = e^{i\omega t} \sum_{n=0}^{\infty} \delta_n j_n(kr) P_n(\cos \theta), \]  

(3.52)

where

\[ \delta_n = \begin{cases} \frac{1}{2} A (-i)^n (2n+1) \left[ e^{-ikh} + (-1)^n e^{ikh} \right] & \text{for standing wave}, \\ A(-i)^n (2n+1) e^{-ikh} & \text{for progressive wave}. \end{cases} \]  

(3.53)

On the other hand, the scattered velocity potential can be expanded as

\[ \Phi_s = e^{i\omega t} \sum_{n=0}^{\infty} A_n \delta_n h_n^{(2)}(kr) P_n(\cos \theta). \]  

(3.54)

The total velocity potential outside the sphere is given by

\[ \Phi = \Phi_i + \Phi_s. \]  

(3.55)

Similarly, the transmitted velocity potential inside the sphere can be written as

\[ \Phi^* = e^{i\omega t} \sum_{n=0}^{\infty} B_n \delta_n j_n(k^*r) P_n(\cos \theta), \]  

(3.56)

where \( k^* \) is the wavenumber of the velocity potential inside the sphere.

The coefficients \( A_n \) and \( B_n \) can be determined from the boundary conditions on the surface of the sphere:

\[ \frac{\partial \Phi}{\partial r} = \frac{\partial \Phi^*}{\partial r} \quad \text{at} \quad r = a, \]  

(3.57)

and

\[ \rho \frac{\dot{\phi}}{\partial r} = \rho^* \frac{\dot{\phi}^*}{\partial r} \quad \text{at} \quad r = a. \]  

(3.58)

Where \( \rho \) is the density of the fluid outside the sphere, and \( \rho^* \) is the density of the fluid inside the sphere.

Hence

\[ A_n = \frac{\lambda (ka) j_{n}(k^*a) j_n^{(i)}(ka) - (k^*a) j_n^{(i)}(k^*a) j_n(ka)}{(k^*a) j_{n}^{(i)}(k^*a) h_n^{(2)}(ka) - \lambda (ka) j_n(ka) h_n^{(2)i}(ka)}, \]  

(3.59)

and

\[ B_n = \frac{i}{(ka) \left( (k^*a) j_{n}^{(i)}(k^*a) h_n^{(2)}(ka) - \lambda (ka) j_n(ka) h_n^{(2)i}(ka) \right)}, \]  

(3.60)
where $\lambda = \frac{\rho}{\rho}$. Here the following identity\(^5\) is used to simplify Eq.(3-60):

$$j'_n(x)h''_n(x) - j''_n(x)h'_n(x) = -\frac{i}{x^2}. \quad (3-61)$$

Consider the special case of a rigid sphere, i.e. $k^*a \to 0$, then

$$A_n \to \frac{j'_n(ka)}{h''_n(ka)}. \quad (3-62)$$

Therefore, the scattered velocity potential is

$$\Phi_s = e^{i\omega t} \sum_{n=0}^{\infty} \frac{j'_n(ka)}{h''_n(ka)} \delta_n h^{(2)}(kr) P_n(\cos \theta). \quad (3-63)$$

Consider a rigid sphere of dimensionless wavenumber $ka = 2.0$ which is located at the anti-node ($kh = 0.0$) of the incident velocity potential of a standing wave. The value of $|\Phi_s|/|\Phi_i|$ at a distance of $r = 5a$ based on Eq.(3-63) (Substitute $\delta_n$ with $\frac{1}{2}A(-i)^n(2n + 1) \left[ e^{-ikh} + (-1)^n e^{ikh} \right]$) versus polar angle is plotted in Figure 3-2. The radial length from the origin is proportional to the value of $|\Phi_s|/|\Phi_i|$ calculated.

![Figure 3-2](image)

**Figure 3-2** $|\Phi_s|/|\Phi_i|$ vs. polar angle for a rigid sphere with $ka = 2.0$, $kh = 0.0$ and $r = 5a$ in a standing velocity potential.

Similarly, consider a rigid sphere of dimensionless wavenumber $ka = 2.0$ which is located at the anti-node ($kh = 0.0$) of the incident velocity potential of a progressive

\(^5\) It is well known in the theory of ordinary differential equation that if both $y_1(x)$ and $y_2(x)$ satisfy the differential equation

$$\frac{d}{dx} \left[ p(x) \frac{dy}{dx} \right] + q(x)y(x) = 0,$$

then the Wronskian of $y_1(x)$ and $y_2(x)$ gives

$$W(y_1(x), y_2(x)) = y_1(x)y'_2(x) - y'_1(x)y_2(x) = -\frac{C}{p(x)},$$

where $C$ is an integration constant.

In the present case, the constant $C$ can be determined by the asymptotic behaviors given in Eq.(3-44) and (3-45).
wave. The value of $|\Phi_s|/|\Phi_i|$ at a distance of $r = 5a$ based on Eq (3-63) (Substitute $\delta_n$ with $A(-i)^n(2n + 1)e^{-ikh}$) versus polar angle is plotted in Figure 3-3. The radial length from the origin is proportional to the value of $|\Phi_s|/|\Phi_i|$ calculated.

![Figure 3-3](image)

Figure 3-3 $|\Phi_s|/|\Phi_i|$ vs. polar angle for a rigid sphere with $ka = 2.0$, $kh = 0.0$ and $r = 5a$ in a progressive velocity potential.

### 3-6 Time-averaged radiation pressure

The frequencies of the acoustic waves used for acoustic levitation are usually beyond 23 kHz to avoid the uncomfortable noise to human hearing. The natural frequencies for drop motion is found to be

$$\omega_n^2 = n(n - 1)(n + 2) \frac{\sigma}{\rho a^3}.$$  \hfill (3-64)

For a small drop in water with radius 0.1 cm at room temperature, the density of water is about 1 g/cm$^3$ and the surface tension coefficient is around 72 dynes/cm. The second mode shape oscillation natural frequency is only 121 Hz. Even for the tenth mode shape oscillation, the natural frequency is only 1403 Hz, which is much slower than the acoustic wave frequency. Therefore the drop motion can not respond to such a high frequency forcing; we can neglect the fast time scale effect of the acoustic wave. However, we have to consider the long term steady contribution of the acoustic wave to the drop motion. To account for the steady contribution of the acoustic wave, a time-averaged radiation pressure $\bar{p}$ is calculated as follows:

$$\bar{p} = \frac{1}{T} \int_0^T (p - p_0) dt.$$  \hfill (3-65)

To calculate the radiation pressure on a sphere, write the total velocity potential $\phi$ corresponding to the scattering of a unit amplitude planar incident wave with velocity potential

$$\phi_i = \begin{cases} \frac{1}{2} e^{i\omega t} \left[ e^{-ik(z+h)} + e^{ik(z+h)} \right] & \text{for standing wave} \\ e^{i\omega t} e^{-ik(z+h)} & \text{for progressive wave} \end{cases}$$
by a scatterer as follows:

$$\phi = e^{i\omega t} \tilde{\phi},$$  \hspace{1cm} (3-66)

where $\omega$ is the angular frequency of the acoustic wave, $k$ the wavenumber corresponding to frequency $\omega$, and asterisk stands for taking the complex conjugates of the corresponding variables.

Since the Helmholtz equation is linear, the total velocity potential $\Phi$ corresponding to the scattering of a planar incident wave with amplitude $A$ by the same scatterer can be expressed as

$$\Phi = Ae^{i\omega t} \tilde{\phi}.$$  \hspace{1cm} (3-67)

Substituting Eq.(3-67) and Eq.(3-22) into Eq.(3-65), we have

$$\tilde{p} = \frac{\rho_0 (kA)^2}{4} [\|	ilde{\phi}\|^2 - \frac{1}{(ka)^2} (|\tilde{\phi}_n|^2 + |\tilde{\phi}_t|^2)],$$  \hspace{1cm} (3-68)

where $\tilde{\phi}_n$ and $\tilde{\phi}_t$ stand for the normal and tangential derivatives of $\tilde{\phi}$ defined on the surface, respectively.

With the same characteristic quantities as chosen in the previous chapter, the dimensionless time-averaged acoustic radiation pressure can be expressed as

$$\tilde{p} = B_a \ast \frac{1}{4} [\|	ilde{\phi}\|^2 - \frac{1}{(ka)^2} (|\tilde{\phi}_n|^2 + |\tilde{\phi}_t|^2)],$$  \hspace{1cm} (3-69)

where $B_a \equiv (ka)^2 \tilde{A}^2$ is defined as the acoustic bond number. Note $B_a$ depends on $\tilde{A}$ quadratically. For convenience, we define

$$\tilde{p} \equiv \frac{1}{4} [\|	ilde{\phi}\|^2 - \frac{1}{(ka)^2} (|\tilde{\phi}_n|^2 + |\tilde{\phi}_t|^2)].$$  \hspace{1cm} (3-70)

It follows that

$$\tilde{p} = B_a \tilde{p}.$$  \hspace{1cm} (3-71)

Note that Eq.(3-70) resembles the Bernoulli's equation in some sense. The first term $\|	ilde{\phi}\|^2$ plays the role of pressure head and the second term $(|\tilde{\phi}_n|^2 + |\tilde{\phi}_t|^2)$ plays the role of dynamic pressure in the Bernoulli's equation. Thus the increase of dynamic pressure will decrease the time-averaged radiation pressure contributed. More interestingly, the $ka$ dependence reveals that the direction of the levitation force depends on the size of the drop. Consequently, the equilibrium position of the drop depends on the size of the drop. If $ka \ll 1$ (The drop size is very small compared with the wavelength.), then the time-averaged radiation pressure is dominated by the kinetic energy term and the pressure term has no significant effect because of little variation experienced over the small drop size. On the other hand, if $ka \gg 1$, then the time-averaged radiation pressure is dominated by the change of the pressure term due to the large variation over large drop size.

Based on Eq.(3-70), the radiation pressure on the surfaces of the rigid scatterers corresponding to Figure 3–2 and 3–3 are plotted against the polar angle in Figure 3–4 and 3–5, respectively.
Figure 3–4  Radiation pressure $\hat{p}$ vs. polar angle for a rigid sphere with $ka = 2.0$ and $kh = 0.0$ in a planar standing wave field. The radial length deviating from the unit circle is proportional to the value of $\hat{p}$ on the surface in that direction. Over regions of increased pressure, the radial length is less than 1, indicating a net inward force on the sphere. Over regions of decreased pressure, the radial length is greater than 1, indicating a net outward force on the sphere.

Figure 3–5  Radiation pressure $\hat{p}$ vs. polar angle for a rigid sphere with $ka = 2.0$ and $kh = 0.0$ in a planar progressive wave field.

Obviously, from the symmetry of radiation pressure the sphere in standing wave, Figure 3–4, experiences zero net force in the z-axis direction at the anti-node of velocity potential. However, the sphere in progressive wave, Figure 3–5, does experience a net upward force in the direction of wave propagation.

Yosioka & Kawasima (1955) further integrated the radiation pressure over the surface to get the net levitation force $N$ acting on small spheres ($ka \ll 1, k^*a \ll 1$ and $\lambda = \frac{\rho^*}{\rho} = O(1)$) as follows:

(1) Planar progressive wave

$$N = \pi a^2 E \cdot 4(ka)^4 F(\lambda, \beta),$$  \hspace{1cm} (3–72)
and
\[ F(\lambda, \beta) = \frac{1}{(1 + 2\lambda)^2} \left[ \left( \lambda - \frac{1 + 2\lambda}{3\lambda\beta^2} \right)^2 + \frac{2}{9}(1 - \lambda)^2 \right], \quad (3-73) \]

where \( E = \frac{1}{2} \rho k^2 A^2 \) is the mean energy density of the incident wave field, and \( \beta \equiv \frac{k}{k_c} \) is the compressibility factor.

(2) Planar standing wave
\[ N = \pi a^2 E \cdot (ka) \sin(2kh) F(\lambda, \beta), \quad (3-74) \]

and
\[ F'(\lambda, \beta) = \frac{\lambda + [2(\lambda - 1)/3]}{1 + 2\lambda} - \frac{1}{3\lambda\beta^2}. \quad (3-75) \]

In summary, the velocity potential of the steady-state acoustic radiation problem is governed by the Helmholtz equation. The Helmholtz equation can be solved analytically for problems with simple geometry. In general, it must be solved numerically. With the velocity potential, King’s formula gives an excellent approximant in calculating the acoustic radiation pressure exerted on the scatterer in an inviscid compressible fluid. This formula will be used to model the acoustic forcing, when studying the dynamics of the acoustically levitated drops in Chapter 6.

REFERENCES

ALEKSEEV, V. N. 1983 Force produced by the acoustic radiation pressure on a sphere. 

BARMATZ, M. 1982 Materials processing in the reduced gravity environment of space. 


HASEGAWA, T. 1977 Comparison of two solutions for acoustic radiation pressure on a 

HASEGAWA, T. 1979 Acoustic-radiation force on a sphere in a quasistationary wave field-

137, 212-240.

LUPI, V. D. 1990 The development of an acoustic levitation test facility for cloud physics 

MAIDANIK, G. & WESTERVELT, P. J. 1957 Acoustic radiation pressure due to incident 


*Acustica* 5, 167-173.
Chapter 4

Theoretical Analysis

4-1 Introduction

Although the equations governing the incompressible, inviscid, and irrotational fluid motion is linear, the kinematic boundary condition on the moving boundary of a bubble, which is unknown a priori, introduces nonlinearity into the problem. Another source of the nonlinearity comes from the Bernoulli's equation also to be imposed on the unknown moving boundary. Principles of superposition is not applicable and no generally applicable methods can be used to solve the nonlinear partial differential equations. However, asymptotic approximation to the original equations are possible for some cases with small parameters. In this chapter, the domain perturbation is used to derive the approximate boundary conditions correct to the second order.

4-2 Small amplitude shape oscillations

In the previous section, the Rayleigh-Plesset equation is derived to describe the dynamics of a spherical bubble in an infinite region of incompressible Newtonian fluid. For a nonspherical bubble, no generally applicable methods can be used to solve Eq.(2–50),(2–51), and (2–52) analytically. However, a regular asymptotic analysis based on the domain perturbation can be used to approximate the dynamics of an aspherical bubble departing slightly from its spherical form.

Consider that a spherical bubble with radius $R(t)$ oscillating in an infinite region of inviscid, incompressible fluid is perturbed slightly away from its spherical form at time $t = 0$ by a small shape perturbation $f(\theta, 0) \ll R(t)$. The moving interface of the a-spherical bubble, $S$, can be expressed as:

$$F(r, \theta, t) = r - R(t) - f(\theta, t) = 0 \quad (4–1)$$

It is interesting to know how this small perturbation will modify the original bubble motion. More interestingly, will this shape perturbation grow or decay in
amplitude? In other words, is the original bubble motion stable subject to the attack of unexpected shape perturbations? In order to answer this question, Eq.(2-50), (2-51), and (2-52) are Taylor expanded about the original bubble motion \( R(t) \), referred as base flow hereafter, and only terms up to the quadratic order are kept. The approximate equations will be analyzed to determine the stability of the base flow.

Let the velocity potential of the base flow be \( \phi_0(r, \theta, t) \) and the correction of the velocity potential due to the small shape perturbation \( f(\theta, t) \) be \( \phi_1(r, \theta, t) \). Then the total velocity potential \( \phi(r, \theta, t) = \phi_0(r, \theta, t) + \phi_1(r, \theta, t) \). The kinematic boundary condition of the base flow gives

\[
\phi_0(r, t) = -\frac{R^2(t) \dot{R}(t)}{r}.
\]  

(4-2)

Kinematic boundary condition (KBC) on the interface, Eq.(2-51), can be rewritten as:

\[
- \left( \frac{dR}{dt} + \frac{\partial f}{\partial t} \right) + \left( \frac{\partial \phi_0}{\partial r} + \frac{\partial \phi_1}{\partial r} \right) - \frac{1}{r^2} \frac{\partial f}{\partial \theta} \frac{\partial \phi_1}{\partial \theta} = 0 \text{ on } r = R(t) + f(\theta, t).
\]  

(4-3)

Note this kinematic boundary condition is imposed on the moving interface, which is not known a priori. To overcome this difficulty, the domain perturbation method is applied to find the appropriate boundary conditions that are to be imposed on the unperturbed interface. From the Taylor expansion, we have

\[
\frac{\partial \phi_0}{\partial r} \bigg|_{r=R+f} \sim \frac{\partial \phi_0}{\partial r} \bigg|_{r=R} + f \frac{\partial^2 \phi_0}{\partial r^2} \bigg|_{r=R} + \frac{1}{2} f^2 \frac{\partial^3 \phi_0}{\partial r^3} \bigg|_{r=R} + O(f^3)
\]  

(4-4)

Thus

\[
\frac{\partial \phi_0}{\partial r} \bigg|_{r=R+f} \sim \dot{R}(t) - 2f \frac{\dot{R}(t)}{R(t)} + 3f^2 \frac{\dot{R}(t)}{R^2(t)} + O(f^3)
\]  

(4-5)

\[
\frac{\partial \phi_1}{\partial r} \bigg|_{r=R+f} \sim \frac{\partial \phi_1}{\partial r} \bigg|_{r=R} + f \frac{\partial^2 \phi_1}{\partial r^2} \bigg|_{r=R} + \frac{1}{2} f^2 \frac{\partial^3 \phi_1}{\partial r^3} \bigg|_{r=R} + O(f^3)
\]  

(4-6)

Similarly, we can derive the following equations easily.

\[
\frac{1}{r^2} \sim \frac{1}{R^2} \left( 1 - 2 \frac{f}{R} + 3 \frac{f^2}{R^2} \right) + O(f^3)
\]  

(4-7)

\[
\frac{\partial \phi_1}{\partial \theta} \bigg|_{r=R+f} \sim \frac{\partial \phi_1}{\partial \theta} \bigg|_{r=R} + f \frac{\partial^2 \phi_1}{\partial r \partial \theta} \bigg|_{r=R} + \frac{1}{2} f^2 \frac{\partial^3 \phi_1}{\partial r^2 \partial \theta} \bigg|_{r=R} + O(f^3)
\]  

(4-8)

\[
- \frac{1}{r^2} \frac{\partial f}{\partial \theta} \frac{\partial \phi_1}{\partial \theta} \bigg|_{r=R+f} \sim - \frac{1}{R^2} \left( \frac{\partial f}{\partial \theta} \frac{\partial \phi_1}{\partial \theta} \bigg|_{r=R} + f \frac{\partial^2 \phi_1}{\partial \theta \partial r} \bigg|_{r=R} - 2 \frac{f}{R} \frac{\partial f}{\partial \theta} \frac{\partial \phi_1}{\partial \theta} \bigg|_{r=R} \right) + O(f^3)
\]  

(4-9)

Define \( \eta = \cos \theta \) then we have

\[
\frac{\partial}{\partial \theta} = -\sin \theta \frac{\partial}{\partial \eta},
\]  

(4-10)
and
\[
- \frac{1}{r^2} \frac{\partial f}{\partial \theta} \frac{\partial \phi_1}{\partial \theta} \bigg|_{r=R+f} \sim - \frac{1 - \eta^2}{R^2} \left( \frac{\partial f}{\partial \eta} \frac{\partial \phi_1}{\partial \eta} \bigg|_{r=R} + f \frac{\partial^2 f}{\partial \eta \partial r} \frac{\partial^2 \phi_1}{\partial r \partial \eta} \bigg|_{r=R} - \frac{2}{R} \frac{f}{\partial \eta} \frac{\partial f}{\partial \eta} \bigg|_{r=R} \right) + O(f^3)
\]  
(4-11)

Substituting Eq.(4-4)-(4-11) into (4-3), we obtain the kinematic boundary condition to be imposed on \( r = R(t) \) as follows:
\[
\frac{\partial f}{\partial t} - \frac{\partial \phi_1}{\partial r} + 2 \frac{\dot{R}}{R} f = f \frac{\partial^2 \phi_1}{\partial r^2} - \frac{(1 - \eta^2)}{R^2} \left( \frac{\partial f}{\partial \eta} \frac{\partial \phi_1}{\partial \eta} + f \frac{\partial f}{\partial \eta} \frac{\partial^2 \phi_1}{\partial r \partial \eta} - \frac{2}{R} \frac{f}{\partial \eta} \frac{\partial f}{\partial \eta} \right) + 
\]
\[+ \frac{1}{2} f^2 \frac{\partial^3 \phi_1}{\partial r^3} + 3 f^2 \frac{\dot{R}}{R^2} + O(f^3) \]  
(4-12)

Likewise, we can perturb the dynamic boundary condition, Eq.(2-52) as follows:
\[
\frac{\partial \phi}{\partial t} \bigg|_{r=R+f} \sim \frac{\partial \phi_0}{\partial t} \bigg|_{r=R} + f \frac{\partial^2 \phi_0}{\partial r \partial t} \bigg|_{r=R} + \frac{1}{2} \frac{f^2}{\partial r^2} \frac{\partial^3 \phi_0}{\partial t} \bigg|_{r=R} + 
\]
\[+ \frac{\partial \phi_1}{\partial t} \bigg|_{r=R} + f \frac{\partial^2 \phi_1}{\partial r \partial t} \bigg|_{r=R} + \frac{1}{2} \frac{f^2}{\partial r^2} \frac{\partial^3 \phi_1}{\partial t} \bigg|_{r=R} + O(f^3) \]  
(4-13)

Perturbation of the kinetic energy term reads as:
\[
\nabla \phi \cdot \nabla \phi = \nabla \phi_0 \cdot \nabla \phi_0 + 2 \nabla \phi_0 \cdot \nabla \phi_1 + \nabla \phi_1 \cdot \nabla \phi_1
\]  
(4-14)
\[
\nabla \phi_0 \cdot \nabla \phi_0 = \frac{R^4 \dot{R}^2}{r^4}
\]  
(4-15)
\[
(\nabla \phi_0 \cdot \nabla \phi_0) \bigg|_{r=R+f} \sim \dot{R}^2 \left( 1 - \frac{4 f}{R} + 10 \frac{f^2}{R^2} \right) + O(f^3)
\]  
(4-16)
\[
\nabla \phi_0 \cdot \nabla \phi_1 = \frac{R^2 \dot{R}}{r} \frac{\partial \phi_1}{\partial r}
\]  
(4-17)
\[
(\nabla \phi_0 \cdot \nabla \phi_1) \bigg|_{r=R+f} \sim \dot{R} \frac{\partial \phi_1}{\partial r} \bigg|_{r=R} + f \dot{R} \left( \frac{1}{2} \frac{\partial^3 \phi_1}{\partial r^3} \bigg|_{r=R} - \frac{3}{R} \frac{\partial^2 \phi_1}{\partial r^2} \bigg|_{r=R} \right) + O(f^3)
\]  
(4-18)
\[
(\nabla \phi_1 \cdot \nabla \phi_1) \bigg|_{r=R+f} \sim (\nabla \phi_1 \cdot \nabla \phi_1) \bigg|_{r=R} + f \frac{\partial}{\partial r} (\nabla \phi_1 \cdot \nabla \phi_1) \bigg|_{r=R} + 
\]
\[+ \frac{1}{2} f^2 \frac{\partial^2}{\partial r^2} (\nabla \phi_1 \cdot \nabla \phi_1) \bigg|_{r=R} + O(f^3)
\]  
(4-19)
Perturbation of the volume term gives
\[ <(R + f)^3>^{-\gamma} \sim (R^3 + 3R^2 <f> + 3R <f^2> + O(f^3))^{-\gamma}. \] (4-20)

From binomial theorem, we know
\[ (1 + x)^{-\gamma} = 1 + C \binom{-\gamma}{1} x + C \binom{-\gamma}{2} x^2 + \cdots. \] (4-21)

Therefore, Eq.(4-20) can be approximated as
\[ <(R + f)^3>^{-\gamma} \sim R^{-3\gamma} \left[ 1 - 3\gamma \left( \frac{<f>}{R} + \frac{<f^2>}{R^2} - \frac{3(\gamma + 1)}{2R^2} <f^2> \right) \right] + O(f^3). \] (4-22)

Calculation of the total curvature \( \kappa \) on the interface is based on the equality
\[ \kappa = \nabla \cdot \mathbf{n}, \] (4-23)

where \( \mathbf{n} \) is the unit normal on the interface. From the differential geometry, the normal vector on the interface can be easily obtained as
\[ \nabla F = (1, -\frac{1}{r} \frac{\partial f}{\partial \theta}, 0). \] (4-24)

Thus the unit normal \( \mathbf{n} \) can be approximated as
\[ \mathbf{n} = \frac{\nabla F}{|\nabla F|} \sim \left( 1 - \frac{1}{2r^2} \left( \frac{\partial f}{\partial \theta} \right)^2 + O(f^3), -\frac{1}{r} \frac{\partial f}{\partial \theta}, 0 \right). \] (4-25)

Substituting Eq.(4-25) into (4-23) gives
\[ \kappa \sim \frac{1}{r^2} \frac{\partial}{\partial t} \left( \frac{1}{2} \left( \frac{\partial f}{\partial \theta} \right)^2 \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( -\frac{\sin \theta \frac{\partial f}{\partial \theta}}{r} \right) + O(f^3) \]
\[ \sim \frac{1}{r} \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + O(f^3) \] (4-26)

Applying Eq.(4-26) at \( r = R + f \) gives
\[ \kappa \bigg|_{r=R+f} \sim \frac{2}{R} \left( 1 - \frac{f}{R} + \frac{f^2}{R^2} \right) - \frac{1}{R^2} \left( 1 - 2\frac{f}{R} \right) \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + O(f^3) \] (4-27)

Define \( \nabla_s^2 \equiv \frac{\partial}{\partial \eta} \left[ (1 - \eta^2) \frac{\partial}{\partial \eta} \right] \), then we have
\[ \kappa \bigg|_{r=R+f} \sim \frac{2}{R} \left( 1 - \frac{f}{R} + \frac{f^2}{R^2} \right) - \frac{1}{R^2} \left( 1 - 2\frac{f}{R} \right) \nabla_s^2 f + O(f^3) \] (4-28)

Substituting Eq.(4-13)-(4-28) into (2-52), we obtain the approximate dynamic boundary condition to be imposed on the unperturbed interface as follows:
\[ \frac{\partial \phi_1}{\partial t} + \frac{1}{R^2} (\nabla_s^2 + 2) f - (\omega_0^2 + 2) R^{-3\gamma - 1} <f> + (\tilde{R} f + R \frac{\partial \phi_1}{\partial R}) \sim -f \frac{\partial^2 \phi_1}{\partial t \partial r} \]
\[ -\frac{1}{2} \nabla \phi_1 \cdot \nabla \phi_1 + \frac{2}{R^3} \left( \nabla_s^2 f + f \right) + (\omega_0^2 + 2) R^{-3\gamma - 2} <f^2> - \frac{3}{2} (1 + \gamma) <f^2> \]
\[- \frac{1}{2} f^2 \frac{\partial^2 \phi_1}{\partial t \partial r^2} - \frac{1}{2} f \frac{\partial}{\partial r} (\nabla \phi_1 \cdot \nabla \phi_1) - \frac{1}{4} f^2 \frac{\partial^2}{\partial r^2} (\nabla \phi_1 \cdot \nabla \phi_1) + f^2 \left( \frac{\dot{R}}{R} - \frac{3 \ddot{R}^2}{R^2} \right) \]

\[- f \dot{R} \left( \frac{\partial^2 \phi_1}{\partial r^2} - \frac{2}{R} \frac{\partial \phi_1}{\partial r} \right) - (p_v - p_{v0}) + O(f^3), \quad (4-29) \]

where

\[\omega_0^2 = 3 \gamma (p_\infty + 2 - p_{v0}) - 2. \quad (4-30)\]

In summary, the truncated series expansions of the kinematic and dynamic boundary conditions obtained thus far are correct to the quadratic order:

**Continuity**

\[\nabla^2 \phi_1 = 0, \quad (4-31)\]

**KBC**

\[\frac{\partial f}{\partial t} - \frac{\partial \phi_1}{\partial r} + 2 \frac{\dot{R}}{R} f = f \frac{\partial^2 \phi_1}{\partial r^2} - \frac{(1 - \eta^2)}{R^2} \frac{\partial f}{\partial \eta} + f^2 \frac{\ddot{R}}{R^2}, \quad (4-32)\]

**DBC**

\[\frac{\partial \phi_1}{\partial t} + \frac{1}{R^2} (\nabla^2_\phi + 2) f - \left( \frac{\omega_0^2 + 2}{R^2} \right) \dot{R} \left( \frac{\partial^2 \phi_1}{\partial r^2} - \frac{2}{R} \frac{\partial \phi_1}{\partial r} \right) + (\ddot{R} f + \dot{R} \frac{\partial \phi_1}{\partial r}) = - f \frac{\partial^2 \phi_1}{\partial t \partial r}, \]

\[- \frac{1}{2} \nabla \phi_1 \cdot \nabla \phi_1 + 2 \frac{f}{R^3} \left( \nabla^2_\phi f + f \right) + (\omega_0^2 + 2) \dot{R} (-3 \gamma - 1) \left[ f^2 > - \frac{3}{2} (1 + \gamma) \right. \left. < f^2 \right], \]

\[f^2 \left( \frac{\dot{R}}{R} - \frac{3 \ddot{R}^2}{R^2} \right) - f \dot{R} \left( \frac{\partial^2 \phi_1}{\partial r^2} - \frac{2}{R} \frac{\partial \phi_1}{\partial r} \right) - (p_v - p_{v0}), \quad (4-33)\]

where

\[\omega_0^2 = 3 \gamma (p_\infty + 2 - p_{v0}) - 2. \quad (4-34)\]

### 4-3 Linear analysis

A linearized version of Eq.(4-31),(4-32), and (4-33) reads as

\[\nabla^2 \phi_1 = 0, \quad (4-35)\]

\[\frac{\partial f}{\partial t} - \frac{\partial \phi_1}{\partial r} + 2 \frac{\dot{R}}{R} f = 0, \quad (4-36)\]

\[\frac{\partial \phi_1}{\partial t} + \frac{1}{R^2} (\nabla^2_\phi + 2) f - \left( \frac{\omega_0^2 + 2}{R^2} \right) \dot{R} \left( \frac{\partial^2 \phi_1}{\partial r^2} - \frac{2}{R} \frac{\partial \phi_1}{\partial r} \right) + (\ddot{R} f + \dot{R} \frac{\partial \phi_1}{\partial r}) = 0. \quad (4-37)\]

Mathematically, the most general solution of a Laplace equation

\[\nabla^2 \phi = 0 \quad (4-38)\]

can be expressed as a series expansion of the spherical harmonics:

\[\phi(r, \theta, \psi, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn}(t) r^n P_n^m(\cos \theta) e^{im\psi} + B_{mn}(t) r^{-(n+1)} P_n^m(\cos \theta) e^{im\psi}. \quad (4-39)\]
Particularly, for an axisymmetric case the series expansion takes a simpler form as follows:

$$\phi(r, \theta, t) = \sum_{n=0}^{\infty} A_n(t) r^n P_n(\cos \theta) + B_n(t) r^{-(n+1)} P_n(\cos \theta).$$  \hspace{1cm} (4-40)

$\phi_1$ is limited to the following expression due to the boundary conditions at infinity:

$$\phi_1(r, \theta, t) = \sum_{n=2}^{\infty} b_n(t) r^{-(n+1)} P_n(\cos \theta).$$  \hspace{1cm} (4-41)

For an axisymmetric bubble, series expansion of $f$ in terms of Legendre polynomials can be written as

$$f(\theta, t) = \sum_{n=0}^{\infty} a_n(t) P_n(\cos \theta).$$  \hspace{1cm} (4-42)

Substitution of Eq. (4-41) and (4-42) into Eq. (4-36) and (4-37) gives

$$a_n + \frac{(n+1)}{R^{(n+2)}} b_n + \frac{2\dot{R}}{R} a_n = 0.$$  \hspace{1cm} (4-43)

$$\frac{1}{R^{(n+1)}} \ddot{b}_n - \frac{(n+1)\dot{R}}{R^{(n+2)}} b_n + \left[ \frac{(2-n-n^2)}{R^2} + \ddot{R} \right] a_n = 0.$$  \hspace{1cm} (4-44)

Combining Eq. (4-43) and (4-44), we have

$$\ddot{a}_n + 3\frac{\dot{R}}{R} \dot{a}_n + \left[ \frac{(n-1)(n+1)(n+2)}{R^3} - (n-1) \frac{\dot{R}}{R} \right] a_n = 0, \hspace{1cm} n \geq 2.$$  \hspace{1cm} (4-45)

It is noted that to the linear order, all shape modes are decoupled with each other due to the assumption of small shape perturbation. However, all the shape modes are strongly affected by the volume mode oscillation, $R(t)$. For a bubble with constant volume (no volume mode oscillation), $R = 1$, the amplitude of $n$-th mode is governed by

$$\ddot{a}_n + \omega_n^2 a_n = 0, \hspace{1cm} n \geq 2,$$  \hspace{1cm} (4-46)

where $\omega_n^2 = (n-1)(n+1)(n+2)$ is the Rayleigh's frequency. Thus the constant volume shape oscillations of all modes ($n \geq 2$) are linearly stable.

For a bubble with volume oscillation $\ddot{R}, \dot{R} \neq 0$, the dynamics is complicated by the variable coefficients and the stability of the system can not be determined easily without actually carrying out the integration of the dynamical system exactly. However, according to Eq. (4-45), the surface tension term has stabilizing effect. The acceleration term $\ddot{R}$ has the destabilizing effect if $\ddot{R} > 0$ and stabilizing if $\ddot{R} < 0$. This is analogous to the Rayleigh-Taylor instability essentially. $\ddot{R}$ plays the role of positive damping if $\ddot{R} > 0$ and negative damping if $\ddot{R} < 0$. Leal (1992) gives an interesting physical explanation to this:

The area of the bubble surface is either increased with time or decreased with time depending on whether $\ddot{R} \geq 0$ or $\ddot{R} < 0$. When the surface area is increased, the magnitude of any surface disturbance is decreased and the wavelength is increased. On the other hand, when the area is decreased, the
magnitude of the disturbance is increased and its wavelength is decreased.
The increase or decrease in amplitude corresponds to a destabilizing or sta-
bilizing effect.

4-4 Second-order analysis

Mode coupling provides a way of exchanging energy between modes. To study
the mode coupling, a second-order analysis is required. Consider now the shape
perturbation, \( f \), consists of two dominant shape perturbations in modes \( m \) and \( n \).
In other words, the amplitudes of modes \( m \) and \( n \) are much larger than all other
shape modes yet they are much smaller than the amplitude of the volume oscillation,
\( R(t) \). Thus to the leading order, the following forms of \( f \) and \( \phi \) are assumed:

\[
    f = a_m(t)P_m(\cos \theta) + a_n(t)P_n(\cos \theta),
\]
\[
    \phi_1 = b_m(t)r^{-(m+1)}P_m(\cos \theta) + b_n(t)r^{-(n+1)}P_n(\cos \theta),
\]

Substituting Eq. (4-47) and (4-48) into Eq. (4-32) and (4-33) yields

\[
    a_m' + \frac{2\dot{R}}{R}a_m + \frac{(m + 1)}{R^{(m+2)}}b_m
    = 3\frac{\ddot{R}}{R^2}(A_m a_m^2 + 2B_{mn}a_m a_n + B_{nm}a_n^2)
    + \frac{1}{2}(m + 1)(m + 4)A_m R^{-(m+3)}a_m b_m
    + \frac{1}{2}(n + 1)(n + 4)B_{mn} R^{-(n+3)}a_m b_n
    + \left[(m + 1)(m + 2) - \frac{1}{2}n(n + 1)\right]B_{mn} R^{-(m+3)}a_n b_m
    + [(n + 1)(n + 2)B_{nm} - C_{nm}] R^{-(n+3)}a_n b_n,
\]

and

\[
    \frac{b_m'}{R^{(m+1)}} - \frac{(m + 1)\dot{R}}{R^{(m+2)}}b_m
    = \left[\frac{2(1 - m - m^2)}{R^3} + \left(\frac{\ddot{R}}{R} - \frac{3\dot{R}^2}{R^2}\right)\right]A_m a_m^2
    + \left[\frac{2}{R^3}(2 - m - n - m^2 - n^2) + 2\left(\frac{\ddot{R}}{R} - \frac{3\dot{R}^2}{R^2}\right)\right]B_{mn} a_m a_n
    + \left[\frac{2(1 - n - n^2)}{R^3} + \left(\frac{\ddot{R}}{R} - \frac{3\dot{R}^2}{R^2}\right)\right]B_{nm} a_n^2
    - (m + 1)(m + 4)\frac{\dot{R}}{R^{(m+3)}}A_m a_m b_m
    - (m + 1)(m + 4)\frac{\dot{R}}{R^{(m+3)}}B_{mn} a_n b_m.
\]
\[-(n+1)(n+4) \frac{\dot{R}}{R^{(n+3)}} B_{mn} a_m b_n - (n+1)(n+4) \frac{\dot{R}}{R^{(n+3)}} B_{nm} a_n b_n \]
\[-\frac{1}{4} (m+1)(3m+2)R^{-2(m+2)} A_m b_m^2 \]
\[- \left[ (m+1)(n+1) + \frac{1}{2} n(n+1) \right] R^{-(m+n+4)} B_{mn} b_m b_n \]
\[- \left[ \frac{1}{2} (n+1)^2 B_{mm} + \frac{1}{2} C_{nm} \right] R^{-2(n+2)} b_n^2 \]
\[+(m+1)R^{-(m+2)} A_m a_m b_m^* + (m+1)R^{-(m+2)} B_{mn} a_n b_n^* \]
\[+(n+1)R^{-(n+2)} B_{nm} a_m b_n^* + (n+1)R^{-(n+2)} B_{nm} a_n b_m^*, \quad (4-50) \]

where

\[ A_n \equiv \int_{-1}^{1} P_n(x) dx, \]
\[ B_{mn} \equiv \int_{-1}^{1} P_m^2(x) P_n(x) dx, \]
\[ C_{mn} \equiv \int_{-1}^{1} (1-x^2) P_m^2 P_n(x) dx, \]
\[ \int_{-1}^{1} (1-x^2) P_m''(x) P_n(x) dx = \frac{1}{2} n(n+1) B_{mn}. \]

The time derivative terms \(b_m\) and \(b_n\) on the right hand side of Eq (4-50) can be eliminated by replacing them with the corresponding linear approximations given in Eq.(4-49) and (4-50). Thus

\[ b_m^* + \left[ \frac{(2-m-m^2)}{R^2} + \frac{\dot{R}}{R} \right] R^{m+1} a_m - \frac{(m+1)\dot{R}}{R} b_m \]
\[ = \left[ (m^3 - 3m)R^{m-2} - m\ddot{R}R^m - 3\dot{R}^2 R^{m-1} \right] A_m a_m^2 \]
\[ + \left[ (m^3 - 3m + n^3 - 3n) R^{m-2} - (m+n)\ddot{R}R^m - 6\dot{R}^2 R^{m-1} \right] B_{mn} a_m a_n \]
\[ + \left[ (n^3 - 3n) R^{m-2} - n\ddot{R}R^m - 3\dot{R}^2 R^{m-1} \right] B_{nm} a_n^2 \]
\[-3(m+1)\dddot{R}R^{-2} A_m a_m b_m - 3(m+1)\dddot{R}R^{-2} B_{mn} a_n b_n \]
\[ -3(n+1)\dddot{R}R^{-n-2} B_{mn} a_m b_n - 3(n+1)\dddot{R}R^{-n-2} B_{nm} a_n b_n \]
\[ -\frac{1}{4} (m+1)(3m+2)R^{-(m+3)} A_m b_m^2 \]
\[- \left[ (m+1)(n+1) + \frac{1}{2} n(n+1) \right] R^{-(n+3)} B_{mn} b_m b_n \]
\[- \left[ \frac{1}{2} (n+1)^2 B_{nm} + \frac{1}{2} C_{nm} \right] R^{m-2n-3} b_n^2 \quad (4-51) \]

Eq.(4-49), and (4-51) together describe how the dynamics of the \(n\)-th mode will couple with the \(m\)-th mode. Again the variable coefficients make it impossible to predict the stability of the dynamical system without integrating the equations exactly. However in the sonoluminescence, the minimum bubble size is only a few microns and its natural frequencies for shape oscillations are much higher than the excitation frequency, usually around 26kHz. Thus within the time scale of shape
oscillation, $R, \dot{R}, \ddot{R}$ can be regarded as constant locally and a local stability analysis is possible. One important factor to achieve the sonoluminescence is the stability of the volume oscillation and $P_2$ mode is the most detrimental one to destabilize the volume oscillation. An interesting question to ask is if $P_2$ mode is close to the linear stability margin how the existence of $n$-th mode will affect the stability of $P_2$ mode? In order to study this, we set $m = 2$ and treat $R, \dot{R}, \ddot{R}$ as constants in Eq. (4-49) and (4-51). Define

$$\omega^2_n = \frac{(n-1)(n+1)(n+2)}{R^3} - \frac{(n+1)\ddot{R}}{R},$$

then the most critically unstable circumstance for $P_2$ mode occurs when

$$\omega^2_2 \sim 0 \quad \text{and} \quad \dot{R} \sim 0,$$

or one can write

$$\frac{\dot{R}}{R} = \beta \quad \text{and} \quad \ddot{R} = \frac{4}{R^2},$$

where $\beta$ is a very small parameter. Rescale the variables $a_2, b_2, a_n, b_n$ as follows:

$$x_1(t) \equiv a_2(t), \quad x_2(t) \equiv -\frac{3}{\omega_n R^4}b_2(t), \quad x_3(t) \equiv a_n(t), \quad x_4(t) \equiv -\frac{(n+1)}{\omega_n R^{n+2}}b_n(t),$$

Substituting Eq. (4-52)-(4-55) into (4-49) and (4-51), we obtain

$$\begin{cases}
\dot{x}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} -2\beta x_1 \\ 3\beta x_2 \\ -2\beta x_3 \\ (n+1)\beta x_4 \end{bmatrix}, \\
\dot{x}_2 = \begin{bmatrix} -2\beta x_1 \\ 3\beta x_2 \\ -2\beta x_3 \\ (n+1)\beta x_4 \end{bmatrix}, \\
\dot{x}_3 = \begin{bmatrix} Q_1(x_1, x_2, x_3, x_4) \\ Q_2(x_1, x_2, x_3, x_4) \\ Q_3(x_1, x_2, x_3, x_4) \\ Q_4(x_1, x_2, x_3, x_4) \end{bmatrix},
\end{cases}$$

where

$$Q_1 = -\frac{3A_2}{R}x_1x_2 - \frac{(n+4)B_{2n}}{2R}x_1x_3 + \frac{\left\{ \frac{1}{2}n(n+1) - 12 \right\} B_{2n}}{3R}x_2x_3,$$

$$Q_2 = \frac{18A_2}{\omega_n^2 R^4}x_1^2 - \frac{3(n^3 - 7n - 6)B_{2n}}{\omega_n^2 R^4}x_1x_3 + \frac{2A_2}{R}x_2^2 + \frac{(\frac{1}{2}n + 3)B_{2n}}{R}x_2x_4,$$

$$Q_3 = \frac{(C_{2n} - 12B_{2n})}{3R}x_1x_2 - \frac{[(n+1)(n+2) - 3]B_{2n}}{(n+1)R}x_1x_4 - \frac{3B_{2n}}{R}x_2x_3,$$

$$Q_4 = \left\{ \frac{1}{2}n + 2 \right\}A_n \frac{R}{x_3x_4},$$

$$Q_4 = \frac{6(n+1)B_{2n}}{\omega_n^2 R^4}x_1^2 - \frac{(n+1)(n^3 - 7n - 6)B_{2n}}{\omega_n^2 R^4}x_1x_3 + \frac{1}{2}(n+1)(9B_{2n} + C_{2n})x_2^2,$$

$$+ \frac{(n+2)B_{2n}}{R}x_2x_4 - \frac{(n+1)(n^3 - 7n)A_n}{4R}x_3^2 + \frac{3(n+2)A_n}{4R}x_4^2.$$
To understand the dynamics of Eq. (4–56), we introduce the normal form concept in the following section to simplify the expression of the dynamical system to its simplest form.

4-5 Normal form

In general, the method of normal form provides a systematic way of finding successive analytic nonlinear coordinate transformation \( x = h(y) \) to reduce the expressions of a dynamical system, \( \dot{x} = f(x) \), to its simplest form. For a dynamical system on the center manifold of a non-hyperbolic fixed point,

\[
\dot{x} = Jx + Q(x),
\]

the linear part of the vector field, \( J \), in its simplest form is a Jordan canonical matrix having eigenvalues with zero real parts thus we need only to search for a near-identity transformation

\[
x = y + P(y),
\]

to reduce the nonlinear parts to its simplest form. First, we Taylor expand the nonlinear parts formally as follows:

\[
Q(x) = Q_2(x) + Q_3(x) + Q_4(x) + \cdots,
\]

where \( Q_i(x) \) represent the \( i \)-th order homogeneous terms in the Taylor expansion of \( Q(x) \). Next, we choose \( P(y) \) as a homogeneous quadratic polynomial so that we can eliminate as many terms in \( Q_2(x) \) as possible. From the near-identity transformation Eq. (4–59), we have

\[
\dot{x} = (I + DP)y.
\]

Making use of the fact that \( P(y) \) is small on a small neighborhood of the center manifold, we can easily invert Eq. (4–61) approximately as

\[
\dot{y} = (I - DP)(Jx + Q_2(x)) + O(|y|^3)
\]

Substituting Eq. (4–59) into the above equation gives

\[
\dot{y} = Jy + JP - (DP)Jy + Q_2(y) + O(|y|^3)
\]

Or we can write the above equation as

\[
\dot{y} = Jy + (DJy)P - (DP)Jy + Q_2(y) + O(|y|^3)
\]

Note that the quantity

\[
[P, Jy] = (DJy)P - (DP)Jy
\]

is known as the Lie bracket of the vector fields \( P \) and \( Jy \). In order to determine the \( P(y) \) that simplifies \( [P, Jy] + Q_2(y) \) to its simplest form, we consider the space of the vector-valued monomials of degree \( k \), \( H_k \). Let \( e_i, i = 1, 2, \cdots, n \) denote a basis
of $R^n$, and $y = (y_1, y_2, \ldots, y_n)$ be coordinates with respect to this basis. We then define the vector-valued monomial of degree $k$ as

$$y_1^{m_1}y_2^{m_2}\cdots y_n^{m_n}e_i, \quad \sum_{j=1}^n m_j = k,$$  \hspace{1cm} (4-66)

where $m_j \geq 0$ are integers. The set of all vector-valued monomials of degree $k$ forms a linear vector space, denoted as $H_k$. For a given matrix $J$, the Lie bracket defines a linear map of $H_k$ into a subspace of $H_k$ represented by $L(H_k)$. And we represent the space complementary to $L(H_k)$ as $G_k$. Thus

$$H_k = L(H_k) \oplus G_k.$$ \hspace{1cm} (4-67)

The terms corresponding to the projection of $Q_2(y)$ to $L(H_2)$ can be eliminated while the terms corresponding to the projection of $Q_2(y)$ to $G_2$ can not be eliminated and are called resonant terms! A small program of finding the normal form of degree 2 corresponding to a given $n \times n$ matrix $J$ written in matlab is shown in Appendix A. The same procedure can be applied successively to simplify the higher-order terms. According to the normal form theorem mentioned above, we are able to simplify Eq.(4-56) as follows:

$$
\begin{align*}
    y_1 &= y_2 + 2c_2 y_1^2 \\
    y_2 &= c_1 y_1^2 + c_2 y_1 y_2 + c_3 (y_3^2 + y_4^2) \\
    y_3 &= y_4 + c_4 y_1 y_4 + c_5 y_1 y_3 \\
    y_4 &= -y_3 - c_4 y_1 y_3 + c_5 y_1 y_4
\end{align*}
\hspace{1cm} (4-68)

where

$$
\begin{align*}
    c_1 &= \frac{18A_2}{(n-1)(n+1)(n+2)R}, \\
    c_2 &= 0, \\
    c_3 &= \frac{(3n+2)A_n}{8R} - \frac{3(n^3-7n)B_{n2}}{(n+1)(n-2)(n+3)R}, \\
    c_4 &= -\frac{3(n^3-n^2+2n-4)R_{n2}}{2(n+1)(n-2)(n+3)R}, \\
    c_5 &= 0.
\end{align*}
\hspace{1cm} (4-69)

Since the normal form is not unique, we can also write

$$
\begin{align*}
    \dot{y}_1 &= y_2 \\
    \dot{y}_2 &= c_1 y_1^2 + 5c_2 y_1 y_2 + c_3 (y_3^2 + y_4^2) \\
    \dot{y}_3 &= y_4 + c_4 y_1 y_4 + c_5 y_1 y_3 \\
    \dot{y}_4 &= -y_3 - c_4 y_1 y_3 + c_5 y_1 y_4
\end{align*}
\hspace{1cm} (4-70)

For convenience, we define

$$y_3 = r \cos \theta, \quad y_4 = r \sin \theta.$$ \hspace{1cm} (4-71)
Then Eq. (4-70) can be rewritten as
\[
\begin{align*}
\begin{cases}
\dot{y}_1 \\
\dot{y}_2 \\
r \\
\dot{\theta}
\end{cases} &= 
\begin{cases}
y_2 \\
c_1 y_1^2 + c_3 r^2 \\
0 \\
-(1 + c_4 y_1)
\end{cases}
\end{align*}
\tag{4-72}
\]
Alternatively, we have
\[
\ddot{y}_1 - c_1 y_1^2 - c_3 r^2 = 0,
\tag{4-73}
\]
and
\[
\begin{align*}
r &= \text{constant}, \\
\dot{\theta} &= -(1 + c_4 y_1),
\end{align*}
\tag{4-74}
\tag{4-75}
\]
Several values for $c_1$, $c_3$ and $c_4$ are listed below

<table>
<thead>
<tr>
<th>$n$</th>
<th>$c_1 R$</th>
<th>$c_3 R$</th>
<th>$c_4 R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.085714</td>
<td>-0.057143</td>
<td>-0.095238</td>
</tr>
<tr>
<td>4</td>
<td>0.029388</td>
<td>-0.026117</td>
<td>-0.064317</td>
</tr>
<tr>
<td>5</td>
<td>0.014286</td>
<td>-0.087413</td>
<td>-0.051476</td>
</tr>
<tr>
<td>6</td>
<td>0.008163</td>
<td>-0.037819</td>
<td>-0.043823</td>
</tr>
<tr>
<td>7</td>
<td>0.005143</td>
<td>-0.074498</td>
<td>-0.038516</td>
</tr>
<tr>
<td>8</td>
<td>0.003463</td>
<td>-0.035470</td>
<td>-0.034525</td>
</tr>
<tr>
<td>9</td>
<td>0.002449</td>
<td>-0.063120</td>
<td>-0.031370</td>
</tr>
<tr>
<td>10</td>
<td>0.001798</td>
<td>-0.031842</td>
<td>-0.028793</td>
</tr>
</tbody>
</table>

It can be easily seen from Eq. (4-74) and (4-75) that the $n$-th mode oscillates at constant amplitude. However its frequency is slowed down by the existence of $P_2$ mode. The dynamics for $y_1$ exhibits the saddle-node bifurcation at $r = 0$. In other words, the existence of the $n$-th mode creates a stable node and an unstable saddle point on the phase diagram of $(y_1, \dot{y}_1)$. More specifically, this means the existence of the $n$-th mode provides a stabilizing effect to the $P_2$ mode for certain initial conditions. Yet, the existence of the small damping terms in Eq. (4-56) containing the parameter $\beta$ tend to change the periodic orbits around the node into stable or unstable spirals locally depending on the sign of $\dot{R}$.

### 4-6 Resonance between volume mode and shape mode

The small amplitude resonant interactions between volume mode and shape oscillation modes have been well explored by Feng & Leal (1993) and Yang, Feng & Leal (1993). Through the method of domain perturbation, they obtained the following systems of approximate equations for Eq. (5-2), (5-3), which are correct to the second order and are to apply at $r = 1$,
\[
\frac{\partial f}{\partial t} - \frac{\partial \phi}{\partial r} = f \frac{\partial^2 \phi}{\partial r^2} - (1 - \eta^2) \frac{\partial f}{\partial \eta} \frac{\partial \phi}{\partial \eta} + O(f^3, f^2 \phi),
\tag{4-76}
\]
\[
\frac{\partial \phi}{\partial t} + (\nabla_s^2 + 2)f - (\omega_0^2 + 2)\langle f \rangle = -f \frac{\partial^2 \phi}{\partial t \partial r} - \frac{1}{2}(\nabla \phi)^2 + 2f(f + \nabla_s^2 f) +
(\omega_0^2 + 2)[(f^2) - \frac{3}{2}(1 + \gamma)(\langle f \rangle)^2] - f \frac{\partial}{\partial r}(\nabla \phi) \cdot \nabla \phi + O(f^3, f^2 \phi), \tag{4-77}
\]

where \( \nabla_s^2 \equiv \frac{\partial}{\partial \eta}((1 - \eta^2)\frac{\partial}{\partial \eta}) \) denotes the surface Laplacian, \( \eta = \cos \theta \), \( \langle \cdot \rangle \) denotes the spherical surface average, and
\[
\omega_0^2 = 3\gamma(p_\infty + 2) - 2 \tag{4-78}
\]
is the natural frequency of the volume mode. Note the same system of equations can be obtained from Eq.(4-32) and Eq.(4-33) by setting \( R(t) = 1 \).

Combined with a multiple time-scale asymptotic expansion, the slowly-varying amplitude equations for the shape modes are derived. Feng & Leal (1993) used these equations to investigate the evolutions of the shape modes in both one-one and one-two resonant interactions.

### 4-6.1 Amplitude Equations

In this section, we consider the one-two resonant interactions between volume mode and the \( n \)-th shape oscillation mode of an ideal gas bubble. Here, we introduce the smallness parameter \( \epsilon \) to expand \( f \) and \( \phi \) by successive orders as follows:
\[
r = 1 + f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \cdots, \tag{4-79}
\]
\[
\phi = \epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots. \tag{4-80}
\]
For one-two resonance, the natural frequency for the volume mode, \( \omega_0 \), is nearly twice that of the \( n \)-th shape oscillation mode, \( \omega_n \). Hence we assume
\[
\omega_0 = 2\omega_n + \epsilon \beta_0, \tag{4-81}
\]
where \( \beta_0 \) is the frequency detuning parameter and \( \beta_0 = 0 \) corresponds to exact resonance. Following Feng’s notation, see Feng & Leal (1993), we assume that
\[
f_1 = \frac{1}{2}[\alpha_{1,0}(\tau)e^{i\omega_n t} + \alpha_{1,n}(\tau)e^{i\omega_n t}P_n(\eta)] + c.c., \tag{4-82}
\]
\[
\phi_1 = -i\omega_n\left[\frac{1}{r}\alpha_{1,0}(\tau)e^{i\omega_n t} + \frac{1}{2(n + 1)r_{n+1}}\alpha_{1,n}(\tau)e^{i\omega_n t}P_n(\eta)\right] + c.c., \tag{4-83}
\]
where \( \tau = \epsilon t \) is the slow time scale. Obviously, the above assumed forms satisfy the governing equation Eq.(2-109) and boundary conditions Eq.(4-76) and (4-77) at \( O(\epsilon) \). The solvability conditions for \( f_2 \) and \( \phi_2 \) to be bounded require that
\[
\frac{d\alpha_{1,0}}{d\tau} = i\beta_0 \alpha_{1,0} + iH_5 \alpha_{1,n}^2, \tag{4-84}
\]
\[
\frac{d\alpha_{1,n}}{d\tau} = iH_6 \alpha_{1,0} \alpha_{1,n}^*, \tag{4-85}
\]
where
\[
H_5 = \frac{(4n - 1)\omega_n}{16(n + 1)(2n + 1)}, \quad H_6 = \frac{(4n - 1)\omega_n}{4}, \tag{4-86}
\]
as defined in Feng & Leal (1993).

4-6.2 Hamiltonian and the first integral

In order to simplify Eq(4-84) and (4-85), we rescale the variables as follows:

\[ \alpha_{1,0} = \frac{2}{H_6} \alpha_0, \quad \alpha_{1,n} = \sqrt{\frac{2}{H_5 H_6}} \alpha_n. \]  

(4-87)

Then Eq(4-84) and (4-85) can be rewritten as

\[ \alpha_0 = i \beta_0 \alpha_0 + i \alpha_n^2, \]  

(4-88)

\[ \alpha_n = 2 i \alpha_0 \alpha_n^*. \]  

(4-89)

Since \( \alpha_0 \) and \( \alpha_n \) are both complex, we introduce the following change of variables to reformulate Eq(4-88) and (4-89) into a dynamical system of real variables:

\[ \alpha_0 = I e^{2i \eta}, \]  

(4-90)

\[ \alpha_n = (x + iy)e^{i \eta}, \]  

(4-91)

where \( I = \sqrt{p - \frac{1}{2} (x^2 + y^2)} \). From Eq(4-82), (4-83), (4-87), and (4-90), we know that the real part of \( \alpha_0 \) stands for the slowly-varying shape perturbation of the volume mode to the order \( O(\epsilon) \), while the imaginary part represents the slowly-varying velocity perturbation. For convenience, we will choose \( \eta(0) = 0 \). With this choice, \( x \) represents the shape perturbation of the \( n \)-th shape oscillation mode and \( y \) the velocity perturbation. And the amplitude of the perturbation is then given by \( \sqrt{x^2 + y^2} \). Combining Eq(4-88), (4-90), and (4-91), we obtain

\[ i \dot{\eta} = -2xy, \]  

(4-92)

\[ \dot{\eta} \quad = \frac{\beta_0}{2} + \frac{1}{2I} (x^2 - y^2). \]  

(4-93)

Combining Eq(4-89), (4-90), and (4-91), we have

\[ \dot{x} = 2Iy + \frac{1}{2} \beta_0 y + \frac{y}{2I} (x^2 - y^2), \]  

(4-94)

\[ \dot{y} = 2Ix - \frac{1}{2} \beta_0 x - \frac{x}{2I} (x^2 - y^2), \]  

(4-95)

Furthermore, we have

\[ \ddot{p} = 2I \dot{\eta} + x \dot{x} + y \dot{y} = 0. \]  

(4-96)

Eq(4-96) gives us a first integral of the dynamical system, and reveals that the Hamiltonian \( H \) of the system can be written as

\[ H = \frac{\beta_0}{4} (x^2 + y^2) - (x^2 - y^2) \sqrt{p - \frac{1}{2} (x^2 + y^2)} - \frac{1}{2} \beta_0 p. \]  

(4-97)
Figure 4-1 Phase diagram for the n-th Legendre mode perturbation imposed on the volume mode oscillation with frequency detuning ratios $\kappa = 0.8, -0.8, 5.0, -5.0$, respectively. Note that the stability of the spherical volume mode oscillation is represented by the stability of the origin. (a),(b) show that the origin is a center and thus stable for $\kappa_0$ small. (c),(d) reveals that the origin is a saddle point and thus unstable when $\kappa_0$ is large.
4-6.3 Stability of shape oscillation modes

In order to study the stability of the shape oscillation modes, we linearize Eq.(5-7) and (4-95) about the origin as follows

\[ \dot{x} = (2\sqrt{p} + \frac{1}{2} \beta_0) y, \quad (4-98) \]

\[ \dot{y} = (2\sqrt{p} - \frac{1}{2} \beta_0) x. \quad (4-99) \]

The two eigenvalues are \( \lambda_{1,2} = \pm \frac{1}{2} \beta_0 \sqrt{\kappa^2 - 1} \), where \( \kappa = \frac{4\sqrt{p}}{\beta_0} \). If \( \kappa^2 < 1 \) or \( \beta_0 > 4\sqrt{p} \), then the two eigenvalues are pure imaginaries and the origin is a center hence stable. If \( \kappa^2 > 1 \) or \( \beta_0 < 4\sqrt{p} \), then the two eigenvalues are real and of opposite signs therefore the origin is a saddle point thus unstable. Therefore, \( \kappa^2 = 1 \) or \( \beta_0^2 = 16p \) marks the boundary between the stable and unstable cases. Let’s go back to the physical space. The stability boundaries are defined by

\[ \beta_0^2 = 4(H_2^2 |\alpha_{1,0}|^2 + H_5 H_6 |\alpha_{1,n}|^2) \quad (4-100) \]

After a proper rescaling of the variables \( x = \sqrt{p} \tilde{x} \) and \( y = \sqrt{p} \tilde{y} \), the Hamiltonian \( H \) can be written as follows:

\[ H = \frac{\beta_0 p}{4} [(\tilde{x}^2 + \tilde{y}^2) - \kappa(\tilde{x}^2 - \tilde{y}^2)\sqrt{1 - \frac{1}{2} (\tilde{x}^2 + \tilde{y}^2)^2} - 2]. \quad (4-101) \]

Note that the topological behavior of the Hamiltonian \( H \) is determined by the parameter \( \kappa \) only, though two parameters are required for the Hamiltonian. With the Hamiltonian given in Eq.(4-101), we can sketch the orbits very easily. See Figure 4-1 (a)-(d). Analytically, the stability of shape oscillation mode is nothing but the local stability of the fixed point at the origin. Therefore, it can be easily identified by the eigenvalues of the linearized system about the origin. But how to characterize the stability of the shape oscillation mode numerically? Intuitively, we might try to impose some small \( n \)-th mode shape perturbation on the bubble and see whether the amplitude of the perturbation (the modulus of the vector \( (x, y) \)) grows or not to determine the stability of that particular mode. If it grows, it is said to be unstable. Otherwise, it is stable. Quite surprisingly, this is not always the case. Figure 4-1 (b) disapproves the naive idea. Since for point \( A \) (on the \( x \)-axis, which means a pure shape perturbation is imposed), the amplitude of the initial perturbation does grow even though it is stable. See Figure 4-1 (b). For point \( B \) (on the \( y \)-axis, which means a pure velocity perturbation is imposed), however, the amplitude of the initial perturbation always decays. See Figure 4-1 (b). This suggests that in order to distinguish the stable and unstable cases we need to choose different kinds of initial perturbation for \( \beta_0 > 0 \) and \( \beta_0 < 0 \), separately. Figure 4-1 (a) and (c) suggest that for \( \beta_0 > 0 \) to distinguish the qualitative difference between stable and unstable cases by the amplification of initial perturbation, the initial perturbation should be given in terms of shape. In Figure 4-1 (a), the initial shape
perturbation (on $x$-axis) is not increased due to the fact that $x$-axis is the major axis. In Figure 4-1 (b), the initial shape perturbation is always amplified due to the fact that the point on the $x$-axis is the closest point to the origin on the orbit. On the contrary, Figure 4-1 (b) and (d) suggest that for $\beta_0 < 0$ the initial perturbation should be given in terms of velocity (on the $y$-axis) in order to differentiate the stable cases from the unstable ones by the amplification of the initial perturbation. Here the analytic approach, though limited by the small amplitude assumption, gives the guideline for choosing the right perturbation. Without the analytic guidelines, we might not be able to get a conclusive numerical results.

REFERENCES


Chapter 5

Numerical Implementation

5-1 Introduction

The perturbation analysis introduced in Chapter 4 is not appropriate for the analysis of the nonlinear dynamics of drops and bubbles with large amplitude oscillations. The goal of this chapter is to take the numerical approach to implement a computer program that is capable of analyzing the nonlinear dynamics of acoustically levitated drops and cavitation bubbles. The major steps starting from formulation through implementation to verification are detailed in this chapter.

In Section 5-2, the problem statement for the physical model proposed in chapter 2 is reformulated and a conceptual algorithm to solve the problem is stated. In Section 5-3, boundary element method is chosen to approximate and solve the governing equation given in Section 5-2 with a discrete model. The fourth order Runge-Kutta method is used to integrate the boundary conditions Eq.(5-2),(5-3) and to update the configuration for the next time step. In Section 5-4, several issues concerning the stability of time integration are addressed. Based on the conservation laws, several examples are used to verify the accuracy of the code in Section 5-5. Dynamics presented in terms of Legendre mode decomposition are tested against the theoretical results also.

5-2 Problem statement

In chapter 2, the system of differential equations governing the velocity potential $\phi$ are derived based on Eulerian description. For the purpose of easy implementation, Lagrangian description is chosen to advance the interface by following the motion of material particles on the interface. A restatement encompassing both the drop case and the bubble case can be written as

$$\nabla^2 \phi = 0 \quad \text{in} \ V,$$

(5-1)
\[ \frac{DR}{Dt} = \nabla \phi \quad \text{on } S, \quad (5-2) \]
\[ \frac{D\phi}{Dt} = \frac{1}{2} \nabla \phi \cdot \nabla \phi + q \nabla \cdot n + p \quad \text{on } S, \quad (5-3) \]
where \( p = -p_o + Bz, q = -1 \) for the drop case and \( p = -p_g + p_\infty, q = +1 \) for the bubble case. \( \frac{D}{Dt} \) stands for the material derivative following a fixed material particle.

For the bubble case, we have an additional boundary condition at infinity:
\[ \phi = O \left( \frac{1}{r} \right) \rightarrow 0, \quad \text{as } r \rightarrow \infty. \quad (5-4) \]

The governing equation, Eq.(5-1), applies to both steady and unsteady flow. Yet no dependence of time appears explicitly in this equation. This implies that as a result of incompressibility, the irrotational flow can be determined by the instantaneous boundary conditions. In other words, any change in the boundary conditions propagates throughout the whole domain immediately. Consequently, the velocity potential can be fully determined from the status quo of boundary data without using any information from the dynamic boundary condition\(^1\).

Accordingly, the problem can be easily solved with the following procedures:

(1) Setup the initial conditions of the problem.

(2) Solve the Laplace equation, Eq.(5-1), subjected to the current boundary conditions by boundary element method.

(3) Use 4-th order Runge-Kutta method to integrate Eq.(5-2) and (5-3) numerically.

(4) Advance the geometric configuration and values of velocity potential on boundary to the next time step.

(5) Repeat steps (2)-(4) till the final time.

5-3 Boundary element method

As a natural consequence of the rapid emergence of powerful computers since 1960, techniques for solving boundary value problems numerically have been well developed due to their importances in many branches of mathematical physics and engineering. More than often, numerical approach is the only resort available to analyze and understand the dynamics of a complicated system, either mathematical or physical.

\(^1\) A wave equation has to be solved, if the compressibility were accounted for. The second order time derivative term in the wave equation relies on information coming from the dynamic boundary condition, thus the wave equation cannot be solved independently. All three equations are coupled and must be solved simultaneously.
Most likely, finite difference is the earliest method ever devised for solving BVP. With values defined only at nodes on the uniform mesh, it approximates the original differential equations with truncated Taylor series expansion of variables locally. A system of linear algebraic equations is thus obtained and solved. Though simple and easy to implement, a serious drawback of the finite difference is the awkwardness in dealing with irregular geometric boundary.

Finite element method based on the weak formulation (balanced form) derived from the variational principle approximates the differential equations with a system of linear algebraic equations through element discretization. The versatility of the element discretization avoids or at least lessens the difficulty of resolving the irregular geometric boundary. Combined with Galerkin's method, the coefficient matrix is sparse, symmetric and positive-definite. Nowadays, any problem for which governing equations can be written down is a candidate for the Galerkin method. This prominent feature leads to the success of FEM in science, engineering, and even in industry.

Boundary element method (henceforth referred to as BEM), however, relies on the inverse formulation (boundary integral equation)\(^2\) resulted from applying the Green's second identity to the original differential equation. A system of linear algebraic equations is also obtained through element discretization. Unlike FEM, the coefficient matrix for BEM is fully populated and asymmetric. Two distinguished features of BEM are that it reduces the dimension of the problem by one and it applies to problems involving not only finite domains but also infinite domains. Furthermore, better accuracy can be achieved since the method is semi-analytic.

5.3.1 Mathematical statement for BEM

The standard BVP to solve in this thesis by BEM is defined below:

Let \( S_0 \) be a closed bounding surface enclosing a finite volume \( V_0 \) in the \( R^3 \) space. We are looking for a solution to the problem governed by

\[
L \phi = 0 \quad \text{in} \ V,
\]

where \( L \equiv \nabla^2 \) for potential problem, and \( L \equiv \nabla^2 + k^2 \) for acoustic radiation problem. \( k \) is the wavenumber in the context of acoustic scattering.

The boundary conditions to satisfy are

\[
\phi = \phi^* \quad \text{on} \ S_1, \quad \text{(Dirichlet)} \tag{5-6}
\]

\[
\frac{\partial \phi}{\partial n} = q \quad \text{on} \ S_2, \quad \text{(Riemann)} \tag{5-7}
\]

\[
c_1 \phi + c_2 \frac{\partial \phi}{\partial n} = 0 \quad \text{on} \ S_3, \quad \text{(Robin)} \tag{5-8}
\]

\(^2\) A very insightful classification of the approximate methods can be found in Brebbia, Telles & Wrobel (1984).
where \( S_1, S_2, \) and \( S_3 \) are disjoint sets and \( S_0 = S_1 \cup S_2 \cup S_3 \).

Two types of problems can be defined according to the choice of \( \mathbf{V} \). If \( \mathbf{V} = \mathbf{V}_0 \), it is called an interior problem (such as drops, Figure 5–1). If \( \mathbf{V} = \mathbf{V}_0^\ast \), it is classified as an exterior problem (such as bubbles, Figure 5–2).

For the exterior problem to be well-posed, additional boundary conditions at infinity need to be specified. Due to the physical setup we are interested in, only the following boundary condition at infinity is considered:

\[
\phi = O \left( \frac{1}{r} \right) \to 0, \quad \text{as } r \to \infty.
\]  \hspace{1cm} (5–9)

5-3.2 Direct formulation

Integral equations have been used to solve the BVP such as potential problem ever since the early 1900's. Two approaches can be taken to transform the original
BVP into integral equations: indirect formulation and direct formulation. Indirect formulation makes use of the fundamental solutions to construct the single-layer or double-layer potentials generated by some unknown continuous source distributions over the Liapunov (smooth) boundary. Numerical methods are then used to find the source distributions so that the prescribed boundary conditions can be satisfied by the potentials constructed.\(^3\) On the other hand, direct formulation applies the Green's second identity to reformulate the original BVP into a powerful boundary integral equation (referred to as BIE), which provides an excellent starting point for solving the BVP numerically.

It was generally believed that indirect formulation is better than the direct formulation because less calculation is required, especially in those days computers were not prevailing. However, there are two advantages of the direct formulation over the indirect one. First, it provides solutions that corresponds to the physical quantities directly, whereas using the simple- and double-layer representations, the source distributions obtained commonly lack physical meaning and post-processing is required to obtain the physical quantities that we are interested. Another advantage is that the restriction for the boundary surface to be Liapunov smooth can be relaxed in direct formulation. It can be applied to the more general Kellogg regular surfaces which allows corners and edges to be included. See Brebbia, Telles & Wrobel (1984). Nowadays, most BEM commercial codes adopt the direct formulation and so does the present thesis.

Green's second identity relates the volume integral over \( \mathbf{V} \) bounded by \( \mathbf{S} \) to the surface integral over \( \mathbf{S} \) for any two functions \( \phi \) and \( \psi \) sufficiently smooth and nonsingular in \( \mathbf{V} \):

\[
\int_{\mathbf{S}} (\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}) dS = \int_{\mathbf{V}} (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dV, \tag{5-10}
\]

where \( \mathbf{n} \) is the outward unit normal of \( \mathbf{S} \) pointing away from \( \mathbf{V} \).

Let \( \phi^* \) be the free-space Green's function (fundamental solution) of the operator \( \mathcal{L} \) with the source point located at point \( \mathbf{x}_0 \) so that

\[
\mathcal{L}\phi^* + 4\pi \delta(\mathbf{x} - \mathbf{x}_0) = 0, \tag{5-11}
\]

where \( \delta \) is the Dirac delta function, and \( \mathbf{x} \) is a generic field point.

The 3D fundamental solution for the Laplacian operator is

\[
\phi^* = \frac{1}{R}, \tag{5-12}
\]

and that for the Helmholtz operator is

\[
\phi^* = \frac{e^{-ikR}}{R}. \tag{5-13}
\]

\(^3\) A brief discussion on the indirect formulation is given in Brebbia, Telles & Wrobel (1984).
In both cases, $R = |x - x_0|$ is the distance between the interested field point $x$ and the source point $x_0$.

Now consider $x_0$ as a source point on the $S_0$. See Figure 5-1, 5-2. Let us choose $\psi = \phi^*$ and apply the Green’s second identity, Eq.(5 10), to reformulate the BVP defined by Eq.(5-5), (5-6), and (5-7) into a BIE. We note that $\phi^*$ is singular at the point $x_0$. Analogous to the complex contour integral, a principal value procedure is taken. To remove the singularity at point $x_0$, we delete a small ball $B(x_0, \epsilon)$ of radius $\epsilon$ centered at $x_0$ from the domain $V$ and denote the new bounding surface as $\tilde{S}$ and the new volume as $\tilde{V}$. Since $L\phi = 0$, and $L\phi^* = 0$ for $x \neq x_0$, we get

$$\phi^* L\phi - \phi L\phi^* = \phi \nabla^2 \phi^* - \phi^* \nabla^2 \phi = 0 \text{ in } \tilde{V}. \quad (5-14)$$

For interior problem, we have

$$\int_{\tilde{S}} (\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n}) dS = 0, \text{ and} \quad (5-15)$$

$$\tilde{S} = S^* \cup \sigma_i. \quad (5-16)$$

For exterior problem, we have

$$\int_{\tilde{S} + \Sigma} (\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n}) dS = 0, \text{ and} \quad (5-17)$$

$$\tilde{S} = S^* \cup \sigma_o. \quad (5-18)$$

Here, $S_\epsilon = S_0 \cap B(x_0, \epsilon)$ is the part of the interface $S_0$ enclosed by the deleted small ball $B(x_0, \epsilon)$, $S^* = S_0 - S_\epsilon$ is the remainder of the interface after the small ball is deleted, and $\Sigma$ is the boundary at infinity for the exterior problem. The additional boundary condition at infinity Eq.(5-9) ensures that

$$\int_{\Sigma} (\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n}) dS = 0. \quad (5-19)$$

Therefore, we can combine Eq.(5-15), and (5-17) into one general BIE:

$$\int_{S^*} (\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n}) dS = - \int_{\sigma} (\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n}) dS. \quad (5-20)$$

For exterior problem, we have $\sigma = \sigma_o$. While for interior problem, $\sigma = \sigma_i$. Taking the limit $\epsilon \to 0$ for the integral over $\sigma$, we obtain

$$(P.V.) \int_{S_\epsilon} (\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n}) dS = - \lim_{R=\epsilon \to 0} \int_{\sigma} (\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n}) dS$$

$$= + \lim_{R=\epsilon \to 0} \int_{\sigma} (\phi \frac{\partial \phi^*}{\partial R} - \phi^* \frac{\partial \phi}{\partial R}) dS \quad (5-21)$$

$$= \lim_{R=\epsilon \to 0} \int_{\sigma} \frac{\partial \phi^*}{\partial R} dS \phi(x_0) \quad (5-22)$$

which results from using a spherical coordinate system with the origin located at $x_0$. Hence, $\partial/\partial n = -\partial/\partial R$. For convenience, we define

$$C^o(x_0) = \lim_{R=\epsilon \to 0} \int_{\sigma_o} \frac{\partial \phi^*}{\partial R} dS, \quad (5-23)$$
\[ C^i(x_0) = \lim_{R=\varepsilon \to 0} \int_{\sigma_i} \frac{\partial \phi^*}{\partial R} dS. \] (5-24)

Then we have
\[ C^o(x_0) + C^i(x_0) = -4\pi. \] (5-25)

Define \( \nu \) to be the outward unit normal of the surface \( S_0 \) pointing away from the enclosure of \( S_0 \). Then for interior problem we have
\[ (\text{P.V.}) \int_{S_0} (\phi \frac{\partial \phi^*}{\partial \nu} - \phi^* \frac{\partial \phi}{\partial \nu}) dS = C^i(x_0) \phi(x_0), \] (5-26)
while for exterior problem, we have
\[ (\text{P.V.}) \int_{S_0} (\phi \frac{\partial \phi^*}{\partial \nu} - \phi^* \frac{\partial \phi}{\partial \nu}) dS = -C^o(x_0) \phi(x_0). \] (5-27)

Note that \( \phi \) can be arbitrary harmonic function in domain \( V \). If we choose \( \phi = 1 \), then from Eq.(5-26) we have
\[ C^i(x_0) = \int_{S_0} \frac{\partial \phi^*}{\partial \nu} dS \] (5-28)

Thus for interior problem, we have the boundary integral defined on the surface \( S_0 \) as follows
\[ (\text{P.V.}) \int_{S_0} (\phi \frac{\partial \phi^*}{\partial \nu} - \phi^* \frac{\partial \phi}{\partial \nu}) dS = \int_{S_0} \frac{\partial \phi^*}{\partial \nu} dS \phi(x_0). \] (5-29)

While for the exterior problem, we have
\[ (\text{P.V.}) \int_{S_0} (\phi \frac{\partial \phi^*}{\partial \nu} - \phi^* \frac{\partial \phi}{\partial \nu}) dS = (4\pi + \int_{S_0} \frac{\partial \phi^*}{\partial \nu} dS) \phi(x_0), \] (5-30)

Eq.(5-29), and (5-30) are called the surface integral formulations for the interior and exterior problems, respectively.

Generally, the above integral formulations do not admit closed form solutions, and numerical approximation must be performed. A common procedure is to discretize the boundary \( S_0 \) and the relevant boundary data \( \phi \) and \( \bar{q} \) as done in the FEM, then numerical integration is done to evolve the surface integrals. System of linear algebraic equations are then obtained and boundary conditions are imposed. Finally, the linear algebraic equations are solved for the unknowns at the nodal points.

### 5-3.3 Special treatment for scattering problem

It is shown in chapter 3 that the spatial part of the total velocity potential for a time-harmonic acoustic field is governed by the Helmholtz equation. However, it belongs to neither interior nor exterior problem. Thus the above integral forms do not apply and special treatment is required. Since the total acoustic field consists of the incident wave and the scattered wave, the total velocity potential \( \phi_T \) is the sum
of the incident wave velocity potential $\phi_{inc}$ and the scattered wave velocity potential $\phi_s$, i.e.

$$\phi_T = \phi_{inc} + \phi_s.$$  \hspace{1cm} (5-31)

For the incident wave, the Helmholtz equation is valid for the whole space. Hence it is an interior problem. We then have

$$\int_{S_0} (\phi_{inc} \frac{\partial \phi^*}{\partial \nu} - \phi^* \frac{\partial \phi_{inc}}{\partial \nu}) dS = C^i(x_0)\phi_{inc}(x_0).$$  \hspace{1cm} (5-32)

However, for the scattered wave the Helmholtz equation is valid for the domain outside the cavity. Therefore, it is an exterior problem. And we obtain

$$\int_{S_0} (\phi_s \frac{\partial \phi^*}{\partial \nu} - \phi^* \frac{\partial \phi_s}{\partial \nu}) dS = -C^o(x_0)\phi_s(x_0),$$  \hspace{1cm} (5-33)

Substituting $\phi_s = \phi_T - \phi_{inc}$ and Eq.(5-32) into Eq.(5-33), we have

$$\int_{S_0} (\phi_T \frac{\partial \phi^*}{\partial \nu} - \phi^* \frac{\partial \phi_T}{\partial \nu}) dS + 4\pi \phi_{inc}(x_0) = (4\pi + \int_{S_0} \frac{\partial \phi^*}{\partial \nu} dS)\phi_T(x_0).$$  \hspace{1cm} (5-34)

It is well known that for acoustic radiation problem the surface Helmholtz integral formulation does not have a unique solution at certain frequencies corresponding to the eigenfrequencies of the interior problem. Though important, this issue is irrelevant to the current thesis, thus will not be pursued in the present work! Interested readers are referred to Schenck (1968), Seybert & Rengarajan (1987) and Chen & Zhou (1992).

5-3.4 Axisymmetric problem

The discussion in this subsection will demonstrate how the surface integral can be simplified for problems with axisymmetry. For this special case part of the surface integral can be reduced to a line integral along the generator while the integral over the azimuthal angle can be expressed in terms of complete elliptic integrals analytically. Consequently, only the integral along the generator needs to be discretized and evaluated numerically.

From differential geometry, the surface element $dS$ on a surface defined by

$$r = r(\xi, \eta)$$  \hspace{1cm} (5-35)

can be written as

$$dS = \left| \frac{\partial r}{\partial \xi} \times \frac{\partial r}{\partial \eta} \right| d\xi d\eta.$$  \hspace{1cm} (5-36)

Thus for a body of revolution with generator described by $r = f(z)$ in the polar coordinates $(r, \theta, z)$, the surface can be expressed as

$$r = f(z) \cos(\theta)e_r + f(z) \sin(\theta)e_\theta + ze_z,$$  \hspace{1cm} (5-37)

and the surface element can be written as

$$dS = F(z)d\theta dz,$$  \hspace{1cm} (5-38)
where

$$F(z) = f(z)\sqrt{1 + f'(z)^2}. \quad (5-39)$$

For the Laplace equation, the surface integral on the left hand side of Eq.(5-29) and (5-30) can be simplified as follows:

$$\int_{S_0} \left[ \phi \frac{\partial}{\partial \nu} \left( \frac{1}{R} \right) - \left( \frac{1}{R} \right) \frac{\partial \phi}{\partial \nu} \right] dS = \int_{0}^{2\pi} \left[ \phi \frac{\partial}{\partial \nu} \left( \frac{1}{R} \right) - \left( \frac{1}{R} \right) \frac{\partial \phi}{\partial \nu} \right] d\theta F(z) dz, \quad (5-40)$$

where \(x_0 = r_0 e_r + z_0 e_z\) is a source point on the generator and

$$R = \sqrt{r^2 + r_0^2 - 2rr_0 \cos \theta + (z - z_0)^2}. \quad (5-41)$$

Due to axisymmetry, \(\phi\) and \(\frac{\partial \phi}{\partial \nu}\) do not depend on \(\theta\), one can further simplify Eq.(5-40) as follows:

$$\int \left[ \bar{q}^* \phi - \bar{u}^* \frac{\partial \phi}{\partial \nu} \right] F(z) dz, \quad (5-42)$$

where

$$\bar{u}^*(r, z, r_0, z_0) = \int_{0}^{2\pi} \frac{d\theta}{R}, \quad \text{and} \quad \bar{q}^*(r, z, r_0, z_0) = \int_{0}^{2\pi} \frac{\partial \phi}{\partial \nu} \left( \frac{1}{R} \right) d\theta. \quad (5-43)$$

Similarly, the surface integral on the right hand side of Eq.(5-29) and (5-30) can be simplified as follows:

$$\int_{S_0} \frac{\partial}{\partial \nu} \left( \frac{1}{R} \right) dS = \int \bar{q}^* F(z) dz. \quad (5-44)$$

Define

$$a = r^2 + r_0^2 + (z - z_0)^2, \quad b = 2rr_0, \quad m = \frac{2b}{a + b}, \quad (5-45)$$

then \(\bar{u}^*\) and \(\bar{q}^*\) can be expressed in terms of complete elliptic integrals of the first kind \(K(m)\) and the second kind \(E(m)\) as follows:

$$\bar{u}^* = \frac{4}{\sqrt{a + b}} K(m), \quad (5-46)$$

$$\bar{q}^* = \frac{4}{\sqrt{a + b}} \left\{ \frac{1}{2r} \left[ \frac{r_0^2 - r^2 + (z_0 - z)^2}{a - b} E(m) - K(m) \right] \nu_r + \frac{z_0 - z}{a - b} E(m) \nu_z \right\}. \quad (5-47)$$

For the Helmholtz equation, the surface integral on the left hand side of Eq.(5-34) can be divided into two parts as follow:

$$\int_{0}^{2\pi} \left[ \phi_T \frac{\partial}{\partial \nu} \left( \frac{1}{R} \right) - \left( \frac{1}{R} \right) \frac{\partial \phi_T}{\partial \nu} \right] d\theta F(z) dz + \quad (5-48)$$

---

\(^4\) By definition,

$$K(m) = \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 - m \cos^2 \phi}}, \quad E(m) = \int_{0}^{\pi/2} \sqrt{1 - m \cos^2 \phi} d\phi.$$

Polynomial approximations of \(K(m)\) and \(E(m)\) with high accuracy is given in Appendix C.
\[ \int_0^{2\pi} \left[ \phi_T \frac{\partial}{\partial \nu} \left( \frac{e^{-ikR} - 1}{R} \right) - \left( \frac{e^{-ikR} - 1}{R} \right) \frac{\partial \phi_T}{\partial \nu} \right] d\theta F(z) dz. \]

Further simplification gives

\[ \int \left[ q^* \phi_T - \vec{u}^* \frac{\partial \phi_T}{\partial \nu} \right] F(z) dz + \int \left[ K_A \phi_T - K_B \frac{\partial \phi_T}{\partial \nu} \right] F(z) dz, \tag{5-49} \]

where

\[ K_A(r, z, r_0, z_0) = \int_0^{2\pi} \frac{\partial}{\partial \nu} \left( \frac{e^{-ikR} - 1}{R} \right) d\theta, \tag{5-50} \]

and

\[ K_B(r, z, r_0, z_0) = \int_0^{2\pi} \left( \frac{e^{-ikR} - 1}{R} \right) d\theta. \tag{5-51} \]

Similarly, the surface integral on the right hand side of Eq.(5-34) can be simplified as follows:

\[ \int \frac{\partial}{\partial \nu} \left( \frac{e^{-ikR}}{R} \right) dS = \int q^* F(z) dz + \int K_A F(z) dz. \tag{5-52} \]

Unfortunately, \( K_A \) and \( K_B \) must be evaluated numerically. Fortunately the kernels are not singular thus can be integrated with Gaussian quadrature without difficulty.

In summary, the BIE's with axisymmetry for the interior and exterior proble, are:

\[ \int \left[ q^* \phi - \vec{u}^* \frac{\partial \phi}{\partial \nu} \right] F(z) dz = \left( \int q^* F(z) dz \right) \phi(x_0) \quad \text{ (interior),} \tag{5-53} \]

and

\[ \int \left[ q^* \phi - \vec{u}^* \frac{\partial \phi}{\partial \nu} \right] F(z) dz = (4\pi + \int q^* F(z) dz) \phi(x_0) \quad \text{ (exterior).} \tag{5-54} \]

And the BIE with axisymmetry governing the total velocity potential for the scattering problem is

\[ \int \left[ q^* \phi_T - \vec{u}^* \frac{\partial \phi_T}{\partial \nu} \right] F(z) dz + \int \left[ K_A \phi_T - K_B \frac{\partial \phi_T}{\partial \nu} \right] F(z) dz + 4\pi \phi_{inc}(x_0) = \left( 4\pi + \int q^* F(z) dz + \int K_A F(z) dz \right) \phi_T(x_0) \quad \text{(scattering).} \tag{5-55} \]

Since there is no possibility of confusion, we will drop the subscript \( T \) hereafter.

### 5-3.5 Discretization

Eq.(5-53)-(5-55) consists of a line integral along the generator only, but numerical evaluations of these BIE's are impeded by the fact that \( \phi, \frac{\partial \phi}{\partial n} \) are still unknown. In order to make the BIE's more amenable to computer routines and conform with its limited storage, discretization is employed to represent the unknowns with values at finite number of nodes deployed on the generator. Although the shape (generator) of the body is explicitly defined, it is also discretized into elements with the same
number of nodes on each. Piecewise polynomial approximation is then used to interpolate the variations of physical quantities along the element locally. Isoparametric quadratic element is used in the present thesis.

Nodes and Elements

Let the generator be partitioned into \( M \) quadratic elements by \( N \) nodes \( x_j, j = 1, 2, \ldots, N \) with polar coordinates \((r_j, z_j)\). There are three nodes labeled with local node number \( \alpha = 1, 2, 3 \) within each element. Hence \( N = (2M + 1) \).

A connectivity matrix \( g \) is defined to relate the local node number \( \alpha \) of the \( i \)-th element with its global node number \( j \) as follows:

\[
j = g[i, \alpha].
\]  

(5-56)

For convenience, we will denote \( \frac{\partial \phi}{\partial \nu} \) by \( \psi \). As a shorthand notation, we use the superscript of a variable to represent the element number and subscript for the node number. Variables with no superscript are considered as global variables (except for shape functions to be mentioned later!). Variables with no subscript are considered as a local function defined within that particular element. Thus \( r^i_\alpha \) denotes the \( r \) coordinate of the \( \alpha \)-th node within \( i \)-th element. Obviously, we have

\[
r_j = r^i_\alpha, \quad \text{if} \; j = g[i, \alpha].
\]  

(5-57)

Shape functions

By piecewise polynomial approximation, the values of \( r, z, \phi, \psi \) of a generic point within \( i \)-th element is assumed to be interpolated in terms of the nodal values \( r^i_\alpha, z^i_\alpha, \phi^i_\alpha, \psi^i_\alpha, (\alpha = 1, 2, 3) \) as follows\(^5\):

\[
\begin{align*}
    r^i(\xi) &= \sum_{\alpha=1}^{3} N_\alpha(\xi)r^i_\alpha, \\
    z^i(\xi) &= \sum_{\alpha=1}^{3} N_\alpha(\xi)z^i_\alpha, \\
    \phi^i(\xi) &= \sum_{\alpha=1}^{3} N_\alpha(\xi)\phi^i_\alpha, \\
    \psi^i(\xi) &= \sum_{\alpha=1}^{3} N_\alpha(\xi)\psi^i_\alpha.
\end{align*}
\]  

(5-58)

\( N_\alpha(\xi) \) are the quadratic shape functions defined as follows:

\[
\begin{align*}
    N_1(\xi) &= \frac{1}{2}\xi(\xi - 1), \\
    N_2(\xi) &= 1 - \xi^2, \\
    N_3(\xi) &= \frac{1}{2}\xi(\xi + 1),
\end{align*}
\]  

(5-59)

where \(-1 \leq \xi \leq 1\) is the local coordinate.

\(^5\) Special consideration must be taken to prevent \( r \) from becoming negative.
Discretized BIE

With the change of variables \( z = z^t(\xi) \) within the \( i \)-th element, it can be easily shown that

\[
F(z)dz = r^t(\xi)J^t(\xi)d\xi,
\]

where \( J^t(\xi) \) is the Jacobian of the transformation and can be written as

\[
J^t(\xi) = \left[ \left( \frac{dr^t}{d\xi} \right)^2 + \left( \frac{dz^t}{d\xi} \right)^2 \right]^{1/2}.
\]

Substituting Eq.(5-58)-(5-60) into Eq.(5-53), we have

\[
\sum_{i=1}^{M} \sum_{\alpha=1}^{3} \left[ a^i_{\alpha} \phi^i_{\alpha} - b^i_{\alpha} \psi^i_{\alpha} \right] = c\phi(x_0), \quad \text{(interior)} \tag{5-62}
\]

where

\[
a^i_{\alpha}(r_0, z_0) \equiv \int_{-1}^{1} \bar{q}^* N_{\alpha}(\xi)r^t(\xi)J^t(\xi)d\xi, \tag{5-63}
\]

\[
b^i_{\alpha}(r_0, z_0) \equiv \int_{-1}^{1} \bar{u}^* N_{\alpha}(\xi)r^t(\xi)J^t(\xi)d\xi, \tag{5-64}
\]

\[
c(r_0, z_0) \equiv \sum_{i=1}^{M} \int_{-1}^{1} \bar{q}^* r^t(\xi)J^t(\xi)d\xi. \tag{5-65}
\]

Note the \((r_0, z_0)\) dependence of \(a^i_{\alpha}, b^i_{\alpha}\), and \(c\) come from the fundamental solutions \(\bar{u}^*\) and \(\bar{q}^*\), where \((r_0, z_0)\) is the coordinates of the corresponding source point \(x_0\).

Similarly, for Eq.(5-54) we have

\[
\sum_{i=1}^{M} \sum_{\alpha=1}^{3} \left[ a^i_{\alpha} \phi^i_{\alpha} - b^i_{\alpha} \psi^i_{\alpha} \right] = (4\pi + c)\phi(x_0). \quad \text{ (exterior)} \tag{5-66}
\]

From Eq.(5-55), we have

\[
\sum_{i=1}^{M} \sum_{\alpha=1}^{3} \left[ a^i_{\alpha} \phi^i_{\alpha} - b^i_{\alpha} \psi^i_{\alpha} \right] + 4\pi \phi_{\text{inc}}(x_0) = (4\pi + c)\phi(x_0), \quad \text{ (scattering)} \tag{5-67}
\]

where

\[
a^i_{\alpha}(r_0, z_0) \equiv \int_{-1}^{1} \left( \bar{q}^* + K_A \right) N_{\alpha}(\xi)r^t(\xi)J^t(\xi)d\xi, \tag{5-68}
\]

\[
b^i_{\alpha}(r_0, z_0) \equiv \int_{-1}^{1} \left( \bar{u}^* + K_B \right) N_{\alpha}(\xi)r^t(\xi)J^t(\xi)d\xi, \tag{5-69}
\]

\[
c(r_0, z_0) \equiv \sum_{i=1}^{M} \int_{-1}^{1} \left( \bar{q}^* + K_A \right)r^t(\xi)J^t(\xi)d\xi. \tag{5-70}
\]

Numerical evaluations of the integrals in this subsection can be done with the standard Gaussian quadrature and the logarithmic Gaussian quadrature depending
on the kernels involved! Due to the nature of the singularity associated with fundamental solutions \( u^* \) and \( q^* \), Gaussian quadratures can not be applied directly. Details of how to evaluate these integrals accurately are given in Su (1994).

5-3.6 Global assemblage

Eq.(5-62), (5-66), and (5-67) are expressed in terms of the local node number. Simplification is possible by rewriting them in terms of the global node number, thus known as global assemblage.

Since by definition\(^6\) we have

\[
\phi_j = \phi_{\alpha}^i \quad \text{and} \quad \psi_j = \psi_{\alpha}^i, \quad \text{if} \quad j = g[i, \alpha]. \tag{5-71}
\]

By collecting terms, one can rewrite Eq.(5-62) as follows:

\[
\sum_{j=1}^{N} (A_j \phi_j - B_j \psi_j) = c \phi(x_0), \quad \text{(interior)} \tag{5-72}
\]

where

\[
A_j(r_0, z_0) \equiv \sum_{i=1}^{M} \sum_{\alpha=1}^{3} \delta_{j}^{g[i, \alpha]} a_{\alpha}^i (r_0, z_0), \tag{5-73}
\]

\[
B_j(r_0, z_0) \equiv \sum_{i=1}^{M} \sum_{\alpha=1}^{3} \delta_{j}^{g[i, \alpha]} b_{\alpha}^i (r_0, z_0). \tag{5-74}
\]

Here \( \delta \) is the Dirac delta function and

\[
\delta_{j}^{i} = \begin{cases} 
1 & \text{if} \quad i = j \\
0 & \text{otherwise}
\end{cases}
\]

Similarly, Eq.(5-66) and (5-67) can be rewritten as:

\[
\sum_{j=1}^{N} (A_j \phi_j - B_j \psi_j) = (4\pi + c) \phi(x_0), \quad \text{(exterior)} \tag{5-75}
\]

\[
\sum_{j=1}^{N} (A_j \phi_j - B_j \psi_j) + 4\pi \phi_{\text{inc}}(x_0) = (4\pi + c) \phi(x_0). \quad \text{(scattering)} \tag{5-76}
\]

5-3.7 Collocation

Let us put the source point \( x_0 \) on the node \( x_k \), and define

\[
A_{kj} = A_j(r_k, z_k), \quad B_{kj} = B_j(r_k, z_k), \quad C_k = c(r_k, z_k). \tag{5-77}
\]

Then from Eq.(5-72), we have

\[
\sum_{j=1}^{N} (A_{kj} \phi_j - B_{kj} \psi_j) = C_k \phi_k. \quad \text{(interior)} \tag{5-78}
\]

\(^{6}\) A special treatment for problems with corners is given in Appendix D.
Let \( k \) vary from 1 to \( N \), then a system of \( N \) linear algebraic equations with \( 2N \) variables \((\phi_k, \psi_k, k = 1, 2, \cdots, N)\) can be obtained from Eq.(5-78).

Similarly, from Eq.(5-73) and (5-74), one have

\[
\sum_{j=1}^{N} (A_{kj} \phi_j - B_{kj} \psi_j) = (4\pi + C_k) \phi_k, \quad \text{(exterior)} \tag{5-79}
\]

\[
\sum_{j=1}^{N} (A_{kj} \phi_j - B_{kj} \psi_j) + 4\pi \phi_{\text{inc}}(x_k) = (4\pi + C_k) \phi_k. \quad \text{(scattering)} \tag{5-80}
\]

Likewise, a system of \( N \) linear algebraic equations can be obtained from Eq.(5-79) and (5-80).

### 5-3.8 Boundary conditions

Since there are \( N \) linear algebraic equations with \( 2N \) variables in the discretized BIE’s, Eq.(5-78), (5-79) and (5-80), the boundary conditions are supposed to provide the rest information needed to solve the problem. Indeed, for the Dirichlet or Riemann types of boundary conditions, either \( \phi \) or \( \psi \equiv \frac{\partial \phi}{\partial \nu} \) is known on each node and thus reduces the number of unknown variables in the discretized BIE’s into \( N \). Subsequently, the solutions can be obtained by solving the system of \( N \) linear algebraic equations with \( N \) unknowns. For the Robin type boundary conditions, however, it provides another \( N \) equations to form a system of \( 2N \) linear equations with \( 2N \) unknowns.

### 5-4 Time integration and stability

In this thesis, the explicit fourth order Runge-Kutta method is used to perform the time integration of Eq.(5-2) and (5-3). Undoubtedly, the first question to ask is “what is the appropriate time step to adopt?”. From the view point of efficiency, exceedingly small time step is undesirable. Due to the stiffness and nonlinearity inherited in the physical system, however, the stability requirement imposes a very stringent restriction on the time step allowed, which will make the calculation unnecessarily expensive. Since if the spatial discretization is crude, it makes no sense to solve the ODE very accurately at all. Several subtle points concerning the stiffness and nonlinearity will be addressed below.

In the celebrated paper by Courant, Friedrichs, & Lewy (1928), it is pointed out that the convergence of finite difference approximations for elliptic equations is independent of the choice of the mesh (Instead, the choice of the mesh is determined by the accuracy.). For the initial value problems of the hyperbolic type equations, however, the convergence is obtained only if the ratio of the mesh sizes in time and spatial dimensions satisfies certain inequalities. This condition is now classic and known as the CFL condition in the numerical analysis.
Is there a CFL condition for the incompressible flows (governed by Laplace equation, which is elliptic type.) associated with the drop or bubble problems? Here, we assert that there is a CFL-like condition for the size of time step to satisfy from a physical interpretation. Obviously, the resolution of the shape is determined by the number of nodes deployed on the interface. The more nodes deployed on the interface, the more mode shapes resolved. It is shown in chapter 2 that the natural frequency of the mode shape grows like \( n^{3/2} \) as the mode number \( n \) increases. Thus, the time step required to resolve the dynamics of these higher modes will certainly decrease. This is a typical phenomenon of stiffness.

Usually the stiffness may vary over the whole time interval of integration and exceedingly small time step is only needed for certain stage of the time integration. For the other stage, the exceedingly small time step is unnecessary. The relaxation phenomenon associated with nonlinearity even worsen this situation. Thus the algorithm of adaptive time step is needed for efficiency but not yet well established to date.

During the time integration, the evaluation of the right hand side of Eq.(5-3) requires the calculation of total curvature \( \nabla \cdot \mathbf{n} \). Curvature calculation is usually the most pathologic source for the simulations of problems with free surfaces. This is especially true for the element methods. With only finite nodes deployed on the interface, how to interpret the surface appropriately for the curvature calculation is not a trivial problem. From the differential geometry, the total curvature \( \nabla \cdot \mathbf{n} \) for an axisymmetric surface with generator described by \( (r(\xi),z(\xi)) \) can be expressed as

\[
\nabla \cdot \mathbf{n} = \frac{r(s) [r''(s)z'(s) - r'(s)z''(s)] - z'(s) [r'(s)^2 + z'(s)^2]}{r(s) [r'(s)^2 + z'(s)^2]^{3/2}}, \tag{5-81}
\]

where \( s \) is the arclength measured from the north pole along the generator.

It is generally believed that node clustering happens at locations with high curvature, e.g. See Tsamopoulos (1985). Since the curvature calculation involves higher order differential operators, it could be very sensitive to the integration error when the spacing between nodes is significantly reduced due to node clustering. In addition, the system may become stiffer locally due to the reduced spacing.

Currently, convergence can only be guaranteed by the repetition of the same simulations with smaller time steps. Theoretical estimation for the maximum size of the time step is not available. Empirically, the step size of 0.005 is required for a drop with 41 nodes and 0.001 for a bubble with 41 nodes oscillating at moderate amplitude. For a bubble with 41 nodes oscillating at large amplitude, the step size required is about 0.0001, which is very expensive.
5-5 Verification

In this section, the computer program implemented based on the above derivation is verified by conducting various simulations and then checking the conservation laws or testing against theoretical results, if available.

Figure 5–3 Second mode oscillations with initial conditions. $r = 1, \phi = 0.6P_3, N = 41, \Delta t = 0.005$. (a) Amplitude of the tip motion, $z(1) = 1.0$. (b) Energy tracing. The upper oscillatory curve is the kinetic energy, the lower oscillatory curve is potential energy, and the nearly straight line is the total energy.

5-5.1 Energy conservation and accuracy

Energy conservation provides the simplest measure to check the accuracy of the simulation in the average sense. Here the case of a drop in free oscillation with radius $r = 1$ and velocity potential $\phi = 0.6P_2$ is purposely chosen to compare with one computed by Lundgren & Mansour (1988). The results are shown in Figure 5–3 and it appears to be identical to Lundgren’s results.

Consider the case of an ideal gas bubble in free oscillation with initial radius $R(0)$ larger than its equilibrium radius $a$, $R(0) = 1.5a$. The gas inside the bubble is assumed to undergo adiabatic process with ratio of specific heats $\gamma = 1.4$. The pressure at infinity is $p_\infty = 2.0$. Figure 5–4 shows the results obtained from the numerical simulation in comparison with the results obtained directly from numerical integration of the Rayleigh-Plesset equation.

As shown in Figure 5–4, the two results almost overlap with each other. In fact, the total energy is conserved up to the fifth digit. This agreement assures us that the accuracy of the simulation is beyond doubt!

5-5.2 Dynamics and Legendre mode decomposition

Another way of verifying the accuracy is to make use of the Rayleigh’s frequencies for shape oscillations with infinitesimal amplitudes. Consider an ideal gas bubble
Figure 5–4 Free oscillation of an ideal gas bubble with initial conditions. $R = 1.5a, \phi = 0, p_{\infty} = 2.0, \gamma = 1.4, N = 41, \Delta t = 0.0001$. (a) Time history of bubble radius $R(t)$. (b) Energy tracing. The oscillatory curves, from the thickest to the thinnest, are the kinetic energy, surface energy, potential energy at infinity, and internal energy respectively and the straight line is the total energy.

with $\gamma = 1.4$ and initial condition $r = 1 + 0.001 \sum_{n=0}^{10} P_n$ oscillates in an unbounded quiescent inviscid fluid with far field pressure $p_{\infty} = 100.0$. The results of the simulation were decomposed into its Legendre mode components and the frequency of each mode is calculated manually by dividing the number of peaks by the time elapsed. The frequencies thus obtained is compared with the Rayleigh's frequencies for infinitesimal shape oscillations. The results are plotted in Figure 5–5. It seems that the simulated frequencies agrees very well with the Rayleigh's frequencies for low modes. For the higher modes, however, the simulated frequencies are a little bit higher than the corresponding theoretical values. As an example, the simulated frequency for the 10-th mode is $5.6358 \pm 0.003$Hz (112 peaks within 19.873 time units) and the theoretical value is 5.4857Hz. It is an increase of about 2.74%. One possible reason for this could be attributed to the curvature calculation based on the discrete model is higher than the continuum model (this is particularly true for the higher
modes!), thus resulted in an increase of the shape oscillation frequencies.

5-6 Concluding remarks

In this chapter, details concerning the derivations of boundary integral formulations for the interior, exterior, and the scattering problems are discussed. The implementation of the Boundary Element Method are discussed in great detail. Time integration is performed with the explicit fourth order Runge-Kutta scheme and stability issues concerning the stiffness and curvature effect are addressed.

Several examples are chosen to verify the accuracy of the code. Energy conservation serves as an excellent global measure of testing the accuracy. Rayleigh's frequencies for infinitesimal-amplitude shape oscillations provides a convenient way of verifying the dynamical response of the simulation results. Computed examples in the literature are also compared with the simulated results. All the verifications indicate that the implemented code is reliable with high accuracy.

However, the capability of the code is limited by the stiffness of the physical system and its accuracy relies on the correct calculation of curvature on the interpolated surface significantly. Nonlinearity worsens the stiffness problem and makes the calculation extremely inefficient. Thus an adaptive step size algorithm is necessary but not yet well established currently. How to correctly interpret the interpolated surface based on finite number of nodes is always a very crucial point in simulating free surface flows.
REFERENCES


Chapter 6

Acoustically Levitated Drops

6-1 Introduction

King's work (1934) on the radiation pressure on a rigid sphere initiates the concept of acoustic levitation. The study of acoustic levitation consists of two essential parts: the understanding of the acoustic forcing and the dynamic behavior of the drop. Because of the significant difference in time scales, the acoustic forcing problem can be reduced to a steady-state scattering problem governed by the Helmholtz equation. Analytic solutions are only available for problems with nearly spherical boundaries. By expanding the planar incident wave velocity potential in terms of spherical harmonics and matching the boundary conditions on the surface, King obtained the total velocity potential for the acoustic field scattered by a rigid sphere. This method was further modified by Hasegawa & Yosioka (1969) to account for the elasticity of the sphere. Yosioka & Kawasima (1955) derived the acoustic radiation force on an elastic sphere in a plane quasistationary wave field. Yosioka, Kawasima & Hirano (1964) also calculated the radiation force on a rigid sphere in a plane quasistationary wave field. By using basic properties of spherical Bessel functions of the first kind and second kind, Hasegawa (1977) simplified the results of the previous work.

Studies on the dynamic response of the drop by Marston (1980; 1981), and Tian, Holt & Apfel (1993) are mainly focused on the static deformations and the shape oscillations of the drops. Both of their methods are based on the same spherical harmonics expansion method. Experiments conducted by Trinh & Wang (1982) and Trinh & Hsu (1986) demonstrate consistent results with the theoretical predictions. Although this series expansion method seems very successful, two disadvantages are inevitable. 1. The evaluation of spherical Bessel functions causes a convergence problem which requires some special treatment. 2. This method can not be applied to more general boundary conditions.

On the other hand, Boundary Element Method (BEM) has provided us another
feasible alternative of solving the Helmholtz equation numerically. It has been rea-
alyzed that there exist the so-called eigenfrequencies at which this method does not
guarantee a unique solution due to the transformation from an exterior integral for-
mulation to a surface integral formulation. This non-uniqueness problem has been
attempted by several researchers; see Schenck (1968), Burton & Miller (1971), and

In this chapter, the Helmholtz equation and the Laplace equation which gov-
erns the incompressible flow inside the drop are solved by the BEM with quadratic
elements. The dynamic boundary conditions are integrated with respect to time
by fourth order Runge-Kutta integration directly. The dynamical behaviors of the
acoustically levitated drops such as global vibrations, shape oscillations and static
equilibrium shapes will be the major interests of the present work.

Current study of the dynamics of the acoustically levitated axisymmetric liquid
drop is motivated by an interesting observation: in an experimental study, Lee,
Anilkumar & Wang (1991) reported that, corresponding to the same physical pa-
rameters, two static equilibrium shapes coexist. These two equilibria coalesce at a
critical acoustic pressure amplitude. Above this pressure amplitude, no equilibrium
shape of the drop exists. The existence of two equilibria is not uncommon. In nonlin-
ear systems, two equilibria are created at a saddle-node bifurcation as what appears
to be the case at the critical acoustic pressure amplitude in the above experiment.
The surprising thing is that both equilibrium shapes seem to be stable which is con-
tradictory to the case of a saddle-node bifurcation since the two equilibria created
at the saddle-node bifurcation cannot be simultaneously stable.

The static shape of an acoustically levitated drop was extensively studied using
analytical and numerical means. The analytical work, based on perturbation from
the spherical shape of the drop, was carried out by Marston (1980). His result was
shown to have very good agreement with the experimental data of Trinh & Hsu
(1986) for slightly flattened drops (the dimensionless equatorial radius \( R^* < 1.1 \)).
If the drop deformation is not small, numerical techniques must be employed. The
erlier work done by Lee, Anilkumar & Wang (1991) and Lee, Anilkumar & Wang
(1994) gave an iteration approach of finding the static shapes for highly deformed
drops. They used the shape of a disk to approximate highly deformed drops. Hence
their results have limited applicability. Tian, Holt & Apfel (1993) attempted a
better approximation for the equilibrium shape. They expressed the drop shape
in terms of Legendre polynomials and substituted the truncated series expansion
into the Helmholtz equation to find the velocity potential of the scattered acoustic
field. This approach allowed them to obtain equilibrium shapes with dimensionless
equatorial radius larger than 2.

More accurate solutions of the equilibrium shapes of a highly deformed drop have
been obtained by Lee, Anilkumar & Wang (1994). The coupling of the drop shape to
the acoustic wave is accounted for by calculating the scattered wave field using the
boundary integral method. The resulting radiation stress acts on the drop surface and the shape of the drop is determined by solving the Young-Laplace equation. The numerical result thus obtained compares favorably with experimental data. In particular, two equilibria are found corresponding to certain identical physical parameters. Since the numerical program is mainly developed for calculating the equilibrium shapes, the question about the stability of the two coexisting equilibria still remains. Thus this issue will be pursued in the present work.

In Section 2, numeric calculation of acoustic radiation pressure is discussed. The translational motion of the drop driven by the acoustic levitation force is investigated in Section 3. Section 4 gives a thorough study of the dynamics of the shape oscillations. In Section 5, static equilibrium shape is obtained by incorporating artificial damping to eliminate the transients. The coupling effect between the translational motion and the shape oscillations is addressed in Section 6. Section 7 concludes the present work about drop dynamics.

![Figure 6-1](image)

Figure 6-1 The geometric configuration of the scattering problem.

### 6-2 Acoustic forcing

As shown in Figure 6-1, an axisymmetric rigid drop with equilibrium radius $a$ is put into a time-harmonic standing wave field. The velocity potential of this standing wave in the reference coordinate system can be described by the velocity potential $\Phi_{\text{inc}}(\vec{x}, t)$ as follows:

$$\Phi_{\text{inc}}(\vec{x}, t) = A \exp(-i\omega t)\tilde{\phi}_{\text{inc}}(\vec{x}; h),$$

(6-1)

where $A$ is the amplitude of the velocity potential, $\omega$ is the angular frequency of the wave, $t$ is time, $\phi_{\text{inc}}$ defines the shape of the incident standing wave, and $h$ is the
distance from the center of the drop to the nodal plane of the standing wave. The associated total velocity potential, $\Phi \equiv A \exp(-i\omega t)\hat{\phi}(\hat{x})$, governed by the Helmholtz equation can be solved by BEM easily as demonstrated in § 5.3.3.

Based on King’s formula, the acoustic radiation pressure exerted on the drop surface is related to the steady-state total velocity potential $\hat{\phi}$ by Eq.(3–68). Since acoustic levitation makes use of the variation of radiation pressure over the drop surface, we expect that $ka$ can not be too small or the drop will be under nearly uniform compression and no levitation force is gained. On the other hand, if $ka$ is far greater than 1, the highly oscillatory pressure variation over the drop surface will cancel each other and only little levitation force can be obtained. For the acoustic levitation to be effective, $0.2 \leq ka \leq 1.0$ would be a reasonable range. This also qualifies the frequency that can be used. For example for a millimeter-sized drop ($a \approx 1$ mm) to be levitated in the air (sound speed $c_0 = 340$ m/s), the corresponding frequency range would be $10$ kHz $\leq f \leq 50$ kHz. In a ground-based experiment, a frequency beyond the human hearing, which is around $23$ kHz, is preferred.

It can be easily shown that for a sinusoidal planar standing wave, $\Phi_{\text{inc}} = A \exp(-i\omega t)\sin(k(z + h))$, the time-averaged levitation force $F_z$ exhibits a $\sin(2kh)$ dependence. Some theoretic work about the calculation of the time-averaged acoustic levitation force on the elastic sphere in planar progressive and standing wave fields has been done in Yosioka & Kawasima (1955), Yosioka, Kawasima & Hirano (1964), Hasegawa & Yosioka (1969), and Hasegawa (1977).

**Numerical calculation of radiation pressure**

Theoretically, BEM can be used to solve the scattering problems for any wave number except for the so-called eigenfrequencies as mentioned in the Introduction. Practically, for a high frequency problem ($ka \gg 1$), the number of elements required to describe the highly oscillatory behavior of the incident wave would increase with the increasing wave number $ka$, which makes BEM impractical. Thus BEM is suitable for intermediate wavenumbers (wavelength that is comparable or longer than the characteristic dimension of the drop, say $ka \leq 10$).

**Acoustic levitation force on a rigid sphere**

In order to investigate the accuracy of BEM, we consider the case for the scattering of standing wave by a rigid sphere with $ka = 2.0$. The levitation coefficient $Y_s$ ($Y_p$) is defined as the time-averaged acoustic levitation force divided by $\pi a^4 \bar{E}$ for $kh = \pi/4$, in which $\bar{E} = \frac{1}{2} \rho_0 k^2 |A|^2$ is the mean energy density of the planar standing (progressive) wave. In Figure 6–2, the levitation coefficient $Y_s$ versus the wavenumber $ka$ for the scattering problem of a rigid sphere given by Hasegawa & Yosioka (1969) and BEM are compared. The BEM solutions show good agreement with the theoretic results. Note that the BEM fails at the eigenfrequency $ka = \pi$,
but even at $ka = 3.14, 3.16$ the BEM still gives accurate results.

Figure 6–2  Levitation coefficients $Y_s$ for a rigid sphere located at $kh = \pi/4$ in a standing wave. The levitation coefficients $Y_s$ obtained from BEM are represented by circles (○) and those obtained from Hasegawa’s method are represented by the solid line. Note the point at $ka = \pi$, which is the eigenfrequency of the BEM formulation, deviates from Hasegawa’s value. But BEM still gives very good results for the neighboring points $ka = 3.14, 3.16$.

Figure 6–2 gives important information about the levitation force. First, for the acoustic levitation to be most efficient, the value of $ka$ ranges between 0.5 and 1.5. Second, the direction of the levitation force depends on $ka$, since the levitation force can be positive or negative for different $ka$. We show in Figure 6–3 (c) and Figure 6–3 (d) the change in levitation force as a function of the vertical position of the drop for $Y_s > 0$ and $Y_s < 0$, respectively. For the purpose of reference, the variations of the velocity potential and the pressure of the acoustic field are also shown in Figure 6–3 (a) and Figure 6–3 (b).

For $Y_s$ positive, Figure 6–3 (c), at pressure node the acoustic levitation force serves as restoring force to pull the drop back to the nodal position. Thus under 0g environment, the drop will be trapped in the pressure nodal plane stably. On the other hand, for $Y_s$ negative, Figure 6–3 (d), the pressure anti-node is a stable equilibrium position while the pressure node is an unstable position. Third, there exist certain values of $ka$ for which acoustic levitation forces are always zero. Table 6.1 gives the numerical data comparisons of the levitation coefficient, $Y_s(Y_p)$, obtained from BEM and Hasegawa (1977) for a rigid immovable sphere in standing waves and progressive waves. Forty quadratic elements are used in the BEM. In Hasegawa’s method, only the first fifty terms are kept when evaluating the spherical
Figure 6–3  Levitation force for a rigid sphere located in a standing wave at various position.(a) Velocity potential of the standing wave. (b) Pressure field of the standing wave. (c) Levitation coefficients for $Y_s > 0$. In this case, the pressure node is a stable equilibrium position, if without gravity. (d) Levitation coefficients for $Y_s < 0$. In this case, the pressure anti-node is a stable equilibrium position, if without gravity.

Bessel functions $j_n(x)$ and spherical Neumann functions $n_n(x)$.

**Acoustic levitation on a prolate or oblate spheroid**

In order to define the relative geometry of a spheroid, we will define the length of the axis parallel to the direction of wave propagation to be $c$ and that of the axis perpendicular to the direction of wave propagation to be $d$. Define the aspect ratio of the spheroid as $\eta = c/d$. Then $\eta > 1$ stands for a prolate spheroid and $\eta < 1$ represents an oblate spheroid. Table 6.2 gives the numerical comparisons. It is not surprising to know that the oblate gains more acoustic levitation force than the prolate, since as one would expect a disk shape drop will experience more levitation force than a cylindrical one due to a larger projected area in the direction of wave propagation.
Table 6.1 Comparison of the levitation coefficients $Y_e(Y_p)$ obtained from BEM (40 quadratic elements) and those obtained from Hasegawa’s method [2] for a rigid immovable sphere in a standing (progressive) wave field.

<table>
<thead>
<tr>
<th>$\kappa a$</th>
<th>$Y_e$(BEM)</th>
<th>$Y_e$(Hasegawa)</th>
<th>Error(%)</th>
<th>$Y_p$(BEM)</th>
<th>$Y_p$(Hasegawa)</th>
<th>Error(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.7319</td>
<td>0.7318</td>
<td>0.01</td>
<td>0.0604</td>
<td>0.0604</td>
<td>0.00</td>
</tr>
<tr>
<td>1.0</td>
<td>0.9099</td>
<td>0.9099</td>
<td>0.00</td>
<td>0.4730</td>
<td>0.4735</td>
<td>0.11</td>
</tr>
<tr>
<td>1.5</td>
<td>0.4265</td>
<td>0.4266</td>
<td>0.02</td>
<td>0.7344</td>
<td>0.7346</td>
<td>0.03</td>
</tr>
<tr>
<td>2.0</td>
<td>-0.0648</td>
<td>-0.0648</td>
<td>0.00</td>
<td>0.7510</td>
<td>0.7514</td>
<td>0.05</td>
</tr>
<tr>
<td>2.5</td>
<td>-0.1343</td>
<td>-0.1339</td>
<td>0.30</td>
<td>0.8070</td>
<td>0.8071</td>
<td>0.01</td>
</tr>
<tr>
<td>3.0</td>
<td>0.0175</td>
<td>0.0180</td>
<td>2.80</td>
<td>0.8260</td>
<td>0.8261</td>
<td>0.01</td>
</tr>
<tr>
<td>$\pi$</td>
<td>-0.0195</td>
<td>0.0469</td>
<td>—</td>
<td>0.4826</td>
<td>0.8354</td>
<td>—</td>
</tr>
<tr>
<td>4.0</td>
<td>-0.00984</td>
<td>-0.00974</td>
<td>1.03</td>
<td>0.8668</td>
<td>0.8670</td>
<td>0.02</td>
</tr>
<tr>
<td>6.0</td>
<td>-0.00551</td>
<td>-0.00527</td>
<td>4.55</td>
<td>0.9109</td>
<td>0.9113</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Table 6.2 Levitation coefficients $Y_e$ for spheroids obtained from BEM with 40 quadratic elements.

<table>
<thead>
<tr>
<th>$\kappa a$</th>
<th>b/a = 0.10</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
<th>1.50</th>
<th>2.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.80</td>
<td>0.1757</td>
<td>0.8038</td>
<td>1.0553</td>
<td>1.1174</td>
<td>0.6769</td>
<td>0.1119</td>
</tr>
<tr>
<td>0.90</td>
<td>0.1702</td>
<td>0.7653</td>
<td>0.9810</td>
<td>1.0061</td>
<td>0.5436</td>
<td>0.0134</td>
</tr>
<tr>
<td>0.95</td>
<td>0.1679</td>
<td>0.7476</td>
<td>0.9481</td>
<td>0.9564</td>
<td>0.4831</td>
<td>-0.0283</td>
</tr>
<tr>
<td>1.00</td>
<td>0.1658</td>
<td>0.7319</td>
<td>0.9176</td>
<td>0.9099</td>
<td>0.4265</td>
<td>-0.0648</td>
</tr>
<tr>
<td>1.05</td>
<td>0.1639</td>
<td>0.7173</td>
<td>0.8889</td>
<td>0.8662</td>
<td>0.3734</td>
<td>-0.0969</td>
</tr>
<tr>
<td>1.10</td>
<td>0.1622</td>
<td>0.7037</td>
<td>0.8619</td>
<td>0.8250</td>
<td>0.3238</td>
<td>-0.1242</td>
</tr>
<tr>
<td>1.20</td>
<td>0.1592</td>
<td>0.6790</td>
<td>0.8122</td>
<td>0.7488</td>
<td>0.2339</td>
<td>-0.1656</td>
</tr>
</tbody>
</table>

6-3 Translational motion

The motion of the liquid drop in an acoustic field consists of translational motion of the drop centroid and the shape oscillation. In this section, attention is focused on the translational motion. Three important issues concerning the translational motion are addressed.

6-3.1 Analytic estimates of minimum trapping pressure and calibration point

The most crucial point of the acoustic levitation experiment is the determination of the minimum trapping pressure (MTP) and the calibration position of the levitated drop. We propose a spherical shape approximation to estimate the MTP and the
calibration position. Since for any fixed-shape drop in the planar standing wave field, the acoustic levitation force $F$ depends only on two parameters $ka$ and $kh$. It can be normalized as follows

$$F(ka, kh, \text{drop shape}) = f(ka, kh, \text{drop shape})(\pi a^2)\frac{1}{2}\rho_0 k^2 \|A\|^2,$$  

(6-2)

where $f(ka, kh, \text{drop shape})$ is a dimensionless function. If $h = h_0$ is an equilibrium position, the acoustic levitation force equals the weight of the drop, i.e.

$$F(ka, kh_0, \text{equilibrium shape}) = \rho_i \left(\frac{4}{3}\pi a^3\right) g,$$  

(6-3)

or

$$f(ka, kh_0, \text{equilibrium shape}) = \frac{8B}{3B_a},$$  

(6-4)

where $B$ is the bond number and $B_a$ is the acoustic bond number as defined in § 2-8 and 3-6.

Since the levitation force exhibits a sine dependence on $2kh$, we can write

$$f(ka, kh, \text{equilibrium shape}) = Y_s(ka, \text{equilibrium shape}) \sin(2kh).$$  

(6-5)

Therefore, the equilibrium position can be obtained by

$$\sin(2kh_0) = \frac{8B}{3B_a Y_s}.$$  

(6-6)

To obtain a preliminary approximation of $h_0$, a starting calibration point, one can assume the equilibrium shape of the drop be spherical. With the spherical shape approximation, $Y_s$ is a constant for a fixed $ka$, which is tabulated in Table 6.1. Thus for a fixed bond number, $B$, Eq.(6-6) imposes a lower limit(MTP) on the acoustic bond number $B_a$ for a fixed $ka$. See Figure 6-4.

Since the value $Y_s$ of an oblate spheroid is larger than that of a sphere, Eq.(6-6) gives a conservative estimate of MTP. For water drops (surface tension 72 dynes/cm) of diameters between 0.5 and 1 mm in a levitator (air specific gravity 0.00122, sound speed 340 m/sec) operating at a fixed frequency of 21.76 kHz ($ka = 0.1 - 0.2$), the MTP estimate predicted by Eq.(6-6) is around 158.4-158.45 dB, which is almost identical to the estimate of 158.5 dB by Lee, Anilkumar & Wang (1994). For a chosen acoustic bond number $B_a$, one can calculate the approximate offset $kh_0$(calibration point) by Eq.(6-6) very easily.

### 6-3.2 Analytic estimate of translational frequency

When a stably levitated drop is subjected to a perturbation so that its centroid is slightly away from the equilibrium position, the acoustic radiation pressure will serve as the restoring force. The drop will oscillate around the stable equilibrium position until this translational motion is damped out by the viscosity of the air. The frequency of this translational motion is a quantitative measure of stability of the drop equilibrium position. In particular, when the translational frequency
becomes zero, the equilibrium is unstable. The translational frequency of the drop can be obtained from the dynamic equation governing drop centroid motion. For the small perturbation about the equilibrium position $h_0$, the equation of motion for the centoidal position, $z$, of the drop can be written as

$$\rho_i \left( \frac{4}{3} \pi a^3 \right) \ddot{z} = F(ka, k(h_0 + z), \text{equilibrium shape}) - \rho_i \left( \frac{4}{3} \pi a^3 \right) g. \quad (6-7)$$

For small $kz$, the function $F(ka, k(h_0 + z), \text{equilibrium shape})$ can be approximated by

$$F(ka, k(h_0 + z), \text{equilibrium shape}) \approx F(ka, kh_0, \text{equilibrium shape}) + \frac{\partial F}{\partial (kh)} (kz). \quad (6-8)$$

Thus we have

$$\rho_i \left( \frac{4}{3} \pi a^3 \right) \ddot{z} = \frac{\partial F}{\partial (kh)} (kz), \quad (6-9)$$

or

$$\frac{d^2 z^*}{dt^2} - \frac{3}{8} B_a \frac{\partial f}{\partial (kh)} (ka, kh_0, \text{equilibrium shape})(ka) z^* = 0. \quad (6-10)$$

From Eq.(6-10) we have

$$\frac{d^2 z^*}{dt^2} - \frac{3}{4} B_a Y_s \cos(2kh_0)(ka) z^* = 0. \quad (6-11)$$

Thus if $Y_s > 0$, for $h_0$ to be a stable equilibrium position $\cos(2kh_0)$ must be less than zero. If $Y_s < 0$, then $\cos(2kh_0)$ must be greater than zero. The translational frequency of the drop of the small vibration about the equilibrium position can be
expressed as

$$\omega^2 = \frac{3}{4} B_a (ka) \| Y_s \cos(2kh_0) \| = (ka) \sqrt{\frac{9}{16} B_a^2 Y_s^2 - B^2}, \quad (6-12)$$

where Eq.(6-6) has been used. Moreover, if bond number $B = 0$, then $\sin(2kh_0) = 0$ and $\| \cos(2kh_0) \| = 1$ we have

$$\omega_1^2 = \frac{3}{4} B_a (ka) \| Y_s \| = \frac{3}{4} (ka)^3 \left( \frac{\rho_0}{\rho_1} \| A^* \| \right)^2 \| Y_s \|. \quad (6-13)$$

Since levitation force plays the role of restoring force, the translational frequency gives us the indication of the stability of the trapping of drops. The higher translational frequency, the better stability of levitation. The levitation will be marginally stable at a zero translational frequency.

![Figure 6-5](image)

**Figure 6-5** Translational frequencies under various acoustic bond numbers. The translational frequencies based on spherical shape approximation are represented by the solid lines. The frequencies obtained from numerical simulations are represented by the cross signs($\times$). (a) $B = 0.0$, (b) $B = 0.1$, (c) $B = 0.2$.

A comparison of the translational frequencies obtained from the spherical shape approximation, Eq(6-12), and that obtained from the numerical simulation are given in Figure 6-5 for bond number $B = 0.0, 0.1, 0.2$. The intersections of these curves with the horizontal axis correspond to the minimum trapping pressures for each case in terms of acoustic bond numbers. The numerical results are simply obtained by
measuring the period on a plot of drop centroidal velocity vs. time. Since the equilibrium position is not known a priori, a test simulation by releasing a spherical drop from the approximate equilibrium position obtained from Eq.(6-6) is performed to find the more accurate equilibrium position. Then another simulation is conducted by releasing a spherical drop near the new equilibrium position with zero initial velocity. The deviation of the spherical shape approximation from the simulation results are observed for large acoustic bond numbers due to the larger shape deformation. To get the translational frequency at small acoustic bond number requires long period of time integration for the numerical simulation. We note that the agreement is better at small acoustic bond number or small wavenumber. This is not surprising since spherical shape approximation is good at those limits.

6-3.3 Stability analysis of translational motion on the phase plane

The shape oscillation affects the scattering of the acoustic wave field. This in turn influences the net levitation force exerted on the drop. Consequently, the drop shape oscillation is coupled to its translational motion. We describe in the following the drop translational motion which occurs simultaneously with the shape oscillations.

We first obtain approximate analytical solutions to the translational motion of the drop. When the shape oscillation is neglected, the levitation force can be calculated by assuming the drop to be a rigid sphere. We know that for a rigid drop, the levitation force simply exhibits a sinusoidal dependence \( \sin(2kZ) \) on the location \( Z \) of the centroid measured from the nodal plane of the acoustic pressure field, Figure 6-1. The variation of the levitation force (solid line) with position for a drop with \( ka = 0.575 \) is schematically shown in Figure 6-6 (a) together with the pressure field (dashed line) of the acoustic standing wave.

We observe that the levitation force is zero at the two pressure nodes located at \( Z = 0 \) and \( Z = \lambda/2 \). In the middle of these two pressure nodes, at the pressure anti-node, the levitation force is also zero. Because the levitation force after time-averaging is a second order term of the acoustic field, the spatial frequency is doubled; therefore it goes through one period between every two adjacent pressure nodes.

The equation of motion for the centroid of the drop under spherical approximation can be expressed as

\[
\rho_i \frac{4\pi}{3} a^3 \ddot{Z} = -Y_s(\pi a^2) \bar{E} \sin(2kZ) - \rho_i \frac{4\pi}{3} a^3 g,
\]

where \( Y_s \) is the levitation coefficient and \( \bar{E} \) is the mean energy density of the standing wave. The equation can be written in the dimensionless form as

\[
\frac{d^2 Z^*}{dt^{*2}} = -P \sin(2kaZ^*) - B,
\]

\[ (6-14) \]

\[ (6-15) \]
Figure 6–6 Orbits in the phase plane \((Z, \dot{Z})\). The pressure field (broken line) and the time-averaged levitation force (solid line) are shown schematically in (a). (b) \(B = 0\), (c) \(B = 0.136\).

where

\[
P = \frac{3}{8} B_a Y_s,
\]

and \(B\) is the Bond number defined earlier in § 2-8; they are two dimensionless parameters. The variables with asterisks stand for the corresponding dimensionless variables.

The dimensionless form of the approximate equation for the drop centroid is analogous to the dynamic equation of a plane pendulum, Nayfeh & Mook (1979).
The Bond number $B$ is similar to a constant torque acting on the pendulum at the pivot. Solutions for the plane pendulum are conveniently described by orbits in the phase plane. In an analogous way, we use the location of the drop centroid relative to the pressure node and the centroidal velocity, i.e. $(Z, \dot{Z})$, as two phase variables to describe the translational motion obtained from our numerical simulation of the drop dynamics.

When the Bond number is zero, orbits in the phase plane similar to those of a plane pendulum are obtained. In Figure 6–6 (b), we show these orbits obtained by simulating the dynamics of an initially spherical drop with $ka = 0.575$, $B = 0$, $Ba = 1.071$ released from various locations in an acoustic field. Since the levitation force is zero at the pressure nodes ($z = 0$ and $z = \lambda$) and the pressure anti-node ($z = \lambda/4$), they are the equilibrium positions of the drop. We observe that the drop centroid oscillates around the two pressure nodes periodically. They are stable equilibrium positions. In between them, at the pressure anti-node, the equilibrium position is unstable similar to the inverted position of the plane pendulum. For all stationary initial conditions (zero initial velocity), the drop centroid oscillates around the pressure nodes. These periodic orbits are bounded by orbits which connect the two neighboring unstable fixed points at the pressure anti-nodes. The orbits connecting the unstable saddle points (labeled as a and b) are called heteroclinic orbits (labeled as $\Gamma_0$). They separate periodic oscillation from unbounded translational motion of the centroid. The oscillatory and translational motions of the drop centroid are analogous to the oscillation and rotation of a simple pendulum. Since we do not restrict the drop shape to remain spherical, the periodic orbits within the heteroclinic orbits thus obtained are smeared by the presence of shape oscillations.

When the Bond number is nonzero, the qualitative behavior of the orbits are substantially different from the above. First of all, at the drop equilibrium position, a net positive levitation force is required to balance the gravity force. Therefore, the stable equilibrium positions are located below the nearest pressure nodes and the unstable one is shifted above the nearest pressure anti-node ($z$ axis points upward). Orbits in the phase plane corresponding to $ka = 0.575$, $B = 0.136$, and $Ba = 1.071$ are shown in Figure 6–6 (c). They are analogous to those of a simple pendulum with a constant torque acting at the pivot of the pendulum. Due to the shift of the equilibrium positions, there are two equilibria between two adjacent nodes of the pressure field ($0 < z < \lambda/2$); one is a center and the other is a saddle. Instead of heteroclinic orbits connecting the neighboring saddles, a homoclinic orbit connects the saddle point to itself. It separates the phase plane into two regions. One region corresponds to periodic translational motion and the other corresponds to drops moving in one direction in the acoustic field if the drop has zero initial centroidal velocity.

The region of the periodic oscillation is associated with the trapping region of the acoustic field. The size of the trapping region in an acoustic field is of practical
Figure 6-7  Equilibrium centroidal positions of a drop in an acoustic field. The unstable and stable equilibrium positions are represented by the crosses (×) and the circles (o) respectively. To depict the size of the trapping zone, the maximum z-coordinates of the homoclinic orbits are also shown in this Figure and are represented by the stars (*). (a)ka=0.575, B=0.136, (b)ka=0.2, B=0.136.

importance. If we release a drop at various positions in an acoustic field with zero initial velocities, the saddle point (labeled as c) of the homoclinic orbit (labeled as Γ₁) together with the extreme point (labeled as d) give the possible region of stable levitation, which is known as the trapping region corresponding to the given acoustic bond number. For a drop with fixed Bond number and wavenumber, the Z coordinates of these two points are denoted by × and * in Figure 6-7 (a). In addition, the stable equilibrium positions are denoted by o. Therefore, there are three points corresponding to a fixed value of Bₐ. The interval between the symbols × and * captures the size of the trapping region. We observe that the size of the trapping region decreases as the acoustic bond number decreases. Moreover, there is a minimum acoustic bond number below which the trapping region does not exist. In the pendulum analogy, this corresponds to the rotation of the pendulum when the applied torque overcomes the restoring force of gravity.

To demonstrate the dependence of the trapping region on the wavenumber, the intervals of Z coordinate for ka = 0.2 is plotted in Figure 6-7 (b). Comparing Figure 6-7 (a) and Figure 6-7 (b), we note that the minimum acoustic bond number for ka = 0.2 is higher than for ka = 0.575. This is because that the levitation coefficient Yₜ as defined in Eq.(6-15) is smaller for ka = 0.2 (cf. Figure 6-2).

6-4 Shape oscillation

In this section, we focus our attention on the shape oscillations of the drop. In the absence of gravity, a drop released at the pressure node initially will remain there forever. The discussion on the free shape oscillation for a spherical drop was given by
Lamb (1932). Under the action of the surface tension force and the acoustic force, the drop oscillates about its equilibrium shape. For shape oscillation which does not deviate too much from the spherical shape, spherical harmonic functions can be used to characterize the drop dynamics. For an axisymmetric drop, Legendre functions are used to describe the shape oscillation modes. By decomposing the axially symmetric shape of a drop into the superposition of modes of the form

\[ r = a + \epsilon_n P_n(\cos \theta) \sin(\omega_n t + \eta), \]  

(6-16)

the linearized solution for the free vibration of a drop can be expressed as follows:

\[ \Phi_i = -\frac{\omega_n a}{n} \left( -\frac{r}{a} \right)^n \epsilon_n P_n(\cos \theta) \cos(\omega_n t + \eta), \]  

(6-17)

\[ (\omega_n)^2 = \frac{n(n - 1)(n + 2)\sigma}{\rho_i a^3}, \]  

(6-18)

where \( \Phi_i \) is the velocity potential inside the drop and \( \omega_n \) is the frequency of the corresponding mode \( n \). \( \sigma \) is the surface tension; \( a \) is the undisturbed radius of the drop; and \( P_n(\cos \theta) \) is the Legendre polynomial of order \( n \) with \( \theta \) the polar angle of the undisturbed drop. After normalizing Eq.(6-18) using characteristic time defined in § 2-8, the angular frequency of mode \( n \) is given by \( \omega'_n = \sqrt{n(n - 1)(n + 2)} \) (with period \( T' = 2\pi/\omega'_n = 2\pi/[n(n - 1)(n + 2)] \)). For example, Rayleigh's second mode dimensionless shape oscillation frequency is \( \omega'_2 = 2.828 \).

The acoustic field surrounding the liquid drop changes the static equilibrium shape of the drop which in turn alters the shape oscillation frequency of the drop. To study the shape oscillation frequency dependence on the acoustic bond number, a spherical drop is placed at its equilibrium position and is allowed to oscillate under a fixed acoustic bond number in the numerical simulation. The time history of the tip velocity of the north pole is plotted. By measuring the time elapsed after 40 peaks, one can easily calculate the shape oscillation frequency. The frequency obtained this way is plotted against the acoustic bond number in Figure 6-8 for various combinations of wavenumbers, \( ka = 0.2, 0.4, 0.6, 0.8 \), and bond numbers, \( B = 0.0, 0.1, 0.2 \). The shape oscillation frequencies are slightly higher than Rayleigh's frequency for small acoustic bond numbers,( Note that there exists a lower limit imposed on the acoustic bond number for cases of bond numbers not equal to zero.) and the shape oscillation frequencies decrease with the increasing acoustic bond number.

To illustrate the dependence of the shape oscillation frequency upon initial conditions, we compare the results of the numerical simulation of a drop released from a shape close to the equilibrium shape to that of a spherical initial shape (we will discuss how to get the equilibrium shape in next section.). The shape oscillation frequencies versus acoustic bond numbers are plotted in Fig(6-9) for drops with wavenumber \( ka = 0.4 \) and bond number \( B = 0.0 \). It is found that the shape oscillation frequencies for static equilibrium shape are slightly higher than that of
Figure 6–8 Shape oscillation frequencies under various acoustic bond numbers. Note that the frequencies have been normalized with respect to Rayleigh's free oscillation frequency. (a) $B = 0.0$ ('+': $ka = 0.2$, 'o': $ka = 0.4$, '*': $ka = 0.6$), (b) $B = 0.1$ ('+': $ka = 0.2$, 'o': $ka = 0.4$, '*': $ka = 0.6$), (c) $B = 0.2$ ('+': $ka = 0.4$, 'o': $ka = 0.6$, '*': $ka = 0.8$).

spherical shapes. We also notice that the shape oscillation frequency increases with the acoustic bond number when acoustic bond number is small. Since both curves must approach the same Rayleigh's free shape oscillation frequency, $\omega_0 = 2.828$. If the bond number is nonzero, there is a minimum acoustic bond number required for the levitation of the drop. Hence the increasing parts of these curves may not exist for a stably levitated drop (c.f. Figure 6–8). This is especially true for drops with large bond numbers.

It is found that, corresponding to a small to medium acoustic bond number, $B_a$, the frequency increases for small acoustic bond number up to a maximum value and then decreases with the increasing acoustic bond number. An intuitive extrapolation of this phenomenon implies that there exists a critical value of the acoustic bond number, $B_{a,c,r}$, at which shape oscillation frequency is zero. Since a zero oscillation frequency often indicates loss of stability of the equilibrium, it is thus very likely that there is no shape oscillation beyond some critical acoustic bond number $B_{a,c,r}$. Therefore, the understanding of the drop shape oscillation will shed light to the understanding of the stability of the drop equilibrium shape.

For the shape oscillation which deviates greatly from the spherical shape, it is
Figure 6–9  Shape oscillation frequencies for drops with different initial conditions. Note that the frequencies have been normalized with respect to Rayleigh's free oscillation frequency. The data points for drop released from spherical shape are represented by circles (o) and the data points for drop released from nearly steady static equilibrium shape are represented by the cross sign (×).

inconvenient to use the Legendre modes to characterize the drop shape. Figure 6–10 shows the drop shapes at equal time intervals after it is released from an initially prolate shape in zero gravity. The drop centroid lies at the node of the acoustic pressure, i.e. $Z = 0$. The acoustic bond number is $B_a = 2.56$, and the ratio of the drop radius to the wavelength, defined as wavenumber henceforth, is $ka = 0.2$. The initial prolate spheroidal drop has an aspect ratio 1.71. At certain instants of time, it appears that shape modes such as the fourth and the sixth Legendre modes are present. The drop also develops into flat disks with dimples. Such drop shapes require a large number of Legendre modes to fully approximate them.

Since the equatorial radius $R$ defined in Figure 6–1 is the most easily measured quantity in experiments, we use it to qualitatively describe the shape oscillations. Motivated by the work of Kang & Leal (1990), we use $R$ together with $\dot{R}$, which is the rate of change of the drop equatorial radius, as two phase variables. We use the methods and the terminology of dynamical systems in Guckenheimer & Holmes (1983) to describe orbits in this phase plane.

Figure 6–11 shows the orbits in the phase plane $(R, \dot{R})$ for a drop in an acoustic field with zero gravity. Here, the parameters are, $ka = 0.575$, $B = 0$, and $B_a = 2.15$. Six orbits are shown in the figure. These six orbits correspond to drops with spheroidal initial shapes of different aspect ratios. The initial dimensionless
Figure 6–10 Large-amplitude shape oscillation of a drop with $ka=0.2$, $Ba=2.56$, $B = 0$. The initial aspect ratio is 1.71 (prolate).

Equatorial radii corresponding to these six orbits A, B, C, D, E, and F as indicated in Figure 6–11 are 0.900, 0.977, 0.978, 1.000, 1.100, and 1.200 respectively.

Two qualitatively different orbits are seen in Figure 6–11. They correspond to two different kinds of motions of the shape modes. Orbits A and B correspond to unbounded growth of the drop equatorial radius while orbits C, D, E, and F correspond to periodic shape oscillations. Among the periodic orbits, we observe that due to the presence of the shape modes with high mode numbers, orbits E and F are only approximately periodic. We point out that the initial conditions for orbits B and C differ only by 0.001 in the initial equatorial radii; yet these two orbits have totally different fate. Furthermore, orbit C divides the phase plane into two separate sets. Inside orbit C, the drop undergoes periodic shape oscillation. Outside orbit C, the drop equatorial radius grows unbounded. Experiments by Anilkumar, Lee & Wang (1993) show that as the drop equatorial radius becomes large, surface waves are parametrically excited by the acoustic wave field near the two poles.
Figure 6-11 Phase plot of the large-amplitude shape oscillation of a drop with $ka = 0.575$, $B = 0$, $B_a = 2.15$. Six orbits (A-F) with different initial equatorial radii, $R = 0.9, 0.977, 0.978, 1.0, 1.1, 1.2$, are shown.

Figure 6-12 Time evolution of the equatorial radii for drops with $ka = 0.575$, $B = 0.0$, and $B_a = 2.15$ starting with different initial equatorial radii $R = 0.9, 0.977, 0.978, 1.0, 1.1, 1.2$. 


The subsequent violent vibration shatters the drop. Therefore, we consider the unbounded growth of the drop equatorial radius as leading to the drop breakup.

In Figure 6–12, we plot the drop equatorial radii as functions of time for the six orbits in Figure 6–11. Orbit F corresponds to shape oscillation with small amplitude. The initial oblate shape with equatorial radius 1.2 is close to the equilibrium shape of the drop. This equilibrium shape is shown in Figure 6–13 and it is found by incorporating artificial damping as done in Feng & Leal (1996) and Lundgren & Mansour (1988). Orbits in the phase plane (R, \( \dot{R} \)) are similar to those of a single degree-of-freedom nonlinear oscillator with quadratic nonlinearity. See Chapter 7 of Guckenheimer & Holmes (1983).

For given initial conditions, R is a periodic function of time if the initial conditions are not too far from the equilibrium. The nonlinearity is exhibited by the amplitude dependence of the oscillation frequency as shown in Figure 6–12. Specifically, we note that the period of orbit C is nearly three times that of orbit F.

In Figure 6–14, we plot the time evolution of the drop equatorial radius for orbit C Figure 6–14 (a) and the corresponding shapes Figure 6–14 (b–h). We observe that between point 3 and point 6, the drop shape hardly changes. That is, for relatively long period of time, the drop shape remains nearly the same. Therefore, we surmise that these drop shapes are very close to an equilibrium shape and since the drop eventually evolves away from this shape, this equilibrium shape is unstable. Orbits in the phase plane shown in Figure 6–11 provide further evidence for the existence of a fixed point of saddle type near where orbits B and C separate from each other as marked by the \( \times \) symbol. Orbits B and C together approximate an orbit which connects the fixed point of saddle type to itself. This orbit is called the homoclinic orbit. It separates the phase plane into two regions; within each region the drop dynamics is qualitatively the same.
The homoclinic orbit encloses a region in the phase space within which the drop undergoes periodic oscillation around a stable equilibrium. This stable equilibrium and the unstable equilibrium occur for the same parameters of $B_a$, $B$, and $ka$. The separation of these two equilibria in the phase space is proportional to the size of the region of stable shape oscillation. To determine how the parameters affect the region of stable shape oscillation, we plot the drop equatorial radii corresponding to these two equilibria as functions of $B_a$. This is shown in Figure 6-15, where the circle and the cross correspond to the stable and the unstable equatorial radii respectively. Here $ka = 0.575$ and $B = 0.0$. We observe that as $B_a$ increases, the two equilibrium radii approach each other. They eventually coalesce at a critical acoustic bond number $B_{a,cr}$ (near 2.4), where a saddle-node bifurcation takes place. Above this critical acoustic bond number, the region of periodic shape oscillation disappears and no equilibrium shape exists; a drop with any initial condition will experience unbounded growth of its equatorial radius and breaks up eventually. Therefore, the critical acoustic bond number corresponds to an upper limit of the strength of the acoustic wave above which no stable levitation of the drop is possible.

The critical acoustic Bond number depends on the wave number $ka$. Such dependence is shown in Figure 6-16, where both the stable and the unstable equilibrium radii are plotted against the acoustic Bond number $B_a$ for drops with $ka = 0.2, 0.575$.
Figure 6-15 Equilibrium radii vs. acoustic bond number for a drop with $ka = 0.575$, and $B = 0.0$. The equatorial radius of a stable equilibrium shape is represented by a circle (o) and that of an unstable equilibrium shape by a cross (x). A saddle-node bifurcation is observed. The saddle-node bifurcation defines an upper threshold, $B_{a, cr}$, for the acoustic bond number. Currently, $B_{a, cr} = 2.482$ and the maximum stable equilibrium radius is 1.443.

and $B = 0.0$. The stable equilibrium radii are shown as open circles and the unstable equilibrium radii are shown as crosses. The circles and crosses meet when the acoustic Bond number reaches the critical acoustic Bond number. The critical acoustic Bond numbers, $B_{a, cr}$, are 3.126 and 2.482 for drops with $ka = 0.2$ and $ka = 0.575$ respectively. The corresponding equatorial radii, $R^*$, are 1.500 and 1.443, respectively. In both cases, there exists an upper threshold on the acoustic Bond number above which no equilibrium shape is possible. Physically, the stability of the shape oscillation is governed by the balance of the radiation pressure, internal liquid pressure, and the surface tension. The variation of the acoustic pressure from the poles to the equator is larger for larger drops than for small ones. Thus the critical acoustic bond number is smaller for $ka = 0.575$ than for $ka = 0.2$.

It is important to note that drop breakup can occur subcritically as indicated by orbits A and B in Figure 6-11 and Figure 6-12. Corresponding to these initial conditions, the drop is flattened by the acoustic pressure and its equatorial radius grows unbounded. Figure 6-17 shows the shape changes for the first 4 time units.

In short, to levitate a liquid drop, the acoustic bond number must exceed the minimum acoustic Bond number defined above. Based on the previous discussion with regard to Figure 6-15 and Figure 6-16, too large acoustic bond number tends to flatten the drop severely and causes it to break. Therefore, the translation motion and the shape oscillation impose the lower and the upper limits on the acoustic bond number for the stable levitation of the liquid drop.
Figure 6–16 Equilibrium radii vs. acoustic bond number for a drop with $ka = 0.2$, and $B = 0.0$. The equatorial radius of a stable equilibrium shape is represented by a circle (○) and that of an unstable equilibrium shape is represented by a cross (×). A saddle–node bifurcation is also observed. The upper threshold, $B_{a,cr}$, in this case is 3.126 and the maximum stable equilibrium radius is 1.5.

Large–amplitude shape oscillation ($ka=0.575, B=0.0, Ba=2.15, A.R.=0.977$)

Figure 6–17 Unbounded growth of the equatorial radii for a drop with $ka = 0.575$, $B = 0$, $Ba = 2.15$, $R = 0.977$. Only the drop shapes at the first 4 time units are shown.
6-5 Static equilibrium shape

In order to obtain the equilibrium shape, we incorporate the viscous normal stress terms into the normal stress boundary condition to damp out the transients as done by Lundgren & Mansour (1988). For comparison, the static equilibrium shape obtained by our program and that obtained by Marston's Formula, Lee, Anilkumar & Wang (1994):

\[ r = 1 - \frac{3aF_A}{64\sigma \rho c_0^2} [1 + \frac{7}{5}(ka)^2](3\cos^2 \theta - 1) \]  \hspace{1cm} (6-19)

for a drop with \( ka = 0.4 \) and \( B = 0 \) under various acoustic bond numbers are plotted in Figure 6–18. The aspect ratio versus acoustic bond number relation is plotted in Figure 6–19. Clearly, the drop shapes obtained from numerical simulations agree with those obtained from Marston's formula for small acoustic bond number and they deviate from each other for large acoustic bond number. Figure 6–20 shows the static equilibrium shape obtained by BEM for a drop with \( ka = 0.4 \) and \( B = 0.1 \) under various acoustic bond numbers. It can be seen from the diagram that the

Figure 6–18 Deformed drop shapes under various acoustic bond numbers for a drop with \( ka = 0.4 \) and \( B = 0.0 \). The drop shapes calculated from Marston's formula are represented by the dash lines. The drop shapes obtained from numerical simulations are represented by the stars(*).
centroids of the drops for the cases with smaller acoustic bond numbers deviate more from the origin than those for the cases with larger acoustic bond numbers due to the difference in acoustic levitation resultants. The corresponding aspect ratio versus acoustic bond number relation for cases with different wavenumbers is plotted in Figure 6–21.

To compare the previous results in Lee, Anilkumar & Wang (1994) and Trinh & Hsu (1986), we reproduce the following two figures. Figure 6–22 shows the variation of drop aspect ratio versus rms sound intensity in the absence of gravity. Here the definition of aspect ratio given in Lee, Anilkumar & Wang (1994) is the inverse of our definition. Figure 6–23 shows the variation of aspect ratio of drop versus the wave number of the drop. Our values range between those in Lee, Anilkumar & Wang (1994) and the result given by Marston's formula. All of these results are in good agreement with the results in Lee, Anilkumar & Wang (1994) and the experimental data given in Trinh & Hsu (1986) for a small acoustic bond number but disagreement occurs for a large acoustic bond number.

6-6 Coupling effect between the translational oscillation and the shape oscillation

Because forces causing shape oscillations are dependent on the position of the drop and the levitation force is dependent on the drop shape, the translational motion and the shape oscillation are coupled to each other.
Figure 6–20 Deformed drop shapes under various acoustic bond numbers for a drop with $ka = 0.4$ and $B = 0.1$.

Figure 6–21 Aspect ratios under various acoustic bond numbers for a drop with $ka = 0.4$ and $B = 0.1$. 
Figure 6–22  Aspect ratios under various sound intensities. A comparison of the results given in Tian, Holt & Apfel (1993) (solid line) and those from BEM numerical simulations (represented by “o”). Note Apfel’s definition of the aspect ratio is different from ours.

Figure 6–23  Comparison of the results given in Tian, Holt & Apfel (1993) (p.3100, Figure 2.) (represented by “×”), the experimental data given in Trinh & Hsu (1986) (p.1337, Figure 4.) (represented by “+”), and the results from the BEM numerical simulations (represented by “o”).
Figure 6–24 Time evolution of the centroidal position(a) and equatorial radius(b) of an initially spherical drop with \( k a = 0.575, B = 0.136, \) and \( B_a = 1.071 \) released at different positions, \( Z_0 = -2.332, -1.00, -0.378, \) in the acoustic field.

We plot Figure 6–24 to illustrate such coupling. Figure 6–24 (a) shows the centroidal positions of an initially spherical drop as functions of time. The three curves correspond to a drop released at different initial positions. Figure 6–24 (b) shows the corresponding shape oscillations. The drop translational motion is accompanied by the shape oscillation and the amplitude of shape oscillation also changes with the position change of the drop centroid. Furthermore, the coupling causes curves in Figure 6–24 (a) to have small ripples superimposed to the large amplitude translational oscillations.

Despite the presence of the above coupling mechanism, in general the translational motion and the shape oscillation can be treated separately. This stems from the following observation about the periods of the oscillations. Comparing Figure 6–24 (a) with Figure 6–24 (b), it is apparent that the period of the prolate-oblate shape oscillation is nearly an order of magnitude shorter than the period of the translational oscillation. Since the prolate-oblate shape mode is the slowest among all possible shape modes, the shape oscillation frequency of all modes are at least an order of magnitude higher than that of the translational motion. This frequency separation makes it less likely that resonant energy exchange occurs between the shape oscillation and the translational motion, Feng & Leal (1996).

Although the shape oscillation and the translational motion cannot interact
through resonant modal interactions, under certain conditions the shape oscillation can have a dramatic effect on the translational motion. In our earlier discussion of Figure 6–6 (c), we make the analogy between the drop translational motion and the motion of a simple pendulum under a constant torque. For both cases, a homoclinic orbit in the phase plane separates the regions of periodic oscillation from unstable motion. For the pendulum problem a small periodic force can lead to the transversal crossing of the stable and the unstable manifolds of the unstable equilibrium (see Chapter 4 of Guckenheimer & Holmes (1983)). As a result, corresponding to an initial condition within the homoclinic orbit, the equatorial axis of the drop may grow unbounded under periodic forcings. Similarly, the fast shape oscillation amounts to a small perturbation on the dynamics of the translational mode. This small perturbation leads to the transversal crossing of the stable and the unstable manifolds of the unstable equilibrium. Consequently, for initial conditions close to the separatrix, shape oscillation can qualitatively change the translational motion.

Figure 6–25 shows numerical simulations of a drop in a physical environment given by $ka = 0.575$, $B = 0.136$, $B_a = 1.071$. The drop centroid and the equatorial radius are given as functions of time for two slightly different initial conditions. The initial conditions are such that they have the same initial centroidal position,
\(Z_0 = -2.3144\), but the initial equatorial radii are \(R = 1.037\) and \(R = 1.040\) respectively. With an initial equatorial radius \(R = 1.040\), the drop will oscillate about the equilibrium position four periods and then fall off. But for \(R = 1.037\) the drop will undergo the translational oscillation stably. Apparently, a slight variation in the initial values of \(R\) leads to a substantial change to the translational motion.

We have also conducted numerical simulations to examine the effect of the translational motion on the shape oscillations. In particular, as the initial shape gets closer to the homoclinic orbit indicated in Figure 6-11, we expect that the period of the shape oscillation gets longer and becomes comparable with the period of the translational oscillation; hence resonant coupling may occur. Interestingly, the coupling of the shape oscillation and the translational motion leads to the excitation of higher modes. The resulting dynamics becomes too complicated to be described by the drop centroidal position and the equatorial radius.

6-7 Conclusion

In this paper, the dynamics of an acoustically levitated drop is studied in terms of its shape oscillation and translational motion. The study of the shape oscillation, first of all, renders information about the equilibrium shapes and stability of the drop. Orbits in the phase plane \((R, \dot{R})\) indicate that there can be two equilibrium shapes for a given set of physical parameters as shown earlier in Lee, Anilkumar & Wang (1994). Moreover, we conclude that for the two equilibria, one is stable and the other one is unstable. As the acoustic Bond number increases, these two equilibria approach each other and coalesce at a critical acoustic Bond number through a saddle-node bifurcation. Above the critical acoustic Bond number, no equilibrium exists. Below the critical acoustic Bond number, a homoclinic orbit can be found. The homoclinic orbit connects the unstable equilibrium point — a saddle point — to itself and separates the phase plane into a region of stable shape oscillation, within which the stable equilibrium point lies, and a region of unbounded growth of the equatorial radius. The critical acoustic Bond number establishes an upper limit on the applicable acoustic pressure for stable drop levitation.

The translational motion is characterized by using the position and the velocity of the drop centroid as two phase variables. Fixed points in the phase plane correspond to the equilibrium positions of the drop centroid. We find that between two neighboring pressure nodes, there are one stable and one unstable equilibrium position. The stable equilibrium position is just below the pressure node. As the acoustic Bond number decreases, these two equilibria approach each other and they coalesce at another critical acoustic Bond number through a saddle-node bifurcation. Above the critical acoustic Bond number, a homoclinic orbit connecting the unstable equilibrium to itself encircles a region of stable drop levitation, in which the drop oscillates around the stable equilibrium periodically. Below the critical
acoustic Bond number, no equilibrium exists. Thus the critical acoustic Bond number constitutes a lower threshold of the acoustic pressure for stable levitation. This lower threshold is related to the minimum trapping pressure (MTP) introduced in Anilkumar, Lee & Wang (1993).

Owing to the order-of-magnitude difference between the periods of the shape oscillation and the translation motion, their coupling can be ignored on a first approximation. In particular, away from the stability boundary, the shape oscillation does not affect the stability of the translational motion. Obviously, the coupling between the two can be significant under certain circumstances. Figure 6–25 shows the effect of a small change in the initial shape on the translational stability of the drop when the initial centroidal position is close to its separatrix in the $(Z, \dot{Z})$ plane. By the same token, a small change in the initial centroidal position can significantly affect the shape oscillation and the breakup of the drop when the initial shape is close to the separatrix in the $(R, \dot{R})$ plane. However, the excitation of high modes makes it insufficient to use a single variable $R$ to characterize the drop shape. Therefore, the complexity of such coupling demands further research.

Our study indicates that the two equilibria which coexist for a subcritical acoustic Bond number cannot be simultaneously stable. This is in apparent contradiction to the experimental findings of Anilkumar, Lee & Wang (1993). We realize that our numerical simulation differs from the situation in an acoustic levitator. Levitators used in experiments are, without exception, resonant levitators: the levitator is carefully tuned to one of its resonance frequencies to achieve maximum levitation force. The position and the shape of the liquid drop significantly affect the acoustic (resonant) characteristics of the levitator chamber. It is reported in Anilkumar, Lee & Wang (1993) that as the drop shape experiences a sudden change, the measured sound pressure shows a downturn in intensity. This differs from our assumption that the acoustic field sufficiently away from the drop is independent of the drop motions. To correctly represent the drop dynamics in a resonant acoustic levitator, the detuning introduced by the drop scattering in the acoustic field must be included. The study of the drop dynamics in resonant levitators is beyond the scope of this paper. Interested readers may refer to the theoretical treatment of Rudnick & Barmatz (1990).

REFERENCES


Chapter 7

Single Bubble

7-1 Introduction

Cavitational phenomena were first documented in the literature by Reynolds (1894). The theoretical analysis on the problem of the collapse of an empty cavity in an incompressible liquid medium given by Rayleigh (1917) initiated the scientific study of cavitation and bubble dynamics. Based on Rayleigh’s equation, Plesset (1949) considered a gas-filled bubble with the effect of viscosity as well as surface tension and derived the so-called Rayleigh-Plesset equation. This equation describes the ‘inertially-dominated’ bubble dynamics in the absence of thermal consideration. More sophisticated models considering the effect of slight compressibility (Prosperetti & Lezzi 1986) of liquid, mass-diffusion (Hsieh & Plesset 1961, Eller & Flynn 1965), and thermal conduction (Scriven 1959, Flynn 1975) are now available. A comprehensive review of researches based on Rayleigh-Plesset equation prior to 1976 was given by Plesset & Prosperetti (1977). Intensive emphasis was placed on the radial motion of spherical bubbles. Recent review of studies on bubble dynamics taking the nonlinear dynamical systems approach was done by Feng & Leal (1997).

Compared to the considerable amounts of researches on the radial oscillation of spherical bubbles, researches concerning the shape oscillation are scant in the literature. Perhaps one of the major reasons for that is the difficulty of solving the PDE’s associated with the shape oscillation, while it is only required to integrate an ODE (Rayleigh-Plesset equation) for the radial motion of a spherical bubble. With the profound influence of nonlinear dynamics, people starts to realize the importance of coupling between the radial oscillation and shape oscillations. Via the nonlinearity, shape oscillations can be excited subharmonically by the volume mode through an internal resonance mechanism analogous to the parametric instability of a mechanical system. In other words, the spherical shape could be unstable under certain combination of parameters and energy can be transferred between the volume mode and the shape modes.
Peak pressure associated with the shock wave following the rebound stage of collapsing bubbles was believed to be the culprit for cavitation damage in earlier investigations such as Rayleigh (1917). The ensuing researches thus put particular emphasis on the effect of the compressibility of the liquid. Current estimates of the peak pressure were obtained from the assumption of a spherical bubble. However, this assumption cannot be justified. Since the peak pressure decreases very rapidly with distance from the center of the bubble, the solid boundary must be in immediate neighborhood of the rebounding bubble so that the cavitation damage can be attributed to the radiated pressure pulse. Under such circumstance, the bubble cannot remain its sphericity. Far less is known about the pressure generated by the non-spherical bubble near the boundary.

Suggested by an experimental paper by Kornfeld & Suvorov (1944), an explanation of cavitation damage is that cavitation is caused by the impingement of liquid jets formed by the evaporation of a collapsing bubble. Later Benjamin & Ellis (1966) provided an indisputable evidence of the formation of liquid jets on bubbles collapsing near a solid wall.

In Section 7-2, free oscillation of a single spherical bubble is simulated and the results are tested against that obtained from numerical integration of the Rayleigh-Plesset equation. The spherical volume oscillation of a bubble could be unstable under the two-to-one resonance conditions between the volume mode and shape modes. This phenomenon is thoroughly investigated in Section 7-3. Instability regions in the parameter space for the shape modes are identified both theoretically and numerically. The transient bubble oscillation within the instability regions is studied in Section 7-4. Liquid jet on a bubble near a wall is a possible cause of the cavitation damage. One mechanism for the jet formation is proposed in Section 7-5.

### 7-2 Free oscillation

Though simple, free oscillation of a single bubble provides the most essential information for predicting the bubble dynamics. Because of its own importance in many applications such as underwater sound emission, sonochemistry and sonoluminescence, the volume mode oscillation governed by the Rayleigh-Plesset equation has been well investigated for years and these results have now become classical in bubble dynamics. They also provide the most convenient and basic way of verifying the accuracy of numerical simulation.

As a demonstration of the accuracy of the code developed in the present work, the motion of an adiabatic ideal gas bubble ($\gamma = 1.4$) with initial conditions $R(0) = 3$ and $\dot{R}(0) = 0$ oscillating in an inviscid fluid with far field pressure $p_\infty = 2.0$ is simulated. 41 nodes are deployed uniformly on the interface and a time step of 0.001 is used. The results are plotted in Figure 7–1. For the purpose of comparison, the
result obtained from numerical integration of the Rayleigh-Plesset equation is also shown. Excellent agreement is seen. Energy is conserved up to the fifth significant digit.

![Graph showing energy evolution](image)

**Figure 7–1** Time evolution of the radius of spherical bubble oscillation with \( p_{\infty} = 2.0, R_0 = 3.0 \). For the purpose of comparison, the result obtained from numerical integration of the Rayleigh-Plesset equation is also shown. Kinetic energy, Surface energy, internal energy, energy flowing to infinity, and total energy are shown below.

The energy focusing effect during the sudden collapsing stage favors the sonochemistry and sonoluminescence. The stiffness inherited in the sudden collapse is responsible for the difficulties of numerical simulations. Furthermore, the distance between nodes decreases rapidly during the collapsing stage. Thus a CFL-like stability condition, Courant, Friedrichs, & Lewy (1928), for numerical integration would require an exceedingly small time step that goes beyond tolerance.

At this point, an important question to ask is whether the spherical bubble is stable or whether one should expect deviations to non-spherical shape as the bubble surface expands or contracts. It is obvious that without the spherical symmetry, the bubble dynamics becomes unpleasantly complex from all perspectives. Unfortunately, it is known experimentally that shape oscillations arise as a bifurcation from the spherical volume oscillation. As a numerical illustration of this phenomenon, a bubble with initial perturbation \( P_2 = 0.02 \) is considered. The far field pressure is set to 180 (N=41, \( \Delta t = 0.001 \)). A sequence of bubble shapes are shown in Figure 7–2.
Initially the bubble is spherical and undergoes pure radial motion. $P_5$ mode starts to grow significantly after $t = 4.0$, which distorts the shape into a pentagon. This phenomenon happens only when certain frequency conditions are satisfied. Naturally, one might ask what causes the $P_5$ mode to grow? This will be answered in the following sections.

Figure 7–2  Spherical bubble oscillation is unstable to small shape perturbation.
($p_\infty = 180, P_2 = 0.02$)
7-3 Two-to-one resonance

Nonlinearity opens up a conduit of energy transfer between linearly uncoupled modes via the modal interaction. In the territory of linear theory, primary resonance happens only when the excitation frequency is commensurate with one of the natural frequencies of the oscillator. However, a prominent characteristic of nonlinear systems is the possibility of the secondary resonance. More resonant mechanisms are made possible by the nonlinear interaction between modes. For the free oscillation of a nonlinear system with certain frequencies commensurate or nearly commensurate to one another, conditions for the strong interaction between modes involved in the internal resonance might exist either superharmonically or subharmonically. The interaction decreases as the detuning of the internal resonance increases.

Although various internal resonances are possible for commensurate frequencies, the energy transfer between modes is the fastest for a quadratic nonlinear system, when the two-to-one subharmonic resonance condition holds. In this section, the two-to-one resonance between the volume mode and one of the shape modes is investigated. Previous works (Ffowcs Williams & Guo 1991, Mei & Zhou 1991, Feng & Leal 1993) on the coupling between the volume mode and the shape modes are based on perturbation analysis of weakly nonlinear cases, where the deviation of the bubble shape from the spherical equilibrium is assumed to be small. Such analyses conclude that the internal resonance results in the instability of volume mode oscillation and leads to the growth of shape modes. For approximation keeping up to quadratic nonlinear terms, the volume mode and the shape mode are found to exchange energy in a periodic fashion. Shape oscillation might reach large amplitude at the expense of the volume mode. When large amplitude shape oscillations occur, the weakly nonlinear theory is obviously inappropriate to predict the bubble dynamics. Thus numerical approach is adopted to investigate this topic.

In two-to-one resonance, the volume mode plays the role of excitation and oscillates at a frequency twice that of the specific shape mode interested. From Eq.(2-114) and Eq.(2-115), we know that for a given bubble in an inviscid fluid only the dimensionless natural frequency of the volume mode can be changed by adjusting the far field pressure, \( P_\infty \), at equilibrium. To achieve the two-to-one resonance, one can make the natural frequency of the volume mode twice that of the mode we are interested by selecting an appropriate equilibrium far field pressure to ensure \( \omega_0 = 2\omega_n \).

Stable bubble oscillation

We first examine the bubble dynamics when a small perturbation is introduced into the small amplitude free oscillation of the volume mode. In Figure 7-3, an adiabatic ideal gas bubble (\( \gamma = 1.4 \)) in an inviscid fluid is slightly perturbed from its equilibrium shape (\( R = 1.0 \)) with small initial shape perturbations in \( P_0 \) mode (0.05)
and $P_2$ mode (0.001). The natural frequencies of these two modes are $\omega_0 = 6.9282$ and $\omega_2 = 3.4641$ by setting $p_\infty = 9.905$. The bubble dynamics is analyzed by plotting the modal amplitudes of the first eight modes as functions of time. As predicted by the theory, the $P_2$ mode grows at the expense of the volume mode. It is quite surprising that a small perturbation in the volume mode gives rise to the large growth in the $P_2$ mode. This significant amplification of $P_2$ mode perturbation demonstrates the striking effect of the two-to-one resonance. However, the initial exponential growth immediately slows down. The $P_2$ modal amplitude reaches a maximum then decreases. If we were to continue the calculation further, the above process will repeat. Such repetition is predicted to be periodic by the theory, resulting in quasiperiodic oscillations of the modal amplitudes. Since the amplitude modulation occurs on a much slower time scale in comparison with the period of each oscillation and the time step in the numerical integration is controlled by the numerical stability (typically chosen to be 0.001), the accumulated errors prevent us from carrying out integrations of several modulation cycles.

![Graphs showing two-to-one resonance between $P_0$ and $P_2$ modes, Initial perturbations: $P_0 = 0.05$, $P_2 = 0.001$](image)

Figure 7–3 Two-to-one resonance between $P_0$ and $P_2$ modes, Initial perturbations: $P_0 = 0.05$, $P_2 = 0.001$

Owing to the quadratic nonlinearity, the excitation of the $P_2$ mode introduces $P_4$ and $P_6$ modes as can be observed in Figure 7–3. However, their amplitudes remained small. The odd modes are not excited. (One might suspect that the odd modes are stable to the even mode perturbations.)

Figure 7–4 shows an even larger amplitude of the initial perturbation in volume mode(0.08) aggravates the effect of resonance. The finite amplification of initial perturbation and its dependence on the amplitude of excitation are the characteristic
phenomena of a nonlinear system.

Figure 7-4 Two-to-one resonance between \( P_0 \) and \( P_2 \) modes, Initial perturbations: \( P_0 = 0.08, P_2 = 0.00 \)

Transient bubble oscillation

Let us consider another nearly spherical bubble released from a compression state (with \( P_0 = -0.2 \) and \( P_2 = 0.01 \)) in a fluid with far field pressure \( p_{\infty} = 9.0 \) (The resonant pressure for \( P_2 \) mode is \( p_{\infty} = 9.9048 \)). Numerical simulations are performed with time steps \( \Delta t = 0.001, 0.0005, 0.0001 \). The Legendre mode decomposition of the bubble shape is plotted in Figure 7-5.

Again odd modes are not excited. In the present case, the large amplitude volume oscillation gives rise to the significant growth of high modes. Note the two-to-one resonance still prevails even with slight detuning from the exact resonant frequency. Physically, this case is different from the previous two cases in nature. In the previous cases, the energy transfer between modes is periodic and stable bubble oscillation persists. However, in this case the energy transfer leads to the breakup of the bubble. Thus it is called a transient bubble.

Note that Legendre mode decomposition breaks down near \( t = 5.0 \). A snapshot of the bubble shape at \( t = 5.0 \), Figure 7-6, explains why the Legendre mode decomposition breaks down! The multi-valuedness of radii at certain angles are the pathological cause for the failure of Legendre mode decomposition. This also alerts us that Legendre mode decomposition is not an ideal tool for studying axisymmetric bubble with non-convex shape generator.
Figure 7–5  Legendre decomposition of a transient bubble with $P_{oo} = 9.0$ and initial perturbations $P_0 = -0.2, P_2 = 0.01$.

Figure 7–6  Bubble shape at $t = 5.0$. Note: The multivalueness of radii at certain angles is the pathological cause for the failure of Legendre mode decomposition.

As shown in Figure 7–7, the total energy is indeed conserved up to $t = 4.5$. The tip velocity is shown in Figure 7–8. Three curves for 3 different time steps are plotted on the same diagram, and the differences are indiscernible. This indicates a good convergence of the simulations. The tip (north pole) starts to vibrate violently near $t = 4.5$. Several bubble shapes after $t = 4.5$ are shown in Figure 7–9.
Figure 7-7  Energy tracing of a transient bubble with $p_{\infty} = 9.0$ and initial perturbations $P_0 = -0.2$, $P_2 = 0.01$.

Figure 7-8  Tip velocity at the north pole. ($p_{\infty} = 9.0$, $P_0 = -0.2$, $P_2 = 0.01$)
Figure 7–9 Transient bubble shapes at $t = 4.4, 4.6, 4.8, 5.0. (p_\infty = 9.0, P_0 = -0.2, P_2 = 0.01)$

Figure 7–10 Instability wedges for detuned two-to-one resonance between volume mode and shape modes.

**Stability wedge for two-to-one resonance**

Since the two-to-one resonance destabilizes the spherical volume oscillation, one interesting question to ask is how the detuning from the exact two-to-one resonant frequency will affect the stability of the spherical volume mode.

Theoretically speaking, the two-to-one resonance can be achieved between the volume mode and any shape oscillation modes by choosing appropriate ambient pressure. Because of the ambient pressure achievable in a laboratory, resonances between the volume and shape modes with moderate mode numbers are more feasible. We thus give more detailed results concerning the resonances with the 5-th and the 6-th modes.

To best describe the dependence of the stability of volume mode on the frequency detuning and the amplitude of initial perturbation, we try to sketch the stability
boundary of the shape oscillation mode on the frequency-amplitude diagram as shown in Figure 7-10. In order to compare with the theoretic work, the inequality in (4-100) defines the boundaries separating the stable from the unstable regions are plotted as solid lines in Figure 7-10.

For a given pair of frequency and amplitude, a corresponding simulation is conducted. The stability is then determined by the evolution of the shape modes. For a small perturbation in the shape mode, we determine the stability by observing whether or not the shape amplitude grows exponentially initially. Note that the mere growth of the shape amplitude does not indicate the instability of the volume mode. This can be observed in Figure 4-1. Note that even though the origin is a center, for an initial perturbation with $\theta = \pi/2$, the function $r(t)$, that is the amplitude in the shape mode, actually grows initially. Therefore this should not be taken as an indication of the instability. This situation is demonstrated in Figure 7-11 when the $P_5$ mode is given an initial perturbation; $P_5$ mode grows but not exponentially and the volume oscillation is actually stable. By selecting either an initial displacement or an initial velocity as perturbations, we can easily determine whether the shape mode is unstable.
Figure 7–12 Two-to-one resonance between $P_0$ mode and $P_0$ mode with different detuning ratio.

For a fixed amplitude of the volume mode oscillation, we identify a pair of points, one stable and the other unstable. Together they define a boundary separating the stable and unstable regions. These are also plotted in Figure 7–10. There is a good agreement between the theory and the numerical results.

For parameters (the amplitude and frequency of the initial volume mode) in the wedge, Figure 4–1 (b) and (c) show the existence of homoclinic orbits. Therefore, the shape mode is unstable.

Figure 7–12 (a) shows the dynamics of a typical case in region (I). The amplitude of initial perturbation of volume mode is 0.01 and the far field pressure is 310.0, which corresponds to a volume mode natural frequency of 36.17 based on Eq.(2–114). From the diagram, there are 57.5 peaks in 10 time unit, which corresponds to a frequency of 36.12. The amplitude of initial perturbation of mode 6, which is 0.001, is not amplified but oscillates at a certain frequency. This frequency gives the indication of the closeness to the stability boundary. The slower the frequency, the closer to the boundary the case is.

Figure 7–12 (b) shows the dynamics of a typical case in region(II) with all the parameters kept the same as the previous case except the equilibrium pressure is lowered down to 300.0, which corresponds to a volume mode natural frequency of 35.59. The initial perturbation of mode 6 is now significantly amplified. In Figure 7–12 (c), the significant interaction between the two modes illustrates the perfect two-to-one resonance. Figure 7–12 (d) and (e) together complete the description of the dynamics as we cross the left boundary of the stability wedge. Similar dynamics are demonstrated in Figure 7–13 (a)-(e) for the stability wedge of mode 5. It has been
shown that the angles formed by the wedges increase as \( n \) increases, Feng (1993). This is consistent with our current diagram.

We have seen that the two-mode theoretical model is completely integrable. Therefore no chaos can occur. In the numerical simulations, the presence of the high modes can be considered as perturbations. In general, these perturbations can lead to chaos. In Figure 7–14, we show a case where strong interactions between the volume mode and shape mode resemble chaos. However, the possible numerical errors associated with long time integration prevent us from computing quantitative proofs that chaos indeed occurs.

7-4 Stable and transient bubble oscillations within the instability wedge

As mentioned earlier, a clear boundary can be found that separates the stability region for spherical volume mode from the unstable region either theoretically or numerically. One interesting issue follows: What happens in the unstable region? Linear theory predicts the exponential growth of the shape mode without bounds! Although the perturbation theory predicts bounded motions with periodic exchange of energy between the volume and shape modes, the perturbation results are applicable only for small amplitude oscillations of both the volume mode and the shape mode. We investigate in the following the transient behavior for a range of finite amplitudes. The dynamics within this region is so complicated that it can not be captured by a few parameters! Therefore, we try to sort things out qualitatively
Figure 7–14  Exact two-to-one resonance between volume mode and $P_6$ mode. $P_0 = 0.01$, $P_5 = 0.001$ and $p_{\infty} = 265.14$.

by numerical simulation! Here we decided to perform the numerical simulation for 10 time units to see whether the bubble will remain stable oscillation or not! In other words, we are trying to pursue the possibility of a boundary that separates the stable bubble oscillation (steady energy exchange) from the transient behavior of a collapsing bubble within the instability wedge of shape modes!

In interpreting these data, it is important to realize that a bubble not collapsing during a chosen computation time does not imply that the bubble will never collapse. In other words, the precise boundary may depend on the length of computational time. We limit our computation to 10 dimensionless units, which roughly covers about 100 cycles of volume oscillations, to minimize the effect of the cumulative computational errors and the numerical stability can be guaranteed by repetitions of calculations with even smaller yet tolerable time steps. Thus we can infer that the bubble collapsing is a physical phenomenon but not a numerical artifact.

Figure 7–15 shows the modal amplitudes when the initial amplitudes of the $P_0$ and $P_6$ modes are 1.028 and 0.01, respectively. The far field pressure $p_{\infty} = 210$ is chosen such that $\omega_0 = 29.8$ ($\omega_5 = 12.96, \omega_6 = 16.73$). For this combination of parameters, the computation stops after a brief time. Simulations with smaller time steps ($\Delta t = 0.0005, 0.0001$) are performed and computation stops at about the same time. Similar behaviors are observed for other combinations of parameters.
Figure 7–15 Legendre mode decomposition for the dynamics of a bubble with initial shape perturbations $P_0 = 0.028, P_6 = 0.01$ subject to pressure at infinity $p_\infty = 210$. The computation stops at $t = 3.2$. The abrupt changes near the end of computation are caused by the failure of Legendre mode decomposition due to the multivalu-ueness of radial length in certain angle $\theta$.

and initial conditions. Note that the abrupt changes of Legendre modes near the end of computation are not caused by the sudden change of the shape, instead they are caused by the failure of Legendre decomposition due to the multivalueness of radial length for certain angle $\theta$.

On closer examination, we find that the computation stops because a liquid jet seems to form at the two poles of the bubble. The development of the interface near the north pole is shown in Figure 7–16 for a few moments just before the computation stops. Notice that the second node is moving very fast toward the pole due to the high curvature at the pole and this causes severe non-uniformity in the nodal distribution. This brings up a very serious and difficult concern in the subsequent calculation. Since the curvature depends on the high order spatic derivatives of the shape geometry, it is very sensitive to how we interpolate the shapes based on the discrete nodal data which in turn makes the dynamical system very stiff. Furthermore, the non-uniformity of the nodal distribution will make the cubic spline interpolation of the bubble shape more unrealistic. Therefore, we are dubious about the results after $t = 3.14$, though it gives a beautiful description of
the jet formation at the two poles!

As an indication of our numerical accuracy, we show the kinetic, potential, and total energy in Figure 7–17. Note that the total energy remains a constant up to the last moment before the jet forms.

Figure 7–18 shows the boundary that divides the instability wedge into stable oscillation and collapse region as expected.

In the subsequent section, we will examine the mechanisms for jet formation. The evidences to be presented there and the accuracy in the total energy of the bubble convince us that the observed formation of liquid jets at the bubble poles are not caused by numerical artifacts but is genuine. Therefore we found that inside the wedge corresponding to the instability of the volume mode oscillations, there are two qualitatively different outcomes: for small amplitude of the volume mode, the periodic exchange of energy between the volume and shape modes will persist; for large amplitude of the volume mode, the onset of shape oscillations leads to the development of liquid jets at the two poles.

7-5 The mechanism for the jet formation

The formation of liquid jets at the poles connected to the development of curvature concentration as shown in Figure 7–16 has also been encountered in dynamic simulations of surface waves and drops. We suspect that this development is caused by a "geometric amplification factor" introduced in Leal (1992). In the following, we provide numerical evidences to such instability mechanism and show that such
Figure 7-17  Energy tracing of the bubble. The heavy dashed line represents the total energy, the heavy solid line is the kinetic energy, the light solid line is the surface energy, and the dot-dashed line stands for the internal energy. For the purpose of clarity, the energy associated with the pressure at infinity is not shown.

Figure 7-18  Boundary separating the stable bubble oscillations from the transient bubble oscillations in the instability wedge of mode 6.)
a mechanism is responsible for the development of local curvature concentration which eventually leads to the jet formation near the surface of a bubble.

7-5.1 Geometric amplification factor

To illustrate the geometric amplification factor, Leal (1992) uses the bubble dynamical equation governing the coefficient \( a_n(t) \) of the \( n \)-th shape perturbation mode:

\[
\ddot{a}_n + 3 \frac{\dot{R}}{R} a_n + \left[ \frac{(n - 1)(n + 1)(n + 2)}{R^3} - (n - 1) \frac{\dot{R}}{R} \right] a_n = 0, \quad n \geq 2. \tag{7-1}
\]

The first term \( (n - 1)(n + 1)(n + 2)/R^3 \) stands for the surface tension and is always stabilizing. The acceleration term in the above equation \(-\dot{R}/R\) is destabilizing when \( \dot{R} > 0 \) but stabilizing when \( \dot{R} < 0 \). Instability happens when the destabilizing effect of \( \ddot{R} \) overcomes the stabilizing effect coming from the surface tension term; this is the Rayleigh-Taylor instability.

The term \( 3\dot{R}/R \) plays the role of negative damping when \( \dot{R} < 0 \) and positive damping when \( \dot{R} > 0 \). When the area of bubble surface decreases \( (\ddot{R} < 0) \), the wavelength of the \( n \)-th mode decreases and the amplitude increases due to the negative damping. The combined effect steepens the local surface perturbation and leads to curvature concentration. Therefore, the change of the bubble surface area provides an alternative mechanism of instability. For this reason, this type of instability is different from the Rayleigh-Taylor instability.

In a collapsing bubble, the geometric amplification factor is destabilizing. In the meantime, when bubble is near its minimum radius, the Rayleigh-Taylor mechanism is also at play. In order to investigate how these two types of instability mechanisms will affect the dynamics of a localized disturbance, we select a spherical bubble, whose initial radius is 1.5 times its equilibrium radius with dimensionless pressure at infinity \( P_\infty = 2.0 \). The localized disturbance on the surface is imposed by moving the node at the north pole by 0.001 (bubble radius is 1.5) along the symmetry axis. The results of numerical simulation of this case are shown in Figure 7–19. Quite surprisingly, the velocity of the node at the north pole starts to oscillate at a very high frequency with an increasing amplitude till the radius reaches its minimum. The tip acceleration at the north pole is higher than 2000. Since the wavelength for this disturbance is very short, positive acceleration of the radius is not large enough to overcome the stabilizing effect of the surface tension.

As another illustration of the geometric amplification factor, we note that such a factor is stabilizing for an expanding bubble. In Figure 7–20, we show another numerical simulation for an initially compressed bubble, whose initial radius is only 0.6 times its equilibrium radius, with the same localized disturbance imposed. In this case, we see the tip disturbance decreases from its initial value. This indicates that the area increases on the bubble surface has a dominant effect over the Rayleigh-Taylor instability for short wavelength disturbances.
Figure 7–19 Geometric amplification for a contracting bubble. Note that the tip acceleration grows when the surface area decreases.
Figure 7-20 Geometric amplification for a expanding bubble. Note that the tip acceleration decays when the surface area grows.
7-5.2 Local geometric amplification factor

In the previous section, the kinematic instability is caused by the global area decrease during the volume mode collapsing. Since the initial disturbance is applied locally (at the north pole), we expect the same amplification effect to occur if the local surface area experiences similar changes as in the above subsection. Here, we choose a bubble with an initial deformation in the 6-th mode $P_6 = 0.1$ and no initial component in the volume mode $P_0 = 0$ with a dimensionless pressure at infinity $p_\infty = 210$ and impose the same localized disturbance by moving the node at the north pole by 0.001 along the symmetry axis. In Figure 7–21, we observe highly oscillatory tip velocity riding on the 6-th mode. Note that the growth or decay of the disturbance is directly associated with the slope of the 6-th mode; this is true at least during the first a few cycles. Interestingly, since the 6-th mode and the 10-th mode are approximately at two-to-one resonance, the 10-th mode gradually grows and thus modifies the nature of the local area change near the two poles.

We believe that the geometric amplification factor is mainly responsible to the development of high curvature that eventually leads to the jet formation. Figure 7–22 shows the tip velocity of the north pole corresponding to the simulation shown in Figure 7–15. From the figure, we see small wiggles between $t = 2.1$ and 2.35. And even larger wiggles occur between $t = 2.55$ and 2.80. In Figure 7–23, we show the average bubble radius, the amplitude of the 6-th mode decomposition, the tip velocity and acceleration corresponding to these two intervals. We can clearly see the tip accelerations grow when 6-th mode has a negative velocity. This indicates that the highly oscillatory tip acceleration is caused by the geometric amplification factor of the localized higher mode disturbances riding on the 6-th Legendre mode.

Interestingly, the acceleration caused by the geometric amplification of the local disturbance may reach such a large value that the interface is locally Rayleigh-Taylor unstable. The critical wave number for Rayleigh-Taylor instability is given by

\[ k_{cr} = \sqrt{\frac{p\vartheta}{\sigma}}. \]  

(7–2)

Based on our choices of characteristic scales, we have the dimensionless critical wavelength

\[ \lambda_{cr}^* = \frac{2\pi}{\sqrt{g^*}}. \]  

(7–3)

where $g^*$ is the local acceleration of the interface. The acceleration at the point just before the jet forms at the two poles is around 15000. Corresponding to this acceleration, the critical wavelength for the instability is 0.0513, which is very close to the length scale of the localized disturbance as estimated by the mesh size.

In the previous section, the kinematic instability is caused by the global area decrease during the volume mode collapsing. Is this the only route to kinematic instability? In order to answer this question, we now choose a bubble with an initial deformation in the 6-th mode $P_6 = 0.1$ and no initial component in the volume
Figure 7-21 Local geometric amplification for a bubble with $P_0 = 0.1$ and pressure at infinity $P_{\infty} = 210$. A local disturbance is imposed by moving the north pole by 0.001 along the symmetric axis. Note that the mode is excited due to the near two-transverse modes.
mode $P_0 = 0$ with a dimensionless pressure at infinity $p_{\infty} = 210$ (which is close to the pressure required for natural frequency of the 6-th mode) and impose the same localized disturbance by moving the node at the north pole by 0.001 along the symmetry axis. Here we are looking for the existence of the same highly oscillatory tip velocity riding on the 6-th mode instead of the volume mode. Figure 7-22 shows that the tip velocity does oscillate at a very high frequency with an increasing amplitude till the end of the 6-th mode collapsing stage. This reveals that not only the area decrease associated with the volume mode collapsing has the destabilizing effect but also the area decrease associated with the higher modes has the same effect. It is worth noting that the 10-th mode is also excited through the one-to-two resonance. Obviously, the tip velocity is also affected by the existence of the 10-th mode. It can be easily seen from Figure 7-22 that the amplitude of the tip acceleration decrease when 10-th mode has the stabilizing effect. With this complication, the kinematic instability must be determined by all the modes with significant amplitudes. One more interesting observation is that geometrically $P_0 < 0$ may have surface area larger than $P_0$. However, the fluid flow towards the symmetry axis associated with $P_0 < 0$ may have the clustering effect which also decreases the wavelength of the local disturbance and amplifies it magnitude.
Figure 7–23  Direct correlation between the growth of tip acceleration and the local surface area decrease caused by the collapsing $P_6$ mode are clearly seen in this figure.
7-6 Conclusion

In this chapter, we have extended small amplitude perturbation analysis on the energy transfer between resonance of volume mode and shape modes to large amplitude shape oscillations. We identified parameter regions in which the shape oscillation gives rise to the amplification of localized disturbances. The large acceleration resulting from such amplification may exceed the threshold for the Rayleigh-Taylor instability and leads to the forming of liquid jets.

Liquid jets in collapsing bubbles have been proposed by Prosperetti (1997) to be the cause of sonoluminescence. Although there are ample experimental and numerical evidence of jet formation in a collapsing bubble near a boundary, Prosperetti provides numerical results that a jet can form for a bubble not near a boundary. For conditions that allows sonoluminescence, he shows that the formation of the jets in the absence of a boundary is possible for a collapsing bubble with a translational velocity or in a pressure gradient.

While this work does not resolve the cause for sonoluminescence, it provides a scenario that an oscillating bubble far from any boundary can lead to the formation of the jet. The energy of the volume oscillation first transfer to a shape mode through internal resonances. This leads to the amplification of the local disturbances of even shorter length scale without resonance conditions among the frequencies.

In real fluids, the geometric amplification factor is also controlled by viscosity. Whether liquid jet can form is determined by these two counteracting forces; a detailed investigation of how the viscosity modifies the local dynamics would certainly be very useful in predicting the jet formation. The occurrence of shape modes in bubble oscillations is similar to the Faraday instability of surface waves. Therefore, it is not completely irrelevant for us to point out that jet formation has been observed in Faraday wave experiments Hogrefe et. al. (1999). If similar geometric amplification factor is at play, measurement of the surface velocities should reveal high frequency oscillations preceding the jet formation.

REFERENCES


Courant, R., Friedrichs, K. & Lewy, H. 1928 On the partial difference equations of
mathematical physics. *Mathematische Annalen* 100, 32-74. (Also republished in *IBM Journal*, 1967)


RAYLEIGH, LORD. 1917 On the pressure developed in a liquid during the collapse of a spherical void. *Philos. Mag.* 34, 94-98.

REYNOLDS, O. 1894 Experiments showing the boiling of water in an open tube at ordinary temperature. *Sci. Pap.* 2, 587. (1901)

RYSKIN, G. & LEAL, L. G. 1984 A Numerical solution of free-boundary problems in fluid


Chapter 8

Two Bubbles

8-1 Introduction

With dilute concentration of bubbles in a liquid medium in which the bubble size is very small compared with the distance between bubbles, the dynamics of bubble motion can be well predicted by the single bubble model. If the concentration of the bubbles in liquid is high, bubble-bubble interaction becomes important and several new physical phenomena which were not observed in the single bubble model can occur. Perhaps, one of the most interesting phenomenon is the translational motion of the two pulsating bubbles induced by the interaction. The forces drive the bubbles to translate are now called 'Bjerknes' forces.

Bjerknes (1906) found that the forces exerting on the two pulsating bodies in liquid are inversely proportional to the square of the distance between them. Based upon the law of kinematic buoyancy, which is analogous to the Archimedian law, Bjerknes further proposed that "the kinematic buoyancy exerted on a body immersed in a fluid is equal to the mass of water displaced by the body times the acceleration of the body". Perhaps Hicks (1879; 1880) is the first one that derived such analytic expressions for the Bjerknes forces between two pulsating spheres. A brief literature review of the Bjerknes forces and related researches was given by Pelekasis & Tsamopoulos (1993 A).

Another interesting example of the Bjerknes force comes from the cavitation bubbles near rigid boundaries. The practical problem of cavitation damage caused by a collapsing bubble near the solid boundary is of essential importance to hydraulic engineering relating to impeller blades, propeller blades and spillways. Inspired by the method of images, one can model the bubble near a rigid boundary as two identical spherical bubbles pulsating in phase with each other. This provides a practical incentive of investigating the bubble-bubble interaction.

In this chapter, the two bubble problem will be explored through examples extensively but not exhaustively. Section 8-2 extends the implementation of the BEM
presented in Chapter 5 to analyze the two bubble problem. In Section 8-3, the motions induced by the Bjerknes forces between two bubbles are investigated with different types of initial conditions. The effect of shape oscillations on the translational motion will be examined through examples in Section 8-4. The dynamics of bubbles of different sizes is presented in Section 8-5. In Section 8-6, the isotropic forcing is applied on the two bubble problem and the dynamics is summarized. Section 8-7 briefly summarizes the results presented in this chapter.

8-2 BEM implementation for two bubble problem

In order to analyze the two bubble problem, Eq.(5-30) needs to be modified according to the geometric configuration shown in Figure 8-1 as follows:

\[
\text{(P.V.)} \int_{S_1+S_2} (\phi \frac{\partial \phi^*}{\partial \nu} - \phi^* \frac{\partial \phi}{\partial \nu}) dS = (4\pi + \int_{S_1+S_2} \frac{\partial \phi^*}{\partial \nu} dS) \phi(x_0),
\]

(8-1)

where the subscripts 1, 2 are used to specify the upper and lower bubbles, respectively.

Consequently, Eq.(5-54) can be rewritten as

\[
\int \left[ \hat{q}^* \phi - \hat{u}^* \frac{\partial \phi}{\partial \nu} \right] F_1(z) dz + \int \left[ \hat{q}^* \phi - \hat{u}^* \frac{\partial \phi}{\partial \nu} \right] F_2(z) dz = \]

...
\[(4\pi + \int \hat{q}^* F_1(z) dz + \int \hat{q}^* F_2(z) dz) \phi(x_0),\]

where \( r = F_1(z) \) and \( r = F_2(z) \) describe the generators of the two bubbles.

Let the generator of the first bubble be partitioned into \( M_1 \) quadratic elements by \( N_1 \) nodes \( x_j, j = 1, 2, \ldots, N_1 \) with polar coordinates \( (r_j, z_j) \) and the generator of the second bubble be partitioned into \( M_2 \) quadratic elements by \( N_2 \) nodes \( x_j, j = N_1 + 1, N_1 + 2, \ldots, N_1 + N_2 \) with polar coordinates \( (r_j, z_j) \). Then Eq.(5–75) can be modified as follows:

\[
\sum_{j=1}^{N_1+N_2} (A_j \phi_j - B_j \psi_j) = (4\pi + c) \phi(x_0). \tag{8-2}
\]

Consequently, Eq.(5–79) can be rewritten as

\[
\sum_{j=1}^{N_1+N_2} (A_{kj} \phi_j - B_{kj} \psi_j) = (4\pi + C_k) \phi_k \quad (k = 1, 2, \ldots, N_1 + N_2). \tag{8-3}
\]

With proper boundary conditions imposed, one can solve the system of \( N_1 + N_2 \) linear equations defined by Eq.(8–3) easily.

### 8-3 Attraction and repulsion

Kornfeld & Suvorov (1944) pointed out that the Bjerknes forces between two pulsating spheres in liquid with the same pulsation frequency are attractive forces when the phases coincide and repulsive when the phases are opposite. By assuming both bubbles undergoing sinusoidal pulsation and neglecting all the nonlinear interaction effects, Crum (1975) followed Eller's approach and derived a quantitative relationship describing the Bjerknes forces \( F_B \) (positive if in repulsion) between two pulsating spheres as follows:

\[
F_B = -2\pi \rho \omega^2 \delta_1 \delta_2 \bar{R}_1 \bar{R}_2 \left( \frac{\cos \phi}{D^2} \right), \tag{8-4}
\]

where \( \rho \) is the density of the fluid, \( \omega \) is the pulsation frequency, \( \bar{R}_1 \) and \( \bar{R}_2 \) are the equilibrium radii of the two spheres with respective pulsation amplitudes \( \delta_1 \) and \( \delta_2 \). The distance between the two bubbles is \( D \equiv D_1 + D_2 \). \( \phi \) is the phase angle between \( R_1(t) \) and \( R_2(t) \). See Figure 8–2. Hence if the pulsations are in phase, the force is negative, which implies attraction.

Eq.(8–4) was tested against the experimental data by Crum (1975) for the cases of bubble attraction. Excellent agreement was obtained between the measured and predicted values of the relative velocity of approach of the air bubbles, provided the drag effect is considered. However, no attempt was made to verify the accuracy of the equation for the case of bubble repulsion. In addition, there is no sufficient justification for treating the oscillating air bubbles as pulsating rigid spheres. Will the shape oscillations of the bubbles affect the dynamics of the translational motion?
How will the two bubbles interact with each other? Will the two-to-one resonance persist even with 2 bubbles? These are the questions we try to answer in the following.

**Attraction of two identical bubbles, \( \phi = 0 \)**

In Crum’s derivation, the secondary Bjerknes force exerting on the first bubble by the second one, while both bubbles undergo the sinusoidal pulsations with the same frequency \( \omega \) and a phase lag \( \phi \) between them, is accounted as a kinematic buoyancy resulting from the acceleration induced by the pulsation of the second bubble neglecting the existence of the first pulsating bubble. Pelekasis & Tsamopoulos (1993 A) thus concluded that the result is asymptotic which holds only in the limit of small perturbation, \( \delta_1, \delta_2 \to 0 \), and very large distances between two bubbles, \( D \to \infty \). The purpose of the subsequent simulations are focused on demonstrating the accuracy of Eq.(8-4).

Small amplitude free oscillation of two bubbles in an inviscid fluid best fits the assumptions laid down in Crum’s derivation. The small amplitude volume oscillation is close to the sinusoidal pulsation. Unfortunately, the shape oscillation during the translational motion complicates the dynamics, which will not be reflected in Crum’s simplified model! However, it is generally believed that the high mode shape oscillations are confined within the near field because of their faster decay rates than that of the volume oscillation. Thus it is quite reasonable to expect that Crum’s model can capture at least the averaged motion of the centroid of the bubble when the two bubbles are at sufficient distance away from each other. Figure 8-3 shows the translational motion of the centroid of the first bubble for the case of two bubbles of equal size, \( \bar{R}_1 = \bar{R}_2 = 1.0 \), with centroids located 3 radii apart initially, \( D = 3.0 \). The gap between the two bubbles is only one radius. The far field pressure is set to 1333.33 (corresponding to the atmospheric pressure nondimensionalized) and
the amplitudes of volume oscillation for the four cases are $\delta_1 = \delta_2 = 0.01, 0.02, 0.03,$ and 0.1, respectively.

Simplified analysis of the average motion of the centroid can be obtained from the Crum’s model. Based on Eq.(8-4), the equation of motion for the first bubble can be written as

$$F_B = \rho \frac{4\pi}{3} (\tilde{R}_1 + d_1 \cos \omega t)^3 \frac{\dot{D}}{2},$$

or

$$\ddot{D} = -\frac{\alpha}{D^2 (1 + \epsilon \cos \omega t)^3},$$

where $\alpha \equiv 3\omega^2 \delta_1 \delta_2 \tilde{R}_1^{-1} \tilde{R}_2 \cos \phi$ and $\epsilon = d_1 / \tilde{R}_1$. The initial conditions are $D(0) = 3.0$ and $\dot{D}(0) = 0$. Eq.(8-6) is highly nonlinear and usually blows up at finite time. The numerical integration results (the light curves) are also shown in Figure 8-3.

The excellent agreement shown in Figure 8-3 is quite surprising. The slight deviations at the end of simulations signifies that the strong interactions can no longer be neglected, when the two bubbles are too close to each other.

The simulations all end up with two touching bubbles. Figure 8-4 shows the shapes of the two bubbles just before touching. Obviously, the shape deformation increases with the increasing amplitude of the volume oscillation. The shapes characterized by severe deformation on the side facing away from the direction of acceleration and smoother on the other side are referred as the spherical cap shapes, $\?$. This can be easily explained with the concept of Rayleigh-Taylor instability,
which states that the interface that separates two immiscible fluids is unstable when
the acceleration of the interface is in the direction of the lighter fluid toward the
heavier fluid. From Figure 8–4, the nodes cluster toward the side away from the
translational acceleration of the bubble. Clearly, the acceleration of the interface on
the side away from the translational acceleration is in the direction of the light fluid
toward the heavy fluid and thus unstable. Subsequently, severe deformation ensues
when the amplitude of the volume oscillation increases. Accordingly, this results in
the spherical cap shape shown in Figure 8–4 (d). For further details, see Pelekasis
& Tsamopoulos (1993 A).

Figure 8–5 demonstrates the good agreement of Eq.(8–6) with the simulation
results for the cases corresponding to different values of far field pressure. A
general observation of the simulations conducted indicates that the volume mode fre-
cuencies of the two bubbles oscillating in phase are always slower than that of the
corresponding cases with a single bubble. The volume mode oscillation for the case
with $p_\infty = 400, \delta_1 = \delta_2 = 0.01, D_1 = D_2 = 1.5$ is shown in Figure 8–6. The
frequency is 5.6042Hz, which is slower than the Rayleigh’s frequency for a single
bubble, 6.5358Hz.
Figure 8–5  Translational motions induced by the pulsation of two in-phase bubbles with various far field pressures. \( \bar{R}_1 = \bar{R}_2 = 1.0, D = 3.0, \delta_1 = \delta_2 = 0.01, p_{oo} = 50, 400, 1333.33 \).

Figure 8–6  Volume mode oscillation for one of the two bubbles oscillating in phase with \( p_{oo} = 400, D_1 = D_2 = 1.5, \delta_1 = \delta_2 = 0.01, \) and \( \bar{R}_1 = \bar{R}_2 = 1.0 \).
Figure 8–7 Large amplitude volume mode oscillation for one of the two bubbles oscillating in phase with $p_{\infty} = 2, 20, 100$, $D_1 = D_2 = 3.0$, $\delta_1 = \delta_2 = 0.3$, and $R_1 = R_2 = 1.0$.

Due to the nonlinearity, the large amplitude volume oscillation is no longer close to the sinusoidal oscillation. As expected, the fluctuations between those two results increase. Quite surprisingly, however, the averaged motion predicted by Eq.(8–4) agrees very well with the simulation results even for large amplitude volume oscillation, see Figure 8–7.

Repulsion of two anti-phase bubbles, $\phi = \pi$

According to Eq.(8–4), two bubbles oscillating out of phase will repel each other. Numerical simulations are also conducted to verify the accuracy. Although excellent agreement is found between the predictions of Eq.(8–4) and results of numerical simulations for attracting bubbles, significant difference exists between those two for repelling bubbles. The difference can be remedied by multiplying the force predicted by Eq.(8–4) with some empirical constant $\beta$ locally. i.e.

$$F_B = -2\beta \pi \rho \omega^2 \delta_1 \delta_2 R_1^2 R_2 \left( \frac{\cos \phi}{D^2} \right). \quad (8–7)$$

Note that $\beta$ approaches 1 as the separation between the two bubbles increases. A possible cause for the difference can be attributed to the fact that for the case of two bubbles oscillating in phase the mid-plane can be considered as a rigid wall. For the case of two bubbles oscillating in anti-phase ($\phi = \pi$), however, the mid-plane should be considered as a pressure release boundary. As is well known in the fluid dynamics, a body translating in an inviscid fluid is subjected to two forces:
the force coming from the pressure gradient and the dragging force due to relative acceleration between the liquid and the body. It is the pressure gradient effect that was not accounted for in the Crum’s model which leads to the difference.

Consider the case of two bubbles with equilibrium radius \( \bar{R}_1 = \bar{R}_2 = 1 \) subjected to \( p_\infty = 400 \), initial perturbations \( \delta_1 = -\delta_2 = 0.01 \), and \( D_1 = D_2 = 1.5 \). After proper correction (\( \beta = 3.0 \) in the current case), excellent agreement between the two results is shown in Figure 8–8 and Figure 8–9. The volume oscillation frequency for the bubbles in this case is 7.8421 Hz, which is faster than the Rayleigh’s frequency for a single bubble, 6.5358 Hz.

**Repulsion of two alternating-phase bubbles**

Though powerful, Eq.(8–4) and (8–7) only account for the two bubbles oscillating independently. More generally, the two bubbles will interact and communicate with each other. Therefore, they can not oscillate independently. Rather, they cooperate with each other. Let us consider the case of two bubbles with equal equilibrium radius separated with a distance of 3 radii between the center of the two bubbles. The far field pressure is \( p_\infty = 400.0 \). Initial perturbation(\( P_0 = 0.01 \)) is imposed on bubble 1, while bubble 2 is stationary. The volume oscillations and translational motions of the two bubbles are shown in Figure 8–10 (a)-(d). Since the volume oscillation is no longer sinusoidal, Eq.(8–4) and (8–7) do not apply in this case. A general observation from the simulations is that one bubble starts with volume
oscillation while the other stationary always leads to the repulsion of the two bubbles. For the purpose of convenience, this will be referred to as the alternating-phase bubbles hereafter. Under such circumstances, one additional observation is that the total volume going to infinity oscillates sinusoidally at a frequency slower than the Rayleigh’s frequency for a single bubble with the corresponding size. In the present case, the frequency is 5.7010 Hz and the Rayleigh’s frequency is 6.5358Hz.

8.4 Effect of shape oscillations on the translational motion

As mentioned in Chapter 2, the concept of mode shape comes from the linearized theory. A corresponding linearized formulation for the two bubble problem was first developed in Pelekasis (1991). Owing to the complexity of the geometric configuration, the mode shape can no longer be expressed in terms of a single or a finite number of Legendre polynomials. Instead, an infinite series of Legendre polynomials has to be used. Usually, the mode shapes can only be obtained by solving an eigenvalue problem numerically. In addition, the concept of mode shape can not be applied for the case of two translating bubbles. Thus the determination of mode shape for two bubble case is of very limited value.

Legendre mode decomposition of the bubble shape, however, is still a useful tool for describing the dynamics of the bubble motion. In the subsequent discussion, the
Figure 8–10 Translational motion (repulsion) induced by two alternating-phase bubbles. The far field pressure $p_\infty$ is set to 400.0. Initial perturbation ($P_0 = 0.01$) is imposed on bubble 1, while bubble 2 is stationary.

Figure 8–11 In-phase shape oscillations result in the attraction of the two bubbles. The initial perturbations $P_0 = 0.1$ are imposed on both bubbles. Due to the symmetry, only the data of bubble 1 are shown. The far field pressure is set to $p_\infty = 2.0$. The distance between the two centroids is 2.5.
In-phase shape oscillations

It has been shown in the previous section that in-phase volume oscillation leads to the attraction of the two bubbles. It is also expected that the interaction between the shape modes is much weaker than that between the volume oscillations. In order to investigate the contribution of in-phase shape oscillations on the translational motion, special precaution is required to avoid the interference of the volume oscillation. The best way to achieve this is to reduce the far field pressure \( p_\infty \) so that the Bjerknes force associated with the volume oscillation is negligible. Here \( p_\infty = 2.0 \) is used. The initial perturbations \( P_6 = 0.1 \) are imposed on both bubbles. In order to obtain a significant amount of translational motion, the two bubbles must be brought close enough to each other. Therefore, \( D = 2.5 \) is used. The initial gap between the two bubbles is only 0.3. Simulation stops when the two bubbles touch each other and the dynamics of the dominant modes are plotted in Figure 8–11.

Obviously, in-phase shape oscillation leads to the attraction of the two bubbles. Quite surprisingly, the two-to-one resonance between the sixth mode and the tenth mode still persists, even the two bubbles are brought so close to each other.

Anti-phase shape oscillations

To investigate the contribution of anti-phase shape oscillation to the translational motion, initial perturbation \( P_6 = 0.1 \) is imposed on the first bubble and \( P_6 = -0.1 \) is imposed on the second bubble. The far field pressure is set to \( p_\infty = 2.0 \) and the distance \( D = 2.5 \) between the centroids of the two bubbles is used. The translational motions of the centroids of the two bubbles are plotted in Figure 8–12. Clearly, anti-phase shape oscillation leads to the repulsion of the two bubbles.
Alternating-phase shape oscillations

Consider the cases of one bubble being imposed upon an initial perturbation \( P_2 = 0.1 \) and the other one is stationary. The two bubbles are separated at a distance \( D = 2.4, 2.6, 2.8, 3.0 \) and \( 4.0 \) between the centroids. The far field pressure is \( p_\infty = 2.0 \). Simulation results are shown in Figure 8–13 (a)-(e).

The first column demonstrates how the dynamics of the \( P_2 \) mode of the first bubble is affected by the introduction of the second bubble. Clearly, the interaction increases significantly with the decreasing distance between the two bubbles. The excitation of the \( P_2 \) mode in the second bubble grows as the distance decreases. The translational motion of the centroids of both bubbles are plotted in the third and the fourth columns, respectively. No significant translational motions are observed for \( D \geq 2.8 \). The translational motions (repulsion) for \( D \leq 2.6 \), though more or less pronounced, are quite small compared with those induced by the volume mode dominated oscillations mentioned in the previous section. For the purpose of more realistic comprehension, the bubble shapes corresponding to the moment of maximum deformation in the second bubble are shown in Figure 8–14.

Effect of shape oscillations on two attracting bubbles

It is also interesting to know that if both the volume mode and shape modes are significant, will the translational motion induced by the volume mode oscillation be significantly affected by the shape oscillations? In Figure 8–15 (a), three initial perturbations \( P_0 = 0.05 \), \( (P_0 = 0.05, P_6 = 0.01) \), and \( (P_0 = 0.05, P_6 = 0.1) \) are imposed on both bubbles. Obviously, the sixth mode shape oscillations of various amplitudes have no significant effect on the translational motion of the bubble induced by the in-phase volume oscillations. (The dash line is predicted by Eq.(8–4).)

As shown in Figure 8–15 (b), three initial perturbations \( P_0 = 0.1 \), \( (P_0 = 0.1, P_4 = 0.1) \), and \( (P_0 = 0.1, P_6 = 0.1) \) are imposed on both bubbles, respectively. Apparently, no significant effect on the translational motion is observed for various shape
Figure 8–13 Alternating-phase shape oscillations (a) $D = 4.0$, (b) $D = 3.0$, (c) $D = 2.8$, (d) $D = 2.6$, and (e) $D = 2.4$. Subscripts are used to specify the bubble under discussion.
Figure 8–14  Bubble shapes at the moment of maximum deformation in the second bubble. (a) $D = 4.0$, (b) $D = 3.0$, (c) $D = 2.8$, (d) $D = 2.6$, and (e) $D = 2.4$. Subscripts are used to specify the bubble under discussion.

Figure 8–15  No significant effect of shape oscillation on the translational motion of two attracting bubbles is observed.
Table 8.1 Parameters for studying the effect of shape oscillations on two repelling bubbles.

<table>
<thead>
<tr>
<th></th>
<th>$p_{oo}$</th>
<th>$P_0$</th>
<th>$P_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>265.14</td>
<td>0.02</td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>265.14</td>
<td>0.02</td>
<td>0.001</td>
</tr>
<tr>
<td>(c)</td>
<td>265.14</td>
<td>0.02</td>
<td>0.01</td>
</tr>
<tr>
<td>(d)</td>
<td>265.14</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>(e)</td>
<td>265.14</td>
<td>0.02</td>
<td>0.05</td>
</tr>
<tr>
<td>(f)</td>
<td>265.14</td>
<td>0.02</td>
<td>0.10</td>
</tr>
</tbody>
</table>

Figure 8–16 Effect of the shape oscillation on the translational motion of two repelling bubbles is quite significant.

modes with significant amplitudes. (The dash line is predicted by Eq.(8–4).)

Effect of shape oscillations on two repelling bubbles

In order to investigate the effect of shape oscillation on two repelling bubbles, we consider the cases of one bubble with parameters given in Table 8.1 while the second bubble is stationary. The two bubbles are expected to expel each other due to the alternating-phase volume oscillations.

As shown in Figure 8–16, the repulsions are not only slowed down but also they change into attractions. This is quite different from what was observed in the case of attracting bubbles, where the shape oscillations have no effect on the translational motion. Due to the complicated dynamics involved, this peculiar phenomenon cannot be explained with the simple argument of the volume mode induced translational motion mentioned above.
Two-to-one resonance

Two-to-one resonance is a nonlinear coupling between two normal modes. Since the normal modes for the two bubble problem are difficult to obtain, the two-to-one resonance between two normal modes is even harder to analyze theoretically. Instead of trying to find the normal modes for the two bubble problem, we consider a more reasonable question as follows: The addition of a stationary bubble near a bubble under two-to-one resonance conditions will certainly modify the surrounding flow structure and thus one may wonder what happens then?

Let us consider a single bubble with initial perturbation $P_0 = 0.02, P_5 = 0.01$ subjected to the far field pressure $p_\infty = 265.14$. (The corresponding natural frequency of volume mode for a single bubble is exactly twice that of the sixth mode.) The dynamics of the $P_5$ mode is shown in Figure 8–17 (a). The exponential growth of $P_5$ mode due to the two-to-one resonance is obvious.

Let us now put a stationary bubble next to the first bubble with a distance of 3 radii between the centroids of the two bubbles. As explained in the previous section, the two bubbles are expected to repel each other due to the alternating-phase volume oscillations. The corresponding dynamics of the $P_5$ mode of the first bubble is shown in Figure 8–17 (b). Clearly, the two-to-one resonance is suppressed and delayed by the introduction of the second stationary bubble.

As another example, let us consider the cases of the first bubble with parameters given in Table 8.2 while the second bubble is stationary. The distance $D$ between the centroids is 4.4.
Figure 8–18 Study of the effect of two-to-one resonance on the translational motions.

The translational motions of both centroids of the two bubbles are plotted in Figure 8–18. The dynamics of the dominant modes for both bubbles of the four cases are shown in Figure 8–19—8–22. For a single bubble, the far field pressure $p_\infty = 84.19$ corresponds to the natural frequency of volume mode which is exactly twice that of the fourth mode. However, the fourth mode is not excited in the two bubble case, see Figure 8–19 and 8–21. Instead, the fourth mode is excited in the second bubble when $p_\infty = 100.0$, Figure 8–20. Moreover, the fourth mode is excited in the first bubble with the help of the sixth mode in the same bubble but at a smaller amplitude than that of the second bubble, Figure 8–22.

Since the shape mode grows at the expense of the volume mode during the two-to-one resonance, one can expect that the translational motion dominated by the volume mode oscillation will be slowed down by the two-to-one resonance between the volume mode and one of the shape modes. From Figure 8–20 and 8–22, the fourth mode is excited in the second bubbles of cases (b) and (d). The corresponding sudden change in the translational motions of the second bubbles of both cases can be easily observed in Figure 8–18. The sixth mode shape oscillations in the first bubbles seem to significantly retard the translational motions of the first bubbles for cases (c) and (d). More interestingly, it seems that the two-to-one resonance between the sixth mode and the tenth mode is not affected by the translational motion, Figure 8–21.

In order to gain a more physical comprehension, the bubble shapes for case (d) in Table(8.2) in contrast to their initial positions (the underlying light curves) are shown in Figure 8–23. The significant upward movement of the second bubble between (C) and (D) can be easily captured from the figure. A jet-like flow at the second bubble can be clearly identified in Figure 8–23 (E), which leads to the termination of the computation.
Figure 8–19 The dynamics of dominant modes for the case with parameters given in Table 8.2 (a).

Figure 8–20 The dynamics of dominant modes for the case with parameters given in Table 8.2 (b).
Figure 8–21 The dynamics of dominant modes for the case with parameters given in Table 8.2 (c).

Figure 8–22 The dynamics of dominant modes for the case with parameters given in Table 8.2 (d).
8-5 Bubbles of Different Sizes

In order to investigate the size effect, small bubbles of six different values of radii (bubble 2) are used; \( \tilde{R}_2 = 0.25, 0.5, 0.6, 0.75, 0.8, 0.9 \). The far field pressure \( p_\infty \) is set to 50.0, the distance between the two centroids \( D = 3.0 \), and initial perturbation \( P_0 = 0.02 \) is imposed on the larger bubble (bubble 1) for all cases. The translational motions of the centroids of both bubbles are shown in Figure 8–24. It is observed that the two bubbles repel each other for \( \tilde{R}_2 \geq 0.9 \), while the two bubbles attract each other when \( \tilde{R}_2 \leq 0.8 \). The bubble with radius \( \tilde{R}_2 = 0.5 \) move fastest toward the large bubble, while bubbles of very small radius \( \tilde{R}_2 \leq 0.25 \) drift sluggishly like a rigid sphere due to the strong surface tension. The second shape mode is excited in the smaller bubble with \( \tilde{R}_2 = 0.4 \), the third mode with \( \tilde{R}_2 = 0.6 \), and the fourth mode with \( \tilde{R}_2 = 0.75 \). See Figure 8–25.

Simple calculation suggests that these shape modes of the smaller bubbles are excited via the harmonic resonance with the volume modes of the larger ones. In
Figure 8–24 Translational motions of the centroids of two bubbles of different sizes. Six different values of the smaller radii (bubble 2) are used; $R_2 = 0.25, 0.5, 0.6, 0.75, 0.8, 0.9$. The far field pressure $p_{\infty}$ is set to 50.0, the distance between the two centroids $D = 3.0$, and initial perturbation $P_0 = 0.02$ is imposed on the larger bubble (bubble 1) for all cases.

Figure 8–25 Shape modes in the small bubbles are excited via direct harmonical resonance with the volume modes of the large bubbles.
order to verify this conjecture, the far field pressures are carefully tuned so that the volume mode frequencies of the larger bubbles are exactly equal to that of the second and third modes of the smaller bubbles, respectively. Therefore, for the case with \( \tilde{R}_2 = 0.4 \) the far field pressure is set to \( p_\infty = 43.12 \) so that the corresponding frequency for the volume mode of the larger bubble is 13.69, which is exactly the natural frequency for the second mode of the smaller bubble. The linear growth of the second mode of the second bubble asserts the possibility of direct harmonic resonance with the volume mode of the large bubble, Figure 8–26 (a). What is most interesting is the translational motion of the smaller bubble. The small bubble approaches the larger one initially and then it undulates around its initial position, Figure 8–26 (b).

Similarly, for the case with \( \tilde{R}_2 = 0.6 \) the far field pressure is set to \( p_\infty = 42.57 \) so that the corresponding volume mode frequency (13.61) of the larger bubble is exactly equal to that of the third mode of the smaller bubble. Again the direct harmonic resonance between the two modes is illustrated by the linear growth shown in Figure 8–27, but with smaller amplitude than that of the previous case.

As another example, the far field pressure is increased to 1333.33 (corresponding to the atmospheric pressure nondimensionalized). The distance between the
centroids is $D = 3.0$ and initial perturbation $P_0 = 0.02$ is imposed upon the first bubble with $\hat{R}_1 = 1.0$. Six different values of the radii for the second bubble are used; $\hat{R}_2 = 0.25, 0.5, 0.84, 0.86, 1.2, 1.5$. The result is shown in Figure 8–28. Obviously, the two bubbles repel each other when $\hat{R}_2 \geq 0.86$ while they attract each other when $\hat{R}_2 \leq 0.84$.

8-6 Isotropic forcing

Since the characteristic dimensions of bubbles of interest are usually much smaller than the acoustic wavelength, the acoustic forcing can be approximated by an oscillatory isotropic pressure field without introducing significant errors. Thus only the isotropic forcing will be considered in the present work. This topic has been
Figure 8–29 Comparison between the simulation results produced from the code developed in the present work and those given in Pelekasis 1993B, Figure 5.

well explored by Pelekasis (1991). As a simple verification of the accuracy of the numerical code developed in the present work, the example case given in Pelekasis & Tsamopoulos (1993 B), Figure 5 (page 511) is reproduced here, Figure 8–29. The excellent agreement is obvious. Bubble shapes, obtained up to breakdown of computations with $R = 1, D = 4, p_{\infty} = 100, \epsilon = 0.6, \omega_f = 6.5$ at (a): $t=0.64(\square), t=0.76(\Diamond), t=0.80(\circ), t=0.84(\triangle)$; (b): $t=0.92(\square), t=0.96(\Diamond), t=1.00(\circ), t=1.04(\triangle)$, are also shown in Figure 8–29. (Note the dimensionless pressure used in the present work is different from that used in Pelekasis & Tsamopoulos (1993 B) by a factor of 2.)
Table 8.3  The natural frequencies of the corresponding eigenmodes.

<table>
<thead>
<tr>
<th>R</th>
<th>Volume</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>14.71</td>
<td>3.46</td>
<td>6.32</td>
<td>9.49</td>
<td>12.96</td>
<td>16.73</td>
<td>20.78</td>
<td>25.10</td>
<td>29.66</td>
<td>34.47</td>
</tr>
<tr>
<td>0.75</td>
<td>22.76</td>
<td>5.33</td>
<td>9.74</td>
<td>14.61</td>
<td>19.96</td>
<td>25.76</td>
<td>32.00</td>
<td>38.64</td>
<td>45.67</td>
<td>53.07</td>
</tr>
</tbody>
</table>

Kornfeld & Suvorov (1944) argued that the primary Bjerknes forces between two bubbles in the acoustic field are repulsive, if the forcing frequency \( \omega_f \) lies between the two eigenfrequencies of the volume oscillations of the two bubbles. Otherwise, the primary Bjerknes forces between the two bubble are attractive. This general statement was verified by Pelekasis & Tsamopoulos (1993 B) numerically using two initially in-phase bubbles. However, the effect of different transients are not considered. In this section, a different kind of initial condition will be used.

Consider two bubbles with different radii \( \bar{R}_1 = 1.0 \) and \( \bar{R}_2 = 0.75 \), and an initial distance \( D = 3.0 \) between the two centroids. Initial perturbation \( P_0 = 0.02 \) is imposed upon the first bubble only, while the second bubble remains stationary. The far field pressure \( p_\infty \) at equilibrium is set to 50.0 and varies as follows:

\[
p_\infty = 50(1 + \epsilon \sin \omega_f t), \tag{8-8}
\]

where \( \epsilon \) is the amplitude of the isotropic forcing and \( \omega \) the frequency. Here \( \epsilon = 0.1 \) is used for all cases.

Under these conditions, the natural frequencies of the corresponding eigenmodes\(^1\) for each bubble are given in Table 8.3. It is interesting to note that the volume mode frequency of the first bubble is almost equal to that of the fourth mode of the second bubble. Can these two modes resonate with each other harmonically? This is indeed observed in all the following cases. Taking \( \omega_f = 3.14 \), the volume of both bubbles oscillate at a frequency of 14.14 and the envelope of these volume oscillations varies with a frequency of 3.14 approximately in response to the slow variation of the far field pressure, Figure 8–30. The fourth mode is excited in the second bubble due to the harmonic resonance with the volume mode of the first bubble, Figure 8–30. Since the forcing frequency \( \omega_f \) is below the natural frequencies of the volume modes of the two bubbles, they attract each other as predicted by Kornfeld’s principle, Figure 8–35.

Increasing \( \omega_f \) to 18, which is roughly twice the natural frequencies of the fourth mode of the first bubble and the third mode of the second bubble, both are excited due to the two-to-one subharmonic resonances between the forcing and the corresponding shape modes. Significant growth of the fourth mode is observed in the first bubble. Several bubble shapes up to the breakdown of computations are shown in Figure 8–31.

Further increasing \( \omega_f \) to 20, the forcing is in harmonic resonance with the volume mode and the fifth mode of the second bubble and in two-to-one subharmonic

\(^1\) The eigenmodes for a single bubble, instead of the two bubble case.
resonance with the fourth mode of the first bubble and the third mode of the second bubble. However the two-to-one resonance does not have enough time to evolve. In Figure 8–32 the volume of the second bubble increases significantly due to the harmonic resonance with forcing, which in turn enhances the translational motion involved, Figure 8–35. The fifth mode is also excited in the second bubble but with a much smaller amplitude than that of volume mode.

For the case of $\omega_f = 25$, it is quite difficult to predict whether the two bubbles will repel or attract each other. The third mode in the second bubble is excited through two-to-one resonance with the forcing. And the sixth mode is in harmonic resonance with the forcing and thus grows. Several bubble shapes up to the breakdown of
Figure 8–32 The volume of the second bubble increases significantly due to the harmonic resonance with forcing, which in turn enhances the translational motion involved.

Figure 8–33 Several bubble shapes up to the breakdown of computations are show here.

computations are shown in Figure 8–33. Since this forcing frequency is too close to one of the natural frequencies of the two bubbles, it is hard to apply Kornfeld’s principle at this blurry boundary. It turns out that the two bubbles repel each other at this forcing frequency.

Moving away from the blurry boundary, the two bubbles attract each other again when \( \omega_f = 30 \), Figure 8–35. It appears that the forcing is in subharmonic resonance with the fourth mode of the second bubble in the present case, Figure 8–34.

Furthermore, consider the case of two bubbles with parameters \( R_1 = 1.0 \), \( R_2 = 0.75 \), \( D = 3.0 \), \( \epsilon = 0.1 \), and \( \omega_f = 6.28 \). Initial perturbation \( P_0 = 0.02 \) is imposed
Figure 8–34  It appears that the forcing is in subharmonic resonance with the fourth mode of the second bubble in the present case.

upon the first bubble only, while the second bubble remains stationary. Six different values of the far field pressure at equilibrium are used; \( p_{\infty} = 50, 80, 100, 110, 120, 200 \). Note the forcing frequency \( \omega_f \) is well below the natural frequencies of the volume modes of both bubbles. Quite surprisingly, the translational motions of the two bubbles switch from attraction to repulsion with the increase of the far field pressure, Figure 8–36. This indicates that the initial conditions of the two bubbles play significant roles in determining the fate of the two bubbles.

8-7 Conclusion

In this chapter, the implementation of the BEM for analyzing the two bubble problem is given. Computer program has been verified to give correct results.

For two bubbles of the same size, the Bjerknes forces between them are attractive if the two bubbles are in-phase and repulsive if anti-phase. Alternating-phase bubbles repel each other always. The magnitudes of the Bjerknes forces can be predicted by Crum’s formula for two attracting bubbles. For two repulsive bubbles, however, an empirical multiplication factor is needed. Similarly, two bubbles under in-phase shape oscillations attract each other, while bubbles in anti-phase or alternating-phase shape oscillations repel each other. Shape oscillation has no effect on the translational motions of two attracting bubbles. On the contrary, shape oscillation has significant effect on the translational motions of two repelling bubbles.

For bubbles of different sizes, direct resonance between the forcing and volume mode and shape modes, subharmonic internal resonance between bubbles, and subharmonic resonance between the isotropic forcing and shape modes seem to be
Figure 8–35 Moving away from the blurry boundary, the two bubbles attract each other again when \( \omega_f = 30 \)
Figure 8–36  The translational motions of the two bubbles switch from attraction to repulsion with the increase of the far field pressure.
possible.

REFERENCES


Chapter 9

Conclusions

9-1 Major contributions

In this thesis, starting from formulating the hydrodynamic model of an incompressible inviscid fluid with the Laplace equation and the steady-state scattering problem of a planar incident wave scattered by an obstacle in a compressible fluid with the Helmholtz equation, BEM code is developed to solve both the Laplace equation and Helmholtz equation numerically. Time marching of the kinematic and dynamic boundary conditions on the interface is performed with the fourth order Runge-Kutta integration. Numerical simulations are conducted to explore the nonlinear dynamics of drops and bubbles. The major contributions of this thesis are briefly summarized below, and possible future work will be suggested, where appropriate.

Acoustically levitated Drops

- To achieve stable drop levitation, the translational motion impose a lower limit on the acoustic bond number, the minimum trapping pressure, while the shape oscillation of drop set up an upper limit to prevent the drop breakup by the strong acoustic radiation pressure.
- Owing to the order-of-magnitude difference between the periods of translational motion and shape oscillation, the coupling effect can be neglected on a first approximation. In particular, away from the stability boundary of trapping region the shape oscillation does not affect the translational motion.
- In our study, two equilibria which coexist for an acoustic bond number between the lower and upper limits can not be simultaneously stable.
- Static equilibrium shape can be found by incorporating artificial damping to eliminate the transients.
- The natural frequency for the shape oscillation increases with the increasing acoustic bond number to a maximum value and then decreases with the increasing
acoustic bond number.

**Single bubble**

- The spherical volume oscillation of a bubble could be unstable due to nonlinear interaction between eigenmodes.
- Two-to-one resonance is studied and the instability wedges are identified numerically. The instability wedges agree with the theoretic prediction for small amplitude volume mode oscillation but deviate from the theoretic prediction for large amplitude volume mode oscillation.
- Within the instability, a qualitative boundary can be found which separates the stable bubble oscillations from the transient bubble oscillations.
- The geometric amplification factor is believed to be the mechanism that leads to the jet formation during the breakup of a collapsing bubble.

**Two bubbles**

- The Bjerknes forces between two attracting bubbles can be well predicted by Crum's formula.
- The Bjerknes forces between two repulsive bubbles should be modified by multiplying the Bjerknes force given by Crum's formula with some empirical constant.
- In-phase bubbles always attract each other, while anti-phase and alternating-phase bubbles repel each other.
- Shape oscillation has significant effect on the translational motions of two repelling bubbles, while no significant effect on those of two attracting bubbles is observed.
- Two-to-one resonance has the effect of slowing down the repulsive motions, while the attraction of two bubbles seems to surpress the two-to-one resonance.
- Our numerical examples suggest that for bubbles of different sizes the direct resonance between volume mode of one bubble with the shape mode of the other is possible. Furthermore, two-to-one resonance between the isotropic forcing and shape modes is also possible.

**9-2 Future work**

Several interesting phenomena are observed with the examples given in Chapter 8. Yet, the program is still limited by the stiffness of the collapsing bubble, and the severe nodal clustering effect associated with the strong translational motion aggravates the stability issue significantly. Nodal redistribution provides a possible alternative, however special caution should be paid to avoid introducing unknown dynamics into the numerical simulation. Due to the nonlinearity, exceedingly small
time step is only necessary for certain critical stage. Thus an adaptive time step algorithm is highly recommended for efficient simulations.
Appendix A

Normal Form Calculation

The following program (nf.m) in matlab takes a $n \times n$ matrix $a$ as an input argument and returns the corresponding normal form as output.

```matlab
function [normal_form,resonance]=nf(a)
    echo off;
    n=length(a(:,1));
    no_monomials = n*(n+1)/2;
    no_monomial_vectors = n*no_monomials;
    %
    % Define monomial terms.
    % order: 1 2 3 4 5 6 7 8 9 10
    % index: 11, 12, 13, 14, 22, 23, 24, 33, 34, 44
    %
    k = 0;
    for i=1:n,
        for j=i:n,
            k = k+1;
            index1(k) = i;
            index2(k) = j;
        end
    end
    for k=1:no_monomial_vectors,
        i = ceil(k/no_monomials);
        j = rem(k,no_monomials);
        if j==0,
            j = no_monomials;
        end
        r = index1(j);
        s = index2(j);
        for p=1:n,
            for q=1:no_monomials,
                c(p,q) = 0.0;
            end
        end
        c(i,j) = 1;
    end
end
```
% Calculate DLY
b = a*c;
normal_form(:,:,k) = reshape(b',no_monomial_vectors,1);

% Calculate DYL
for l=1:n,
    for m=1:n,
        dym(m,l) = 0;
    end

% Differentiation
if r==l,
    if r==s,
        dym(s,l) = 2;
    else
        dym(s,l) = 1;
    end
elseif s==l,
    dym(r,l) = 1;
end
end
d = dym*a;
t = 0;
for p=1:n,
    for q=p:n,
        t = t+1;
        if p==q,
            tmp(t) = d(p,q);
        else
            tmp(t) = d(p,q)+d(q,p);
        end
    end
end
for p=1:no_monomials,
    index = (i-1)*no_monomials+p;
    normal_form(index,k) = normal_form(index,k)-tmp(p);
end
resonance = null(normal_form, r);
for p=1:n,
    sprintf('Equation %d ',p)
    RHS = sprintf('t');
    for q=1:n,
        if a(p,q)==0
            RHS = sprintf('%s%f+%s%d ',RHS,a(p,q),q);
        end
    end
end
for u=1:length(resonance(1,:)),
    for q=1:no_monomials,
        h = (p-1)*no_monomials+q;
        if resonance(h,u)==0,
            RHS = sprintf('%s%f+%s%d+%s%d ',RHS,a(p,q),h,x(k),k);
        end
    end
end
end
end
display(RHS);
end
Appendix B

Numerical Integration Formulas

B-1 Complete elliptic integrals of the first and second kinds

The complete elliptic integrals of the first kind $K(m)$ and second kind $E(m)$ can be approximated accurately with a series given in Abramowitz & Stegun (1964) as follows:

$$K(m) = \left[ a_0 + a_1m_1 + a_2m_1^2 + a_3m_1^3 + a_4m_1^4 \right] +$$
$$\left[ b_0 + b_1m_1 + b_2m_1^2 + b_3m_1^3 + b_4m_1^4 \right] \ln(1/m_1) + \epsilon(m), \quad (B-1)$$

where $m_1 = 1 - m$, $|\epsilon(m)| \leq 2 \times 10^{-8}$ and

\[
\begin{align*}
a_0 &= 1.38629 \ 436112 & b_0 &= 0.50000 \ 00000 \\
a_1 &= .09666 \ 344259 & b_1 &= .12498 \ 593597 \\
a_2 &= .03590 \ 092383 & b_2 &= .06880 \ 248576 \\
a_3 &= .03742 \ 563713 & b_3 &= .03328 \ 355346 \\
a_4 &= .01451 \ 196212 & b_4 &= .00441 \ 787012
\end{align*}
\]

$$E(m) = \left[ 1 + a_1m_1 + a_2m_1^2 + a_3m_1^3 + a_4m_1^4 \right] +$$
$$\left[ b_1m_1 + b_2m_1^2 + b_3m_1^3 + b_4m_1^4 \right] \ln(1/m_1) + \epsilon(m), \quad (B-2)$$

where $m_1 = 1 - m$, $|\epsilon(m)| \leq 2 \times 10^{-8}$ and

\[
\begin{align*}
a_1 &= .44325 \ 141463 & b_1 &= .24998 \ 368310 \\
a_2 &= .06260 \ 601220 & b_2 &= .09200 \ 180037 \\
a_3 &= .04757 \ 383546 & b_3 &= .04069 \ 697526 \\
a_4 &= .01736 \ 506451 & b_4 &= .00526 \ 449639
\end{align*}
\]
B-2 One-dimensional standard Gaussian quadrature

The integral in this case can be written as

\[ I = \int_{-1}^{1} f(\xi) d\xi \sim \sum_{i=1}^{n} w_i f(\xi_i), \]  

(B-3)

where \( n \) is the number of integration points, \( \xi_i \) is the coordinate of the \( i \)th integration point, \( w_i \) is the associated weighting factor. Values of \( \xi_i \) and \( w_i \) are listed in Table B.1.

Table B.1 Coordinates of integration points and weighting factors for Standard Gaussian Quadrature

<table>
<thead>
<tr>
<th>( \pm \xi_i )</th>
<th>( w_i )</th>
<th>( \pm \xi_i )</th>
<th>( w_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 2 )</td>
<td></td>
<td>( n = 8 )</td>
<td></td>
</tr>
<tr>
<td>0.57735 02691 89626 1.00000 00000 00000</td>
<td>0.18343 46424 95650 0.36268 37833 78362</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.77459 66692 41483 0.55555 55555 55555</td>
<td>0.52553 24099 16329 0.31370 66458 77887</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00000 00000 00000 0.88888 88888 88888</td>
<td>0.79666 64774 13627 0.22238 10344 53374</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.86113 63115 94053 0.34785 48451 37454</td>
<td>0.96028 98564 97536 0.10122 85362 90376</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n = 3 )</td>
<td></td>
<td>( n = 9 )</td>
<td></td>
</tr>
<tr>
<td>0.33998 10435 84856 0.65214 51548 62546</td>
<td>0.00000 00000 00000 0.33023 93550 01260</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.86113 63115 94053 0.34785 48451 37454</td>
<td>0.32425 34234 03809 0.31234 70770 40003</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00000 00000 00000 0.56888 88888 88889</td>
<td>0.61337 14327 00590 0.26061 06964 02935</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n = 4 )</td>
<td></td>
<td>( n = 10 )</td>
<td></td>
</tr>
<tr>
<td>0.53846 93101 05683 0.47862 86704 99366</td>
<td>0.01487 43389 81631 0.29552 42247 14753</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.90617 98459 38664 0.23692 68850 56189</td>
<td>0.43339 53941 29247 0.26926 67193 09996</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00000 00000 00000 0.56888 88888 88889</td>
<td>0.67940 95682 99024 0.18064 81606 94857</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n = 5 )</td>
<td></td>
<td>( n = 11 )</td>
<td></td>
</tr>
<tr>
<td>0.23861 91860 83197 0.46791 39345 72691</td>
<td>0.06506 33666 88985 0.14945 13491 50581</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.66120 93864 66265 0.36076 15730 48139</td>
<td>0.97390 65285 17172 0.06667 13443 08688</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.93246 95142 03125 0.17132 44923 79170</td>
<td>( n = 12 )</td>
<td>0.12523 34085 11469 0.24914 70458 13403</td>
<td></td>
</tr>
<tr>
<td>( n = 7 )</td>
<td></td>
<td>0.36783 14989 98180 0.23349 25365 38355</td>
<td>0.58731 79542 86617 0.20316 74267 23066</td>
</tr>
<tr>
<td>0.00000 00000 00000 0.41795 91836 73469</td>
<td>0.76990 26741 94305 0.16007 83285 43346</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.40584 51513 77397 0.38183 00505 05119</td>
<td>0.90411 72563 70475 0.10693 93259 95318</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.74153 11855 99394 0.27970 53914 89277</td>
<td>0.98156 06342 46719 0.04717 53363 86512</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### B-3 One-dimensional logarithmic Gaussian quadrature

Kernels including a logarithmic singularity at one end of the integration domain can be integrated using the following formula:

\[
I = \int_0^1 \ln \left( \frac{1}{\xi} \right) f(\xi) \, d\xi \sim \sum_{i=1}^{n} w_i f(\xi),
\]

where integration point coordinates \( \xi_i \) and weighting factor \( w_i \) are given in Table B.2.

#### Table B.2 Coordinates of integration points and weighting factors for Logarithmic Gaussian Quadrature

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \pm \xi_i )</th>
<th>( w_i )</th>
<th>( n )</th>
<th>( \pm \xi_i )</th>
<th>( w_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.11200880</td>
<td>0.71853931</td>
<td>8</td>
<td>0.13320243(-1)</td>
<td>0.16441660</td>
</tr>
<tr>
<td></td>
<td>0.60227691</td>
<td>0.28146068</td>
<td></td>
<td>0.79750427(-1)</td>
<td>0.23752560</td>
</tr>
<tr>
<td>3</td>
<td>0.63890792(-1)</td>
<td>0.51340455</td>
<td></td>
<td>0.19787102</td>
<td>0.22684198</td>
</tr>
<tr>
<td></td>
<td>0.36899706</td>
<td>0.39198004</td>
<td></td>
<td>0.35415398</td>
<td>0.17575408</td>
</tr>
<tr>
<td></td>
<td>0.76688030</td>
<td>0.94615406(-1)</td>
<td></td>
<td>0.52945857</td>
<td>0.11292402</td>
</tr>
<tr>
<td>4</td>
<td>0.41448480(-1)</td>
<td>0.38346406</td>
<td></td>
<td>0.70181452</td>
<td>0.57872212(-1)</td>
</tr>
<tr>
<td></td>
<td>0.24527491</td>
<td>0.38687532</td>
<td></td>
<td>0.84937932</td>
<td>0.20979074(-1)</td>
</tr>
<tr>
<td></td>
<td>0.55616545</td>
<td>0.19043513</td>
<td></td>
<td>0.95332645</td>
<td>0.36864071(-2)</td>
</tr>
<tr>
<td>5</td>
<td>0.84898239</td>
<td>0.39225487(-1)</td>
<td></td>
<td>0.10869338(-1)</td>
<td>0.14006846</td>
</tr>
<tr>
<td>6</td>
<td>0.29134472(-1)</td>
<td>0.29789346</td>
<td></td>
<td>0.64983682(-1)</td>
<td>0.20977224</td>
</tr>
<tr>
<td></td>
<td>0.17397721</td>
<td>0.34977622</td>
<td></td>
<td>0.16222943</td>
<td>0.21142716</td>
</tr>
<tr>
<td></td>
<td>0.41170251</td>
<td>0.23448829</td>
<td></td>
<td>0.29374996</td>
<td>0.17715622</td>
</tr>
<tr>
<td></td>
<td>0.67731417</td>
<td>0.98930460(-1)</td>
<td></td>
<td>0.44663195</td>
<td>0.12779920</td>
</tr>
<tr>
<td></td>
<td>0.89477136</td>
<td>0.18911552(-1)</td>
<td></td>
<td>0.60548172</td>
<td>0.78478879(-1)</td>
</tr>
<tr>
<td>7</td>
<td>0.216344005(-1)</td>
<td>0.23876366</td>
<td></td>
<td>0.75411017</td>
<td>0.39022490(-1)</td>
</tr>
<tr>
<td></td>
<td>0.12958339</td>
<td>0.30828657</td>
<td></td>
<td>0.87726585</td>
<td>0.13867290(-1)</td>
</tr>
<tr>
<td></td>
<td>0.31402045</td>
<td>0.24531742</td>
<td></td>
<td>0.96225056</td>
<td>0.24080402(-2)</td>
</tr>
<tr>
<td></td>
<td>0.53865721</td>
<td>0.14200875</td>
<td></td>
<td>0.90425944(-2)</td>
<td>0.12095474</td>
</tr>
<tr>
<td>8</td>
<td>0.75691533</td>
<td>0.55454622(-1)</td>
<td></td>
<td>0.53971054(-1)</td>
<td>0.18636310</td>
</tr>
<tr>
<td></td>
<td>0.92266884</td>
<td>0.10168958(-1)</td>
<td></td>
<td>0.13531134</td>
<td>0.19566066</td>
</tr>
<tr>
<td>10</td>
<td>0.16719355(-1)</td>
<td>0.19616938</td>
<td></td>
<td>0.24705169</td>
<td>0.17357723</td>
</tr>
<tr>
<td></td>
<td>0.10018568</td>
<td>0.27030264</td>
<td></td>
<td>0.38021171</td>
<td>0.13569597</td>
</tr>
<tr>
<td></td>
<td>0.24629424</td>
<td>0.23968187</td>
<td></td>
<td>0.52379159</td>
<td>0.93647084(-1)</td>
</tr>
<tr>
<td></td>
<td>0.43346349</td>
<td>0.16577577</td>
<td></td>
<td>0.66577472</td>
<td>0.55787938(-1)</td>
</tr>
<tr>
<td></td>
<td>0.63235098</td>
<td>0.88943226(-1)</td>
<td></td>
<td>0.79419019</td>
<td>0.27159893(-1)</td>
</tr>
<tr>
<td></td>
<td>0.81111862</td>
<td>0.33194304(-1)</td>
<td></td>
<td>0.89816102</td>
<td>0.95151992(-2)</td>
</tr>
<tr>
<td>9</td>
<td>0.94084816</td>
<td>0.59327869(-2)</td>
<td></td>
<td>0.96884798</td>
<td>0.16381586(-2)</td>
</tr>
</tbody>
</table>

Note: Numbers are to be multiplied by the power of 10 in parentheses.
REFERENCES