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TRANSPORT THEORY: MICROSCOPIC REVERSIBILITY
AND SYMMETRY

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ABSTRACT

The general theory of Fokker-Planck equation symmetries and their relation to derived transport theory symmetries is developed. The property of microscopic reversibility implies a symmetry in the Fokker-Planck equation for processes obeying detailed balance. It is shown that this symmetry is not sufficient to guarantee Onsager reciprocity for the full matrix of transport coefficients. The general transport matrix has broken symmetry. A partial symmetry can be identified. The theory is compared to the different formulation given by Onsager.
I. Introduction

Observations on systems near thermal equilibrium have shown that irreversible processes are described by linear phenomenological laws which relate the fluxes to the forces by a matrix of transport coefficients. Furthermore, as suggested by a large body of experimental evidence and running back to Kelvin’s analysis of the thermoelectric effect in 1854, these transport coefficients possess a certain set of symmetries – the Onsager reciprocity relations. Of course, no completely general theoretical proof of these relations can be made. Onsager showed the connection between these relations and microscopic reversibility in the underlying dynamics, implying their universal validity. His proof, however, was essentially empirical and based on two key assumptions. First, that the fluxes of the extensive variables specifying the thermodynamic state depend linearly on the forces that cause them. Second, that the average regression of fluctuations of these variables from a given nonequilibrium state obey the same linear law. This limitation in the proof of the Onsager relations was emphasized in the review by Casimir, who noted that this second assumption was, “really a new hypothesis . . . we do not think it can be rigorously proved without referring in some way or another to kinetic theory.”

In short, the preponderance of observational experience and the general nature of Onsager’s empirical proof suggest that all systems governed by linear force-flux relationships have obey the reciprocity relations. But in the absence of a first principles proof of these relations one might still ask if there are systems that have linear force-flux relations without Onsager symmetry. The present paper will develop a general kinetic theory that implies that such systems do, in fact, exist. The theory does not contradict Onsager’s original arguments, but simply restricts their applicability. A large portion of the transport matrix results from reciprocal scattering processes and obeys the Onsager relations as a consequence of microscopic reversibility. In general, however, there is a contribution to the transport arising from essentially kinematic flows that do not fundamentally obey reciprocal relations. Flows of this type (notably the Ware pinch) arise in toroidal plasma confinement configurations but do not occur in the types of systems first studied by Kelvin and toward which Onsager’s arguments were directed.
The necessity of separating out the kinematic flows before identifying the symmetric transport fluxes is in principle well known\textsuperscript{5–9}. Perhaps the most familiar example of this occurs in Enskog's\textsuperscript{5} derivation of the hydrodynamic equations from kinetic theory where the transport calculation is performed in a frame of reference moving with the macroscopic fluid velocity. One does not have a symmetric transport theory in the laboratory frame. An analogous separation of the kinematic flows occurs in the theories of Van Kampen\textsuperscript{6}, Graham and Haken\textsuperscript{7}, Zwanzig\textsuperscript{8}, and Mori\textsuperscript{9}. What makes the example in the present paper interesting is that these kinematic flows turn out to be linearly proportional to the forces, so that one could consider them as transport fluxes. Indeed, they would be indistinguishable, empirically. However, when the kinematic flows are included in the transport matrix, the "Onsager" symmetry can be broken\textsuperscript{10}.

In the original formulation of Neoclassical transport theory\textsuperscript{11}, no distinction was made between kinematic and collisional fluxes and a symmetry was identified and termed "Onsager" symmetry that included both. The Neoclassical symmetry, however, is due to a specific mathematical perturbation theory. It cannot be traced to a physical reciprocity of the basic microscopic scattering events and is not fundamentally valid.

The practical purpose of this paper is to develop a general formulation of a transport theory for confined plasmas capable of treating in a common framework, for a complex magnetic geometry, collisions and turbulence. This is important because of the dominance, in practise, of turbulent transport. The general formulation makes it possible to establish which transport matrix symmetries are fundamentally linked to microscopic reversibility and therefore are generally valid.

In Section II, a general Fokker–Planck kinetic equation is derived and the symmetries in this equation that arise from microscopic reversibility are identified. This kinetic equation is solved, in Section III, by an expansion which generates the transport equations. In Section IV, the symmetries of the transport equations are displayed. These are compared in Section V to the symmetries that result from a different type of transport theory based on the regression of fluctuations about a global thermal equilibrium. This latter type of transport is what Onsager considered. It can be obtained from our Fokker–Planck equation in certain
special cases. By finding the conditions for deriving Onsager's results from this same kinetic theory, it will become quite clear why the relations fail in the more general case.
II. Kinetic Theory: Generalized Fokker–Planck Equation

A formulation of transport theory for fusion plasmas that does not depend on the details of the scattering processes but only some very general properties has already been developed\textsuperscript{12}. This Lagrangian formulation is based upon an action–space kinetic equation that in Ref. 12 was obtained for a specific scattering process (Coulomb collisions) in the ideal banana regime limit. In this section we derive a kinetic equation of the same type that is more general with respect to both scattering process and collisionality. The derivation will link certain general mathematical properties of the kinetic equation to fundamental physical principles. In particular the scattering operator, including turbulence, will be shown to be self-adjoint as a consequence of microscopic reversibility. The kinematic flows that will make additional contributions to the transport matrix can be identified and shown to be separate processes whose conjugate parts are not related by time reversal. The friction coefficient which is associated with the kinematic flows will be shown to be time reversible (the name "kinematic" was chosen on the basis of this property).

A general kinetic equation of the Chapman–Kolmogorov type can now be written for the distribution function, $f(\mathbf{X}, t)$ provided that the underlying scattering processes (collisional and turbulent) can be considered as Markov processes.

$$f(\mathbf{X}, t) = \int \{d\mathbf{X}'\} P(\mathbf{X}, t|\mathbf{X}', t') f(\mathbf{X}', t'),$$  \hspace{1cm} (1)

where the conditional probability density $P(\mathbf{X}, t|\mathbf{X}', t')$ is the probability of a transition from the state described in a generalized manner by the vector $\mathbf{X}'$ at time $t'$ to the new state described by $\mathbf{X}$ at time $t$. This equation can describe: kinematic, reversible flows, where $P$ is a delta function along the deterministic orbit; binary collisions where $P$ will depend on $f$ so that Eq. (2) is non-linear; and turbulent scattering processes where the statistics of the turbulent fluctuations appear in the transition probability function. We restrict the class of turbulent fluctuations somewhat by requiring that they have saturated and that all the statistical functions relax on a time scale short compared to the evolution time scale of the one particle distribution function, $f$. Then $P$ will
depend only on \( f \) and not on higher statistical functions, so that Eq. (2) is closed.

We now proceed to the Fokker-Planck limit of Eq. (2), appropriate to the turbulent and collisional scattering processes found in fusion plasmas, where the state vector scatters in small relative steps and in a continuous fashion to a good approximation. Following the usual Fokker-Planck equation derivation procedure we transform Eq. (2) into a differential equation by using the Kramers-Moyal expansion and subsequently assume that the transition probability function, \( P \), is very peaked with respect to \( \Delta X \equiv X - X' \); both the the transition probability function, \( P \), and the distribution function \( f(X, t) \) are also assumed to vary slowly with \( X \). Then Eq. (2) reduces to the Fokker-Planck equation,

\[
\frac{\partial f(X, t)}{\partial t} = - \frac{\partial}{\partial X^\nu} [K^\nu(X)f(X, t)] + \frac{1}{2} \frac{\partial^2}{\partial X^\nu \partial X^\mu} [D^{\nu \mu}(X)f(X, t)] \tag{2}
\]

where the summation convention over repeated indices is used. The friction vector \( K^\nu(X) \) and the diffusion tensor \( D^{\nu \mu}(X) \) are given by

\[
K^\nu(X) \equiv \lim_{\epsilon \to 0^+} \epsilon^{-1} \int \{d\Delta X\} \Delta X^\nu w(X, \Delta X, \epsilon, t) \tag{3}
\]

\[
D^{\nu \mu}(X) \equiv \lim_{\epsilon \to 0^+} \epsilon^{-1} \int \{d\Delta X\} \Delta X^\nu \Delta X^\mu w(X, \Delta X, \epsilon, t) \tag{4}
\]

where

\[
w(X, \Delta X, \epsilon, t) \equiv P(X + \Delta X, t + \epsilon|X, t) \tag{5}
\]

The transition probability, \( P(X, t|X', t') \), considers the system running forward in time from \( t' \) to \( t \). It is obtained by computing microscopic trajectories in each realization of the scattering fields starting at \( X' \) at time \( t' \), and averaging the results over the statistical ensemble for the scattering process. If, in each realization, time is reversed at time \( t \) and the system run backward to time \( t' \), one recovers the time reversed initial data. This is a statement of microscopic
reversibility. For stationary processes microscopic reversibility implies the following condition for the transition probability

\[ P(X, t|X', t') J(X') = P(\mathcal{T}X', -t'|\mathcal{T}X, -t) J(X) \]  

(6)

where \( \mathcal{T} \) is the time reversal operator and \( J(X') \) is the Jacobian, or volume element in the space \( X \). A detailed proof of this relation is given in the Appendix. Equation (6) expresses the physical condition of detailed balance or equality for forward and inverse rates for each transition. Not all systems obey detailed balance. If certain types of radiative processes and external forces are included, the system can still be described by the Fokker-Planck equation, but the full transition probability will not have the symmetry, Eq. (6), primarily because stationarity is lost. Nevertheless, we expect that some of the processes, the scattering processes, to have the detailed balance property, and that the corresponding piece of the transition probability will obey Eq. (6).

We now write the Fokker-Planck equation in an alternative form to exploit the symmetry relation, Eq. (6). This follows from expanding

\[ w(X - \Delta X, \Delta X, \epsilon, t)J(X - \Delta X) = w(X, \Delta X, \epsilon, t)J(X) 
- \frac{1}{2} \Delta X^\nu \frac{\partial w(Y, \Delta X, \epsilon, t)J(Y)}{\partial Y^\nu} 
+ \cdots \]  

(7)

which is sufficiently accurate for the Fokker-Planck description. Using Eq. (7), for \( w(X, \Delta X, \epsilon, t) \), to rewrite the friction vector, \( K^\nu(X) \), gives,

\[ K^\nu(X) = J^{-1}(X)H^\nu(X) + \frac{1}{2} J^{-1}(X) \frac{\partial}{\partial X^\mu} [J(X)D^{\mu\nu}(X)] \]  

(8)

where,
Upon inserting Eq. (8) into Eq. (2) yields the alternate form Fokker-Planck equation.

\[
\frac{\partial f(X, t)}{\partial t} = - \frac{\partial}{\partial X^\nu}[H^\nu(X)f(X, t)] \\
+ \frac{1}{2} \frac{\partial}{\partial X^\nu}\left[J(X)D^\nu(X)\frac{\partial f(X, t)}{\partial X^\mu}f(X, t)\right].
\] (10)

Equation (10) is a valid form whether or not detailed balance, Eq. (6), holds. Now one can also define an invariant distribution function (invariant upon any transformation of the vectors which describes the state of the system) \(g(X, t)\) as follows

\[
g(X, t) = \frac{f(X)}{J(X)}. \tag{11}
\]

Utilizing the invariant distribution function \(g\) in Eq. (10), the Fokker-Planck equation takes the covariant form

\[
\frac{\partial g(X, t)}{\partial t} = -J^{-1}(X)\frac{\partial}{\partial X^\nu}[J(X)H^\nu(X)g(X, t)] \\
+ \frac{1}{2}J^{-1}(X)(X)\frac{\partial}{\partial X^\nu}\left[J(X)D^\nu(X)\frac{\partial g(X, t)}{\partial X^\mu}\right]. \tag{12}
\]

We now examine the symmetry properties of the vector \(H^\nu\), of Eq. (9). These are based on the time reversal properties of the vector \(h^\nu(X, \epsilon, t)\), defined by
\[ H^\nu(X) \equiv \lim_{\epsilon \to 0^+} h^\nu(X, \epsilon, t) \]  

The time reversed form of the function \( h^\nu \), since \( J(\tau X) = J(X) \), can be written.

\[ h^\nu(\tau X, -\epsilon, -t) = -\epsilon^{-1} \int \{d\Delta X\} \Delta X^\nu \]

\[ P(\tau X + \frac{\Delta X}{2}, t - \epsilon, \tau X - \frac{\Delta X}{2}, t) J(\tau X - \frac{\Delta X}{2}) \]  

Using the symmetry relation for the transition probabilities, Eq. (6), and Eq. (A28) yields

\[ h^\nu(\tau X, -\epsilon, -t) = -\epsilon^{-1} \int \{d\Delta X\} \Delta X^\nu \]

\[ P(X - \tau^{-1} \frac{\Delta X}{2}, t; X + \tau^{-1} \frac{\Delta X}{2}, t + \epsilon) J(X + \tau^{-1} \frac{\Delta X}{2}) \]

\[ + \epsilon^{-1} \int \{d\Delta X\} \Delta X^\nu P_{ns}(X - \tau^{-1} \frac{\Delta X}{2}, t; X + \tau^{-1} \frac{\Delta X}{2}, t + \epsilon) \]  

The time reversal transformation rule for the infinitesimals \( dX^\nu \) can now be approximated by introducing the matrix \( \epsilon^\nu_\mu \) (in the most cases \( \epsilon^\nu_\mu = \pm \delta^\nu_\mu \) as follows

\[ \tau dX^\nu \equiv \epsilon^\nu_\mu dX^\mu \]  

Since upon, time reversal, the difference operator \( \Delta \) (final state – initial state) changes sign, one has, by virtue of the previous equation

\[ \tau \Delta X^\nu = -\epsilon^\nu_\mu \Delta X^\mu \]  

Upon change of variables in Eq. (15) and taking into account Eq. (17) and the properties of the operator \( \tau \), Eq. (A5), yields upon taking the limit \( \epsilon \to 0^+ \)
\[ \tau H^{\nu}(X) = H^{\nu}(\tau X) = -\epsilon_{\mu}^{\nu} H^{\mu}(X) \]

+ \epsilon_{\mu}^{\nu} \lim_{\epsilon \to 0^+} \epsilon^{-1} \int \{d\Delta X\} \Delta X^{\nu} P_{\nu\mu}(X + \frac{\Delta X}{2}, t; X - \frac{\Delta X}{2}, t + \epsilon) \quad (18) \]

One can now introduce the concept of "reversible", \( R^{\nu} \), and "irreversible" \( I^{\nu} \), parts of a vector function \( A^{\nu}(X) \) of the space of states as follows

\[ R^{\nu}(X) = \frac{1}{2} [A^{\nu}(X) - \epsilon_{\mu}^{\nu} A^{\mu}(\tau X)] \quad (19) \]

\[ I^{\nu}(X) = \frac{1}{2} [A^{\nu}(X) + \epsilon_{\mu}^{\nu} A^{\mu}(\tau X)]. \quad (20) \]

As a consequence of Eqs. (19-20) as well as of the identity \( \epsilon_{\mu}^{\nu} \epsilon_{\rho}^{\rho} = \delta_{\mu}^{\nu} \) (\( \delta_{\mu}^{\nu} \) is the Kronecker delta) one has

\[ R^{\nu}(\tau X) = -\epsilon_{\mu}^{\nu} R^{\mu}(X) \quad (21) \]

\[ I^{\nu}(\tau X) = \epsilon_{\mu}^{\nu} I^{\mu}(X) \quad (22) \]

The differential operators \( \partial / \partial X^{\nu} \) transform, upon time reversal, like the infinitesimals \( dX^{\nu} \) in Eq. (16); therefore, taking into account Eqs. (21-22) and (A5) yields the following transformation rules for the operators \( J^{-1} \partial / \partial X^{\nu} J R^{\nu} \) and \( J^{-1} \partial / \partial X^{\nu} J I^{\nu} \)

\[ \tau \left( J^{-1} \frac{\partial}{\partial X^{\nu}} J R^{\nu} \right) = -J^{-1} \frac{\partial}{\partial X^{\nu}} J R^{\nu} \quad (23) \]

\[ \tau \left( J^{-1} \frac{\partial}{\partial X^{\nu}} J I^{\nu} \right) = J^{-1} \frac{\partial}{\partial X^{\nu}} J I^{\nu} \quad (24) \]

that is the operator associated with the "reversible" part of the vector \( A^{\nu} \) transforms like the time variable \( t \). The transformation relation, Eq. (18), readily implies that for stationary processes (i.e. \( P_{ns} = 0 \)) one has
\[ \mathcal{T} H^\nu(X) = H^\nu(\mathcal{T} X) = -\epsilon^\nu_\mu H^\mu(X) \]  

that is, the vector \( H^\nu(X) \) is a purely "reversible" vector and, therefore, its associated term in the covariant Fokker–Planck equation, Eq. (12), changes sign upon time reversal. For general processes, of course, the second term of the right side of Eq. (18) clearly corresponds to the irreversible part of the vector \( H^\nu(X) \).

Concluding, we saw that the Fokker Planck equation, given by Eqs. (10) or (12) consists of two parts. The last term in these equations, which involves the operator for the scattering processes, turbulent and collisional, is in self-adjoint form. The term associated with the flow vector \( H^\nu \), on the other hand, can directly drive flows that do not have a reciprocal process and are not subject to the Onsager relations. This term, for stationary processes only, transforms like a reversible quantity and is associated with kinematic flows. If the variables \( X^\nu \) are even under time reversal (i.e. \( \epsilon^\nu_\mu = \delta^\nu_\mu \)) then Eq. (25) implies that, for stationary processes, the vector \( H^\nu \) is identically zero. This form of the Fokker–Planck equation has been previously derived in a different context and with a different way by Van Kampen, who also identified the terms on the right of Eq. (10) as "reversible" and "irreversible" flows respectively. Van Kampen then assumed some special properties of the Fokker–Planck coefficients which lead directly to a type of transport equation. His assumptions, however apply only for systems which are close to thermal equilibrium; in this paper we are interested in states far from thermal equilibrium as it happens with confined plasmas. His transport equations are compared in Sec. IV to those derived in the following Sec. III. Graham also developed a theory which applies for stationary steady states far from thermal equilibrium.
III. Transport Equations

We consider a magnetic confinement geometry where the equilibrium particle motion in the absence of fluctuations can be described by action–angle variables \( X = (J, \theta) \). In practical terms, this means that the collisionless orbits are contained to a good approximation (for longer than a collision time). Although the use of canonical action–angle variables is not necessary it greatly simplifies the mathematical symmetries needed to establish the reciprocal relations.

For the tokamak example of Ref. 12, the actions are,

\[
J_1 = \pi m (b \times \nu)^2 / \Omega \equiv \text{MAGNETIC MOMENT} \tag{28}
\]

\[
J_2 = m \int dsa \equiv \text{PARALLEL INVARIANT} \tag{27}
\]

\[
J_3 = 2\pi q \langle \psi \rangle / c \equiv \text{DRIFT CENTER FLUX COORDINATE} \tag{28}
\]

while the conjugate angles \( \theta_1, \theta_2, \) and \( \theta_3, \) are, respectively, bounce average gyrophase, bounce phase and drift phase. The associated frequencies, \( \omega_i \equiv \partial H / \partial J_i, \) are then the bounce averaged gyrofrequency, bounce (or transit) frequency and bounce averaged drift frequency.

Generally, one is interested in systems where the distribution function, \( f, \) evolves on time scales much longer than the correlation time of the basic statistical processes. Moreover, it often happens that the angular dependence of \( f \) is destroyed at a faster rate than the transport processes, so that one need work only with

\[
f(J, t) = \int d^3 \theta f(J, \theta, t) \tag{29}
\]

Thus, in our tokamak example, the \( \theta_1 \) dependence of \( f \) is phase mixed away on the gyroperiod time scale, the \( \theta_2 \) dependence on a bounce time scale [the boundary later between trapped and circulating particles is resolved by collisions so that there is a minimum effective transit frequency of \( \omega_{tmin} = \omega_0 2\pi \)]
Both angular mixing processes are much faster than the collision frequency for fusion plasmas. The $\theta_\nu$ dependence can be eliminated by axisymmetry. Formally, what happens in Eq. (2) is that for time intervals, $|t - t'|$, exceeding these mixing times, the probability function, $P$, becomes independent of the angles, implying the same for $f$. This gives an effective correlation time for action scattering. We can then write a kinetic equation of Chapman-Kolmogorov type for the distribution function of Eq. (29)

$$f(J, t) = \int d^3 J' P_f(J, t|J', t') f(J', t'),$$  \hspace{1cm} (30)

where,

$$P_f(J, t|J', t') = \int d^3 \theta d^3 \theta' P(J, \theta, t|J', \theta', t'),$$  \hspace{1cm} (31)

is the angle averaged transition probability. Following the same procedure as in Sec II. yields a Fokker-Planck equation of the form

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial J} \cdot H f + \frac{1}{2} \frac{\partial}{\partial J} \cdot D \cdot \frac{\partial}{\partial J} f,$$  \hspace{1cm} (32)

where the explicit forms of the kinematic flow vector $H$ and the diffusion tensor $D$ are

$$H(J) \equiv \lim_{\epsilon \to 0^+} \epsilon^{-1} \int d^3 \Delta J \Delta J P_f(J + \frac{\Delta J}{2}, t + \epsilon|J - \frac{\Delta J}{2}, t) J(J - \frac{\Delta J}{2})$$  \hspace{1cm} (33)

$$D(J) \equiv \lim_{\epsilon \to 0^+} \epsilon^{-1} \int d^3 \Delta J \Delta J \Delta J P_f(J + \Delta J, t + \epsilon|J, t)$$  \hspace{1cm} (34)

For stationary processes the detailed balance symmetry property, Eq. (6), reads

$$P_f(J, t|J', t') = P_f(J', -t'|J, -t)$$  \hspace{1cm} (35)

since the action-angle description is canonical (i.e. $J(X' \to X) = 1$) and $\tau J = J$. 

13
The last term in Eq. (32), which is self-adjoint, describes turbulent and collisional scattering processes in action space. The term associated with kinematic flows can, through the coefficient, \( \langle \Delta J_3 / \Delta t \rangle = H_3 \), directly drive radial flows that do not have, as we already mentioned, a reciprocal process and are not subject to the Onsager relations.

Binary collisions from like-particle encounters are described by a bilinear operator with more complicated symmetries that, as expected, also imply reciprocal relations for the transport coefficients. In the interest of simplicity, we refer the reader to Ref. 12, and will not discuss such processes further here.

The kinetic equation, Eq. (32), can be solved by an expansion about local thermal equilibrium to generate the transport equations provided that three general conditions are met:

1. Radial scattering is slow compared to velocity scattering.
2. Velocity scattering is dominated by collisions.
3. Kinematic flows driven by external forces are small compared to velocity space flows driven by collisions.

Condition (1) implies, among other things, that the radial extent of the collisionless particle trajectories, \( \Delta \), be small compared to the characteristic macroscopic scale length, \( a \). We assume it is then possible to choose the three actions such that one of them, \( J_3 \), has two properties which allow it to be used as a radial variable in constructing a transport theory. First, it must be the actual radial position of a particle, to order \( \Delta / a \), (apart from constant multiplicative factors). Secondly, the Hamiltonian function, \( H(J) \), must depend weakly on this radial action such that

\[
\omega_1, \omega_2 \gg \partial H / \partial J_3
\]

\[
\partial \ln H / \partial J_3 \ll \partial \ln n / \partial J_3, \quad \partial \ln T / \partial J_3,
\]

where \( n \) and \( T \) are the particle density and temperature. These conditions assure that the radial position does not significantly affect a particle's energy and eliminates from consideration systems with spatially localized global thermal equilibrium states. The actions, \( J_1 \) and \( J_2 \), are the velocity like variables.
Usually, in a plasma, these will be taken to be the magnetic moment and the parallel invariant.

To formalize these condition, we write the kinetic equation (32) in the form

\[
\frac{\partial f}{\partial t} + \frac{\partial}{\partial J} \cdot [a(J)Vf] = C(f),
\]

(37)

where, \(C(f) = \partial/\partial J \cdot D \cdot \partial/\partial J f\), is the scattering operator and \(V\) is a coefficient parameterizing the external forces that drive the kinematic flows.

For example, in the tokamak problem, \(V\) is the toroidal loop voltage giving a force, \(F = e\xi qV/2\pi R\), in the \(e\xi\) direction. The instantaneous increment in action due to this force is, \(\Delta J = (qV/2\pi R)e\xi \cdot \partial/\partial J\), so that the acceleration coefficient becomes,

\[
a(J) = \frac{1}{\Delta t} \langle (q/2\pi R)e\xi \cdot \partial/\partial J \rangle_k,
\]

(38)

The action vector is written, \(J = J_1e_1 + J_2e_2 + J_3e_3\), and we employ the notation,

\[
J_\perp = J_1e_1 + J_2e_2,
\]

(39)

for the velocity–like part of the action vector.

In this formulation, quantities such as particle number and energy are expressed as densities per unit of action, \(J_3\), and arise as reduced moments such as \(n_3 = \int dJ_1dJ_2 f\). Thus, when the distribution \(f\) is known, one obtains a particle transport equation from Eq. (37) by simply integrating out the velocity like actions. Since \(J_3\) is a (Lagrangian) particle label and not the (Eulerian) spatial position, the density \(n_3\) does not correspond exactly to the spatial density. This difference arises from the finite orbital widths of the collisionless orbits, and is small by a factor of order \(\Delta/a\). It will turn out that this distinction is, to the order one works in transport theory, irrelevant as far as the density is concerned. However, since kinematic flows within a surface can result from such effects (e.g., the banana current), it is necessary to develop to the relationship between the action space moments and the various spatial densities required.
As spatial variables we will use the conventional magnetic flux coordinates, 
\((\psi, \beta, s)\) discussed in Ref. 12. The volume element is then,

\[
\nabla_s \cdot \nabla \beta \times \nabla \psi d^3x = dsd\beta d\psi = B d^3x,
\]

and the specific volume, \(dV/d\psi\), between flux surfaces is

\[
dV/d\psi = \oint \frac{|d\beta|}{B} = 2\pi \oint \frac{|ds|}{B},
\]

where the line integral covers one circuit of the poloidal circumference. The path length, \(|ds|\), in Eq. (41) is taken to be positive. The flux surface average of any function, \(F\), can be written

\[
\langle F \rangle_\psi \equiv \frac{d\psi}{dV} \int dAF/|\nabla \psi| = \frac{d\psi}{dV} \oint |d\beta F/B.
\]

The subscript, \(\psi\), distinguishes this from the bounce average, natural to the Lagrangian representation, and denoted by

\[
\langle F \rangle = \int d\theta_2 F = \omega_2 \oint dsF/w,
\]

where the path element, \(ds\), in Eq. (43) has the same sign as \(w\).

Given any distribution \(f(J, \theta)\) in the action–angle variables, the spatial density \(n(x)\) can be obtained by projection, viz.

\[
n(x) = \int d^3Jd^3\theta \delta(x - x(J, \theta))f(J, \theta),
\]

with \(x, (J, \theta)\) determined by the transformation \(x, v \rightarrow J, \theta\). In magnetic flux coordinates this becomes, using Eq. (39),

\[
n(x) = \int d^3Jd^3\theta B \delta(s - s(J, \theta))\delta(\beta - \beta(J, \theta))\delta(\psi - \psi(J, \theta))f(J, \theta).
\]

The flux surface average density takes a particularly simple form. We apply Eq. (267) to Eq. (45). There results,
\( \langle n(x) \rangle_\psi = \frac{d\psi}{dV} \int d^3 J \int d^3 \theta \delta(\psi - \psi(J, \theta)) f(J, \theta), \) \hspace{1cm} (46)

For present purposes, we write the third action as

\( J_3 \equiv \frac{2\pi q}{c} \psi + \Delta J_3(\theta). \) \hspace{1cm} (47)

Thus, to order \( \Delta J_3/J_3 \sim \Delta/a \), \( J_3 = (2\pi q/c)\psi \) and to second order in \( \Delta/a \) for \( f \) independent of \( \theta \), the flux surface average density becomes

\[ \langle n(x) \rangle_\psi = \frac{2\pi q}{c} \frac{d\psi}{dV} \int d^2 J_\perp f \left( J_\perp, \frac{2\pi q}{c} \psi \right) = \frac{dJ_3}{dV} n_3, \] \hspace{1cm} (48)

so that \( \langle n \rangle_\psi /n_3 \) is just the specific volume between surfaces \( J_3 = \text{const.} \). Actually, the local density \( n(x) \), not flux surface averaged, and \( n_3 \) have the same relationship although only to first order in \( \Delta/a \).

Now, we also need an expression for the flow conjugate to the external force, \( \mathbf{F} \), that is parameterized by the flux function, \( V = V(\psi) \). The actual current in the \( e_\psi \) direction is given by

\[ J = \int d^3 J d^3 \theta \delta(x - x(J, \theta)) q e_\psi \cdot \mathbf{v}(J, \theta) f(J), \] \hspace{1cm} (49)

which in general depends on the position, \( (s, \beta) \), within the magnetic flux surface in addition to \( \psi \). To obtain a flux function, \( I = I(\psi) \), for this flow, conjugate to, \( V = V(\psi) \), we multiply \( J \) by \( 1/2\pi R \) and \( dV/dJ_3 \), to give an angular current per unit \( J_3 \) and then flux surface average. There results

\[ I \equiv \frac{c}{2\pi q} \frac{dV}{d\psi} \langle J/2\pi R \rangle_\psi, \]

\[ = \int d^3 J d^3 \omega_2 \int \frac{ds}{u} \frac{c}{2\pi q} \delta \left( \psi - \frac{c}{2\pi q} (J_3 - \Delta J_3) \right) \frac{q}{2\pi R} e_\psi \cdot \mathbf{v} f, \]

\[ = \int d^2 J_\perp \omega_2 \int \frac{ds}{u} \frac{q}{2\pi R} e_\psi \cdot \mathbf{v} f \left( J_\perp, \frac{2\pi q}{c} \psi + \Delta J_3 \right). \] \hspace{1cm} (50)
Expanding in powers of, $\Delta J_3/J_3 = \Delta/a$, this becomes,

$$I = \int d^2 J_\perp \omega_2 \int ds \frac{q}{u} \frac{c}{2\pi R} e_\perp \cdot \mathbf{v} \left[ f \left( J_\perp, \frac{2\pi q}{c} \psi \right) + \Delta J_3 \frac{\partial f}{\partial J_3} \left( J_\perp, \frac{2\pi q}{c} \psi \right) + \cdots \right].$$

(51)

Here the first term gives a moment over the bounce averaged velocity in the $e_\perp$ direction in a flux surface, while the second term, arising from the radial excursion of the orbits is of the diamagnetic type. We will label these two contributions as implicit, $I^i$, and explicit, $I^e$, respectively. To be more general, a coefficient,

$$a_\perp^0 = e_2 \int ds \frac{q}{u} \frac{c}{2\pi R} e_\perp \cdot \mathbf{v},$$

(52)

can be defined such that,

$$I^i = \int d^2 J_\perp \omega_\perp \cdot a_\perp^0 f \left( J_\perp, \frac{2\pi q}{c} \psi \right),$$

(53)

is the implicit current. The ordering of Eq. (37), in terms of the small parameter, $\Delta/a$, follows from the general conditions stated above. In particular, one seeks solutions where the distribution function, $f$, evolves on the transport time scale or at rate of order $(\Delta/a)^2$ slower than the basic collision frequency. Other parameters are related to $\Delta/a$ to achieve a maximal ordering where as many processes as possible appear in the transport equations. Thus, the coefficient, $V$, is treated as order, $\Delta/a$, so that the kinematic flow operator is ordered,

$$\frac{\partial}{\partial J_\perp} \cdot a_\perp^0 V f = \mathcal{O}(\Delta/a f),$$

(54)

$$\frac{\partial}{\partial J_3} a_3 V f + \frac{\partial}{\partial J_\perp} \cdot a_\perp^1 V f = \mathcal{O}((\Delta/a)^2 f).$$

(55)

In this ordering, the power dissipated in the system by the external forces, $V$, balances thermal conduction in the energy transport equation. Here the
kinematic flow coefficient, defined in Eq. (38), has, for its velocity space components, been decomposed into an ordered sum, \( a_\perp = a_\perp^0 + a_\perp^1 \), such that the leading order piece is equal to the purely kinematic coefficient, Eq. (52). This means that the statistical aspects of the average on Eq. (38) are relatively weak for this term.

Similarly, the scattering operator, \( C(f) \), is decomposed into the ordered sum,

\[
C(f) = C_0(f) + C_1(f) + C_2(f) = \frac{\partial}{\partial J} \cdot D^0 \cdot \frac{\partial}{\partial J} f + \frac{\partial}{\partial J} \cdot D^1 \cdot \frac{\partial}{\partial J} f + \frac{\partial}{\partial J} \cdot D^2 \cdot \frac{\partial}{\partial J} f, \tag{56}
\]

where \( D^0 \) contains only velocity components,

\[
D^0 = D_{11}^0 e_1 e_1 + D_{12}^0 (e_1 e_2 + e_2 e_1) + D_{22}^0 e_2 e_2, \tag{57}
\]

\( D^1 \) contains off-diagonal, radius-velocity components,

\[
D^1 = D_{13}^1 (e_1 e_3 + e_3 e_1) + D_{23}^1 (e_2 e_3 + e_3 e_2), \tag{58}
\]

and the diagonal radial components, \( D_{33} \), as well as other small contributions to the other components, appear in \( D^2 \). In addition, \( D^0 \) and \( D^1 \) have the property that

\[
D^0 \cdot \omega_\perp = D^1 \cdot \omega_\perp = 0, \tag{59}
\]

where \( \omega_\perp = \partial H / \partial J_\perp \). This condition, Eq. (59), assures that all scattering processes which do not conserve the energy of this particular particle species do not appear until second order and can be accounted for in the transport equation.

The Fokker–Planck coefficients, \( D = (\Delta J^2 \Delta J / \Delta t) \), are Lagrangian functional integrals averaged over the various stochastic processes and cannot, in general, be separated into a sum of turbulent and collisional parts. Nonetheless, it may be useful for physical interpretation to assume such as separation can be made, at least approximately. In that case, \( D^0 \) and \( D^1 \) will include only
collisional scattering, whereas $D^2$ will include turbulent scattering of all action components in addition to collisional radial diffusion, and collisional processes involving energy exchange like ion–electron temperature equilibration.

Following this ordering of the kinetic equation, we expand $f$ in powers of $\Delta/a$. The zero order equation is,

$$0 = C_0(f_0).$$  \hfill (60)

Now, with the property given by Eq. (59), $f_0$ can be any function of $H$ and $J_3$ [provided the $J_3$ dependence of $H$ is negligible as assumed by condition Eq. (36)]. This gives an arbitrariness to the solutions that cannot be properly resolved by the linear scattering operator given by Eq. (56). However, if like particle collisions were included in the analysis, the local H–theorem of Ref. 12 would apply and uniquely require $f_0$ to be a local Maxwellian, $f_0 = f_m = N(J_3) \exp(-H_0(J_\perp)/T(J_3))$. To conform with the more general theory we utilize this specific equilibrium. For reasons that will be clear shortly, we normalize $f_m$ as follows:

$$f_0 = f_m \equiv n(2\pi m T)^{-3/2} \exp(-H_0/T),$$  \hfill (61)

where $n = n_3 \partial J_3 / \partial V = n_3 \frac{2\pi a}{c} \partial \psi / \partial V$ is the density per unit spatial volume.

Having solved Eq. (60), we proceed to the first order equation

$$\frac{\partial}{\partial J_\perp} \cdot [V \mathbf{a}_\perp^0 f_0] - \frac{\partial}{\partial J} \cdot \left[ D^1 \cdot \frac{\partial}{\partial J} f_0 \right] = C_0(f_1)$$  \hfill (62)

The second term on the left side of Eq. (62) becomes, using Eq. (59),

$$\frac{\partial}{\partial J} \cdot D^1 \cdot \frac{\partial}{\partial J} = -\frac{\partial}{\partial J} \cdot D^1 \cdot \omega_\perp \frac{f_0}{T}$$

$$+ \frac{\partial}{\partial J_\perp} \cdot \left( D^1 \cdot e_3 \frac{\partial f_0}{\partial J_3} \right) = \left( \frac{\partial}{\partial J_\perp} \cdot D^1 \cdot e_3 \right) \frac{\partial f_0}{\partial J_3}$$  \hfill (63)

In the external force term, since $\mathbf{a}_\perp^0$ describes a kinematic flow in a canonical phase space, $\partial / \partial J_\perp \cdot \mathbf{a}_\perp^0 = 0$, one can write
\[
\frac{\partial}{\partial J_\perp} \cdot V a_\perp f_0 = V a_\perp \cdot \frac{\partial f_0}{\partial J_\perp} = -\frac{V}{T} a_\perp \cdot \omega f_0
\]  

(64)

Using these properties, the first order equation can be written as either

\[
C_0(f_1) = \frac{\partial}{\partial J_\perp} \cdot a_\perp V f_0 - \frac{\partial}{\partial J_\perp} \cdot \left( D^1 \cdot e_3 \frac{\partial f_0}{\partial J_3} \right)
\]  

(65)

or

\[
C_0(f_1) = -a_\perp \cdot \omega f_0 \frac{V}{T} - \left( \frac{\partial}{\partial J_\perp} \cdot D^1 \cdot e_3 \right) \frac{\partial f_0}{\partial J_3}
\]  

(66)

Inversion of the collision operator, \(C_0\), is now required to determine \(f_1\), subject to certain integrability conditions. In the present formulation these conditions are trivially satisfied, as we now show.

The integrability conditions are determined by the annihilators of \(C_0\), in this case the particle and energy moments. It is evident that the particle moment, \(\int d^2 J_\perp\), also annihilates the source terms from Eq. (65). For the energy moment, we compute \(\int d^2 J_\perp H_0(J)\), from Eq. (65), and integrate the right side by parts. There results the condition,

\[
0 = -\int d^2 J_\perp a_\perp \cdot \omega f_0 V + \int d^2 J_\perp \omega \perp \cdot D^1 \cdot e_3 \frac{\partial f_0}{\partial J_3}
\]  

(67)

The first term vanishes since the equilibrium distribution, \(f_0\), carries no current in the sense of Eq. (53). Using Eq. (59) the second term is zero.

For use in the ensuing transport equation, it is convenient to express the driving terms for \(f_1\) is terms of thermodynamic forces \(A_1, A_2, A_3\),

\[
A_1 = d \ln n/d J_3
\]  

(68)

\[
A_2 = d \ln T/d J_3
\]  

(69)

\[
A_3 = V/T
\]  

(70)
Using Eq. (61) for \( f_0 \), the first order Eq. (66) becomes \( C_0(f_1) = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 \), where the coefficients \( \alpha_i \), are given by

\[
\alpha_1 = -\frac{\partial}{\partial J_\perp} \cdot [D^1 \cdot e_3] f_0 \quad (71)
\]

\[
\alpha_2 = -\frac{\partial}{\partial J_\perp} \cdot \left[ D^1 \cdot e_3 \left( \frac{H_0}{T} - \frac{3}{2} \right) \right] f_0 \quad (72)
\]

\[
\alpha_3 = -a_\perp^0 \cdot \omega_\perp f_0 \quad (73)
\]

It is now convenient to write \( f_1 = f_0 \hat{f}_1 \), and to define the linearized collision operator, \( C_\ell \), by

\[
C_0(f_0 \hat{f}_1) = C_\ell(\hat{f}_1), \quad (74)
\]

where, by virtue of Eq. (59),

\[
C_\ell(\hat{f}_1) = \frac{\partial}{\partial J_\perp} \cdot D_0 f_0 \cdot \frac{\partial}{\partial J_\perp} \hat{f}_1, \quad (75)
\]

and \( C_\ell \) is clearly self-adjoint. Moreover, this factorization of \( f_1 \) remains useful when binary collisions are included in \( C_0 \), so that the operator, although more complicated, retains the self-adjoint form. If the perturbed distribution, \( \hat{f}_1 \), is expressed as a sum over the thermodynamic forces,

\[
\hat{f}_1 = \sum_i g_i A_i, \quad (76)
\]

then the individual responses, \( g_i \), are determined from

\[
C_\ell(g_i) = \alpha_i \quad (77)
\]

To second order in \( \Delta/a \), Eq. (62) becomes,
The integrability conditions for the solution of $f_2$ now provide the transport equations. Accordingly, although the transport equations themselves are second order, they require knowledge of $f$ only through first order.

The particle moment of Eq. (76) is,

\[
\frac{\partial f_0}{\partial t} + \frac{\partial}{\partial J_3} a_3(J) f_0 V + \frac{\partial}{\partial J_\perp} \cdot a_0^0(J) f_1 V + \frac{\partial}{\partial J_\perp} \cdot a_1 f_0 V - C_1(f_1) - C_2(f_0) = C_0(f_2)
\]  

(78)

The second term in brackets can be integrated by parts,

\[
\frac{\partial n_3}{\partial t} + \frac{\partial}{\partial J_3} \left[ \int d^2 J_\perp a_3 f_0 V - \int d^2 J_\perp e_3 \cdot D^1 \cdot \frac{\partial}{\partial J_\perp} f_1 - \int d^2 J_\perp D_{33} \frac{\partial f_0}{\partial J_3} \right]
\]

(79)

\[
= 0
\]

The second term in brackets can be integrated by parts,

\[
- \int d^2 J_\perp e_3 \cdot D^1 \cdot \frac{\partial}{\partial J_\perp} f_1 = \int d^2 J_\perp f_1 \frac{\partial}{\partial J_\perp} \cdot (e_3 \cdot D^1)^t = \int d^2 J_\perp f_1 \frac{\partial}{\partial J_\perp} \cdot D^1 \cdot e_3 = - \int d^2 J_\perp \alpha_1 \dot{f}_1
\]

(80)

using the symmetry of $D^1$ and the definitions Eqs. (71)-(73) to give,

\[
\frac{\partial n_3}{\partial t} + \frac{\partial}{\partial J_3} \left[ \int d^2 J_\perp \alpha_3 f_0 V - \int d^2 J_\perp \alpha_1 \dot{f}_1 - \int d^2 J_\perp D_{33} \frac{\partial f_0}{\partial J_3} \right] = 0
\]

(81)

This is in the form of a transport equation.
\[
\frac{\partial n_3}{\partial t} + \frac{\partial \Gamma}{\partial J_3} = 0
\]  
(82)

with the flux, \( \Gamma \), being proportional to the thermodynamic forces, \( A_i \). Note that the thermodynamic force, \( A_1 \), is defined in terms of the spatial number density, \( n \), while the transport equation itself is expressed per unit \( J_3 \). One can now define the transport coefficients, \( T_{1i} \), as follows,

\[
\Gamma = T_{1i} \frac{d\ln n}{dJ_3} + T_{12} \frac{d\ln T}{dJ_3} + T_{13} \frac{V_T}{T}
\]  
(83)

We use an inner product notation, \( (g, h) \equiv \int d^2 J \, g h \), to write the coefficients. The second term in Eq. (81) by virtue of Eqs. (76) and (77), yields implicit contributions of the form \( T_{12} \equiv - (\alpha_1, g_1) \). In addition, there are explicit contributions to the flow that do not require the calculation of \( f_1 \). We denote these by \( T_{1j} \). Collecting terms, the particle transport coefficients, \( T_{1j} \), become,

\[
T_{11} = (-D_{33}, f_0) - (\alpha_1, g_1) = T_{11}^e + T_{11}
\]  
(84)

\[
T_{12} = \left(-D_{33}, f_0 \left( \frac{H_0}{T} - \frac{3}{2} \right) \right) - (\alpha_1, g_2) = T_{12}^e + T_{12}
\]  
(85)

\[
T_{13} = (T \alpha_3, f_0) - (\alpha_1, g_3) = T_{13}^e + T_{13}
\]  
(86)

The coefficient, \( T_{11} \), is the particle diffusion coefficient, appropriately normalized, and contains both explicit and implicit parts.

The procedure for obtaining the energy transport equation from the energy moment of Eq. (78) is similar. The first order term in the external force, \( V \), now contributes to give power input. The energy transport equation is in the form

\[
\frac{\partial}{\partial t} \left( \frac{3}{2} n_3 T \right) + \frac{\partial}{\partial J_3} \left( \frac{3}{2} \frac{\Gamma T + q}{T} \right) = T^i V + P^e
\]  
(87)

with the heat flux,
\[ q/T = T_{21} \frac{d \ln n}{d J_3} + T_{22} \frac{d \ln n}{d J_3} + T_{32} V/T \]  \hspace{1cm} (88)

and explicit heating power

\[ P^e = - \int d^2 J_\perp H_0(J) \frac{\partial}{\partial J_\perp} \cdot a^1 f_0 V \]  \hspace{1cm} (89)

The transport coefficients are again decomposed into implicit and explicit parts, as

\[ T_{21} = \left( -D_{33}, f_0 \left( \frac{H_0}{T} - \frac{3}{2} \right) \right) - (\alpha_2, g_1) = T_{21}^e + T_{21}^i \]  \hspace{1cm} (90)

\[ T_{22} = \left( -D_{33}, f_0 \left( \frac{H_0}{T} - \frac{3}{2} \right)^2 \right) - (\alpha_2, g_2) = T_{22}^e + T_{22}^i \]  \hspace{1cm} (91)

\[ T_{23} = \left( T \alpha_3, f_0 \left( \frac{H_0}{T} - \frac{3}{2} \right) \right) - (\alpha_2, g_3) = T_{23}^e + T_{23}^i \]  \hspace{1cm} (92)

Finally, the current, \( I^i \), appearing in the power input term \( I^i V \) of Eq. (87), can also be expressed as a flux driven by the thermodynamic forces

\[ I^i = T_{31}^i \frac{d \ln n}{d J_3} + T_{32}^i \frac{d \ln n}{d J_3} + T_{33}^i V/T \]  \hspace{1cm} (93)

where the transport coefficients in Eq. (318) are only of the implicit type,

\[ T_{31}^i = -\langle \alpha_3, g_1 \rangle \]  \hspace{1cm} (94)

\[ T_{32}^i = -\langle \alpha_3, g_2 \rangle \]  \hspace{1cm} (95)

\[ T_{33}^i = -\langle \alpha_3, g_3 \rangle \]  \hspace{1cm} (96)

When the equilibrium Hamiltonian, \( H_0(J) \), is known, transport equations Eqs. (82) and (87), together with the coefficients in Eqs. (84)-(86), (90)-(92), and
(94)-(96) are a closed system evolving the density and temperature. The particle density and current are expressed naturally per unit $J_3$ in this representation. However, the Hamiltonian, $H_0(J)$, depends implicitly on the magnetic field. This requires knowledge of the current density in real space and includes the kinematic contribution, $I^e$, in addition to $I^i$. Were it not for this circumstance the real space densities would never be needed in the transport theory. The appearance of $n$ in the thermodynamic force, $A_1$, is a consequence of the normalization, Eq. (61), and not fundamental.

In addition to the dissipative current, $I^i$, there are diamagnetic currents, $I^e$, computed above, for which no work is done by the external force. The flux surface averaged current, Eq. (51), to first order in $\Delta/a$ has an implicit piece, as given by Eq. (93) and an explicit piece in the form

$$I_T^e = T_{31}^e \frac{d \ln n}{d J_3} + T_{32}^e \frac{d \ln T}{d J_3}$$

obtained from the second term of Eq. (51) using $f = f_0$. 
IV. General symmetries of the Transport Matrix

Consider first the matrix, $T_{ij}$, of transport coefficients derived in Sec. III. This is decomposed, $T_{ij} = T_{ij}^i + T_{ij}^s$, into an implicit part, $T_{ij}^i$, which requires the calculation of a first order distribution function, and a part, $T_{ij}^s$, which is given explicitly in terms of the Maxwellian distribution, $f_0$, and the Fokker–Planck coefficients.

The symmetry of the implicit coefficients follows immediately from the self-adjointness of $C_{2}$,

$$T_{ij}^i = -(\alpha_i, g_j) = -(g_i, C_{2}(g_j)) = -(g_i, \alpha_j) = T_{ji}^i \quad (98)$$

Equality of the explicit coefficients, $T_{12}^s$ and $T_{21}^s$, is evident by inspection of Eqs. (84)-(86) and (90)-(92). This leaves the explicit coefficients $T_{13}, T_{31}$ and $T_{23}, T_{32}$ without any general symmetry relation. Noting that these coefficients represent kinematic flows, we can define a kinematic transport matrix, $T^k$, consisting of these coefficients with all other elements zero. The rest of the matrix, $T - T^k$, will be designated, $T^*$. To summarize,

$$T = T^* + T^k \quad (99)$$

where the part due to scattering processes

$$T^* = \begin{pmatrix} T_{11} & T_{12} & T_{13}^i \\ T_{21} & T_{22} & T_{23}^i \\ T_{31}^i & T_{32}^i & T_{33} \end{pmatrix} \quad (100)$$

is symmetric, and the part due to kinematic flows,

$$T^k = \begin{pmatrix} 0 & 0 & T_{13}^s \\ 0 & 0 & T_{23}^s \\ T_{31}^s & T_{32}^s & 0 \end{pmatrix} \quad (101)$$

is, in general, not symmetric.
The symmetry of the matrix $T^*$ can be traced back directly to microscopic reversibility. Mathematically, this gives rise to symmetric Fokker–Planck equation as demonstrated in Sec. II, which accounts for both the self-adjoint property of $C_\ell$, and the equality of the forcing functions, $\alpha_\ell$, for $f_1$, in Eq. (77), with the moments, $\alpha_i$, defining the fluxes in Eqs. (82) and (87). Physically, the fluxes at conjugate locations in the matrix $T^*$ correspond to scattering processes that are microscopic inverses of one another.

On the other hand, the fluxes at conjugate locations in the matrix, $T^k$, correspond to independent kinematic flows. The coefficient $T^i_{13}$ describes a radial flow driven directly by the force, $V/T$, while $T^5_{31}$ comes from a current within the flux surface due to finite orbital widths and the density gradient. Such processes cannot be linked by time reversal and are therefore not conjugate in the Onsager sense. This is why the matrix, $T^k$, does not possess a general symmetry.

It is enlightening to compare these symmetries to those arising in a different type of transport theory that corresponds more closely to the empirical theory of Onsager. The comparison is considered in the following section.
V. Onsager's Transport Theory and Restricted Symmetries

Onsager's transport theory\(^2\) is based on the regression or relaxation of fluctuations about a global thermal equilibrium. The transport equations are not directly derived but result from an empirical construction (described below) that relates the relaxation of the global moments to the fluxes. The fluctuations can be described by a Fokker-Planck equation where the independent variables correspond to the global parameters. The interpretation is then quite different from the Fokker-Planck equation on the single particle phase space employed in Sec. III above. The Fokker-Planck equation symmetries remain, nevertheless, and lead to the usual Onsager relations, when this type of transport theory is appropriate. In the discussion below we will attempt to compare and contrast the two transport theories, particularly as to their regimes of validity. The principal difference is the requirement, in the Onsager type theory, of a nearly global thermal equilibrium. The theory of Sec. III, in contrast, requires only nearness to a local equilibrium and is not in many cases of practical interest close to a global steady state.

We now give a specific, but readily generalizable, example of Onsager's formulation of transport theory. Consider a medium bounded at \(x = \pm a\), by perfectly insulating walls and homogeneous in the \(y\) and \(z\) directions. The system is characterized by spatial distributions of energy \(E(x)\), density \(n(x)\), and a variety of other possible quantities, \(Q'^i(x)\) such as magnetization, charge etc. In equilibrium, these distributions are, on average, constant functions of \(x\), but undergo fluctuations about this constant value, as illustrated in Fig. 1. The spatial distribution of energy can be characterized by the global moment,

\[
\alpha^1 = \int_{-a}^{+a} dxxE(x),
\]

and similar moments characterizing the other quantities,

\[
\alpha^i = \int_{-a}^{+a} dxxQ'^i(x).
\]

If one averages statistically over all distributions, when the system is in equi-
librium, the average value of these moments is zero, $\bar{\alpha}^i = 0$.

Now one can take a subset of the equilibrium ensemble (equivalent to a non-equilibrium state) where for each $i$, all the $\alpha^i$'s are non-zero and identical. For this non-equilibrium ensemble, each distribution, $Q^i(x)$, has an average gradient, $\overline{dQ^i/dx}$, which is non-zero. For small departures from equilibrium, $\alpha^i$ small in some sense, this average gradient should be proportional to the moment, $\alpha^i$, thus

$$\frac{dQ^i}{dx} = c\bar{\alpha}^i,$$  \hspace{1cm} (104)  

where $\overline{Q^i}$ is the equilibrium value, and $c$ is a constant. The system will now relax to equilibrium as the moments $\alpha^i$ approach zero with rate of transport, or flux $J^i$ given by

$$J^i = \frac{d\bar{\alpha}^i}{dt}.$$  \hspace{1cm} (105)  

Again, this average is taken over a non-equilibrium ensemble. Equations (104), (105) are the relations necessary to relate the regression of fluctuations to transport theory. Now, if there is a dynamical theory expressing the relaxation of the moments, one can write,

$$\frac{d\bar{\alpha}^i}{dt} = \sum_j T^i_j \overline{\bar{\alpha}^j}.$$  \hspace{1cm} (106)  

This is clearly a global dynamical theory in which the coefficients depend only on the equilibrium values, $\overline{Q^i}$, and are independent of $x$. To construct a transport theory one first uses Eq. (106) to rewrite the flux as,

$$J^i = -\frac{1}{c} \sum_j T^i_j \frac{d\overline{Q^j}}{dx} \equiv -\sum_j T^i_j \frac{d\overline{Q^j}}{dx}.$$  \hspace{1cm} (107)  

Then the isolated system of Fig. 1 is embedded in a larger system, allowing slow spatial variations of the equilibrium values, $\overline{Q^i} \rightarrow \overline{Q^i}(x)$. By assuming that Eq. (107) is still valid for this slice of the larger system then, using the conservation equation,
\[ \frac{\partial Q^i}{\partial t} + \frac{\partial J^i}{\partial x} = 0, \quad (108) \]

together with Eq. (107), one can obtain local transport equations,

\[ \frac{\partial Q^i}{\partial t} - \sum_j \frac{\partial}{\partial x} T^i_j \frac{\partial Q^j}{\partial x} = 0 \quad (109) \]

This, in essence is Onsager's argument for obtaining the transport coefficients, \( T^i_j \), from the regression coefficients, \( T^j_i \), of Eq. (106).

The linear regression theory for the fluctuations can be obtained from the Fokker-Planck equation, where the independent variables are the global moments, \( \alpha^i \),

\[ \frac{\partial f}{\partial t} = -\frac{\partial}{\partial \alpha^i} H^i f + \frac{1}{2} \frac{\partial}{\partial \alpha^i} J D^i_j \frac{\partial}{\partial \alpha^j} f \quad (110) \]

The equilibrium distribution, \( f_0 \), in the absence of kinematic flows, \( H^i = 0 \), is given by, \( f_0 = J \), the phase volume. For the system considered here, we expect \( J \) to be Gaussian in the fluctuations, \( \alpha^i \), for (macroscopically) small values of \( \alpha^i \),

\[ J = A \exp(-g_{ij}\alpha^i\alpha^j), \quad (111) \]

where \( A \) is a normalization constant, and \( g_{ij} \) is a symmetric matrix. Equation (111) is the form expected from statistical mechanics where the exponent, \( -g_{ij}\alpha^i\alpha^j \), represents the entropy and takes its maximum value for the equilibrium position, \( \alpha^i = 0 \). By the same considerations, we expect the diffusion tensor, \( D^i_j \), to be independent of the \( \alpha^i \), for the values of \( \alpha^i \) considered. Except in certain special cases, the kinematic flow term precludes an equilibrium solution of Eq. (110) and would have to be excluded from a regression theory. Here we include it as a small additive and non-symmetric contribution to the flux derived below.

We now assume the system to be perturbed to some non-equilibrium state \( f \), where the \( \alpha^j \)'s have non-vanishing mean values,
\[ \bar{\alpha}^j \equiv \int \{ d\alpha \} \alpha^j f \]  

(112)

The regression of these mean values, \( \bar{\alpha}^j \), is obtained from the first moment of Eq. (110),

\[
\frac{\partial \bar{\alpha}^j}{\partial t} = \int \{ d\alpha \} H^j f + \frac{1}{2} \int \{ d\alpha \} f D^{ji} \frac{\partial}{\partial \alpha^i} \ln J
\]

\[ = \int \{ d\alpha \} H^j f - \int \{ d\alpha \} f D^{jk} g_{ki} \alpha^i \]

(113)

Defining the kinematic flow,

\[ w^j \equiv \int \{ d\alpha \} H^j f, \]

(114)

and the regression coefficients,

\[ T^j_i \equiv D^{jk} g_{ki}, \]

(115)

we have,

\[ \frac{d\bar{\alpha}^j}{dt} - w^j = -T^j_i \bar{\alpha}^i \]

(116)

The matrix \( T^j_i \) is clearly symmetric, since both \( D^{jk} \) and \( g_{ki} \) are symmetric. Note that it is necessary even here to extract the kinematic flows before symmetric regression equations can be obtained. This fact has been noted previously by several authors\(^5\)\(^-\)\(^9\).
VI. Conclusion

The general theory of the Fokker-Planck equation and its symmetries has been reviewed. The equation possesses a partial group of symmetries resulting from microscopic reversibility for processes obeying detailed balance. These symmetries in the kinetic theory hold whether or not the more restrictive conditions required to derive a transport theory are obeyed. The transport theory symmetries depend critically on specific system properties.

Onsager considered systems near a global thermal equilibrium state and had a Fokker-Planck equation describing the evolution of the distribution function of global variables. The assumption of a global equilibrium eliminates certain non-stationary, kinematic and radiative processes. A symmetric regression theory for the relaxation of these global parameters can then be derived. By regarding the isolated system as a slice embedded in a larger system, one gets a transport theory obeying the Onsager reciprocal relations.

A more general transport theory, derived in the present paper, uses a Fokker-Planck equation for a distribution function on a single particle phase space. Here, the development of a transport theory requires only closeness to local thermal equilibrium and the transport equations are directly derived. Non-stationary kinematic and radiative processes are allowed. The transport theory has broken symmetry in general although a symmetric portion corresponding to Onsager's reciprocity can be identified.

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APPENDIX: Microscopic Reversibility and Detailed Balance Conditions

A turbulent plasma is characterized by the occurrence of enhanced fluctuations due to some unstable collective modes the plasma can support. These enhanced fluctuations always affect to some degree the rates of the various relaxation processes. A growing in amplitude electromagnetic perturbation, for instance, can easily change or even destroy the structure of the phase space orbits which corresponds to the integrable part of the problem. If the latter has a Hamiltonian denoted by $H_0$, then the overall Hamiltonian can be modeled as,

$$H = H_0 + \xi H_1$$  \hspace{0.5cm} (A1)

where $\xi$ is a small parameter of statistical nature. The basis of this model is the sensitivity of the unperturbed orbits to the initial conditions; a growing electromagnetic perturbation can turn them completely irregular. In low level turbulence cases, $\xi H_1$ can easily be considered as a perturbation to the Hamiltonian $H_0$. One usually includes in $\xi H_1$ scattering processes due to particle-to-particle interactions (Coulomb collisions). Therefore $\xi H_1$ can be modeled to be responsible for all the relaxation processes occurring in the system as they are being modified by a low level turbulence and collisionality.

The dynamical equations of motion in canonical form are

$$\dot{Y} = F(Y, t; \xi) = J \frac{\partial H}{\partial Y}$$ \hspace{0.5cm} (A2a)

satisfying the initial conditions

$$Y(t = 0) = Y_0$$ \hspace{0.5cm} (A2b)

where $Y$ corresponds to a phase space point and $J$ the symplectic matrix. Although the parameter $\xi$ is going to be modeled as a statistical quantity, it expresses the various couplings occurring in the system in a deterministic way. Equation (A2) does not need to be integrable of course; however, we assume the existence and uniqueness of the solution. The formal solution in generalized coordinates (not necessarily canonical) $X$ as well as in the original canonical coordinates $Y$ is
\[ X = U(t; \xi)X_0 \]  \hspace{1cm} (A3a)

\[ Y = V(t; \xi)Y_0 \]  \hspace{1cm} (A3b)

with

\[ X = OY \]  \hspace{1cm} (A3c)

where \( U(t; \xi) \) (\( V(t; \xi) \)) is the operator which propagates the system from the point \( X_0 \) (\( Y_0 \)) to the point \( X \) (\( Y \)) during the time interval \( t \) taking into account all the various interactions (\( \xi \)); \( O \) is the transformation operator \( X \rightarrow Y \). The relation between the operators \( U(t; \xi) \) and \( V(t; \xi) \) is,

\[ U(t; \xi) = O^{-1} V(t; \xi) O \]  \hspace{1cm} (A4)

where \( O^{-1} \) is the inverse operator of \( O \). One can now introduce the time reversal operator \( T \) which has the following properties

\[ T t = -t \]  \hspace{1cm} (A5a)

\[ T g(X_1, t_1; X_2, t_2; \cdots; X_n, t_n) = g(T X_n, -t_n; \cdots; T X_2, -t_2; T X_1, -t_1) \]  \hspace{1cm} (A5b)

\[ T^2 = I \]  \hspace{1cm} (A5c)

where \( I \) is the identity operator and \( t_1 \geq t_2 \geq \cdots \geq t_n \); as a consequence, one has for the Jacobian determinant \( J(X \rightarrow T X) \) of the mapping \( X \rightarrow T X \)

\[ J(X \rightarrow T X) = 1 \]  \hspace{1cm} (A5d)

Invariance of Hamilton's equations under time reversal implies that the principle of microscopic reversibility can be expressed as follows,
\[ \mathcal{T} \mathcal{V}(t; \xi) = \mathcal{V}^{-1}(t; \xi) \quad (A6) \]

where \( \mathcal{V}^{-1} \) denotes the inverse operator of \( \mathcal{V} \) and \( \xi \). Equations (A4-A6) then imply

\[ \mathcal{T} \mathcal{U}(t; \xi) = \mathcal{U}^{-1}(t; \xi) = \mathcal{U}(-t; \xi) \quad (A7) \]

where \( \mathcal{U}^{-1} \) denotes the inverse operator of \( \mathcal{U} \). Equation (A2) describes an orbit in phase space which originates from the point \( X_0 \).

Now consider an ensemble of initial points distributed according to a phase space density

\[ N(X, t = 0) = \delta^6(X - X_0) \quad (A8) \]

As all points move according to Eq. (A2) the density function is being conserved since no orbits are being added or eliminated. Therefore, one has,

\[ \frac{\partial N}{\partial t} + \frac{\partial}{\partial X} \cdot [F_{nc}(X, t; \xi)N] = 0 \quad (A9) \]

where the generalized force \( F_{nc} \) is related to the force \( F_c \) of the canonical description through

\[ F_{nc} = \frac{\partial X}{\partial Y} \cdot F_c \quad (A10) \]

The formal solution of Eq. (A9) is,

\[ N(X, t; \xi, X_0) = J(X)N(\mathcal{U}^{-1}(t; \xi)X, t = 0) \quad (A11) \]

The factor \( J(X) \) is the Jacobian determinant \( \text{det}(\partial \mathcal{U}^{-1}X/\partial X) \) of the mapping \( X_0 \rightarrow X \).

We are now in position to model the perturbation as a stochastic process with \( \xi \) being distributed according to the probability density function \( P(\xi) \). The joint probability, \( f_2(X, t; X', t') \) is defined as follows,
\[ f_2(X, t; X', t') = \int \{dX_0\} P_0(X_0) \int d\xi P(\xi) J(X) \delta [\mathcal{U}^{-1}(t; \xi)X - X_0] \]

where the probability \( P_0(X_0) \) refers to an ensemble of initial conditions. Using Eqs. (A8) and (A12) one has,

\[ f_2(X, t; X', t') = \int \{dX_0\} P_0(X_0) \int d\xi P(\xi) J(X) \delta [\mathcal{U}^{-1}(t; \xi)X - X_0] \]

\[ J(X') \delta [\mathcal{U}^{-1}(t'; \xi)X' - X_0] \quad (A13) \]

Upon using the transformation property of the delta functions: \( \delta [\mathcal{U}^{-1}(t; \xi)X - X_0] = \delta \{\mathcal{U}^{-1}(t; \xi)[X - \mathcal{U}(t; \xi)X_0]\} = \det(\partial \mathcal{U}^{-1}X/\partial X) \delta [X - \mathcal{U}(t; \xi)X_0] = \delta [X' - \mathcal{U}(t'; \xi)X_0] J^{-1}(X) \)

Eq. (A13) becomes

\[ f_2(X, t; X', t') = \int \{dX_0\} P_0(X_0) \int d\xi P(\xi) \delta [X - \mathcal{U}(t; \xi)X_0] \delta [X' - \mathcal{U}(t'; \xi)X_0] \quad (A14) \]

Under time reversal the above equation takes the form (Eq. (A5))

\[ f_2(\tau X', -t'; \tau X, -t) = \int \{dX_0\} P_0(X_0) \int d\xi P(\xi) \delta [\tau X' - \mathcal{U}(-t'; \xi)X_0] \delta [\tau X - \mathcal{U}(-t; \xi)X_0] \quad (A15) \]

Using Eq. (A7) for microscopic reversibility in Eq. (A15) yields

\[ f_2(\tau X', -t'; \tau X, -t) = \int \{dX_0\} P_0(X_0) \int d\xi P(\xi) \delta [\tau X' - \mathcal{U}^{-1}(t'; \tau^{-1} \xi)X_0] \delta [\tau X - \mathcal{U}^{-1}(t; \tau^{-1} \xi)X_0] \quad (A16) \]
which by virtue of Eqs. (A5) and (A7) becomes

\[ f_2(\mathcal{T}X', -t'; \mathcal{T}X, -t) = \int \{dX_0\} P_0(X_0) \int d\xi P(\xi) \]

\[ \delta[X' - \mathcal{U}(t'; \mathcal{T}^{-1}\xi)\mathcal{T}^{-1}X_0] \delta[X - \mathcal{U}(t; \mathcal{T}^{-1}\xi)\mathcal{T}^{-1}X_0] \]  

(A17)

or, since \( d\xi P(\xi)\{dX_0\} P_0(X_0) = d\mathcal{T}^{-1}\xi P(\mathcal{T}^{-1}\xi)\{d\mathcal{T}^{-1}X_0\} P_0(\mathcal{T}^{-1}X_0) \), upon changing of integration variables in Eq. (A17) one obtains

\[ f_2(\mathcal{T}X', -t'; \mathcal{T}X, -t) = \int \{dX_0\} P_0(X_0) \int d\xi P(\xi) \]

\[ \delta[X' - \mathcal{U}(t'; \xi)X_0] \delta[X - \mathcal{U}(t; \xi)X_0] \]  

(A18)

Comparing Eqs. (A18) and (A14) gives the symmetry relation for the joint probability under time reversal,

\[ f_2(X, t; X', t') = f_2(\mathcal{T}X', -t'; \mathcal{T}X, -t) \]  

(A19)

For processes with external influences (time independent on the time scale of interest) this condition can be written as follows

\[ f_2(X, t; X', t'; \{\lambda\}) = f_2(\mathcal{T}X', -t'; \mathcal{T}X, -t; \{\mathcal{T}\lambda\}) \]  

(A20)

where \( \{\lambda\} \) corresponds to a set of externally controlled parameters or constraints (magnetic field in particular for systems of interest).

For stationary Markov processes the joint probabilities are simply related to the conditional probability density \( P(X, t|X', t') \), which is called transition probability and depends only on \( t-t' \), and the one point stationary distribution function \( f(X') \):

\[ f_2(X, t; X', t') = P(X, t|X', t')f(X') \]  

(A21)
Then Eq. (A19) yields for the transition probability

$$P(X, t|X', t')f(X') = P(\mathcal{T}X', -t'| \mathcal{T}X, -t)f(\mathcal{T}X)$$  \hspace{1cm} (A22)

Taking now into account the relation between the volume elements \{dX\} and \{dX'\}

$$\frac{J(X)}{J(X')} = \frac{\{dX'\}}{\{dX\}} = \frac{f(X)}{f(X')}$$  \hspace{1cm} (A23)

and since \(J(\mathcal{T}X) = J(X)\) by virtue of Eq. (A5), one finally has

$$P(X, t|X', t')J(X') = P(\mathcal{T}X', -t'| \mathcal{T}X, -t)J(X)$$  \hspace{1cm} (A24)

Equation (A24) expresses the **detailed balance condition** for stationary Markov processes.

In general, for non-stationary processes, one can formally add a piece, \(P_{ns}\), in the right hand side of Eq. (A24) which accounts for all those processes associated with non-stationary behavior (accelerating particles, for example)

$$P(X, t|X', t')J(X') = P(\mathcal{T}X', -t'| \mathcal{T}X, -t)J(X)$$

$$+ P_{ns}(X, t; X', t')$$  \hspace{1cm} (A25)

Time reversing the above equation yields

$$P(\mathcal{T}X, -t| \mathcal{T}X', -t')J(X') = P(X', t'| \mathcal{T}X, t)J(X)$$

$$+ \mathcal{T}P_{ns}(X, t; X, t)$$  \hspace{1cm} (A26)

where the time reversed \(P_{ns}(X, t; X, t)\) is to be taken according to the rule expressed by Eq. (A5b). Interchanging \((X, t) \rightarrow (X', t')\) in Eq. (A25) yields

$$P(X', t'| X, t)J(X) = P(\mathcal{T}X, -t| \mathcal{T}X', -t')J(X')$$

$$+ P_{ns}(X'; t', X, t)$$  \hspace{1cm} (A27)
Comparing now the last two equations yields the following property for $P_{ns}$

$$\mathcal{T} P_{ns}(X, t; X', t') = -P_{ns}(X', t'; X, t)$$  \hspace{1cm} (A28)
FIGURE CAPTION

Fluctuations about equilibrium in an isolated system. The moment $\alpha^1 = \int_{-a}^{+a} dx x E(x)$ describes the asymmetry of the distribution for a specific fluctuation.