IDEAL MAGNETOHYDRODYNAMIC STABILITY LIMITS
OF A HIGH $\beta$ TOKAMAK
WITH SUPERIMPOSED HELICAL FIELDS

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ABSTRACT

The stability of a high $\beta$ tokamak with a superimposed single helicity stellarator field is investigated in the context of the sharp boundary surface current model. The analysis treats external ideal magnetohydrodynamic modes including both pressure-driven ballooning effects and current-driven kink instabilities. The results show that the addition of an $\ell = 2$ field is unfavorable, lowering the allowable ohmic heating current and decreasing the maximum stable $\beta$. The addition of an $\ell = 3$ field however, is favorable. Both the allowable ohmic heating current and maximum stable $\beta$ are increased as the $\ell = 3$ amplitude increases.

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I. Introduction

There has been recent renewed interest in stellarators because of a number of favorable experimental results.1 One particularly encouraging result from the W VII-A group is that in a stellarator with ohmic heating current, major disruptions can be suppressed if sufficient helical transform is present.2

This is a clear advantage of the stellarator concept. Equally important, however, the results suggest that the superposition of a helical field on a tokamak should be effective in eliminating major disruptions in that device as well. The question of disruption control is one of a few crucial physics issues regarding the ultimate desirability of a tokamak as a fusion reactor.

Experimental2,3 and theoretical2,4,5 evidence indicates that helical transforms on the order of $i_{\alpha}/2\pi \equiv \nu_{\alpha} \sim 0.1 - 0.15$ are sufficient to suppress disruptions. Such fields effectively move the $q = 2$ resonant surface outside the plasma boundary while leaving the current profile unchanged. This prevents the $m = 2, n = 1$ tearing mode, which is thought to be the dominant mode in the formation of disruptions, from being excited. (Here, $m$ and $n$ are the poloidal and toroidal mode numbers, respectively.)

Assuming that the situation just described prevails a vital remaining question is to determine the effects of the additional helical fields on the ideal magnetohydrodynamic $\beta$ limits. Specifically, since the maximum $\beta$ value thus far achieved in tokamaks3 is somewhat marginal with respect to reactor requirements, any further decrease due to the helical fields would be very serious.

It is this question that is addressed in the present paper; that is, we consider the equilibrium and stability of a toroidal high-$\beta$ tokamak with superimposed helical fields. The calculation is carried out within the context of the stellarator expansion7-8 so that the effects of current driven kinks and pressure driven ballooning modes are simultaneously included.

It should be noted that a substantial theoretical effort has been carried out in connection with the early stellarator work at Princeton.1,7,8,9 These studies which include internal and external modes in both the ideal and resistive models are quite helpful in understanding the present results. However, because of the low $\beta$ values observed in experiments relatively little effort was devoted to determining $\beta$ limits. More recently, a number of large scale numerical computations have been carried out to address this question.1,10,11 While these calculations are the most realistic in principle, the numerical problems associated with a three-dimensional geometry and the large parameter space to be explored imply that many such studies will be required before a comprehensive understanding is achieved.

The present work, which is largely analytical should be viewed as complementary to the numerical cal-
calculations. Since the configurations of interest are three-dimensional and are characterized by no less than six dimensionless parameters, a great simplification occurs by carrying out the calculation using the sharp boundary surface current model. This relatively simple model makes quite reliable predictions for the stability of low m number external modes, which are ultimately shown to be the worst modes in the regime of interest. Furthermore, recent studies\textsuperscript{12,13,14} with diffuse profiles have shown that such modes give rise to the most severe $\beta$ limits in an axisymmetric high $\beta$ tokamak.

An additional simplifying feature of the analysis is that the superimposed stellarator field is assumed to correspond to a pure, single helicity field; no supplementary vertical fields and/or helical sideband fields are allowed. Such additional fields can generate a vacuum magnetic well from the basic anti-well of the single helicity field. However, for reasons of mathematical simplicity these effects are not considered in the present calculation.

Although our analysis is mathematically valid over a wide class of configurations, ranging from the pure high $\beta$ tokamak to the pure stellarator and including any arbitrary intermediate hybrid, the results become physically unreliable when the stellarator field dominates. The reasons are as follows. First, when the helical transform is much larger than the ohmic transform the most unstable modes correspond to high $m$ values. In this regime profile effects, as well as finite ion Larmor radius effects become important and the sharp boundary profile is likely to be a poor approximation. Second, from the point of view of stellarator physics, the configuration is far from optimized since no vertical fields and/or helical sideband fields are allowed. This leads to an incomplete and pessimistic description of the stellarator. For these reasons, we consider the most appropriate way to view the present calculation is as one which describes the stability of a high $\beta$ tokamak with superimposed helical fields.

The basic results of our analysis are summarized as follows.

1. As in the case of a pure high $\beta$ tokamak, if the ohmic heating current is held fixed, there is an equilibrium $\beta$ limit above which a separatrix moves onto the plasma surface.

2. If the total rotational transform on the surface is held fixed, rather than the ohmic heating current, there is no equilibrium $\beta$ limit. This is the sharp boundary analog of the flux conserving tokamak.

3. For any helical multiplicity, $\ell$, the helical transform should be parallel rather than anti-parallel to the ohmic heating transform. If not, the $n = 0, m = 1$ rigid shift displacement becomes unstable at essentially zero $\beta$.

4. The addition of a parallel $\ell = 2$ field has a negative overall effect on stability. Both the critical $\beta$ and the maximum allowable ohmic heating current decrease as the helical transform increases. This is a
consequence of the fact that a pure $\ell = 2$ field has no vacuum magnetic well and very little shear. The situation may improve if the configuration is optimized by adding vertical fields and/or helical sideband fields. This question will be addressed in a future paper.

5. The addition of a parallel $\ell = 3$ field has a favorable effect on stability. Both the critical $\beta$ and the maximum allowable ohmic heating current increase as the helical transform increases. This is a consequence of the simultaneous presence of helical shear (in the $\ell = 3$ field) and a parallel ohmic heating transform. The resulting stabilization dominates even in the absence of a vacuum magnetic well.

6. In most cases of interest the maximum critical $\beta$ occurs at the intersection of the stability boundary with the equilibrium $\beta$ limit.

7. However, the critical $\beta$ for stability of a sequence of flux conserving equilibria, for which there is no equilibrium $\beta$ limit, is lower than that corresponding to (6). This follows because flux conservation appears as an additional constraint on the region of available parameter space.

In deriving these results the paper is organized as follows. Section II contains a description of the sharp boundary model and the corresponding toroidal equilibria of an arbitrary hybrid stellarator/tokamak. In Section III, the stability analysis is carried out by means of the Energy Principle. The end result is a form of $\delta W$ which is a function only of a single scalar quantity of one variable (i.e., poloidal angle). It is this quantity, proportional to the normal displacement of the plasma surface, over which the minimization is performed. Section IV presents the results. Included are a number of special cases (e.g., infinitely long straight system, finite length straight system, low $\beta$ toroidal system, and pure high $\beta$ tokamak) which help form a foundation for understanding the full high $\beta$ toroidal hybrid system.
II. Model and Equilibrium

We consider the equilibrium and stability of a toroidal stellarator/tokamak hybrid as described by the sharp boundary surface current model. In this model, all currents flow on the plasma surface and the plasma pressure profile is constant. The geometry is illustrated in Fig. 1. Here, the cylindrical coordinates \((R, \phi, Z)\) are related to the toroidal coordinates \((r, \theta, z)\) by the familiar relations

\[
\begin{align*}
R &= R_o + r \cos \theta \\
Z &= r \sin \theta \\
\phi &= -z/R_o
\end{align*}
\] (1)

As stated previously the configurations of interest are characterized by a single helical field and zero vertical field. A more general calculation will be forthcoming. For the configurations under consideration six independent dimensionless parameters are required to provide a description of the physics: inverse aspect ratio, \(\epsilon \equiv a/R_o\), plasma \(\beta \equiv 2p/B_o^2\), helical rotational transform, \(\iota_H\), ohmic rotational transform, \(\iota_I\), number of helical periods \(N\), and helical winding multipolarity, \(\ell\).

The calculation is carried out in the context of the stellarator expansion\(^{(7,8)}\)

\[
\beta \sim \epsilon \sim \frac{1}{N} \sim \delta^2 \ll 1
\] (2)

\[
\iota_I \sim \iota_H \sim \ell \sim 1
\]

Here, \(\delta\) is an additional parameter, which is not independent but which serves as a convenient expansion parameter. Specifically, it represents the helical distortion of the flux surfaces and is related to \(\iota_H\) and \(N\) as follows: \(\delta^2 \sim \iota_H/N\).

Equation (2) represents a maximal ordering with respect to \(\beta\) limits due to ballooning effects. Also, note that the scaling \(\beta \sim \epsilon\) is identical to that in a high \(\beta\) tokamak.
A. Plasma Equilibrium

Since all the currents are assumed to flow only on the plasma surface, the magnetic field in the plasma region satisfies $\nabla \times \mathbf{B} = 0$ and can thus be expressed as

$$B = B_0 \left[ \frac{R_0}{R} e_x + \frac{1}{h} \nabla \psi_1 \right]$$

with $\psi_1$ regular in the interior region and satisfying

$$\nabla^2 \psi_1 = 0 \quad n \cdot B|_{r_p} = 0$$

Here $B_0$ is the average internal toroidal field, $h = N/R_0$ is the pitch number of the helical field, and $r_p = r_p(\theta, z)$ is the plasma surface. Within the stellarator expansion and assuming a single helical field and zero vertical field, $r_p$ has the form

$$r_p(\theta, z) = a[1 + \delta \cos \alpha \ldots]$$

where $a$ is the average plasma radius and $\delta$ is the amplitude of the helical surface distortion. The quantity $\delta$ is directly related to the helical rotational transform at the plasma surface. The precise relation is given shortly. Note that in the stellarator expansion $\psi_1$ and $\delta$ are ordered as follows:

$$\psi_1 \sim \delta \sim \epsilon^{1/2}$$

This ordering gives rise to a helical transform per helical period of order $\delta^2$ and a total helical transform $\ell_H \sim N \delta^2 \sim 1$, comparable to the ohmic heating transform.

In calculating $\psi_1$ it is useful to note that to the order required for the stability analysis, it is not necessary to include the toroidal corrections on the helical fields. Thus, the solution for $\psi_1$ which is regular at the origin is given by

$$\psi_1 = AI_v(hr) \sin \alpha$$

The amplitude $A$ is related to $\delta$ by the boundary condition $n \cdot B|_{r_p} = 0$. The outward normal $n$ can easily be calculated from Eq. (5). The result is
\( n = e_r + \ell \delta \sin \alpha e_\theta + x \delta \sin \alpha e_z \ldots \)  

where \( x = h a \sim 1 \). This yields the following expression for \( A \)

\[
A = \frac{x \delta}{\vec{u}(x)}
\]

and completes the specification of the magnetic fields in the plasma. In order to carry out the stability analysis, only \( B_\theta \) and \( B_z \) are required on the plasma surface. These are summarized by writing

\[
B = B_t b
\]

with \( b \), correct to second order, given by

\[
\begin{align*}
    b_\theta &= -x \delta L \cos \alpha - x \delta^2 (\ell - L) \cos^2 \alpha \\
    b_z &= 1 - (x^2 \delta^2 L/\ell) \cos \alpha - \left[ x^2 \delta^2 \cos^2 \alpha + \epsilon \cos \theta \right]
\end{align*}
\]

Here, \( L \equiv \ell L(x)/x \ell_p \) and \( L \approx 1 \) for \( x \ll 1 \).

The final point worth noting is that the surface distortion \( \delta \) is directly related to the vacuum helical rotational transform. A straightforward calculation shows that the transform for the full torus, evaluated at the plasma surface, can be written as

\[
\frac{\dot{\iota}_{HF}}{2\pi} \equiv \iota_{HF} = \left( \frac{x \delta^2 \ell}{2\epsilon} \right) \left[ 1 - \frac{2L}{\ell} + (\ell^2 + x^2) \left( \frac{L}{\ell} \right)^2 \right]
\]

For \( x \ll 1, \dot{\iota}_{HF}/2\pi \approx (\ell - 1)x \delta^2/\epsilon \). In Eq. (12), \( \dot{\iota}_{HF} > 0 \) when \( h/\ell > 0 \). The actual transform has the opposite sign, but we will maintain this sign convention for the convenience of considering \( \dot{\iota}_{HF} > 0 \).

B. Vacuum Equilibrium

In the vacuum region it is convenient to write the fields as

\[
\dot{B} = B_0 (b + \dot{b})
\]

where \( B_0 \) is the average external toroidal field. Consistent with the assumption \( \beta \sim \epsilon \) in the stellarator expansion, the ratio \( B_t/B_0 \) must be ordered so that
The quantity $\hat{b}$ is the normalized plasma magnetic field, extended to the vacuum region and $\hat{b}$ is the yet to be determined diamagnetic response of the plasma resulting from the surface currents. Since $\hat{b}$ corresponds to a vacuum field, it follows that

$$\hat{b} = \frac{1}{h} \nabla \hat{\psi}.$$  

(15)

After some analysis, we find that the appropriate scaling for $\hat{\psi}$ is given by $\hat{\psi}(r, \theta, z) = \hat{\psi}_2(r, \theta) + \hat{\psi}_3(r, \theta, z)$. Observe that $\hat{\psi}_2$ is independent of $z$.

The calculation of the vacuum fields is simplified by noting that the stability analysis requires only $\hat{B}_0$ and $\hat{B}_z$ on the plasma surface. These quantities can be calculated just from the jump conditions across the plasma surface.

$$[n \cdot B]_{r_p} = 0$$  

(16a)

$$[p + \frac{B^2}{2}]_{r_p} = 0$$  

(16b)

Consider the first of these conditions. Since $n \cdot \hat{b}_{r_p} = 0$, Eq. (16a) implies that $n \cdot \hat{b}_{r_p} = 0$. The first non-vanishing contribution occurs in second order and requires

$$\hat{b}_{r_2}(\theta) = \frac{1}{h} \frac{\partial \hat{\psi}_2}{\partial \theta} = 0.$$  

(17)

The third order contribution expresses the normal derivative of $\hat{\psi}_3$ in terms of $\hat{b}_{r_2}(\theta) = (1/x) \partial \hat{\psi}_2(a, \theta)/\partial \theta$.

$$\frac{\partial \hat{\psi}_3(a, \theta, z)}{h \partial \theta} = \frac{\partial \theta}{\partial \theta}(\hat{b}_{r_2} \cos \alpha)$$  

(18)

Turning to the pressure balance jump condition we note that this is a nonlinear relation which must be expanded to fourth order to complete the specification of the equilibrium. In terms of $\hat{b}$ the jump condition can be expressed as

$$\beta = (1 - B_i^2/B_0^2)\hat{b}^2 + \hat{b}^2 + 2 \hat{b} \cdot \hat{b}$$  

(19)
where \( \beta = 2p/B_0^2 \sim \delta^2 \).

We now expand \( 1 - B_0^2/B_0^2 = c_2 + c_3 + c_4 \ldots \). The first non-vanishing contribution to Eq. (19) occurs in second order and yields

\[
c_2 = \beta
\]

indicating that radial pressure balance is similar to that in a \( \theta \) pinch. Upon evaluating the third order contribution we find \( c_3 = 0 \) and obtain an expression for \( \hat{\psi}_3(a, \theta, z) \) given by

\[
\hat{\psi}_3(a, \theta, z) = \left( \frac{x^2 \delta L}{\epsilon} \right) (\beta + \frac{\epsilon \delta}{x}) \sin \alpha
\]

The final equation that is required is the fourth order pressure balance relation. Actually, only the average value of this equation over one helical period, \( 0 \leq \rho z \leq 2\pi \), is required. The full equation determines \( \hat{\psi}_4(a, \theta, z) \) and averaging annihilates the explicit \( \hat{\psi}_4 \) dependence. What remains is the parallel current constraint which determines \( b_0 \). This is more conveniently expressed in terms of a new variable \( b_p(\theta) = \epsilon \epsilon_0(\theta)/\epsilon \sim 1 \).

After a lengthy calculation, we find

\[
b_p^2 - 2\eta l b_p - 4(\beta/\epsilon) [ \lambda - \sin^2(\theta/2) ] = 0
\]

where \( \eta l = \xi_0/2\pi \) and \( \lambda \) is a new normalized constant replacing \( c_4 \). Solving for \( b_p \) yields

\[
b_p(\theta) = \xi_0 \pm \left( \frac{4\beta}{\xi \xi_0^2} \right)^{1/2} \left[ 1 - \frac{\xi_0^2 \sin^2(\theta/2)}{1/2} \right]^{1/2}
\]

where \( \lambda \) itself has been redefined in terms of \( k_0^2 : k_0^2 = (4\beta/\epsilon)/(\xi_0^2 + 4\lambda \beta/\epsilon) \).

At this point, the equilibrium is effectively complete except that the constant \( k_0 \) is undetermined. As in other surface current calculations, the constant \( k_0 \) is related to the net toroidal current flowing in the plasma, \( I_p = \oint B_{pol} \cdot dl \). We now define a quantity \( \eta_0 \), proportional to \( I_p \), as follows:

\[
\frac{\xi_0}{2\pi} = \eta_0 \equiv \frac{\eta l I_p}{2\pi \xi^2 \xi_0^2} = -\frac{1}{2\pi} \int_{0}^{2\pi} b_p(\theta) d\theta
\]

(As with the helical transform we have introduced a negative sign for convenience.) Note that \( \eta_0 \sim I_p \) represents the ohmic heating transform only in the limit of low \( \beta \). The precise transform will be given shortly.

Substituting Eq. (23) into Eq. (24) leads to a relation for \( k_0 \) in terms of \( \eta_0, \eta l, \) and \( \beta \).
\[(\frac{\beta}{\epsilon})^{1/2} = (\epsilon_I + \epsilon_H) \pi \frac{k_o}{4 E(k_o)}\]  

(25)

with \(E(k_o)\) the elliptic integral. This relationship enables us to write \(b_p(\theta)\) in the following simple form

\[b_p(\theta) = \epsilon_H - (\epsilon_H + \epsilon_I) I(I)\]  

(26)

where \(I(\theta) = (1 - k_o^2 \sin^2(\theta/2))^{1/2}\) and \(I = \int_0^{2\pi} I d\theta / 2\pi = 2E(k_o)/\pi\).

Summarizing, by using the jump conditions across the plasma surface we have been able to compute \(b\) on the plasma surface to the order required for the stability analysis. If we assume \(\ell, \beta, \epsilon, x, \epsilon_I\) and \(\epsilon_H\) (or equivalently \(\delta\)) is given, these fields can be written as

\[
\begin{align*}
\delta_0 &= \epsilon b_p + \delta \left\{ xL \beta + (\ell L - 1)eb_p \right\} \cos \alpha + L \frac{db_p}{d\theta} \sin \alpha \\
\delta_z &= \left( \frac{2 \delta L}{\ell} \right) \left[ \beta + \frac{\ell \epsilon b_p}{x} \right] \cos \alpha + x^2 \delta^2 \left[ \beta + \frac{\ell \epsilon b_p}{x} \right] \cos^2 \alpha 
\end{align*}
\]  

(27)

where \(b_p(\theta)\) is given by Eq. (26) and \(k_o\) is determined transcendentally from Eq. (25). There is also an additional fourth order contribution to \(\delta_z\) proportional to \(\partial \delta / \partial z\) which is as yet undetermined. However, since this term has zero average value over one helical period, it is ultimately not required in the stability analysis and hence is omitted here.

C. Equilibrium \(\beta\) Limit

One interesting feature of the surface current model is that there is a \(\beta\) limit above which no equilibria exist. A similar situation occurs for the high \(\beta\) tokamak. The equilibrium limit follows from Eq. (25) by noting that \(b_p(\theta)\) is real only if \(k_o^2\) lies in the range \(0 < k_o^2 < 1\). Furthermore, within this range the ratio \(k_o/E(k_o)\) has its largest value when \(k_o^2 = 1\). Thus, in arbitrary hybrid stellarator/tokamaks, equilibria are possible only if

\[\frac{\beta}{\epsilon} \leq \frac{\pi^2}{16} (\epsilon_I + \epsilon_H)^2\]  

(28)

When \(\epsilon_H = 0\), this reduces to the high \(\beta\) tokamak limit.\(^{18}\)

The source of the difficulty can be understood by examining the total rotational transform just outside the plasma surface.
For $\beta/\epsilon \ll 1$ it follows that $k_0^2 \ll 1$ and $\iota \approx \iota_I + \iota_H$. However, as $k_0^2 \to 1$, $\iota \to 0$ even though the vacuum helical transform and plasma current are finite. It is the vanishing of the rotational transform on the plasma surface that corresponds to the equilibrium $\beta$ limit.

By now it is well known that the equilibrium limit in a high $\beta$ tokamak is somewhat artificial;\textsuperscript{15,16} that is, the limit occurs when the toroidal current is held fixed as $\beta$ increases, thereby requiring increasingly larger vertical fields to maintain toroidal force balance. For a sufficiently large vertical field the separatrix moves onto the plasma surface and $\iota = 0$. The $\beta$ limit is avoided by considering a sequence of flux conserving equilibria in which the rotational transform profile is held fixed rather than the current. In this case, equilibria exist even as $\beta \to \infty$ although their qualitative features change dramatically (i.e., current peaking on the outside of the torus, large toroidal shifts, etc.) when $\beta/\epsilon \gg 1$.

A similar situation prevails for the hybrid stellarator/tokamak. In fact, as a crude approximation to flux conservation, assume that the transform at the surface, $\iota$ given by Eq. (29) is held constant as $\beta$ increases. It then follows from Eq. (28) that the $\beta$ limit is given by

$$\frac{\beta}{\epsilon \iota^2} \ll \frac{\pi^2 K^2(k_0)}{16 E^2(k_0)}$$

Consequently, for fixed $\iota$ and $\epsilon$, $\beta \to \infty$ as $k_0 \to 1$ there is no equilibrium $\beta$ limit.

Keeping these considerations in mind, it is interesting to note that with regard to equilibrium $\beta$ limits, the transition from a pure stellarator to a pure high $\beta$ tokamak is a smooth one depending only on the total transform and inverse aspect ratio. Stated differently, there are no additional geometric scaling factors which imply that one configuration has more favorable equilibria than the other.

The final point worth stating is that the configurations under consideration do not possess a vacuum magnetic well. This follows because the vertical field is zero in the plasma and only a single helicity is permitted. Thus, in the context of the stellarator expansion, the vacuum magnetic well is identical to that of a straight helix, which is known to have unfavorable average curvature leading to localized interchange instabilities. However, the influence of a magnetic well on the stability of the surface current model is not clear, since $\gamma' = 0$, thus eliminating localized interchanges everywhere except perhaps on the surface.
III. Stability Analysis

We investigate the stability of the hybrid stellarator/tokamak configuration by means of the Energy Principle. The surface current model provides a reasonable description of long wavelength, low poloidal mode number instabilities. A simplifying feature of the analysis follows from the fact that in minimizing $\delta W$ the most unstable modes are incompressible: $\nabla \cdot \xi = 0$ with $\xi$ the plasma displacement. In particular, the determination of $\delta W$ ultimately requires a minimization, only with respect to a single scalar quantity, $n \cdot \xi$, and this evaluated only on the plasma surface.

For the incompressible surface current model, the potential energy of the system is given by the familiar plasma, surface and vacuum contributions.

$$\delta W = \delta W_p + \delta W_s + \delta W_v$$

(31)

where

$$\delta W_p = \frac{1}{2} \int_P |B_1|^2 d\rho$$

$$\delta W_s = \frac{1}{2} \int_S |\xi|^2 n \cdot [\nabla (p + B^2/2)] dS$$

(32)

$$\delta W_v = \frac{1}{2} \int_v |\hat{B}_1|^2 d\rho$$

Here $B_1$ and $\hat{B}_1$ are the perturbed magnetic fields in the plasma and vacuum, respectively, $\xi(\theta, z) \equiv n \cdot \xi|_p$ is the normal component of plasma displacement evaluated on the surface and the notation $[A]$ denotes the jump in $A$ from vacuum to plasma.

D. The Perturbation

The first step in the analysis is the specification of the perturbation $\xi$. The most general form of $\xi$ can be quite complicated in an arbitrary three-dimensional geometry. However, if we restrict attention to long wavelength modes and make use of the stellarator expansion, then the most general form of $\xi$ reduces to

$$\xi(\theta, z) = \{ \xi_0(\theta) + \delta [\xi_0'(\theta) \cos \alpha + \xi_0(\theta) \sin \alpha] + \delta^2 \xi_0(\theta, z) \} e^{ikz}$$

(33)

Here, $k \equiv n/R$, where $n$ is the toroidal wavenumber of the perturbation and "long wavelength" implies $ka = \epsilon n \sim \delta^2$. In fact, it is the large difference in length scales between the equilibrium helical period, $ha \sim 1$, and
the perturbation wavelength, \( \kappa a \sim \delta^2 \) that permits the simple decomposition given in Eq. (33). The quantities \( \xi_s(\theta), \xi_c(\theta), \xi_s(\theta) \) and \( \xi_2(\theta, z) \) are each of order unity and it is these functions which must be varied to minimize \( \delta W \). Eventually, \( \xi_s, \xi_c \) and \( \xi_2 \) are expressed analytically in terms of \( \xi_o \) so that the final minimization requires the variation of only a single scalar quantity of one variable.

Physically, the quantity \( \xi_o \) represents the basic "flute-like" contribution to the perturbation and \( \xi_s \) and \( \xi_c \) represent helical sideband distortions resulting from the stellarator fields. Because of the toroidal coupling each of the amplitudes \( \xi_o, \xi_c \) and \( \xi_s \) are functions of \( \theta \).

F. Surface Energy

The second step in the analysis is the evaluation of \( \delta W_s \), which is the only term that can give rise to instability. A straightforward calculation shows that the surface element \( ndS \) can be expressed as

\[
ndS = n_o(r_p R/hR_o)d\theta d\alpha \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \alpha \leq 2\pi N
\]

where

\[
n = n_o/|n_o| \quad \text{and} \quad n_o = e_r - \frac{1}{r_p} \frac{\partial_r r_p \partial_\theta}{r_p} e_\theta - \frac{R_o \partial_r r_p}{R} e_z
\]

This formulas is valid for an arbitrary, unexpanded, three-dimensional surface, \( r = r_p(\theta, z) \).

In the next step, we note that \( \nabla \times B = 0 \) and \( n \cdot B|_{r_p} = 0 \) in both the plasma and vacuum region. Consequently, we can write \( n \cdot \nabla(p + B^2/2) = -B \cdot (B \cdot \nabla)n \). After a lengthy calculation, this term can be evaluated and substituted into \( \delta W_s \). The result is

\[
\delta W_s = -\frac{1}{2} \int |\xi|^2 \left( \frac{r_p R}{hR_o} \right) d\theta d\alpha \left\{ \frac{\|D_0^2\|}{r_p} \left[ 1 - r_p \frac{\partial}{\partial \alpha} \left( \frac{1}{r_p^2} \frac{\partial r_p}{\partial \theta} \right) \right] \right. \\
- 2\|D_0 B_2\| \left[ \frac{\partial}{\partial \theta} \left( \frac{R_o \partial r_p}{r_p R} \right) \right] \\
- \left. \|D_0^2\| \left[ \frac{\partial}{\partial z} \left( \frac{R_o^2 \partial r_p}{R^2} \right) - \frac{1}{r_p R \partial \theta} (r_p \sin \theta) \right] \right\}
\]

Equation (36) is also valid for arbitrary \( r_p(\theta, z) \).
We now substitute the expanded form of the equilibrium and the perturbation into the expression for \( \delta W_s \). The first non-vanishing term is of order \( \delta^4 \) and can be written as

\[
\frac{\delta W_s}{2\pi R_o} = -\frac{B_o^2}{2} \int_0^{2\pi} d\theta (f_1 \xi_0^2 + f_2 \xi_0 \xi_0)
\]

\[
f_1(\theta) = \varepsilon\beta \cos \theta + \frac{1}{2} \delta^2 \beta [x^2(1 - L^2)] + \varepsilon^2 b_p^2 + \varepsilon^3 b_p^2 [x(2L - \ell - \ell L^2)]
\]

\[
f_2(\theta) = x^2 \delta^2 \beta + 2 \varepsilon^3 b_p^2 [x(\ell - L)]
\]

Note that at this point in the calculation the relation between \( \xi_0 \) and \( \xi_e \) is as yet undetermined.

F. Plasma and Vacuum Energies

We consider next the plasma and vacuum energies. The perturbations which minimize \( \delta W_P \) and \( \delta W_S \), subject to the constraints \( \nabla \cdot B_1 = \nabla \cdot \hat{B}_1 = 0 \) must satisfy \( \nabla \times B_1 = \nabla \times \hat{B}_1 = 0 \). The minimizing magnetic fields are such that all of the perturbed currents flow on the plasma surface. As a result, we can write.

\[
B_1 = \nabla V, \quad \hat{B}_1 = \nabla \hat{V}
\]

with \( V \) and \( \hat{V} \) satisfying

\[
\nabla^2 V = 0, \quad \nabla^2 \hat{V} = 0
\]

We require \( V \) regular at the origin and \( \hat{V} \) regular at infinity (for no conducting wall). Under these conditions, the plasma and vacuum terms can be converted to surface integrals in the usual way

\[
\delta W_P = \frac{1}{2} \int \left( \frac{r_p R}{\hbar R_o} \right) (V \cdot n \cdot \nabla V) d\theta d\alpha \quad (40)
\]

\[
\delta W_V = -\frac{1}{2} \int \left( \frac{r_p R}{\hbar R_o} \right) (\hat{V} \cdot n \cdot \nabla \hat{V}) d\theta d\alpha
\]

The problem now is to express \( V, \hat{V} \) and \( n \cdot \nabla V, n \cdot \nabla \hat{V} \) in terms of \( \xi(\theta, z) \). This is accomplished in two steps. First \( n \cdot \nabla V, n \cdot \nabla \hat{V} \) are related to \( \xi \) by means of the linearized contributions to the boundary conditions \( n \cdot \hat{B}_{r_p} = n \cdot \hat{B}_{r_p} = 0 \). Second, \( V, \hat{V} \) are then expressed in terms of \( n \cdot \nabla V, n \cdot \nabla \hat{V} \) through the solution to Laplace's equation.
G. Boundary Conditions

The boundary conditions on the plasma surface are given by

\[ n \cdot \nabla V |_{r_p} = n \cdot \nabla \times (\xi \times B) |_{r_p} \]
\[ n \cdot \nabla \hat{V} |_{r_p} = n \cdot \nabla \times (\xi \times \hat{B}) |_{r_p} \]  

and can be expressed solely in terms of \( \xi(\theta, z) \) as follows:

\[ n \cdot \nabla V |_{r_p} = \left[ B \cdot \nabla \xi - \xi n \cdot (n \cdot \nabla) B \right] |_{r_p} \]
\[ n \cdot \nabla \hat{V} |_{r_p} = \left[ \hat{B} \cdot \nabla \xi - \xi n \cdot (n \cdot \nabla) \hat{B} \right] |_{r_p} \]  

After a lengthy calculation Eq. (42) can be further reduced by writing it as an explicit function only of surface quantities. We find

\[ n \cdot \nabla V |_{r_p} = \frac{1}{r_p R Q} \left[ \frac{D}{D\theta} (R Q B \xi) + R_o \frac{D}{Dz} (r_p Q B z \xi) \right] \]
\[ n \cdot \nabla \hat{V} |_{r_p} = \frac{1}{r_p R Q} \left[ \frac{D}{D\theta} (R Q \hat{B} \xi) + R_o \frac{D}{Dz} (r_p Q \hat{B} z \xi) \right] \]  

where

\[ Q \equiv |n_s| = \left[ 1 + \left( \frac{1}{r_p} \frac{\partial r_p}{\partial \theta} \right)^2 + \left( \frac{R_o}{R} \frac{\partial r_p}{\partial z} \right)^2 \right]^{1/2} \]  

and

\[ \frac{D}{D\theta} = \frac{\partial}{\partial \theta} + \frac{\partial r_p}{\partial \theta} \frac{\partial}{\partial r} \]
\[ \frac{D}{Dz} = \frac{\partial}{\partial z} + \frac{\partial r_p}{\partial z} \frac{\partial}{\partial r} \]  

are surface derivatives at fixed \( z \) and fixed \( \theta \), respectively. In its present form Eq. (43) is valid for arbitrary \( r_p(\theta, z) \).

The next step is to substitute the stellarator expansion into Eq. (43). The first non-vanishing contribution is of order \( \delta \). Since \( \delta W_p \) and \( \delta W_V \) are quadratic in \( V \) and \( \hat{V} \), this gives rise to stabilizing terms \( \delta W_p \sim \delta W_V \sim \delta^2 \). However, the potentially destabilizing term \( \delta W_\xi \) is of order \( \delta^4 \). Thus, instability can occur only if the leading order contribution to \( n \cdot \nabla V \) and \( n \cdot \nabla \hat{V} \) can be made to vanish simultaneously.
This is accomplished as follows. Within the stellarator expansion, the order \( \delta \) contributions to \( n \cdot \nabla V \) and \( n \cdot \nabla \hat{V} \) are identical and are of the form

\[
n \cdot \nabla V|_{r_p} = n \cdot \nabla \hat{V}|_{r_p} = [A_c \cos \alpha + A_s \sin \alpha] e^{ikz} \tag{46}
\]

where the coefficients \( A_c \) and \( A_s \) are functions of \( \xi_c, \xi_s, \) and \( \xi \). Consequently, setting \( A_c = A_s = 0 \) gives a relationship for the minimizing perturbations \( \xi_c \) and \( \xi_s \) in terms of \( \xi \). The result is

\[
\xi_c = (LL - 1 + \frac{1}{2}L/L)e \xi \tag{47}
\]

\[
\xi_s = L \frac{d\xi}{d\theta}
\]

We must now calculate the order \( \delta^2 \) contributions to \( n \cdot \nabla V \) and \( n \cdot \nabla \hat{V} \) in order to evaluate \( \delta W_F \) and \( \delta W_V \). A straightforward calculation yields expressions of the form

\[
n \cdot \nabla V|_{r_p} = B [A_o + A_1 \cos \alpha + A_2 \sin \alpha + A_3 \cos 2\alpha + A_4 \sin 2\alpha + \frac{\partial \xi_2}{\partial z}] e^{ikz} \tag{48a}
\]

\[
n \cdot \nabla \hat{V}|_{r_p} = n \cdot \nabla V|_{r_p} + \frac{\epsilon B_o}{A} \frac{d}{d\theta} (b_p \xi) e^{ikz} \tag{48b}
\]

where the \( A_j \) are functions of \( \xi_c, \xi_c, \xi_s, \) and \( \theta \). Note that this is the only place where \( \xi_2 \) explicitly appears in the calculation.

The boundary condition can be viewed as a Fourier series in \( \alpha \) containing zeroth, first and second harmonics. Since the harmonics are orthogonal and each harmonic gives rise to a separate stabilizing contribution in \( \delta W_F \) and \( \delta W_V \), the minimizing perturbation is one in which \( \partial \xi_2 / \partial z \) is chosen to cancel as many harmonics as possible. \( \partial \xi_2 / \partial z \) can in fact cancel all harmonics except the zeroth which would give rise to a non-periodic (in \( z \)) \( \xi_2 \). With this choice we need only maintain the \( A_o \) contribution in Eq. (48a). This leads to the following expression for \( n \cdot \nabla V \) and \( n \cdot \nabla \hat{V} \)

\[
n \cdot \nabla V|_{r_p} = \frac{\epsilon B_o}{a} \left[ \xi_3 - (d\xi_3/d\theta) e^{ikz} \right] \tag{49}
\]

\[
n \cdot \nabla \hat{V}|_{r_p} = \frac{\epsilon B_o}{a} \left[ (\xi_3 + (db_p/d\theta)) \xi_0 - (u_{II} - b_p)(d\xi_0/d\theta) e^{ikz} \right]
\]

where \( n = kR_o \sim 1 \) is the toroidal wavenumber.

At this point it is convenient to Fourier analyze the perturbations with respect to \( \theta \)
\[ \xi_0 = \sum_{m} \xi_m e^{im\theta} \]
\[ n \cdot \nabla V|_{r_p} = \frac{ieB_0 e^{ikz}}{a} \sum_{m} a_m e^{im\theta} \tag{50} \]
\[ n \cdot \nabla \hat{V}|_{r_p} = \frac{ieB_0 e^{ikz}}{a} \sum_{m} \hat{a}_m e^{im\theta} \]

Here, the \( \hat{\xi}_m \) are unspecified coefficients over which the final minimization is to be performed. Note that the \( m = 0 \) coefficients are absent, a consequence of the incompressibility condition and long wavelength assumption.

Equation (49) relates the field amplitude coefficients \( a_m \) and \( \hat{a}_m \) to the \( \hat{\xi}_m \) as follows:

\[ a = G \cdot \xi, \quad \hat{a} = \hat{G} \cdot \hat{\xi} \tag{51} \]

where the matrices \( G \) and \( \hat{G} \) are given by

\[ G_{\xi m} = [n - \omega I] \delta_{\xi - m} \]
\[ \hat{G}_{\xi m} = G_{\xi m} + \ell \hat{G}_{\xi m} \tag{52a} \]

and

\[ \hat{G}_{\xi m} = \frac{1}{\pi} \int_0^{\pi} b_p(\theta) \cos((\ell - m)\theta) d\theta \tag{52b} \]

The final step in the evaluation of \( \delta W_{\xi} \) and \( \delta W_{\hat{V}} \) is to relate \( V, \hat{V} \) to \( n \cdot \nabla V, n \cdot \nabla \hat{V} \) on the plasma surface through the solution of Laplace's equation. Since the non-zero contribution in \( n \cdot \nabla V|_{r_p} \) and \( n \cdot \nabla \hat{V}|_{r_p} \) correspond only to the zeroth harmonic in \( a \), the solution for \( V \) and \( \hat{V} \) in the long wavelength limit can be written as

\[ V = i eB_0 e^{ikz} \sum_{m} c_m(\frac{r}{a})^{[m]} e^{im\theta} \tag{53} \]
\[ \hat{V} = i eB_0 e^{ikz} \sum_{m} \hat{c}_m(\frac{r}{a})^{[m]} e^{im\theta} \]

We now note that to the order required \( n \cdot \nabla V|_{r_p} \approx \partial V/\partial r|_a \) and \( n \cdot \nabla \hat{V}|_{r_p} \approx \partial \hat{V}/\partial r|_a \). Thus, differentiating Eq. (53) and equating the result with Eq. (50) yields

\[ c = D \cdot a = (D \cdot G) \cdot \hat{\xi} \tag{54} \]
\[ \hat{c} = -D \cdot \hat{a} = -(D \cdot \hat{G}) \cdot \hat{\xi} \]
where $D$ is a diagonal matrix whose elements are given by

$$D_{m\ell} = \frac{1}{|\ell|} \delta_{m-\ell}$$

(55)

This essentially completes the determination of the perturbed magnetic fields in terms of the minimizing coefficients, $\xi_m$.

H. Final Form of $\delta W$

In this section we combine the previous results and obtain a final form for $\delta W$ expressed solely in terms of the minimizing coefficients, $\xi_m$.

The surface energy is completed by substituting $\xi_c$ from Eq. (47) into $\delta W_S$ given by Eq. (37). The resulting expression which is only a function of $\xi_c(\theta)$ is then written in terms of $\hat{\xi}_m$ by Eq. (50).

The plasma energy is evaluated by substituting $n \cdot \nabla V|_{r_p}$ from Eq. (50) and $V|_{r_p}$ from Eq. (53) into $\delta W_P$, given by Eq. (40). The coefficients $a_m$ and $c_m$ which then appear are expressed in terms of $\hat{\xi}_m$ by Eq. (51) and (54). A similar calculation follows for the vacuum energy.

The last step is to introduce an appropriate normalization for the energy. A convenient choice is

$$K = \frac{\pi \rho_o a^2 (\xi^* \cdot \hat{\xi})}{2\pi R_0}$$

(56)

where $\rho_o$ is the mass density.

Upon carrying out these operations we obtain an energy principle

$$\omega^2 = \frac{\delta W}{K} = \frac{v_o^2 (\xi \cdot W \cdot \hat{\xi})}{R_0^2 (\xi^* \cdot \hat{\xi})}$$

(57)

where $v_o = B_0 / \rho_o^{1/2}$ is the Alfven speed and the energy matrix $W$ is given by

$$W = -H + G^T \cdot D \cdot G + \hat{G}^T \cdot D \cdot \hat{G}$$

(58)

In this equation $H$ is the matrix associated with the surface energy and can be written as
\[ H_{tm} = \beta \left[ \frac{2(2 - k_0^2)}{k_0^2} \delta_{t-m} + \frac{3}{2} (\delta_{t-m-1} + \delta_{m-t-1}) + K_2 \nu_{\parallel} \delta_{t-m} \right] - \nu_{\parallel} [\nu_{\parallel} \delta_{t-m} - 2(1 + K_1) \bar{G}_{tm}] \]  

(59)

where the coefficients \( K_1 \) and \( K_2 \) are given by

\[
K_1 = \frac{2L \ell^3 - 3(L^2 + 1) \ell^2 + 2L(2 + x^2-1)\ell - 2\ell x^2 - L}{(L^2 + 1)\ell^2 - 2L\ell + \ell^2}
\]

\[
K_2 = \frac{2L \ell^2 - (L^2 + 1) \ell + 2\ell x^2}{(L^2 + 1)\ell - 2L\ell + \ell^2}
\]

When \( z \ll 1 \) these complicated expressions reduce to

\[
K_1 = \begin{cases} 
5x^2/16 & \ell = 1 \\
x^2/2 & \ell = 2 \\
\ell - 2 & \ell \geq 3
\end{cases}
\]

(60)

It is useful to note that the coefficient \( K_2 \) is directly related to the condition for the existence of a vacuum magnetic well. In particular, if we define the magnetic well quantity as follows

\[
U = \frac{B_0}{2\pi R_0} \int \frac{d\ell}{B}
\]

then a straightforward calculation shows that

\[
K_2 = \frac{N}{2\nu_{\parallel}} \left( \frac{1}{x} \frac{dU}{dx} \right)
\]

Similarly, it can be shown that the coefficient \( K_1 \) is directly proportional to the shear in the vacuum helical field

\[
K_1 = \frac{1}{2} \left( \frac{x}{\nu_{\parallel}} \frac{d\nu_{\parallel}}{dx} \right)
\]

An important property of \( W \) is that it is real and symmetric. Hence, the minimization of \( \delta W \) is equivalent to examining the sign of the lowest eigenvalue of \( W \), denoted by \( \lambda_{\min} \). If \( \lambda_{\min} < 0 \) the system is unstable with an approximate growth rate \( \gamma \approx (\nu_{\parallel}/R_0)(-\lambda_{\min})^{1/2} \). If \( \lambda_{\min} > 0 \), the system is stable. The critical condition for marginal stability occurs when \( \lambda_{\min} = 0 \).
The eigenvalues of $W$ are found numerically using standard techniques. In these calculations $\ell$, $\ell_f$, $\ell_H$, $x$, $k_0$, and $n$ are specified. We then compute $\beta/e$ and $b_p$ which in turn enables us to calculate the elements of $W$. The infinite matrix $W$ is truncated symmetrically about the $m$-th harmonic where $m$ is the predominant poloidal mode number [i.e., $m$ is chosen so that $n - m(\ell_f + \ell_H) \approx 0$. Numerical tests indicate that 12 poloidal harmonics and 100 grid points over the range $0 < \theta \leq \pi/2$ are sufficient to obtain three figure accuracy in all cases of interest.

The numerical values of critical $\beta$ presented in the next section were obtained by carrying out the procedure described above and then iterating on $\beta$ (i.e., $k_0$) until the lowest eigenvalue $\lambda_{\text{min}} = 0$. 
IV. Results

The configurations under consideration are rather complicated in that they are characterized by a large number of dimensionless parameters. As a consequence it is useful to present the results as a series of simple analytic limits to help form our intuition and provide a basis for understanding the full system. Hence, the results are presented in the following sequence: (1) infinitely long straight hybrid configuration; (b) finite length straight hybrid system; (c) low $\beta$ toroidal hybrid system; (d) pure high $\beta$ tokamak; and (e) full high $\beta$ toroidal hybrid configuration.

I. Infinitely Long Straight Hybrid Configuration

The first case of interest is the infinitely long straight hybrid system. In this configuration we assume that the major radius of the torus becomes infinite (i.e., $e \to 0$). Since the total transform then also becomes infinite it is necessary to introduce the transform per helical period: $u_h \equiv \int_{i}^{i+1}/hR_o$, and $u_i = \int_{i}^{i+1}/hR_o$ where $hR_o = N$ is the number of helical periods. Similarly, in an infinite system, $n = kR_o$ where $k$ is not quantized, but is a continuously variable wave number.

The corresponding asymptotic ordering is given by

$$\beta \sim u_h \sim \frac{u_i}{h} \sim \frac{k}{h} \ll 1$$
$$\frac{k^2}{\varepsilon} \sim \frac{e}{u_h} \to 0$$

In this limit the matrix $W$ is diagonal; that is, all the poloidal mode numbers, $m$, are decoupled. Specifically, the matrices $G$, $\hat{G}$, and $H$ reduce to

$$G_{\ell m} = hR_o[k_h/h - mu_h] \delta_{\ell-m}$$
$$\hat{G}_{\ell m} = hR_o[k_h/h - m(u_i + u_h)] \delta_{\ell-m}$$
$$H_{\ell m} = h^2R_o^2[(u_i + u_h)^2 + K_2\beta u_h - u_h^2 - 2(1 + K_1)\varphi_{\phi_h}] \delta_{\ell-m}$$

and give rise to the dispersion relation

$$\omega^2 = h^2\nu_a^2 \left\{ -\nu_i^2 + 2K_1\varphi_{\phi_h} - K_2\beta \varphi_h \right\}$$
$$+ \frac{1}{|m|} \left[ \frac{k}{h} - m(u_h - u_i) \right]^2 + \frac{1}{|m|} \left[ \frac{k}{h} - mu_h \right]^2$$
1. Pure Tokamak, $\psi_h = 0$

The case $\psi_h = 0$ corresponds to an infinitely long straight tokamak. The most unstable wavenumber is 
$k/h = (m/2)\psi_t$ and the resulting dispersion relation has the form

$$\omega^2 = h^2v_a^2\left(\frac{|m|}{2} - 1\right)\psi_t^2$$  \hspace{1cm} (64)

Equation (64) agrees with early sharp boundary calculations$^{(19)}$ and indicates that the system is unstable for $m = 1$ and stable for $m \geq 2$. This result is not surprising in that one cannot make $q = 1/\psi_t > 1$ in an infinitely long system with no toroidal periodicity constraints.

2. Pure Stellarator, $\psi_t = 0$

For the case of a pure stellarator, $\psi_t = 0$, the dispersion relation reduces to

$$\omega^2 = h^2v_a^2[-K_2\beta_{th} + \frac{2}{|m|}(\frac{k}{h} - m\psi_t)^2]$$  \hspace{1cm} (65)

Thus, setting $k/h = m\psi_t$ gives rise to a pressure-driven instability for any $m$. These modes are the macroscopic analogs (i.e., zero radial nodes) of localized interchange perturbations and are driven by the average unfavorable curvature of the magnetic field (i.e., a straight helical system with a straight magnetic axis does not possess a magnetic well, $K_2 > 0$). Note that the localized interchanges themselves do not occur in the sharp boundary model since $\psi' = 0$.

3. Hybrid Stellarator/Tokamak

For the general hybrid configuration, the most unstable wavenumber is given by

$$\frac{k}{h} = \frac{m}{2}(\psi_t + 2\psi_h)$$  \hspace{1cm} (66)

and leads to the following dispersion relation

$$\omega^2 = h^2v_a^2\left[\left(\frac{|m|}{2} - 1\right)\psi_t^2 + 2K_1\psi_t\psi_h - K_2\beta_{th}\right]$$  \hspace{1cm} (67)
As in the pure tokamak, the most unstable case is \( m = 1 \). We assume now that the configuration is a tokamak with given \( t_i \) and examine the influence of adding helical fields. Equation (67) indicates that the hybrid system can be stabilized for values of \( \beta \) satisfying

\[
\frac{\beta}{|t_i|} < \frac{2}{K_2} \left[ K_1 - \frac{1}{4} \frac{t_i}{|t_i|} \right]
\]

which is illustrated in Fig. 2.

For \( t_i \) and \( t_h \) anti-parallel the system is always unstable. However, for a sufficiently large parallel transform

\[
t_h > \frac{t_i}{4K_1}
\]

the system is stable for positive, non-zero values of \( \beta \). In the limit of very large helical fields, \( \beta \) approaches an asymptote

\[
\frac{\beta}{|t_i|} \rightarrow \frac{2K_1}{K_2}
\]

Summarizing, we note that in an infinitely long straight system, both the pure tokamak and the pure stellarator are unstable. However, if a sufficiently large parallel helical transform is superimposed on the ohmic transform the resulting hybrid system can then be stabilized below some critical value of \( \beta \). In the sharp boundary hybrid model the most dangerous mode corresponds to \( m = 1 \). It is driven by both the pressure gradient and the parallel current and is stabilized by the interaction of the shear in the helical field with the ohmic heating field.

J. Finite Length Straight Hybrid Configurations

In the second case of interest, we approximate the full toroidal system by a finite section of the infinite straight hybrid configuration just described. The corresponding dispersion relation is identical to that given by Eq. (63) with the additional requirement that the modes be periodic over the length of the system. Thus, if we denote the equivalent length of the torus by \( 2\pi R_s \) then we must choose \( k = n/R_s \) with \( n \) an integer. Likewise, we re-introduce the full transform \( \omega_{II} = hR_s \beta_{II}, t_i = hR_s t_i, N = hR_s \). The dispersion relation can now be written as

\[
\omega_0^2 = -\frac{t_i^2}{2} + 2K_1 t_i \omega_{II} - K_2 N \beta_{II}
\]

\[
+ \frac{1}{|m|} \left[ n - m(t_i + \omega_{II}) \right]^2 + \left[ \frac{1}{|m|} \left[ n - m \omega_{II} \right] \right]^2
\]
where \( \omega_0^2 = \omega_0^2 R_0^2 / \nu_0^2 \).

Since the finite length system has additional toroidal periodicity constraints it must be more stable than the infinite length case.

1. Pure Tokamak, \( \phi_{II} = 0 \)

The special case \( \phi_{II} = 0 \) represents a finite length straight tokamak whose dispersion relation reduces to

\[
\omega_0^2 = (m - 1)\nu_1^2 - 2n\nu_1 + 2n^2 / m \tag{72}
\]

Here, without loss in generality we consider \( m \) and \( \nu_1 \) positive but allow \(-\infty < n < \infty\). This expression agrees with early sharp boundary calculations (19) and can be rewritten in the following convenient form

\[
\omega_0^2 = \begin{cases} 
2n(n - \nu_1) & m = 1 \\
(m - 1)\left[\nu_1 - n/(m - 1)\right]^2 + (m - 2)n^2 / m(m - 1)^2 & m \geq 2
\end{cases}
\]

We see that \( m \geq 2 \) modes are stable for any \( n \). Thus, \( m = 1 \) is the most unstable case. For the \( m = 1, n = 0 \) rigid shift the system is neutrally stable. When \( n \neq 0 \) the most dangerous case is \( m = 1, n = 1 \) and requires

\[
\nu_1 < 1 \tag{74}
\]

for stability. This is the well-known Kruskal-Shafranov limit. For a finite length straight system, there is a current limit but no \( \beta \) limit for any \( \beta \) within the ordering when Eq. (74) is satisfied.

2. Pure Stellarator, \( \phi_{II} = 0 \)

Setting \( \phi_{II} = 0 \) yields the following dispersion relation for the pure stellarator

\[
\omega_0^2 = -K_0N\beta_{II} + \left( \frac{2}{m} \left| n - m\phi_{II} \right| \right)^2 \tag{75}
\]

Even in a finite length system \( n - m\phi_{II} \) can always be made arbitrarily small by appropriate choices of \( m \) and \( n \). Consequently, the straight stellarator is always unstable. We note again that without a vertical field and/or helical sideband fields the average curvature of a straight helix is always unfavorable, leading to pressure-driven instabilities.
3. Hybrid Stellarator/Tokamak

The results for the full hybrid system are somewhat complicated because the $m$ and $n$ values describing the most unstable mode change as $\iota_T$ and $\iota_H$ vary. Even so, it is still possible to ascertain the general features of the stability behavior as follows.

For systems with $\iota_T < 1$ the optimum straight system is a pure tokamak. In such cases setting $\iota_H = 0$ eliminates the pressure-driven term and leads to a configuration which is stable to all $\beta$'s within the ordering.

Helical fields can, however, help to stabilize tokamaks with $\iota_T > 1$ and $\ell \geq 3$. In the regime of interest the most unstable modes are usually $m = 1, n = 0, 1, 2$. For $m = 1, n = 0$, the dispersion relation reduces to

$$\omega^2 = \iota_H [2\iota_H + 2(1 + K_1)\iota - K_2 N \beta]$$  \hspace{1cm} (76)

We see that for small helical transforms anti-parallel to the ohmic transform (i.e., $-\iota_H/\iota_T < 1 + K_1$) the system is unstable to the $m = 1, n = 0$ rigid shift perturbation. Hence, if the helical fields are to improve stability, the corresponding transform should be parallel to the ohmic transform.

We now assume $\iota_H/\iota_T > 0$ and consider $m = 1, n \neq 0$ modes. The resulting dispersion relation gives rise to a critical $\beta$ for stability which can be written as

$$N \beta \leq \frac{2}{K_2} \left[ \frac{\iota_H - n \iota}{\iota_H} + (1 + K_1)\iota - 2n \right]$$  \hspace{1cm} (77)

Equation (77) describes a series of stability curves which are a function of $n$. A typical case is illustrated in Fig. 3 for $\ell = 3, \iota_T = 1.5, x \ll 1$ and $n = 1, 2$, the most dangerous modes. Also illustrated is the equivalent infinitely long result given by Eq. (68) which predicts lower $\beta$ limits as expected.

We see that there is a critical helical transform associated with the $m = 1, n = 1$ mode above which the system is stable for positive $\beta$, even though $\iota_H > 1$. This value of $\iota_H$ is given by

$$\iota_H = 1 - \frac{1}{2}(1 + K_1)\iota_T + \frac{1}{2}[(1 + K_1)^2 - 4K_1\iota_T]^{1/2}$$  \hspace{1cm} (78)

For $x \ll 1$ we can use the simplified form for $K_1$ [i.e., Eq. (60)] in which case

$$\iota_H \approx \begin{cases} 
1 & \text{for } \ell = 2 \\
1 - \iota_T + [\iota_T(\ell - 1)]^{1/2} & \text{for } \ell = 3 
\end{cases}$$  \hspace{1cm} (79)

Equation (79) shows that as a practical matter this stabilization is difficult to achieve for $\ell = 2$ as a large helical transform is required. For $\ell \geq 3$ small to moderate transform is sufficient in many cases of interest.
In analogy with the infinitely long system helical fields can have a positive stabilizing influence on the operation of tokamaks with high current, $q_I > 1$. In general the helical and ohmic transforms must be parallel to avoid $n = 0$, $m = 1$ modes. For sufficiently large $\psi_H > 0$ the system can be stabilized to all modes below some critical value of $\beta$ even though $q_I > 1$. In practice this stabilization is effective for $\ell \geq 3$, the $\ell = 2$ system requiring large helical transforms, $\psi_H \geq 1$.

Despite this stabilization, within the context of the straight, finite length, sharp boundary model the pure tokamak with $q_I < 1$ is the optimum configuration with respect to $\beta$ limits. In this regime, any helical field will lower the critical $\beta$.

K. Low $\beta$ Toroidal Hybrid System

In the third special case we consider the full toroidal system in the limit where $\beta \to 0$. Specifically we assume

$$\psi_l \sim \psi_H \sim \ell \sim 1, \quad \epsilon \sim 1/N \sim \delta^2 \ll 1 \quad (80)$$

and take the limit

$$\beta/\epsilon \sim k_0^2 \to 0 \quad (81)$$

The resulting $W$ matrix is diagonal and the corresponding dispersion relation is identical to Eq. (71) except that the $N\beta$ term is eliminated:

$$\omega^2 = -\epsilon^2 + 2K_1\psi_H + \frac{1}{|m|}[n - m(\psi_l + \psi_H)]^2 + \frac{1}{|m|}[n - m\psi_H]^2 \quad (82)$$

Equation (82) implies that the critical rotational transforms for marginal stability in the limit $\beta \to 0$ are identical in the toroidal and finite length straight systems. However, although the $\beta \neq 0$ effects are in general destabilizing, as indicated in the straight system, they are qualitatively different and even more destabilizing in the toroidal system because of ballooning effects.

In this section we focus attention on Eq. (82) and determine the low $\beta$ marginal stability boundaries as a function of $\psi_l$ and $\psi_H$. As before, without loss in generality we can assume $m > 0$ and $\psi_H > 0$. In the regime of interest the dominant unstable modes correspond to $m = 1$, low $n$. For this case the condition for stability reduces to
\[ \ell_I \geq \frac{(n - \ell_H)^2}{n - (1 + K_1)\ell_H}, \quad n - (1 + K_1)\ell_H \geq 0 \]  

(83)

Equation (83) is illustrated in Fig. 4 for \( \ell = 2 \) and \( \ell = 3 \) stellarator fields. The following points should be noted.

For both \( \ell = 2 \) and \( \ell = 3 \) when the helical and ohmic transforms are anti-parallel, the \( m = 1, n = 0 \) rigid shift perturbation is unstable unless the helical transform is substantial:

\[ \ell_H > -(1 + K_1)\ell_I \]  

(84)

From a practical point of view this essentially restricts the transforms to be parallel and also corresponds to the desired direction for the suppression of tearing modes.

A second point concerning anti-parallel transforms is related to \( m \geq 2 \) modes. These modes become unstable if \( \ell_H \) greatly exceeds the value given by Eq. (84) (i.e., the configuration becomes too much like a pure stellarator). This can be seen by assuming \( \ell_H = n/m + \delta_{II} \) and \( \ell_I = \delta_{II} \) with \( \delta_{II}, \delta_{I} \ll 1 \). It then follows that the system is unstable whenever

\[ \delta_{II} \leq -(m^2/nK_1)(\ell_H)^2 \]  

(85)

The envelope of these instabilities can easily be estimated by setting \( \ell_H = n/m \) and then calculating the range of unstable \( \ell_I \)'s. We find instability when

\[ -\frac{2K_1n}{m(m - 1)} \leq \ell_I \leq 0 \]  

(86)

The boundary \( \ell_I \leq 0 \) is also plotted in Fig. 4 and corresponds to the region labelled "high \( m \) instabilities".

Consider now the situation where the transforms are parallel. The addition of an \( \ell = 2 \) field has a destabilizing effect in that as the helical transform is increased the critical ohmic transform for stability (i.e., the Kruskal current) decreases. This trend continues up to relatively large values of \( \ell_H \approx 1 \). This unfavorable behavior is not unexpected since a single \( \ell = 2 \) field has neither a magnetic well nor shear (when \( x \ll 1 \)) to suppress pressure driven interchange instabilities.

On the other hand, the addition of a parallel \( \ell \geq 3 \) helical transform has a positive stabilizing effect. As the helical transform increases the critical ohmic transform for stability also increases. This is a consequence of the non-zero shear in \( \ell \geq 3 \) fields, even when \( x \ll 1 \).
In summary, a parallel $\ell \geq 3$ field improves the macroscopic stability of a tokamak at low $\beta$ and allows operation at higher ohmic currents. A single $\ell = 2$ field, however, decreases stability. This pessimistic conclusion may be modified when a vertical field and/or helical sidebands are allowed to create a magnetic well and will be discussed in a future paper. For both $\ell = 2$ and $\ell = 3$ the most dangerous modes in the parameter regimes of interest correspond to $m = 1$, low $n$.

L. High $\beta$ Tokamak

The case of the high $\beta$ tokamak self-consistently treats $\beta/\epsilon \sim 1$ and includes the effects of toroidal ballooning but assumes no helical fields are present. The corresponding dispersion relation is obtained by setting $\iota_f = 0$ in the $W$ matrix [Eq. (58)] and assuming $\beta/\epsilon \sim \kappa_0^2 \sim 1$

$$W_{\ell m} = \frac{\beta}{\epsilon} \left[ \frac{2(2 - \kappa_0^2)}{\kappa_0^2} \delta_{\ell - m} + \frac{3}{2} (\delta_{\ell - m - 1} + \delta_{m - \ell - 1}) \right] + \frac{2n^2}{|m|} \delta_{\ell - m} \tag{87}$$

$$\left( \frac{\ell}{|\ell|} + \frac{m}{|m|} \right) n \tilde{G}_{\ell m} + \sum_{\rho \neq \nu} |\rho| \tilde{G}_{\ell \rho} \tilde{G}_{m \rho}$$

$$\tilde{G}_{\ell m} = -\frac{\iota_f}{E(k_0)} \int_0^{\pi/2} \cos[2(\ell - m)y] \left[ 1 - \kappa_0^2 \sin^2 y \right]^{1/2} dy$$

Since $W$ is no longer diagonal, its eigenvalues must in general be determined numerically. This was carried out by Freidberg and Haas(18). Their results are summarized in Fig. 5 which illustrates critical $\beta$ versus $\iota_f$ for the most dangerous mode, $n = 1$. The poloidal modes are now coupled and the minimizing eigenfunctions are dominated by $m = 2$ with significant $m = 1$ and $m = 3$ sidebands. Also illustrated is the equilibrium limit given by Eq. (82).

We see that at large transforms, $\iota_f > \iota_c$, the critical $\beta$ is limited by stability considerations. As the current increases above $\iota_c$, the critical $\beta$ for stability decreases. Above $\iota_f = 1$, the system is unstable for any $\beta/\epsilon$.

For low transforms, $\iota_f < \iota_c$, the system is stable up to the equilibrium limit. However, as the current decreases below $\iota_c$, the value of $\beta/\epsilon$ that can be held in toroidal equilibrium also decreases.

Summarizing, in a toroidal high $\beta$ tokamak described by the sharp boundary model there is an optimum current, $\iota_f = \iota_c \approx 0.58$ at which the critical $\beta$ for stability is maximized. This maximum critical $\beta$ is given by $\beta/\epsilon \approx 0.21$. We note that in comparison with the straight tokamak, toroidal effects are destabilizing: that is, a straight tokamak with $\iota_f < 1$ is stable for any $\beta$ contained in the ordering. In the toroidal case both equilibrium and stability requirements limit $\beta/\epsilon$ to some finite value, even if $\iota_f < 1$. 28
M. High $\beta$ Toroidal Hybrid Configurations

The stability of the high $\beta$ toroidal hybrid system is described by the full matrix $W$ given by Eq. (58). In discussing the results it is useful to consider the two cases of interest, $\ell = 2$ and $\ell = 3$ separately.

1. The $\ell = 2$ Hybrid System

The stability diagram for a high $\beta$ tokamak with a superimposed $\ell = 2$ helical field is shown in Fig. 6. Plotted here are curves of critical $\beta/\epsilon$ versus the "total transform", $\epsilon_l + \epsilon_H$. The sequence of curves correspond to different values of the parameter $\eta = \epsilon_H/\epsilon_l$ which measures the relative amount of helical field to ohmic heating current. Also shown is the equilibrium limit $\beta/\epsilon = (\pi/4)^2(\epsilon_l + \epsilon_H)^2$.  

The case illustrated corresponds to $ha \equiv x = .2$. Higher values have been computed. The results are only weakly dependent on $ha$ and in general indicate slightly higher values of $\beta/\epsilon$ for increasing $ha$.

The curves in Fig. 6 correspond to $n = 1$ which is usually the worst mode in the parameter regimes of interest. The other important mode is $n = 0$ which gives rise to instability when $\epsilon_H$ and $\epsilon_l$ are anti-parallel. For this reason only positive $\epsilon_l$ and $\epsilon_H$ are considered in Fig. 6.

As a reference note that the curve $\eta = 0$ corresponds to the pure high $\beta$ tokamak. There are two important points to note as $\epsilon_H$ is increased. First, the maximum allowable "total transform", which occurs for $\beta/\epsilon \to 0$ is given by $\epsilon_l + \epsilon_H \approx 1$. Thus, as the helical field is increased the allowable ohmic heating current must decrease. This unfavorable result is identical to that discussed in the section on the "Low $\beta$ Toroidal Hybrid System".

Secondly, consider the effects of the helical field on the $\beta/\epsilon$ limits. For small, increasing $\epsilon_H$ the critical $\beta/\epsilon$, corresponding to the intersection of the stability curves with the equilibrium limit, decreases from the pure tokamak value, $\beta/\epsilon = .21$. This trend continues as $\epsilon_H/\epsilon_l$ increases to unity. However, for larger values of $\epsilon_H/\epsilon_l$ a second window of stability appears. This region is characterized by the maxima in the $\beta/\epsilon$ curves which lie considerably below the equilibrium limit. The $\beta/\epsilon$ values in the second region can approach that of the pure high $\beta$ tokamak but require substantial helical transforms, $\epsilon_H/\epsilon_l \gtrsim .5$.

The existence of the two stability regions can be understood in terms of the structure of the eigenfunction. In the first region near the equilibrium limit the eigenfunction has a strong $m = 2$ component. This is a consequence of the fact that in a straight tokamak the $m = 1$, $n = 1$ mode has a stability threshold at $\epsilon_l = 1$ (i.e., $\delta W_{11} \sim 1 - \epsilon_l$) while the $m = 2$, $n = 1$ mode has a stable neutral point at $\epsilon_l = 1$ [i.e., $\delta W_{21} \sim (1 - \epsilon_l)^2$]. In the toroidal case these two modes are strongly coupled giving rise to an eigenfunction with a large destabilizing
component of ballooning but one which pays only a small penalty in additional linebending because of the simultaneous neutrality at $\zeta_l = 1$. As $\zeta_{II}$ increases the $m = 2$ mode is detuned (i.e., the simultaneous neutrality vanishes). Consequently the ballooning nature of the eigenfunction decreases giving rise to increased stability. However, as $\zeta_{II}$ increases there is also an increased destabilizing influence due to the unfavorable curvature of the helical field. These two effects compete and at sufficiently large $\zeta_{II}$ the stabilizing influence dominates. What remains is a second widow of stability in which the most unstable mode is now predominantly $m = 1, n = 1$. In this regime the behavior is somewhat similar to that of the straight hybrid system whose stability curve is also illustrated in Fig. 6. A comparison of the maximum $\beta/\epsilon$ values and corresponding $\zeta_l$ values for the two stability regions is shown in Fig. 8 for the case of $n = 1$. We see that in the parameter regime of interest the highest values of $\beta/\epsilon$ occur for the pure high $\beta$ tokamak.

In summary the addition of a single $\ell = 2$ helical field to a high $\beta$ tokamak has an unfavorable effect on the ideal magnetohydrodynamic stability of external modes. The allowable ohmic heating currents and $\beta/\epsilon$ limits decrease as the helical field increases. A second window of stability is found giving rise to $\beta/\epsilon$ limits comparable to the pure high $\beta$ tokamak, but requires relatively large helical transforms $\zeta_{II}/\zeta_l \gtrsim 0.5$. As stated previously these pessimistic predictions are related to the fact that a single $\ell = 2$ field has neither shear nor a magnetic well. The results suggest that further studies be carried out which allow the possibility of a vertical field and/or helical sideband fields. Such calculations aimed at determining the properties of an "optimized" $\ell = 2$ system are currently being carried out and hopefully will shed some light on the ultimate desirability of using $\ell = 2$ in a hybrid configuration.

2. The $\ell = 3$ Hybrid System

We now consider the case of a high $\beta$ tokamak with a superimposed $\ell = 3$ helical field. The corresponding stability diagram is illustrated in Fig. 9 for $ha = 0.2$. This diagram is similar to Fig. 6 except that here it is more convenient to characterize the sequence of curves by the parameter $\zeta_{II}$ rather than $\zeta_{II}/\zeta_l$. In comparison the $\ell = 3$ results are simpler and in general favorable.

As before, we restrict attention to the case when $\zeta_{II}$ and $\zeta_l$ are parallel to avoid $n = 0$ modes. The first point to note is that as $\zeta_{II}$ increases, the maximum stable "total transform", $\zeta_{II} + \zeta_l$, (which occurs for $\beta/\epsilon \to 0$), also increases. A more detailed examination indicates that $\zeta_{II} + \zeta_l$ increases faster than $\zeta_{II}$. Thus, the addition of an $\ell = 3$ helical field permits a larger ohmic heating current, consistent with the "Low $\beta$ Toroidal Hybrid" analysis. This trend is opposite that predicted for an $\ell = 2$ helical field.
The next point to consider is the $\beta/\epsilon$ limits. Increasing $\iota_H$ gives rise to a monotonic increase in $\beta/\epsilon$, (also the opposite of the $\ell = 2$ case). In all cases, the maximum $\beta/\epsilon$ corresponds to the intersection of the stability curve with the equilibrium limit. The gains in $\beta/\epsilon$ and $\iota_I$ over the pure $\beta$ tokamak are substantial. This can be seen in Fig. 10 where we have illustrated maximum $\beta/\epsilon$ and corresponding $\iota_I$ versus $\iota_H$. For example, when $\iota_H = .2$ the critical $\beta/\epsilon$ increases from .21 to .61 while $\iota_I$ increases from .58 to .78.

The favorable behavior just described can be understood qualitatively in terms of the structure of the eigenfunction. With the exception of $\iota_H = 0$ the most unstable mode corresponds to $m = 1$, $n = 1$. The higher $i$ value is more effective in detuning the $m = 2$ mode and has a weaker destabilizing influence because of the non-zero shear. Hence, the competition between these two effects is dominated by a net stabilizing influence and the resulting stability diagram is essentially an enlarged version of the second stability window found for $\ell = 2$. Further evidence of this point can be seen by examining the $\iota_H = .4$ curve in Fig. 9 and noting how similar it is to the "Straight Hybrid" result.

In summary, the addition of a single $\ell = 3$ field to a high $\beta$ tokamak improves the ideal magnetohydrodynamic stability against external modes in that higher values of both $\beta/\epsilon$ and $\iota_I$ are allowed. This result is apparently closely related to the simultaneous presence of helical shear and ohmic heating current. The helical shear is only weakly present in the case of $\ell = 2$. The improved results occur without optimization of the stellarator field; that is, neither a vertical field nor a helical sideband field is required to generate a vacuum magnetic well. The ultimate desirability of choosing $\ell = 3$ as opposed to $\ell = 2$ involves a judgement between the increased technological difficulties of generating a given transform for higher $\ell$ versus the consequences that external modes will have in an actual experiment.

F. The Effect of "Flux Conservation"

The final point to be discussed concerns the effect of flux conservation. Specifically, we address the following question. In many of the stability results just presented, the maximum $\beta$ values occur at the intersection of the stability boundary with the equilibrium $\beta$ limit. If we instead consider a sequence of flux conserving equilibria, for which there is no equilibrium limit would the resulting stability limits on $\beta$ be higher?

The answer is negative. To see this, recall that in the sharp boundary model flux conservation corresponds to the condition that the total transform on the surface, $i(a)$, [given by Eq. (29)] be held constant as $\beta$ varies. In terms of the stability diagrams this appears as an additional constraint. For example, in Fig. 11, we repeat the stability diagram for $\ell = 3$, $ha = .2$ and $\iota_{II} = .4$. Superimposed are several curves of $i(a) = \text{const}$. Flux conservation implies that as $\beta$ is increased, the plasma is constrained to move along one of these curves.
At sufficiently high $\beta$ there is an intersection between the stability curve and any given $\epsilon(a) = \text{const.}$ curve. The point of intersection corresponds to the maximum critical $\beta$. The main consequence is that in all cases, the intersections lie beneath the equilibrium limit, indicating that the $\beta$ limits from flux conservation are lower than those previously obtained for the same values of $\epsilon_1$ and $\epsilon_H$.

This result is presumably true for arbitrary diffuse profiles as well. In principle, the maximum $\beta$ values are higher when the flux function $F(\psi) \equiv RB_\psi$ can be optimized freely rather than constrained by the condition of flux conservation. In practice, however, many simple, numerically convenient choices of $F(\psi)$ are far from the optimum and lead to artificially low $\beta$ limits compared to a sequence of flux conserving equilibria.

In summary, the $\beta$ limits calculated in previous sections should be considered slightly optimistic in that they are always higher than those corresponding to a more realistic sequence of flux conserving equilibria.
V. Conclusions

We have considered the stability of a high $\beta$ tokamak with superimposed helical fields to external ideal magnetohydrodynamic modes.

The results indicate that the addition of a single $\ell = 2$ field has a detrimental effect on both the maximum $\beta$ limit and the maximum allowable ohmic heating current. This is a consequence of the fact that a single $\ell = 2$ field has no vacuum magnetic well and very little shear. Further studies are suggested in order to include the effect of a vertical field and/or helical sideband fields to improve the stability properties of the basic $\ell = 2$ configuration.

On the other hand, the addition of a single $\ell = 3$ field has a favorable effect on stability. Both the maximum allowable $\beta$ values and ohmic heating current are higher than for the pure high $\beta$ tokamak.

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FIGURE CAPTIONS

Fig. 1. The toroidal geometry.

Fig. 2. Stability boundary for the infinitely long straight system. The case shown corresponds to an $\ell = 3$ system with $x \ll 1$ so that $K_1 = K_2 = 1$. The shaded region is stable.

Fig. 3. Stability diagram for the finite length straight system for $m = 1$ and various $n$. The case shown corresponds to $\ell = 3, \tau_f = 1.5$, and $x \ll 1$ so that $K_1 = K_2 = 1$. The dashed curve represents the corresponding $m = 1$ diagram for the infinite length system. The shaded region is stable.

Fig. 4. Stability diagram for the low $\beta$ toroidal system for $m = 1$ and various $n$. Case (a) corresponds to $\ell = 2, x = .2, K_1 = .02$. Case (b) corresponds to $\ell = 3, x \ll 1, K_1 = 1$. The shaded region is stable.

Fig. 5. Stability diagram for the high $\beta$ tokamak for the $n = 1$ mode. The shaded region is stable.

Fig. 6. Stability diagram for a high $\beta$ tokamak with a superimposed $\ell = 2$ field for various values of $\eta = \tau_{HF}/\tau_f$. The stable regions lie beneath the curves.

Fig. 7. Harmonic content of the $n = 1$ mode at marginal stability for an $\ell = 2$ field with $x = .2$ and $\tau_{HF} = .2$. For $\tau_{HF} + \tau_f \approx 1$ the dominant mode is $m = 1, n = 1$. As $\tau_{HF} + \tau_f \to 0$, the dominant $m$ increases.

Fig. 8. Maximum $\beta/\epsilon$ and corresponding $\tau_f$ as a function of $\tau_{HF}$ for the $n = 1$ mode in an $\ell = 2, x = .2$ system. The curve labeled $m = 2, n = 1$ corresponds to the intersection of the stability curves with the equilibrium limits. The $m = 1, n = 1$ curve corresponds to the peak in the second stability window.

Fig. 9. Stability diagram for a high $\beta$ tokamak with a superimposed $\ell = 3$ field for various values of $\tau_{HF}$. The stable regions lie beneath the curves.

Fig. 10. Maximum $\beta/\epsilon$ and corresponding $\tau_f$ as a function of $\tau_{HF}$ for the $n = 1$ mode in an $\ell = 3, x = .2$ system.

Fig. 11. Stability diagram for the $n = 1$ mode in an $\ell = 3, x = .2, \tau_{HF} = .4$ system. Superimposed are curves of constant rotational transform corresponding to flux conserving operation.
Figure 1
Figure 2
(a) MI

(b) MI

Figure 4
Figure 6
Figure 9
Figure 10
EQUILIBRIUM LIMIT

STABILITY BOUNDARY

\[ \frac{\beta}{\varepsilon} \]

\[ t_I + t_H \]

Figure 11