Observation of self-binding Turbulent Fluctuations in Simulations Plasma and there Relevance to Plasma Kinetic Theories

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Non-wave-like fluctuations of the phase-space density are observed in simulations of turbulent plasma. During decay from an initial state, the mean square fluctuation level decays at a much slower rate than that of an individual fluctuation. The distribution function of the fluctuation amplitudes becomes non-Gaussian (skewed) in favor of negative fluctuations. An enhancement in the aggregate fluctuation lifetime is also observed when the turbulence is driven by an external source. A model based on a collection of self-binding negative fluctuations, called phase-space density holes, can explain the observations. Collisions between holes produce hole fragments and lead to fluctuation decay. However, the hole fragments are self-binding and tend to recombine into new holes. The implications of these results for kinetic theories of plasma turbulence are discussed. In particular, it is shown that the theory of clumps, when suitably modified to include fluctuation self-binding, can explain many features of the nonlinear instability recently observed in computer simulations.

I. INTRODUCTION

We present a series of numerical simulations which were designed to test specific aspects of kinetic theories of plasma turbulence and to probe the behavior of fluctuations moving at speeds less than the thermal velocity (i.e., non-wave-like phenomena). The simulations are of a one-dimensional, one-species plasma. Besides the reason of simplicity, we have focused on the one-dimensional problem because the large number of particles required for phase-space diagnostics would be prohibitive in a three-dimensional simulation with present-generation computers. However, we believe the phenomena we report here will occur in three-dimensional plasma with a magnetic field. The simulation results are both novel and surprising, especially in the light of the conventional wisdom that small-amplitude fluctuations in a one-dimensional, one-species plasma have been thoroughly studied and understood with both simulation and theory of the past twenty years. Apparently, the reasons the novel phenomenon we discuss has been overlooked for so long are because of the historical preoccupation with waves and preconceived ideas as to what phenomena are important. The simulations discussed here give compelling evidence for the existence and importance of non-wave-like fluctuations in turbulent plasma. These fluctuations are small-scale (on the order of a Debye length and a tenth of the thermal velocity in size), self-binding depressions (holes) in the phase-space density rather than waves. The turbulence can be understood in terms of a superposition of these phase-space holes rather than a superposition of waves. This state of affairs is reminiscent of fluid turbulence where the turbulence is described as a superposition of eddies (vortices). It is noteworthy that the importance of nonlinear, non-wave-like phenomena in fluids has been known for decades.

Early efforts at deriving a renormalized theory of plasma turbulence involved a one-point description or theory of the coherent response.1-3 This produces a model in which the turbulent plasma is described by a nonlinear dielectric function. However, further work shows that it is essential to derive a renormalized two-point equation (correlation function equation) in order to include an important class of nonlinear random fluctuations called clumps.4,5 These random fluctuations result from a fine scale granulation of the plasma phase-space density. Clumps can be enhancements (δf>0) or depletions (δf<0) in the phase-space density f. The depletions have been referred to as phase-space holes.5 Here δf = f - ⟨f⟩, where ⟨f⟩ is the ensemble-averaged phase-space density. This type of fluctuation appears to be an important and prevalent component of plasma turbulence and would appear to be present in any plasma containing turbulent transport processes. For example, recent work has shown that clump fluctuations can grow in amplitude (are unstable) in a variety of linearly stable plasma configurations.6-9 The renormalized analytical theories of these fluctuations are necessarily approximate and it has been argued that they fail to describe an important aspect of the dynamics of these fluctuations, namely the self-interaction or tendency of some of these fluctuations to form into bound states.5 In this paper, we present simulation results that appear to confirm the existence of these self-bound fluctuations in turbulent plasma. These results have important implications for the clump instability as well as plasma kinetic theory in general.

We have devised novel diagnostics in order to investigate the self-interaction feature of the fluctuations. In the past, both theory and simulations of plasma turbulence have focused on the mean square fluctuation level. Conventional thinking has held that when the distribution of fluctuation amplitudes is sufficiently Gaussian, the fluctuation correlation function is adequate to describe the turbulence. However, the diagnostics show that the self-binding fluctuations—because they are negative and have enhanced lifetimes—produce a pronounced skewness to the fluctuation distribution.

At present, renormalized kinetic theories—which are
essentially perturbative in the electric field—do not describe the self-binding or particle-trapping tendency of the fluctuations. This self-binding feature is responsible for the tendency of the negative fluctuations to coalesce into new (negative) fluctuations. In the absence of an external source (i.e., the system decays from an initial state), the coalesced fluctuations become progressively more isolated from each other in phase space as time evolves. We have observed this reduction in the density of the fluctuations in phase space—sometimes referred to as intermittency—in the simulations. It is difficult to see how kinetic theories, which assume that fluctuations are uniformly distributed in phase space (rather than concentrated in local regions), can treat this intermittent feature of the fluctuations.

In addition to these general implications for kinetic theories, the one-species simulation results described in this paper have direct relevance to the lump instability in a two-species plasma. The enhanced lifetime of the self-bound clumps that we observe in the simulation implies that the threshold for the lump instability is lower than that previously calculated. The magnitude of the enhancement observed here (approximately a factor of 3) brings the theory of the lump instability threshold into agreement with observation.

At present, renormalized theories of plasma fluctuations such as clumps focus on the two-point fluctuation correlation function 
\[ \langle \delta f(1) \delta f(2) \rangle = \langle \delta f(x_1,v_1,t) \delta f(x_2,v_2,t) \rangle. \]
For detailed discussions of these theories, the reader is referred to Ref. 4 and the review article of Ref. 10. These theories derive a generic evolution equation for 
\[ \langle \delta f(1) \delta f(2) \rangle \]
which can be written as
\[ \left( \frac{\partial}{\partial t} + T_{12} \right) \langle \delta f(1) \delta f(2) \rangle = S, \tag{1} \]
where \( T_{12} \) and \( S \) depend only on \( \langle \delta f(1) \delta f(2) \rangle \) so that Eq. (1) is indifferent to the sign of \( \delta f \). Here \( S \) is a source term of the clumps and arises from the rearrangement of regions of different phase-space density by the turbulence. In its simplest form, the operator \( T_{12} \) describes the destruction of the fluctuations by velocity dispersion and diffusion [as in Eq. (7)]. In such a theory, the fluctuations decay because the orbits of their constituent particles undergo stochastic instability.

The operator \( T_{12} \) determines the characteristic time \( \tau_{cl}(x_-,v_-) \) for two particles to diverge stochastically given that their initial separations were \( x_-=x_1-x_2, v_-=v_1-v_2 \). Consequently, when \( S=0 \) in Eq. (1), the phase-space density will be torn up into smaller and smaller grains as time elapses: \( \langle \delta f(1) \delta f(2) \rangle = 0 \) for \( t \gg \tau_{cl}(x_-,v_-) \). For \( S \neq 0 \), this destruction process is compensated by the production of new fluctuations. In the steady state, \( \langle \delta f(1) \delta f(2) \rangle = \tau_{cl}(x_-,v_-) S \).

In order to test Eq. (1), we measured various characteristics of the fluctuations for both \( S=0 \) and \( S \neq 0 \). The \( S=0 \) case, which we refer to as decay turbulence, involved the decay of an initial distribution of fluctuations. In the \( S \neq 0 \) experiments, which we refer to as driven turbulence, we generated fluctuations by imposing a given external electric field spectrum on the plasma. The fluctuation correlation function was measured at equal time intervals during the experiments. We also followed particle orbits in time in order to test the \( \tau_{cl}(x_-,v_-) \) model of \( T_{12} \). In addition, we investigated one of the basic assumptions of Eq. (1), i.e., the fluctuation amplitudes have a Gaussian distribution. This assumption implies that fluctuations with \( \delta f<0 \) are equally probable as those with \( \delta f>0 \). In order to test this, we divided the phase space into cells of size \( \Delta x_w \) by \( \Delta v_w \) and made a histogram of the average fluctuation \( \bar{\delta f} \) found in each cell. We denote the probability density of finding a fluctuation \( \bar{\delta f} \) in a cell as \( P(\bar{\delta f}) \).

The principal results of these measurements can be stated as follows. In the decay experiments [where \( S=0 \) in Eq. (1)] we found:

A1. The correlation function does not decay as Eq. (1) predicts: \( \langle \delta f(1) \delta f(2) \rangle \neq 0 \) for \( t \gg \tau_{cl}(x_-,v_-) \).

A2. The characteristic velocity and spatial widths of the correlation function increase with time while \( \langle \delta f(1)^2 \rangle \) decreases with time.

A3. The particle orbits undergo stochastic instability in agreement with the \( \tau_{cl}(x_-,v_-) \) model.

A4. \( P(\bar{\delta f}) \) becomes non-Gaussian (skewed) in favor of fluctuations with \( \delta f<0 \). The skewness \( \langle \delta f(1)^3 \rangle / \langle \delta f(1)^2 \rangle ^{3/2} \) was typically of order \( -1 \).

A5. \( P(\bar{\delta f}) \) becomes more skewed as time elapses.

A6. The skewness decreases with increasing \( \Delta x_w \) and \( \Delta v_w \) [the phase-space cell used to measure \( P(\bar{\delta f}) \)].

In the driven experiments where \( S=0 \neq 0 \) in Eq. (1), we found:

B1. \( \langle \delta f(1) \delta f(2) \rangle \sim 3 \tau_{cl}(x_-,v_-) S_{\text{ext}} \), so that Eq. (1) predicts \( \langle \delta f(1) \delta f(2) \rangle = \tau_{cl}(x_-,v_-) S_{\text{ext}} \).

B2. \( P(\bar{\delta f}) \) is Gaussian.

Item A4 is probably the most striking evidence of deficiencies in the existing kinetic theories of plasma turbulence. It implies that during decay turbulence, the system is composed of a few deep phase-space "holes" (\( \delta f \) is large and negative) with \( \delta f > 0 \) (but small) between the holes. To see this, consider a system composed of a large number of identical holes—each with amplitude \( \delta f \leq 0 \) and phase-space area \( A = \Delta x_w \Delta v_w \). Let \( \delta f_+ \) denote the amplitude of \( \delta f < 0 \) between the holes and let the fraction of phase-space area occupied by holes (the hole-packing fraction) be denoted by \( p \). Then, charge conservation implies
\[ \rho \delta f_- + (1-p) \delta f_+ = 0 \tag{2} \]
or
\[ \delta f_+ - \delta f_- = (p-1)/p. \tag{3} \]
Now consider \( P(\bar{\delta f}) \) for this simple system with a phase-space cell or window of area \( A_w = \Delta x_w \Delta v_w \). The quantity \( P(\bar{\delta f}) \) will be significantly skewed toward \( \delta f<0 \) if two conditions are satisfied. First, we must have \( -\delta f_- \gg \delta f_+ \), which implies that the packing fraction must be small (\( p \ll 1 \)). Second, the average number of holes (\( \bar{n} \)) in the window must be small. If \( \bar{n} \gg 1 \) and the holes are randomly located in phase space, then \( P(\bar{\delta f}) \) would be Gaussian. Therefore, we must have \( p \ll 1 \) and \( \rho A_w < A \) [see A6] in order to have significant skewness.

We believe that the \( \delta f<0 \) fluctuations implied by the skewed \( P(\bar{\delta f}) \)'s that we observe are related to so-called
phase-space density holes. A single isolated phase-space hole is a self-bound equilibrium which behaves like a self-gravitating fluid element. This equilibrium is characterized by a definite relationship between its amplitude ($f$) and its phase-space dimensions ($\Delta x, \Delta v$). Such a hole is a state of maximum entropy subject to constant mass, momentum, and energy. It is a fluctuation in the phase-space density for which the potential energy fluctuation is negative and of the order of the thermal energy fluctuation. It is a Bernstein-Greene-Kruskal mode.

The binary interaction between two equilibrated phase-space holes has been investigated in Refs. 5 and 11. It has been shown that the two holes (with $\Delta x$ less than a Debye length and small relative velocity) will attract each other and coalesce into a new hole of larger phase-space dimensions (cf. Fig. 6 of Ref. 11). In doing so, the average-or coarse-grained fluctuation level is reduced because some of the original hole material gets "mixed" with the interstitial background material ($\delta f > 0$) between the holes. Collisions between two equilibrated holes with larger relative velocity lead to tidal deformations of the holes and the production of hole fragments (cf. Fig. 7 of Ref. 11). In a turbulent plasma with many phase-space "holes," we would expect similar processes to occur. Indeed, the authors of Ref. 5 imply that any group of fluctuations with $\delta f < 0$ would tend to form into a collection of phase-space density holes. Any fluctuations with $\delta f > 0$—being local enhancements in the phase-space density—would tend to blow apart and form the interstitial background between holes. Because of the random interactions of a turbulent plasma, a $\delta f < 0$ fluctuation will only approximate an equilibrium (Bernstein-Greene-Kruskal) phase-space hole. In a turbulent plasma, a phase-space density hole retains its tendency to self-bind but, because of collisions with other holes, it never equilibrates (virializes). Any hole fragments produced in these collisions will subsequently tend to recombine (coalesce) with other holes. The process of collision and recombination of holes is an essential feature of the turbulent state.

A simple physical model—based on the interactions of a collection of holes—can apparently explain the essential features of the simulation results. Consider the decay experiments. Any given initial distribution of fluctuations will tend to form into a collection of phase-space density holes. Any fluctuations with $\delta f > 0$ background material. The fluctuation level will decay as collisions between holes produce hole fragments which then mix into the background. For $p \sim 1$, these turbulent collisions occur on the $\tau_{eq} (x, v)$ time scale so that an individual hole lasts only for a time $\tau_{eq} (x, v)$ (cf. A3). However, the hole fragments produced during these collisions will, because of their self-binding feature, tend to recombine (coalesce) into new holes. Therefore, the mean square fluctuation level decays at a much slower rate than that predicted by the stochastic instability model (cf. A1). The correlation function will increase in width even as $\langle \delta f(1) \rangle^2$ decays in time because hole coalescence will produce holes with larger scale lengths (cf. A2). Consequently, the holes with small scale lengths will decrease in number as they coalesce into the new (larger-scale-length) hole structures. This reduction in the packing fraction of the (smaller) holes produces a skewed $P(\delta f)$ (cf. A4). As time elapses, these processes will continually reduce the hole-packing fraction so that $P(\delta f)$ will become more skewed (cf. A5). This decay of the mean square fluctuation level ($\langle \delta f^2 \rangle$) can be described by the following model equation for the hole-packing fraction $p$:

$$\frac{dt}{dP} = \frac{\Delta n}{b \Delta x}.$$  (4)

Equation (4) assumes that the net effect of hole collisions and hole fragment coalescence is to keep hole size and depth constant while the number of holes decreases with time. The last term on the right if Eq. (4) is the aggregate fluctuation (hole) decay rate. If the holes were closely packed in phase space ($p \sim 1$), then the hole-hole collision rate would be approximately $\Delta v / \Delta x \sim \tau_{eq}^{-1}$ where $\Delta v$ and $\Delta x$ are the hole dimensions. However, for $p \ll 1$, collisions between holes will be less frequent because the holes will be more isolated from each other in phase space. The factor of $b > 1$ in Eq. (4) accounts for the fact that hole recombination will reduce the aggregate hole decay rate below the hole-hole collision rate. The simulation results imply that $b \sim 3$. In the case of the driven experiments, the packing fraction (and $\langle \delta f^2 \rangle$) does not decay with time because the externally imposed waves are continually creating holes (clumps) by rearranging the phase-space density (cf. B2). The fluctuation amplitude produced by this rearrangement is larger than that predicted by Eq. (1), since the hole self-binding effect not included in $T_{12}$ will increase the fluctuation lifetime (cf. B1).

We designed the experiments to simulate the features typical of turbulent plasma. In the decay experiments, turbulence developed from an initial state of local depletions and enhancements of the phase-space density. The initial fluctuations were distributed throughout the phase space and were of size $\Delta x = \lambda_D$ by $\Delta v = 0.1 v_B$. $\Lambda_D$ and $v_B$ denote the Debye length and thermal velocity, respectively. Figure 1(c) shows the $P(\delta f)$ that we observed at the end of the simulation when $\omega_p t = 220 (\omega_p = v_B / \lambda_D$ is the plasma frequency). The Gaussian curve (dashed line) in Fig. 1(c) represents $P(\delta f)$ if the fluctuation amplitudes had a Gaussian distribution. The skewed "tail" of $P(\delta f)$, where $\delta f < 0$, is due to the remnants of deep holes initially present. The shifted peak of $P(\delta f)$ for $\delta f > 0$ is due to the background material that rapidly decayed (from $\delta f / \langle f \rangle = 1$) as it blew itself apart by charge repulsion. The $P(\delta f)$ extends further to the left than to the right because the fluctuations with $\delta f < 0$ decay more slowly than those with $\delta f > 0$. The hole fluctuations decay more slowly because of their tendency to self-bind and recombine with other holes. The correlation function at $\omega_p t = 220$ (Figs. 1[a] and 1[b]) shows the existence of fluctuations with phase-space dimensions of the same order as those initially present. However, we show in Sec. IV that the $\tau_{eq}(x, v)$ model predicts that no fluctuations with sizes larger than $\Delta x_0 = 0.06 \lambda_D$ by $\Delta v_0 = 0.05 v_B$ (the smallest measurement cell) should exists after $\omega_p t = 60$.

At a qualitative level, interpreting the fluctuations as a superposition of holes seems to provide a simple explanation of the simulation results. In Sec. VI we attempt to make a more quantitative comparison by deriving a formula relating
We have therefore focused on a simplified model that appears to contain some of the essential features of the problem. Throughout the paper we have made a conscious effort to separate the simulation results from the hole model used to interpret them. We believe that the simulation results (Sec. III) stand independently. However, we also believe that the hole model and the conclusions inferred from it, are a meaningful first step toward a more complete analytical theory of the self-binding effects.

It is clear that the existing kinetic theories of plasma turbulence are inadequate to explain the simulation results. We believe that there are several salient features of our findings that have particular significance for these theories.

1. The hole ($\delta f < 0$) fluctuations appear to dominate the turbulence. In the decay turbulence simulations, the correlation function decays at a slower rate than Eq. (7) predicts. This is clearly due to the mutual attraction and coalescing of hole material rather than the repulsive tendency of the $\delta f > 0$ background material. However, Eq. (1) treats $\delta f > 0$ and $\delta f < 0$ fluctuations on an equal footing. This indiscernibility of Eq. (1) to the sign of $\delta f$ implies a distribution of fluctuation amplitudes, $P(\delta f)$, that is Gaussian. However, for packing fractions less than $\frac{1}{2}$, $P(\delta f)$ is not Gaussian. In this case, the fluctuations are concentrated in isolated local regions—a situation referred to in fluid turbulence as intermittency. The neglect of higher-order correlation functions, which is the basis of Eq. (1), is generally thought to be valid only when $P(\delta f)$ is approximately Gaussian. However, it is difficult to see how any theory which depends only on $\langle \delta f \delta f' \rangle$ can describe fluctuation self-trapping, which intrinsically depends on the sign of $\delta f$.

2. An individual fluctuation undergoes stochastic instability and, therefore, decays at the rate $\tau_x(\ldots, \ldots)^{-1}$ for $p = 1$ or $\Delta n/\Delta x$ for $p < 1$. However, the aggregate fluctuation level decays at a much slower rate because the hole-packing fraction decreases with time and the holes tend to recombine into new holes. This is reminiscent of fluid turbulence where the aggregate fluctuation level decays at a slower rate than that for the dispersion of a passive scalar contaminant. The existing theories of $T_{12}$ [e.g., Eq. (7)] treat orbit stochasticity but neglect self-binding and thus predict that the individual and aggregate fluctuation levels decay at the same rate.

3. The trapping tendency of particles which leads to the self-binding tendency of the holes is an important feature of the turbulence. In the existing theories, it is assumed that trapping can be ignored if the electric field correlation time $\tau_{o_e}$ is less than the trapping time $\tau_T$. $T_{12}$ is then constructed by a renormalized perturbation expansion in the electric fields. Since such expansions use unperturbed orbits as the lowest-order approximation, it does not seem possible to describe particles trapped in holes by these conventional expansion techniques.

Though a proper theory of these effects is lacking, the simulation results discussed in this paper give an indication of the modifications needed in existing theories. For example, consider the recently reported nonlinear instability in a one-dimensional simulation ion--electron plasma. The instability has been identified with the electron--ion clump instability. However, the observed threshold for the onset of the instability is lower than that calculated from the existing theory of the electron--ion clump instability. The discrepan-
In the limit $x_\rightarrow 0$, the functions $D_{12}(x_-)$ and $F_{12}(x_-)$ become $D$ and $F$, where $D$ and $F$ are single-point diffusion and dynamical drag coefficients. In a one-species plasma, as is the case here, the $D$ and $F$ terms due to the plasma self-consistent fields cancel exactly. However, $S$ can be nonzero in a two-species plasma or if an external source of electric field fluctuations is imposed on the plasma. For the case of externally applied fields,

$$S = 2D_{12}(x_-)\left(\frac{\partial(f(1))}{\partial v_1} - \frac{\partial(f(2))}{\partial v_2}\right),$$

where $D_{12}(x_-)$ is defined by

$$D_{12}(x_-) = \left(\frac{e}{m}\right)^2 \int \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} \left|\langle E(2,k,\omega)\rangle\right|_{\text{ext}}$$

$$\times \pi \delta(\omega - ku) \exp(ikx_-).$$

Here, $\langle E(2,k,\omega)\rangle_{\text{ext}}$ is the external electric field spectrum for wavenumbers $k$ and frequency $\omega$, and $e(k,\omega)$ is the plasma dielectric function that shields the external fields.

The decay of the fluctuations described by the left-hand side of Eq. (7) is due to the random increase in the relative separation of particle orbits. For small separations, $D = k_0^2 D^2 \rightarrow 0$ so that two particles diffuse together since they feel approximately the same forces. Here $D$ is the one-point velocity diffusion coefficient of quasilinear theory and $k_0$ is an average wavenumber characteristic of the turbulence,

$$k_0^2 = \frac{1}{2D} \left(\frac{\partial D}{\partial x_-}\right)_{x_- = 0}$$

If two particles have large separations, $D \rightarrow 2D$ since they feel different forces and diffuse independently. The smaller the initial separations, the larger is the relative orbit diffusion time. Let us define $\tau_{ci}(x_-,v_-)$ as the time for two phase-space points initially separated by $x_- v_-$ to be separated spatially by $k_0^{-1}$. The ensemble-averaged orbits defined by the operator $T_{12}$ of Eq. (7) imply

$$\frac{\partial}{\partial t} \langle x_-^2 \rangle = 4 \langle D_- \rangle = 4k_0^2 D \langle x_-^2 \rangle,$$

for $|k_0 x_-| < 1$. For steady-state turbulence, the time asymptotic solution of Eq. (12) is

$$\langle x_-^2(t) \rangle = \frac{1}{4} (x_-^2 - 2x_- v_\tau_0 + 2v_-^2 \tau_0^2) \exp(t/\tau_0),$$

where

$$\tau_0 = (4k_0 D)^{-1/3} = (12)^{-1/3} \tau_{ci}.$$  

The particle orbits described by Eq. (12) diverge exponentially until $t = \tau_{ci}(x_- v_-)$, where

$$\tau_{ci}(x_- v_-) = \tau_0 \ln \left(\frac{3}{k_0^2 (x_-^2 - 2x_- v_\tau_0 + 2v_-^2 \tau_0^2)}\right),$$

when the argument of the logarithm is larger than unity and $\tau_{ci} = 0$ otherwise. For $t > \tau_{ci}(x_- v_-)$, the orbits diffuse independently.

In the decay turbulence experiments, we will focus on Eq. (7) with $S = 0$. In these experiments, $G(x_- v_-)$ and $D$ decay from an initial state. However, because the time dependence of $D$ appears only weakly in $\tau_0 = (4k_0 D)^{-1/3}$, the
steady-state expression for \( \tau_{el}(x_-,\nu_-) \) given by Eq. (15) is still approximately the characteristic time for an individual fluctuation to decay. With this in mind, we can easily obtain the decay of an initial correlation \( G(x_-,\nu_-) \) when \( S = 0 \). We can express \( G(x_-,\nu_-) \) in terms of the time-reversed orbits \( x_-(t) \) and \( \nu_-(t) \) of Eq. (7) and obtain

\[
G(x_-,\nu_-) = G[x_-(t),\nu_-(t),0],
\]

where \( x_-(0) = x_- \) and \( \nu_-(0) = \nu_- \). Let us assume that \( G(x_-,\nu_-) = 0 \) for \( k(x_-,\nu_-) \). Since two particles (initially separated by \( x_-,\nu_- \)) will be correlated only if \( t < \tau_{el}(x_-,\nu_-) \), Eq. (16) implies that \( G(x_-,\nu_-) = 0 \) for \( t > \tau_{el}(x_-,\nu_-) \), i.e., fluctuations will be "chopped up" from the outside (\( x_- \) large) toward the inside (\( x_- \) small) as time elapses.

For the driven experiments, we consider a steady-state spectrum due to internal charge fluctuations which yields the result

\[
\langle \delta f^2 \rangle = \langle f(v - \delta v) - f(v) \rangle^2.
\]

Since the particles are diffusing in velocity, \( \langle \delta v \rangle^2 = 2\tau_{el}(x_-,\nu_-)D^{ext} \). Assuming that \( \delta v \partial \ln f(v)/\partial v \ll 1 \), Eq. (18) yields the result Eq. (17). The fluctuation level will produce a plasma electric field spectrum due to internal charge generated by

\[
\langle E^2(k,\omega) \rangle = \frac{(4\pi)^2}{|\epsilon(k,\omega)|^2} |k|^{-1} \int d\nu_\perp \int d\nu_\parallel \exp(-ikx_\parallel) \overline{G}(x_-,\nu_-),
\]

where

\[
\overline{G}(x_-,\nu_-) = G(x_-,\nu_-) - \overline{G}(x_-,\nu_-).
\]

\( \overline{G}(x_-,\nu_-) \) follows from Eq. (17), but with \( D_{(x-)}(x_-) \) replaced by \( 2D_e \),

\[
\overline{G}(x_-,\nu_-) = 2D^{ext}(\nu_-) \frac{\partial \langle f(1) \rangle}{\partial \nu_\perp} \frac{\partial \langle f(2) \rangle}{\partial \nu_\parallel},
\]

where \( \tau_{el}(x_-,\nu_-) = 2\tau_{el}(x_-,\nu_-)[(\nu_\perp k_\parallel \sigma_{el})^2]^{-1} \). The plasma diffusion coefficient induced by \( D^{ext} \) is

\[
D^{ind} = \left( \frac{e}{m} \right)^2 \int \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} \langle E^2(k,\omega) \rangle \pi \delta(\omega - kv).
\]

Using Eqs. (16)–(21), this becomes

\[
D^{ind} = RD^{ext}.
\]

Here \( R \) is defined by

\[
R = \int \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} |\epsilon(k,\omega)|^2 A(k) \frac{\langle \epsilon(k,\omega) \rangle^2}{|\epsilon(k,\omega)|^2 + |k| |\epsilon(k,\omega)|^2/|\pi\nu_0^2|},
\]

where

\[
\langle \epsilon(k,\omega) \rangle = -\pi \frac{\omega}{k^2} \frac{\partial \langle f \rangle}{\partial \nu}.
\]

is the imaginary part of the dielectric \( \epsilon(k,\omega) \) and

\[
A(k) = \int d\nu_\perp \int d\nu_\parallel e^{-ikx_\parallel - \tau_{el}(x_-,\nu_-)}
\]

\[
= 2\pi \frac{[1 - J_0(6\nu_0^2k)]}{k^2}.
\]

(\( J_0 \) is in an ordinary Bessel function and the factor \( 6\nu_0^2 \) is approximate). The total plasma diffusion coefficient is then

\[
D = D^{ind} + D^{ext} = (1 - R)D^{ext}.
\]

The parameter \( R \) is less than unity in a stable plasma.13

### III. Diagnostics

In the plasma simulation, we treated a periodic system of length \( L = 10\pi\lambda_D \) (\( \lambda_D \) is the Debye length) in which we integrate \( N_p \) particle trajectories. The trajectories were obtained using a finite time step of \( 0.2\omega_0^{-1} \) (\( \omega_0 \) is the plasma frequency) and by solving Poisson's equation on a grid divided into 512 zones. The value of \( N_p \lambda_D/L = 65 \) was necessary for an accurate determination of the one-particle distribution function and the two-point correlation function and closely approximates a collisionless plasma. We measured the single-particle distribution function by counting the number of particles in a phase-space cell. At \( v = 0 \) in an initial Maxwellian distribution, the average number of particles in the smallest cell of size \( L/512 \) by \( v_h/200 \) was 8. Throughout the calculations, the spatially averaged distribution function remained Maxwellian (i.e., \( f_0(v) = (2\pi)^{-3/2} \exp[-v^2/(2v_h^2)] \)). Therefore, we generally achieved an ensemble-averaged measurement by averaging over the system length. However, for the two-particle correlation function, we performed additional averaging over the velocity region \( |v| < v_h \), where the turbulence was assumed to be homogeneous in velocity. We found that statistical accuracy was not reliable when using small numbers of particles or small numbers of phase-space cells for ensemble averaging. When \( N_p \lambda_D/L \) or the number of particles per cell is small, integrals of the correlation function can be measured instead.14

During the experiments, we performed a number of diagnostics on the system every \( 200\omega_0^{-1} \) (up to \( 2200\omega_0^{-1} \)). To do so, we divided the phase space into cells of size \( \Delta x_w \) by \( \Delta v_w \) and counted the number of particles per cell, \( N \). We then calculated the mean number of particles per cell, \( \langle N \rangle \), and the fluctuation about the mean, \( \delta N = N - \langle N \rangle \). The particle distribution (averaged over a cell) is \( \overline{f} = \overline{N}/(\Delta x_w \Delta v_w) \) and the fluctuation is \( \overline{\delta f} = \delta N/(\Delta x_w \Delta v_w) \). The bar indicates the average over the cell size.

In order to measure the two-point correlation function as accurately as possible, we used a phase-space cell of size \( \Delta x_w = \Delta x_s = L/512 \) and \( \Delta v_w = \Delta v_s = v_h/200 \). Let \( \langle ij \rangle \) de-
note the coordinates of this cell in the phase space. Then, the correlation function we measured is
\[
\langle \delta f(1) \delta f(2) \rangle = \frac{1}{M_c} \sum_{i,j} \delta f \left( |\Delta x_i + x_\perp | \Delta v_i + v_\perp \right)
\]
which the \(i\) sum is over the system length \(|i\Delta x_i| < L\), and the \(j\) sum is over the small region \((\Delta x_a)\) of velocity space where \(\Delta v_a \delta f \ln \langle f(\partial \varepsilon / \partial v_x) | \Delta v_i | < 2\Delta v_a / 200\), and \(M_c\) is the total number of cells averaged over.

We determined the probability distribution \(P(\delta f)\) of finding a fluctuation \(\delta f = \Delta N / (\Delta x_a \Delta v_a)\) in a phase-space cell or window of size \(\Delta x_a\) by \(\Delta v_a\). This was done by making a histogram of the particle number fluctuation, \(\Delta N\), found in each window. The phase-space window sizes varied according to \(\Delta x_a < \lambda_D\) and \(\Delta v_a < v_\infty\). The fluctuations described by \(P(\delta f)\) satisfy
\[
\int d\delta f P(\delta f) = 1
\]
and overall charge neutrality
\[
\int d\delta f \delta f P(\delta f) = 0
\]
throughout the experiments.

As a further diagnostic, we followed particle orbits in both the decay and driven turbulence experiments. We selected \(M_p\) particles with velocities \(|u(t)| < 0.1v_\infty\) at time \(t_0\). Tracking these particles in time, we constructed the mean square velocity change
\[
\langle \Delta v^2 \rangle = \frac{1}{M_p} \sum_{i=1}^{M_p} \left[ u_i(t) - u_i(t_0) \right]^2,
\]
where \(u_i(t)\) is the velocity of the \(i\)th particle at time \(t\). We also followed the relative velocity between two particles. Let \(M_\perp\) be the number of particle pairs that have \(v_1(t_0) - v_2(t_0) = v_\perp \pm 0.002v_\infty\) and \(x_1(t_0) - x_2(t_0) = x_\perp \pm 0.1\lambda_D\) at time \(t_0\). Tracking these pairs in time, we constructed the mean square relative velocity change
\[
\langle \Delta v_\perp \rangle = \frac{1}{M_\perp} \sum \left[ (v_1(t) - v_2(t_0)) - (v_1(t_0) - v_1(t_0)) \right]^2,
\]
where the sum is over the \(M_\perp\) pairs.

In all the experiments, we are interested in fluctuations in addition to the thermal (discrete-particle) level. According to discrete-particle fluctuation theory, the correlation function of a thermal plasma is
\[
\langle \delta f(1) \delta f(2) \rangle = n_0^{-1} \delta(x_\perp) \delta\langle v_\perp \rangle \langle f(1) \rangle
\]
\[
+ \langle f(1) \rangle \langle f(2) \rangle \exp \left[ |x_\perp| / \lambda_D \right] / 2n_0 \lambda_D,
\]
where \(n_0\) is the particle number density \(N_p / L\). The first term in Eq. (33) is the self-correlation of discrete particles and the second term is due to the shielding of the discrete particles by the spatially averaged plasma distribution \(\langle f \rangle\). In the simulation, we could not resolve scales less than the smallest measurement cell \((\Delta x = 10\pi\lambda_D / 512, \Delta v = v_\infty / 200)\) so that we measured the discrete particle distribution function to be
\[
\langle \delta f(1) \delta f(2) \rangle = \frac{3262 \langle f \rangle}{\lambda_D v_\infty n_0} \left( A + \frac{\exp \left[ -|x_\perp| / \lambda_D \right]}{3262} \right),
\]
where \(A = 1\) for \(x_\perp < \lambda_D, v_\perp < v_\infty\) and is zero otherwise. We neglect the shielding term, since it is beyond the statistical resolution of the small measurement cells. Here \(\langle f \rangle\) remained Maxwellian throughout the simulations, consistent with a collisional self-heating time of order \(10^3\omega_T^{-1}\). Subtracting out the discrete particle fluctuations, the correlation function we focused on is
\[
C(x_\perp,v_\perp) = \langle \delta f(1) \delta f(2) \rangle - \frac{3262 / n_0 \lambda_D v_\infty}{A},
\]
where \(f_0\) is the initial Maxwellian.

Particle discreteness will contribute to the \(P(\delta f)\) measurements. We can estimate this discreteness contribution as follows. In thermal equilibrium, the particles will be randomly distributed in a phase-space cell. Therefore, \(\langle\Delta N^2\rangle = \langle N\rangle\), so that the probability of finding a discrete particle fluctuation \(\Delta N\) in the cell is
\[
P_{th}(\Delta N) = \left( 2\pi \langle N \rangle \right)^{-1/2} \exp \left[ -\Delta N^2 / (2\langle N \rangle) \right].
\]
In terms of \(\delta f\), we can write this as
\[
P_{th}(\delta f) = \left( \frac{2\pi f_0^2}{\langle N \rangle} \right)^{-1/2} \exp \left[ - \frac{1}{2} \left( \frac{\delta f}{f_0} \right)^2 \right].
\]
Therefore, the width of the discrete particle contribution to \(P(\delta f)\) will be small for large \(\langle N \rangle\), i.e., large windows. This also implies that the smaller windows—where the contributions to \(P(\delta f)\) will be due almost entirely to discrete particle fluctuations—will have \(P(\delta f)\)'s that are Gaussian.

IV. OBSERVATIONS

A. Decay turbulence

In the decay experiments, we followed the decay of an initial phase-space distribution of fluctuations. Because linear plasma waves are not strongly Landau-damped for wavelengths greater than \(\lambda_D\), we prepared an initial phase-space configuration that minimizes the excitation of long-wavelength plasma waves. We set up a periodic "checkerboard" pattern of local excesses and depletions of particles in phase space. The correlation function at \(t = 0\) (Fig. 2) and its contours [Fig. 3(a)] show the periodicity of the checkerboard. Initially, the holes are phase-space regions devoid of particles. The particles removed from these holes were put randomly in the phase-space regions between the holes. This explains why initially \(P(\delta f) = f_0\) is sharply defined while \(P(\delta f) > 0\) is randomly distributed about \(\delta f = + f_0\) [cf. Fig. 2(c)].

As time elapsed, we performed frequent diagnostics on the system. All measurements discussed in this section were made in the velocity region \(|v| < v_\infty\), where \(\delta f / \partial v \approx 0\) (measurements made in the region \(v = -v_\infty\) yielded similar results). We found that the statistical properties of the system were not sensitive to details of the initial conditions. After a few
tens of plasma times, a fully turbulent, random distribution of fluctuations evolved. This can be seen in the contour lines of the correlation function at \( \omega_p t = 40 \), where there is only slight evidence for any remnant of the initial phase-space periodicity [cf. Fig. 3(c)]. By the time \( \omega_p t = 60 \), complete destruction of the initial checkerboard periodicity is achieved. Also, by this time, the half-width and height of the correlation function have decreased significantly (cf. Fig. 4). New randomly distributed phase-space structures of size \( \Delta x \leq \lambda_D \) by \( \Delta v \leq 0.1 v_{th} \) have formed. The steady increase in the velocity width of the correlation function for \( 60 < \omega_p t < 220 \) is evident from Fig. 4. This implies that fluctuations with larger velocity scales are being produced. This occurs even as the mean square fluctuation level [measured by the peak of the correlation function \( C(\Delta x, \Delta v, t) \)] is decreasing with time (cf. Figs. 4 and 5).

A kinetic theory described by an equation such as Eq. (7) predicts that the initial checkerboard fluctuations are torn apart by ballistic motion and diffusion. This is the process of orbit stochastic instability (cf. Sec. II). Two phase-space points in a check will be diffused by the turbulent electric fields caused by the large \( |\vec{D}| \) initially present. The time \( \tau_{st}(x, v) \) for two phase-space points (initially separated by \( x, v \)) to be diffused from each other spatially by a distance \( k^{-1} \sim \lambda_D \) is given by Eq. (15). The diffusion coefficient in \( \tau_0 \) is

\[
D = \frac{\langle e/m \rangle^2 \langle E^2 \rangle}{4 \pi n_0 \mu m^2} \lambda_D^2 v^4 \omega_p^2, \quad (38)
\]

where \( \langle E^2 \rangle \) is the mean square electric field. In thermal equilibrium, \( \langle E^2 \rangle / \langle 4 \pi n_0 \mu m^2 \rangle = (n_0 \lambda_D)^{-1} \). During the initial decay, the measured value of \( C(\Delta x, \Delta v, t) \) was approximately seven times the discrete-particle function level [see Eq. (35)] so that \( \tau_0 \approx 10 \omega_p^{-1} \). We recall from Sec. II that orbit stochastic instability will destroy a fluctuation of size \( k^{-1} \) down to the scales \( x, v \) in a time \( \tau_{st}(x, v) \). Therefore, using Eq. (15), we calculate that the initial checkerboard fluctuations (of size \( \lambda_D, 0.1 v_{th} \)) should be "chopped up" or destroyed down to the smallest measurement cell (of size \( 0.061 \lambda_D, 0.005 v_{th} \)) in a time \( \tau_{st}(0.061 \lambda_D, 0.005 v_{th}) \approx 60 \omega_p^{-1} \). However, we observe fluctuations of size \( \Delta x \approx 0.1 v_{th} \), even up to \( \omega_p t = 220 \).

In addition to the fluctuation decay time, the qualitative features of the decay process are in disagreement with Eq. (7). We have observed that \( C(x, v, t) \) for all \( x < 0.2 \lambda_D \) and \( v < 0.02 v_{th} \) decayed at the same rate (cf. Fig. 6). The shape of this inner "core" of \( C(x, v, t) \) persists through the run (compare Fig. 1 with Fig. 7). However Eq. (7) implies that \( C(x, v, t) \) would decay with a time scale \( \tau_{st}(x, v) < 60 \omega_p^{-1} \) (larger \( x \) or \( v \) decaying faster). The data of Fig. 6 were not accurate enough to determine the exact decay rate. However, the decay of the core is consistent with a power law decay (time scale \( \approx 3 \tau_0 \)) or an exponential decay (time scale \( \approx 10 \tau_0 \)). (A model for the time decay is discussed in Sec. VIII.) We have also observed that, for \( v \) values outside the core region (\( |v| > 0.2 v_{th} \)), the correlation function decays (decreases) for \( \omega_p t < 60 \), but increases thereafter (cf. Fig. 4). This leads to a \( v \) correlation function, \( C(0, v, t) \), composed of two distinct \( v \) regions. This is particularly evident in Fig. 1(c). The inner core (\( \Delta v \leq 0.02 v_{th} \)) forms by \( \omega_p t \approx 60 \) and persists through the simulation. The main body (\( |v| > 0.02 v_{th} \)) of \( C(0, v) \) expands to larger \( v \) scales for \( \omega_p t > 60 \). This increase in scale size apparently occurs in the \( v \) dimension only and not in the \( x \) dimension. From Fig. 5 we see that \( C(x, 0) \) retains its form during \( 60 < \omega_p t < 220 \), implying a smooth distribution of spatial scales whose maximum is approximately \( \lambda_D \). We can define a characteristic velocity scale \( \Delta v \) and spatial scale \( \lambda_D \) by

\[
C(\Delta x, \Delta v, t) = C(\Delta x, 0), \quad (39)
\]

From Fig. 7 we see that \( \Delta_D / \lambda_D \approx 10 \omega_p^{-1} \). For small \( x \) and \( v \), this appears to be the case for most of the simulation. However, the production of the \( v \) tail at later times leads to \( \Delta_D / \lambda_D < 10 \omega_p^{-1} \). For example, in the extreme case of Fig. 1, \( \Delta_D / \lambda_D \approx 10 \omega_p^{-1} \) in the core region, whereas \( \Delta_D / \lambda_D \approx 5.3 \omega_p^{-1} \) in the tail region.

The decay of the initial checkerboard pattern can also be seen in the time dependence of \( P(\vec{D}) \). Initially, there is a peak in \( P(\vec{D}) \) at \( \vec{D} = f_0 \) for the background material and at \( \vec{D} = -f_0 \) for the hole material [cf. Fig. 2(c)]. In other words, the most likely hole fluctuation is \( \vec{D} = +f_0 \) and the most likely background level is \( \vec{D} = -f_0 \). Figure 8(a) shows a typical \( P(\vec{D}) \) at a later time. The amplitude for \( \vec{D} > 0 \) has been reduced significantly. The reduction of the maximum amplitude for fluctuations with \( \vec{D} < 0 \) is less dra-
mastic. During the time $\omega \mu t < 80$, we observed that the $\delta f > 0$ material decays faster than the $\delta f < 0$ material. For $\omega \mu t > 80$, the maximum $\delta f < 0$ is roughly constant in time, while that of $\delta f > 0$ decreases.

The contribution of discrete-particle fluctuations to the observed $P(\delta f)$ is small for all but the smallest windows. For example, using Eq. (37), we calculate that the width of the discrete-particle Gaussian contributing to Fig. 8(a) is $\delta f / f_0 = 0.015$. The width of $P(\delta f)$ in Fig. 8(a) is substantially larger than this. However, the discrete-particle contribution is not small for all window sizes. The $P(\delta f)$ for the smallest window ($\Delta x, \Delta v$) was Gaussian and due almost entirely to discrete-particle fluctuations. Also, as time elapses and the turbulent fluctuation level approached the thermal level, discrete-particle fluctuations make a larger and larger contribution to the observed $P(\delta f)$ for the same window size.

Most of the $P(\delta f)$ curves are non-Gaussian or skewed toward $\delta f < 0$. The magnitude of the skewness is frequently of order $-1$. The degree of skewness depends on the window size $A_w$ and time. Consider a fixed time (greater than $\omega \mu t = 80$). As $A_w$ is increased from the smallest window of size $A_w = A = 6.136 \times 10^{-5} \lambda_D v_{th}$ to $A_w = 0.05 \lambda_D v_{th}$, $P(\delta f)$ goes from being Gaussian to being skewed. As $A_w$ is
increased further, the skewness decreases, i.e., its absolute magnitude decreases. This dependence of the measured skewness on window size is shown in Fig. 9. The skewness is small for very small $A_w$ because the discrete-particle contribution to $P(\bar{y})$ becomes dominant in accordance with Eq. (36). The decrease in the skewness for large $A_w$ (where discrete-particle effects becomes negligible) can also be seen from Figs. 10(b) and 10(c). For a given window size, $P(\bar{y})$ becomes more skewed as time elapses. This can be seen by comparing Fig. 10. Figure 8(b) shows a typical plot of the measured skewness against time.

Figure (11) shows the results of orbit tracking during the decay of the fluctuations. Measurements were made after the fully developed turbulent state had emerged (after $\omega_w t = 80$). It is clear from Fig. 11(a) that the particles diffused in velocity $\langle \Delta v^2 \rangle$ varies linearly with $t$. The mean square relative velocity change for an initial particle separation of $v_0 = 0.02v_{th}, x_0 = 0.12A_D$ is shown in Fig. 11(b). Up to $\omega_w t - 80 = 40$, the relative diffusion coefficient $D_\perp$ is clearly much less than the one-particle diffusion coefficient $D_0$ of Fig. 11(a). However, $D_\perp \approx 2D$ thereafter. For the larger initial separation of $x_0 = 1.47A_D, v_0 = 0.02v_{th}$, we see from Fig. 11(c) that $D_\perp \approx 2D$. These results are in agreement with the stochastic instability model. Recall that $k_0^{-1} \approx \lambda_D$ for the decaying fluctuations. Therefore, $D_\perp \approx 2D$ for

![Graph](image-url)
\[ x_\text{,} = 0.12 \lambda_\text{D}, \text{but } D_\text{,} = 2D \text{ for } x_\text{,} = 1.47 \lambda_\text{D}. \]  

We also note that \( \tau_0 \approx \omega_p^{-1} \) during the decay, so that Eq. (15) gives \( \tau_0 \approx \omega_p^{-1} \) for initial values of \( x_\text{,} = 0.12 \lambda_\text{D}, \) \( v_\text{,} = 0.02 \nu_\text{wh} \). This is consistent with Fig. 11(b) where the particles diffuse independently after a time interval \( \tau_0 \approx \omega_p^{-1} \). We conclude that the particle orbits are undergoing stochastic instability at the rate \( \tau_0 \approx \omega_p^{-1} \). An individual fluctuation therefore decays at the rate \( \tau_0 \approx \omega_p^{-1} \).

The decay of an individual fluctuation can also be inferred from the contour lines of constant \( C(x_\text{-}, v_\text{-}) \) [cf. Fig. 3]. The contours resemble nested ellipses whose major axis is tilted toward \( x_\text{-}, v_\text{-} > 0 \). This tilt is consistent with a decaying fluctuation since \( x_\text{-} > 0 \) and \( v_\text{-} > 0 \) regions are correlated. Moreover, the angle which the major axis of the contours of Fig. 3 makes with the \( x_\text{-} \) axis is consistent with that for stochastic instability. To see this, we note that the contours of constant \( \tau_0 \approx \omega_p^{-1} \) can be written as

\[
\frac{(x_\text{-})^2}{\lambda_\text{D}^2} - 2 \left( \frac{x_\text{-}}{\lambda_\text{D}} \right) \left( \frac{v_\text{-}}{0.1 \nu_\text{wh}} \right) (0.1 \omega_p \tau_0) + 2 \left( \frac{v_\text{-}}{0.1 \nu_\text{wh}} \right)^2 (0.1 \omega_p \tau_0)^2 = \text{const}.
\]

For \( \omega_p \tau_0 \approx 10 \), these contours approximate the tilted contours of Fig. 3. We conclude from this and the \( \langle \Delta v^2 \rangle \) measurements that the fluctuations producing the closed contours of Fig. 3 are being torn apart at a rate \( \tau_0 \approx \omega_p^{-1} \). This rate is faster than the decay rate we have observed for the aggregate fluctuation level \( \langle \Delta \bar{f} \rangle^2 \). We believe that this occurs because an individual fluctuation decays at the rate \( \tau_0 \approx \omega_p^{-1} \), but also tends to recombine with other fluctuations.

### B. Driven turbulence

In this series of experiments, we started with a spatially homogeneous, Maxwellian distribution of particles. We then imposed a group of 300 waves with a wide phase velocity spectrum \( \langle \Delta v_\text{ph} \rangle = 3 \nu_\text{ph} \). In order to minimize transient effects, the waves were turned on adiabatically from \( t = 0 \) and reached constant amplitude by \( \omega_p t = 60 \). All the diagnostic measurements were made after this time. The wavelengths were chosen randomly between \( \lambda_\text{D} \) and \( 2 \lambda_\text{D} \) in order to avoid mode coupling between the waves and any long-lived plasma oscillations. The wave amplitudes were chosen such that their turbulent trapping width\(^2 \) would \( \approx 0.1 \nu_\text{wh} \).

Consistent with the clump theory, fluctuations were not produced near \( v = 0 \) (where \( \delta f_\text{ph}/\delta v \approx 0 \)), but were generated in the region near \( v = \nu_\text{wh} \) (where \( \delta f_\text{ph}/\delta v \neq 0 \)). Figure 12 shows the correlation function generated at \( v = \nu_\text{wh} \) when the particle orbits were determined with only the external waves present (Poisson's equation was not invoked). Figure 13 shows the result when the particle orbits were determined by the external waves and the plasma self-consistent fields (Poisson's equation was invoked). The peak values of the \( C(x_\text{-}, v_\text{-}) \) were enhanced in the self-consistent case by 1.8 over the non-self-consistent case. A typical \( P(\Delta \bar{f}) \) curve for the self-consistent case is shown in Fig. 14(a). It is evidently Gaussian, as were the \( P(\Delta \bar{f}) \) curves for the non-self-consistent case [cf. Fig. 14(b)].

These results can be directly related to the renormalized kinetic theory described by Eq. (7). We recall from Sec. II that an externally applied electric field spectrum of waves, \( \langle E^2(k, \omega) \rangle \text{, ext} \), will randomly rearrange the particle distribution and create clumps of phase-space density. In the absence of the plasma self-consistent fields, the external waves...
will produce a fluctuation level given by

\[ \langle \delta f(1) \delta f(2) \rangle = 2\tau_0 D_{12}(x_\perp) \frac{\partial f(1)}{\partial v_1} \frac{\partial f(2)}{\partial v_2} , \]

where we have evaluated Eq. (17) at the \( x_\perp, v \) values where \( \tau_C(x_\perp, v) = \tau_0 \). As we shall see, this choice of \( x_\perp, v \) is convenient for comparisons of the self-consistent and non-self-consistent cases, i.e., the weakly varying logarithm factor in \( \tau_C(x_\perp, v) \) can be set equal to unity. Moreover, we will focus on the ratio of Eq. (40) to an analogous self-consistent expression, in which case the logarithm factor no longer enters. In Eq. (40), \( D_{12}(x_\perp) \) is given by

\[ D_{12}(x_\perp) = \left( \frac{e^2}{m} \right) \int \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} (E^2(k,\omega))^{\text{ext}} \times \pi \delta(\omega - kv) \exp(ikx_\perp) , \]

and \( \tau_0 = (4k_n^2 D)^{-1/2} \) where \( D = D_{12}(0) \), and \( k_n \) characterizes the driven fields, i.e.,

\[ k_n^2 = \frac{1}{2D} \left( \frac{\partial^2 D}{\partial x^2} \right)_{x=0} . \]

Estimates of \( \tau_0 \) and \( D \) show that Eq. (40) reasonably approximates the fluctuation level of Fig. 13. In the presence of the plasma self-consistent fields, \( D \) in Eq. (41) will be reduced by the plasma dielectric, i.e., the external waves will be shielded by the plasma. For measurements at \( v = v_n \), Re \( \varepsilon = 1 + 0.28(k\lambda_D)^{-2} \) and Im \( \varepsilon = 0.76(k\lambda_D)^{-2} \). We take \( k = \lambda_D^{-1} \) since the wavelengths of the driven fields are weighted more heavily in \( D^{\text{ext}} \) (we have verified this in the orbit tracking experiments). Therefore, \( |\varepsilon|^2 = 2.2 \) and \( \varepsilon \to D^{\text{ext}} = \varepsilon D^{\text{ext}} \). Then, according to Eq. (1), the fluctuation level due to \( D \) and the plasma self-consistent fields is

\[ \langle \delta f(1) \delta f(2) \rangle = 2\tau_{\text{sc}} (2.2)^{-1} D_{12}(x_\perp) \frac{\partial f(1)}{\partial v_1} \frac{\partial f(2)}{\partial v_2} , \]

where \( \tau_{\text{sc}} \) is the characteristics lifetime of the mean square fluctuation level in the presence of \( D \) and the self-consistent fields, i.e., \( T_{\text{sc}} \) on the left-hand side of Eq. (7) is assumed to include hole self-binding as well as the velocity streaming and diffusion terms. Since the data gives a value of 1.8 for the ratio of the left-hand side of Eq. (43) to Eq. (40), we obtain \( \tau_{\text{sc}}/\tau_0 = 3.96 \).

Let us now compute the value of \( \tau_{\text{sc}}/\tau_0 \) predicted by the theory of Sec. II in order to compare it with the value 3.96. According to Eq. (7), \( \tau_{\text{sc}} \) can be determined from Eq. (11) and Eq. (14) by using \( D_{\perp} \) due to the total (external and induced) fluctuation field [cf. Eq. (27)]. Therefore, for small \( x_\perp \),

\[ D_{\perp} = (k_n^2 D^{\text{ind}} + k_n^2 D^{\text{ext}})x_\perp = (k_n^2 R + k_n^2)D^{\text{ext}}x_\perp \]

where \( k_n^2 \) is derived from the induced fields \( [D_\perp = D^{\text{ind}} + \text{induced fields}] \).

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\[ \langle \delta f(1) \delta f(2) \rangle = 2\tau_{\text{sc}} (2.2)^{-1} D_{12}(x_\perp) \frac{\partial f(1)}{\partial v_1} \frac{\partial f(2)}{\partial v_2} , \]

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\[ D_{\perp} = (k_n^2 D^{\text{ind}} + k_n^2 D^{\text{ext}})x_\perp = (k_n^2 R + k_n^2)D^{\text{ext}}x_\perp \]

where \( k_n^2 \) is derived from the induced fields \( [D_\perp = D^{\text{ind}} + \text{induced fields}] \).
The stochastic instability model for $T_{12}$ would give

$$
\tau_{m} = \left( \frac{k_{+}^{2} R + k_{-}^{2}/k_{a}^{2}}{|\epsilon|} \right)^{-1/3} .
$$

From Figs. (12) and (13), we find that $k_{+}/k_{a} \approx 0.67$ so that

$$
\tau_{m}/\tau_{0} = 1.21 .
$$

Therefore, the measured mean square fluctuation level was 3.2 times larger than that predicted by the renormalized theories, i.e.,

$$
\langle f^{2}(1) \rangle \approx 3.2 \left( 2\tau_{m} D_{12}^{m} \frac{\partial \langle f(1) \rangle}{\partial v_{1}} \frac{\partial \langle f(2) \rangle}{\partial v_{2}} \right) .
$$

Note that the enhancement factor of 3.2 is more accurate than the approximate solutions given by Eqs. (40) and (43) might imply, since we have used only the ratio of the solutions.

Figure 15 shows the result of the orbit tracking measurements of $\langle \Delta v^{2} \rangle$ in the presence of the external waves only. We made the measurements in the region $|v|<v_{th}$ (where $\partial f_{eq}/\partial v=0$) so that the production of clumps would not interfere with the analysis. The mean square velocity change increased linearly with time, consistent with particle diffusion [cf. Fig. 15(a)]. Figure 15(b) shows the results for the mean square relative velocity change of particle pairs, whose initial separation was $x_{-}=0.12\lambda_{D}$, $v_{-}=0.08v_{th}$. The results for an initial separation of $x_{-}=1.47\lambda_{D}$, $v_{-}=0.02v_{th}$ are shown in Fig. 15(c). As in the decay turbulence case, the results are consistent with the predictions of the orbit stochastic instability model. Actually, this is a case where the model should most unambiguously apply, i.e., in an externally applied turbulent wave spectrum. It is clear that $D_{-} < D$ for $t<80\omega_{p}^{-1}$, $k_{+}(x_{-},v_{-}) \approx 40\omega_{p}^{-1}$ and $k_{-}(x_{+},v_{+}) \approx 1$. $D_{-}=2D$ for $t>80\omega_{p}^{-1}$.

Comparison of Figs. 11 and 15 show that the orbit exponential times $\tau_{st}(x_{-},v_{-})$ are comparable for the self-consistent and non-self-consistent field cases. We note that $k_{+}$ and $D$ (cf. Figs. 11 and 15) are also comparable in the two cases. Therefore, even though one case involves the plasma self-consistent fields while the other does not, they are essentially indistinguishable as far as the orbit stochastic instability $[\tau_{st}(x_{-},v_{-})]$ is concerned. However, as we have seen, the mean square fluctuation level in the driven run is affected by the plasma self-consistent fields [cf. Eq. (46)]. This result is consistent with the results of the decay turbulence experiments. An individual fluctuation decays at the rate $\tau_{st}(x_{-},v_{-})^{-1}$ in accordance with the orbit stochastic instability model of $T_{12}$ in Eq. (7). However, the aggregate fluctuation level decays at a much slower rate. We believe that, though an individual fluctuation decays at the rate $\tau_{st}(x_{-},v_{-})^{-1}$, the mean square (aggregate) fluctuation level decays more slowly because of the tendency for fluctuations to recombine.

V. HOLE MODEL AND INTERPRETATION

In this section, we show that the essential features of the simulation results can be understood in terms of a collection of Bernstein–Greene–Kruskal-like holes. An isolated phase-space hole is a Bernstein–Greene–Kruskal equilibrium, and has been shown to be a state of maximum entropy subject to constraints of mass, momentum and energy. In an ion–electron plasma with immobile ions (as in these simulations), an electron phase-space hole is self-binding. The phase-space density surrounding the hole, being of opposite charge from the hole, is attracted to the region of depleted charge. Alternatively, we note that the holes have negative mass so that two phase-space holes will attract each other. For an isolated, self-bound hole in equilibrium, the hole depth $(-\tilde{f})$, and the hole potential must be sufficient to trap (bind) the hole. If the hole is $\Delta x$ by $\Delta v$ in width, then an approximate calculation based on the maximum entropy argument yields

$$
\tilde{f} = \Delta v (6\omega_{p}^{2}\lambda_{D}^{2})^{-1} g{(\Delta x/\lambda_{D})}^{-1} ,
$$

where

$$
g(z) = (1+2z)[1-\exp(-z)] - 2
$$

and

$$
\lambda^{-2} = \omega_{p}^{2} P.V. \int du (u-v)^{-1} \frac{\partial f}{\partial u}.
$$

(P.V. means principal value and $\lambda \rightarrow \lambda_{D}$ as $v \rightarrow 0$). As $\Delta x/\lambda$ approaches 0 or $\infty$, $g$ has the limits $-{(\Delta x/\lambda_{D})}^{2}/6$ and $-1$ respectively. At $v=0$ of the Maxwellian distributions, $f=1/2\pi\rho_{th}^{-1} = f_{0}$, so that Eq. (47) becomes

$$
\tilde{f} = (2\pi)^{1/2}\Delta v \rho_{th}^{-1} g{(\Delta x/\lambda_{D})}^{-1} .
$$

For $\Delta v<\rho_{th}$ and $\Delta x \approx \lambda_{D}$, Eq. (50) implies an equilibrium hole depth $\tilde{f} \ll f_{0}$. 

Berman, Tetreault, and Dupree
Of course, turbulent fluctuations cannot be exact equilibria since they are continuously colliding or interacting with each other. Berk et al. have investigated some of the properties of binary collisions between two phase-space holes. In a collision between two holes with small relative velocity, the two holes can merge into a single-hole structure (cf. Fig. 6 of Ref. 11). For larger relative velocities, tidal deformations of the holes occur (cf. Fig. 7 of Ref. 11). They explain the tendency of holes to coalesce by a useful gravitational analogy in which holes may be regarded as gravitating masses. In Ref. 5, hole coalescence is shown to be the result of the plasma's tendency to attain states of higher entropy. The entropy is increased as the phase-space density is mixed during the hole-hole coalescence. In this way, the "coarse-grained" average of $f$ (and thus, the electrostatic fields) is reduced. We would expect both hole coalescence and tidal deformation to occur simultaneously in a turbulent plasma. We therefore model turbulent fluctuations as a collection of interacting holes with the interaction consisting of two competing processes—the coalescing of holes with the concomitant increase of $\Delta x$ and $\Delta v$ scales and the breaking up of the holes into smaller fragments due to $D_z$ diffusion.

The tendency of holes to coalesce and increase their $\Delta x$ and $\Delta v$ scales can be understood by considering the hole mass $M$, energy $T_p$, and entropy $-\sigma$. When two holes (1) and (2) coalesce that have nonzero relative velocity, the self-energy of the resulting hole (3) is always less than the sum of the original self-energies, i.e., $T_{03} < T_{01} + T_{02}$. The total mass, however, is conserved ($M_3 = M_1 + M_2$). For holes with $\Delta x/\lambda < 1$, the entropy $-\sigma$ is an increasing function of $\Delta x/\lambda$ which in turn is an increasing function of $M^2/T_p$. Therefore, the holes with $\Delta x/\lambda < 1$ tend to coalesce since this will lead to increased $M^2/T_p \Delta x/\lambda$, and $-\sigma$. Furthermore, one can also show that for $\Delta x/\lambda < 1$, $(\Delta v/\nu_h)^2 = (\Delta x/\lambda) M (\mu m \lambda)^{-1}$, so that $\Delta v$ also increases as the holes coalesce. On the other hand, the entropy (constant for $T_0$) decreases with $\Delta x/\lambda$ for $\Delta x/\lambda > 1$ and is proportional to $(T_0 M)^{1/2}$. For this case, the most probable final state for interacting holes is to concentrate all the energy in a hole of length $\Delta x/\lambda \approx 4$ (with a corresponding mass of $M^2/T_0 \approx 5$) and put the remaining mass in a hole of zero energy, i.e., mix it into the positive $\delta f$ background. This means that hole coalescing will produce larger $\Delta x$ scales but not greater than a few $\lambda_D$. However, $\Delta v$ could increase without limit. Since $\tilde{f}$ is determined by $\Delta x$ and $\Delta v$ (cf. Eq. [50]), the fact that the $x_-$ and $\nu_-$ widths of $C(x_-,\nu_-)$ remain approximately constant as $C(x_-,\nu_-)$ decreases with time implies that $\tilde{f}$ for an individual fluctuation remains constant but that the packing fraction of the fluctuation must decrease.

For comparisons between the observations and the hole model, it is convenient to express Eq. (50) in terms of the hole area $A = \Delta x \Delta v$ and the parameter $\tau = \Delta x/\Delta v$. Therefore,

$$\tilde{f} = \frac{\sqrt{\pi}}{6} \left( \frac{1}{\omega_v \nu_h} \frac{A}{A_D} \right)^{1/2} g \left[ \left( \frac{A}{A_D} \right)^{1/2} \right]^{-1},$$

where $\Delta x = (\tau \Delta v)^{1/2}$, $\Delta v = (\tau/\nu_h)^{1/2}$, and $A_D = \tau^{-1} \lambda_D^2$ is the area of a Debye-length-long hole. If we identify $\tau$ with the characteristic scale ratio $\Delta_x/\lambda_D$ defined in Sec. IV, then we find that $\tau \approx 10 \omega_v^{-1}$ at $\omega_v t = 120$. With this value of $\tau$, $A_D = 0.1 \lambda_D \nu_h$. The independence of $\tau$ can be understood by the following intuitive argument. Consider a steady-state distribution of holes. As the holes collide with each other, "tidal" electric field effects will limit the scales that can coexist simultaneously in the system. Coexistence of disparate scales demands that holes with different $\Delta x$, or $\Delta v$, do not destroy each other by tidal forces. The electric field of hole (1) must be comparable to the self-binding field of another hole (2) across which it acts, i.e.,

$$\left( \frac{\partial E}{\partial x} \right)_1 \Delta x_1 \approx \left( \frac{\partial E}{\partial x} \right)_2 \Delta x_2$$

or

$$\left( \frac{\partial v}{\partial x} \right)_1 \approx \left( \frac{\partial v}{\partial x} \right)_2.$$
only. Here \( C(x,0) \) retains its form during \( 60<\omega_c t<220 \), implying a smooth distribution of spatial scales whose maximum is \( \approx A_D \). These features of \( C(x,\rho) \) are consistent with the maximum hole entropy arguments of Ref. 4. Fluctuation energy tends to flow into holes with larger and larger velocity scales. The most probable holes, however, are a few Debye lengths long.

As we have seen in the decay experiments, the production of large \( \rho \) scale fluctuations occurs even as the mean square fluctuation level decays. We interpret this fact as the result of the tendency of holes to coalesce. As two holes coalesce, hole and interstitial material between the holes mix so that the coarse-grained phase-space density is reduced. Alternatively, we can view this as a reduction in the hole-packing fraction due to hole coalescing. The driving force behind this coalescing and mixing process is the tendency of the holes to attract each other or self-bind. It is also this tendency which explains the slow decay rate of the aggregate fluctuation level. As we have seen from the tilt of the correlation function contours and the orbit tracking experiments for \( \rho = \frac{1}{2} \), an individual fluctuation decays at the rate \( \tau_{ci}^{-1}(x,\rho) \); as the holes collide with each other, their constituent particles undergo stochastic instability. However, the hole fragments of \( \delta f < 0 \) material resulting from these collisions tend to recombine (because of the self-binding tendency) into new holes composed of different (or unrelated) hole fragments. The self-binding or recombination effect is somewhat less than the \( \tau_{ci}^{-1}(x,\rho) \) collision rate. For \( \rho = \frac{1}{2} \), there is a net decay rate of order of \( [3\tau_{ci}(x,\rho)]^{-1} \) (cf. item B1 of Sec. I).

The tendency of the \( \delta f < 0 \) hole fluctuations to attract each other and coalesce also provides a simple interpretation of the \( P(\delta f) \) observation in the decay experiments. Consider the \( t = 0 \) curve of Fig. 2(c). As time elapses, the background material tends to repel itself and mix with the hole material. Alternatively, the reduction in the number of holes (due to hole coalescence) creates more room in phase space into which the background material can expand. The production of holes with different phase-space area [and according to Eq. (51), different \( f \)] will broaden the initial \( P(\delta f > 0) \) peak. Equivalently, as the holes tend to coalesce into new holes, the mixing of hole and background material reduces the coarse-grained phase-space density \( \delta f \). The rate at which the mixing reduces the \( \delta f > 0 \) fluctuations is less than that for \( \delta f < 0 \) since the holes tend to self-bind. These considerations imply a distribution \( P(\delta f) \) such as that shown in Figs. 8(a) and 1(c).

As we discussed in Sec. I, one requirement for a strong negatively skewed \( P(\delta f) \) is that the hole-packing fraction \( \rho \) be much less than \( \frac{1}{2} \). Equivalently, fluctuations with \( \delta f \) large and negative must be more likely than \( \delta f \) large and positive. Clearly, this will occur in the decay experiments if the holes tend to self-bind. If there is an external source of clumps—as in the driven experiments—the hole-packing fraction will be maintained at one-half. Though any individual fluctuation will tend to decay (and reduce the hole-packing fraction), new clumps are being created by the rearrangement of \( f_d(\mathbf{v}) \) throughout phase space. This explains why we observed \( P(\delta f) \) to be Gaussian in the driven experiments—both with and without self-consistent fields.

**VI. A THEORETICAL MODEL OF** \( \mathcal{P}(\delta f) \)

As we have shown in Sec. V, the essential features of the observations can be explained by a simple model of interacting Bernstein–Greene–Kruskal-like holes. Although the discussion of Sec. V is sufficient to understand the qualitative features of the model, it is important to determine if the model can explain any of the quantitative aspects of the simulation results. Therefore, we have developed a mathematical model for \( P(\delta f) \). As we shall see, the \( P(\delta f) \) model we propose has many simplifying assumptions. However, reasonable agreement is obtained with the observations (cf. Sec. VII).

Recall that \( P(\delta f) \) was obtained by making a histogram of the particle number fluctuation in a phase-space window of area \( A_w \). As we have shown above, the \( P(\delta f) \) observations are consistent with a distribution of interacting Bernstein–Green–Kruskal-like phase-space holes. We will assume that each hole is identified by the area \( A \) in which it fits. Because of hole coalescence and decay, it seems reasonable to assume a continuous distribution of hole sizes \( A \). Therefore, the theoretical model of \( P(\delta f) \) that we propose has two essential ingredients: the number (it will be negative) of particles, \( N_h(A) \), in a hole of area \( A \); and the number of holes, \( N(A) \), in a window of size \( A_w \). We define \( \mathcal{P}(\delta f,A) \) as the probability of observing a fluctuation \( \delta f \) in a window that has \( h \) holes in it on the average. Using \( N_h(A) \) and \( F \), we solve a model equation that relates \( \mathcal{P}(\delta f,A) \) to the average number of holes in a window \( h = [f_{da} F(A)] \). Then, we adjust \( \delta f \) so that the solution \( \mathcal{P}(\delta f,A) \) agrees with the observed \( P(\delta f) \). The idea is to model \( P(\delta f) \) with a distribution of holes \( F \) where each hole has a specific phase-space structure \( [N_h(A)] \). We assume that there is a smallest \( A_{min} \) and largest \( A_{max} \) hole size at any given time so that \( h_{min} = [A_{min} F(A)] \). We therefore model \( P(\delta f) \) with the probability \( \mathcal{P}(\delta f,h_{min}) \) of observing the fluctuation \( \delta f \) in a window that has \( h_{max} \) holes in it on the average.

Lacking a detailed theory, we are forced to make intuitive and simplifying assumptions about the expressions for \( N_h(A) \) and \( F(A) \) in a turbulent plasma. We first consider the case where the window area is (much) larger than any of the holes \( A_h > A \). We will ignore the possibility (unimportant if \( A_h > A \) that the window overlaps only a portion of a hole. Then, \( N_h(A) = \hat{f}(A) A \), where \( \hat{f}(A) \) is the depth of a hole of area \( A \). In order to obtain a model of \( \delta f(A) \) for a turbulent plasma, we resort to a modified version of the isolated hole expression (cf. Sec. V). This seems to be a reasonable approach since the effects of diffusion and hole binding (coalescence) are comparable, i.e., the aggregate fluctuation lifetime is of order of three times the lifetime of an individual fluctuation. First, we will assume that small \( [A_h A_D] \) holes have the same ratio of \( \Delta x / \Delta \). This assumption is motivated by the simulation results as discussed in the previous section. We therefore consider Eq. (51) with \( \tau = 10 \omega_c A \). A second modification of the isolated hole expression (50) is also required. We note that Eq. (50) describes an approximate hole

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2451 Phys. Fluids, Vol. 26, No. 9, September 1983 Berman, Treitault, and Dupree 2451
in which all the hole mass is concentrated within an area \( A = \Delta x \Delta v \) and a uniform depth \( \tilde{f} \). However an actual hole of the same mass with \( \Delta x \ll \lambda_D \) will be more spread out with a \( \Delta x \) and \( \Delta v \) approximately twice as large. Therefore, the average \( \tilde{f} \) over the larger area will be about half as large as that given by Eq. (50). On the other hand, when \( \Delta x \gg \lambda_D \), Eq. (50) is reasonably accurate but the simulation results indicate that \( \tau = \frac{\Delta x \Delta v}{2 \pi} \approx 2 \pi \tilde{f} \) compared with its value of \( 10 \omega_p^{-1} \) for \( \Delta x \ll \lambda_D \). Both the large and small \( \Delta x / \lambda_D \) limits are reproduced by the expression

\[
\tilde{f} = \frac{\sqrt{6 \pi}}{\omega_p} \left( \frac{\alpha A}{A_D} \right)^{1/2} \left( \frac{\alpha A}{A_D} \right)^{1/2} - 1,
\]

(55)

where \( \alpha \approx 6, \tau = 10 \omega_p^{-1} \), and \( A_D = 0.1 \nu \lambda_D \) is the area of the Debye-length-long hole. A simple analytical expression for \( \tilde{f} / \tilde{f}_0 \) can be written which agrees well with Eq. (55) in the limit of large \( A > A_D \) and small \( A < A_D \) hole dimensions:

\[
\tilde{f} = \frac{1}{\omega_p} \left( \frac{A}{A_D} \right)^{1/2} \left( 1 + \frac{A}{A_D} \right).
\]

(56)

We will find it useful to use Eq. (56) instead of Eq. (55) for analytical calculations in Sec. VI.

In addition to \( N_h(A) \), we also require a model equation for the hole distribution \( F(A) \). For this, we consider the simple expression

\[
F(A) dA = \left( \frac{p/A}{A} \right) dA,
\]

where \( p \) depends on \( A \) in general. Equation (57) implies that holes with larger \( A \) are fewer in number than those with small \( A \). If \( p(A) \) is a slowly varying function of \( A \), then Eq. (57) has a simple physical meaning. Using Eq. (57), the sum of the areas of the holes with areas between \( A \) and \( eA \) (\( e \) is Euler's constant \( e = 2.718 \ldots \)) is \( A_{\text{total}} = A_w p(A) \) so that \( p(A) = A_{\text{total}} / A_w \). Therefore, \( p(A) \) can be interpreted as the fractional phase-space area of a window that is occupied by the holes of area between \( A \) and \( eA \). We refer to \( p(A) \) as the hole-packing fraction. If \( p(A) \) is independent of \( A \), then all holes have the same packing fraction. We point out that this would imply that large holes would be composed of small holes. We assume that \( p(A) = p \) is independent of \( A \). With this restriction, we use Eq. (57) to calculate the average number \( \bar{n}(A) \) of holes with areas \( A \) or less in a window of size \( A_w \). If \( A_1 \) is the area of the smallest hole, then

\[
\bar{n}(A) = \int_{A_1}^{A_w} dA' F(A') = n_t \left( 1 - \frac{A_1}{A} \right),
\]

(58)

where \( n_t = p A_w / A_1 \) is the average number of holes of area \( A_1 \) in the window. The maximum value that \( \bar{n} \) takes is \( \bar{n}_m = n_t - p A_w / A_m \) where \( A_m \) is the area of the largest-size hole.

\( \mathcal{P}(\bar{s}, \bar{n}) \) can be related to the average number of holes in the window \( \bar{n} \). To obtain this relation, we first consider the simplified case where the phase space is populated with holes—all of which have the same area. If these holes are randomly distributed, the probability of observing \( n_h \) holes in a window, when there are \( \bar{n} \) holes per window on the average, is

\[
\mathcal{P}(n_h, \bar{n}) = \left( \frac{\bar{n}}{n_h} \right) \bar{n}^{n_h - 1} e^{-\bar{n}}.
\]

(59)

This distribution is skewed unless \( \bar{n} \gg 8 \), in which case, \( \mathcal{P}(n_h, \bar{n}) \) approximates the Gaussian distribution \( \exp[-(n_h - \bar{n})^2/(2\delta_n^2)] \). Because the holes are randomly distributed in the window, \( \langle \delta n^2 \rangle = \langle (n_h - \bar{n})^2 \rangle = \bar{n} \). The single hole size distribution (59) satisfies the following difference-differential equation

\[
\frac{d}{d\bar{n}} \mathcal{P}(n, \bar{n}) = \mathcal{P}(n, \bar{n}) - \mathcal{P}(n - 1, \bar{n}).
\]

(60)

Equation (60) can be readily generalized to a distribution of different hole sizes. For this, we define \( N \) to be the number of particles per window that have been removed from the holes (in that window). We also define \( N_{bg} \) as the average number of particles in a window due to the \( \delta f^0 \) background material. Clearly, \( \langle N \rangle = \langle N_{bg} \rangle + \langle N \rangle \). We also define \( \bar{n} = \bar{n}(A) \) to be the average number of holes of size \( A \) or less in the window. Then, the probability \( \mathcal{P}(N, \bar{n}, A) \) of finding \( N \) particles removed from the holes in a particular window by counting only holes of size \( A \) or less, satisfies

\[
\frac{d}{d\bar{n}} \mathcal{P}(N, \bar{n}, A) = \mathcal{P}(N, \bar{n}) - \mathcal{P}(N - N_h, \bar{n}, A).
\]

(61)

We can derive Eq. (61) by relating the probabilities for two distributions that differ only by a small set of holes with \( \Delta \bar{n} < 1 \). The probability \( \mathcal{P}(N, \bar{n}, A) \) can be written as the sum of two independent probabilities:

\[
\mathcal{P}(N, \bar{n}, A) = \mathcal{P}(N, \bar{n}) (1 - \Delta \bar{n}) + \Delta \bar{n} \mathcal{P}(N - N_h, \bar{n}, A).
\]

(62)

The first term in Eq. (62) is the probability of observing the original hole distribution with no additional holes contained in the set \( \Delta \bar{n} \). The second term is the probability of observing one hole in the set \( \Delta \bar{n} \) multiplied by the probability of correspondingly observing \( N_h \) fewer particles (one hole less) in the original distribution. Equation (62) is just Eq. (61) expanded to lowest order in \( \Delta \bar{n} \). In order to obtain \( \mathcal{P}(\bar{s}, \bar{n}, A) \) from Eq. (61)---which we then identify with the observed \( \mathcal{P}(\bar{s}, \bar{n}) \)---we note that \( \bar{s} = (N - \langle N \rangle) / A_w = (N - \langle N \rangle) / A_w \). In addition, we require \( N_h(\bar{n}) = N_h(\bar{n}) = f_0(\bar{n}) A_w \), which, with Eqs. (56) and (58), can be written as

\[
N_h(\bar{n}) = f_0 A_w (\omega_p)^{-1} \left( \frac{A_D}{A_w} \right)^{1/2} \left( 1 + \frac{A}{A_D} \right),
\]

(63)

for \( A < A_D \).

In general, Eq. (61) is difficult to solve. As in the simplified case of Eq. (60), \( \mathcal{P}(N, \bar{n}, A) \) of Eq. (61) is very skewed and sensitive to variations in \( \bar{n} \) when \( \bar{n} \) is small. Moreover, the singular behavior of \( N_h(\bar{n}) \) [cf. Eq. (63) as \( \bar{n} \to 0 \)] makes \( \mathcal{P}(N, \bar{n}, A) \) of Eq. (61) sensitive to \( \bar{n} \) as \( \bar{n} \to \bar{n} \). As we shall see, however, there is an intermediate region of \( \bar{n} \) where \( \mathcal{P}(N, \bar{n}, A) \) of Eq. (61) is only weakly skewed and thereby changes slowly with \( \bar{n} \). For this reason, we choose to solve Eq. (61) as an initial value problem in \( \bar{n} \) where the initial value \( \bar{n} \) lies within this intermediate region of \( \bar{n} \). We can write the formal solution to Eq. (61) as

\[
\mathcal{P}(N, \bar{n}, A) = e^{-\mathcal{A}} \mathcal{P}(N, \bar{n}, A) + e^{-\mathcal{A}} \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\bar{n}}^{\bar{n}} \cdots \int_{\bar{n}}^{\bar{n}} \mathcal{P}(N - N_h, \bar{n}, A) \cdots \mathcal{P}(N - N_h, \bar{n}, A),
\]

(64)
where $\bar{n} = \bar{n}_i + \Delta \bar{n}$. The average value of $N$ is
\[
\langle N \rangle = \int_0^\infty dN N \mathcal{P}(N, \bar{n}) = \int_0^\infty d\bar{n}' N_\bar{h}(\bar{n}'),
\]
whereas the mean square value is
\[
\langle \delta N^2 \rangle = \int_0^\infty dN (N - \langle N \rangle)^2 \mathcal{P}(N, \bar{n}) = \int_0^\infty d\bar{n}' N_\bar{h}(\bar{n}')^2.
\]
Note that $\langle N \rangle$ and $\langle \delta N^2 \rangle$ are defined as functions of $\bar{n}$.

In order to evaluate Eq. (64), we consider the process of adding holes with increasing $A$ [i.e., $h(A)$] into the window. First, we note that if $\partial N_\bar{h}(\bar{n})/\partial \bar{n} = 0$, then Eq. (61) reduces to Eq. (60) and $\mathcal{P}(N, \bar{n})$ will be Gaussian for $\bar{n} > 8$. Therefore, we suspect that $\mathcal{P}(N, \bar{n})$ of Eq. (61) will be Gaussian if
\[
8 \frac{\partial \ln N_\bar{h}(\bar{n})}{\partial \bar{n}} < 1,
\]
and $\bar{n} > 8$. In order to verify this, we consider the skewness
\[
s(\bar{n}) = \frac{\int_0^\infty dN(N - \langle N \rangle)^3 \mathcal{P}(N, \bar{n})}{\left[ \int_0^\infty dN N^2 \mathcal{P}(N, \bar{n}) \right]^{3/2}},
\]
where
\[
\langle N \rangle = \int_0^\infty dN N \mathcal{P}(N, \bar{n}).
\]
Using Eq. (64), we find that
\[
s(\bar{n}) = \frac{\int_0^\infty d\bar{n}' N_\bar{h}(\bar{n}')} {\left[ \int_0^\infty \bar{n}' d\bar{n}' N_\bar{h}(\bar{n}') \right]^{3/2}}.
\]
Figure 16 is a plot of Eq. (70) using $N_\bar{h}(\bar{n})$ obtained from Eq. (55). If $\bar{n}$ is small (few holes in the window), the skewness is large because $\bar{n} < 8$. For larger $\bar{n}$, the skewness decreases and depends only weakly on $\bar{n}$. In this intermediate region of $\bar{n}$, Eq. (67) is satisfied. If $\bar{n}$ is increased further so that Eq. (67) is violated, then the skewness increases again. This latter increase of skewness is rapid and occurs in a region $\Delta \bar{n} < 1$. We are therefore, led to the following prescription for the evaluation of Eq. (64). We choose the initial value $\bar{n}_1$ to be an intermediate value of $\bar{n}$ where the condition Eq. (67) begins to be violated or equivalently where $s(\bar{n})$ is at its minimum. Let this value of $\bar{n}$ be denoted by $\bar{n}_g$. The Gaussian $s(\bar{n})$ is due to those holes with area $A_1$ and are, on the average, $h(A_1) > 8$ in number. We can evaluate $s(\bar{n})$ by splitting up the hole number distribution into regions of width $\Delta \bar{n} > 8$. Then, $s(\bar{n})$ will be Gaussian in each region if Eq. (67) is satisfied. We can obtain $s(\bar{n})$ by combining (convolving) all such regions together. For example, combining two regions together, we obtain
\[
\mathcal{P}(N, \bar{n}_g(2)) = \left[ 2\pi (\delta N^2) g \right]^{-1/2} \exp \left[ \frac{(N - \langle N \rangle)^2}{2(\delta N^2) g} \right],
\]
where $\langle N \rangle_{12} = \langle N \rangle_{1} + \langle N \rangle_{2}$ and $\langle \delta N^2 \rangle_{12} = \delta N_1 N_2^2(\delta n_1) + \delta n_2 N_2^2(\delta n_2)$. Convolution all regions leads to the result
\[
\mathcal{P}(N, \bar{n}_g) = \left[ 2\pi (\delta N^2) g \right]^{-1/2} \exp \left( \frac{(N - \langle N \rangle)^2}{2(\delta N^2) g} \right),
\]
where the mean square value of $N$ for the Gaussian is
\[
\langle \delta N^2 \rangle_g = \int_0^\infty \bar{n} d\bar{n}' N_\bar{h}(\bar{n}'),
\]
and the mean value can be written as
\[
\langle N \rangle_g = \langle N \rangle - \int_{\bar{n}_g}^{\bar{n}_g + \Delta \bar{n}} \bar{n}' d\bar{n}' N_\bar{h}(\bar{n}').
\]

![FIG. 16. Theoretical skewness $s(\bar{n}_g)$ for the model discussed in Sec. VI.](image)
\( \delta N = 0 \). Note that from Eq. (66), \( \langle \delta N^2 \rangle \) is given by Eq. (75). Note also that if \( \Delta \mathcal{P} = 0 \) (no skewness), then \( \Delta (N) = 0 \) so that \( \mathcal{P}(\delta N,\bar{n}_m) \) would be an unshifted Gaussian. Qualitatively, the skewness-producing holes will shift the Gaussian to the right (\( \delta N > 0 \)) of the origin and enhance (stretch) the \( \delta N < 0 \) tail of the Gaussian. Using the conservation of total probability, we can calculate the ratio \( r \) of the area under the "tail" portion \( \Delta \mathcal{P}(\delta N,\bar{n}_m) \) to the area under the Gaussian portion \( \mathcal{P}(\delta N,\bar{n}_m) \). We find

\[
 r = \Delta \bar{n},
\]

for \( \Delta \bar{n} \ll 1 \). Once \( r \) is known, Eq. (79) becomes a relation for \( \Delta \bar{n} \). Here \( \Delta \bar{n} = \bar{n}_m - \bar{n}_g \) so that Eq. (58) gives an additional determination of \( \Delta \bar{n} \)

\[
\Delta \bar{n} = p(A_w/A_g - A_w/A_m).
\]

We note that Eq. (80) is more model-dependent than Eq. (79), since \( r \) of Eq. (79) can be obtained directly from the measured \( P(\delta f) \) curves.

In the interest of simplicity, we have up to this point restricted the model \( \mathcal{P}(\delta N,\bar{n}_m) \) to holes with areas less than the window area. We can now relax this restriction by redefining \( N_h(A) \) and \( F(A) \) so that they include holes with arbitrary \( A \). If \( A > A_w \), the number of holes with area between \( A \) and \( A + dA \) that contribute to the window is \( pA^{-1}dA \). Therefore, we redefine \( F(A) \) to be

\[
F(A)dA = (1 + A_w/A) (p/A) dA.
\]

We point out that this modified version of Eq. (57) is probably only qualitatively correct. In particular, it still suffers from the assumption that \( p(A) = p = \text{const} \) over the entire range of \( A \). Using Eq. (81), we find that the average number of holes that the window sees that have area less than or equal to \( A \) is

\[
\bar{n}(A) = n_s (1 - A_w/A) + p \ln (A/A_1),
\]

where \( n_s = pA_w/A_1 \). The correction to Eq. (58) for \( A > A_w \) holes will be small if \( n_s (1 - A_w/A) > p \ln (A_w/A_1) \) or \( A_w/A_1 > \ln (A_w/A_1) \) for \( A > A_w \). This is satisfied for all but the smallest windows. We redefine \( N_h(A) \) as the number of particles in a hole that contribute to the window of area \( A_w \). We neglect again the possibility that the hole and window overlap. For \( A > A_w \), this is a serious omission—especially for holes whose shape differs significantly from that of the window. However, since the \( A < A_w \) holes are more numerous than those with \( A \geq A_w \) and have \( \Delta x/\Delta \bar{n} = \tau \), we should apply the \( \mathcal{P}(\delta f,\bar{n}_m) \) model to windows that have \( \Delta x/\Delta \bar{n}_m = \sqrt{\tau} \). This will tend to minimize the occurrence of hole–window overlap. With this restriction in mind, we write the following (very) approximate expression for \( N_h(A) \) which includes holes with arbitrary \( A \):

\[
N_h(A) = \bar{f}(A) A A_w / (A + A_w).
\]

Equation (83) correctly gives the number of particles in a hole that contribute to the window for the limiting cases of \( A > A_w \) and \( A < A_w \).

Equations (71), (72), and (77) can be evaluated in conjunction with Eqs. (55), (75), (78), and (83) to yield an approximate solution for \( \mathcal{P}(\delta f,\bar{n}_m) \). As we have seen, the solution procedure assumes that any skewness in \( \mathcal{P}(\delta f,\bar{n}_m) \) will occur only in the small region \( \Delta \bar{n} = \bar{n}_m - \bar{n}_g \lesssim 1 \) beyond \( \bar{n}_g \). Though this assumption is strongly supported by Eq. (70) for the skewness (cf. Fig. 16), we have attempted to verify the approximation procedure directly. We have, therefore, solved Eq. (61) explicitly as an initial value problem. We chose the window \( \Delta x_w = 1.27 \lambda_D, \Delta \bar{n}_w = 0.125 \nu_v, \) so that \( \Delta x_w/\Delta \bar{n}_w \approx \tau \approx 10 \nu_v^{-1} \). Using a packing fraction \( p = 0.1494, A_w/A_D = 1.329 \times 10^{-2}, \) and \( A_w/A_D = 1 \) (see Sec. VII), we calculated from Eq. (70) that the minimum (absolute) value of the skewness \( s(\bar{n}) \) occurred for \( \bar{n}_g = 16.930 \). Using Eqs. (75), (77), and (78), we then obtained the initial value \( \mathcal{P}(\delta f,\bar{n}_m) \) for Eq. (61). Equation (61) was then solved numerically for \( \bar{n}_g > \bar{n}_g \). The result for \( \bar{n}_m - \bar{n}_g = \Delta \bar{n} = 0.249 \) is shown in Fig. 17. The distribution clearly exhibits the characteristic features of the observed \( P(\delta f) \) [cf. Fig. 8(a)]. The measured skewness of Fig. 17 is \( s = -1.05 \) and agrees well with Eq. (70). We also find that choosing the initial condition such that \( 8 < \bar{n}_m < \bar{n}_g \) led to \( \mathcal{P}(\delta f,\bar{n}_m) \) that was approximately Gaussian until \( \bar{n}_g > \bar{n}_g \). Moreover, the skewness of \( \mathcal{P}(\delta f,\bar{n}_m) \) was small and weakly dependent on \( \bar{n} \) in the region \( 8 < \bar{n}_m < \bar{n}_g \). This result is consistent with Eq. (70) (and Fig. 16) and justifies the procedure of constructing Eq. (77) by convolution. The small values of \( \Delta \bar{n} \) justify the retention of only the \( m = 1 \) term of Eq. (64) for the approximate expression Eq. (72) for \( \Delta \mathcal{P} \).

We have used the \( \mathcal{P}(\delta f,\bar{n}_m) \) model to calculate some of the prominent features of the observed \( P(\delta f) \). In doing so, we have used Eq. (55) for numerical calculation and the approximate expression (56) for analytical work. The shift and mean square width are readily identifiable features of the \( P(\delta f) \) curves. We can easily calculate these quantities from the \( \mathcal{P}(\delta f,\bar{n}_m) \) model. Since the \( P(\delta f) \) curves are plotted against \( \delta f/\nu_0 \) we calculate the normalized shift \( q = \Delta (N) / (f_0 A_w) \). Using Eqs. (78), (83), (58), and (56), we obtain

\[
q = 2p \left[ \left( A_1 / A_g \right)^{1/2} - \left( A_1 / A_m \right)^{1/2} \right]
\]

\[
+ \frac{A_1}{A_D} \left( \frac{A_m}{A_1} \right)^{1/2} - \frac{A_1}{A_D} \left( \frac{A_g}{A_1} \right)^{1/2} \right].
\]

For \( A_1 < A_D \) and \( A_g < A_D \), Eq. (84) can be simplified to

\[
q = 2 \left( \frac{A_1}{A_w} \right)^{1/2} \left( \Delta \bar{n} + p \frac{A_w}{A_m} \right) \mathrm{.}
\]

where we used Eq. (58) to express \( A_g \) in terms of \( n_s - \bar{n}_g = \bar{n}_m - \bar{n}_g + \Delta \bar{n} + p A_w / A_m \). Similarly, the normalized mean square width of the Gaussian, \( \sigma_q^2 = (\delta N^2) \xi / P(\delta f) \) is
\( f_0 \), follows from Eq. (75),

\[
\sigma^2_{w^2}(A_w) = \int A_w \ f_0 ^2 \frac{A}{A + A_w} \left( \frac{f(A)}{f_0} \right)^2 \ dA.
\]  

(86)

Using Eq. (54) and the approximate analytical expression (56), Eq. (86) becomes

\[
\sigma^2_{w^2}(A_w) = p \left[ \frac{A_1}{A_w} \ln \frac{1 + A_w/A_1}{1 + A_1/A_w} + \left( \frac{A_1}{A_D} \right)^2 \left( \frac{A_w}{A_1} - 1 \right) + \frac{A_1}{A_D} \left( 2 - \frac{A_w}{A_D} \right) \ln \frac{A_w + A_w}{A_1 + A_w} \right] \]

or

\[
\sigma^2_{w^2}(A_w) = p \left[ \frac{A_1}{A_w} \ln \left( \frac{A_1 + A_w}{A_w} \right) \right],
\]

(87)

(88)

when \( A_1 < A_w < A_D \). Similarly, the mean square width of the full distribution is

\[
\sigma^2_{m^2}(A_w) = \int A_w \ A_1 \ f_0 ^2 \frac{A}{A + A_w} \left( \frac{f(A)}{f_0} \right)^2 \ dA.
\]

(89)

This is just Eq. (86) with \( A_1 \) replaced by \( A_w \). The first term in \( \sigma^2_{m^2} \) is due to holes of size \( A < A_D \), while the other terms come from holes with \( A > A_D \). The scaling \( \sigma^2_{m^2}(A_w) \sim A_w^{-1} \) for \( A_w > A_1 \) and \( A_w > A_m \) is consistent with a random distribution of holes with sizes \( A < A_{m,1} \), i.e., \( \langle N^2 \rangle \sim \langle N \rangle \). The symmetrized correlation function Eq. (35) for the one-parameter hole model (cf. Ref. 13) is

\[
C(A_\Delta, A_V) = C(A_w) = \frac{1}{4 \ A_\Delta \ A_V} \ \frac{\partial^2}{\partial A_\Delta ^2} \ \langle N^2 \rangle .
\]

(90)

For \( A_w = A_\Delta \), \( C(A_\Delta, A_V) \) is due mainly to the deepest, smallest holes, i.e., those for which \( f(A) \approx f(A_w) = f_0 \). Larger holes contribute to the width of the correlation function. For \( A_m > A_1 \) and \( A_1 < A_D < A_m \), the \( A_m \) and \( A_1 \) scale lengths in the logarithm factors of \( \sigma^2_{m^2}(A_m) \) indicate that \( C(A_w) \) will qualitatively be composed of two regions: one with width \( \approx A_m \), due to deep holes, and one of width \( \approx A_m \) due to holes of size \( A_m \).

VII. COMPARISON OF \( P(0) \) MODEL WITH OBSERVATIONS

In order to compare the hole model of \( P(0) \) with the decay turbulence observations, we consider the typical curve shown in Fig. 8(a) \( (\Delta x_w = 1.23 \lambda_D, \ \Delta u_w = 0.1172 v_{ish}, \ \omega_p t = 120) \). We calculate from Eq. (37) that discrete-particle fluctuations do not contribute significantly to the \( P(0) \) of Fig. 8(a).

Using the measured values \( \sigma^2_{m^2}(A_w) = 0.0137 \) and \( \sigma^2_{m^2}(A_m) = 0.18 \) from Fig. 8(a), we solve the exact Eqs. (89) and (55) for \( \sigma^2_{m^2}(A_w) \) and \( \sigma^2_{m^2}(A_m) \) as simultaneous integral equations for \( p \) and \( A_m \). We find that \( p = 0.18 \) and \( A_m = 1.7 A_D \). \( = 1.4 A_w \). "Completing the Gaussian" in Fig. 8(a), we find that the ratio of areas for the tail to the completed Gaussian is \( r = 0.26 \). Equation (79) then gives \( \Delta n = 0.26 \). We note that this value of \( \Delta n \) for this window is consistent with the numerical solution of Eq. (61) discussed in Sec. VI. With this value of \( \Delta n \), we use Eq. (80) to obtain \( A_w = 0.43 A_w \) or \( \Delta u_w = 0.65 \Delta u_w \). Then, Eq. (89) for \( \sigma^2_{m^2}(A_w) \) allows us to compute \( \sigma^2 \). The measured value from Fig. 8(a) is \( \approx 0.07 \). Using Eq. (84) and the values \( \Delta n, A_s, \) and \( A_m \) just inferred above, we find that \( q = 0.03 \) whereas the measured value from Fig. 8(a) is 0.05. At \( \omega_p t = 120 \), we use the values \( p = 0.18 \) and \( A_m = 1.7 A_D \) to find that Eq. (90) is consistent with the qualitative shape of the correlation function. These points of approximate agreement show that the hole model is reasonably successful in describing the "lower-order" moments of \( P(0) \) represented by \( q \) and \( \sigma^2 \).

The higher-order moments of \( P(0) \) are more difficult to describe by the \( P(0) \) model. For example, the skewness is very sensitive to \( \Delta n \) and, therefore, to \( p \) and \( A_m \). This is to be expected from Eqs. (63) and (70) since \( \Delta n \) is due mainly to the deepest, smallest holes, and \( \Delta n \) is due to the tail of \( f(A) \). Larger holes contribute to the width of the correlation function. For \( A_m > A_1 \) and \( A_1 < A_D < A_m \), the \( A_m \) and \( A_1 \) scale lengths in the logarithm factors of \( \sigma^2_{m^2}(A_m) \) indicate that \( C(A_w) \) will qualitatively be composed of two regions: one with width \( \approx A_m \), due to deep holes, and one of width \( \approx A_m \) due to holes of size \( A_m \).

FIG. 18. Measured skewness (-) and theoretical skewness (x) in time.
where we note that approximately as and width of the completed Gaussian ing windows where \( A > A_D \) [cf. Eq. (55)]. This helps to explain the fact that the model and observed skewness agree better in Fig. 18 (where \( A_m < A_D \)) than in Fig. 8(b) (where \( A_m > A_D \)). The approximative model is more accurate in treating the holes with \( A < A_D \). We note from Fig. 8(a) (where \( A_m > A_D \)) that the value \( \bar{v} \approx 0.4 \) implies a hole with \( \Delta v \approx 0.4 V_m \) and \( \Delta x \approx A_D \) [see Eq. (50)]. We recall that this value of \( \Delta x/\Delta v \approx 2.5 \omega_p^{-1} \) is consistent with the value \( \tau < 5.3 \omega_p^{-1} \) inferred from the \( v_\perp \) tail of \( C(0,v_\perp) \) (cf. Sec. III). These large holes imply a value for \( A_m \) that is larger than the one we obtained from \( \sigma_m \), above. This discrepancy can be attributed to the fact that the width of \( P(\bar{v}) \) is less sensitive to the tail of \( P(\bar{v}) \)(and therefore, to holes with large \( A \)) than the higher-order moments such as the skewness and, therefore, predicts a smaller value of \( A_m \).

A property of the \( \mathcal{P}(\bar{v},\bar{n}) \) model that is less sensitive to the hole distribution is the scaling with window size. Using windows where \( \Delta x_w/\Delta v_w \approx \tau = 10 \omega_p^{-1} \), and \( A_m \approx A_m \approx A \), we have observed that the normalized shift \( q \) and width of the completed Gaussian \( \sigma_m \) decrease approximately as \( A \sim 1/2 \). This is in agreement with Eqs. (84) and (88) where we note that \( \Delta \bar{n} > p A_w/A_m \) for these windows. The model skewness [derived from Eqs. (55), (70), (83), and (58)] decreases with \( A_w \) as is shown in Fig. 19. The scaling is seen to be in qualitative agreement with the measured skewness for windows where \( \Delta x_w/\Delta v_w \approx 10 \omega_p^{-1} \) and \( A_w > A_D/2 \) (cf. Fig. 9). The initial increase in the skewness of Fig. 9 is due to the particle discreteness (see Sec. III).

The scaling of the \( \mathcal{P}(\bar{v},\bar{n}) \) model with time can be easily obtained for the case \( A_m < A_D, \Delta \bar{n} > p A_w/A_m \), and \( A_m \approx A_D \). Equations (85)–(91) then imply that \( q^2, \sigma_m^2(A_m) \), \( \sigma_m^2(A_w) \), and \( \sigma_m^2(A_m) \) should scale in time with the packing fraction \( p(t) \). The measured values for these quantities (normalized to their value at \( \omega_p t = 120 \)) are shown in Fig. 20 for the window \( \Delta x_w = 0.2 A_D, \Delta v_w = 0.3 V_m \). The values of \( p(t) \) plotted in Fig. 20 were obtained by solving \( \sigma_m^2(A_w) \) and \( \sigma_m^2(A_m) \) [Eq. (89) and Eq. (55)] simultaneously for \( p(t) \) and \( A_m(t) \) (cf. Fig. 21). The quantities \( q^2, \sigma_m^2(A_w), \sigma_m^2(A_m) \), and \( \sigma_m^2(A_m) \) appear to decay in time with \( p(t) \) in agreement with the model. Figure 18 shows that the scaling of the skewness with time agrees with the model for \( A_m > 80 \), i.e., when the hole model is valid. The skewness agreement is not due to \( p(t) \) alone. Though \( N_A = p \) (so that \( s \sim p^{1/2} \)), \( s \) also depends on \( A_m(t) \). This explains why the skewness varies linearly with time from Fig. 18 whereas \( p(t)^{1/2} \sim t^{1/2} \) from Fig. 21.(a). \( s(t) \) increases faster than \( p(t)^{1/2} \sim t^{1/2} \) because \( A_m(t) \) increases with time. We also note that \( \sigma_m^2(A_m) \) appears to decay with \( p(t) \) alone in Fig. 20, even though \( \sigma_m^2(A_m) \) depends on \( A_m(t) \) as well as \( p(t) \). Again, this is consistent with the fact that the lower-order moments of \( \mathcal{P}(\bar{v},\bar{n}) \) (such as \( \sigma^2 \)) are less sensitive to the large holes \( \langle A_m \rangle \) than the higher moments (such as \( s \)).

FIG. 20. Time dependence of various moments of \( P(\bar{v}) \) inferred from the theoretical model discussed in Sec. VI: the normalized quantities \( q^2(A_w) \) (dot), \( \sigma_m^2(A_w) \) (times), \( \sigma_m^2(A_m) \) (circle), \( \sigma_m^2(A_m) \) (box), and \( \rho \) (triangle) for a window where \( A_w = 0.6 A_D \) (0.05 \( V_m \)).
Reference to $A_m(t)$ in Fig. 22 shows that the first term in Eq. (91) (i.e., the 1) dominates the term proportional to $A/A_D$ throughout the simulation. It is for this reason that $p(t)$ in Fig. 21 closely approximates $C(\Delta x, \Delta v)$ displayed in Fig. 6. We conclude that the unbound debris [which is not included in Eq. (91)] makes little contribution to the correlation function. This is consistent with the rapid mixing of these fluctuations into the background plasma, i.e., the unbound debris should have a mixing rate described by Eq. (15) since little or no recombination should occur for these fluctuations. Therefore, the core region of $C(x, v)$ is due to the internal structure of the holes of area $A$. The width of the core is constant in time since the deepest holes always have $A = A_i$, i.e., $f(A_i) = f_0$. That these holes exist even at $\omega_r t = 220$ is clear from Figs. 1(a) and 23. However, the significant skewness of Fig. 23 implies that these deepest holes are few in number. This is consistent with the time decay of the peak of the correlation function.

VIII. A MODEL OF FLUCTUATION DECAY

A simple but intuitively appealing model can be constructed for the time behavior of fluctuations during the decay experiments. We begin by recalling the hole model interpretation for the correlation function depicted in Fig. 1. The core region $\{A < A_i\}$ is due to the deepest holes [i.e., those satisfying $f(A_i) = f_0$] whereas the main body $\{A_i < A < A_D\}$ is due to larger holes formed by the coalescing of the deep holes. This interpretation of the peak of the correlation function is consistent with Eq. (91) since $A_i/A_D \approx 10^{-2}$ and $A_m < A_D$ throughout the experiment, i.e.,

$$C(A) \approx p f(A)^2, \tag{92}$$

where we have used Eq. (56) for the deepest holes. These deep holes maintain their characteristic scale sizes $\{A > A_i\}$ even as their mean square fluctuation level decreases. This is easily verified by comparing the core region of $C(x, v)$ at $\omega_r t = 60, 120,$ and 220. If the scale sizes remain approximately constant in time, Eq. (50) implies that their depths remain constant also (cf. Fig. 23). Therefore, their mean square fluctuation level decays because their packing fraction decays, i.e.,

$$\frac{d \ln \langle \delta f^2 \rangle}{dt} = \frac{dp}{dt}. \tag{93}$$

Figure 21 shows that the holes are closely packed at $\omega_r t = 60$, i.e., $p \approx 0.4$. The holes will interact strongly with each other: hole–hole collisions $\{D\}$ will lead to the breakup of holes into fragments and to the recombination (coalescence) of the fragments into new holes. The resulting decay of the mean square fluctuation level can be modeled by the following equation for the packing fraction:

$$\frac{dp}{dt} = -vp. \tag{94}$$

The quantity $v$ in Eq. (94) is a phenomenological decay rate. If the holes are closely packed ($p \approx 0.4$), then $v \approx \Delta v/\Delta x$, where $\Delta x$ and $\Delta v$ are the approximate hole dimensions. More appropriate to a turbulent system of holes where $p = 1/2$ (e.g., the driven experiments) is the identification $v \approx [\tau_0 \Delta x, \Delta v]^{-1} \approx \tau_0^{-1}$. In either case, we have $v \approx 10 \omega_p^{-1}$ since $\tau_0^{-1} \approx \Delta v/\Delta x \approx 10 \omega_p^{-1}$ as discussed in previous sections. However, as the holes are destroyed in the decay experiments, the collision frequency between holes will decrease because the phase-space separation between the remaining holes will increase, i.e., the hole-packing fraction will decrease. Therefore, the hole–hole collision frequency is $p \Delta v/\Delta x$. However, hole fragment recombination will reduce this hole–hole collision rate by the factor $b^{-1}$ where $b > 1$. Consequently, the decay rate of the aggregate fluctuation (hole) level is

$$v = 2p \left[\frac{\Delta x}{\Delta v}\right]^{-1}. \tag{95}$$

We stress that the decay rate is determined by both the decrease of the packing fraction in time and the tendency of the holes to recombine. Solving Eqs. (93)–(95) for $t > t_0$ gives

$$p(t) = p(t_0) / \left[1 + 0.2\omega_p(t - t_0) p(t_0) b^{-1}\right]. \tag{96}$$

Presumably, we must choose $t_0 > 80 \omega_p^{-1}$ since it is only after this time that Bernstein–Greene–Kruskal-like holes have formed. Equation (96) fits the data well if $b \approx 2.4$ (cf. Fig. 21). We note that this value of $b$ implies that the effects of hole recombination (self-binding) and turbulent decay (collisions between holes) are comparable. Note also that when $p = 1/2$, the net decay rate of $[2.4\tau_0]^{-1}$ is approximately the same as the result of the driven experiments (cf. Sec. VI and item B1 of Sec. I).

It is interesting to use the model described by Eqs. (93)–(95) in an effort to rectify the discrepancy between the existing clump theory and the observations of Ref. 6. In Ref. 6, an instability was observed in a one-dimensional plasma simulation in which an electron Maxwellian distribution $f_{e_0} = (2\pi v_e^2)^{-1/2} \exp \left[-(v - v_{e_0})^2/(2v_e^2)\right]$ drifts with a velocity $v_p$ relative to an ion Maxwellian distribution $f_{i_0} = (2\pi v_i^2)^{-1/2} \exp \left[-v^2/(2v_i^2)\right]$. The observed growth rate of $\langle \delta f^2 \rangle$ for the ions as a function of $v_p/v_t$ is shown in Fig. 24. The corresponding linear growth rate (i.e., $2\gamma_L$) is also shown. (Note that $\gamma_L$ was shown in Fig. 4 of Ref. 6.) The observed instability has been identified with the clump instability.7 The instability of clumps in a one-dimensional electron–ion plasma has been investigated theoretically in Ref. 7 (where the point of marginal stability is computed) and in Ref. 8 (where the growth rate is calculated). If one neglects the self-binding feature of the clumps, such calculations predict a higher stability threshold and a lower growth rate than those observed in the simulations. It has been suggested that these discrepancies would be eliminated if self-binding were included in the theory.

In order to see how this might be done, let us first consider the theory of electron–ion clumps in the absence of self-binding. The electron clumps are described by an equation of
Theoretical growth rate of the nonlinear clump instability when hole self-binding is included (2γ) and linear (2γL) growth rate of (δf*2) as a function of νD/νi.

The growth rate satisfies

\[
\frac{d}{dt} G^* = S^* ,
\]

where \( T_{ij} \) and \( \tau_{ij} \) in accord with the stochastic instability model, \( G^* = \langle \delta f_0^* \delta f_i^* \rangle \) is the electron fluctuation correlation function, and \( S^* \) is the electron clump source term. As in Eq. (7), \( S^* \) is composed of two terms. It is shown in Ref. 8 that Eq. (97) can be written schematically as

\[
\frac{d}{dt} G^* = \frac{R^{ee} G^*}{\tau_{ee}} + \frac{R^{ie} G^*}{\tau_{ie}},
\]

where \( R^{ee} \) and \( R^{ie} \) are the two-species analogs of the parameter \( R \) defined in Eq. (24) of Sec. II. The \( R^{ee} \) term in Eq. (98) is due to the diffusion term in \( S^* \) [cf. the first term of Eq. (8)]. The ion electric fields rearrange the electron phase-space density and create electron clumps: \( D \langle f'_0 \rangle^2 \sim \langle E^2 \rangle \langle f'_0 \rangle^2 \sim \tilde{G} \langle f'_0 \rangle^2 / \tau_{ee} \sim R^{ee} / \tau_{ee} \). The \( R^{ie} \) term in Eq. (98) is due to the second term in \( S^* \) and transfers momentum between electrons and ions, i.e., \( R^{ie} \sim -f'_0 f'_i \). The ion clumps satisfy an equation analogous to Eq. (98):

\[
\frac{d}{dt} G^i = \frac{R^{ii} G^i}{\tau_{ii}} + \frac{R^{ei} G^i}{\tau_{ei}},
\]

where \( R^{ii} \sim \langle f'_0 \rangle^2 \). For \( \gamma_{ee} < 1 \) and scale lengths less than the typical clump dimension, \( G^* \approx G^i \) and \( G^i \approx G^i \). If we now set \( \partial / \partial t \sim 2 \gamma \), the simultaneous solutions of Eq. (98) and Eq. (99) yield

\[
(1 + 2 \gamma_{ee} - R^{ii}) (1 + 2 \gamma_{ee} - R^{ii}) = R^{ee} R^{ii} \sim 1. (100)
\]

Detailed calculations show that the growth rate satisfies (when \( \gamma_{ee} < 1 \) and \( \gamma_{ee} < 1 \))

\[
(1 + 2 \gamma_{ee} - R^{ii}) (1 + 2 \gamma_{ee} - R^{ii}) = (R^{ii})^2 , \quad (101)
\]

where \( \gamma_{ee} \) and \( \tau_{ee} \) are Eq. (14) for the electrons and ions, respectively. For \( \gamma_{ee} < 1 \), the parameter \( c \sim 3 \) and reflects the fact that \( \tau_{ee} < \tau_{ee} \). Here \( R^{ee}, R^{ii}, \) and \( R^{ie} \) are similar to Eq. (24) and are defined as

\[
R^{ab} = \int dk \left[ \frac{N^{ab}(k)}{1 - 2 \pi |k| N^{ab}(k) / k_0} \right] A(k) , \quad (102)
\]

where \( A(k) \) is defined by Eq. (26),

\[
N^{ab}(k) = \frac{\text{Im} \left( \epsilon(k, k_0) \right) \text{Im} \left( \epsilon(k, k_0) \right)}{\pi^2 |\epsilon(k, k_0)|^2} , \quad (103)
\]

and the turbulence length scale \( k_{0}^{-1} \) satisfies

\[
k_{0}^{-1} = \frac{1}{2 R^{ee}} \int dk \left| \frac{N^{ee}(k)}{1 - 2 \pi |k| N^{ee}(k) / k_0} \right| A(k) . \quad (104)
\]

Because of Eqs. (102) and (103), \( [R^{ee}]^2 = R^{ee} R^{ee} \) in Eq. (101) [cf. Eq. (100)]. In order to obtain the growth rate \( \gamma \) from Eq. (101), we must solve Eqs. (102) and (104) simultaneously. This can be done numerically. The point of marginal stability follows from \( 2R^{ee} = 1 \) [cf. Eq. (101) with \( \gamma = 0 \)] and Eq. (104). For the parameters used in the simulation of Ref. 6, the numerical solution of these coupled equations gives a threshold drift of \( \nu_D \sim 2.5 \nu_i \) at \( \nu_D = \nu_i \). However, the instability threshold in the simulation was observed at \( \nu_D = 1.5 \nu_i \). Note that the \textit{linear} stability boundary occurs at \( \nu_D = 3.924 \nu_i \) for the parameters in Ref. 6.

We can include the effects of self-binding in the clump theory in the following way. First, we recall that in Eqs. (98) and (99), the \( \tau_{ee}^{-1} \) term describes the decay of fluctuations due to \( D \_ \) and ballistic streaming. However, the results of the driven and decay experiments imply that the net decay rate of the mean square fluctuation level is less than \( \tau_{ee}^{-1} \). In the driven experiments (where \( p = \frac{1}{2} \)), we observed a decay rate of \( \nu \approx \nu_{R} \) when \( b \approx 3 \). Therefore, the simulation results imply that Eqs. (98) and (99) should be written as

\[
\left( \frac{\partial}{\partial t} + \frac{1}{b \tau_{ee}} \right) G^* = \frac{R^{ee} G^*}{\tau_{ee}} + \frac{R^{ie} G^*}{\tau_{ie}} , \quad (105)
\]

and

\[
\left( \frac{\partial}{\partial t} + \frac{1}{b \tau_{ee}} \right) G^i = \frac{R^{ii} G^i}{\tau_{ii}} + \frac{R^{ei} G^i}{\tau_{ei}} . \quad (106)
\]

The factor \( b \) does not appear on the right-hand side of Eq. (105) and Eq. (106) because the \( R \)'s are proportional to \( \tau_{ee} \). Consequently, the growth rate expression Eq. (101) is replaced by

\[
(1 + 2 \gamma_{ee} \tau_{ee} - R^{ii}) (1 + 2 \gamma_{ee} \tau_{ee} - R^{ii}) = (R^{ii})^2 . \quad (107)
\]

The point of marginal stability now follows from

\[
2 R^{ee} = 1/b . \quad (108)
\]

Using the value \( b \approx 3 \) inferred from both the decay and driven experiments, we expect the stability threshold to occur when \( 2 R^{ee} \approx 0.33 \) and Eq. (104) are satisfied simultaneously. Numerical solutions show that this occurs for \( \nu_D = \nu_i \) when \( \nu_D = 1.5 \nu_i \) in agreement with the observations of Ref. 6. Figure 24 shows the growth rate observed in Ref. 6 and the growth rate \( \gamma \) calculated from Eq. (107). In accord with Ref. 6, we used \( \tau_{ee} = \tau_{ee} (m_e / m_i)^{1/2} = 0.5 \tau_{ee} = 4 \tau_{ee} \). The parameter \( c \) was chosen to fit the data at \( \nu_D = 2.5 \nu_i \), i.e., \( c = 2.9 \). (Note that this value is consistent with the theoretical prediction for \( c \).) We have used \( \nu_D = \nu_i \) for all drift values except \( \nu_D = \nu_i \) when \( \nu_D = 1.5 \nu_i \) in accord with the measurements of Ref. 6 we used \( \nu_D = 0.5 \nu_i \). Equation (107) reproduces the essential features of the data for all but the larger drifts. However, we note that the \( \gamma_{ee} < 1 \) and \( \gamma_{ee} < 1 \) restrictions assumed in the derivation of Eq. (101) are violated for these larger drifts. Moreover, Eq. (101) excludes the existence of linear plasma waves that would become nonlinearly
unstable (via resonance broadening) at these larger values of $v_D$ (note that linear stability occurs at $v_D = 3.924 v_i$). Note also that we have assumed that the effect of self-binding on Eqs. (105) and (106) is independent of $v_D$. This may be incorrect. However, use of the approximate value $b = 3$ seems appropriate near $v_D \approx 0$ since for this drift, the two-species plasma approximates the stable (one-species) plasma discussed in this paper. Indeed, if we had used $b = 1$ for $v_D = 0$, thereby omitting self-binding, Eq. (107) would give $2\gamma \approx -9.2 \times 10^{-3} \omega_{pe}^{-1}$ rather than the value $2\gamma \approx -3.3 \times 10^{-3} \omega_{pe}^{-1}$ for $b = 3$. Another note of caution concerns the neglect of collisions in Eq. (101). In principle, collisions can lead to the destruction and production of clump fluctuations. The production of clumps results when discrete particle fluctuations rearrange the phase-space density [as in an analogous fashion to $D^{\ast i}$ in Eq. (17)]. Therefore, both $S$ and $T_{12}$ of Eq. (1) will be enhanced by collisions. The net effect of these competing collisional corrections is difficult to calculate. However, estimates show that they are comparable at the marginal point observed in the simulation of Ref. 6, i.e., at $v_D \sim 1.5 v_i$. This conclusion would tend to justify the use of Eq. (101) where collisions are neglected in interpreting the marginal point observed in Ref. 6.

For $p < 1$, the clump instability picture is replaced by one of isolated growing holes. A hole that is isolated in phase space (i.e., $\rho \rightarrow 0$), has been shown to be unstable in an ion-electron plasma with opposing (velocity space) density gradients.\(^{26,27}\) For instance, an electron hole can exchange momentum with the ions that are reflected by the hole. As a result, the hole moves up the density gradient, thereby getting deeper. This instability is the single (isolated) fluctuation analog of the clump instability. We can see this by writing Eqs. (105) and (106) in the limit of an isolated hole. The ions act coherently ($\gamma \tau_0 \rightarrow \infty$) so that only the momentum transfer term (proportional to $R^{\ast i}$) survives in Eq. (105). Also, for an isolated hole, $p \rightarrow 0$, so that the $\tau_0^{-1}$ fluctuation collision term vanishes. The isolated electron hole grows at the rate

$$\gamma_{bb} \sim \frac{\Delta v}{\Delta x} R^{\ast i},$$

(109)

since $\tau_0^{-1} \rightarrow \infty$, $\Delta x / \Delta v$ for an isolated hole. The marginal point for the isolated hole instability follows from $R^{\ast i} = 0$. However, for a collection of holes, i.e., nonzero packing fraction, instability is more difficult to achieve. The source term ($R^{\ast i}$) must be large enough to overcome the decay of fluctuations due to hole–hole collisions. These collisions are more frequent for larger packing fractions. Therefore, for a collection of electron holes, we would expect a growth rate satisfying

$$\gamma \sim \frac{\Delta v}{\Delta x} R^{\ast i} - \frac{2p}{b} \frac{\Delta v}{\Delta x}. $$

(110)

Equation (110) follows from Eqs. (105) and (106) by making the replacement $\tau_0^{-1} \rightarrow \rho \Delta v / \Delta x$ and noting that the $\tau_0^{-1}$ factors on the right-hand sides of Eqs. (105) and (106) are replaced by $\Delta v / \Delta x$ since the $R^{\ast i}$ are proportional to $\tau_0^{-1}$. Moreover, if the ions act coherently ($\gamma \tau_0 \rightarrow \infty$), only the $R^{\ast i}$ term survives in Eq. (105). Equation (110) implies that $R^{\ast i} = 2p / b$ replaces Eq. (108) for the point of marginal stability. Therefore, as $p \rightarrow 0$, the threshold $v_D$ for the onset of the instability will approach zero. The effect of the fluctuation packing fraction on the instability is now under study and will be published elsewhere.

**Note added in proof**: We have recently become aware of two-dimensional computer simulations of fluid turbulence (J. C. McWilliams, Workshop on Chaos and Coherent Structures in Fluids, Plasmas, and Solids, Los Alamos, 1983) where similar intermittent phenomena have been observed—namely, the emergence of isolated vortices and a decrease in packing fraction generating intermittency.

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