MODE COUPLING AND ANOMALOUS DISSIPATION
IN MHD TURBULENCE

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ABSTRACT

An energy conserving model of MHD turbulence is described that predicts both mode coupling and turbulent dissipation. The dissipation takes the form of anomalous resistivity and viscosity due to turbulent magnetic fields. The model predicts a dual cascade of energy to large wave numbers and magnetic flux to small wave numbers. Turbulent rearrangement of equilibrium magnetic shear generates resonant fluctuations via a mixing length process. The effect of tearing mode turbulence on the disruptive instability in tokamaks is discussed.
A. INTRODUCTION

Recently there has been much interest in the importance of turbulent magnetic field fluctuations to fusion plasmas. Theoretical investigations have focused on magnetic field line stochasticity in space (1-3) as well as its effects on finite-$\beta$ drift wave (4), resistive interchange (5), and disruptive instabilities (6) in laboratory plasmas. It has been suggested that turbulent magnetic field line reconnection may be important to such varied phenomena as solar flares (7), the earth's magneto-tail (8), and the disruptive instability in tokamaks (9,10).

Computer studies have been particularly useful in revealing the dynamics of turbulent MHD phenomena. The dual cascade of energy to large wave-numbers and vector potential (magnetic flux) to small wave numbers has been observed for the two dimensional case (11). This cascade has also been observed in three dimensional, magnetic shear-driven tearing mode turbulence (10). In this latter case, the mode coupling proceeds after the overlap of two low mode number magnetic islands. The production of many modes leads to a "fully developed" turbulent state characterized by anomalous dissipation (9,10). It seems plausible that this anomalous dissipation is responsible for the nonlinear (exponential) growth of the low mode number tearing modes observed during this turbulent phase (9,10).

We report here on an investigation of the turbulence described by the resistive MHD equations. Our purpose is to apply some established concepts of Vlasov turbulence...
to the problem of turbulent MHD. Because of the mathematical complexity of the equations, we have emphasized the physics that we believe to be essential to any final theory. Our main result is a self-consistent, energy conserving model that predicts both mode coupling and turbulent dissipation. The dissipation takes the form of anomalous resistivity and viscosity due to the turbulent magnetic fields. Large wave number energy fluctuations are produced via a mixing length process from the small wavenumber modes. This is a resonant mode coupling process that is similar to the production of clumps in Vlasov turbulence.\(^{(12)}\)

The paper is organized as follows. Section B deals with the energy conservation and symmetry properties of the MHD equations. It is shown in Section C how the two point correlation function is a useful quantity to describe the conservation and symmetry properties of two dimensional MHD turbulence. In Section D we consider the case of three dimensional turbulence driven by magnetic shear. The mixing length model is discussed in Section E. The effects of turbulent resistivity and viscosity on tearing mode growth are discussed in Section F. In Section G, the diffusion equations describing the correlation functions are derived.

B. Conservation Laws

The resistive MHD equations provide a fluid description of the self-consistent magnetic and velocity fields of a tokamak plasma.\(^{(13)}\) Self-consistency is achieved by combining Ampere's Law with momentum balance, and using Faraday's Law in conjunction with Ohm's law to obtain
\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \frac{1}{4\pi} \mathbf{B} \cdot \nabla \mathbf{v} - \mathbf{v} \times (\mathbf{B} B_{\perp} + \nabla (p + B_{\perp}^2 / 8\pi)) \tag{1a}
\]

\[
\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) B_{\perp} = \mathbf{B} \cdot \mathbf{v} + \frac{c^2}{4\pi} \eta \nabla^2 B_{\perp} \tag{1b}
\]

Here, \( \mathbf{v} \) and \( B_{\perp} \) are components of fluid flow velocity and magnetic field perpendicular to the \( z \) direction. The total magnetic field is \( \mathbf{B} = B_z \hat{z} + B_{\perp} \), where \( B_z \) is assumed constant. Other quantities in Eq. (1) are resistivity \( \eta \), scalar pressure \( p \), mass density \( \rho \), and the speed of light \( c \). Below we will find it useful to use normalized variables, where resistivity is normalized to an average value \( \bar{\eta} \), lengths are in units of minor radius \( a \), time is in units of the resistive time \( \tau_R = 4\pi a^2 / c^2 \bar{\eta} \), and magnetic field is in units of \( B_z \). Then Eqn. (1) becomes

\[
\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = S \mathbf{B} \cdot \nabla \mathbf{v} - \mathbf{v} \times (\mathbf{B} B_{\perp} + \nabla p^*) \tag{2a}
\]

\[
\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) B_{\perp} = \mathbf{B} \cdot \mathbf{v} + \eta \nabla^2 B_{\perp} \tag{2b}
\]

where \( S = \tau_R / \tau_H \) and \( \tau_H = a (B_z^2 / 4\pi \rho)^{-1/2} \) is the Alfvén time.

In Eqn. (2a), \( p^* \) is the normalized value of the total pressure \( p + B_{\perp}^2 / 8\pi \). Equation (2b) satisfies \( \mathbf{v} \cdot B_{\perp} = 0 \) automatically. We impose \( \mathbf{v} \cdot \mathbf{v} = 0 \) on Eq. (2a) so that \( p^* \) is a known functional of \( B_{\perp} \) and \( \mathbf{v} \).
Equation (2) embodies several conservation properties of physical significance. Defining the poloidal flux, \( \psi \), through \( B_\perp = \nabla \psi \times \hat{z} \) and the velocity stream function, \( \phi \), through \( \mathbf{V} = \nabla \phi \times \hat{z} \), Eqn. (2b) can be written as:

\[
\frac{\partial \psi}{\partial t} + \mathbf{V} \cdot \nabla \psi = \eta \nabla^2 \psi - \frac{\partial \phi}{\partial z} \tag{3}
\]

The nonlinear term in Eq. (3) generates the \( \mathbf{B} \cdot \mathbf{V} \) and \( \mathbf{V} \cdot \mathbf{V} \) terms in Eq. (2b). Since the \( \mathbf{V} \cdot \mathbf{V} \) term in Eqn. (3) conserves poloidal flux, it is a mode coupling effect on \( \psi \). However, this nonlinear term is both mode coupling and dissipation in Eq. (3). To see this, we derive the energy conservation theorem from Eqn. (2) and find:

\[
\frac{\partial}{\partial t} [\langle B_\perp^2 \rangle + S^{-2} \langle \mathbf{V}^2 \rangle] = -\eta \langle J^2 \rangle \tag{4}
\]

where \( \langle \rangle \) denotes an ensemble average and \( \mathbf{J} = \nabla \times \mathbf{B} \) is the current density. In the derivation of Eqn. (4.), the \( \mathbf{V} \cdot \mathbf{V} \) terms vanish separately (i.e., a mode coupling effect), whereas the \( \mathbf{B} \cdot \mathbf{V} \) terms cancel between Eqn. (2a) and Eqn. (2b). This latter cancellation is due to conversion of magnetic (velocity flow) energy into velocity flow (magnetic) energy (i.e., a dissipation effect).

Because of the symmetry of the self-consistent Eqn. (2), the cross helicity \( \langle \mathbf{B} \cdot \mathbf{V} \rangle \) is also conserved.
\[ \frac{\partial}{\partial t} \langle B \cdot V \rangle = -\eta \langle UJ \rangle \] (5)

where \( U_x = \nabla \times V \) is the vorticity. In Eqn. (5) the \( B \cdot V \) terms vanish separately (i.e., a mode coupling effect) whereas the \( V \cdot V \) terms cancel between Eqn. (2a) and Eqn. (2b) (i.e., a dissipation effect).

An interesting form of Eqn. (2) that manifestly shows the symmetry that results from self-consistency is

\[ \left\{ \frac{\partial}{\partial t} + (SB_\perp + V) \cdot V \right\} (V - SB_\perp) = -\eta SV^2 B_\perp - \frac{\mathbf{V} \cdot \mathbf{P}^*}{\perp} \]

This form eliminates the dissipation effect explicitly and thus conserves \( \langle (V - SB_\perp)^2 \rangle \) by mode coupling.

A simple example of these conservation and symmetry properties of the self-consistent case is that of Alfvén waves. When \( \mathbf{V} \cdot \mathbf{P}^* = 0 \), and \( \eta \) is neglected, the linearized form of Eqn. (2) gives perturbations \( \delta V \) and \( \delta B_\perp \) that satisfy

\[ \frac{\partial^2}{\partial t^2} (\delta V, \delta B_\perp) = B_z^2 S^2 \frac{\partial^2}{\partial z^2} (\delta V, \delta B_\perp) \]

(7)

where the Alfvén speed is \( V_A = S = \frac{V_A}{\tau_R} \). The symmetrical behavior
of $\delta B_\perp$ and $\delta V$ give the equipartition property of Alfven waves, i.e., $\delta B_\perp^2 = S^{-2} \delta V^2$. The "plucked string" analogy for Alfven waves follows from the balance between the $B'V B_\perp$ restoring force of the magnetic field lines in Eqn. (2a) and the $B'VV$ frozen-in property of the velocity flow in Eqn. (2b). This balance converts magnetic and flow energy into each other to satisfy energy conservation and produce equipartition.

In this paper we are interested in the nonlinear effects of Eqn. (2) due to a turbulent spectrum of self-consistent $\delta B_\perp$ and $\delta V$ fluctuations. Any model that approximates the nonlinear terms in Eqn. (2) must retain the symmetry and conservation properties of these terms. In particular the two different effects of mode coupling and dissipation must be distinguished. We show below that this distinction can be achieved by considering the correlation functions of $B_\perp$ and $V$. The time evolution of the correlation function is a physically appealing way to describe the cascade of fluctuation energy to different spacial scales during the mode coupling process. Having treated the mode coupling in this way, the symmetry and conservation properties of Eqn. (2) are satisfied in a natural way.

C. Two Dimensional Turbulence

The equations for the correlation functions are derived in Section G. Here and in the next sections we preview some of their consequences. First we consider the case of two dimensional,
homogeneous turbulence where the correlation functions depend only on \( \mathbf{r}_- = \mathbf{r}_1 - \mathbf{r}_2 \). For simplicity we treat the case where the cross helicity is zero. Retention of the \( \langle \mathbf{V} \cdot \mathbf{B}_\perp \rangle \) cross terms is straightforward and leads to equations that preserve the helicity. We also suppress \( \eta \) and \( p^* \) for brevity and focus our attention on the nonlinear convection terms. Also we assume for simplicity that \( \langle \mathbf{B}_\perp \rangle = 0 \). Then, denoting \( \langle \mathbf{B}_\perp (\mathbf{r}_1,t) \cdot \mathbf{B}_\perp (\mathbf{r}_2,t) \rangle \) and \( \langle \mathbf{V}(\mathbf{r}_1,t) \cdot \mathbf{V}(\mathbf{r}_2,t) \rangle \) as \( \langle \mathbf{B}_1 \cdot \mathbf{B}_2 \rangle \) and \( \langle \mathbf{V}_1 \cdot \mathbf{V}_2 \rangle \) respectively, we find that

\[
\frac{3}{\partial t} - \mathbf{V}_- \cdot \mathbf{D} - \mathbf{V}_- - 2S^2 \mathbf{V}_- \cdot \mathbf{D} \cdot \mathbf{V}_- \langle \mathbf{B}_1 \cdot \mathbf{B}_2 \rangle
\]

\[
= -2\mathbf{V}_- \cdot \mathbf{D}_{12} \cdot \mathbf{V}_- \langle \mathbf{V}_1 \cdot \mathbf{V}_2 \rangle
\]

(8)

\[
\frac{3}{\partial t} - \mathbf{V}_- \cdot \mathbf{D} - \mathbf{V}_- - 2S^2 \mathbf{V}_- \cdot \mathbf{D} \cdot \mathbf{V}_- \langle \mathbf{V}_1 \cdot \mathbf{V}_2 \rangle
\]

\[
= -2S^4 \mathbf{V}_- \cdot \mathbf{D}_{12} \cdot \mathbf{V}_- \langle \mathbf{B}_1 \cdot \mathbf{B}_2 \rangle
\]

(9)

where \( \mathbf{V}_- \) denotes a derivative with respect to \( \mathbf{r}_- \), i.e. \( \partial / \partial \mathbf{r}_- \).

Here, \( \mathbf{D}_\perp (\mathbf{r}_-) = 2 \left[ \mathbf{D}_\perp - \mathbf{D}_\perp (\mathbf{r}_-) \right] \) describes the relative diffusion
of two fluid elements. \( D_{12}^V \) describes the correlated diffusion of two fluid elements and has the property that

\[
D_{12}^V (r \to 0) = D^V \quad \text{as} \quad r \to 0 \tag{10}
\]

This occurs because two fluid elements that are close together experience the same forces and, therefore, diffuse together \((D^V \to 0)\). Conversely, if \( L \) is the size of the largest spatial scale, then

\[
D_{12}^V (r \to L) = 0 \quad \text{as} \quad r \to L \tag{11}
\]

so that the two fluid elements diffuse independently \((D^V \to 2D^V)\). Similar statements can be made for the magnetic field line diffusion coefficients \( D_{12}^B \) and \( D^B \).

The separate vanishing of the \( V \cdot V \) terms in Eqn. (4) for the mode coupling of energy is represented in the correlation function equations by \( D_{12}^V \). Property (10) ensures that the \( V \cdot V \) nonlinear terms conserve energy by mode coupling in Eqn. (8) and Eqn. (9). The dissipation of energy by the \( B \cdot V \) terms in Eqn. (2) is represented in the correlation function equations by the \( D^B \) and \( D_{12}^B \) terms. In Eqn. (8) the \( D^B \) term is an anomalous resistivity driven by the self-consistent, stochastic magnetic fields. It is an anomalous viscosity in Eqn. (9).
Along with the $D_{12}^B$ terms, the $D^B$ terms describe the dissipation (conversion) of magnetic and fluid flow energy into each other. Adding Eqn. (8) and (9) and noting that $D^B = 2[D^B - D_{12}^B]$ we find that

$$\left\{ \frac{\partial}{\partial t} - \nabla \cdot [D^V_\perp + S^2D^B_\perp \cdot \nabla] \right\} E = 0$$

(12)

where $E(r_\perp, t) = \langle B_1 \cdot B_2 \rangle + S^{-2}\langle V_1 \cdot V_2 \rangle$ is the total energy correlation function. Property (10) ensures that the total energy is conserved in Eqn. (12). In forming Eqn. (12) we see that the dissipation of magnetic energy by $D^B$ in Eqn. (8) is a source for fluid flow energy through $D_{12}^B$ in Eqn. (9); and the dissipation of fluid flow energy by $D^B$ in Eqn. (9) is a source of magnetic energy through $D_{12}^B$ in Eqn. (8). This is just the energy dynamics of Eqn. (2).

Property (11) implies that the predominant effect of the nonlinear terms on the large spatial scales is dissipation. However, for small scales, $D_{12}^B \approx D^B$, so that there are sources as well as sinks of energy. The nonlinear terms on the right hand side of Eqn. (8) and (9) are incoherent sources of fluctuation energy. Viewed as negative diffusion operators in $r_\perp$, they pile energy up at small $r_\perp$. These considerations imply that the energy flows from large to small scales. This property can be
seen in a more quantitative way by considering the mode coupling process (14), described by Eqn. (12). Property (10) ensures that the total energy \( E(0,t) \) is constant in time. However, for \( \bar{r} > 0 \) there will be a diffusion in \( E(\bar{r},t) \) to large \( \bar{r} \). Also, the relative diffusion process preserves the area under \( E(\bar{r},t) \) i.e. \( \frac{\partial}{\partial t} \int d\bar{r} E(\bar{r},t) = 0 \). Therefore, the spreading of \( E(\bar{r},t) \) to larger \( \bar{r} \) with time must be accompanied by a peaking of \( E(\bar{r},t) \) for small \( \bar{r} \) so that the area is preserved. Hence, as time elapses, the total energy is deposited into small (\( \bar{r} \)) scales. The spreading of \( E(\bar{r},t) \) to large \( \bar{r} \) produces small wave number (\( k \)) components to the spectrum. Since \( \langle B^2 \rangle = (2\pi)^{-2} \int dk k^2 \left| \psi_k \right|^2 \), the poloidal flux flows to large scales. This cascade process, as well as its analogy to two dimensional Navier-Stokes turbulence is well known (15).

Intuitively, the flow of magnetic energy from large to small scales can be seen as follows: the \( \frac{\partial}{\partial t} B \) terms in Eqn. (8) and (9) follow from the nonlinear tension of the magnetic field lines. The anomalous resistivity is due to this random self-consistent magnetic field line restoring force. The bending and twisting of a large scale magnetic field line generates a small scale fluctuation in the field line and requires energy to overcome the field line tension. Therefore, the magnetic energy must flow to the smaller spacial scale.
We note that the Eqn. (8) and (9) appear to violate the time reversibility and nondissipative nature of the MHD Eqn. (2) for \( \eta = 0 \). However, Eqn. (8) and (9) describe the ensemble average, statistical properties of a system whose microstate equations are Eqn. (2) with \( \eta = 0 \). An analogous situation occurs with the Vlasov equation, which is also time reversible and non-dissipative. However, quasilinear theory introduces inversibility and dissipation.

The diffusion coefficients in Eqn. (8) and (9) are fluid-like, i.e. nonresonant. For instance, the velocity diffusion coefficients can be written approximately as

\[
\frac{1}{\mathbb{D}^v} \sim (2\pi)^{-2} \frac{1}{dk} \langle \overline{V}_{k} \overline{V}_{k}^{*} \rangle \tau_{N}(k)
\]

\[
\frac{1}{\mathbb{D}^{v'}} \sim (2\pi)^{-2} \frac{1}{dk} \langle \overline{V}_{k} \overline{V}_{k}^{*} \rangle \tau_{N}(k) (1-\cos k \cdot \mathbf{r}_{-})
\]

where \( \overline{V}_{k} \) is the Fourier component of \( V \) and \( \tau_{N}(k) \) is the correlation time of a velocity fluctuation with scale \( k^{-1} \). In Section G we point out that \( \tau_{N}(k) \sim [k \cdot \mathbb{D}^{v} (k/k^{2}) \cdot k]^{-1} \). The occurrence of \( \mathbb{D}^{v} \) in \( \tau_{N}(k) \) rather than \( \mathbb{D}^{v'} \) ensures that, while the small scales can dissipate the larger scales, the large scales cannot dissipate the smaller scales. This property is known as
Galilean Invariance. (14)

The Galilean Invariance property implies that, as far as the small scale fluctuations are concerned, a large scale fluctuation can be treated as a near spacially uniform background field. If the large \( k \) fluctuations have amplitudes smaller than the small \( k \) fluctuations, then perturbation theory is valid for the large \( k \) components. Consequently, the large \( k \) components will behave like Alfvén waves. (16) We expect equipartition of energy for the large \( k \) components of the spectrum.

These considerations allow an estimate to be made on the relative importance of \( p^* \) in the correlation equations. Incompressibility in Eqn. (2a) gives \( p^* - S^2 \langle \delta B \rangle^2 - \langle \delta V \rangle^2 \). The \( p^* \) term is small on the right hand side of Eqn. (9) if

\[
S^4 r_{-}^{-2} \langle \delta B \rangle^2 D_{-12}^B > \delta V r_{-1}^{-1} p^*, \text{ or } SD_{-12}^B / \delta B^B > \langle S^2 \delta B^2 - \delta V^2 \rangle / S^2 \delta B^2. \]

But from Eqn. (13), \( D^V \cdot r_{-} \delta V \cdot r_{-} \delta B \) so that \( p^* \) is negligible if

\[
[S^2 \langle \delta B \rangle^2 - \langle \delta V \rangle^2] / 2 \langle \delta B \rangle^2 < 1, \text{ i.e., if equipartition of energy is approximately satisfied. We note that this condition is important to the energy flow discussed above. In particular, if the magnetic perturbations were set equal to zero, Eqn. (9) would reduce to the case of two dimensional Navier-Stokes turbulence. One could not conclude, however, that the energy, } <\delta V^2>, \text{ flows to small scales as before. The reason is that } p^*
is not negligible in this case, i.e. \( p^* (\delta V)^2 \).

D. Driven Three Dimensional Turbulence

An equilibrium magnetic field with shear alters the spatially homogeneous case described by Eqn. (12) in two important ways. The mode coupling becomes resonant and the turbulent fluctuations are driven by the free energy source of the magnetic field gradient.

The effects of magnetic shear are most clearly seen by considering Eqn. (6). In Section G, we derive the governing equation for the correlation function \( \langle (B_1 - S^{-1} V_1) \cdot (B_2 - S^{-1} V_2) \rangle \). We consider helicity-free tearing mode turbulence where \( \phi \) and \( \psi \) have opposite parity for an individual tearing mode. Defining \( \delta B_\perp = B_\perp - \langle B_\perp \rangle \) and \( \delta V = V - \langle V \rangle = V \), then \( \delta E(x, t) = \langle \delta B_\perp \cdot \delta B_\perp \rangle + S^{-2} \langle \delta V_1 \cdot \delta V_2 \rangle \) satisfies

\[
\frac{\partial}{\partial t} + S (\langle B_1 \rangle V_1 + \langle B_2 \rangle V_2) - S^2 V_\perp \cdot D_\perp \cdot V_\perp \delta E = S^2 V_\perp \cdot (D_{12} + D_{21}) \cdot V_2 \langle B_1 \rangle \cdot \langle B_2 \rangle \tag{15}
\]

where \( D_\perp = B_\perp + S^{-2} D^V \) and \( D_{12} = B_{12} + S^{-2} D_{12}^V \).

When magnetic shear causes \( \langle B \rangle \cdot V \) to vanish at a mode
rational surface, the correlation function in Eqn. (15) becomes singular for weak fields. The fluctuations produced by the mode coupling are thus resonant at the mode rational surfaces. For finite amplitude fields, this resonance is broadened by the dissipation $S^2 D$. In Section G, it is shown that the diffusion coefficient is also resonant in the presence of magnetic shear, and for weak fields, can be written as

$$D = S^{-1} \int \frac{dk}{(2\pi)^2} \frac{b^2}{k^2} \pi \delta(k_z - k_y\langle B_y \rangle')x) \tag{16}$$

where $\langle B_y \rangle' = \partial\langle B_y \rangle/\partial x$ is the gradient of the average poloidal magnetic field, $x$ is the position of the mode rational surface, $\langle b^2 \rangle_k = \langle \delta B^2 \rangle_k + S^{-2} \langle \delta V^2 \rangle_k$, and $\int dk = \int dk_z \int dk_y$. For finite amplitude fields, the resonance in Eqn. (16) becomes broadened by the effect of $S^2 D$ in Eqn. (15) so that

$$D_{xx} = (2\pi)^{-2} \int \frac{dk}{k^2} \int b^2_k \pi \delta(k_z - k_y \langle B_y \rangle')x) \tag{17}$$

where

$$R(\lambda) = \int_0^\infty dt \exp[i\lambda t - \frac{1}{3}S^4(k_y \langle B_y \rangle')^2D_{xx} t^3] \tag{18}$$

Equation (17) implies that a magnetic field line will
resonate and diffuse if it is within $x_d$ of a mode rational surface, where

$$x_d = \left[ \frac{1}{3} \frac{S D_{xx}}{k_Y \langle B_Y \rangle} \right]^{1/3}. \quad (19)$$

This is also the width of the correlation function of a large spacial scale, since, from Eqn. (15), $\delta E$ peaks for

$$S x_k \langle B_Y \rangle' \propto S^2 D_{xx}/x^2. \quad (19)$$

It is interesting to note that Eqn. (15) retains its form under the redefinition $S^t + t$ and $S^d + D$. Then, for the time stationary case, Eqn. (15) is independent of $S$ and time, and can be interpreted as describing the diffusion of the field lines in $x$ as one moves along $z$. This describes a "stochastic equilibrium". (3) Relaxing the condition $\partial / \partial t = 0$ allows the mode coupling to occur in the regions where $k^* \langle B \rangle = 0$. We show below that this mode coupling rate is exceedingly fast and raises the question as to whether a stochastic equilibrium is possible.

The physics described by Eqn. (15) - (17) can be made more intuitive by writing $D_{xx}$ as

$$D_{xx} \sim <b_x^2> \cdot r_d \quad (20)$$

where

$$t_d = \left[ \frac{1}{3} (k_Y \langle B_Y \rangle')^2 S^4 D_{xx} \right]^{-1/3}. \quad (21)$$

Here, $<b_x^2> \cdot r_d$ is composed by those fluctuations whose resonance
widths are within $x_d$ of $x$, and is defined as the resonant portion of the spectrum. Combining Eqn. (20) and Eqn. (21) gives
\[ D_{xx} \langle b_x^2 \rangle^{3/4} (k_y \langle B_y' \rangle)^{-1/2} \text{s}^{-1} \]
so that $x_d$ and $t_d$ can be written as
\[ x_d \sim \frac{\langle b_x^2 \rangle_{\text{res}}^{1/2}}{k_y \langle B_y' \rangle} \quad (22) \]
and
\[ t_d^{-1} \sim \langle b_x^2 \rangle_{\text{res}}^{1/4} (k_y \langle B_y' \rangle)^{1/2}. \quad (23) \]

First, we note that, in the limit of one resonant fluctuation, Eqn. (22) is the usual expression for the magnetic island width. In the turbulent—many fluctuation—case, the resonance broadening of Eqn. (17) allows modes adjacent to the mode rational surface to contribute to the resonance width $x_d$ and to the diffusion $D_{xx}$. Therefore, $D_{xx}$ will become nonzero when two magnetic islands overlap. Subsequent to this, a nonzero $D_{xx}$ in Eqn. (15) will cause mode coupling and dissipation on the $t_d^{-1}[S^2D_{xx}/x_d^2]^{-1}$ time scale. In this turbulent phase, Eqn. (16) is a valid expression for $D_{xx}'$, since the overlap criteria will be satisfied throughout the spectrum.
E. Mixing Length Model

The nonlinear term on the right hand side of Eqn. (15) is the shear-driven, free energy source for fluctuation energy. It is an incoherent source of fluctuation energy due to the mixing of the average magnetic field gradient by the diffusion process. Because of properties (10) and (11) the incoherent source in Eqn. (15) mainly drives small scale (large k) fluctuations. This is consistent with the mode coupling process where energy flows from large to small scales.

The $p^*$ term neglected on the right hand side of Eqn. (15) is small if $S^4D^B<b^2_\perp> <p^*x^*-1 \delta V>$ which gives

$$[S^2(\delta B_\perp)^2-(\delta V)^2]/S^2(\delta B_\perp)^2(x^d/B)<(\delta B_\perp)/<B_\perp>^2$$

For reasonable parameters ($\delta B_\perp/B_\perp \leq 10^{-1}$ and $x_d \geq 10^{-2}$), the $p^*$ term is negligible if equipartition is approximately satisfied. Hence, for the large k components of the field, the shear driving term dominates the right hand side of Eqn. (15). This ensures the validity of a "mixing length" model.\(^{(14,19)}\) The large k fluctuations are created by the turbulent rearrangement (mixing) of the small k field components on a scale $\delta x$. Then

$$\delta E \sim <b^2_x> \sim (\delta x)^2(\partial <B_\perp>/\partial x)^2 - \tau^2 S^2 D(\partial <B_\perp>/\partial x)^2$$

where $\tau$ is the mixing time that follows from inverting the left hand side of Eqn. (15).

The spatial scale of these modes and their rate of production can be seen in a physically appealing way by an approx-
imate inversion of Eqn. (15). In the resonant region

$$\langle b_x^2 \rangle \sim S^2 \tau_{D} \langle \partial \langle B_x \rangle / \partial x \rangle^2$$

becomes

$$S^2 \tau_{(k_y \delta \psi)} \langle \partial \langle B_x \rangle / \partial x \rangle^2$$

so that

$$\tau \sim (S \delta \psi) \langle B_x \rangle \sim \langle B_y \rangle \sim 1.$$ But from the resonance condition

for \( \delta E \langle x < x_d \rangle \) and Eqn. (19) we find that \( \tau \sim t_d \) where \( t_d \) is
given by Eqn. (21). Therefore, the scales \( x_d \) and \( t_d \) characterize the incoherent fluctuations driven by the shear.

This mixing length model is analogous to the mixing length model of Navier-Stokes turbulence. There, the turbulence produces an eddy viscosity that mixes up the large spacial scale velocity shear to create incoherent turbulent fluctuations. In the MHD case presented here, the magnetic field turbulence produces a turbulent resistivity that "tears" up the large spacial scale magnetic shear to create incoherent turbulent fluctuations.

We suggest that the mixing length model may describe the tearing mode turbulence that is thought to be responsible for some disruptions in tokamaks. During computer simulations study of the disruption, the plasma current profile develops small spacial scale fluctuations at the mode rational surfaces and expands radially outward to induce a drop in the plasma loop voltage ("negative voltage spike"). This phenomena is consistent with that predicted by Eqn. (15). The magnetic shear is the source of free energy that drives the mode coupling process at the mode rational
surfaces. The turbulent diffusion mixes regions of large and small poloidal field. Since $\partial <B_\perp>/\partial x <0$, there is a net transfer of current density radially outward.

The phenomena of resonant mixing occurs also in Vlasov turbulence. (12) There, "clumps" of phase space density, $f$, are produced by the turbulence at the wave-particle resonance, because, from the Vlasov equation,

$$
\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial z} + \frac{e}{m} E_z \frac{\partial}{\partial v} \right) f = 0
$$

(24)

$f$ is singular at $\omega = k_z v$ for weak electric fields $E_z$. These resonant modes are not normal modes of the system, but are more like ballistic modes. The spacial modulations produced at these resonances are due to a resonant mode coupling process.

In the MHD turbulence model presented here, magnetic shear allows a resonant, ballistic-like mode to exist. As in the Vlasov case, this mode is not a normal mode of the system—such as a tearing mode—but is an incoherent, resonant fluctuation produced by the turbulence. This resonance can be seen from Eqn. (6) where it is evident that when $\omega = k_z - k_y <B_y>' x = k_y <B_y>' (x_0 - x)$ is satisfied, singular fluctuations develop. We suggest that these incoherent, resonant modes cause the "clumping" and radial expansion of the plasma current during
computer simulation studies of the disruptive instability.

F. Nonlinear Tearing Mode Growth

In Section C it was shown that the predominant effect of the nonlinear terms in Eqn. (2) for a large spacial scale was anomalous dissipation. Here we outline the effects of anomalous resistivity and viscosity on tearing mode growth. We expect that the nonlinear terms in Eqn. (2) for a large spacial scale can be represented by $S^2D_{xx}$. To derive this from the one point equations, we neglect the mode coupling $V\cdot V$ terms and approximate the $B\cdot V$ terms as pure diffusion (dissipation). Again, this is only valid for the large scales, since the small scales have sources as well as sinks (dissipation).

Fourier transforming Eqn. (2) in $y$ and $t$ gives

$$-i\omega V_k = S^2 \sum_{k'} B_{k'}^* \frac{3}{\partial x} B_{k-k'} + S^2 i k_{yy} B_k + S^2 B_k \frac{3}{\partial x} <B_y>$$

$$-i\omega B_k = \sum_{k'} B_{k'}^* \frac{3}{\partial x} V_{k-k'} + i k_{yy} V_k + \eta \frac{3}{\partial x^2} B_k$$

(25a) (25b)

where $k_{yy} = k_z - k_y <B_y>'x$. Because of Galilean Invariance, we include only $k^y > k$ in the $k'$ sums of Eqn. (25). Then, iterating the perturbations in the nonlinear terms gives
\[
-i\omega k = S^2 \frac{\partial}{\partial x} V_k + iS^2 k_B k + 2S^2 V_k X k + iS^2 k_B k
\]

\[+ S^2 B_k \frac{\partial}{\partial x} <B_y> \quad (26a)\]

\[
-i\omega B_k = \frac{\partial}{\partial x} S^2 k^2 + \frac{1}{\omega} \frac{\partial}{\partial x} B_k + i k W_k + \eta^2 \frac{\partial^2}{\partial x^2} B_k \quad (26b)
\]

Since we are interested in the turbulent effects of the incoherent, resonant \( k' \) fluctuations on the mode \( k \), we set \( \omega' = k_z' - k_y' <B_y>'x \) in Eqn. (26). (Note that this is the resonant-mode coupling rate since \( \omega = k_y <B_y>'(x_0 - x) \). The large scale fluctuations thus satisfy

\[
\frac{\partial}{\partial t} \delta V = \frac{\partial}{\partial x} S^2 D_{xx} \frac{\partial}{\partial x} \delta V + S^2 <B> \cdot \nabla \delta B_{\perp}
\]

\[+ S^2 \delta B_{\perp} \cdot \nabla <B_{\perp}> \quad (27a)\]

\[
\frac{\partial}{\partial t} \delta B_{\perp} = \frac{\partial}{\partial x} (\eta + S^2 D_{xx}) \frac{\partial}{\partial x} \delta B_{\perp} + <B> \cdot \nabla \delta V \quad (27b)\]

where \( D_{xx} \) is given by Eqn. (16).

For finite amplitude fields, the nonlinear resonance in Eqn. (6) will broaden the \( k' \) modes in Eqn. (27) by \( S^2 D_{xx} \) so that \( D_{xx} \) in Eqn. (27) will be given by Eqn. (17).
Taking the $\hat{z}$ component of the "inverse curl" of Eqn. (27a) and
the curl of Eqn. (27b) gives

$$\begin{align*}
S^{-2} \frac{\partial}{\partial t} \delta U &= D_{xx} \frac{\partial^2}{\partial x^2} \delta U - \langle B \rangle \cdot \nabla \delta J + \delta B \cdot \nabla \langle J \rangle \\
\frac{\partial}{\partial t} \delta \psi &= (\eta + S^2 D_{xx}) \frac{\partial^2}{\partial x^2} \delta \psi - \langle B \rangle \cdot \nabla \delta \phi
\end{align*}$$

(28a)

Displaying the shear terms explicitly in the resonant region

gives

$$\begin{align*}
\frac{\partial}{\partial t} \psi_k &= (\eta + S^2 D_{xx}) \frac{\partial^2}{\partial x^2} \psi_k - k_y \langle B_y \rangle' (x-x_0) \phi_k \quad (29a) \\
S^{-2} \frac{\partial}{\partial t} U_k &= D_{xx} \frac{\partial^2}{\partial x^2} U_k - k_y \langle B_y \rangle' (x-x_0) \frac{\partial^2}{\partial x^2} \psi_k \\
\end{align*}$$

(29b)

In the resonant region $|x-x_0| < x_d$, the nonlinear terms dominate
the linear terms. For $S^2 D_{xx} > \eta$ a small $k$ tearing mode will grow
at the rate

$$\gamma_{\text{NL}} \sim S^2 D_{xx} \frac{\Lambda'}{x_d}$$

(30)

where $\Lambda'$ is the usual tearing mode stability parameter. (17)

We note that this is of the same form as the linear growth
rate: $x_d$ replaces the tearing layer width, and $S^2 D_{xx}$ replaces
the collisional resistivity.

It is interesting to compare the physics of the growth described here with that of Rutherford's nonlinear theory of one mode. We integrate Eqn. (29) over the resonant region to obtain

\[ x_d \frac{\partial}{\partial t} \psi_k(x_o) \sim (\bar{\eta} + S^2 D_{xx}) \Delta' \psi_k(x_o). \]  

(31)

In the Rutherford case of one mode, we've shown in Section D that \( x_d \) reduces to the expression for the island width for \( \psi_k \). Hence, Eqn. (31) would lead to the algebraic growth with time of the Rutherford theory. However, after island overlap and the production of many modes, \( x_d \) would be only weakly dependent on the mode \( \psi_k \). Then \( x_d \) would be due to the many \( \psi_k' \) that are within resonance with \( \psi_k \) and would be a number characteristic of the turbulence through \( D_{xx} \) (see Eqn. (19)). For \( S^2 D_{xx} \geq \bar{\eta} \), then Eqn. (31) leads to the exponential growth rate of Eqn. (30). This transition from algebraic to exponential growth after island overlap is an important consequence of the turbulent state.

The scaling of the growth rate \( \gamma_{NL} \) is of interest. Since the anomalous resistivity is normalized to the average resistivity \( \bar{\eta} \), then \( R_m = S^2 D_{xx} \) is an effective Reynolds number.
characterizing the turbulence. We, therefore, write $\gamma_{NL}$ as

$$\gamma_{NL} \sim R_m^{2/3} A'[S k_B <B'_y>]^{1/3}. \quad (32)$$

For the early stages of the turbulent process ($R_m < 1$), then

$$\gamma_{NL} \sim S^{1/3}. \quad (13, 17)$$

In present day tokamaks, $S \sim 10^7$ so that $\gamma_{NL} \sim S^{2/3}$, which scales like the linear growth rate, $\gamma_{2/1}$, of the "2/1" mode. For reasonable parameters ($S \sim 10^7$, $R_m > 1$), $[k_B <B'_y>]^{1/3} \sim 1$, $\alpha \sim 10$, $\tau_R \sim 2$ sec), Eqn. (32) implies a time scale $\leq 900$ $\mu$sec. However, during the fully turbulent phase ($R_m > 1$), $\gamma_{NL}$ is sensitive to variations in $R_m$. From Eqn. (30) we see that

$$\gamma_{NL} \sim (S^2 D_{xx})^{2/3} S^{1/3} B_{res}^{2/3} \sim \gamma_{2/1} \sim \gamma_{NL}$$

Thus, $\gamma_{NL}$ in "real" units depends only on $\tau_H$. Therefore, in the $R_m > 1$ phase, $\gamma_{NL}$ is greater than $\gamma_{2/1}$ and is independent of resistivity $\eta$. (10)

We have argued in Section C that the turbulent deflection of magnetic field lines from unperturbed flux surfaces leads to anomalous resistivity and perpendicular viscosity. Here we show how the deflection of pressure surfaces leads to anomalous parallel viscosity.

Transfer of parallel momentum occurs in the fluid equations via perturbations in pressure surfaces. We note that the electron parallel momentum balance, expressed in real units, is

$$B_z <n_e> e E_z = T_e B_z \cdot \nabla n_e \quad (33)$$

where $T_e$ is the electron temperature, $n_e$ is the electron particle density, $e$ is the electric charge, $E_z$ is the parallel electric field,
and we've assumed that the electron pressure is $n_e T_e$. Magnetic fluctuations will cause deflections in the pressure surfaces in Eqn. (33). If the deflections are random, a parallel viscosity results. To show this, we note that in the tearing mode the ion density responds to the polarization drift

$$n_i = (4\pi e)^{-1} c^2 v_A^{-2} v_\perp^2 \phi$$

(34)

Using quasi-neutrality, Eqn. (33) gives

$$E_z = \lambda_D^2 \frac{c^2}{v_A^2} \frac{\lambda_D^2}{(B \cdot \nabla) U}$$

(35)

where all are quantities are normalized (see Section B) and $\lambda_D$ is the debye长度 normalized to the minor radius. Equation (35) adds a correction to Faraday's Law (Eqn. (28a))

$$\frac{3}{\partial t} \psi = (n + S^2 D_{xx}) \frac{3}{\partial x^2} \psi - B \cdot \nabla \phi + \lambda_D^2 \frac{c^2}{v_A^2} B \cdot \nabla U$$

(36)

Using the curl of Eqn. (2a) to renormalize the $B \cdot \nabla U$ term in Eqn. (36) as was done at the beginning of this section, we find that Eqn. (36) becomes

$$\frac{3}{\partial t} \psi = (n + S^2 D_{xx}) J - B \cdot \nabla \phi - \mu \nabla^2 J$$

(37)
where the anomalous parallel viscosity \( \mu_{\parallel} = \lambda_D \frac{c_s^2}{V_{\text{e}}^2} S^2 D_{xx} \). Converting this to "real" units, we define the electron thermal velocity \( V_{\text{e}}^2 = T_e / m_e \) and the plasma frequency \( \omega_p^2 = 4\pi n_e e^2 / m_e \) so that \( \mu_{\parallel} = \frac{c_s^2}{\omega_p^2} \bar{\mu} \) where \( \bar{\mu} = \frac{V_{\text{e}}}{V_A} (V_{\text{e}} D_m) \). Here \( D_m = a S D_B \) is the stochastic magnetic field diffusion coefficient. An anomalous parallel viscosity \( \bar{\mu} = V_{\text{e}} D_m \) has been suggested as a cause of disruptive instability in tokamaks. Our result differs by the factor \( V_{\text{e}} / V_A \) from the suggested value. This difference is due to frequency response of the current fluctuations. In the model presented here, the electron fluid responds on the Alfven time scale. However, in the model of Kaw, et. al, the field lines are fixed in time (but are stochastic in space) and the electrons respond on the V/\( V_{\text{e}} \) time scale.

From Eqn. (37) the ratio of anomalous viscosity to anomalous resistivity effect is \( k_{\parallel}^2 \rho_s^2 \) where \( \rho_s \) is the ion gyro radius at the electron temperature. We note that the fluid equations used here break down when \( k_{\parallel}^2 \rho_s^2 > 1 \).

G. Derivation of the Diffusion Equations

In deriving the diffusion equations, we follow Reference 14. We consider Eqn. (15) first, and define \( L = B_{\perp} - S^{-1} V_N \), \( N = B_{\perp} + V S^{-1} \) so that \( L_1 \cdot L_2 \) follows from Eqn. (6)

\[
\left\{ \frac{\partial}{\partial t} + S N_1 \cdot V_1 + S N_2 \cdot V_2 \right\} L_1 \cdot L_2 = 0 \tag{38}
\]
Since we are interested here in the nonlinear terms on the left hand side of Eqn. (6), we have surpressed the \( \eta \) and \( p^* \) terms in Eqn. (38). We define

\[
L_1 \cdot L_2 = \langle L_1 \cdot L_2 \rangle + \delta(L_1 \cdot L_2) \tag{39}
\]

\[
\langle N \rangle = \langle N \rangle + \delta N \tag{40}
\]

where \( \delta \) denotes a fluctuation about the average. We now average Eqn. (38) and use Eqn. (39) and (40) to obtain

\[
\{ \frac{\partial}{\partial t} + S\langle N_1 \rangle \cdot \nabla_1 + S\langle N_2 \rangle \cdot \nabla_2 \} \langle L_1 \cdot L_2 \rangle = \sum_{i=1,2} S_i \langle \delta N_i \rangle \langle L_1 \cdot L_2 \rangle \tag{41}
\]

Next, we substitute Eqn. (39) and (40) into Eqn. (38) and subtract Eqn. (41) to obtain

\[
\{ \frac{\partial}{\partial t} + S\langle N_1 \rangle \cdot \nabla_1 + S\langle N_2 \rangle \cdot \nabla_2 \} \delta(L_1 \cdot L_2) = -S [\delta N_1 \cdot \nabla_1 + \delta N_2 \cdot \nabla_2] \langle L_1 \cdot L_2 \rangle + \sum_{i=1,2} S_i \langle \delta N_i \rangle \langle L_1 \cdot L_2 \rangle \tag{42}
\]
In the Markovian approximation, we neglect the last term on the right hand side of Eqn. (42). This allows us to obtain a diffusion equation for \( \langle L_1 \cdot L_2 \rangle \).

With this approximation, we integrate Eqn. (42) to find the response of \( \delta(L_1 \cdot L_2) \)

\[
\delta(L_1 \cdot L_2) = -S \int_0^\infty d\tau G_{12}(t, \tau) \sum_{i=1,2} \delta N(x_i, \tau) \cdot \nabla_i \times \langle L_1(\tau) \cdot L_2(\tau) \rangle
\]

where the propagator \( G_{12} \) satisfies

\[
\left\{ \frac{\partial}{\partial \tau} + SN_1 \cdot \nabla_1 + SN_2 \cdot \nabla_2 \right\} G_{12}(t, \tau) = 0
\]

and \( G_{12}(t, t) = 1 \). If we now substitute the expression \( \delta(L_1 \cdot L_2) \) of Eqn. (43) into Eqn. (41) we obtain

\[
\left\{ \frac{\partial}{\partial \tau} + S \sum_{i=1,2} \langle N_i \rangle \cdot \nabla_i - \sum_{i,j=1,2} \nabla_i \cdot S^2 D_{ij} \nabla_j \right\} \times \langle L_1 \cdot L_2 \rangle = 0
\]

where

\[
D_{ij} = \int_0^\infty d\tau \langle \delta N(x_i, \tau) G_{12}(t, \tau) \delta N(x_j, \tau) \rangle
\]
In arriving at Eqn. (45) we have set \( \langle L_1(\tau) \cdot L_2(\tau) \rangle \sim \langle L_1(t) \cdot L_2(t) \rangle \) in Eqn. (43). This is the usual Markovian approximation, and is valid if the spectrum correlation time is short compared to the relaxation time of \( \langle L_1 \cdot L_2 \rangle \).

Following a similar procedure for the one point equation, we see that \( \langle L_1 \rangle \cdot \langle L_2 \rangle \) satisfies

\[
\left\{ \frac{\partial}{\partial t} + S \sum_{i=1,2} \langle N_i \rangle \cdot \nabla_i - S^2 \sum_{i=1,2} \nabla_i \cdot \mathbf{D}(\mathbf{r}_i, \mathbf{r}_i) \cdot \nabla_i \right\} \times \langle L_1 \rangle \cdot \langle L_2 \rangle = 0 \tag{47}
\]

This equation also follows from Eqn. (45) where we put \( L_1 = \langle L_1 \rangle \) which we note diffuses independently of \( \langle L_2 \rangle \) (i.e., \( D_{12} = 0 \)).

Now we subtract Eqn. (47) from Eqn. (45) to obtain

\[
\left\{ \frac{\partial}{\partial t} + S \sum_{i=1,2} \langle N_i \rangle \cdot \nabla_i - S \sum_{i=1,2} \nabla_i \cdot \mathbf{D}(\mathbf{r}_i, \mathbf{r}_i) \cdot \nabla_i \right\} \delta\langle L_1 \rangle \cdot \langle L_2 \rangle = 2S^2 \nabla_1 \cdot \mathbf{D}_{12} \cdot \nabla_2 \langle L_1 \rangle \cdot \langle L_2 \rangle \tag{48}
\]

where \( \delta\langle L_1 \rangle \cdot \langle L_2 \rangle = \langle L_1 \cdot L_2 \rangle - \langle L_1 \rangle \cdot \langle L_2 \rangle \), the fluctuation correlation function, depends on the relative coordinate \( \mathbf{r}_- = \mathbf{r}_1 - \mathbf{r}_2 \).

We assume that \( \langle \mathbf{V} \rangle = 0 \) so that \( \langle L \rangle = \langle N \rangle = \langle \mathbf{B} \rangle \). In the helicity-free case, \( \langle \mathbf{V} \cdot \mathbf{B} \rangle = 0 \). We assume generally that \( \langle \mathbf{V}_1 \cdot \mathbf{B}_2 \rangle \)
is less than $\langle B_1 \cdot B_2 \rangle$ and $S^{-1} \langle V_1 \cdot V_2 \rangle$. With these considerations, Eqn. (48) reduces to Eqn. (15) of Section D.

The quantity $G_{12}(t,\tau) \delta_N(x_j, t)$ in Eqn. (46) can be evaluated by using Eqn. (44). We write

$$
D_{ij} = \frac{1}{(2\pi)^2} \int \frac{dk}{dk} \int d\tau \ g_N(k) \ e^{-ik \cdot x_j} \langle G_{12}(t,\tau) e^{ik \cdot x_j} \rangle
$$

(49)

where $g_N(k)$ is the Fourier transform of the spectral correlation function

$$
g_N(k) = \int dx e^{-ik \cdot x} \langle \delta_N(x_1, t) \cdot \delta_N(x_2, \tau) \rangle
$$

(50)

We again make use of the Markovian approximation, so that the cut-off of the $\tau$ integral in Eqn. (49) is due to the orbit correlation time of $G_{12}(t,\tau)$ rather than the relaxation of the correlation function $g_N(k)$. We, therefore, set $\tau = t$ in Eqn. (50).

Now, since $\langle G_{12}(t,\tau) \rangle$ satisfies the same equation as $\langle L_1(t) \cdot L_2(t) \rangle$ (i.e., Eqn. (45)), we can write

$$
\{ \frac{3}{3t} + S \langle B_2 \rangle \cdot \dot{V}_2 - S^2 \langle V_2 \cdot D_{22} \cdot V_2 \rangle \} \langle G_{12}(t,\tau) e^{ik \cdot x_2} \rangle = 0
$$

(51)

In the presence of shear, the operator in braces becomes
where $D_{xx}(2)$ is the $x_2$ component of $D_{22}$. Then with $k \cdot r_2 = k_z z_2 + k_y y_2$ we find that

$$e^{-ik \cdot r_2} e^{ik \cdot r_2} =$$

$$\exp[-iS(k_z - k_y z_2 <B_y>_t, t) - T/3] S^4(k_y <B_y>^2) D_{xx} (2) t^3] (53)$$

With this expression in Eqn. (49) we can explicitly evaluate $D_{ij}$. In particular we see that $D_{xx}$ is given by Eqn. (17) of Section D. For weak fields, $D_{12}$ becomes

$$D_{x_1 x_2} = \frac{1}{(2\pi)^2} \int dk \langle b_k^2 \rangle e^{ik \cdot (y_1 - y_2)}$$

(54)
\[ D_- = (2\pi)^{-2} \int \frac{dk}{d\mathbf{k}} \langle b_x^2 \rangle_k \pi \delta(k_z - \mathbf{x} \cdot \mathbf{B}_y) \]

\[ \times 2(1 - \cos k_y (y_1 - y_2)) \]  

(55)

is the x component of the relative diffusion coefficient

\[ D_- = 2(D - D_{12}). \]

The correlation equation for the two dimensional, homogeneous case with no equilibrium field (i.e., Eqn. (12) of Section C) follows from Eqn. (45) if we note that \( \langle N \rangle = 0 \) and assume again that \( \langle V_1 \cdot B_2 \rangle \) is less than \( S \langle B_1 \cdot B_2 \rangle \) and \( S^{-1} \langle V_1 \cdot V_2 \rangle \) for the helicity-free case. The evaluation of \( D_{ij} \) for the \( \langle N \rangle = 0 \) case is more difficult. In the \( \langle N \rangle \neq 0 \) case, there is a lowest order contribution to \( G(t, \tau) \) that is phase independent. This shear effect allows us to put \( G(t, \tau) = \langle G(t, \tau) \rangle \) in writing Eqn. (49). In the \( \langle N \rangle = 0 \) case, however, the lowest order contribution to \( G(t, \tau) \) comes from the perturbations \( \delta N \), so particular care must be taken in evaluating the phase average in Eqn. (46). Although we don't present the details here (see Ref. 14), the expression corresponding to Eqn. (55) can approximately be written as

\[ D_- = 2(2\pi)^{-2} \int \frac{dk}{d\mathbf{k}} \langle b_x^2 \rangle_k \frac{1 - \cos k \cdot r}{k \cdot D_-(k/k^2) \cdot k} \]  

(56)

so that now the fluctuation lifetime \( \tau_N(k) \sim (k \cdot D_- \cdot k)^{-1} \) plays the
role of \( t_d \). This is evident by comparing Eqn. (12) and (15).

The derivation of Eqn. (8) and (9) of Section C is straightforward. Consider first the \( \mathbf{V} \cdot \mathbf{V} \) term in Eqn. (2a). Proceeding as in the beginning of this section we see that the \( \mathbf{V} \cdot \mathbf{V} \) term generates \( \mathbf{V} \cdot \mathbf{D} \cdot \mathbf{V} \) in the equation for \( \langle \mathbf{V}_1 \cdot \mathbf{V}_2 \rangle \). This can be done also by identifying \( \mathbf{SN} \cdot \mathbf{V} \) and \( \mathbf{L} \cdot \mathbf{V} \) in Eqn. (38) and (45). Similarly, the \( \mathbf{V} \cdot \mathbf{V} \) term in Eqn. (2b) generates \( \mathbf{V} \cdot \mathbf{D} \cdot \mathbf{V} \) in the equation for \( \langle \mathbf{B}_1 \cdot \mathbf{B}_2 \rangle \). The \( \mathbf{B} \cdot \mathbf{V} \) terms in Eqn. (8) and (9) are more interesting since they couple the two equations and lead to the anomalous viscosity and resistivity. Supressing the \( \mathbf{V} \cdot \mathbf{V} \) terms now, we write

\[
S^{-2} \frac{\partial}{\partial t} \mathbf{V}_1 \cdot \mathbf{V}_2 = \mathbf{B}_2 \cdot \mathbf{V}_2 \mathbf{V}_1 \cdot \mathbf{B}_2 + \mathbf{B}_1 \cdot \mathbf{V}_1 \mathbf{V}_2 \cdot \mathbf{B}_1
\]  

(57a)

\[
\frac{\partial}{\partial t} \mathbf{B}_1 \cdot \mathbf{B}_2 = \mathbf{B}_2 \cdot \mathbf{V}_2 \mathbf{V}_1 \cdot \mathbf{B}_1 + \mathbf{B}_1 \cdot \mathbf{V}_1 \mathbf{V}_2 \cdot \mathbf{B}_2
\]  

(57b)

But

\[
\frac{\partial}{\partial t} \mathbf{V}_1 \cdot \mathbf{B}_2 = S^2 \mathbf{B}_1 \cdot \mathbf{V}_1 \mathbf{B}_1 \cdot \mathbf{B}_2 + \mathbf{B}_2 \cdot \mathbf{V}_2 \mathbf{V}_1 \cdot \mathbf{V}_1
\]  

(58a)

\[
\frac{\partial}{\partial t} \mathbf{V}_2 \cdot \mathbf{B}_1 = S^2 \mathbf{B}_2 \cdot \mathbf{V}_2 \mathbf{B}_2 \cdot \mathbf{B}_1 + \mathbf{B}_1 \cdot \mathbf{V}_1 \mathbf{V}_2 \cdot \mathbf{V}_2
\]  

(58b)
Consider Eqn. (57b). We expand the correlation functions in Eqn. (58) into average and fluctuating parts. We then integrate to obtain

\[
\delta(V_1 \cdot B_2) = S^2 \int_0^\infty d\tau \hat{G}_{12}(t, \tau) B_1 \cdot V_1 <B_1 \cdot B_2>
\]

\[
+ \int_0^\infty d\tau \hat{G}_{12}(t, \tau) B_2 \cdot V_2 <V_1 \cdot V_2>
\]

and a similar equation for \(\delta(V_2 \cdot B_1)\). The operator \(\hat{G}_{12}\) solves Eqn. (58). If we now substitute the expressions for \(\delta(V_1 \cdot B_2)\) and \(\delta(V_2 \cdot B_1)\) into Eqn. (57b) and average we find that

\[
\frac{\partial}{\partial t} <B_1 \cdot B_2> = S^2 \frac{V_2}{V_2} \cdot \frac{D_{22}}{V_2} \cdot V_2 <B_1 \cdot B_2> + \frac{V_1}{V_2} \cdot \frac{D^B_{21}}{V_1} \cdot V_1 <V_1 \cdot V_2>
\]

\[
+ S^2 \frac{V_1}{V_1} \cdot \frac{D^B_{11}}{V_1} \cdot V_1 <B_1 \cdot B_2> + \frac{V_1}{V_2} \cdot \frac{D^B_{12}}{V_2} \cdot V_2 <V_1 \cdot V_2>
\]

(60)

where

\[
\frac{D^B_{ij}}{V_1} = \int_0^\infty d\tau <B_1(t) \hat{G}_{ij}(t, \tau) B_j(\tau)>.
\]

(61)

For the spatially homogeneous case, the correlation functions
depend only on \( \mathbf{r}_2 - \mathbf{r}_2 \), so that Eqn. (60) reduces to Eqn. (8) of Section C. In similar fashion, Eqn. (9) can be derived.

Moreover, by integrating Eqn. (57) and inserting in Eqn. (58) and averaging, we find that \( \hat{G}_{12}(t,\tau) \) satisfies

\[
\left\{ \frac{\partial}{\partial t} - \mathbf{\nabla} \cdot \mathbf{D} \mathbf{\nabla} \cdot \mathbf{\nabla} - 2s^2 \mathbf{\nabla} \cdot \mathbf{D} \mathbf{\nabla} \right\} <G_{12}(t,\tau)> = 0
\]

(62)
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