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Ronald C. Davidson
Jonathan S. Wurtele

Plasma Fusion Center
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

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Ronald C. Davidson
Science Applications International Corporation, Boulder, CO 80302
Jonathan S. Wurtele
Plasma Fusion Center, Massachusetts Institute of Technology, Cambridge, MA 02139

ABSTRACT

The linearized Vlasov-Maxwell equations are used to investigate detailed free electron laser (FEL) stability properties for a tenuous relativistic electron beam propagating in the z-direction through the planar wiggler magnetic field $\underset{\sim}{B}^0(\underset{\sim}{x}) = -B_w \cos k_0 z \hat{\underset{\sim}{e}}_x$. Here, B_w = const. is the wiggler amplitude, and $\lambda_0 = 2\pi/k_0$ = const. is the wiggler wavelength. The theoretical model neglects longitudinal perturbations ($\delta\phi = 0$) and transverse spatial variations ($\partial/\partial x = 0 = \partial/\partial y$). Moreover, the model is based on the Vlasov-Maxwell equations for the class of self-consistent beam distribution functions of the form $f_b(z,\underset{\sim}{p},t) = \hat{n}_b \delta(p_x)\delta(P_y)G(z,p_z,t)$, where $\underset{\sim}{p} = \gamma m \underset{\sim}{v}$ is the mechanical momentum, and P_y is the canonical momentum in the y-direction. For low or moderate electron energy, there can be a sizeable modulation of beam equilibrium properties by the wiggler field and a concomitant coupling of the k'th Fourier component of the wave to the components $k \pm 2k_0$, $k \pm 4k_0$, $\cdots$ in the matrix dispersion equation. In the diagonal approximation, investigations of detailed stability behavior range from the regime of strong instability (monoenergetic electrons) to weak resonant growth (sufficiently large energy spread). In the limit of ultrarelativistic electrons and very low beam density, the kinetic dispersion relation is compared with the dispersion relation obtained from a linear analysis of the conventional Compton-regime FEL equations. Finally, assuming ultrarelativistic electrons and a sufficiently broad spectrum of amplifying waves, the quasilinear kinetic equations appropriate to the planar wiggler configuration are presented.

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ABSTRACT

Use is made of the linearized Vlasov-Maxwell equations to investigate detailed free electron laser (FEL) stability properties for a tenuous relativistic electron beam propagating in the z-direction through the constant-amplitude planar wiggler magnetic field $\underset{\sim}{B}^0(\underset{\sim}{x}) = -B_w \cos k_0 z \hat{\underset{\sim}{e}}_x$. Here B_w = const. is the wiggler amplitude, and $\lambda_0 = 2\pi/k_0$ = const. is the wiggler wavelength. The theoretical model neglects longitudinal perturbations ($\delta\phi = 0$) and transverse spatial variations ($\partial/\partial x = 0 = \partial/\partial y$), and the beam density and current are assumed to be sufficiently low that equilibrium self fields have a negligible effect. The radiation field is assumed to be plane polarized, and the theoretical model is based on the Vlasov-Maxwell equations for the class of self-consistent beam distribution functions of the form $f_b(z,\underset{\sim}{p},t) = \hat{n}_b \delta(p_x)\delta(P_y)G(z,p_z,t)$, where $\underset{\sim}{p} = \gamma m \underset{\sim}{v}$ is the mechanical momentum, and P_y is the canonical momentum in the y-direction. The linear stability analysis makes no apriori restriction to ultrarelativistic electrons. Indeed, for low or moderate electron energy, there can be a sizeable modulation of beam equilibrium properties by the wiggler field and a concommitant coupling of the k'th Fourier component of the wave to the components $k \pm 2k_0$, $k \pm 4k_0, \cdots$. In the diagonal approximation, the matrix dispersion equation is used to investigate the detailed dependence of free electron laser growth rate on the choice of distribution function $G_0^+(\gamma_0)$. Investigations of stability behavior range from the regime of strong instability (monoenergetic electrons) to weak resonant growth (sufficiently large energy spread). In the limit of ultrarelativistic electrons and very low beam density, the kinetic dispersion relation is compared with the dispersion relation obtained from a linear analysis of the conventional Compton-regime FEL equations. This comparison is made for general beam equilibrium $G_0^+(\gamma_0)$. Differences between the two dispersion relations are traced to the eikenol approximation and the assumption of very narrow energy spread made in the derivation of the conventional Compton-regime FEL equations. Finally, assuming ultrarelativistic electrons and a sufficiently broad spectrum of amplifying waves, the quasilinear kinetic equations appropriate to the planar wiggler configuration are presented.

I. INTRODUCTION

Free electron lasers,[1-4] as evidenced by the growing theoretical[5-33] and experimental[34-47] literature on this subject, can be effective sources for the generation of coherent radiation by intense electron beams. Recent experimental investigations[44-47] have been very successful over a wide range of beam energy and current ranging from experiments at low energy (150-250keV) and low current (5A-45A),[45] to moderate energy (3.4MeV) and high current (0.5kA),[46,47] to high energy (20MeV) and low current (40A).[44] Theoretical studies have included investigations of nonlinear effects[5-15] and saturation mechanisms, the influence of finite geometry on linear stability properties,[16-21] novel magnetic field geometries for radiation generation,[21-26] and fundamental studies of stability behavior.[27-33] Because of the increased experimental emphasis on planar wiggler geometry,[44,46,47] in the present analysis we make use of the linearized Vlasov-Maxwell equations to investigate detailed free electron laser (FEL) stability properties for a tenuous relativistic electron beam propagating in the z-direction (Fig. 1) through the constant-amplitude planar wiggler magnetic field [Eq.(1)]

$$\underset{\sim}{B}^0(\underset{\sim}{x}) = -B_w \cos k_0 z \hat{\underset{\sim}{e}}_x \quad .$$

Here, B_w = const. is the wiggler amplitude, and $\lambda_0 = 2\pi/k_0$ = const. is the wiggler wavelength. The theoretical model neglects longitudinal perturbations ($\delta\phi=0$) and transverse spatial varizations ($\partial/\partial x = 0 = \partial/\partial y$), and the beam density and current are assumed to be sufficiently low that equilibrium self fields have a negligible effect ($\underset{\sim}{E}_s^0 = 0 = \underset{\sim}{B}_s^0$). As illustrated in Fig. 1, the radiation field is assumed to be plane polarized with electric and magnetic field components $\delta\underset{\sim}{E} = \delta E_y(z,t)\hat{\underset{\sim}{e}}_y$ and $\delta\underset{\sim}{B} = \delta B_x(z,t)\hat{\underset{\sim}{e}}_x$. Moreover, the theoretical model is based on the Vlasov-Maxwell equations for the class of self-consistent beam distribution functions of the form [Eq.(9)][30,31]

$$f_b(z,\underset{\sim}{p},t) = \hat{n}_b\delta(p_x)\delta(P_y)G(z,p_z,t),$$

where $\underset{\sim}{p} = \gamma m\underset{\sim}{v}$ is the mechanical momentum, and $P_y = p_y - (eB_w/ck_0)\sin k_0 z - (e/c)\delta A_y(z,t)$ is the canonical momentum in the y-direction, which is exactly conserved. Note from Eq.(9) that the transverse motion of the beam electrons in the x- and y-directions is assumed to be "cold." The kinetic stability analysis in Secs. II-V is based on a detailed investigation of the linearized Vlasov-Maxwell equations for the perturbed distribution function $\delta G(z,p_z,t) = G(z,p_z,t) - G_0(z,p_z)$ and the perturbed vector potential $\delta A_y(z,t)$. Althoguh the principal emphasis is on temporal growth (FEL oscillator), extension of the analysis to spatial growth (FEL amplifier) is relatively straightforward.

As motivation for this article, we remind the reader that conventional treatments[8,9] of the Compton-regime free electron laser instability for a planar magnetic wiggler are based on an analysis of the single-particle orbit equations assuming a monochromatic waveform. The self-consistent evolution of the wave amplitude and phase are then calculated[8,9] from Maxwell's equations, where the wiggler-induced current is determined by a statistical average over the single-particle orbits. While such an approach[8,9] has appealing features (e.g., the model is nonlinear and incorporates trapped-electron dynamics), there are also some shortcomings. For example, the analyses in Refs. 8 and 9 assume a monochromatic waveform for the radiation field, ultrarelativistic electrons, and an eikenol approximation to the wave field. Moreover, the statistical averaging procedure is partially based on an intuitive superposition of particle orbits. The present kinetic analysis, based on the Vlasov-Maxwell equations, is intended to investigate linear stability properties for a planar wiggler FEL from a different perspective. The outline and objectives of the article can be summarized as follows:

(a) We make use of the linearized Vlasov-Maxwell equations (Secs. II and III) to provide a thorough examination of free electron laser stability properties for perturbations about the general class of self-consistent beam equilibria $G_0(z,p_z) = U(p_z)G_0^+(\gamma_0)$ [Eq.(16)]. Here, $U(p_z)$ is the Heaviside step function defined by $U(p_z) = +1$ for $p_z>0$, and $U(p_z)=0$ for $p_z<0$. Moreover, $\gamma_0 mc^2 = [m^2c^4 + c^2p_z^2 + (e^2B_w^2/k_0^2)\sin^2 k_0 z]^{1/2}$ is the electron energy in the equilibrium wiggler field. The basis for performing statistical averages is well established in the Vlasov-Maxwell formalism.

(b) In Sec. III, to evaluate the perturbed distribution function $\delta G(z,p_z,t)$, use is made of the exact particle trajectories in the equilibrium wiggler field $-B_w \cos k_0 z \hat{e}_x$. The analysis makes no apriori restriction to ultrarelativistic electron. Indeed, for low or moderate electron energy, there can be a sizeable modulation of beam equilibrium properties by the wiggler field and a concommitant coupling of the k'th Fourier component of the wave field to the components $k \pm 2k_0$, $k \pm 4k_0, \cdots$. This is evident from the formal matrix dispersion equation (58) and the definition of electron susceptibility $\chi(k,\omega,k_0 z)$ in Eq.(63).

(c) In the diagonal approximation, Eq.(58) reduces to the dispersion relation (77). In Sec. IV, we make use of Eq.(77) to investigate the detailed dependence of the free electron laser growth rate on the choice of distribution funtion $G_0^+(\gamma_0)$. Investigations of stability behavior range from the regime of strong instability (monoenergetic electrons) to weak resonant growth (sufficiently large energy spread). For the case of weak resonant growth, the growth rates are calculated numerically for parameter regimes characteristic of the Los Alamos experiment,[44] and the Livermore experiments planned on the Advanced Test Accelerator (ATA).[47]

(d) The limiting case of ultrarelativistic electrons and very low beam density is considered in Sec. V. We compare the resulting kinetic dispersion relation (106) with the dispersion relation (127) obtained from a linear analysis of the conventional Compton-regime FEL equations.[8,9] This comparison is made for a general beam equilibrium $G_0^+(\gamma_0)$. Differences between the two dispersion relations can be traced to the eikenol approximation and the assumption of very narrow energy spread made in Refs. 8 and 9.

(e) Finally, assuming ultrarelativistic electrons and a sufficiently broad spectrum of amplifying waves, in Sec. V we summarize the quasilinear kinetic equations appropriate to the planar wiggler configuration considered in the present analysis. This represents a straightforward extension of the quasilinear theory development for the case of a helical magnetic wiggler field.[15] The quasilinear dispersion relation (128), the kinetic equation (129) for the distribution of beam electrons $G_0^+(\gamma_0,t)$, and the kinetic equation (131) for the wave spectral energy density $\mathcal{E}_k(t)$ describe the self-consistent nonlinear evolution of the beam electrons and radiation field in circumstances where the wave autocorrelation time is short in comparison with the characteristic growth time [Eq.(92)].

$II + III$

II. THEORETICAL MODEL AND ASSUMPTIONS

A. Theoretical Model

In the present analysis, we consider a relativistic electron beam propagating in the z-direction through the planar wiggler magnetic field (Fig. 1)

$$\underset{\sim}{B}^0(\underset{\sim}{x}) = -B_w \cos k_0 z \, \hat{\underset{\sim}{e}}_x \ . \tag{1}$$

Here, B_w = const. is the wiggler amplitude, and $\lambda_0 = 2\pi/k_0$ is the wiggler wavelength. The electron beam is assumed to have uniform cross section, and the beam density and current are assumed to be sufficiently small that the effects of equilibrium self-electric and self-magnetic fields can be neglected. Moreover, for a tenuous electron beam, the analysis is carried out in the Compton regime; thus longitudinal perturbations are neglected ($\delta\phi \simeq 0$).

We consider transverse electromagnetic fields with one-dimensional spatial variations, where $\partial/\partial x = 0 = \partial/\partial y$, and $\partial/\partial z$ is generally non-zero. Introducing the perturbed vector potential

$$\delta\underset{\sim}{A}(\underset{\sim}{x},t) = \delta A_y(z,t)\hat{\underset{\sim}{e}}_y \ , \tag{2}$$

the electromagnetic field perturbations, $\delta\underset{\sim}{E}(\underset{\sim}{x},t)$ and $\delta\underset{\sim}{B}(\underset{\sim}{x},t)$, can be expressed in the Coulomb guage as

$$\begin{aligned} \delta\underset{\sim}{E}(\underset{\sim}{x},t) &= -\frac{1}{c}\frac{\partial}{\partial t}\delta\underset{\sim}{A}(\underset{\sim}{x},t) = -\frac{1}{c}\frac{\partial}{\partial t}\delta A_y(z,t)\hat{\underset{\sim}{e}}_y \ , \\ \delta\underset{\sim}{B}(\underset{\sim}{x},t) &= \nabla\times\delta\underset{\sim}{A}(\underset{\sim}{x},t) = -\frac{\partial}{\partial z}\delta A_y(z,t)\hat{\underset{\sim}{e}}_x \ . \end{aligned} \tag{3}$$

There are two exact single-particle constants of the motion in the combined equilibrium and perturbed field configuration described by Eqs. (1) and (3). These are the mechanical momentum p_x and the canonical momentum P_y transverse to the beam propagation direction. Here, P_y is defined by

$$P_y = p_y - \frac{e}{c} A^0_{yw}(z) - \frac{e}{c} \delta A_y(z,t) \ , \tag{4}$$

where $-e$ is the electron charge, c is the speed of light in vacuo, and $A^0_{yw}(z)$ is the vector potential for the equilibrium wiggler field in Eq. (1), i.e.,

$$A^0_{yw}(z) = \frac{B_w}{k_0} \sin k_0 z \ . \tag{5}$$

In general, the beam distribution function $f_b(z,\mathbf{p},t)$ evolves according to the nonlinear Vlasov equation[30,31]

$$\left\{ \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} - e \left(\delta \mathbf{E} + \frac{\mathbf{v} \times (\mathbf{B}^0 + \delta \mathbf{B})}{c} \right) \cdot \frac{\partial}{\partial \mathbf{p}} \right\} f_b(z,\mathbf{p},t) = 0 \ . \tag{6}$$

The particle velocity $\mathbf{v}$ and momentum $\mathbf{p}$ are related by

$$m\mathbf{v} = \frac{\mathbf{p}}{(1+\mathbf{p}^2/m^2c^2)^{1/2}} \ , \tag{7}$$

where m is the electron rest mass. In Eq. (6), the field polarization is prescribed by Eq. (3), and $\delta A_y(z,t)$ is determined self-consistently from the Maxwell equation

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} \right) \delta A_y(z,t) = - \frac{4\pi e}{c} \int d^3p \ v_y \left[f_b(z,\mathbf{p},t) - f^0_b(z,\mathbf{p}) \right] \ . \tag{8}$$

Here, $f^0_b(z,\mathbf{p})$ is the equilibrium distribution function ($\partial/\partial t=0$) in the absence of perturbed fields ($\delta A_y=0$).

B. Nonlinear Vlasov-Maxwell Description

In the present analysis, we consider the class of exact solutions to the nonlinear Vlasov equation (6) of the form

$$f_b(z,\mathbf{p},t) = \hat{n}_b \delta(p_x) \delta(P_y) G(z,p_z,t) \ , \tag{9}$$

where P_y is defined in Eq. (4), $\hat{n}_b$=const. is the ambient electron density, and $G(z,p_z,t)$ is the one-dimensional distribution function in the phase space (z,p_z). In Eq. (9), the effective transverse motion of the beam electrons is "cold". Making use of the fact that p_x and P_y are exact constants of the motion in the combined equilibrium and perturbed fields [Eqs. (1) and (3)], we substitute Eq. (9) into Eq. (6) and integrate over p_x and p_y. This readily gives for the nonlinear evolution of the one-dimensional distribution function $G(z,p_z,t)$[30,31]

$$\left\{\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} - mc^2\left(\frac{\partial}{\partial z}\gamma_T\right)\frac{\partial}{\partial p_z}\right\} G(z,p_z,t) = 0 \ . \tag{10}$$

In Eq. (10), $\gamma_T(z,p_z,t)mc^2$ is the particle energy evaluated for p_x=0 and $P_y=p_y-(e/c)(A^0_{yw}+\delta A_y)=0$,

$$\gamma_T(z,p_z,t) = \left(1 + \frac{p_z^2}{m^2c^2} + \frac{e^2}{m^2c^4}\left[A^0_{yw}(z)+\delta A_y(z,t)\right]^2\right)^{1/2} , \tag{11}$$

and v_z is the axial velocity defined by $v_z=\partial(\gamma_T mc^2)/\partial p_z$, i.e.,

$$v_z = \frac{p_z}{\gamma_T m} \ . \tag{12}$$

Moreover, substituting Eq. (9) into Eq. (8), the Maxwell equation for $\delta A_y(z,t)$ becomes[30,31]

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right)\delta A_y(z,t) = -\frac{\hat{\omega}_p^2}{c^2}\left[(A^0_{yw}+\delta A_y)\int\frac{dp_z}{\gamma_T}G - A^0_{yw}\int\frac{dp_z}{\gamma_0}G_0\right] , \tag{13}$$

where $G(z,p_z,t)$ evolves according to Eq. (10), $\hat{\omega}_p^2=4\pi\hat{n}_b e^2/m$ is the nonrelativistic plasma frequency-squared, and $\gamma_0(z,p_z)$ is defined by [see Eq. (11) with δA_y=0],

$$\gamma_0(z,p_z) = \left(1 + \frac{p_z^2}{m^2c^2} + \frac{e^2B_w^2}{m^2c^4k_0^2}\sin^2 k_0 z\right)^{1/2} . \tag{14}$$

In Eq. (13), $G_0(z,p_z)$ is the beam equilibrium distribution that satisfies the steady-state Vlasov equation (10) with $\partial/\partial t=0$ and $\delta A_y=0$. That is, $G_0(z,p_z)$ solves

$$\left\{v_z \frac{\partial}{\partial z} - mc^2\left(\frac{\partial}{\partial z}\gamma_0\right)\frac{\partial}{\partial p_z}\right\} G_0(z,p_z) = 0 , \tag{15}$$

where $\gamma_0(z,p_z)$ is defined in Eq. (14), and v_z is defined by $v_z=\partial(\gamma_0 mc^2)/\partial p_z = p_z/\gamma_0 m$.

C. <u>Beam Equilibrium Properties</u>

Any distribution function $G_0(z,p_z)$ that is a function of the single-particle constants of the motion in the equilibrium field configuration described by Eq. (1) is a solution to the steady-state Vlasov equation (15). Unlike the case of a helical wiggler,[31] the axial momentum p_z is not an exact invariant in the planar wiggler described by Eq. (1). However, the particle energy $\gamma_0 mc^2$ defined in Eq. (14) <u>is</u> an exact invariant in the equilibrium field configuration. Therefore, in the present analysis, we consider the class of equilibrium solutions to Eq. (15) where the particles are moving in the positive z-direction ($p_z>0$) and $G_0(z,p_z)$ has the general form

$$G_0(z,p_z) = U(p_z)G_0^+(\gamma_0) . \tag{16}$$

Here, $U(p_z)$ is the Heaviside step function defined by

$$U(p_z) = \begin{cases} +1 , & p_z>0 , \\ 0 , & p_z<0 . \end{cases} \tag{17}$$

It is assumed that none of the electrons are "trapped" by the equilibrium wiggler field. That is, the form of $G_0^+(\gamma_0)$ in Eq. (16) is such that

$$\gamma_0^2 > 1 + a_w^2 , \tag{18}$$

where a_w is defined by

$$a_w = \frac{eB_w}{mc^2 k_0} . \tag{19}$$

Otherwise, the choice of $G_0^+(\gamma_0)$ in Eq. (16) and in the stability formalism developed in Secs. II and III is quite general.

To illustrate the spatial modulation (in z) of beam equilibrium properties by the wiggler field, we consider the example of a monoenergetic electron beam where

$$G_0^+(\gamma_0) = \frac{(\hat{\gamma}^2-1)^{1/2}}{\hat{\gamma}mc}\,\delta(\gamma_0-\hat{\gamma}) \tag{20}$$

and the constant $\hat{\gamma}$ satisfies $\hat{\gamma}^2 > 1 + a_w^2$. Making use of $dp_z = (\gamma_0 m^2c^2/p_z)d\gamma_0$, the equilibrium electron density $n_b^0(z)=\hat{n}_b\int dp_z G_0(z,p_z)$ can be expressed as

$$\begin{aligned} n_b^0(z) &= \hat{n}_b \frac{(\hat{\gamma}^2-1)^{1/2}}{\hat{\gamma}mc}\int_0^\infty dp_z\,\delta(\gamma_0-\hat{\gamma}) \\ &= \hat{n}_b mc\,\frac{(\hat{\gamma}^2-1)^{1/2}}{\hat{\gamma}}\int_1^\infty d\gamma_0\,\frac{\gamma_0}{p_z}\,\delta(\gamma_0-\hat{\gamma}) , \end{aligned} \tag{21}$$

where use has been made of Eqs. (16) and (20). From Eq. (14), we substitute $p_z=+mc\left[\gamma_0^2-1-a_w^2\sin^2 k_0 z\right]^{1/2}$ in the integrand in Eq. (21) and obtain

$$n_b^0(z) = \frac{\hat{n}_b}{(1-\hat{\kappa}^2\sin^2 k_0 z)^{1/2}} , \tag{22}$$

where $\hat{\kappa}^2<1$ is defined by

$$\hat{\kappa}^2 = \frac{a_w^2}{\hat{\gamma}^2-1} \, . \tag{23}$$

Depending on the size of $\hat{\kappa}^2$, we note from Eq. (22) that there can be a substantial modulation of the equilibrium beam density by the wiggler field. For example, if $k_0=1\ \mathrm{cm}^{-1}$, $\hat{\gamma}=3$, and $B_w=1.7$ kG, then $a_w=1$ and $\hat{\kappa}^2=1/8$, and the peak-to-minimum density modulation in Eq. (22) is about 6%.

Other equilibrium properties calculated from Eqs. (16) and (20) are also modulated as a function of z. For example, the average beam velocity in the z-direction is defined by $V_{zb}^0(z) = \left[\int dp_z (p_z/\gamma_0 m) G_0\right] / (\int dp_z G_0)$. Following the procedure used in the previous paragraph, it is readily shown from Eq. (20) that

$$V_{zb}^0(z) = \hat{V}_b (1-\hat{\kappa}^2 \sin^2 k_0 z)^{1/2} \, , \tag{24}$$

where $\hat{V}_b = c(\hat{\gamma}^2-1)^{1/2}/\hat{\gamma}$. Combining Eqs. (22) and (24), it follows that $n_b^0(z) V_{zb}^0(z) = \hat{n}_b \hat{V}_b = \mathrm{const.}$ (independent of z). This corresponds to a constant flux of electrons, which is expected from the continuity equation under steady-state conditions.

D. Linearized Vlasov-Maxwell Equations

We now make use of Eqs. (10) and (13) to derive the linearized Vlasov-Maxwell equations for small-amplitude perturbations, $\delta G(z,p_z,t)$ and $\delta A_y(z,t)$, about the beam equilibrium described by Eq. (16) for general choice of $G_0^+(\gamma_0)$. In this regard, it is useful to introduce the normalized perturbed vector potential $a_y(z,t)$ defined by

$$a_y(z,t) = \frac{e}{mc^2} \delta A_y(z,t) \, , \tag{25}$$

and express Eq. (11) in the equivalent form,

$$\gamma_T(z,p_z,t) = \left[1 + \frac{p_z^2}{m^2c^2} + a_w^2\sin^2 k_0 z + 2a_w \sin k_0 z a_y(z,t) + a_y^2(z,t)\right]^{1/2} , \tag{26}$$

where $a_w = eB_w/mc^2k_0$. For small-amplitude perturbations, Eq. (26) can be expanded to give the approximate results

$$\begin{aligned} \gamma_T &= \gamma_0 + \frac{a_w \sin k_0 z}{\gamma_0} a_y(z,t) , \\ \frac{1}{\gamma_T} &= \frac{1}{\gamma_0} - \frac{a_w \sin k_0 z}{\gamma_0^3} a_y(z,t) , \end{aligned} \tag{27}$$

where $\gamma_0(z,p_z)$ is defined by

$$\gamma_0 = \left(1 + \frac{p_z^2}{m^2c^2} + a_w^2 \sin^2 k_0 z\right)^{1/2} . \tag{28}$$

Equation (27) is valid to leading order in the perturbed vector potential $a_y(z,t)$, assuming $2|a_w a_y| << \gamma_0^2$.

We now express the distribution function $G(z,p_z,t)$ as its equilibrium value plus a perturbation,

$$G(z,p_z,t) = G_0(z,p_z) + \delta G(z,p_z,t) , \tag{29}$$

and make use of Eqs. (27) and (29) to simplify Eqs. (10) and (13). Retaining only the linear terms proportional to $\delta G(z,p_z,t)$ and $a_y(z,t)$, the Vlasov equation (10) gives

$$\begin{aligned} &\left\{\frac{\partial}{\partial t} + \frac{p_z}{\gamma_0 m}\frac{\partial}{\partial z} - mc^2 \frac{\partial \gamma_0}{\partial z}\frac{\partial}{\partial p_z}\right\} \delta G(z,p_z,t) \\ &\qquad = \frac{p_z}{\gamma_0^3 m} a_w \sin k_0 z \; a_y(z,t) \frac{\partial}{\partial z} G_0(z,p_z) \\ &\qquad\quad + mc^2 a_w \frac{\partial}{\partial z}\left(\frac{\sin k_0 z}{\gamma_0} a_y(z,t)\right) \frac{\partial}{\partial p_z} G_0(z,p_z) , \end{aligned} \tag{30}$$

which describes the evolution of the perturbed distribution function. Making use of Eq. (15) to simplify the right-hand side of Eq. (30), the linearized Vlasov equation can be expressed as

$$\left\{\frac{\partial}{\partial t} + \frac{p_z}{\gamma_0 m}\frac{\partial}{\partial z} - mc^2\left(\frac{\partial\gamma_0}{\partial z}\right)\frac{\partial}{\partial p_z}\right\}\delta G(z,p_z,t)$$

$$= \frac{mc^2 a_w}{\gamma_0}\frac{\partial}{\partial z}\left[\sin k_0 z a_y(z,t)\right]\frac{\partial}{\partial p_z}G_0(z,p_z) \ , \tag{31}$$

where $\gamma_0(z,p_z)$ is defined in Eq. (28), and $G_0(z,p_z)$ has the general form in Fq. (16). Finally, the perturbed vector potential $a_y(z,t)$ evolves according to Eq. (13). Linearizing Eq. (13) and making use of Eqs. (27) and (29), we obtain

$$\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} + \frac{\hat{\omega}_p^2}{c^2}\left(\int\frac{dp_z}{\gamma_0}G_0(z,p_z) - a_w^2\sin^2 k_0 z\int\frac{dp_z}{\gamma_0^3}G_0(z,p_z)\right)\right\}a_y(z,t)$$

$$= -\frac{\hat{\omega}_p^2}{c^2}a_w\sin k_0 z\int\frac{dp_z}{\gamma_0}\delta G(z,p_z,t) \ , \tag{32}$$

where $\hat{\omega}_p^2=4\pi\hat{n}_b e^2/m$, and $\delta G(z,p_z,t)$ evolves according to Eq. (31).

Equations (31) and (32) are the final versions of the linearized Vlasov-Maxwell equations used in the formal stability analysis in Sec. III. Keep in mind that Eqs. (31) and (32) are valid for small-amplitude perturbations about the general class of spatially-modulated beam equilibria $G_0(z,p_z) = U(p_z)G_0^+(\gamma_0)$ [Eq. (16)]. No a priori restriction has been made to a specific choice of $G_0^+(\gamma_0)$, nor has $\kappa^2=a_w^2/(\gamma_0^2-1) << 1$ been assumed.

III. DERIVATION OF THE GENERAL EIGENVALUE EQUATION

In this section, we make use of the linearized Vlasov-Maxwell equations (31) and (32) to investigate free electron laser stability properties. First a formal solution for $\delta G(z,p_z,t)$ is obtained from Eq. (31) using the method of characteristics (Sec. III.A), and then the particle orbits are calculated in the equilibrium field configuration (Sec. III.B). Following a derivation of the exact eigenvalue equation for $a_y(z,t)$ (Sec. III.C), we then simplify the eigenvalue equation in the diagonal approximation (Sec. III.D), where the coupling of the k'th Fourier component of a_y to the $k\pm 2k_0$, $k\pm 4k_0$,... components is neglected.

A. Solution for δG by the Method of Characteristics

The formal solution for $\delta G(z,p_z,t)$ can be obtained from the linearized Vlasov equation (31) by using the method of characteristics. For the case of temporal growth (single-pass FEL oscillator), the solution to Eq. (31) is given by

$$\delta G(z,p_z,t) = mc^2 a_w \int_{-\infty}^{t'} dt' \frac{\partial}{\partial z'} [\sin k_0 z' a_y(z',t')] \frac{1}{\gamma_0'} \frac{\partial}{\partial p_z'} G_0(z',p_z') \tag{33}$$

where the initial value (at $t'=-\infty$) has been neglected. Here, $z'(t')$ and $p_z'(t')=\gamma_0' m dz'/dt'$ are the phase-space trajectories in the equilibrium wiggler field $-B_w \sin k_0 z' \hat{e}_z$. Since $\gamma_0'=\gamma_0$=constant (independent of t'), the axial orbit $z'(t')$ satisfies

$$\gamma_0 \frac{dz'}{dt'} = +c(\gamma_0^2-1-a_w^2\sin^2 k_0 z')^{1/2} , \tag{34}$$

where $p_z'>0$ is assumed, and use has been made of Eq. (28). In Eqs. (33) and (34), the boundary conditions on $z'(t')$ and $p_z'(t')$ are $z'(t'=t)=z$ and $p_z'(t'=t)=p_z=\gamma_0 mv_z$. That is, the phase-space trajectory (z',p_z') pass through (z,p_z) at time $t'=t$.

We examine Eq. (33) for $p_z>0$ and make use of $G_0(z,p_z)=U(p_z)G_0^+(\gamma_0)$ [Eq. (16)]. It readily follows that

$$\frac{\partial}{\partial p_z'} G_0^+(\gamma_0') = \frac{v_z'}{mc^2}\frac{\partial G_0^+(\gamma_0')}{\partial \gamma_0'} , \tag{35}$$

in the integrand in Eq. (33). Since $\gamma_0'=\gamma_0=$const. along an equilibrium trajectory, the factor $\partial G_0^+/\partial\gamma_0$ can be taken outside of the t'-integral in Eq. (33). This gives

$$\delta G(z,p_z,t) = \frac{a_w}{\gamma_0}\frac{\partial G_0^+(\gamma_0)}{\partial\gamma_0}\int_{-\infty}^{t'} dt' v_z' \frac{\partial}{\partial z'}\left[\sin k_0 z' a_y(z',t')\right] , \tag{36}$$

for $p_z>0$. We further simplify Eq. (36) by making use of

$$\frac{d}{dt'}\left[\sin k_0 z' a_y(z',t')\right] = \left(\frac{\partial}{\partial t'} + v_z'\frac{\partial}{\partial z'}\right)\left[\sin k_0 z' a_y(z',t')\right] , \tag{37}$$

where d/dt' is the time derivative along an equilibrium orbit. Substituting Eq. (37) into Eq. (36) and integrating by parts with respect to t', we find (for $p_z>0$)

$$\begin{aligned}\delta G(z,p_z,t) &= \frac{a_w \sin k_0 z a_y(z,t)}{\gamma_0}\frac{\partial G_0^+}{\partial\gamma_0} \\ &\quad - \frac{a_w}{\gamma_0}\frac{\partial G_0^+}{\partial\gamma_0}\int_{-\infty}^{t} dt' \sin k_0 z' \frac{\partial}{\partial t'} a_y(z',t') ,\end{aligned} \tag{38}$$

where use has been made of $z'(t'=t)=z$, and $a_y(z',t'\rightarrow-\infty)=0$ is assumed.

In Sec. III.C, the formal solution for $\delta G(z,p_z,t)$ in Eq. (38) is substituted into the linearized Maxwell equation (32), and properties of the resulting eigenvalue equation for $a_y(z,t)$ are investigated. Although the principal emphasis in the present analysis is on temporal growth (FEL oscillator), for future reference, we conclude this section by stating the generalization of Eq. (38) to the case of spatial growth (FEL amplifier). Some straightforward algebra gives (for $p_z>0$)

$$\delta G(z,p_z,t) = \frac{a_w \sin k_0 z a_y(z,t)}{\gamma_0} \frac{\partial G_0^+}{\partial \gamma_0} - \frac{a_w}{\gamma_0} \frac{\partial G_0^+}{\partial \gamma_0} \int_{-\infty}^{z} \frac{dz'}{v_z'} \sin k_0 z' \frac{\partial}{\partial t'} a_y(z',t') \ , \tag{39}$$

where $a_y(z' \rightarrow -\infty, t') = 0$ is assumed, and $t'(z')$ is the inverse solution of Eq. (34) with boundary conditions $t'(z'=z)=t$ and $v_z'(z'=z)=v_z$.

B. Particle Orbits in the Equilibrium Wiggler Field

The orbit integral on the right-hand side of Eq. (38) requires a determination of the particle trajectory $z'(t')$ in the equilibrium wiggler field $\underset{\sim}{B}^0(\underset{\sim}{x})=-B_w \cos k_0 z \hat{e}_x$. Defining

$$\beta_0 = \left(1 - \frac{1}{\gamma_0^2}\right)^{1/2} \ , \tag{40}$$

and

$$\kappa^2 = \frac{a_w^2}{\gamma_0^2 - 1} \ , \tag{41}$$

the equation of motion (34) can be expressed as

$$\frac{d}{dt'} (k_0 z') = k_0 \beta_0 c \left[1 - \kappa^2 \sin^2 k_0 z'\right]^{1/2} \ . \tag{42}$$

The solution to Eq. (42) can be expressed in terms of the elliptic integral of the first kind

$$F(\eta,\kappa) = \int_0^{\eta} \frac{d\eta'}{\left[1-\kappa^2\sin^2\eta'\right]^{1/2}} . \tag{43}$$

In this regard, we introduce the shorthand notation

$$\begin{aligned} F &= F(\pi/2,\ \kappa) , \\ F' &= F\left[\pi/2,\ (1-\kappa^2)^{1/2}\right] , \\ F_z &= F(k_0\ z,\kappa) . \end{aligned} \tag{44}$$

Integrating Eq. (42) from $t'=t$ to time t' gives

$$F(k_0 z',\kappa) - F(k_0 z,\kappa) = \beta_0 c k_0 (t'-t) , \tag{45}$$

where $z'(t'=t)=z$. Moreover, Eq. (45) can be inverted to give the explicit solution for $z'(t')$. We find[48]

$$z'(t') = z + \beta_F c(t'-t) + \sum_{n=1}^{\infty} z_n\left[\sin 2n(\phi_z+\beta_F c k_0 \tau) - \sin 2n\phi_z\right] . \tag{46}$$

Here, $\tau=t'-t$, and the phase ϕ_z and average speed β_F are defined by[48]

$$\begin{aligned} \phi_z &= \frac{\pi}{2F} F_z , \\ \beta_F &= \frac{\pi}{2F} \beta_0 . \end{aligned} \tag{47}$$

Moreover, the oscillation amplitude z_n in Eq. (46) is defined by[48]

$$z_n = \frac{2}{k_0} \frac{1}{n} \frac{a^n}{1+a^{2n}} , \tag{48}$$

where

$$a = \exp(-\pi F'/F) . \tag{49}$$

Equation (46) is a very useful representation of $z'(t')$ for the subsequent simplification of the orbit integral [Eq. (38)] in Sec. III.C and Sec. IV. In this regard, no a priori assumption has been made that

$\kappa^2 = a_w^2/(\gamma_0^2-1) \ll 1$ in deriving Eq. (46) from Eq. (42). That is, depending on the size of κ^2, the oscillatory modulation of the axial orbit in Eq. (46) can be strong.

In the special limiting case where $\kappa^2 \ll 1$, the oscillatory modulation in (46) is weak, and the various elliptic integral factors defined in Eq. (44) and Eqs. (47) - (49) can be approximated by[48]

$$
\begin{aligned}
2F/\pi &= 1+\kappa^2/4 \ , \\
F' &= \ln(4/\kappa) \ , \\
F_z &= (1+\kappa^2/4)k_0 z \ , \\
\phi_z &= k_0 z \ , \\
a &= \kappa^2/16 \ , \\
z_n &= \frac{2}{nk_0}\left(\frac{\kappa^2}{16}\right)^n
\end{aligned}
\tag{50}
$$

when $\kappa^2 \ll 1$. Of course, Eq. (50) leads to a corresponding simplification in the expression for $z'(t')$ in Eq. (46). In particular, when $\kappa^2 \ll 1$, Eq. (50) can be approximated by

$$
z'(t') = z + \beta_F c(t'-t) + \sum_{n=1}^{\infty} z_n\left[\sin 2n(k_0 z+\beta_F ck_0\tau) - \sin 2nk_0 z\right] \ , \tag{51}
$$

where $\beta_F = (1-\kappa^2/4)\beta_0$ and $z_n = \kappa^2/8k_0 n$. With $n=1$, Eq. (51) is the familiar approximate expression for longitudinal motion in a planar wiggler.

C. General Eigenvalue Equation for $a_y(z,t)$

We now examine the linearized Maxwell equation (32) for the case of temporal growth (FEL oscillator). Substituting Eq. (39) for $\partial G(z,p_z,t)$

into Eq. (32) gives the equation for $a_y(z,t)$,

$$\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} + \frac{\hat{\omega}_p^2}{c^2} S(k_0 z)\right\} a_y(z,t)$$
$$= \frac{\hat{\omega}_p^2}{c^2} a_w^2 \sin k_0 z \int_0^\infty \frac{dp_z}{\gamma_0^2}\frac{\partial G_0^+}{\partial \gamma_0}\int_{-\infty}^{t} dt' \sin k_0 z' \frac{\partial}{\partial t'} a_y(z',t') \ , \tag{52}$$

where $\omega_p^2 = 4\pi\hat{n}_b e^2/m$, $z'(t')$ is the axial orbit defined in Eq. (46) for $\kappa^2 < 1$, and $S(k_0 z)$ is the spatially modulated form function defined by

$$S(k_0 z) = \int_0^\infty \frac{dp_z}{\gamma_0}\left[G_0^+(\gamma_0) - a_w^2 \sin^2 k_0 z\left(\frac{G_0^+(\gamma_0)}{\gamma_0{}^2} - \frac{1}{\gamma_0}\frac{\partial G_0^+(\gamma_0)}{\partial \gamma_0}\right)\right] . \tag{53}$$

While the formal stability analysis in Secs. III.C and III.D is presented for general $G_0^+(\gamma_0)$, for future reference, we state here the explicit functional form obtained for $S(k_0 z)$ for the special case where $G_0^+(\gamma_0)$ corresponds to monoenergetic electrons [Eq. (20)]. Substituting Eq. (20) into Eq. (53) and carrying out the integration over p_z gives

$$S(k_0 z) = \frac{1}{\hat{\gamma}}\cdot\frac{1}{(1-\hat{\kappa}^2 \sin^2 k_0 z)^{3/2}} \ , \tag{54}$$

where $\hat{\kappa}^2 = a_w^2/(\hat{\gamma}^2 - 1)$, and use has been made of $dp_z = (\gamma_0 m^2 c^2/p_z) d\gamma_0$ (see Sec. II.C). As expected, the strength of the spatial modulation of $S(k_0 z)$ depends on the size of $\hat{\kappa}^2$.

Equation (52) is analyzed using a normal mode approach, where $a_y(z,t)$ is assumed to be of the form[31]

$$a_y(z,t) = a_y(z)\exp(-i\omega t) \ , \quad \mathrm{Im}\omega > 0 . \tag{55}$$

Substituting Eq. (55) into Eq. (52) gives the eigenvalue equation for $\hat{a}_y(z)$,

$$\left\{\omega^2 + c^2 \frac{\partial^2}{\partial z^2} - \hat{\omega}_p^2 S(k_0 z)\right\} \hat{a}_y(z)$$

$$= i\omega\hat{\omega}_p^2 a_w^2 \sin k_0 z \int_0^\infty \frac{dp_z}{\gamma_0^2} \frac{\partial G_0^+}{\partial \gamma_0} \int_{-\infty}^{t} dt' \exp[-i\omega(t'-t)] \sin k_0 z' \hat{a}_y(z') , \tag{56}$$

where $z'(t')$ is defined in Eq. (46). In general, Eq. (56) should be solved numerically for the eigenfunction $\hat{a}_y(z)$ and eigenfrequency ω. For present purposes, it is useful to represent $\hat{a}_y(z)$ as the Fourier series

$$\hat{a}_y(z) = \sum_k \hat{a}_{yk} \exp(ikz) , \tag{57}$$

where $k=2\pi n/L$, n is an integer, L is the periodicity length in the z-direction, and the summation extends from $n=-\infty$ to $n=+\infty$. Substituting Eq. (57) into Eq. (56), we obtain

$$\sum_k \left\{\omega^2 - c^2 k^2 - \hat{\omega}_p^2 S(k_0 z) - \hat{\omega}_p^2 \chi(k,\omega,k_0 z)\right\} \hat{a}_{yk} \exp(ikz) = 0 . \tag{58}$$

Here, $\chi(k,\omega,k_0 z)$ is the dimensionless wiggler-induced susceptibility defined by

$$\chi(k,\omega,k_0 z) = -\frac{1}{4} i\omega a_w^2 \int_0^\infty \frac{dp_z}{\gamma_0^2} \frac{\partial G_0^+}{\partial \gamma_0}$$

$$\times \int_{-\infty}^{0} d\tau \exp(-i\omega\tau) \Big\{\exp[i(k+k_0)(z'-z)] [\exp(2ik_0 z)-1]$$

$$+ \exp[i(k-k_0)(z'-z)] [\exp(-2ik_0 z)-1]\Big\} , \tag{59}$$

where $\tau=t'-t$, and the axial orbit $z'(t')$ is defined in Eq. (46). Substituting Eq. (46) into Eq. (59) readily gives

$$\chi(k,\omega,k_0z) = -\frac{1}{4}\, i\omega a_w^2 \int_0^\infty \frac{dp_z}{\gamma_0^2} \frac{\partial G_0^+}{\partial \gamma_0} \int_{-\infty}^0 d\tau$$
$$\times \Bigg(\left[\exp(2ik_0z)-1\right] \exp\left\{-i\left[\omega-(k+k_0)\beta_F c\right]\tau\right\}$$
$$\times \exp\left\{ \sum_{n=1}^\infty i(k+k_0)z_n\left[\sin 2n(\phi_z+k_0\beta_F c\tau)-\sin 2n\phi_z\right]\right\}$$
$$+ \left[\exp(-2ik_0z)-1\right]\exp\left\{-i\left[\omega-(k-k_0)\beta_F c\right]\tau\right\}$$
$$\times \exp\left\{\sum_{n=1}^\infty i(k-k_0)z_n\left[\sin 2n(\phi_z+k_0\beta_F c\tau)-\sin 2n\phi_z\right]\right\}\Bigg), \tag{60}$$

where β_F, ϕ_z, and z_n are defined in Eqs. (47) and (48). To simplify the exponential factors $\exp\left\{\sum_{n=1}^\infty \cdots\right\}$ in Eq. (60), we make use of the identity

$$\exp(ib\sin\alpha) = \sum_{m=-\infty}^{\infty} J_m(b)\exp(im\alpha) \ , \tag{61}$$

where $J_m(b)$ is the Bessel function of the first kind of order m. Defining

$$b_n^\pm = (k\pm k_0)z_n = \frac{k\pm k_0}{k_0}\,\frac{2}{n}\,\frac{a^n}{1+a^{2n}} \ , \tag{62}$$

where a is defined in Eq. (49), the expression for the susceptibility $\chi(k,\omega,k_0z)$ in Eq. (60) can be expressed in the equivalent form

$$\chi(k,\omega,k_0z) = -\frac{1}{4}\, i\omega a_w^2 \int_0^\infty \frac{dp_z}{\gamma_0^2} \frac{\partial G_0^+}{\partial \gamma_0} \int_{-\infty}^0 d\tau$$
$$\times \Bigg(\left[\exp(2ik_0z)-1\right]\exp\left\{-i\left[\omega-(k+k_0)\beta_F c\right]\tau\right\}$$
$$\times \prod_{n=1}^\infty \left[\sum_{m=-\infty}^\infty \sum_{m'=-\infty}^\infty J_m(b_n^+)J_{m'}(b_n^+)\exp\left[im(2nk_0\beta_F c\tau)\right]\exp\left[i(m-m')2n\phi_z\right]\right]$$
$$+ \left[\exp(-2ik_0z)-1\right]\exp\left\{-i\left[\omega-(k-k_0)\beta_F c\right]\tau\right\}$$
$$\times \prod_{n=1}^\infty \left[\sum_{m=-\infty}^\infty \sum_{m'=-\infty}^\infty J_m(b_n^-)J_{m'}(b_n^-)\exp\left[im(2nk_0\beta_F c\tau)\right]\exp\left[i(m-m')2n\phi_z\right]\right]\Bigg). \tag{63}$$

The expression for $\chi(k,\omega,k_0z)$ in Eq. (63) can be further simplified, depending on the parameter regime and frequency range under investigation (Sec. IV).

To summarize, Eqs. (58) and (63) are the final results of this section, and are fully equivalent to the eigenvalue equation (56) for $\hat{a}_y(z)$. In this regard, several points are noteworthy. First, in the limit of zero wiggler amplitude, Eqs. (58) and (63) give the familiar dispersion relation $\omega^2=c^2k^2+\hat{\omega}_p^2\alpha_1$ for electromagnetic waves propagating in the z-direction. [Here, $\alpha_1 = \int_0^\infty dp_zG_0^+(\gamma_0)/\gamma_0$ follows from Eq. (53) for $a_w^2=0$.] Second, the susceptibility $\chi(k,\omega,k_0z)$ defined in Eq. (63) depends on k_0z. This spatial modulation occurs through the factors $\exp(\pm 2ik_0z)$, through the dependence of ϕ_z on k_0z [Eq. (47)], and through the integration over p_z in Eq. (63) (see also Sec. II.C). As a consequence, the k'th Fourier component wave amplitude $\hat{a}_{yk}$ in Eq. (58) is generally coupled to the wave components $\hat{a}_{y,k\pm 2k_0}$, $\hat{a}_{y,k\pm 4k_0}$, etc. Third, in deriving Eqs. (58) and (63), no a priori assumption has been made that the spatial modulation (k_0z dependence) of $S(k_0z)$ and $\chi(k,\omega,k_0z)$ is weak or that the parameter $\kappa^2 = a_w^2/(\gamma_0^2-1)$ is small. Finally, Eqs. (58) and (63) have been derived for perturbations about the general beam equilibrium $G_0(z,p_z) = U(p_z)G_0^+(\gamma_0)$, and the formalism can be used to investigate detailed free electron laser stability properties over a wide range of system parameters consistent with the assumptions and theoretical model described in Sec. II.

D. Diagonal Approximation to the Dispersion Relation

The simplest approximation to Eq. (58) is where we retain diagonal terms and neglect the coupling of $\hat{a}_{yk}$ to the $k\pm 2k_0$, $k\pm 4k_0,\ldots$ Fourier components. In this regard, the quantities $S(k_0z)$ and $\chi(k,\omega,k_0z)$ can formally be expressed as average values plus terms that depend explicitly on k_0z. That is,

$$
\begin{aligned}
S(k_0z) &= \langle S\rangle + \sum_{\ell\neq 0} S_\ell \exp(i\ell 2k_0z) \ , \\
\chi(k,\omega,k_0z) &= \langle\chi\rangle(k,\omega) + \sum_{\ell\neq 0} \chi_\ell \exp(i\ell 2k_0z) \ ,
\end{aligned}
\tag{64}
$$

where the average values $\langle S\rangle$ and $\langle\chi\rangle(k,\omega)$ are defined by

$$
\begin{aligned}
\langle S\rangle &= \int_0^{2\pi} \frac{d(k_0z)}{2\pi} S(k_0z) \ , \\
\langle\chi\rangle(k,\omega) &= \int_0^{2\pi} \frac{d(k_0z)}{2\pi} \chi(k,\omega,k_0z) \ .
\end{aligned}
\tag{65}
$$

Substituting Eqs. (64) and (65) into Eq. (58) and retaining only the diagonal terms gives the dispersion relation

$$
D(k,\omega) = \omega^2 - c^2k^2 - \hat{\omega}_p^2\langle S\rangle - \hat{\omega}_p^2\langle\chi\rangle(k,\omega) = 0 \ . \tag{66}
$$

In Eq. (66), the average quantities $\langle S\rangle$ and $\langle\chi\rangle(k,\omega)$ are calculated from Eq. (65), making use of the definitions of $S(k_0z)$ and $\chi(k,\omega,k_0z)$ given in Eqs. (53) and (63) for general $G_0^+(\gamma_0)$.

The diagonal dispersion relation in Eq. (66) is used in Sec. IV to investigate free electron laser stability properties over a wide range of system parameters. It is important to emphasize that neglecting the coupling to off-diagonal terms in Eq. (58) is likely to be a good approximation insofar as the parameter $\kappa^2=a_w^2/(\gamma_0^2-1)$ is sufficiently small.

IV. FREE ELECTRON LASER STABILITY PROPERTIES

In this section, we make use of Eqs. (53), (63), (65) and the diagonal dispersion relation (66) to investigate detailed free electron laser stability properties.

A. Simplified Dispersion Relation

For present purposes, two main approximations are made in evaluating $\langle\chi\rangle(k,\omega)$ from Eqs. (63) and (65). First, it is assumed that $\kappa^2=a_w^2/(\gamma_0^2-1)$ is sufficiently small that ϕ_z can be approximated by

$$\phi_z = k_0 z \tag{67}$$

in the expression for $\chi(k,\omega,k_0 z)$ in Eq. (63). Refering to Sec. III.B and Eqs. (47) and (50), it is evident that $\kappa^2/4 << 1$ is the appropriate small parameter for validity of Eq. (66). Second, the τ-dependence in the integrand in Eq. (63) is generally of the form

$$\exp\left\{-i\left[\omega-(k+k_0)\beta_F c-m(2nk_0)\beta_F c\right]\tau\right\} , \tag{68}$$

and

$$\exp\left\{-i\left[\omega-(k-k_0)\beta_F c-m(2nk_0)\beta_F c\right]\tau\right\} . \tag{69}$$

In the subsequent stability analysis, we retain contributions to the τ-integral in Eq. (63) that exhibit resonant behavior at the simple upshifted FEL resonance[8,9]

$$\omega \approx (k+k_0)\beta_F c . \tag{70}$$

That is, in contributions to Eq. (63) associated with the factor in Eq. (68), we retain only the m=0 term, and in contributions to Eq. (63) associated with the factor in Eq. (69), we retain only the m=1, n=1 term.

Making use of Eqs. (63) and (65) and the assumptions in the preceding paragraph, the susceptibility $<\chi>(k,\omega)$ can be expressed as

$$<\chi>(k,\omega) = -\frac{1}{4} i\omega a_w^2 \int_0^{2\pi} \frac{d(k_0 z)}{2\pi} \int_0^{\infty} \frac{dp_z}{\gamma_0^2} \frac{\partial G_0^+}{\partial \gamma_0} \int_{-\infty}^{0} d\tau \exp\left\{-i\left[\omega-(k+k_0)\beta_F c\right]\tau\right\}$$
$$\times\left(\left[\exp(2ik_0 z)-1\right] \prod_{n=1}^{\infty} \sum_{m'=-\infty}^{\infty} J_0(b_n^+)\, J_{m'}(b_n^+) \exp[-im'(2n)k_0 z] \right. \tag{71}$$
$$\left. + \left[\exp(-2ik_0 z)-1\right] J_1(b_1^-) \sum_{m'=-\infty}^{\infty} J_{m'}(b_1^-) \exp\left[i(1-m')2k_0 z\right]\right) .$$

We carry out the τ-integration in Eq. (71) and average over the (fast) $k_0 z$ oscillations in the integrand. For example, the first term in the factor $[\exp(2ik_0 z)-1]$ combines with the $m'=1$, $n=1$ term to give a non-zero average value, whereas the -1 term in the factor $[\exp(2ik_0 z)-1]$ combines with the $m'=0$ term to give a non-zero average value.
After some straightforward algebra, we obtain

$$<\chi>(k,\omega) = -\frac{1}{4} \omega a_w^2 \int_0^{2\pi} \frac{d(k_0 z)}{2\pi} \int_0^{\infty} \frac{dp_z}{\gamma_0^2} \frac{\partial G_0^+/\partial \gamma_0}{\omega-(k+k_0)\beta_F c} K(\gamma_0) , \tag{72}$$

where $K(\gamma_0)$ is defined by

$$K(\gamma_0) = \prod_{n=1}^{\infty} J_0^2(b_n^+) - J_0(b_1^+)J_1(b_1^+)$$
$$- J_0(b_1^-)J_1(b_1^-) + J_1^2(b_1^-) . \tag{73}$$

Here, for small values of $a \simeq \kappa^2/16$, it follows from Eqs. (50) and (62) that $b_n^\pm$ can be approximated by

$$b_n^\pm = \frac{k \pm k_0}{k_0} \frac{2}{n} \left(\frac{\kappa^2}{16}\right)^n . \tag{74}$$

For the range of k-values of interest for the free electron laser instability, $(k \pm k_0)/k_0 >> 1$, $b_1^\pm$ is typically of order unity, and $b_n^\pm << 1$ for $n \geq 2$. Therefore, an excellent approximation to Eq. (73)

is given by

$$K(\gamma_0) = J_0(b_1^+)\left[J_0(b_1^+)-J_1(b_1^+)\right]+J_1(b_1^-)\left[J_1(b_1^-)-J_0(b_1^-)\right]. \tag{75}$$

Moreover, if we approximate $b_1^+ \simeq b_1^- \simeq b_1 \equiv (k/k_0)(\kappa^2/8)$ for $k \gg k_0$, then Eq. (75) reduces to the familiar factor $K(\gamma_0)=\left[J_0(b_1)-J_1(b_1)\right]^2$, which occurs in standard single-particle analyses[8] of the free electron laser instability in the Compton regime. As a further point, it should be noted that we have retained the spatial integral $\int_0^{2\pi} \frac{d(k_0 z)}{2\pi} \cdots$ in the expression for $\langle\chi\rangle(k,\omega)$ in Eq. (72). This averages over the (weak) dependence on $k_0 z$ of the momentum integral

$$\int_0^\infty \frac{dp_z}{\gamma_0^2} \cdots = \int_1^\infty \frac{d\gamma_0}{\gamma_0(\gamma_0^2-1)^{1/2}} \frac{mc}{(1-\kappa^2 \sin^2 k_0 z)^{1/2}} \cdots \tag{76}$$

that occurs in Eq. (72).

Substituting Eq. (72) into Eq. (66) gives

$$0=D(k,\omega)=\omega^2-c^2k^2-\hat{\omega}_p^2\langle S\rangle + \frac{1}{4}\hat{\omega}_p^2 a_w^2 \omega \int_0^{2\pi} \frac{d(k_0 z)}{2\pi} \int_0^\infty \frac{dp_z}{\gamma_0^2} \frac{\partial G_0^+/\partial\gamma_0}{\omega-(k+k_0)\beta_F c} K(\gamma_0) \;, \tag{77}$$

which is the final form of the dispersion relation used in the remainder of this paper. Here, $K(\gamma_0)$ is defined in Eq. (75) with $b_1^\pm \simeq \left[(k \pm k_0)/k_0\right]\kappa^2/8$, and β_F is defined by $\beta_F=\pi\beta_0/2F \simeq (1-\kappa^2/4)\beta_0$ where $\beta_0=(1-1/\gamma_0^2)^{1/2}$ [see Eqs. (47) and (50)].

B. Resonant Free Electron Laser Instability

The dispersion relation (77) can be used to investigate detailed free electron laser stability properties over a wide range of system parameters when $\kappa^2 = a_w^2/(\gamma_0^2-1)$ is sufficiently small. In this section, we calculate the growth rate $\gamma_k = \mathrm{Im}\,\omega$ in circumstances corresponding to weak resonant instability. In particular, it is assumed that the growth rate is sufficiently small and the energy spread of the beam electrons is sufficiently large that the inequality

$$\left|\frac{\gamma_k}{(k+k_0)\Delta v_z}\right| << 1 \tag{78}$$

is satisfied. Here, Δv_z is the axial velocity spread characteristic of $G_0^+(\gamma_0)$ over the range of unstable phase velocities. Of course, Δv_z is also related to the beam emittance. In Eq. (77), we express $\omega = \omega_k + i\gamma_k$ and expand for small growth rate γ_k. This gives

$$\begin{aligned} 0 = D(k,\omega_k+i\gamma_k) &= D_r(k,\omega_k) \\ &+ i\left[D_i(k,\omega_k)+\gamma_k \frac{\partial}{\partial\omega_k} D_r(k,\omega_k)\right] + \ldots \end{aligned} \tag{79}$$

where $D_r(k,\omega_k) = \lim_{\gamma_k\to 0_+} \mathrm{Re}\, D(k,\omega_k+i\gamma_k)$ and $D_i(k,\omega_k) = \lim_{\gamma_k\to 0_+} \mathrm{Im}\, D(k,\omega_k+i\gamma_k)$. Making use of

$$\lim_{\gamma_k \to 0_+} \frac{1}{\omega_k - (k+k_0)\beta_F c + i\gamma_k} = \frac{P}{\omega_k - (k+k_0)\beta_F c}$$

$$- i\pi\delta[\omega_k - (k+k_0)\beta_F c] ,$$

where P denotes Cauchy principal-value, we set the real and imaginary parts of Eq. (79) separately equal to zero. This readily gives

$$0 = D_r(k,\omega_k) = \omega_k^2 - c^2k^2 - \hat{\omega}_p^2 \langle S\rangle$$

$$+ \frac{1}{4}\hat{\omega}_p^2 a_w^2 \omega_k \int_0^{2\pi} \frac{d(k_0 z)}{2\pi} \int_0^{2\pi} \frac{dp_z}{\gamma_0^2} \frac{P\partial G_0^+/\partial\gamma_0}{\omega_k - (k+k_0)\beta_F c} K(\gamma_0) , \tag{80}$$

and

$$\gamma_k = - \frac{D_i(k,\omega)}{\partial D_r/\partial\omega_k}$$

$$= \frac{\pi\hat{\omega}_p^2 a_w^2 \omega_k}{4\partial D_r/\partial\omega_k} \int_0^{2\pi} \frac{d(k_0 z)}{2\pi} \int_0^{\infty} \frac{dp_z}{\gamma_0^2} K(\gamma_0) \frac{\partial G_0^+}{\partial\gamma_0} \delta[\omega_k - (k+k_0)\beta_F c] , \tag{81}$$

where $\partial D_r/\partial\omega_k$ denotes $\partial D_r(k,\omega_k)/\partial\omega_k$. Equation (80) is the dispersion relation that determines the real oscillation frequency $\mathrm{Re}\,\omega = \omega_k$, whereas Eq. (81) determines the growth rate $\mathrm{Im}\omega = \gamma_k$ for specified beam distribution function $G_0^+(\gamma_0)$.

Note from Eq. (81) that the instability is driven by resonant electrons with velocity

$$\beta_F c = \frac{\omega_k}{(k+k_0)} . \tag{82}$$

Here, for small κ^2, β_F is defined in terms of γ_0 by $\beta_F^2 = \beta_0^2(1-\kappa^2/2)$, where $\beta_0^2 = 1-1/\gamma_0^2$. That is, β_F^2 can be expressed as

$$\beta_F^2 = 1 - \frac{1+a_w^2/2}{\gamma_0^2} . \tag{83}$$

We denote by $\gamma_0 = \gamma_r$ the resonant energy where $\beta_F(\gamma_r)c = \omega_k/(k+k_0)$. Making use of Eq.(83) then gives for γ_r

$$\gamma_r = \left(\frac{1+a_w^2/2}{1-\omega_k^2/c^2(k+k_0)^2}\right)^{1/2} . \tag{84}$$

To simplify Eq.(81), we convert the p_z integral to an integral over γ_0 [Eq.(75)] and make use of the identity

$$\delta[\omega_k-(k+k_0)\beta_F c] = \frac{\gamma_r^3|\omega_k|}{(1+a_w^2/2)c^2(k+k_0)^2}\ \delta(\gamma_0-\gamma_r) , \tag{85}$$

where use has been made of Eqs.(83) and (84). Moreover, in typical parameter regimes of interest, the principal-value term in Eq.(80) makes a negligibly small contribution to $\partial D_r/\partial\omega_k$, and it is valid to approximate $\partial D_r/\partial\omega_k = 2\omega_k$. Carrying out the integration over $k_0 z$ in Eq.(81) for small κ^2, and making use of Eq.(85), we obtain after some straightforward algebra

$$\gamma_k = \mathrm{Im}\omega = \frac{\pi}{8}\,\frac{\hat{\omega}_p^2}{|k+k_0|c}\,\frac{a_w^2}{(1+a_w^2/2)}\ K(\gamma_r)\gamma_r mc \left[\frac{\partial G_0^+}{\partial\gamma_0}\right]_{\gamma_0=\gamma_r} \tag{86}$$

Here, the resonant energy γ_r is defined in Eq.(84), and $K(\gamma_0)$ is defined in Eq.(74).

The expression for the growth rate in Eq.(86) has a wide range of validity, subject to the inequality in Eq.(78). Note from Eq.(86) that instability exists ($\gamma_k > 0$) over the entire range of γ_r for which $\partial G_0^+/\partial\gamma_0|_{\gamma_0=\gamma_r} > 0$ (Fig. 2). The corresponding real oscillation frequency ω_k of course is determined self-consistently from Eq.(80).

We now make use of Eq.(78) to determine the range of validity of Eq.(86). From Eq.(83), the characteristic velocity spread Δv_z is related to the characteristic energy spread $\Delta\gamma$ by $\beta_F\Delta v_z/c = (1+a_w^2/2)\Delta\gamma/\gamma_0^3$. For $\beta_F \simeq \hat{\beta}$ and $\gamma_0 \simeq \hat{\gamma}$, where $\hat{\gamma}mc^2$ and $\hat{\beta}c$ are the mean energy and mean axial velocity, respectively, of the beam electrons, we obtain the estimate for Δv_z,

$$\Delta v_z = c\,\frac{(1+a_w^2/2)}{\hat{\beta}\hat{\gamma}^2}\,\frac{\Delta\gamma}{\hat{\gamma}} \quad . \tag{87}$$

Moreover, for $\hat{\omega}_p^2 << c^2k^2$, the characteristic wavenumber of the instability (denote by $\hat{k}$) can be estimated from the simultaneous solution to $\omega_k = kc$ [Eq.(80)] and $\omega_k = (k+k_0)\hat{\beta}c$. This gives the familiar result

$$\hat{k} = \frac{\hat{\beta}(1+\hat{\beta})}{(1-\hat{\beta}^2)}\,k_0 = \frac{\hat{\beta}(1+\hat{\beta})\hat{\gamma}^2}{(1+a_w^2/2)}\,k_0, \tag{88}$$

where use has been made of Eq.(83). Finally, if we further estimate $\partial G_0^+/\partial\gamma|_{\gamma_0=\gamma_r} \approx 1/mc(\Delta\gamma)^2$ in Eq.(86), then the inequality $|\gamma_{\hat{k}}/(\hat{k}+k_0)\Delta v_z| << 1$ in Eq.(78) can be expressed in the equivalent form

$$\frac{\pi}{8}\,\frac{\hat{\omega}_p^2}{c^2k_0^2}\,\frac{\hat{\beta}K(\hat{\gamma})a_w^2}{\hat{\gamma}^3(1+\hat{\beta})^2} << \left(\frac{\Delta\gamma}{\hat{\gamma}}\right)^3 . \tag{89}$$

Equation (89) is equivalent to Eq.(78) and can be satisfied by relatively modest values of fractional energy spread $\Delta\gamma/\hat{\gamma}$.

It should also be noted that the instability bandwidth Δk is readily estimated from the simultaneous resonance conditions $kc = \omega_k = (k+k_0)\beta_F$. This gives

$$(\Delta k)(1-\hat{\beta}) = (k+k_0)\Delta v_z/c, \tag{90}$$

where we have approximated $\beta_F \simeq \hat{\beta}$ and $k \simeq \hat{k}$ [Eq.(88)]. Making use of Eqs.(87), (88), and (90), the normalized bandwidth $\Delta k/\hat{k}$ can be expressed as

$$\frac{\Delta k}{\hat{k}} = \frac{1}{\hat{\beta}^2}\,\frac{\Delta\gamma}{\hat{\gamma}} . \tag{91}$$

Equation (91) gives a simple estimate of $\Delta k/\hat{k}$ in terms of the fractional energy spread $\Delta\gamma/\hat{\gamma}$.

To summarize, the expression for weak resonant growth rate in Eq.(86) is valid within the context of Eq.(89). In Sec. IV.C, we make use of Eq.(86) to investigate numerically the linear growth properties in parameter regimes characteristic of the Los Alamos FEL experiment,[11,44] and the Livermore FEL experiment planned on the Advanced Test Accelerator (ATA).[47]

In conclusion, the analysis in this section also has fundamental implications for the range of validity of different nonlinear models for describing the evolution of the free electron laser instability. For a very narrow wave spectrum, the nonlinear development is coherent, and the dynamics of electrons trapped in the pondermotive potential play a critical role in determining the evolution of the system.[8,9] On the other hand, if the instability is sufficiently broad band that the wave autocorrelation time (denote by τ_{ac}) is short in comparison with the characteristic growth time (γ_k^{-1}), then a multi-wave quasilinear model[15] is appropriate, and particle trapping is unimportant. The basic condition for validity of the quasilinear description is that the wave spectrum be sufficiently broad that[15,49,50]

$$\tau_{ac} \gtrsim |\Delta[\omega_k-(k+k_0)v_z]|^{-1} << \gamma_k^{-1} , \tag{92}$$

where $\Delta[\omega_k-(k+k_0)v_z] \simeq (\Delta k)c(1-\hat{\beta})$ is the characteristic spread of $[\omega_k-(k+k_0)v_z]$ over the extent of the amplifying wave spectrum. Equation (92) can then be expressed in the equivalent form

$$\frac{\gamma_k}{c(\Delta k)(1-\hat{\beta})} << 1. \tag{93}$$

Making use of $c(\Delta k)(1-\hat{\beta}) = (\hat{k}+k_0)\Delta v_z$, Eq.(93) reduces to the inequality $\gamma_k/(k+k_0)\Delta v_z << 1$, which is identical to Eq.(78). That is, the condition [Eq.(78) or Eq.(89)] for weak resonant instability and validity of the expression for γ_k in Eq.(86) is identical to the condition [Eq.(92) or Eq.(93)] that the unstable wave spectrum be sufficiently broad that quasilinear theory gives a valid description of the nonlinear evolution of the system. This of course assumes that the bandwidth of the initial (input) signal is comparable to Δk defined in Eqs. (90) or (91).

C. Stability Properties for Weak Resonant Growth

In this section, we make use of Eqs.(86) and (89) to investigate numerically the stability properties for weak resonant growth. As one example, which corresponds to the parameter range planned for the Livermore FEL experiments[47] on the Advanced Test Accelerator (ATA), we consider the case where the beam energy is $\hat{\gamma} = 100$, the beam current is $I_b = |-e|\hat{n}_b \pi r_b^2 \hat{\beta} c = 1.9$ kA, the beam radius is $r_b = 0.45$ cm., the wiggler amplitude is $B_w = 2.3$ kG, and the wiggler wavelength is $\lambda_0 = 2\pi/k_0 = 8.0$ cm. This gives $\hat{n}_b = 6.3\times 10^{11}\text{cm}^{-3}$, $a_w = eB_w/mc^2k_0 = 1.7$, $\hat{\omega}_p^2/c^2k_0^2 = 4\pi\hat{n}_b e^2/mc^2k_0^2 = 3.6$, $b_1^{\pm} = 0.23$ [from Eqs.(73) and (88)], and $K(\hat{\gamma}=100) = 0.78$ [from Eq. (74)]. The inequality in Eq.(89) then reduces to $(\Delta\gamma/\hat{\gamma})^3 \gg 6 \times 10^{-7}$, which requires a fractional energy spread in excess of 0.9% for the growth rate expression in Eq.(86) to be valid. The total effective value of $\Delta\gamma/\hat{\gamma}$ for the Livermore FEL experiment on ATA may be in the range of 1-2%. As a second example, which corresponds to typical operating parameters for the Los Alamos FEL experiment,[11,44] we consider the case where $\hat{\gamma} = 41$, $I_b = 40$A, $r_b = 0.09$cm, $B_w = 3$kG, and $\lambda_0 = 2.73$cm. This gives $\hat{n}_b = 3.3\times 10^{11}\text{cm}^{-3}$, $a_w = 0.76$, $\hat{\omega}_p^2/c^2k_0^2 = 0.21$, $b_1^{\pm} = 0.113$, and $K(\hat{\gamma}=41) = 0.89$. The inequality in Eq.(89) then reduces to $(\Delta\gamma/\hat{\gamma})^3 \gg 2.0\times 10^{-6}$, which requires a fractional energy spread in excess of 1.3%. The effective value of $\Delta\gamma/\hat{\gamma}$ in the Los Alamos FEL experiments is typically 1-2%.

Typical numerical results obtained from Eq.(86) are presented in Figs. 3 and 4, where γ_k/k_0c is plotted versus k/k_0 for the two choices of beam and wiggler parameters given in the previous paragraph.[11,44,47] Here, Eq.(80) has been approximated by $\omega_k=kc$, and the beam distribution function $G_0^+(\gamma_0)$ is assumed to be gaussian with

$$G_0^+(\gamma_0) = \frac{1}{(\pi)^{\frac{1}{2}}mc\Delta\gamma} \frac{(\gamma_0^2-1)^{\frac{1}{2}}}{\gamma 0} \exp\left\{- \frac{(\gamma_0-\hat{\gamma})^2}{2(\Delta\gamma)^2}\right\}, \qquad (94)$$

where $\hat{\gamma} >> 1$ and $\Delta\gamma/\hat{\gamma} << 1$ are assumed. In both Figs. 3 and 4, the growth rate has been plotted for values of $\Delta\gamma/\hat{\gamma}$ corresponding to fractional energy spreads of 1%, 2% and 3%. Note that as the energy spread is increased, the decrease in maximum growth rate is proportional to $(\Delta\gamma)^{-2}$, and the increase in instability bandwidth Δk is proportional to $\Delta\gamma$ [Eq.(91)]. Furthermore, in Figs. 3 and 4, we have chosen the energy spread to be consistent with the validity criterion in Eq.(89), with $\Delta\gamma/\hat{\gamma}$ = 1% corresponding to the limit of the range of validity.

Some further comments are appropriate with regard to the FEL experiments planned on ATA,[47] which will operate in both the amplifier and (single-pass) oscillator modes. Since the input signal in the amplifier configuration will be provided by a laser with very narrow bandwidth, Equation (91) is not the appropriate estimate of $\Delta k/\hat{k}$ for the amplifying wave spectrum, nor will the criterion in Eq.(93) [required for validity of quasilinear theory] be satisfied. Application of the present analysis should therefore be restricted to the oscillator configuration in which the signal grows from low-level broadband noise. Furthermore, the energy spread applicable to ATA should be estimated by including transverse beam emittance, which has been assumed to vanish in the present analysis. Therefore, the present model, which assumes weak resonant instability, should be applied only if the total effective energy spread exceeds 1%. From Fig. 3, for $\Delta\gamma/\hat{\gamma} = 1\%$, we note that the maximum growth rate corresponds to an e-folding distance of $c/[\gamma_k]_{MAX} = 3$m.

D. Stability Properties for Monoenergetic Electrons

We now consider free electron laser stability properties in circumstances where the beam energy spread $\Delta\gamma$ is sufficiently small that the inequality in Eq. (89) is not satisfied. In particular, we make use of the diagnonal dispersion relation (77) to investigate stability properties for monoenergetic beam electrons where $G_0^+(\gamma_0) = (\hat{\gamma}mc)^{-1}(\hat{\gamma}^2-1)^{\frac{1}{2}}\delta(\gamma_0-\hat{\gamma})$ [Eq (20)]. In this regard, it is assumed that $\hat{\gamma}$ is sufficiently large that $\hat{\kappa}^2 = a_w^2/(\hat{\gamma}^2-1)$ can be treated as a small parameter. Therefore, β_F can be approximated by $\beta_F = (1-\kappa^2/4)\beta_0$ in Eq. (76), where $\beta_0 = (1-1/\gamma_0^2)^{\frac{1}{2}}$. Moreover, from Eq. (54), $\langle S\rangle$ can be approximated by

$$\langle S\rangle = \frac{1}{\hat{\gamma}}\left(1+\frac{3}{4}\hat{\kappa}^2\right), \tag{95}$$

for $\hat{\kappa}^2 \ll 1$ and $G_0^+(\gamma_0)$ specified by Eq. (20). We substitute Eq. (20) into Eq. (77), convert the integral over p_z to an integral over γ_0 [Eq. (75)], and integrate by parts with respect to γ_0. This gives

$$\omega^2 - c^2k^2 - \frac{\hat{\omega}_p^2}{\hat{\gamma}}\left(1+\frac{3}{4}\hat{\kappa}^2\right) = \frac{1}{4}\,\frac{\hat{\omega}_p^2 a_w^2\omega}{\hat{\gamma}^2}$$

$$\times\left\{\frac{K(\hat{\gamma})(1+\hat{\kappa}^2/4)}{[\omega-(k+k_0)\hat{\beta}c]^2}\,(k+k_0)c\left[\frac{\partial\beta_F}{\partial\gamma_0}\right]_{\gamma_0=\hat{\gamma}}\right. \tag{96}$$

$$\left.+\frac{\hat{\gamma}(\hat{\gamma}^2-1)^{\frac{1}{2}}}{[\omega-(k+k_0)\hat{\beta}c]}\left[\frac{\partial}{\partial\gamma_0}\left(\frac{K(\gamma_0)(1+\kappa^2/4)}{\gamma_0(\gamma_0^2-1)^{\frac{1}{2}}}\right)\right]_{\gamma_0=\hat{\gamma}}\right\},$$

where $K(\gamma_0)$ is defined in Eq.(74), β_F is defined in Eq.(83), and $\hat{\beta} = \beta_F(\gamma_0=\hat{\gamma})$ is given by

$$\hat{\beta} = \left(1 - \frac{1+a_w^2/2}{\hat{\gamma}^2}\right)^{\frac{1}{2}} . \tag{97}$$

Making use of Eq.(83), we obtain

$$\left[\frac{\partial \beta_F}{\partial \gamma_0}\right]_{\gamma_0=\hat{\gamma}} = \frac{(1+a_w^2/2)}{\hat{\beta}\hat{\gamma}^3} . \tag{98}$$

The dispersion relation (96) can then be expressed in the compact form

$$\begin{aligned} \omega^2 - c^2k^2 - \frac{\hat{\omega}_p^2}{\hat{\gamma}}\left(1 + \frac{3}{4}\hat{\kappa}^2\right) \\ &= \frac{1}{4}\frac{\hat{\omega}_p^2}{\hat{\gamma}^3}a_w^2 \frac{\omega N_1(\hat{\gamma})}{[\omega-(k+k_0)\hat{\beta}c]} , \\ &+ \frac{1}{4}\frac{\hat{\omega}_p^2}{\hat{\gamma}^5}a_w^2 \frac{\omega(k+k_0)c\ N_2(\hat{\gamma})}{[\omega-(k+k_0)\hat{\beta}c]^2} , \end{aligned} \tag{99}$$

where $N_1(\hat{\gamma})$ and $N_2(\hat{\gamma})$ are defined by

$$N_1(\hat{\gamma}) = \left[\gamma_0^2(\gamma_0^2-1)^{\frac{1}{2}}\frac{\partial}{\partial\gamma_0}\left(\frac{K(\gamma_0)(1+\kappa^2/4)}{\gamma_0(\gamma_0^2-1)^{\frac{1}{2}}}\right)\right]_{\gamma_0=\hat{\gamma}} \tag{100}$$

and

$$N_2(\hat{\gamma}) = \frac{1}{\hat{\beta}} K(\hat{\gamma})(1+\hat{\kappa}^2/4)(1+a_w^2/2). \tag{101}$$

Note that both $N_1(\hat{\gamma})$ and $N_2(\hat{\gamma})$ are typically of order unity.

Equation (99), supplemented by the definitions in Eqs.(74), (97), (100) and (101), constitutes the final dispersion relation for monoenergetic electrons. Equation (99) is a fourth-order algebraic equation for the complex eigenfrequency ω, and can be used to investigate detailed stability properties over a wider range of the dimensionless parameters $\hat{\omega}_p^2/c^2k_0^2$, a_w^2 and $\hat{\gamma}$. For purposes of obtaining a simple estimate of the characteristic growth rate, we examine Eq.(99) for $(\hat{\omega}_p^2/c^2k_0^2)(a_w^2/\hat{\gamma}^3) << 1$ and (ω,k) closely tuned to $(\hat{\omega},\hat{k})$ satisfying the simultaneous resonance conditions

$$\hat{\omega} = \left[c^2\hat{k}^2 + \frac{\hat{\omega}_p^2}{\hat{\gamma}}(1+\tfrac{3}{4}\hat{\kappa}^2)\right]^{\frac{1}{2}}, \tag{102}$$

$$\hat{\omega} = (\hat{k}+k_0)\hat{\beta}c.$$

Note that $(\hat{\omega},\hat{k})$ determined from Eq.(102) differs slightly from Eq.(88) because of the inclusion of the $\hat{\omega}_p^2$ contribution in Eq.(102). We now examine Eq.(96) for (ω,k) close to $(\hat{\omega},\hat{k})$. Expressing $\omega = \omega + \delta\omega$ and $k = \hat{k}+\delta k$, then for $\delta k = 0$ and $|\delta\omega/\hat{\omega}| << 1$, Eq.(96) gives

$$(\delta\omega)^3 = \frac{1}{8}\frac{\hat{\omega}_p^2 a_w^2}{\hat{\gamma}^5} N_2(\hat{\gamma})c(\hat{k}+k_0). \tag{103}$$

Equation (103) can be used to estimate the characteristic (maximum) growth rate for $\delta k = 0$. This gives

$$\mathrm{Im}\delta\omega = \frac{(3)^{\frac{1}{2}}}{4}\left[\frac{\hat{\omega}_p^2 a_w^2}{\hat{\gamma}^5} N_2(\hat{\gamma}) c(\hat{k}+k_0)\right]^{1/3} \qquad (104)$$

Keep in mind that Eq.(104) is valid only for negligibly small energy spread, and the range of validity of Eq.(99) does not overlap with the range of validity of Eq.(86). In circumstances where the approximation $\kappa^2 << 1$, $|\hat{k}/k_0| >> 1$, and $\hat{\beta} \simeq 1$ are valid, we find $N_2(\hat{\gamma}) \approx K(\hat{\gamma})(1+a_w^2/2)$. Equation (104) becomes, after use of Eq.(88),

$$\mathrm{Im}\delta\omega = \frac{(3)^{\frac{1}{2}}}{2}\left[\frac{\hat{\omega}_p^2 a_w^2 K(\hat{\gamma}) c k_0}{2\hat{\gamma}^3}\right]^{1/3}, \qquad (105)$$

which corresponds to the familiar expression for the cold-beam Compton-regime growth rate.

V. STABILITY PROPERTIES FOR ULTRARELATIVISTIC ELECTRONS

In this section, we consider Eq.(77) in the limit of an ultrarelativistic, tenuous electron beam and compare the resulting dispersion relation (Sec. V.A.) with the dispersion relation obtained from a linear analysis[51] of the standard Compton-regime FEL equations[8,9] based upon a superposition of single-particle orbits (Sec. V.B.). Finally, in Sec. V.C., we extend the quasilinear kinetic equations derived by Dimos and Davidson[15] for a helical wiggler magnetic field to the case of an ultrarelativistic electron beam propagating through a planar magnetic wiggler.

A. Kinetic Dispersion Relation for Ultrarelativistic Electrons

For an ultrarelativistic, tenuous electron beam with $\hat{\gamma} \gg 1$ and $\hat{\omega}_p^2 \ll c^2k^2$, we approximate $\kappa^2 \ll 1$ and $\int_0^\infty \frac{dp_z}{\gamma_0^2} \ldots = mc\int_1^\infty d\gamma_0 \ldots$ in Eq.(77). In this case, the dispersion relation (77) can be approximated by

$$0 = D(k,\omega) = \omega^2 - c^2k^2 + \tfrac{1}{4}\hat{\omega}_p^2 a_w^2 \; \omega mc \int_1^\infty \frac{d\gamma_0}{\gamma_0^2} \frac{\partial G_0^+/\partial\gamma_0}{\omega - (k+k_0)\beta_F c} K(\gamma_0) , \quad (106)$$

where $K(\gamma_0)$ and $\beta_F(\gamma_0)$ are defined in Eqs.(75) and (83). Here, $G_0^+(\gamma_0)$ is centered about $\gamma_0 = \hat{\gamma} \gg 1$ with characteristic energy spread $\Delta\gamma \ll \hat{\gamma}$. For $\beta_F \simeq 1$ and $\gamma_0 \gg 1$, the axial velocity $\beta_F c$ occuring in Eq.(106) can be approximated by [see Eq.(83)]

$$\beta_F = 1 - \frac{1 + a_w^2/2}{2\gamma_0^2} . \quad (107)$$

Integrating by parts with respect to γ_0 in Eq.(106) and making use of Eq.(107), the dispersion relation (106) can be expressed in the equivalent form

$$\omega^2 - c^2k^2 = \frac{1}{4}\hat{\omega}_p^2 a_w^2 \omega mc \int_1^\infty d\gamma_0 G_0^+(\gamma_0)$$

$$\times \left\{ \frac{(k+k_0)c\ K(\gamma_0)}{[\omega-(k+k_0)\beta_F c]^2} \frac{1}{\gamma_0^5} \left(1+\frac{a_w^2}{2}\right) \right. \tag{108}$$

$$\left. - \frac{[2\gamma_0^{-3}K(\gamma_0) - \gamma_0^{-2}\partial K/\partial\gamma_0]}{[\omega-(k+k_0)\beta_F c]} \right\} .$$

In Eq.(108), for temporal growth (FEL oscillator case), the wavenumber k is real and the oscillation frequency ω is complex. It is convenient to express

$$\omega = kc + \delta\omega , \tag{109}$$

where $\delta\omega$ is complex and corresponds to the wiggler-induced modification to the vacuum dispersion relation $\omega=kc$ [see Eq.(108) with $a_w=0$]. We also introduce the quantity $\Delta\omega(\gamma_0)$ defined by

$$\Delta\omega = -k_0 c + (k+k_0)c \frac{1+a_w^2/2}{2\gamma_0^2} . \tag{110}$$

Making use of Eqs.(107), (109) and (110), it is readily shown that

$$\omega - (k+k_0)\beta_F c = \delta\omega + \Delta\omega , \tag{111}$$

and the dispersion relation (108) can be expressed as

$$2kc\delta\omega+(\delta\omega)^2 = \frac{1}{4}\hat{\omega}_p^2 a_w^2 (kc+\delta\omega) mc \int_1^{\infty} d\gamma_0 \; G_0^+(\gamma_0)$$

$$\times \left\{ \frac{(k+k_0)cK(\gamma_0)}{(\delta\omega+\Delta\omega)^2} \frac{1}{\gamma_0^5} \left(1+\frac{a_w^2}{2}\right) - \frac{[2\gamma_0^{-3}K(\gamma_0) - \gamma_0^{-2}\,\partial K/\partial\gamma_0]}{(\delta\omega+\Delta\omega)} \right\} . \tag{112}$$

Here, $\Delta\omega(\gamma_0)$ is defined in Eq.(110), and the dispersion relation (112) is fully equivalent to Eq.(108) with β_F approximated by Eq.(107).

B. Linearized Compton-Regime FEL Equations

For purposes of comparison, we now investigate linear stability properties within the context of the standard Compton-regime FEL equations[8,9] which describe the interaction of the beam electrons with a monochromatic electromagnetic wave with wavenumber k and frequency kc. For the j'th electron, with energy γ_j, the phase function θ_j and frequency shift $\Delta\omega_j$ are defined by

$$\theta_j = (k+k_0)z_j - kct,$$

$$\Delta\omega_j = -k_0c + \frac{c(k+k_0)}{2\gamma_j^2}\left(1+\tfrac{1}{2}a_w^2\right) . \tag{113}$$

In the notation of this paper, assuming ultrarelativistic electrons, the Compton-regime equations[8,9] are given by

$$\frac{d}{dt}\gamma_j = -\frac{kca_w g}{2\gamma_j}\,\mathrm{Im}[a_y \exp(i\theta_j)]\,, \tag{114}$$

$$\frac{d}{dt}\theta_j = -\Delta\omega_j + \frac{ck}{2\gamma_j^2}\,a_w g\,\mathrm{Re}[a_y \exp(i\theta_j)]\,, \tag{115}$$

$$\frac{d}{dt}\,a_y = \frac{i\hat{\omega}_p^2 a_w g}{2kc}\left\langle \frac{\exp(-i\theta_j)}{\gamma_j} \right\rangle . \tag{116}$$

Equations (114)-(116) describe the coupled nonlinear evolution of the electrons and the radiation field (assumed monochromatic). In Eq.(116), $<\Psi_j>$ denotes the ensemble average over N_T electrons,

$$<\Psi_j> = \frac{1}{N_T}\sum_{j=1}^{N_T} \Psi_j\,, \tag{117}$$

and the amplitude factor g is defined by

$$\begin{aligned} g &= J_0(\hat{b})-J_1(\hat{b}) \\ &= [K(\hat{\gamma})]^{1/2} \end{aligned} \tag{118}$$

Here, $\hat{b}$ is defined by $\hat{b} = b_1^{\pm}(\gamma_0=\hat{\gamma}) \simeq a_w^2/(4+2a_w^2)$, which is a valid approximation for $\hat{\gamma}>>1$, $\hat{\kappa}^2<<1$ and $k/k_0 \simeq 2\hat{\gamma}^2/(1+a_w^2/2)$ [Eqs.(74) and (89)]. Moreover, the identification $g = [K(\hat{\gamma})]^{1/2}$ has been made in the ultrarelativistic limit [Eqs.(75) and (118)].

In the small-signal regime, we linearize Eqs.(114)-(116) and express[51]

$$\theta_j = \theta_{j0} - \Delta\omega_{j0} t + \delta\theta_j,$$

$$a_y = \delta a_y, \tag{119}$$

$$\gamma_j = \gamma_{j0} + \delta\gamma_j,$$

where

$$\Delta\omega_{jo} = -k_0 c + \frac{(k+k_0)c}{2\gamma_{j0}^2}\left(1+\frac{1}{2}a_w^2\right). \tag{120}$$

Here, subscript "zero" labels unperturbed values in the absence of the radiation field ($\delta a_y=0$). Substituting Eq.(119) into Eqs.(114)-(116) and retaining terms which are linear in the perturbation amplitudes, we obtain

$$\frac{d}{dt}\delta\gamma_j = -\frac{kca_w g}{2\gamma_{j0}} \operatorname{Im}[\delta a_y \exp(i\theta_{j0} - i\Delta\omega_{j0} t)], \tag{121}$$

$$\frac{d}{dt}\delta\theta_j = \frac{c(k+k_0)}{\gamma_{j0}^3}\left(1+\frac{1}{2}a_w^2\right)\delta\gamma_j$$

$$+ \frac{ck}{2\gamma_{j0}^2} a_w g \operatorname{Re}[\delta a_y \exp(i\theta_{j0} - i\Delta\omega_{j0} t)], \tag{122}$$

$$\frac{d}{dt}\delta a_y = -\frac{i\hat{\omega}_p^2 a_w g}{2kc}\left\langle \frac{\exp(-i\theta_{j0} + i\Delta\omega_{j0} t)}{\gamma_{j0}}\left(\frac{\delta\gamma_j}{\gamma_{j0}} + i\delta\theta_j\right)\right\rangle. \tag{123}$$

In Eq.(123), the ensemble average < > denotes

$$\langle\Psi(\theta_{j0},\gamma_{j0})\rangle = mc\int_0^{2\pi}\frac{d\theta_0}{2\pi}\int_1^{\infty}d\gamma_0 G_0^+(\gamma_0)\Psi(\theta_0,\gamma_0)\,, \qquad (124)$$

where $G_0^+(\gamma_0)$ is the energy distribution, and θ_{j0} is the initial phase [Eq.(119)]. In Eq.(124), we have converted the summation over discrete particles in Eq.(117) to a continuum integral over the distribution $G_0^+(\gamma_0)$. In obtaining Eq.(123), use has been made of $\langle\gamma_{j0}^{-1}\exp(-i\theta_{j0}+i\Delta\omega_{j0}t)\rangle = 0$.

In Eqs.(121)-(123), the vector potential $\delta a_y(t)$ is expressed as $\delta a_y = \delta\hat{a}_y \exp(-i\delta\omega t)$, where $\mathrm{Im}\delta\omega>0$ corresponds to instability (temporal growth). Integrating Eq.(121) from $t=-\infty$ to time t, and neglecting "initial" values (for $t\to-\infty$), we obtain for $\delta\gamma_j(t)$

$$\delta\gamma_j = -\frac{kca_w g}{2\gamma_{j0}}\,\mathrm{Im}\left[\frac{i\delta\hat{a}_y \exp(i\theta_{j0} - i\Delta\omega_{j0}t - i\delta\omega t)}{(\delta\omega+\Delta\omega_{j0})}\right]\,, \qquad (125)$$

where $\mathrm{Im}(\delta\omega)>0$ has been assumed. Similarly, making use of Eq.(125), we obtain for $\delta\theta_j(t)$ from Eq.(122)

$$\delta\theta_j = \frac{ck}{2\gamma_{j0}^2}a_w g\,\mathrm{Re}\left\{i\delta\hat{a}_y \exp(i\theta_{j0}-i\Delta\omega_{j0}t-i\delta\omega t)\right.$$
$$\left.\times\left[\frac{1}{(\delta\omega+\Delta\omega_j)} - \frac{1}{\gamma_{j0}^2}\left(1+\frac{1}{2}a_w^2\right)\frac{c(k+k_0)}{(\delta\omega+\Delta\omega_j)^2}\right]\right\}\,. \qquad (126)$$

Substituting Eqs.(125) and (126) into Eq.(123), and making use of Eq.(124), we find after some straightforward algebra

$$2kc\delta\omega = \frac{1}{4}\hat{\omega}_p^2 a_w^2 (kc) mc \int_1^{\infty} d\gamma_0 G_0^+(\gamma_0) \times \left\{ \frac{(k+k_0)c\ K(\hat{\gamma})}{(\delta\omega+\Delta\omega)^2} \frac{1}{\gamma_0^5} \left(1 + \frac{a_w^2}{2}\right) - \frac{2\gamma_0^{-3}\ K(\hat{\gamma})}{(\delta\omega+\Delta\omega)} \right\}. \tag{127}$$

In obtaining Eq.(127), use has been made of $(2\pi)^{-1}\int_0^{2\pi} d\theta_0 \exp(2i\theta_0) = 0$, and the factor $\delta\hat{a}_y \exp(-i\delta\omega t)$ has been cancelled from both sides of the equation.

We now compare the kinetic dispersion relation (112) with the dispersion relation (127) obtained from a linear analysis of the standard Compton-regime FEL equations.[8,9] First, comparing Eqs.(107), (109)-(111) and (120), it is evident that $\delta\omega+\Delta\omega = \omega-(k+k_0)\beta_F c$ in both dispersion relations. Moreover, the kinetic dispersion relation (112) reduces directly to Eq.(127) provided we make the following approximations in Eq.(112):

(a) $2kc\delta\omega + (\delta\omega)^2 \simeq 2kc\delta\omega$ on the left-hand side of Eq.(112).

(b) $kc + \delta\omega \simeq kc$ on the right-hand side of Eq.(112).

(c) $K(\gamma_0) \simeq K(\hat{\gamma})$ and $\partial K(\gamma_0)/\partial\gamma_0 \simeq 0$ on the right-hand side of Eq.(112).

Approximations (a) and (b) are associated with the fact that the eikenol approximation has been made in deriving the Compton-regime equations (114)-(116). These approximations are indeed justified because $|\delta\omega| << |\hat{k}c|$ in the ultra-relativistic, tenuous beam limit. Moreover, Approximation (c) is also a reasonably good approximation because $G_0^+(\gamma_0)$ is strongly peaked around $\gamma_0 = \hat{\gamma}$, and the variation of $K(\gamma_0)$ with energy γ_0 is relatively weak.

This completes the proof of equivalence of the two dispersion relations in the limit of an ultrarelativistic, tenuous electron beam.

C. Quasilinear Kinetic Equations for a Planar Wiggler

For completeness, making use of the ultrarelativistic tenuous electron beam assumptions enumerated at the beginning of Sec. V.A., we conclude this paper with a summary of the appropriate quasilinear kinetic equations for the planar wiggler configuration considered in the present analysis. This represents a straightforward extension of the quasi-linear kinetic equations developed by Dimos and Davidson for the case of a helical wiggler magnetic field.[15,50] In this regard, for the quasilinear analysis to be valid, it is important to recognize that the amplifying wave spectrum must be sufficiently broad that $\tau_{ac} << \gamma_k^{-1}$, where τ_{ac} is the wave autocorrelation time defined in Eq. (92).

In quasilinear theory, the average background distribution function $G_0^+(\gamma_0,t)$ is allowed to vary slowly with time in response to the amplifying wave perturbations. The complex oscillation frequency $\omega_k(t) + i\gamma_k(t)$ is then determined adiabatically in time from the linear dispersion relation [see Eq.(106)].

$$(\omega_k+i\gamma_k)^2 - c^2k^2 = -\frac{1}{4}\hat{\omega}_p^2 a_w^2 (\omega_k+i\gamma_k)mc \times \int_1^{\infty} \frac{d\gamma_0}{\gamma_0{}^2} \frac{K(\gamma_0) \ \partial G_0^+(\gamma_0,t)/\partial\gamma_0}{\omega_k-(k+k_0)\beta_F c+i\gamma_k} \quad , \tag{128}$$

where $\hat{\gamma} >> 1$, $\hat{\omega}_p^2 << c^2k^2$ and $\kappa^2 << 1$ have been assumed, and β_F is defined in Eq.(107). For ultrarelativistic electrons with $p_z \simeq \gamma_0 mc$, the appropriate extension of the particle kinetic equation (30) in Ref. 15 [or Eq.(12) in Ref. 49] to the case of a planar magnetic wiggler is

$$\frac{\partial}{\partial t} G_0^+(\gamma_0,t) = \frac{\partial}{\partial\gamma_0}\left[D(\gamma_0,t) \frac{\partial}{\partial\gamma_0} G_0^+(\gamma_0,t)\right] \quad , \tag{129}$$

where the quasilinear diffusion coefficient $D(\gamma_0,t)$ is defined by

$$D(\gamma_0,t) = \frac{1}{4} \frac{\hat{\omega}_p^2}{\hat{n}_b mc^2} a_w^2 \frac{K(\gamma_0)}{\gamma_0{}^2} \sum_{k=-\infty}^{\infty} \frac{i\,\mathcal{E}_k(t)}{\omega_k-(k+k_0)\beta_F c+i\gamma_k} \quad . \tag{130}$$

Here, for $k^2 >> k_0^2$, $\mathcal{E}_k(t) = k^2|\delta A_y(k,t)|^2/8\pi$ is the effective spectral energy dnesity of the magnetic field perturbations, and $\mathcal{E}_k(t)$ evolves according to the wave kinetic equation

$$\frac{\partial}{\partial t} \mathcal{E}_k(t) = 2\gamma_k(t)\,\mathcal{E}_k(t) \quad , \tag{131}$$

where $\gamma_k(t)$ is determined from Eq.(128).

Equations (128) - (131) constitute a closed description of the nonlinear evolution of the system in circumstances where the amplifying wave spectrum is sufficiently broad-band that the inequality in Eq.(92) is satisfied. To summarize, as the wave spectrum amplifies [Eq.(131)], there is a corresponding redistribution of electrons in γ_0-space [Eqs. (129) and (130)] and a concommitant modification of the growth rate $\gamma_k(t)$ [Eq.(128)]. The details of the time evolution and the stabilization process of course depend on the specific parameter regime, the initial distribution function $G_0^+(\gamma_0,t=0)$, and the input spectrum $\mathcal{E}_k(t=0)$. It is sufficient for present purposes simply to note that Eqs.(128) and (130) can be simplified considerably in circumstances corresponding to weak resonant instability[15] (see also Secs. IV.B. and IV.C.), and have been integrated numerically[50] for certain simple functional forms of $G_0^+(\gamma_0,t)$.

VI. CONCLUSIONS

In this paper, we have mde use of the linearized Vlasov-Maxwell equations (Secs. II and III) to investigate detailed free electron laser stability properties for a tenuous relativistic electron beam propagating through a constant-amplitude helical wiggler magnetic field [Eq.(1)]. The analysis was carried out for perturbations about the general class of self-consistent beam equilibria $G_0(z,p_z) = U(p_z)G_0^+(\gamma_0)$ [Eq.(16)]. To evaluate the perturbed distirubtion function $\delta G(z,p_z,t)$, use was made of the exact particle trajectories in the equilibrium wiggler field, and there was no apriori restriction to ultrarelativistic electrons. Indeed, for low or moderate electron energy, it was shown that there can be a sizeable modulation of beam equilibrium properties by the wiggler field and a concommitant coupling of the k'th Fourier component of the wave field to the components $k \pm 2k_0$, $k \pm 4k_0, \cdots$. This is evident from the formal matrix dispersion equation (58) and the definition of electron susceptibility $\chi(k,\omega,k_0z)$ in Eq.(63). In the diagonal approximation, it was shown that Eq.(58) reduces to the dispersion relation (77). In Sec. IV, we made use of Eq.(77) to investigate the detailed dependence of free electron laser growth rate on the choice of distribution function $G_0^+(\gamma_0)$. Investigations of stability behavior ranged from the regime of strong instability (monoenergetic electrons) to weak resonant growth (sufficiently large energy spread). For the case of weak resonant growth, the growth rates were calculated numerically for parameter regimes characteristic of the Los Alamos experiment,[44] and the Livermore experiments planned on the Advanced Test Accelerator (ATA).[47]

The limiting case of ultrarelativistic electrons and very low beam density was considered in Sec. V. We compared the resulting kinetic dispersion relation (106) with the dispersion relation (127) obtained from a linear analysis of the conventional Compton-regime FEL equations.[8,9] This comparison was made for general beam equilibrium $G_0^+(\gamma_0)$. Differences between the two dispersion relations were traced to the eikenol approximation and the assumption of very narrow energy spread in Refs. 8 and 9. Finally, assuming ultrarelativistic electrons and a sufficiently broad spectrum of amplifying waves, in Sec. V we presented the quasilinear kinetic equations appropriate to the planar wiggler configuration considered in the present analysis. This represented a straightforward extension of the quasi-linear theory developed for the case of a helical magnetic wiggler field.[15,50] The quasilinear dispersion relation (128), the kinetic equation (129) for the distribution of beam electrons $G_0^+(\gamma_0,t)$, and the kinetic equation (131) for the wave spectral energy density $\mathcal{E}_k(t)$ describe the self-consistent non-linear evolution of the beam electrons and radiation field in circumstances where the wave autocorrelation time is short in comparison with the characteristic growth time [Eq.(92)].

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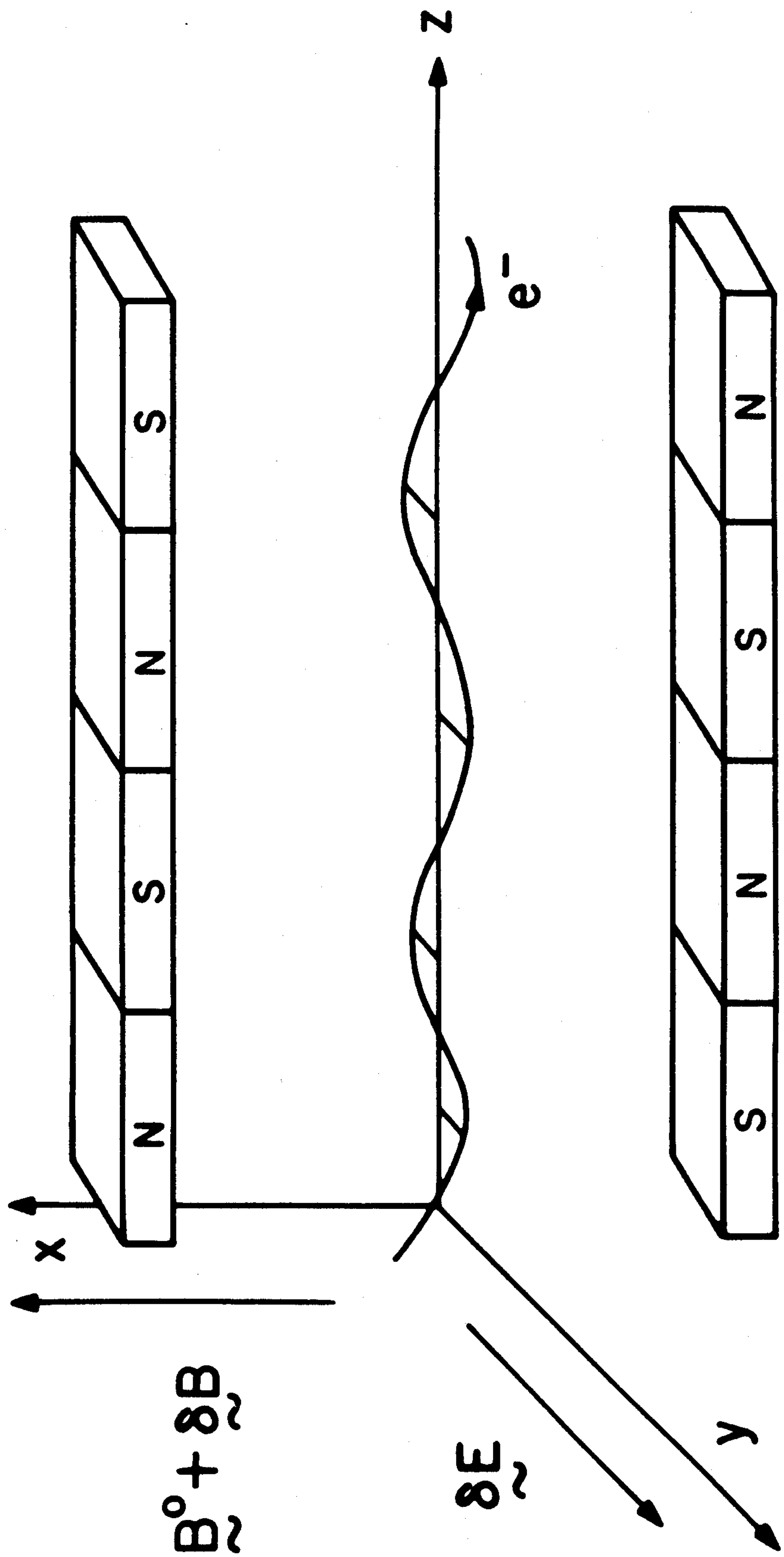
FIGURE CAPTIONS

Fig. 1 Planar wiggler configuration and coordinate system

Fig. 2 Schematic of $G_0^+(\gamma_0)$ versus γ_0. The region of positive slope with $[\partial G_0^+/\partial\gamma_0]_{\gamma_0=\gamma_r}>0$ corresponds to instability [Eq.(86)].

Fig. 3 Plot of normalized growth rate γ_k/k_0c versus k/k_0 obtained from Eqs.(86) and (94) for parameters characteristic of the Livermore FEL experiments planned on ATA.[47] Here the dimensionless parameters $\hat{\gamma}=100$, $\hat{\omega}_p^2/c^2k_0^2 = 3.6$ and $a_w = 1.7$ correspond to beam current $I_b = 1.9$kA, beam radius $r_b = 0.45$cm, wiggler amplitude $B_w = 2.3$kG, wiggler wavelength $\lambda_0 = 8$cm and beam density $\hat{n}_b = 6.3 \times 10^{11}$ cm^{-3}. The figure illustrates the dependence of the growth rate on fractional energy spread for $\Delta\gamma/\hat{\gamma} = 1\%$, 2%, and 3%.

Fig. 4 Plot of normalized growth rate γ_k/k_0c versus k/k_0 obtained from Egs.(86) and (94) for parameters characteristic of the Los Alamos FEL experiment.[11,44] Here, the dimensionless parameters $\hat{\gamma} = 41$, $\hat{\omega}_p^2/c^2k_0^2 = 0.21$ and $a_w = 0.76$ correspond to $I_b = 40$A, $r_b = 0.09$cm, $B_w = 3$kG, $\gamma_0 = 2.73$cm and $\hat{n}_b = 3.3\times10^{11}$cm^{-3}. The figure illustrates the dependence of the growth rate on fractional energy spread for $\Delta\gamma/\hat{\gamma} = 1\%$, 2% and 3%.


z
x
y
e^-
S
N
S
N
N
S
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$\underset{\sim}{B}^o + \delta\underset{\sim}{B}$
$\delta\underset{\sim}{E}$

Fig. 1

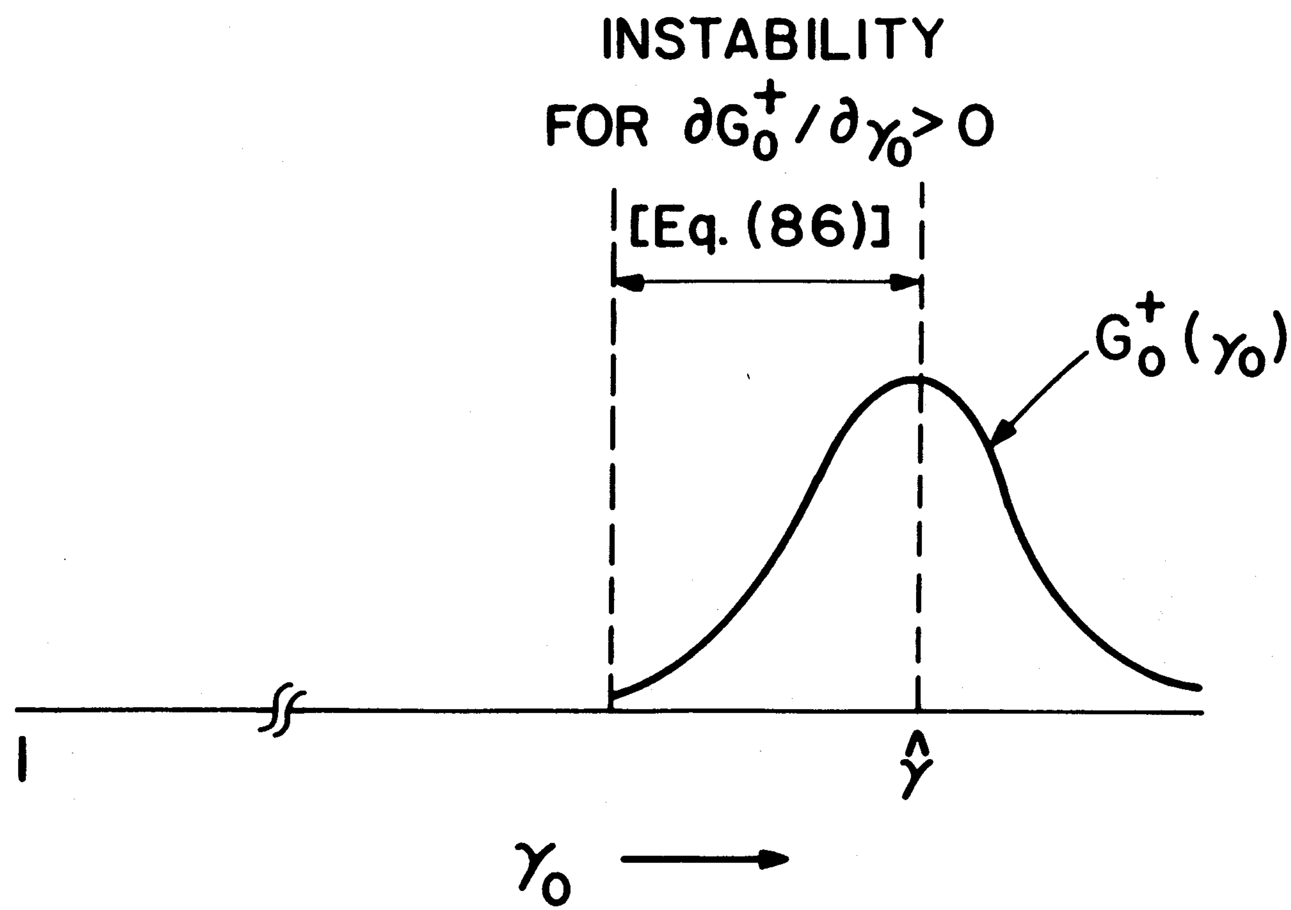

INSTABILITY
FOR $\partial G_0^+/\partial \gamma_0 > 0$
[Eq. (86)]
$G_0^+(\gamma_0)$
1
$\hat{\gamma}$
$\gamma_0 \longrightarrow$

Fig. 2

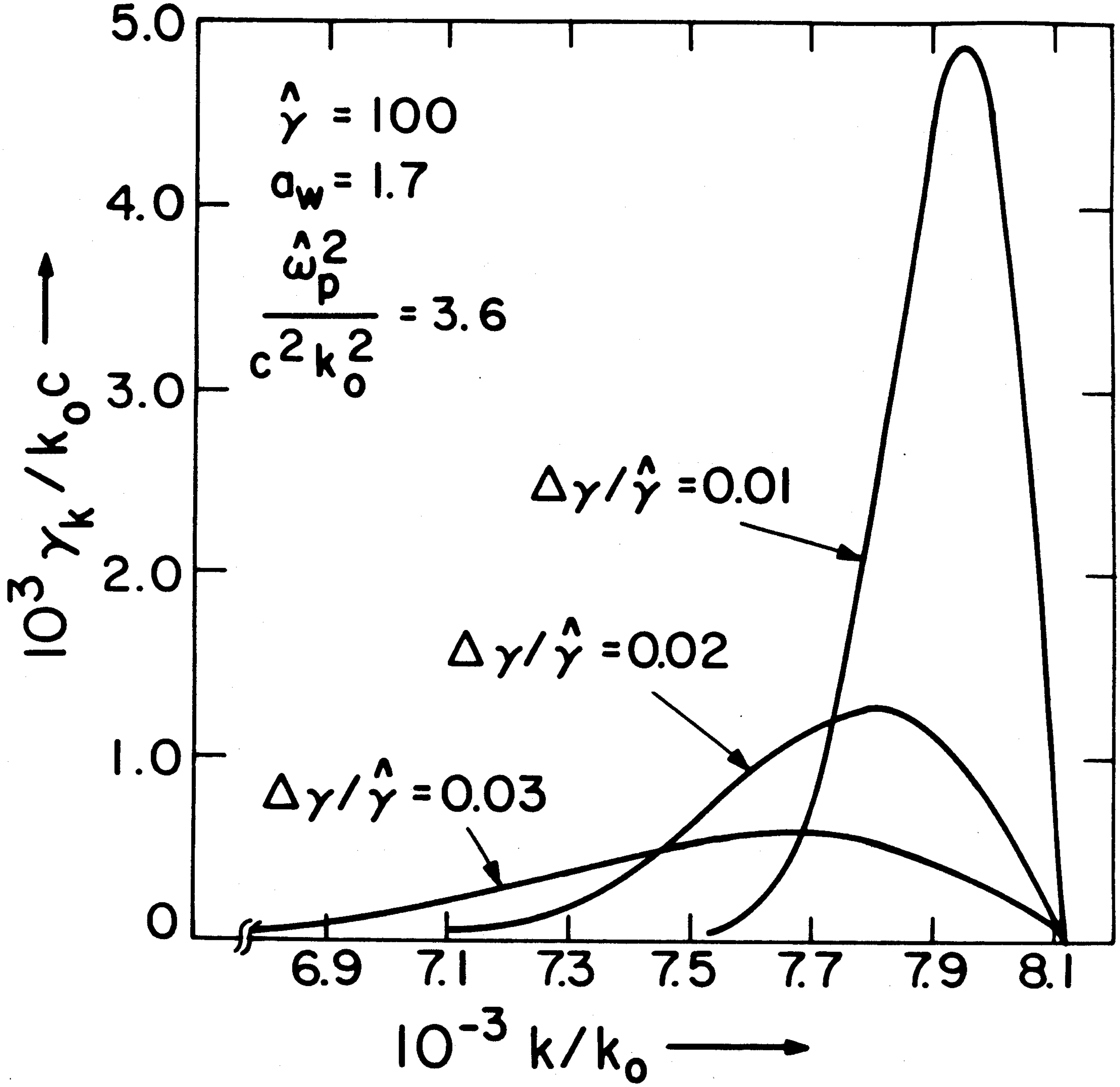
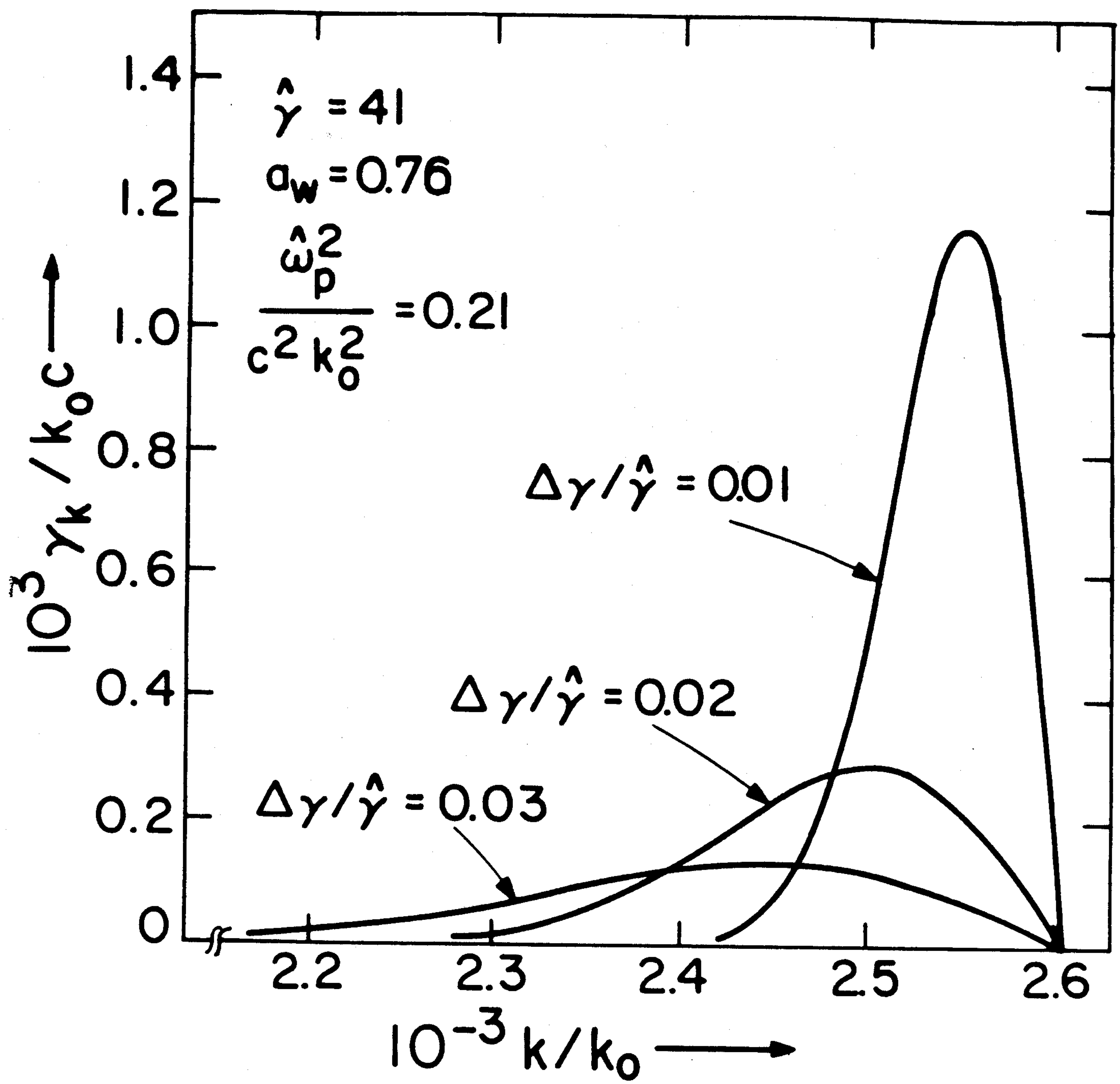

5.0
4.0
3.0
2.0
1.0
0
$\hat{\gamma} = 100$
$a_w = 1.7$
$\frac{\hat{\omega}_p^2}{c^2 k_o^2} = 3.6$
$\Delta\gamma/\hat{\gamma} = 0.01$
$\Delta\gamma/\hat{\gamma} = 0.02$
$\Delta\gamma/\hat{\gamma} = 0.03$
6.9
7.1
7.3
7.5
7.7
7.9
8.1
$10^3 \gamma_k / k_o c$
$10^{-3} k/k_o$

Fig. 3



$\hat{\gamma} = 41$
$a_w = 0.76$
$\frac{\hat{\omega}_p^2}{c^2 k_0^2} = 0.21$
$\Delta\gamma/\hat{\gamma} = 0.01$
$\Delta\gamma/\hat{\gamma} = 0.02$
$\Delta\gamma/\hat{\gamma} = 0.03$
$10^3 \gamma_k / k_0 c$
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
2.2
2.3
2.4
2.5
2.6
$10^{-3} k/k_0$

Fig. 4