EXACT ANALYTICAL MODEL OF THE CLASSICAL WEIBEL
INSTABILITY IN RELATIVISTIC ANISOTROPIC PLASMA

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ABSTRACT

Detailed properties of the Weibel instability in a relativistic unmagnetized plasma are investigated for a particular choice of anisotropic distribution function \( F(p^2, p_z) \) that permits an exact analytical solution to the dispersion relation for arbitrary energy anisotropy. The particular equilibrium distribution function considered in the present analysis assumes that all particles move on a surface with perpendicular momentum \( p_\perp = \hat{p}_\perp = \text{const.} \) and are uniformly distributed in parallel momentum from \( p_z = -\hat{p}_z = \text{const.} \) to \( p_z = +\hat{p}_z = \text{const.} \) (Here, the propagation direction is the z-direction.) The resulting dispersion relation is solved analytically, and detailed stability properties are determined for a wide range of energy anisotropy.
The classical Weibel instability\textsuperscript{1,2} in a uniform plasma is a transverse electromagnetic instability driven by an anisotropy in the average kinetic energy of the constituent electrons and/or ions. This instability has a wide range of applicability to astrophysical plasmas,\textsuperscript{3,4} and to laboratory plasmas\textsuperscript{4} with intense microwave heating. For nonrelativistic anisotropic plasma, detailed properties of the Weibel instability are readily calculated\textsuperscript{1,2} for a wide range of equilibrium distribution functions $F\rho_j(p^2, p_z)$. For relativistic anisotropic plasma, however, because of the coupling of the perpendicular and parallel particle motions through the relativistic mass factor $\gamma = (1 + p^2/m^2c^2 + p^2_2m^2_3c^4)^{1/2}$, stability properties are usually calculated in the limit of extreme energy anisotropy, or long perturbation wavelength,\textsuperscript{4} which allow approximate analytical solutions to the electromagnetic dispersion relation. Here, "perpendicular" and "parallel" refer to directions relative to the propagation direction (the z-direction).

The purpose of the present brief communication is to investigate detailed properties of the Weibel instability in a relativistic unmagnetized plasma for a particular choice of anisotropic distribution function that permits an exact analytical solution to the dispersion relation for arbitrary energy anisotropy. This calculation is intended to provide qualitative insights regarding stability behavior for more general choices of equilibrium distribution function. The particular distribution function [Eq.(3)] considered in the present analysis assumes that all particles move on a surface with perpendicular momentum $p_\perp = \hat{p}_\perp = \text{const.}$ and are uniformly distributed in parallel momentum between $p_z = -\hat{p}_z = \text{const.}$ and $p_z = +\hat{p}_z = \text{const.}$. The resulting dispersion relation [Eq.(8)] can be solved analytically, and detailed stability
properties are determined for a wide range of energy anisotropy. The dispersion relation [Eq.(15)] is also derived for the case where the particles have a thermal equilibrium distribution in parallel momentum \( p_z \) [Eq.(13)], and the corresponding range of unstable wavenumbers is determined in closed form [Eq.(18)]. Extension of the present analysis to include a uniform applied magnetic field \( B_0 \hat{z} \) will be the subject of a future investigation.

A. Theoretical Model

We investigate the electromagnetic stability properties of relativistic anisotropic plasma for wave perturbations propagating in the z-direction with wavevector \( k = k_z \hat{z} \). Perturbations are about the class of uniform, field-free equilibria with distribution function

\[
f_j^0(p) = \hat{n}_j F_j(p^2, p_z),
\]

where \( \hat{n}_j = \text{const.} \) is the ambient density of the j'th plasma component, \( p = (p_x^2 + p_y^2)^{\frac{1}{2}} \) is the particle momentum perpendicular to the propagation direction, and \( p_z \) is the parallel momentum. In the absence of applied magnetic field, the linear dispersion relation for transverse electromagnetic wave perturbations propagating in the z-direction is given by

\[
0 = D_T(k_z, \omega) = 1 - \frac{c^2 k^2}{\omega^2} - \sum_j \frac{\omega_{pj}^2}{\omega^2} \int d^3p \frac{(p_{\perp}/2)}{\gamma (\omega - k_z p_z / \gamma m_j)}
\]

\[
\times \left\{ \left( \omega - \frac{k_z p_z}{\gamma m_j} \right) \frac{a}{\partial p_{\perp}} + \frac{k_z p_z}{\gamma m_j} \frac{a}{\partial p_z} \right\} F_j(p^2, p_z).
\]

Here, \( \omega_{pj}^2 = 4\pi\hat{n}_j e_j^2/m_j \) is the nonrelativistic plasma frequency-squared; \( e_j \) and \( m_j \) are the charge and rest mass, respectively, of a j'th component particle; \( c \) is the speed of light in vacuo; \( \gamma = (1 + p_{\perp}^2/m_j c^2 + p_z^2/m_j c^2)^{\frac{1}{2}} \) is the relativistic mass factor; the range of integration is \( \int d^3p \cdots = 2\pi \int dp_{\perp} p_{\perp} \int dp_z \cdots \).
and the normalization of \( F \) is \( \int d^3p F(p_{\perp}, p_z) = 1 \). In obtaining Eq.(2), the perturbations are assumed to have \( z \)- and \( t \)-dependence proportional to \( \exp[i(k_z z - \omega t)] \), where \( k_z \) is the wavenumber and \( \omega \) is the complex oscillation frequency with \( \text{Im} \omega > 0 \), which corresponds to instability (temporal growth).

For relativistic anisotropic plasma, we note that the perpendicular and parallel particle motions in Eq.(2) are inexorably coupled through the relativistic mass factor \( \gamma = (1 + p_z^2/m_j c^2 + p_{\perp}^2/m_j c^2)^{\frac{1}{2}} \).

In the analysis that follows, we specialize to the case of stationary ions \((m_i \rightarrow \infty)\) and consider a single active component of relativistic anisotropic electrons. Moreover, for simplicity of notation, the electron species labels are omitted from \( \omega_{pe}, m_e, F_e(p_{\perp}, p_z) \), etc. The resulting dispersion relations [Eqs.(8) and (15)] are readily generalized to the multi-component case.

**B. Waterbag Distribution in Parallel Momentum**

The dispersion relation (2) can be used to investigate detailed electromagnetic stability properties for a wide range of anisotropic distribution functions \( F(p_{\perp}, p_z) \). For purposes of elucidating the essential features of the Weibel instability in relativistic anisotropic plasma, we make a particular choice of \( F(p_{\perp}, p_z) \) for which the momentum integrals in Eq.(2) can be carried out in closed analytical form. In particular, it is assumed that the electrons move on a surface with perpendicular momentum \( p_{\perp} = \hat{p}_{\perp} = \text{const.} \) and are uniformly distributed in parallel momentum between \( p_z = -\hat{p}_z = \text{const.} \) and \( p_z = +\hat{p}_z = \text{const.} \). That is, \( F(p_{\perp}, p_z) \) is specified by

\[
F(p_{\perp}, p_z) = \frac{1}{2\pi p_{\perp}} \delta(p_{\perp} - \hat{p}_{\perp}) \frac{1}{2\hat{p}_z} H(p_z^2 - \hat{p}_z^2),
\]

where \( H(x) \) is the Heaviside step function defined by \( H(x) = -1 \) for \( x > 0 \), and \( H(x) = 0 \) for \( x < 0 \). Note from Eq.(3) that \( \int d^3p F(p_{\perp}, p_z) = 1 \). Because
the electrons are uniformly distributed in parallel momentum for $|p_z| < \hat{p}_z$, we refer to the $p_z$-dependence of the distribution function in Eq.(3) as a "water-bag" distribution in $p_z$. For future reference, it is useful to introduce the maximum energy $\hat{\gamma}mc^2$, parallel speed $c\hat{\beta}_z$, and perpendicular speed $c\hat{\beta}_\perp$ defined by

$$\hat{\beta}_z = \frac{\hat{p}_z}{\hat{\gamma}mc}, \quad \hat{\beta}_\perp = \frac{\hat{p}_\perp}{\hat{\gamma}mc},$$

$$\hat{\gamma} = \left(1 + \frac{\hat{p}_\perp^2}{m^2c^2} + \frac{\hat{p}_z^2}{m^2c^2}\right)^{\frac{1}{2}} = \left(1 - \hat{\beta}_\perp^2 - \hat{\beta}_z^2\right)^{-\frac{1}{2}}. \quad (4)$$

We further introduce the effective perpendicular and parallel temperatures defined by

$$T_\perp = \int d^3p \frac{p_\perp^2}{2\hat{\gamma}m} F(p_\perp, p_z), \quad (5)$$

$$\frac{1}{2} T_{\parallel} = \int d^3p \frac{p_z^2}{2\hat{\gamma}m} F(p_\perp, p_z).$$

Substituting Eq.(3) into Eq.(5) and carrying out the required integrations over $p_\perp$ and $p_z$ give

$$T_\perp = \frac{1}{2} \hat{\gamma}mc^2 2^2 G(\hat{\beta}_z), \quad (6)$$

$$T_{\parallel} = \frac{1}{2} \hat{\gamma}mc^2 \left[1 - G(\hat{\beta}_z) + \hat{\beta}_z^2 G(\hat{\beta}_z)\right],$$

where $G(\hat{\beta}_z)$ is defined by

$$G(\hat{\beta}_z) = \frac{1}{2\hat{\beta}_z} \ln \left(\frac{1 + \hat{\beta}_z}{1 - \hat{\beta}_z}\right). \quad (7)$$

From Eq.(7), we note that $G(\hat{\beta}_z)$ is a slowly increasing function of $\hat{\beta}_z$ with $G(\hat{\beta}_z) = 1 + \hat{\beta}_z^2/3\cdots$ for $\hat{\beta}_z^2 \ll 1$. Moreover, in the limit of a nonrelativistic
plasma with \( \beta_z^2 \ll 1 \) and \( \beta_\perp^2 \ll 1 \), Eq.(7) reduces to the expected results, \( T_\perp = (1/2)mc^2\beta_\perp^2 \) and \( T_\parallel = (1/3)mc^2\beta_z^2 \). Depending on the relative values of \( \beta_\perp \) and \( \beta_z \), it is clear that the choice of distribution function in Eq.(3) can cover a wide range of energy anisotropy.

For the choice of distribution function \( F(p_\perp,p_z) \) in Eq.(3), the \( p_\perp \) - and \( p_z \) - integrations required in Eq.(2) can be carried out in closed analytical form. Some straightforward algebra that makes use of Eqs.(2), (3), (4) and (7) gives the dispersion relation

\[
0 = D_T(k_\perp,\omega) = 1 - \frac{c^2k^2}{\omega^2} - \frac{\omega^2}{\omega^2} \left[ G(\hat{\beta}_z) \right] + \frac{\beta_\perp^2}{2(1 - \beta_z^2)} \left( \frac{c^2k_z^2 - \omega^2}{\omega^2 - c^2k_z^2}\beta_z^2 \right)
\]

Equation (8) is readily extended to the case of a multicomponent plasma by making the replacements \( \omega_2 \gamma \cdots + \sum_j (\omega_{p_j}/\hat{\gamma}_j) \cdots, \beta_z + \beta_{zj}, \beta_\perp + \beta_{\perp j}, \) etc.

For a single active (electron) component, Eq.(8) can be expressed in the equivalent form

\[
0 = \omega^4 - \omega^2 \left\{ c^2k_z^2(1 + \beta_z^2) + \frac{\omega^2}{\hat{\gamma}} \beta_\perp^2 - \frac{\omega^2}{\hat{\gamma}} \left[ \frac{\beta_\perp^2}{2\beta_z^2(1 - \beta_z^2)} - G(\hat{\beta}_z) \right] \right\}
\]

\[
+ c^2k_z^2\beta_z^2 \left\{ c^2k_z^2 - \omega^2 \left[ \frac{\beta_\perp^2}{2\beta_z^2(1 - \beta_z^2)} - G(\hat{\beta}_z) \right] \right\}
\]

which is a quadratic equation for \( \omega^2 \). In Eq.(9), \( \omega_P^2 = 4\pi n_e e^2/m \) is the non-relativistic electron plasma frequency-squared, and \( G(\hat{\beta}_z) \) is defined in Eq.(7).

The dispersion relation (9) can be solved exactly for the complex oscillation frequency \( \omega \). In this regard, a careful examination of Eq.(9) shows that there are two classes of solutions for \( \omega^2 \), namely, a fast-wave
branch corresponding to stable oscillations with \( \text{Im}\omega = 0 \) and \((\text{Re}\omega)^2 > c^2k_z^2\), and a slow-wave branch which may or may not exhibit instability, depending on the degree of energy anisotropy. It is readily shown that the necessary and sufficient condition for the slow-wave branch to exhibit instability \((\text{Im}\omega > 0)\) is given by

\[
\frac{\hat{\beta}_z^2}{2\hat{\beta}_z^2} > (1 - \hat{\beta}_z^2)G(\hat{\beta}_z) .
\] (10)

Moreover, when Eq.(10) is satisfied, the corresponding range of \(k_z^2\) corresponding to instability is given by

\[
0 < k_z^2 < k_0^2 \equiv \frac{\omega^2}{\gamma c^2} \left[ \frac{\hat{\beta}_z^2}{2\hat{\beta}_z^2(1 - \hat{\beta}_z^2)} - G(\hat{\beta}_z) \right] .
\] (11)

When Eq.(10) is satisfied, and \(k_z^2\) is in the range specified by Eq.(11), the real oscillation frequency of the slow-wave branch satisfies \(\text{Re}\omega = 0\) and the growth rate of the unstable mode is given by

\[
\text{Im}\omega = \frac{1}{\sqrt{2}} \left( \left[ c^2k_z^2 \hat{\beta}_z^2 + \frac{\omega^2}{\gamma} \frac{\hat{\beta}_z^2}{2\hat{\beta}_z^2} - c^2(k_0^2 - k_z^2) \right]^2 + 4c^4k_z^2\hat{\beta}_z^2(k_0^2 - k_z^2) \right)^{\frac{1}{2}}
\] (12)

\[
- \left[ c^2k_z^2 \hat{\beta}_z^2 + \frac{\omega^2}{\gamma} \frac{\hat{\beta}_z^2}{2\hat{\beta}_z^2} - c^2(k_0^2 - k_z^2) \right] \right) \right)^{\frac{1}{2}}
\]

Note from Eq.(12) that \(\text{Im}\omega = 0\) for \(k_z = 0\) and \(k_z^2 = k_0^2\), and that \(\text{Im}\omega\) passes through a maximum for some value of \(k_z^2\) intermediate between 0 and \(k_0^2\).

In the nonrelativistic limit with \(\hat{\beta}_\perp, \hat{\beta}_z^2 \ll 1\), the necessary and sufficient condition for instability in Eq.(10) becomes \(\hat{\beta}_\perp^2/2\hat{\beta}_z^2 > 1\), and the range of instability is given by \(0 < k_z^2 < k_0^2 \equiv (\omega^2/c^2)(\hat{\beta}_\perp^2/2\hat{\beta}_z^2 - 1)\). In the relativistic regime, however, the instability criterion in Eq.(10) is more complicated, which is illustrated in Fig. 1. In Fig. 1, the region of \((\hat{\beta}_\perp^2, 2\hat{\beta}_z^2)\) parameter space corresponding to instability is above the contour connecting the origin to \((\hat{\beta}_\perp^2, 2\hat{\beta}_z^2) = (0.580, 0.840)\), which corresponds to \(\gamma = \infty\). While the detailed
form of the instability criterion in Eq.(10) differs from the nonrelativistic case, it is evident from Fig. 1 that the condition for instability in the relativistic regime is qualitatively the same, i.e., the Weibel instability exists for sufficiently large values of $\delta^2 / 2 \beta^2_z$. Put another way, for the choice of distribution function in Eq.(3), the Weibel instability in a relativistic anisotropic plasma can be completely stabilized by increasing the thermal anisotropy $2 \beta^2_z / \beta^2_\perp$ to sufficiently large values.

As a numerical example, shown in Fig. 2 are plots of normalized real frequency $\Re \omega / [\omega_p / \gamma^{\frac{1}{4}}]$ [Fig. 2(a)] and growth rate $\Im \omega / [\omega_p / \gamma^{\frac{1}{4}}]$ [Fig. 2(b)] versus normalized wavenumber $c k_z / [\omega_p / \gamma^{\frac{1}{4}}]$ for $\gamma = 9$ and several values of energy anisotropy $\beta^2_z / 2 \beta^2_\perp$. The real oscillation frequencies for both the fast-wave and slow-wave branches are presented in Fig. 2(a). Moreover, for $\gamma = 9$, the slow-wave branch becomes completely stable ($\Im \omega = 0$ and $k_0^2 = 0$) for $\beta^2_z / 2 \beta^2_\perp = 0.697$. It is evident from Fig. 2(b) that the strongest instability occurs for the largest energy anisotropy, i.e., $\beta^2_z = 0$ and $\beta^2_\perp = 80/81$ (for $\gamma = 9$). Moreover, depending on the value of $\beta^2_z / 2 \beta^2_\perp$, the maximum growth rate in Fig. 2(b) can be a substantial fraction of $\omega_p / \gamma^{\frac{1}{4}}$.

C. Thermal Equilibrium Distribution in Parallel Momentum

As a second example, we consider the case where the electron distribution function is specified by

$$ F(p^2_\perp, p_z) = \frac{1}{2\pi p_\perp} \delta(p_\perp - \hat{p}_\perp) \frac{\exp(-\gamma mc^2/T_{11})}{2 \gamma_{\perp} m c \hat{\gamma}_{\perp} c^2 / T_{11}}. \tag{13} $$

Here, the ions are treated as a fixed background $(m_i \rightarrow \infty)$, and a single component of relativistic, anisotropic electrons is assumed. In Eq.(13), $K_n(x)$ is the modified Bessel function of the second kind of order $n$; $\gamma = (1 + p^2_\perp / m^2 c^2 + p^2_z / m^2 c^2)^{\frac{1}{2}}$ is the relativistic mass factor; the constant $\hat{\gamma}_\perp$ is defined by $\hat{\gamma}_\perp = (1 + p^2_\perp / m^2 c^2)^{\frac{1}{2}}$; and $T_{11}$ is a positive constant. Note that the exponential
factor in Eq. (13) corresponds to a thermal equilibrium distribution in parallel momentum $p_z$. Some straightforward algebra that makes use of Eqs. (5) and (13) shows that $T_{\parallel}$ can be identified with the parallel temperature $\int d^3p (p_z^2/\gamma m) \times F(p_{\perp}^2, p_z)$, and that the effective perpendicular temperature $T_{\perp} = \int d^3p (p_{\perp}^2/2\gamma m) F(p_{\perp}^2, p_z)$ is given by

$$T_{\perp} = \frac{1}{2} \frac{\gamma_{\perp} mc^2}{\gamma_{\parallel} mc^2} \left( \frac{\hat{p}_{\perp}}{\hat{\gamma}_{\perp} mc^2} \right)^2 K_0(\gamma_{\perp} mc^2/T_{\parallel}) \frac{K_0(\gamma_{\perp} mc^2/T_{\parallel})}{K_1(\gamma_{\perp} mc^2/T_{\parallel})}. \quad (14)$$

Substituting Eq. (13) into Eq. (2) and carrying out the integrations over $p_{\perp}$ and $p_z$ give the dispersion relation

$$0 = D_T(k, \omega) = 1 - \frac{c^2 k_z^2}{\omega^2} - \frac{\left(\omega p_{\perp}/\hat{\gamma}_{\perp}\right)}{\omega} \left\{ \frac{K_0(\gamma_{\perp} mc^2/T_{\parallel})}{K_1(\gamma_{\perp} mc^2/T_{\parallel})} \right\} + \frac{T_{\perp}}{\gamma_{\perp} mc^2} \frac{\omega^2 - c^2 k_z^2}{K_0(\gamma_{\perp} mc^2/T_{\parallel})} \int_0^\infty d\tau K_0(\tau). \quad (15)$$

In Eq. (15), the complex argument of $K_0(\tau)$ is defined by

$$\zeta^2 = \left( \frac{\gamma_{\perp} mc^2}{T_{\parallel}} - i\omega \right)^2 + c^2 k_z^2 \tau^2, \quad (16)$$

where $\text{Im} \omega > 0$ corresponds to instability, and $T_{\parallel}$ is defined in Eq. (14). The $\tau$-integral in Eq. (15) must generally be evaluated numerically, or in the context of asymptotic expansions for large or small values of $|\zeta|$. Unlike Eq. (8) [or Eq. (9)], the dispersion relation (15) generally incorporates the effects of collisionless dissipation (Landau damping) by the $p_z$-distribution in Eq. (13). For the slow-wave branch, it can be shown from Eq. (15) that the necessary and sufficient condition for instability is given by

$$\frac{T_{\perp}}{T_{\parallel}} > \left[ \frac{K_0(\gamma_{\perp} mc^2/T_{\parallel})}{K_1(\gamma_{\perp} mc^2/T_{\parallel})} \right]^2, \quad (17)$$

where $T_{\parallel}$ is defined in Eq. (14). Moreover, when Eq. (17) is satisfied, it is
found that $\text{Re} \omega = 0$ (for the slow-wave branch) over the range of unstable wave-numbers specified by

$$0 < k_z^2 < k_0^2 = \frac{\omega_p^2}{\hat{\gamma}_c^2} \left[ \frac{T_{\perp}}{T_{\parallel}} K_1(\hat{\gamma}_c mc^2/T_{\parallel}) - \frac{K_0(\hat{\gamma}_c mc^2/T_{\parallel})}{K_1(\hat{\gamma}_c mc^2/T_{\parallel})} \right].$$  \hfill (18)

Note from Eq.(18) that the marginal stability point $k_0^2$ (where $\text{Im} \omega = 0 = \text{Re} \omega$) can be calculated in closed analytical form. This follows from the identity

$$\lim_{\text{Re} \omega = 0} \left[ \frac{k_0^2}{c^2} \int_0^\infty d\tau K_0(\xi) \right] = \frac{\hat{\gamma}_c mc^2}{T_{\perp}} K_1(\hat{\gamma}_c mc^2/T_{\parallel}).$$  \hfill (19)

Finally, shown in Fig. 3 is a plot of the stability boundary in the parameter space $(T_{\perp}/\hat{\gamma}_c mc^2, T_{\parallel}/\hat{\gamma}_c mc^2)$ calculated numerically from Eq.(17). The region above the curve in Fig. 3 corresponds to instability, which requires sufficiently large thermal anisotropy $T_{\perp}/T_{\parallel}$.

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FIGURE CAPTIONS

Fig. 1. Region of \((\hat{\beta}^2_1, 2\hat{\beta}^2_Z)\) parameter space corresponding to instability [Eq.(10)].

Fig. 2. Plots of (a) normalized real frequency \(\text{Re}(\omega/\gamma^\frac{1}{2})\) and (b) normalized growth rate \(\text{Im}(\omega/\gamma^\frac{1}{2})\) versus \(ck_z/(\omega_0/\gamma^\frac{1}{2})\), for \(\gamma = 9\) and several values of \(\hat{\beta}^2_1/2\hat{\beta}^2_Z\) [Eq.(9)].

Fig. 3. Regions of \((T_\perp/\gamma^2_\perp mc^2, T_\parallel/\gamma^2_\parallel mc^2)\) parameter space corresponding to instability [Eq.(17)].
Fig. 1
Fig. 2(a)
Fig. 2(b)

The graph shows the relationship between $\Im \omega / \omega_p / \gamma^{1/2}$ and $ck_z / \omega_p / \gamma^{1/2}$ with various lines representing different values of $\gamma$. The lines are labeled with values of $\gamma = 9.0$, $\gamma = 5.0$, $\gamma = 2.0$, $\gamma = 10.0$, $\gamma = 20.0$, and $\gamma = 50.0$. The graph also highlights $\beta_{\perp}^2 / 2 \beta_{\perp}^2 = \infty$. 

In the context of the figure:

- $\gamma$ is a parameter representing a frequency or angular frequency.
- $\omega_p$ is a frequency parameter, typically the plasma frequency.
- $\gamma^{1/2}$ is a dimensionless parameter.
- $ck_z$ is a wave number parameter, where $c$ is the speed of light and $k_z$ is the wave number in the $z$ direction.
- $\beta_{\perp}^2 / 2 \beta_{\perp}^2 = \infty$ indicates a condition for the wave propagation.
Fig. 3