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ABSTRACT

Two parallel, relativistic electron streams with different velocities support unstable, exponentially growing space-charge waves, and efficient electron bunching may be achieved in the submillimeter wavelength range. Injection of the prebunched streams into a static, periodic (wiggler) magnetic field enhances the intensity of the parametrically excited, back-scattered electromagnetic wave. Calculations are presented of wave enhancement and wave growth in the Raman regime, for the case of two cold, perfectly intermingled prebunched streams.

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1. INTRODUCTION

The generation of coherent electromagnetic radiation in magnetically pumped free electron lasers (FEL) is connected intimately with axial bunching of the electrons in their passage through a transverse, periodic (wiggler) magnetic field. Calculations and experiments suggest that injection of a prebunched electron beam greatly enhances the radiation intensity of the parametrically excited, back-scattered wave. Moreover, the threshold beam current required to initiate wave generation is much lower with prebunching than in the case of the conventional FEL configuration, where prebunching is not used.

The frequency of the prebunching space-charge wave must be equal to the desired radiation frequency of the FEL. In systems designed to operate at low (microwave) frequencies, prebunching is readily achieved, as for example by passing the electron stream through the modulating gap of a resonant cavity, or by allowing the beam to interact with a neighboring slow-wave structure (for example, a helical transmission line). Prebunching in the submillimeter wavelength range where many FEL's are designed to operate is difficult, and use of magnetic wigglers fields has been suggested. Recently, a different prebunching technique has been proposed which exploits the well-known fact that two parallel electron streams with different velocities support unstable, exponentially growing space-charge waves over a wide frequency range. In contrast to other prebunching methods, the present one has the virtue that no physical medium or other wave-supporting structures are required.

In this paper we examine the wave characteristic of a device
illustrated schematically in Fig. 1. It is comprised of a prebunching section and a wiggler-field section. Two relativistic electron streams with velocities $v_1$ and $v_2$ travel along the axis of the system. A space-charge wave grows exponentially out of noise with a frequency determined by the electron number densities of the two streams, and their velocity difference $|v_1 - v_2|$. The prebunched streams then traverse the periodic (wiggler) magnetic field where the electromagnetic wave is generated. In section 2 of the paper we discuss the dispersion characteristics of the space charge waves in the prebunching region. Some of these calculations represent extensions to relativistic velocities of early studies$^{6,7,8}$ on two-stream instabilities made for electrons traveling at nonrelativistic speeds. In section 3 we illustrate the enhancement of the electromagnetic wave generated in the magnetic wiggler field due to prebunching. And in section 4 we solve the magnetic wiggler dispersion equation for the growing electromagnetic and space-charge waves. We assume throughout that the beams are monoenergetic (no thermal spread) and are completely intermingled spatially. Our dispersion equations are applicable to beams with arbitrary number densities and velocities. However, for purposes of illustration, we shall take the beam densities to be equal and their velocity difference, $(v_1 - v_2)$, small.

2. PREBUNCHING REGION

Consider two monoenergetic electron streams with number densities $N_1$ and $N_2$ and velocities $v_1$ and $v_2$ propagating along the positive z axis. The beams are superposed spatially, they are of
uniform density, and of infinite extent in the x and y directions. Solving the linearized particle and relativistic momentum conservation equations, and Poisson's equation, and assuming that all rf quantities vary as exp[jωt−jkz], one arrives at the wave equation\(^8,9\) for the electric field

\[ [1 + \chi_1 + \chi_2] E_z = 0 \]  
\[ \text{(1)} \]

in terms of the electrical susceptibilities \(\chi_1\) and \(\chi_2\) of the two beams:

\[ \chi_{1,2} = -\frac{\omega_{p1,2}^2}{\gamma_{1,2}^3 (\omega - kv_{1,2})^2} \]  
\[ \text{(2)} \]

Here \(\omega_{p1,2} = (N_{1,2} e^2/m_0 e_0)^{1/2}\) are the nonrelativistic plasma frequencies, with \(e\) and \(m_0\) as the electron charge and rest mass, respectively. And \(\gamma_{1,2}\) are the beam energy parameters

\[ \gamma_{1,2} = \left[ 1 - \left( \frac{v_{1,2}}{c} \right)^2 \right]^{-1/2} \]
\[ = 1 + \left( \frac{eV_{1,2}}{m_0 c^2} \right) \]  
\[ \text{(3)} \]

with \(V_{1,2}\) as the beam voltages. We note that for beams of infinite transverse dimensions assumed here (a condition that will be relaxed later), the electric field of the wave is purely axial, and the magnetic field is zero \((E_x=E_y=B_x=B_y=B_z=0)\). Thus Poynting's flux is zero, and all the wave energy\(^9\) resides in axial particle oscillations.

Combining Eqs. (1) and (2) yields the dispersion equation

\[ 1 = \frac{\omega_{p1}^2}{\gamma_1^3 (\omega - kv_1)^2} + \frac{\omega_{p2}^2}{\gamma_2^3 (\omega - kv_2)^2} \]  
\[ \text{(4)} \]
which is a fourth-order polynomial in the frequency $\omega$. Thus, four coupled waves exist; these are the slow (negative energy) and fast (positive energy) waves supported by each electron stream, whose individual dispersion equations, in the absence of coupling, are

$$\omega = kv_1 + \frac{\omega_p}{\gamma^{3/2}}$$

$$\omega = kv_2 + \frac{\omega_p}{\gamma^{3/2}}$$

Wave growth and thus electron bunching occur because of coupling between the negative energy wave on one stream and the positive energy wave on the other stream. To find the wave characteristics, we solve Eq. (4) for complex frequencies $\omega$, and real wave numbers $k$. For the sake of simplicity we assume that the densities of the beams are the same, so that $\omega_p = \omega_p = \omega_p$, and that the difference in the beam velocities $(v_1 - v_2)$ is small, so that $\gamma_1 = \gamma_2 = \gamma$. Under these assumptions, Eq. (4) exhibits a growing root in the range of frequencies given by,

$$0 < \text{Re}(\omega) < \sqrt{2} \frac{\omega_p}{\gamma^{3/2}} \left[ \frac{v_1 + v_2}{v_1 - v_2} \right]$$

This is illustrated in Fig. 2. In this entire range the wave is dispersionless, namely

$$\text{Re}(\omega) = \frac{1}{2} k(v_1 + v_2)$$

with group and phase velocities equal to the arithmetic mean of the beam velocities. The growth rate is maximum (see Fig. 2) at a frequency
\[
\text{Re}(\omega) = \frac{\sqrt{3}}{2} \frac{\omega_p}{\gamma^{3/2}} \left( \frac{v_1 + v_2}{v_1 - v_2} \right)
\]

\[= \sqrt{3} \beta^2 \gamma^{3/2} \omega_p / \Delta \gamma \quad \text{for } |\gamma_1 - \gamma_2| \ll (\gamma_1 + \gamma_2)
\]  

(8)

where the second form of the equation comes from expressing \(v_1\) and \(v_2\) in terms of \(\gamma_1\) and \(\gamma_2\), and defining \(\Delta \gamma \equiv |\gamma_1 - \gamma_2| < \gamma\), \(\beta = (v_1 + v_2)/2c\).

Observe that the wave frequency is proportional to the square root of the beam current density \([J=Ne^+\gamma]\) and inversely proportional to the difference in the beam velocities. Thus, with present-day accelerators fairly high frequencies may be achieved. Take for example a 2MV machine capable of delivering beams with a current density of 1kA/cm\(^2\) and a beam voltage difference of 20kV (that is \(\Delta \gamma/\gamma=10^{-2}\)). It then follows from Eq. (8) that \(\omega/2\pi = 1.52 \times 10^{12}\), which corresponds to a wavelength equal to 197\(\mu\)m. As a second example, consider the ATA accelerator which will deliver \(\sim 10\)kA/cm\(^2\) at a voltage of \(\sim 30\)MV. With these values and an assumed \(\Delta \gamma/\gamma=10^{-2}\), one obtains a wavelength of 17\(\mu\)m.

Note, however, that \(\omega\) cannot be made arbitrarily large by making \(\Delta \gamma\) arbitrarily small. Cold plasma theory neglects beam temperature, and for it to be valid, \((v_1 - v_2)\) must be larger than \(v_{\text{thermal}}\); otherwise the two-stream instability is quenched. Therefore, good, cold beams produced by low emittance electron guns are required. In practice it is difficult to achieve beams with \((\Delta \gamma/\gamma)_{\text{thermal}}\) smaller than \(\sim 10^{-3}\).

At the frequency given by Eq. (8), the temporal growth rate of the wave amplitude is maximum (see Fig. 2) and has the value

\[\text{Im}(\omega) = \frac{\omega_p}{2\gamma^{3/2}}.\]  

(9)
Thus, the spatial growth rate of the wave intensity is $2\text{Im}(\omega)/v_g$ where $v_g$ is the group velocity, and the gain

$$G = 4.34 \omega_p/\beta \gamma^{3/2}c \quad \text{dB/meter.} \quad (10)$$

Using the numerical values of the previous example, we find that $G=36\text{dB/m}$ for the 2MV accelerator and $G=2.6 \text{ dB/m}$ for the 30MV accelerator. We see that in order to obtain reasonably large growth rates, high current, intermediate voltage accelerators are desirable.

The efficiency of converting beam kinetic energy into wave energy can be estimated in the nonlinear limit, that is when exponential growth ceases and the electrons become trapped at the bottoms of the potential wells of the longitudinal wave. At this position along the drift tube the wave reaches its maximum amplitude and the beam kinetic energy is minimum. The change in kinetic energy suffered by one of the beams is approximately

$$\delta(\text{KE}) \approx 2\delta v \left( \frac{\partial \gamma}{\partial v} \right) m_0 c^2 \quad (11)$$

where $\delta v=(\omega/k)-v$ equals the difference between the phase velocity of the wave and the beam velocity. Evaluating $\left( \partial \gamma/\partial v \right)$ from $\gamma = \left[ 1-(v/c)^2 \right]^{-1/2}$ and $\delta v$ from Eq. (7), one finds that the conversion efficiency (at saturation) is given by

$$\eta = \frac{\delta(\text{KE})}{(\gamma-1)m_0 c^2} = \frac{\beta \gamma^3}{(\gamma-1)c} (v_1-v_2) = \frac{\Delta \gamma}{\gamma-1} \quad (12)$$

where the last expression follows from the fact that, $(v_1-v_2) \approx (c/\beta \gamma^2)(\Delta \gamma/\gamma)$ for small velocity differences. When $\gamma \gg 1$, $\eta=\Delta \gamma/\gamma$. If $\Delta \gamma/\gamma = 0.1$, the conversion efficiency is 10% which is reasonably large; however, the oscillation frequency varies as $(\Delta \gamma)^{-1}$ (see Eq. (8)) and therefore the operating parameters for millimeter
and submillimeter wave generation are constrained to a fairly narrow range of $\Delta \gamma / \gamma$.

Hitherto, we have assumed that the electron streams have infinite transverse dimensions, and one needs to inquire to what extent finite geometry affects the dispersion characteristics of the waves. We shall examine the following more realistic model. Two cold, uniform, spatially superposed streams completely fill a cylindrical metal pipe of radius $a$, which acts both as an evacuated drift tube and the waveguide structure. An axial magnetic field guides the electrons along the $z$ axis; it is sufficiently strong so that all steady state and oscillatory electron motions can be assumed to be entirely axial. Under these conditions the wave equation for the axial component of the rf electric field $E_z$ for a beam-filled pipe is

$$\nabla_\perp^2 E_z - \left[ k^2 - \left( \frac{\omega}{c} \right)^2 \right] \left[ 1 + \chi_1 + \chi_2 \right] E_z = 0 \quad (13)$$

where $\nabla_\perp^2$ is the transverse Laplacian operator and $\chi_1, \chi_2$ are the electron susceptibilities defined by Eq. (2). Note that in finite geometry, the space charge wave has, in addition to the axial electric field $E_z$, components of $\vec{E}$ and $\vec{B}$ in the transverse directions (which are found from Maxwell's equations, once $E_z$ is known).

Solving Eq. (13) subject to the boundary condition that $E_z=0$ at the metal waveguide wall situated at $r=a$, one obtains the dispersion equation for the $TM_{nm}$ family of waveguide modes,

$$\left( \frac{p_{mn}}{a} \right)^2 + \left[ k^2 - \left( \frac{\omega}{c} \right)^2 \right] \left[ 1 + \chi_1 + \chi_2 \right] = 0 \quad (14)$$

where $p_{mn}$ is the $n$th zero of the $m$th order Bessel function, $J_m \left( \frac{p_{mn}}{a} \right) = 0$. We see that when $a\rightarrow \infty$, Eq. (14) has roots: $\omega = \pm kc$ which represents the electromagnetic waves, and $(1+\chi_1+\chi_2)=0$ which re-
presents the space-charge waves.

Figure 3 illustrates the dispersion characteristics of the \( \text{TM}_{01} \) mode, for two electron streams with voltages of 0.5MV and 0.3MV traveling in a cylindrical waveguide of 1cm radius. A large energy separation of the streams has been chosen in order to make the real part of the dispersion clearly visible. A maximum of six distinct waves exists; two of these exhibit growth in the wave-number range \( 0 < k < \sim 5 \text{cm}^{-1} \), and they have very similar characteristics to the waves in the absence of the waveguide (see Eq. (7) and Fig. 2). Indeed, the growth rate is insensitive to the waveguide radius as is illustrated in Fig. 4, as long as the waveguide radius \( a \) is sufficiently large compared with the axial wavelength \( 2\pi/k \) at maximum growth rate.

The rf power flow in unbounded electron streams is carried entirely in the oscillatory motion of the electrons, and the flux can be computed from the expression \( \mathbf{P} = m_0 c^2 \left( \gamma' - 1 \right) \mathbf{J} / e \) where \( \mathbf{J} \) is the rf current density and \( \gamma' = \gamma + \tilde{\gamma} \) is the sum of the steady state, \( \gamma \), and oscillatory, \( \tilde{\gamma} \), contributions to the energy parameter \( \gamma' \). In finite geometry, transverse electric and magnetic fields exists, and some of the rf power flow resides in Poynting's flux \( \mathbf{S} = \mathbf{E} \times \mathbf{B} / \mu_0 \). The ratio \( R = \left| \text{Re} \left[ \mathbf{P} \cdot d\mathbf{A} \right] / \text{Re} \left[ \mathbf{S} \cdot d\mathbf{A} \right] \right| \) where the integration is over the waveguide cross-section, is a measure of the energy content in the longitudinal (bunching) field, relative to the energy in the transverse electromagnetic, wave field. Calculating \( R \) for the \( \text{TM}_{01} \) mode one finds that

\[
R = 1.008 \left( \frac{ka}{P_{01}} \right)^2 \left[ \text{Re} \left( \chi_1 + \chi_2 \right) \right] \left[ 1 - (\omega/kc)^2 \right]^2 \left\{ \frac{kv_1}{\omega-kv_1} + \frac{kv_2}{\omega-kv_2} \right\}
\]

For a waveguide radius of 1cm and beam parameters specified in the caption to figure 4, the quantity \( R \) is typically a
few hundred, and therefore the rf power is overwhelmingly in the longitudinal, oscillatory electron motion. The conversion of this power to transverse electromagnetic radiation is the subject matter of section 3 below, where we inject the prebunched streams into a transverse wiggler magnetic field. However, other conversion schemes come to mind, and we shall discuss one of these briefly.

Instead of passing the streams through an empty waveguide, suppose that the latter is filled with a dielectric material of refractive index $n$. In this situation, the oscillatory electron energy may be converted into electromagnetic radiation through the Čerenkov mechanism. The dispersion equation is the same as that given by Eq. (14) except that the term $[k^2-(\omega/c)^2]$ is replaced by $[k^2-(n\omega/c)^2]$. The results of the computations are shown in Fig. 5 in which the growth rate of the unstable root is plotted as a function of wavenumber $k$. We see that the bell shaped curve corresponding to the two-stream instability is virtually unchanged from that when $n=1$. However, the presence of the dielectric medium introduces a new growing wave. It represents unstable Čerenkov radiation due to a single beam alone, and is precisely the emission mechanism that has been subjected to detailed scrutiny elsewhere.13

In practice the dielectric medium cannot permeate the entire waveguide volume because it causes electron scattering, and dielectric lined waveguides must be used instead.13 This introduces a serious problem in that, to ensure good coupling of the waves on the beam with the waves in the dielectric, the beam must skirt the dielectric within a distance $x-\gamma\lambda$ where $\lambda$ is the radiation wavelength. At submillimeter wavelengths, the technical problems
this introduces are very severe. It is important to note that the magnetic wiggler method of energy extraction discussed in the next section does not suffer from the above limitation.

3. MAGNETIC WIGGLER REGION

In the magnetic wiggler region, the electrons are subjected to a static, transverse, periodic magnetic field of the form

\[ B_1 = B_0 \sin(k_0 z) \]

where \( k_0 = \frac{2\pi}{\lambda} \) is the wavenumber and \( \lambda \) the period. Electromagnetic wave energy is generated through the coupling of a positive energy electromagnetic wave

\[ \omega^2 = k^2 c^2 + \omega_{p1,2}^2 / \gamma_{1,2}^2 \]  \hspace{1cm} (17)

with a negative energy, space-charge wave (see Eqs. (5))

\[ \omega = (k + k_0) v_{1,2} - \omega_{p1,2} / \gamma_{1,2}^3 / 2 \]  \hspace{1cm} (18)

upshifted in frequency by the presence of the wiggler magnetic field, which imparts a transverse quiver velocity to the electrons. The subscripts 1 and 2 refer to quantities associated with streams 1 and 2, respectively.

The wave equation for the transverse component of the vector potential \( A_\perp \) of the rf fields is given by

\[ \left[ \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] A_\perp = - \mu_0 \int^1 J_\perp \]  \hspace{1cm} (19)

where \( J_\perp \) is the transverse rf current density associated with the wave, and \( \int^1 \) represents a summation over the two electron streams. For each stream, \( J_\perp \) is, after linearization, of the form

\[ J_\perp = - e(n + n_p) v_\perp - eNv_\perp \]  \hspace{1cm} (20)

Here \( N, v_\perp \) are the time independent number density and velocity, respectively; and \( n \) and \( v_\perp \) are the corresponding time varying quantities; \( n_p \) is the prebunch number density, and \( v_p \) the associated axial prebunch velocity, both of which are, likewise, time
dependent. Combining Eqs. (19) and (20), using the conservation of transverse canonical momentum, \( p_\perp = e A \), and neglecting small terms in the expression for \( J_\perp \), yields,

\[
\left( c^2 \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} - \sum \frac{\omega_p^2}{m_0 c} \right) \left( e A_\perp \right) = - \frac{\omega_c^2}{k_0 c} \cos(k_0 z) \sum \frac{\omega_p^2}{\gamma} \left( \frac{n + n_p}{N} \right) \tag{21}
\]

where \( \omega_c = (e B_0 / m_0) \) is the nonrelativistic cyclotron frequency associated with a wiggler field of amplitude \( B_0 \), and \( \omega_p = (Ne^2 / m_0 \varepsilon_0) \) is the nonrelativistic plasma frequency.

In the cold-beam approximation, fluid equations can be used to describe the electron kinetics. For each stream then, one obtains a linearized particle conservation equation

\[
\frac{\partial}{\partial t}(n + n_p) + \nabla \cdot (n + n_p) + \frac{N}{m_0 \gamma^3} \frac{\partial p_z}{\partial z} + N \frac{\partial v_p}{\partial z} = 0 \tag{22}
\]

and a linearized momentum conservation equation for the axial momentum \( p_z \):

\[
\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial z} \right) p_z = - e E_z + \frac{e \omega_c}{\gamma k_0} \left[ \cos(k_0 z) \frac{\partial A_\perp}{\partial z} - k_0 A_\perp \sin(k_0 z) \right] \tag{23}
\]

where \( v \) is the time-average axial electron velocity. Finally, Poisson's equation gives the first order, rf space-charge electric field in terms of the number density perturbations of the two streams:

\[
\frac{\partial E_z}{\partial z} = - e \left( \frac{n + n_p}{\varepsilon_0} \right) \tag{24}
\]

Note that this relation couples the dynamics of the two streams.

As a result of the periodicity of the equilibrium state, Floquet's theorem requires that all quantities such as \( A_\perp \), \( n \), \( p_z \), and \( E_z \) vary as
\[ Z = \exp(j\omega t - jkz) \sum_{m=-\infty}^{\infty} Z(m) \exp(jmk\vartheta z) \]  

(25)

where \( Z(m) \) represent the series expansion coefficients of the quantity \( Z \). However, our primary interest concerns the lowest order coupling of waves and therefore only terms \( m=0 \) and \( m=\pm 1 \) of Eq. (25) will be retained. Allowing \( A_1, n, n_p \) and \( v_p \) of Eq. (21) to vary in accordance with Eq. (25), one finds that

\[
\varepsilon(\omega, k) \frac{eA(0)}{m_0 c} = -\frac{\omega_c}{2k_0 c} \sum \frac{\omega^2}{Y} \left[ \frac{n(1) + n_p(1)}{N} + \frac{n(-1) + n_p(-1)}{N} \right] 
\]

(26a)

\[
\varepsilon(\omega, k-k_0) \frac{eA(1)}{m_0 c} = -\frac{\omega_c}{2k_0 c} \sum \frac{\omega^2}{Y} \left[ \frac{n(0) + n_p(0)}{N} \right] 
\]

(26b)

\[
\varepsilon(\omega, k+k_0) \frac{eA(-1)}{m_0 c} = -\frac{\omega_c}{2k_0 c} \sum \frac{\omega^2}{Y} \left[ \frac{n(0) + n_p(0)}{N} \right] 
\]

(26c)

where once again the summation sign \( \sum \) signifies an addition of contributions from beams 1 and 2, and where

\[
\varepsilon(\omega, k + mk_0) = \omega^2 - (k + mk_0)^2 c^2 - \sum \frac{\omega^2}{Y} 
\]

(26d)

is the dielectric function of the uncoupled electromagnetic wave.

The quantities \( A(0), A(1), A(-1), n(0), n(1), n(-1), \) etc. refer to the series expansion coefficients \( A(m), n(m) \). Also, substituting Eq. (25) in Eqs. (22), (23), and (24), and eliminating \( E_Z \) and \( p_Z \), yields

\[
\sum \frac{\omega^2}{Y} \frac{n(0) + n_p(0)}{N} = \frac{1}{\gamma_1 \gamma_2} \left( \frac{\chi_1(0)\chi_2(0)}{1+\chi_1(0)+\chi_2(0)} \right) \left[ \frac{\gamma_1 R_2}{\chi_1(0)} + \frac{\gamma_2 R_1}{\chi_2(0)} + \right.
\]

\[
+ \left( \gamma_1 - \gamma_2 \right) \left( R_2 - R_1 \right) \right] 
\]

(27a)

where \( \chi_1(0) \) and \( \chi_2(0) \) are the electron susceptibilities define by Eq. (2), and
Equations (26b), (26c), and (27a) can be combined to give a relationship for the amplitude $A(0)$ of the electromagnetic wave in terms of the velocity amplitude $v_p(-1)$ of the prebuncher. The result simplifies considerably in the special case of small velocity differences, such that $|\gamma_1 - \gamma_2| << |\gamma_1 + \gamma_2|$, in which case the last term on the right-hand side of Eq. (27a) can be neglected. We shall do so here. We then perform upshifts and downshifts in the $m$ values of Eqs. (26b), (26c), and (27a) and find that

$$
\varepsilon(\omega, k) \frac{eA(0)}{m_0 c} = -\frac{\omega_c}{2k_0 c} \sum_{\nu} \frac{\omega^2 P}{\gamma} \left[ \frac{n(-1) + n_p(-1)}{N} \right] \quad (28a)
$$

$$
\varepsilon(\omega, k+2k_0) \frac{eA(-2)}{m_0 c} = -\frac{\omega_c}{2k_0 c} \frac{\omega^2 P}{\gamma} \left[ \frac{n(-1) + n_p(-1)}{N} \right] \quad (28b)
$$

$$
\sum_{\nu} \frac{\omega^2 P}{\gamma} \left[ \frac{n(-1) + n_p(-1)}{N} \right] = \frac{\chi_1(-1) + \chi_2(-1)}{\gamma^2 (1 + \chi_1(-1) + \chi_2(-1))} \left[ \frac{eA(0)}{m_0 c} + \frac{eA(-2)}{m_0 c} \right] \frac{\omega_c}{2} \frac{(k+k_0)^2}{k_0} \left[ \frac{\varepsilon(\omega, k)}{\varepsilon(\omega, k+2k_0)} \right] ^2 \quad (28c)
$$

where

$$
\chi_{1, 2}(-1) = -\frac{\omega^2 P_{1, 2}}{\gamma^3_{1, 2} \left[ \omega - (k+k_0) v_{1, 2} \right]^2} \quad (28d)
$$

Eliminating $[n(-1) + n_p(-1)]$ from the above equations yields

$$
\left( 1 + \left[ 1 + (\chi_1(-1) + \chi_2(-1))^{-1} \right] \right)^{-1} \left( \frac{\varepsilon(\omega, k)}{k_0} \right)^2 \frac{\omega_c}{4\gamma^2} \left[ \frac{1}{\varepsilon(\omega, k)} + \frac{1}{\varepsilon(\omega, k+2k_0)} \right] \left[ eA(0) \right] \frac{m_0 c}{m_0 c} \quad (29)
$$

$$
= \left( \frac{\omega_c}{2k_0 c \varepsilon(\omega, k)} \right) \left[ \frac{\gamma^2 (k+k_0)}{1 + \chi_1(-1) + \chi_2(-1)} \right] \sum_{\nu} \chi(-1) \left[ \omega - (k+k_0) v \right] v_p(-1) \quad (29)
$$
A resonance occurs when the dielectric function of the un-coupled electromagnetic wave \( \varepsilon(\omega, k) = 0 \), as will be discussed below. Since the function \( \varepsilon(\omega, k+2k_0) \) cannot be resonant simultaneously, it makes little contribution to Eq. (29) and the term \( \varepsilon(\omega, k+2k_0)^{-1} \) can be safely neglected. As a result we obtain the sought-after result,

\[
D(k, \omega) \left[ \frac{eA}{m_0 c} \right] = \gamma^2 \frac{\omega}{2(\chi_1 + \chi_2)} \left[ \frac{k+k_0}{k_0} \right] \chi \{ \omega - (k + k_0)v \} \frac{v_P}{c} \tag{30}
\]

Here \( D(k, \omega) \) is a "dielectric function" associated with the coupled waves, and is defined as

\[
D(k, \omega) \equiv \left[ 1 + (\chi_1 + \chi_2)^{-1} \right] \left[ \omega^2 - k^2 c^2 - \sum \frac{\omega^2}{\gamma} \right] + \frac{\omega^2}{4\gamma^2} \left[ \frac{k+k_0}{k_0} \right]^2 \tag{31}
\]

For the sake of brevity, we have written \( A \) for \( A(0) \), \( v_P \) for \( v_P(-1) \) and \( \chi_1, 2 \) for \( \chi_1, 2(-1) \).

We see from Eq. (30) that the amplitude \( A \) of the vector potential of the electromagnetic wave is directly proportional to the product of the amplitude \( B_0 \) of the wiggler magnetic field and the amplitude of the velocity modulation \( v_P \) of the prebuncher. More important, \( A \) varies inversely as the dielectric function \( D(k, \omega) \), which exhibits a minimum near a frequency and wave number given by

\[
\omega = (1 + \beta)\gamma^2 k_0 v \quad ; \quad v = (v_1 + v_2)/2
\]

\[
k = \beta (1 + \beta)\gamma^2 k_0 \tag{32}
\]

results which follow from solving Eqs. (17) and (18), subject to the assumption that \( \omega >> \omega_P \), \( |v_1 - v_2| << v \). Electron prebunching thus enhances the amplitude of the electromagnetic wave at and near the radiation frequency given by Eq. (32). This is illustrated in Fig. 6 which gives a plot of \( |D^{-1}(\omega, k)| \) as a function of
\[ \omega/\omega_p. \] In the example, the electron streams have voltages \( V_1 = 1.50 \text{ MV} \) and \( V_2 = 1.41 \text{ MV} \), and equal plasma densities, \( \omega_p = \omega_p = 2 \times 10^{10} \text{ sec}^{-1} \). The velocity difference \((v_1 - v_2)\) is chosen with the aid of Eq. (8) so that the prebunch frequency of Eq. (8) is equal to the radiation frequency given by Eq. (32). The wiggler magnetic field \( B_0 = 2 \text{kG} \) and the periodicity \( \lambda = 4 \text{ cm} \).

4. WAVE DISPERSION IN THE WIGGLER MAGNETIC FIELD

The wave dispersion equation is obtained by setting \( D(k, \omega) = 0 \) in Eq. (31) with the result that

\[
\left[ 1 + (x_1 + x_2)^{-1} \right] \left[ \frac{\omega^2}{\gamma^2} - k^2 c^2 - \sum \frac{\omega_p^2}{\gamma} \right] = - \frac{\omega^2 c}{4 \gamma^2} \left( \frac{k + k_0}{k_0} \right)^2 \quad (33)
\]

It shows that the space-charge waves are coupled to the electromagnetic waves through the rippled magnetic field. When the latter is set equal to zero we obtain \( 1 + x_1 + x_2 = 0 \) for the space charge waves (see Eq. (1)), and \( \omega^2 - k^2 c^2 - \sum \omega_p^2 / \gamma = 0 \) for the electromagnetic waves (see Eq. (17)). Equation (33) is valid as long as \( |\gamma_1 - \gamma_2| \ll \gamma \). We note in passing that the dispersion equation for arbitrary values of \( \gamma_1 \) and \( \gamma_2 \) is readily obtained by setting \( n_p = v_p = 0 \) in Eqs. (26), and (27) and combining them. The result is,

\[
\left[ 1 + (x_1 + x_2)^{-1} \right] \left[ \frac{\omega^2}{\gamma^2} - k^2 c^2 - \sum \frac{\omega_p^2}{\gamma} \right] = - \frac{\omega^2 c}{4 \gamma_1 \gamma_2} \left( \frac{k + k_0}{k_0} \right)^2 M \quad (34)
\]

where

\[
M = \frac{\gamma_1^2 x_2 + \gamma_2^2 x_1 + x_1 x_2 (\gamma_1 - \gamma_2)^2}{\gamma_1 \gamma_2 (x_1 + x_2)} \quad (35)
\]

\[
= 1 + \frac{\Delta \gamma}{\gamma} \left( \frac{x_1 - x_2}{x_1 + x_2} \right) \quad (\Delta \gamma \ll \gamma)
\]

and where \( x_1 \equiv x_1 (-1), x_2 \equiv x_2 (-1) \) and are given by Eqs. (28d).
In the case of a single electron stream traversing the wiggler magnetic field we set $\chi_2 = \omega p_2 = 0$, $\gamma_1 = \gamma_2 = \gamma$ in Eqs. (33) or (34) and thereby recover the cold-beam, Raman dispersion relation.\textsuperscript{15}

Substituting in Eq. (33) for $\chi_1$ and $\chi_2$ from Eq. (28d), setting $\omega_1 = \omega_2 = \omega_0$, and rearranging terms gives the dispersion relation

$$\left\{ \left( \omega - D \right)^2 - A^2 \right\}^2 - 2L^2 \left\{ (\omega - D)^2 + A^2 \right\} \left[ \omega^2 - T^2 \right] = 2Q \left( \omega - D \right)^2 + A^2$$

where

$$D = \frac{1}{2} (k + k_0) (v_1 + v_2)$$

$$A = \frac{1}{2} (k + k_0) (v_1 - v_2)$$

$$L = \omega_p / \gamma^{3/2}$$

$$T = (k^2 c^2 + 2\omega_p^2 / \gamma)^{1/2}$$

$$Q = \frac{\omega_c^2 \omega_p^2}{4\gamma^5} \left( \frac{k + k_0}{k_0} \right)^2$$

A slightly more accurate result than that given by Eq. (36) is

$$\left\{ \left( \omega - D \right)^2 - A^2 \right\}^2 - 2L^2 \left\{ (\omega - D)^2 + A^2 \right\} \left[ \omega^2 - T^2 \right] = 2Q \left( \omega - D \right)^2 + A^2 \left[ 1 - \left( \beta^2 \omega / D \right) \right]^2 \gamma^4$$

To solve this equation we assume that $\omega$ is complex and $k$ is real.

The sixth order polynomial is then evaluated on the IBM 370 computer. Two unstable, growing roots are obtained: one for the electromagnetic wave, and the other for the space charge oscillations supported by the mutual interaction of the waves on the streams (see section 2).

Figures 7 and 8 show plots of the imaginary parts of $\omega$ as a function of $k$ for the electromagnetic and space charge waves, re-
spectively, for several values of the wiggler magnetic field $B_0$ and a fixed periodicity $\lambda=4$cm. The beam voltages $V_1=1.50\text{MV}$ and $V_2=1.41\text{MV}$ and the beam densities $\omega_{p_1}=\omega_{p_2}=2\times10^{10}\text{sec}^{-1}$ are chosen in conformity of Eq. (8) so that the prebunch frequency $\omega$ of Eq. (8) at which the space charge wave has its maximum growth rate equals the frequency (Eq. (32)) at which the electromagnetic wave has its maximum growth rate. This requires that in Eq. (37)

$$ (v_1-v_2) = \sqrt{3} \omega_p \left[ k_0 \gamma^7 / 2 \beta (1+\beta) \right]^{-1} \text{ where } \beta = (v_1+v_2) / 2c. $$

The functional behavior of the electromagnetic wave growth rate versus $k$ (Fig. 7) is very similar to that obtained when a single beam alone traverses the wiggler magnetic field. Indeed, for a given value of $B_0$, the magnitude of $\text{Im}(\omega)$ at maximum growth for two streams is just slightly lower than the corresponding value of $\text{Im}(\omega)$ when but a single stream with twice the density traverses the wiggler. Thus, the growth rate of the electromagnetic wave is virtually unaffected by the presence of the two-stream instability. Of course the two-stream instability has a dramatic effect on the amplitude of the electromagnetic wave because it causes prebunching, as is discussed in section 3.

On the other hand, the growth of the two-stream instability is influenced quite considerably by the presence of the wiggler magnetic field, as is shown in Fig. 8. Not only is the shape of the curve modified, but the growth rate at maximum growth is larger than in the absence of the wiggler field ($B_0=0$).

5. CONCLUSIONS

The free electron laser is an amplifier (or oscillator) device in which passage of an electron stream through a periodic
magnetic (wiggler) field leads to stimulated emission of cyclotron radiation. The electrons experience a spatially periodic, longitudinal bunching force and thus a velocity modulation, as a result of which coherent emission occurs. The bunching can be made more rapid by prebunching techniques as for example by use of another wiggler or by a high-power laser. In this paper we have examined the possibility of radiation enhancement of a magnetically pumped, two-stream, free electron laser by using the two-stream instability as a bunching mechanism. This technique does not require either additional wave sources or wave supporting structures. In our idealized theory, the streams are assumed to be cold and spatially superposed. Satisfactory growth rates and efficiencies are obtained for wavelengths in the submillimeter range. The requisite beam voltages are typically in the range 1-10 MV. The beam current densities are in the range of kiloamperes per square centimeter. Electron guns with low beam emittances and low energy spreads ($\Delta\gamma/\gamma$)$_{\text{thermal}}^2 < 10^{-3}$ are a prerequisite. By properly "tuning" the beam velocities relative to one another, the radiation intensity is strongly enhanced, although the spatial growth rate of the radiation is largely unaffected by the presence of the two-stream instability.
REFERENCES


14. The choice of a linearly polarized wiggler like that given by Eq. (16) induces longitudinal electron oscillations. To make the longitudinal excursions small compared to the radiation wavelength, and thus minimize the possibility of particle untrapping, the wiggler amplitude $B_0$ must be chosen so that $(\omega_c/k_0c) < \sqrt{2}$ where $\omega_c = eB_0/m_0$. (See H. Boehmer, M.Z. Caponi, J. Münch, G. Neil, and N. Schoen, Proceedings Erice, Sicily, Workshop Aug. 1980). The analysis of section 3 does not take account of these longitudinal oscillations.


CAPTIONS TO FIGURES

Fig. 1. Schematic drawing of the two-stream free electron laser.

Fig. 2. Normalized growth rate of the two-stream instability in the prebunch region as a function of the normalized wave number \( k \). Beam radius \( a = \infty \).

Fig. 3. Real and imaginary parts of the dispersion characteristics for the \( \text{TM}_{01} \) waveguide mode of the two-stream instability. \( V_1 = 0.50 \text{MV}; V_2 = 0.30 \text{MV}; \omega_p = \omega_p = 1.0 \times 10^{10} \text{ rad/sec}; \) waveguide radius \( a = 1.0 \text{cm} \).

Fig. 4. Imaginary part of the dispersion characteristics for the \( \text{TM}_{01} \) mode for different waveguide radii \( a \). \( V_1 = 1.50 \text{MV}; V_2 = 1.45 \text{MV}; \omega_p = \omega_p = 1.0 \times 10^{10} \text{ rad/sec} \). For \( 1 \leq a \leq \infty \text{cm} \), the curves are virtually indistinguishable from one another.

Fig. 5. Imaginary part of the dispersion characteristics for the \( \text{TM}_{01} \) mode of a dielectric-filled waveguide for different values of the refractive index \( n \). \( V_1 = 1.50 \text{MV}; V_2 = 1.45 \text{MV}; \omega_p = \omega_p = 1.0 \times 10^{10} \text{ rad/sec}; a = 1.0 \text{cm} \). The two peaked curves on the left represent the stimulated Čerenkov mode; the bell-shaped curve on the right is the two-stream instability, which is insensitive to \( n \) in the range shown.

Fig. 6. Reciprocal of the dielectric function (Eq. (31)) of the coupled waves as a function of frequency, for \( k/k_0 = 29.0 \). \( V_1 = 1.50 \text{MV}; V_2 = 1.41 \text{MV}; \omega_p = \omega_p = 2.0 \times 10^{10} \text{ rad/sec}; B_0 = 2.0 \text{kG}; l = 4.0 \text{cm} \). The broad peak on the left is associated with the growing electromagnetic wave of the two-stream free electron laser. The sharp peak on the right represents a stable wave mode.
Fig. 7. Growth rate of the electromagnetic wave of the two stream free electron laser, as a function of the wave-number \( k \), for different values of the wiggler magnetic field \( B_0 \). \( V_1 = 1.50 \text{MV}; \ V_2 = 1.41 \text{MV}; \ \omega_{p1}^{\omega_{p2}} = 2.0 \times 10^{10} \text{rad/sec}; \ z = 4 \text{cm}. \)

Fig. 8. Growth rate of the two-stream space charge instability modified by the presence of the wiggler magnetic field \( B_0 \). \( V_1 = 1.50 \text{MV}; \ V_2 = 1.41 \text{MV}; \ \omega_{p1}^{\omega_{p2}} = 2.0 \times 10^{10} \text{rad/sec}; \ z = 4 \text{cm}. \)
Fig. 2
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ELECTROMAGNETIC MODE

SPACE CHARGE MODES

Fig. 3
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Fig. 5
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Fig. 6
Bekefi & Jacobs