Quasilinear theory revisited: General kinetic formulation of wave-particle interactions in plasmas

K. Hizanidis, a Y. Kominis, a and A.K. Ram

October 2010

Plasma Science and Fusion Center, Massachusetts Institute of Technology
Cambridge, MA 02139  U.S.A.

a National Technical University of Athens
Association EURATOM-Hellenic Republic
Zografou, Athens 15773. Greece

This work was supported by the U.S. Department of Energy, This work is supported by DoE grants DE-FG02-99ER-54521 and DE-FG02-91ER-54109, and by Association EURATOM, Hellenic Republic. Reproduction, translation, publication, use and disposal, in whole or in part, by or for the United States government is permitted.

Submitted to Plasma Physics and Controlled Fusion (2010).
Quasilinear theory revisited: 
General kinetic formulation of wave-particle interactions in plasmas

Kyriakos Hizanidis,¹ Yannis Kominis,¹ and Abhay K. Ram²

¹National Technical University of Athens, Association EURATOM-Hellenic Republic
Zografou, Athens 15773. Greece
²Plasma Science and Fusion Center, Massachusetts Institute of Technology
Cambridge, MA 02139. U.S.A.

In laboratory fusion devices electromagnetic waves in the radio frequency regime are routinely used for heating the plasma and for controlling the current profile. The evolution of particle distribution function in the presence of electromagnetic waves is derived from fundamental equations using the action-angle variables of the dynamical Hamiltonian. Unlike conventional quasilinear theories, the distribution function is evolved concurrently with the particle motion. Since the particle dynamics is time reversal invariant, the master equation for the evolution of the distribution function is also time reversal invariant. When the master equation is sequentially averaged over the angles, there emerges a hierarchy of diffusion equations. The diffusion operator in the equation obtained after averaging over all angles is time dependent, in direct contrast to time independent diffusion operator in quasilinear theories. The evolution of the distribution function with time dependent diffusion operator is markedly different from quasilinear evolution and is illustrated for a spectrum of coherent waves. A proper description of wave-particle interactions is important for fusion plasmas since the velocity space gradients of the distribution function decisively affect collisional relaxation and the associated transport processes.

Introduction

The interaction of coherent electromagnetic waves with charged particles in plasmas is a universal phenomenon. In present day experimental fusion devices radio frequency (RF) waves are routinely used for heating the plasma and for generating plasma currents. In the International Thermonuclear Experimental Reactor (ITER [1]) RF waves, particularly in the electron cyclotron range of frequencies, will be used for controlling instabilities deleterious to the confinement of the plasma. RF waves will also play a prominent role as ITER heads towards a steady state operation. The interaction of RF waves with the plasma constituents leads to transfer of energy and momentum from the waves to the particles. Considering the role of RF waves in fusion plasmas it is imperative to have a proper description of the wave-particle interactions. Besides laboratory plasmas, wave-particle interactions manifest themselves in all sorts of plasma environments, including terrestrial, solar and astrophysical plasmas. The waves in the natural plasmas are usually generated by instabilities. A feature common to RF waves and waves in space is that these waves are coherent and not some sort of statistically random fluctuations. So the wave-particle interaction is between coherent plasma waves and charged particles. In this paper we formulate a kinetic theory for the collective behavior of charged particles interacting with coherent electromagnetic waves. The waves modify the particle distribution functions which, in turn, through Maxwell’s equations, modify the electromagnetic fields.

The usual formalism for wave-particle interactions is the quasilinear theory (QLT), in which the evolution of the distribution function is through a diffusion operator acting in velocity (action) space [2]. In deriving this operator it is assumed that the electromagnetic waves act continuously on the particles randomizing their motion, with respect to the phase of the wave, after one interaction time. The interaction time is a measure of the time it takes a particle to go through one phase cycle of the wave spectrum. For a sinusoidal plane wave the interaction time is essentially the time over which the phase of
the waves changes by $2\pi$. The de-phasing with respect to the phase of the wave is assumed to lead to 
random motion of the particle akin to Brownian motion. This is the Markovian assumption and is 
characterized by completely uncorrelated particle orbits, phase-mixing, loss of memory, and ergodicity. 
These statistical properties lead to an important advantage – the long time behavior of particle dynamics 
is the same as that after one interaction time with the wave.

The Markovian assumption for particle orbits has some significant drawbacks. The corresponding 
diffusion coefficient is singular, with a Dirac delta function singularity [2], and, consequently, not 
amenable to implementation in numerical codes. More importantly, the dynamical behavior of particles 
interacting with coherent waves is not ergodic and does not satisfy the Markovian assumption [3]. The 
dynamical phase space of particles interacting with coherent waves is inhomogeneous with phase space 
islands embedded in a chaotic sea. Furthermore, in all wave-particle interactions, the phase space is 
bounded with the effect of the wave being limited to particles having a resonant interaction with the 
waves [4]. Near the boundaries of the bounded phase space, or near islands, particles can get stuck and 
undergo coherent, correlated motion for times very much longer than the interaction time. Even when the 
amplitude of the waves is assumed to be impractically large, so that the entire phase is chaotic (as in the 
standard map) the quasilinear diffusion operator fails to give an appropriate description of the evolution 
of the distribution function [5]. The persistence of long time correlations invalidates the Markovian 
assumption.

In this paper, we formulate a kinetic description for the evolution of a distribution function of 
particles that is commensurate with the dynamical phase space of particles interacting with coherent 
electromagnetic waves in plasmas. In the absence of electromagnetic fields the motion, in a prescribed 
steady state magnetic field, is assumed to be integrable. The Hamiltonian for this unperturbed motion can 
then be expressed in terms of the action-angle variables corresponding to the constants of motion. In an 
avisymmetric tokamak, the action variables are the magnetic moment, toroidal flux, and the parallel (to 
the magnetic field) angular momentum. The corresponding, canonically conjugate, angle variables are the 
gyro phase, the poloidal angle, and the toroidal angle. The effect of the electromagnetic wave is assumed 
to be perturbative – the magnitude of the wave magnetic field being small compared to the ambient 
magnetic field. We then make use of the Lie perturbation method to advance the distribution function in 
time consistent with the motion of particles in the waves. There is no separation of time scales so that the 
distribution function evolves along with the dynamics of the particles. The resulting master evolution 
equation is time reversible just as the equation of motion of the particles is time reversible. If we average 
the master equation sequentially over the angle variables, averaging first over the fastest varying angle, 
we obtain a hierarchy of diffusion equations. The equation where all the angle variables have been 
averaged out is the diffusion equation that can be directly compared with the result from the usual 
quasilinear analysis. In contrast to the quasilinear diffusion operator, the diffusion operator in our kinetic 
equation is time dependent. In the limit when time approaches infinity, the time dependent diffusion 
operator tends to the quasilinear operator. The evolution of the distribution function obtained from our 
kinetic theory is different from that in the quasilinear case. This is demonstrated by considering diffusion 
in a continuous spectrum of waves.

**Detailed formalism**

Let us consider a general form of the perturbed Hamiltonian system:

$$H(J,\theta,t) = H_0(J) + \varepsilon H_1(J,\theta,t)$$

(1)
where $H_0(\mathbf{J})$ corresponds to an integrable Hamiltonian depending only on the actions $\mathbf{J}$,

$$H_1(\mathbf{J},0,t) = \sum_{m=0} A_m(\mathbf{J}) e^{i(\omega_m t - \delta_m)} + \text{cc}$$

(2)

is a perturbation that makes the full Hamiltonian non-integrable, and $\varepsilon$ is an ordering parameter indicating the perturbative effect of $H_1$. The perturbation $H_1$ is expressed as a Fourier series in the periodic angle variables $\theta$, with amplitudes $A_m$, real frequencies $\omega_m$, and a growth or damping rate of the wave given by the magnitude and sign of $\gamma_m$. The action-angle variables $\mathbf{X}=(\mathbf{J},\theta)$ are obtained from $H_0$ and the time evolution of $\mathbf{X}$ from an initial time $t_0$ to a later time $t$ is governed by Hamilton's equations of motion. The time evolution of any function $f(\mathbf{X},t)$ from $t_0$ to $t$ is given by:

$$f(\mathbf{X},t) = S_H(t \rightarrow t) \circ f(\mathbf{X}_0,t_0)$$

(3)

where $\mathbf{X}_0 = \mathbf{X}(t_0)$ are the initial conditions, and $S_H(t \rightarrow t)$ is the time evolution operator. The evaluation of $S_H(t \rightarrow t)$, which is equivalent to solving the equations of motion, may not be possible for the original choice of variables. Then we make use of the Lie transform theory to map the phase space in $\mathbf{X}$ onto a phase space spanned by a new set of variables $\mathbf{Y}$. The canonical transformation $T(\mathbf{X},t)$ for this mapping is such that $\mathbf{Y} = T(\mathbf{X},t) \cdot \mathbf{X}$, where $T(\mathbf{X},t) = \exp[-L(\mathbf{X},t)]$ with $L(\mathbf{X},t)$ being the Lie operator. $L(\mathbf{X},t)$ is obtained from the generating function $w(\mathbf{X},t)$ such that $L \cdot f = [w,f]_{PB}$, where $[\cdot , \cdot]_{PB}$ denotes the Poisson bracket in $\mathbf{X}$ phase space. The transformation is chosen in such a way that the new Hamiltonian $K(\mathbf{Y},t)$ with the corresponding time evolution operator $S_K(t \rightarrow t)$ is easier to evaluate. An important and basic property of the Lie transform operator is that it generates canonical transformations and that it commutes with any function of the phase space variables. The latter property implies that the evolution of $f(\mathbf{X}_0,t_0)$ can be obtained by transforming to the new variable set $\mathbf{Y}_0$, applying the time evolution operator $S_K(t \rightarrow t)$ to the transformed function, and then transforming back to the original variables $\mathbf{X}$ [6],

$$f(\mathbf{X},t) = T(\mathbf{X}_0,t_0) \circ S_K(t \rightarrow t) \circ T^{-1}(\mathbf{X}_0,t_0) \circ f(\mathbf{X}_0,t_0)$$

(4)

For a nearly integrable Hamiltonian system represented by Eq. (1), with $\varepsilon << 1$, a perturbation scheme can be developed in which $T$ is expressed as a power series in $\varepsilon$ [7]. According to this scheme, the old Hamiltonian $H$, the new Hamiltonian $K$, the transformation operator $T$, and the Lie generator $w$ are expanded in power series of $\varepsilon$. We can set $w_0=0$, such that $T_0$ and $T_0^{-1}$ are both the identity operator $I$. Up to second order in $\varepsilon$ we obtain,

$$T = I - \varepsilon L_1 + \frac{\varepsilon^2}{2}(L_1^2 - L_2), \quad T^{-1} = I + \varepsilon L_1 + \frac{\varepsilon^2}{2}(L_1^2 + L_2)$$

(5)

Then the transformed zero order Hamiltonian is $K_0 = H_0$, while the Lie generators, up to second order in $\varepsilon$ are given by

$$\frac{dw_1}{dt} = K_1 - H_1, \quad \frac{dw_2}{dt} = 2(K_2 - H_2) - L_1 \cdot (K_1 + H_1)$$

(6)

The time derivative is along the unperturbed orbit given by $H_0$,

$$\frac{d}{dt} = \partial_t + [\cdot , H_0]_{PB}$$

(6a)

with $\partial_t$ denotes the partial derivative. The solutions to Eqs. (6) and (6a) are obtained by integrating the right hand side along known unperturbed orbits. The $K_n$'s $(n \geq 1)$ are conveniently chosen so that only the
slowly varying terms appear. We choose $K_n$'s so as to eliminate the dependence on $\theta$ up to the second order in $\varepsilon$. Thus, $K_n=0$ for $n=1,2$. We can then calculate the evolution of particles that is accurate up to $\varepsilon^2$. The first order generator $w_1$ is readily obtained from Eqs.(2) and (6) by integration from $t_0$ to $t$:

$$w_1(J, t) = i \sum_{m>0} \frac{A_m(J) e^{im\theta} + A_m(J) e^{-im\theta}}{\Omega_m(J) - i\gamma_m} + cc$$  \hfill (7)

where $\omega_0(J) = \frac{\partial H_0}{\partial J}$ is the frequency vector of the unperturbed system, and $\Omega_m(J) = m \cdot \omega_0(J) - \omega_m$. If the dynamical system is periodic with respect to all angles $\theta$, $w_2=0$ and $L_2=0$.

The evolution of $f(X, t)$ over a small time interval $\Delta t = t - t_0$ can be obtained by the same perturbation scheme. Then, to second order in $\varepsilon$,

$$f(X, t) - f(X_0, t_0) = \left[ T^{-1}(X_0, t_0 + \Delta t) - I \right] \ast f(X_0, t_0) + O(\varepsilon^n; n > 2)$$  \hfill (8)

where

$$T^{-1}(X_0, t_0 + \Delta t) = S_K(t_0 \rightarrow t_0 + \Delta t) \ast T^{-1}(X_0, t_0) + O(\varepsilon^n; n > 2)$$  \hfill (9)

Given that $\Delta t \ll 1$, and using Eq. (50), a Taylor series expansion of the left hand side of Eq.(9) results in

$$T^{-1}(X_0, t_0 + \Delta t) = I + \Delta t \frac{\partial T^{-1}(X, t)}{\partial t} = \Delta t \frac{\partial}{\partial t} \left( \varepsilon L_1 + \varepsilon^2 L_2^2 / 2 \right)$$  \hfill (10)

From Eq. (8),

$$f(X, t) - f(X_0, t_0) = \Delta t \frac{\partial}{\partial t} \left( \varepsilon L_1 + \varepsilon^2 L_2^2 / 2 \right) \ast f(X_0, t_0) + O(\varepsilon^n; n > 2)$$  \hfill (11)

In the limit $\Delta t \rightarrow 0$ this yields

$$\frac{df}{dt} = \varepsilon \left[ \frac{\partial w_1(f)}{\partial f} \right]_{PB} + \frac{\varepsilon^2}{2} \left[ \left[ \frac{\partial}{\partial t}, w_1(f) \right]_{PB} + \left[ w_1(f), \frac{\partial}{\partial t} \right]_{PB} \right] + O(\varepsilon^n; n > 2)$$  \hfill (12)

where $w_1 = w_1(J, \theta, t)$ is given by Eq.(7). It is important to note that in this scheme the new Hamiltonian is independent of $\theta$ - the $\theta$ dependence having been transformed away to terms which are of higher order than $\varepsilon^2$.

Let $f(X, t)$ be the particle distribution function and let us define angle-averaged distribution functions $F_i(J, \theta_{m \neq i}, \theta_{n \neq i}, t)$, $F_{lm}(J, \theta_{m \neq l}, \theta_m, t)$ and $F_{lmm}(J, t)$ with $\{i, m, n\} = \{1, 2, 3\}$, such that

$$F_i(J, \theta_{m \neq i}, \theta_{n \neq i}, t) \equiv \left\langle f(J, J) \right\rangle_{i} = \frac{1}{2\pi} \oint d\theta_i f(J, \theta_i, t)$$  \hfill (13a)

$$F_{lm}(J, \theta_{m \neq l}, \theta_m, t) \equiv \left\langle f(J, J, t) \right\rangle_{lm} = \frac{1}{(2\pi)^2} \oint d\theta_l \oint d\theta_m f(J, \theta_l, \theta_m, t)$$  \hfill (13b)

$$F_{lmm}(J, t) \equiv \left\langle f(J, J, t) \right\rangle_{lmm} = \frac{1}{(2\pi)^2} \oint d\theta_l \oint d\theta_m \oint d\theta_3 f(J, \theta_l, \theta_m, \theta_3, t)$$  \hfill (13c)

If the system is periodic with respect to all angles $\theta$, than an averaging Eq. (12) over one or more angles yields:
\[
\partial_t F + [F, H_0]_{PB} = \nabla_Q \cdot (\tilde{D} \cdot \nabla_Q F)
\]

(14)

where \( F \) is any one of the three defined in Eq.(13), \( Q \) represents the respective reduced phase spaces \((J, \theta_{m\neq l}), (J, \theta_{n\neq m})\) and \((J)\), and the diffusion tensor \( D \) is given by

\[
\tilde{D}(Q,t) = \frac{\varepsilon^2}{2} \partial_{\theta} \left[ \begin{array}{c}
\langle \nabla_\theta w_i \nabla_\theta w_i \rangle - \langle \nabla_\theta w_i \nabla_j w_i \rangle \\
-\langle \nabla_\theta w_i \nabla_j w_i \rangle + \langle \nabla_\theta w_i \nabla_j w_i \rangle
\end{array} \right]
\]

(15)

Here \( \langle \rangle \) denotes averaging over the angles complementary to \( Q \). The Poisson bracket term in the left hand side of Eq.(14) exists as long as the distribution \( F \) is a function of one or more angles. Equations (12) and (14) form a hierarchy of four Fokker-Planck types of equations whose dimensionality depends upon the number of angles retained in the appropriate description. When all the angles are retained, we obtain the evolution of \( F \) in the complete 6-dimensional phase space. As each angle is averaged out, we reduce the phase space by projecting \( F \) onto lower dimension. When all the angles are averaged out the evolution equation is for \( F \) projected onto a 3-dimensional phase space corresponding to space spanned by only action variables. When wave-particle interactions do not exist, the right hand side of Eq.(14) is zero and we obtain the Vlasov equation for the unperturbed particle motion in reduced phase space.

Corresponding to the canonical transformation, \( X \rightarrow X' \), there exists a transformation between the respective subspace, \( Q \rightarrow Q' \). Let \( M_{Q \rightarrow Q'} \) be the corresponding Jacobian matrix of the subspace transformation and \( |M_{Q \rightarrow Q'}| \) be its determinant. Even though the transformation is canonical, the sub-matrix of the transformation is not necessarily unitary. Then, in the new subspace \( Q' \), Eq.(14) becomes:

\[
\partial_t F' + [F', H_0']_{PB} = -F' \left[ \ln |M_{Q \rightarrow Q'}| , H_0' \right]_{PB} + [M_{Q \rightarrow Q'}]^{-1} \nabla_{Q'} \cdot \left[ (\tilde{D} \cdot \tilde{M}_{Q \rightarrow Q'}) \cdot \nabla_{Q'} \right] F'
\]

(16)

The first term in the right hand side can be interpreted as a "non-inertial" term which is due to the evolution of the transformation itself. This term vanishes if \( Q=J \), that is, when all the angular dependence has been averaged out. In this case \( |M_{Q \rightarrow Q'}| \) depends only on \( J \).

**Comparison with the quasilinear theory**

Consider the evolution equation of the angle-averaged distribution function in action space. In Eq.(14) \( Q=J \), and the diffusion tensor in Eq.(15) takes on the following dyadic form

\[
\tilde{D}(J,t) = \sum_{m \neq 0} \frac{\varepsilon A_m(J)}{\Omega_m^2(J)} e^{\gamma_m t} \left\{ \gamma_m \left[ 1 - e^{-\gamma_m t} \cos \left( \Omega_m(J) t \right) \right] + e^{-\gamma_m t} \Omega_m(J) \sin \left( \Omega_m(J) t \right) \right\}
\]

(17)

In contrast with the traditional QLT [2], the action-space Fokker-Planck equation given in Eqs.(14), with the diffusion tensor in Eq. (17), possesses a time-dependent diffusion tensor that treats resonant and non-resonant interactions on equal footing. The Fokker-Planck equation evolves on the same time scale as the averaged action-space distribution function does. In contrast to QLT, our formalism is the same for growing \( (\gamma_m>0) \) and damped \( (\gamma_m<0) \) waves. Even for \( \gamma_m=0 \), \( D \) is continuous and non-singular in the vicinity of resonances \( (\Omega_m=0) \). In this case, the width of a resonance decreases with time. Although \( D \) resembles the "running diffusion tensor" introduced by Balescu [8], it is fundamentally different. The \( D \) given above depends on the action variables and it fully incorporates the inhomogeneous resonant
structure of the phase space. The running diffusion tensor by Balescu is for a chaotic, Markovian-type, phase space.

For very long times $t \to \infty$, and with $\gamma_m = 0$, we readily obtains the quasilinear result

$$\lim_{t \to \infty} \mathbf{D}(J; t) = \sum_{m=0}^{\infty} \mathbf{m} m A_m(J) \delta[\Omega_m(J)]$$

(18)

where $\delta$ is the Dirac's delta function. However, this limit is questionable since it assumes statistically random, or Markovian, processes [3]. The limit of extending time to infinity in the canonical perturbation theory, such as in the evaluation of $w_1$, can only be justified for statistical process of the Markovian type in which there is phase mixing and rapid de-correlation of the particle orbits [5]. The loss of memory in the Fokker-Planck equation with the time and action space dependent inhomogeneous diffusion tensor, of Eq.(17) is not of a statistical nature. It is rather a loss of memory of the initial phases (angles) over which we have averaged. The singular delta function in QLT completely ignores the short time effects that are important in the evolution the distribution function. While the diffusion operator is evolving so is the distribution function. The particles that determine the diffusion operator are part of the distribution function. Additionally, the delta function requires ad-hoc smoothing remedies in numerical implementations.

The consequences of our kinetic formalism for the distribution function expressed as an inhomogeneous, time dependent, Fokker-Planck equation are illustrated for the case of a one-dimensional unperturbed particle Hamiltonian $H_0(J) = J^2/2$. The averaged distribution function in action space is initially taken to be a Maxwellian distribution $F(J, t=0) = \exp(-J^2/2)/(2\pi)^{1/2}$. We assume that the particle motion is perturbed by a set of three discrete modes. The evolution of $F(J, t)$ is obtained from Eqs. (15) and (16) and shown in Fig. 1. In Fig. 1a the discrete modes are assumed to be in steady state while in Fig. 1b they are assumed to be damped. In the case of the damped modes, $F(J, t)$ approaches a steady state. For discrete modes, the usual QLT is not applicable since smoothing of the delta function in the diffusion coefficient in Eq.(17) cannot be justified. The particle motion is not chaotic for discrete waves with arbitrarily small amplitudes.

In Fig. 2 we plot $D(J, t)$, from Eq.(17), as a function of $J$ and $t$. It is evident that as $t \to \infty$, $D(J; t)$ approaches, as expected, the quasilinear form. The difference between this model and the usual QLT becomes clear when we look at the evolution of the angle averaged $F(J, t)$. In Fig. 3a we plot the evolution of $F(J, t)$ from its initial Maxwellian state for $D(J, t)$ given in Eq.(17). The corresponding evolution for a time-independent quasilinear diffusion coefficient is plotted in Fig. 3b. The time-dependent $D$ of Eq.(17) leads to early time effects that persist for all times. These effects are not at all present in the QLT result. Consequently, the long time behavior of the two distribution functions differs significantly.
Fig. 2: (a) $D(J,t)$, from Eq.(17), as a function of $J$ and $t$ for a continuous spectrum of waves, $A_m = 3 \times 10^{-3} \exp \left[ -\frac{(m-1)^2}{0.18} \right]$, $\omega_m = 1$, and $\gamma_m = 0$. (b) $D(J,t)$ as a function of $J$ for $t = 3\pi/4$ (blue), $3\pi/2$ (green), $3\pi$ (red), and $\infty$ (black dashed). The black dashed curve corresponds to the quasilinear diffusion coefficient.

The implication of this difference is very important. If we take the limit $t \to \infty$ of Eq.(14), then this limit cannot be commuted through the derivative on the right hand side. Otherwise, the long time behavior in Fig. 3a would have been the same as in Fig. 3b. Thus, the usual QLT is incapable of accounting for diffusive effects at early times which affect the long time behavior of the particle distribution function.

Fig. 3: Evolution of $F(J)$ for the case of continuous spectrum shown in Fig. 2. (a) shows the evolution as obtained from (14) using (17); (b) shows the evolution According to the quasilinear diffusion coefficient (black dashed curve in Fig. 2b).

**Conclusion**

In magnetic fusion devices, electromagnetic (radio frequency, RF) waves are used for heating and for generating plasma currents in magnetic confinement devices. The plasma waves are collective particle oscillations which, in turn, interact with the plasma particles. From first principles, we have derived a kinetic theory for the evolution of the particle distribution function when interacting with electromagnetic waves inside plasmas. The interaction of particles with coherent waves is non-Markovian so that the complete structure of the dynamical phase space is included in our formalism. The kinetic theory leads to a time dependent diffusion operator which evolves on the same time scale as the particle orbits. It is, thus, markedly different from the diffusion operator which one gets in quasilinear theories based on the Markovian assumption. The statistical assumptions in quasilinear theories are in contradiction to the particle dynamics in coherent electromagnetic waves as obtained from the equations of motion. Consequently, based on our kinetic description, the evolution of a distribution function of particles interacting with radio frequency plasma waves leads to considerably different results when compared to the conventional quasilinear theories. This is not only relevant to present day experiments but also to the International Thermonuclear Experimental Reactor [1] in which electron cyclotron waves will be used for local current generation and control of plasma instabilities. An additional complexity is related to the fact that, in practice, particles do not continuously interact with the same spectrum of waves. In tokamaks, the waves fields used for heating and current drive, for example, the electron cyclotron waves, are spatially confined inside the plasma. Any given particle, during its toroidal excursion, will interact with the fields over a short fraction of its single transit path length. On its next transit, it will most likely have drifted,
due to the inhomogeneity of the magnetic field, away from the location where the previous interaction took place. Thus, the interaction of particles with electromagnetic waves encompasses interesting and complex physics, which in most of the realistic cases are not within the domain of validity of the commonly used statistical assumptions in quasilinear theories. The kinetic formalism developed in this paper accounts for this complex phase space structure.

In the high temperature fusion plasma the time scale for particle collisions is much longer than any time scale for wave-particle interactions between radio frequency waves and the charged particles. Thus, we can justify the neglect of collisions in the kinetic description of wave-particle interaction. However, when we consider the steady state attained on collisional time scales we do need to have a proper description of the particle distribution function, since the velocity space gradients of the distribution function decisively affect the collisional relaxation and the associated transport processes. This could also be the case in the cooler plasma edge, where coherent electromagnetic waves are generated by instabilities. Then, in the left hand side of Eq.(14) an angle-averaged collisional term would have to be added [9]. A perturbation scheme with collisional relaxation in the presence of coherent electromagnetic waves could then be formulated. This is a topic for future research.

In the guiding center approximation for an axisymmetric tokamak, the three actions correspond to the magnetic moment, \( \mu \), the canonical toroidal momentum, \( p_\phi \) and the poloidal canonical momentum, \( J_p \). These three actions can always be defined even when the guiding center approximation is not valid, for example, when frequencies and wave numbers are comparable to the gyro-frequencies and gyro radii, respectively. The conjugate angles correspond to the gyrophase \( \Theta_g \), the toroidal angle \( \Phi \), and the poloidal angle \( \Theta \). Of the three actions, two are velocity-like, namely the magnetic moment and the toroidal momentum. The poloidal canonical momentum is mainly space-like and it is associated with the flux surface near which the motion either the circulating or banana trapped particles is confined. The canonical toroidal and poloidal momenta can be evaluated from the toroidal flux function \( \psi \) and the poloidal flux function \( \psi_p \). In magnetic field coordinates, the toroidal flux is commonly used to express, in covariant form, the equilibrium magnetic field [10],

\[
B = G(\psi) \nabla \Phi + I(\psi) \nabla \Theta + A(\psi, \Theta) \nabla \psi. 
\]

Here \( G(\psi) \) and \( I(\psi) \) are related to the poloidal and the toroidal currents, respectively, while \( A(\psi, \Theta) \) is related to the non-orthogonality of the coordinate system. The poloidal flux is determined from the safety factor \( q(\psi_p) = d\psi/d\psi_p \). Averaging over the gyrophase leads to an equation of the type given in Eq.(14), which determines the evolution of a distribution function for time scales longer than the gyration period. Such a description is suitable for waves or perturbing magnetic fields with frequencies small compared to the gyro-frequencies. The description includes the dynamics of both circulating and banana-trapped particles. This property is lost when we average over the toroidal angle, which is similar to bounce averaging. This averaging provides separate equations for trapped and passing particles. These equations incorporate the spatial dynamics in the poloidal plane and the velocity-like dynamics. If collisions are important at this stage, collisional trapping and de-trapping will couple these equations together. Averaging over the poloidal angle, which is the same as flux surface averaging, lead to a kinetic equation which captures the radial dynamics, across flux surfaces while, at the same time, preserving the velocity space dynamics. This kinetic equation includes the long time behavior of the local distribution functions when radio frequency waves are used for heating and current drive, and for stabilizing the neoclassical tearing mode. By including collisions we can obtain an action space kinetic equation which can be used for evaluating
the transport coefficients in the presence of wave-particle interactions due to coherent electromagnetic waves.

Acknowledgements

This work has been supported by the European Fusion Program (Association EURATOM-Hellenic Republic), the Hellenic General Secretariat of Research and Technology, and by US DOE Grant number DE-FG02-91ER-54109.

References