Damping of the Fast Alfvén Wave  
In Ion-Cyclotron Resonance Heating  

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DAMPING OF THE FAST ALFVÉN WAVE
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ABSTRACT
We determine the conditions under which the damping of the fast Alfvén wave, used in the heating of tokamak plasmas, can be described by the geometrical optics approximations. Also, for these conditions, we derive an analytical formula which explicitly determines the amount of energy which is damped onto the plasma particles by the fast wave. The conditions for the validity of the geometrical optics approximations cover a wide range of $k_\parallel$'s (the component of the wave-vector parallel to the total magnetic field). There is a narrow range of $k_\parallel$'s over which our analysis is not valid; this is the regime where mode-conversion is dominant. The results from our theory are in excellent agreement with those obtained from numerical solutions of fourth and sixth order differential equations that are commonly to describe this type of heating. However, our theory has a distinct advantage of lending itself to detailed scaling studies as explicit expressions for the damping of the fast Alfvén wave are obtained.
I. INTRODUCTION

One of the successful ways of heating tokamak plasmas is to inject radio-frequency waves in the ion-cyclotron range of frequencies (ICRF). Usually, in this scenario, the fast Alfvén wave (FAW) is excited at the edge of the plasma. As it propagates into the plasma it encounters cyclotron harmonic resonances and singular layers where other normal modes of the plasma can be excited. Such regions are commonly designated as “mode-conversion regions.” In the conventional treatment of ICRF heating, starting with a $k_\parallel$ (component of the wave vector parallel to the total magnetic field) spectrum imposed by the antenna, the FAW couples to the ion-Bernstein wave (IBW) near the cyclotron harmonic resonances in a single ion-species plasma, or near the two ion-hybrid resonance layer for a two ion-species plasma. In a single ion-species plasma the second ion-cyclotron harmonic is located inside the plasma; in a two ion-species plasma the second harmonic of the majority ions and the two-ion hybrid resonance layer are located in the plasma. ICRF heating in a two ion-species plasma has generally led to more effective results than heating in a single ion-species plasma.

In the mode-conversion regions wave propagation and damping cannot be treated by geometrical optics techniques. In general, the wave-couplings in such regions must be described by (integro) differential equations for the fields [1,2]; their solution, if at all possible, require complicated numerical techniques. Since the mode-conversion region is very narrow compared to the plasma (minor) radius, a frequent simplification is to model the problem in slab geometry. The solutions outside the mode-conversion region are matched to the WKB modes in the plasma which satisfy the geometrical optics criteria. Then, the mode-conversion region is characterized by transmission, reflection, mode-conversion, and dissipation power densities. Even in the slab geometry model, the solution to the approximate differential formulation requires intense computations [3,4,5,6]. Hence, recent efforts have concentrated on developing simpler models of mode-conversion regions. These simpler models lend themselves to an easier understanding of this type of heating. The construction of such models requires the reduction in the order of the differential equation description of the mode-conversion process [7,8].

In this paper we focus on obtaining the dissipation of the linear wave energy which
results in heating of the plasma. Such dissipation is dependent on $k_{||}$ which will be treated as a parameter as it is determined mainly by the antenna. We show that, except for a narrow region of $k_{||}$'s where mode-conversion is important, energy dissipation can be accurately determined from appropriate local analyses. A closed form analytical result for the dissipation is obtained that lends itself to a scaling analysis of heating. In the region of validity of our model, the results are in excellent agreement with those obtained from a numerical analysis of higher order model differential equations [1] that are commonly used to describe the mode-conversion region.

We examine in detail the local dispersion relation for a slab geometry model in the vicinity of the cyclotron harmonic resonance and the two-ion hybrid resonance. The dispersion relation is the one derived from the Vlasov-Maxwell set of equations assuming a Maxwellian plasma. Over the region of interest, we assume a linear variation in the magnetic field in which the plasma is immersed. The variation is incorporated in a WKB sense into the dispersion relation through the cyclotron frequencies. We find that for $k_{||} \approx 0$ there exists another kinetic mode near the cyclotron harmonic resonance which couples to the FAW. The existence of this mode has been noted before [9] but it has been ignored in analyses of the coupling of the FAW to the IBW. We study the consequences of this mode on the coupling to the IBW. For larger values of $k_{||}$ this new mode does not affect the coupling of the FAW to the IBW. However, the coupling between the FAW and IBW is affected when $k_{||}$ is increased. For small $k_{||}$, the cyclotron resonance layer and the ion-ion hybrid resonance are distinctly separated and the FAW and the IBW are strongly coupled. As $k_{||}$ is increased the coupling between the modes weakens and eventually the FAW is decoupled from the IBW. By approximating the dispersion relation in the region of interest, we determine the conditions under which the ion-cyclotron absorption region is distinctly separated from the mode-conversion region and also the condition when the FAW is decoupled from the IBW. In either of these two cases we use the geometrical optics approximations to determine the damping of the FAW. An algebraic expression for this damping is determined and is found to be in excellent agreement with the results obtained from a more detailed analysis of the mode-conversion regions.
II. THE KINETIC DISPERSION RELATION

We assume that the local dispersion relation describing ICRF waves in a magnetized Maxwellian plasma is given by [10]

$$D = (\epsilon_{zz} - n_\parallel^2)(\epsilon_{yy} - n^2) - \epsilon_{zy}^2 = 0,$$

where

$$n^2 = n_\parallel^2 + n_\perp^2 = c^2 k^2 / \omega^2,$$

$$\epsilon_{zz} = \sum_{\alpha} \frac{\omega_{pa}^2}{\sqrt{2} k_\parallel v_{T\alpha} \omega} \sum_{m=-\infty}^{\infty} m^2 \Gamma_m(\lambda_\alpha) \frac{\lambda_\alpha}{\lambda_\alpha} Z(\zeta_\alpha^m),$$

$$\epsilon_{yy} = \sum_{\alpha} \frac{\omega_{pa}^2}{\sqrt{2} k_\parallel v_{T\alpha} \omega} \sum_{m=-\infty}^{\infty} \left[ m^2 \Gamma_m(\lambda_\alpha) - 2 \lambda_\alpha \Gamma'_m(\lambda_\alpha) \right] Z(\zeta_\alpha^m),$$

$$\epsilon_{zy} = \sum_{\alpha} \frac{\omega_{pa}^2}{\sqrt{2} k_\parallel v_{T\alpha} \omega} \sum_{m=-\infty}^{\infty} m \Gamma'_m(\lambda_\alpha) Z(\zeta_\alpha^m) - \frac{\omega_{pe}^2}{\omega \Omega_e},$$

$$\Gamma(\lambda_\alpha) = e^{-\lambda_\alpha} I_m(\lambda_\alpha),$$

$$\lambda_\alpha = k_\parallel^2 \rho_\alpha^2, \quad \rho_\alpha = \frac{v_{T\alpha}}{\Omega_\alpha},$$

$$\zeta_\alpha^m = \frac{\omega - m \Omega_\alpha}{\sqrt{2} k_\parallel v_{T\alpha}}, \quad v_{T\alpha}^2 = \frac{k T_\alpha}{m_\alpha}.$$

$T_\alpha$, $\Omega_\alpha$, $\omega_{pa}$ and $m_\alpha$ are the temperature, cyclotron frequency, plasma frequency, and mass of species $\alpha$ respectively. The majority species deuterium (D) will be denoted by $\alpha = 1$ and the minority species hydrogen (H) by $\alpha = 2$. The ratio of the minority to majority concentration is $\eta = n_2 / n_1$ and is assumed to be much less than one. This dispersion relation is valid in the limit of zero electron mass. For waves in the ion-cyclotron regime this is a good approximation.

We assume that in the region of interest near the center of the plasma, the plasma can be considered a rectangular slab. The magnetic field is taken to be along the $z$-direction (toroidal direction) with a slow variation in $z$ (the radial direction) given by $B = B_0(1 - x/R_0)$ where $R_0$ is the major radius and $x = 0$ is the center of the plasma.
For a fixed frequency $\omega$ the cyclotron frequencies become functions of $x$ through the $x$ dependence of $B$. Equivalently, $x$ is just the frequency separation from the second harmonic assuming that at $x = 0$ we have $\omega = 2\Omega_1$. i.e. $x = (\omega - 2\Omega_1) R_0 / 2\Omega_1$. The density and the temperature in this slab are assumed to be constant and both species have the same temperature.

A. Pure Perpendicular Propagation

We solved the dispersion relation numerically for $k_\perp$ as a function of $x$ for the case $k_\parallel = 0$ and $\eta = 0$. The results are plotted in fig. 1 in the form $x$ vs. $k_\perp$. In addition to the familiar fast Alfvén wave (FAW) and ion Bernstein wave (IBW) there is a third wave which remains very near to the second harmonic. This wave will be referred to as the kinetic electromagnetic wave (KEW). Fig. 2 gives a closer view of the KEW. It couples to the FAW below the second harmonic at a point $x = x_m$. The separation from the second harmonic for the KEW increases as $k_\perp$ increases until it reaches some maximum separation. The KEW then approaches the second harmonic asymptotically. The KEW was noticed as early as 1964[9] but has been ignored since.

The plot can be divided into four regions as shown in fig. 3. For clarity we have plotted the results in the form $n_\perp$ vs. $x$. For a fixed $\omega$, the index of refraction $n$ and the wavevector $k$ differ only by a numerical factor. The KEW is not drawn in. On this scale it would simply sit at $x = 0$. In region 1 the FAW is propagating except very near to $x = 0$ where it couples to the KEW. The IBW is highly evanescent with a zero real part of $n_\perp$ and an imaginary part of $n_\perp$ which goes to zero at the low field edge of region 2. In region 2 the FAW and the IBW $n_\perp$ are both real. The IBW Re$n_\perp$ is zero at the low field edge ($x = x_c = 0$) and increases until it becomes degenerate with that of the FAW at the high field edge ($x = x_-$). The real parts of $n_\perp$ of the FAW and IBW remain degenerate in region 3. The imaginary parts of $n_\perp$ have equal magnitudes but opposite signs. This is the mode coupling region. In region 4 the two waves are real. If $\eta$ is slowly increased from zero the picture changes in region 2. We find that $x_c$ and $x_-$ both move towards the high field side with $x_-$ approaching $x_c$. Region 3 also enlarges toward the high field side. When $x_c = x_-$ the IBW no longer exists in region 2 and the FAW simply goes to a cutoff at $x = x_c$. If $\eta$ is further increased $x_c$ and $x_-$ separate again and $n_\perp$ of the IBW and FAW
are purely imaginary in region 2. The progression for increasing $\eta$ is shown in figs. 4 and 5.

B. Oblique Propagation

For small $k_\parallel$ the $n_\perp$ of the FAW acquires a small imaginary part in region 1 located about $z = 0$ with the peak Doppler shifted to the high field side. In region 1 the KEW gets decoupled from the FAW and moves into the complex $n_\perp$ plane as $n_\parallel$ increases. The FAW propagates through $z = 0$ with $|\text{Re } n_\perp| >> |\text{Im } n_\perp|$. We also have $|n'_\perp/n_\perp| << 1/R_0$ and $|n''_\perp/n'_\perp| << 1/R_0$ where prime denotes differentiation with respect to $x$. In this case the geometrical optics (g.o.) approximation holds and Im $n_\perp(x)$ gives a damping profile. Fig. 6 shows a numerical solution of (1) for PLT type parameters with $\eta = 0.05$ and for $n_\parallel = 3$. The damping coefficient is given by

$$\mu = \int_{-\infty}^{\infty} \text{Im } k_\perp(z) dz$$

(2)

and the fraction of the power dissipated is

$$P_d = 1 - e^{-2\mu}.$$  

(3)

For sufficiently large $n_\parallel$, Re $n_\perp$ of the FAW and the IBW are no longer degenerate in region 3 (fig. 7). The Im $n_\perp$ of the FAW that arose from the non-zero $n_\parallel$ first spreads into region 3 then into region 4. The resulting Im $n_\perp$ in region 3 looks like a superposition of the $n_\parallel = 0$ bubble and the spreading damping profile. The contribution from the bubble decreases as $n_\parallel$ is increased. The Im $n_\perp$ of the IBW in region 3 appears to peel off to join the large Im $n_\perp$ that was previously found only in regions 1 and 2 for $n_\parallel = 0$. The Re $n_\perp$ of the FAW is no longer slowly varying so that the g.o. approximation is no longer valid. For even larger $n_\parallel$, Im $n_\perp$ of the FAW continues to spread (fig. 8). When it has spread into region 4 to the extent that the peak of the Im $n_\perp$ profile is near to the where high field edge of region 3 was for $n_\parallel = 0$, full decoupling is attained. The Re $n_\perp$ of the FAW smoothens out and becomes slowly varying in the g.o. sense. We then can assume that Im $n_\perp$ gives the damping.
III. THE APPROXIMATE DISPERSION RELATION

In order to describe the numerically observed behaviour of the dispersion relation, we construct an approximate dispersion relation. Since we are interested only in the region \( \omega \approx 2\Omega_1 \), we only need to keep \( m = 0, \pm 1, 2 \) terms for the majority and \( m = 0, 1 \) terms for the minority. We assume that \( \lambda_1 << 1 \) and expand \( \Gamma_n(\lambda_1) \) to third order in \( \lambda_1 \) and to first order in \( \lambda_2 \) so that the KEW can be included in the description. This gives

\[
\begin{align*}
\epsilon_{zz} & \approx -\frac{1}{3} c^2 + \frac{7}{12} \lambda_2 \frac{c^2}{\sqrt{\pi} \delta v_a^2} Z(\zeta) + \frac{1}{2 \sqrt{\pi} \delta v_a^2} Z(\zeta/2), \\
\epsilon_{yy} & \approx -\frac{1}{3} c^2 + \frac{1}{12} \lambda_1 \frac{c^2}{\sqrt{\pi} \delta v_a^2} Z(\zeta) + \frac{1}{2 \sqrt{\pi} \delta v_a^2} Z(\zeta/2), \\
\epsilon_{xy} & \approx -\frac{2}{3} c^2 + \frac{7}{6} \lambda_1 \frac{c^2}{\sqrt{\pi} \delta v_a^2} Z(\zeta) + \frac{1}{2 \sqrt{\pi} \delta v_a^2} Z(\zeta/2),
\end{align*}
\]

where

\[
\delta = 2 \sqrt{\frac{2}{\pi}} k_{||} \rho_1, \\
\zeta = \zeta_1^2 \approx \frac{4x}{\sqrt{\pi} \delta}.
\]

We assume \( x, \eta << 1 \). The elements (4) are then used in (1) to obtain the approximate dispersion relation.

A. Perpendicular Propagation

For the case where \( k_{||} = 0 \), the asymptotic expansion of the \( Z \) functions gives

\[
D \approx \left( 192 - 96 \frac{\lambda_1}{\beta_1} \right) \left( \frac{x}{R_0} \right)^2 + \left( 48 - 36 \frac{\lambda_1}{\beta_1} \right) \left( \lambda_1 + 2\eta \right) \left( \frac{x}{R_0} \right) - 27\lambda_3^3 \eta - \frac{3}{4} \lambda_1^4 = 0
\]

where

\[
\beta_1 = \frac{8\pi n_1 k T_1}{B^2} = 2 \frac{v_{T1}^2}{c_A^2}.
\]

Very near to \( x = 0 \) we can ignore the \( x^2 \) term. The resulting equation describes the KEW. The point where the FAW adjoins the KEW \( (x = x_m) \) is a local maximum of \( x = x(k_{||}) \) and is given by

\[
|x_m| \approx -\frac{18\beta_1^3 \eta + \beta_1^4}{4(\eta + \beta_1)} R_0.
\]

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Ignoring the last two terms of (5) gives a good description of the FAW and the IBW without the KEW. We solved for the roots and have listed them in Appendix 1. The four regions arise naturally. We find that the low field edge of region 2 is at \( x = -\left(\eta/2\right)R_0 \) and the FAW will go to a cutoff when \( \eta \geq 2\beta_1 \). This criterion on \( \eta \) is satisfied for most plasmas of interest so an ion-ion hybrid scenario refers generally to a dispersion relation with a FAW cut off.

B. Oblique Propagation

For \( k_\parallel \neq 0 \) we must substitute (4) directly into (1) and keep terms to order \( \lambda_1^4 \). The resulting dispersion relation has a real and an imaginary part. It is dependent on six small independent parameters: \( \text{Re}(\lambda_1) \), \( \text{Im}(\lambda_1) \), \( \zeta \), \( \beta_1 \), \( \eta \) and \( \delta \). We are interested in calculating \( \text{Im}k_\perp \) as a function of \( x \) when g.o. is applicable, and from that the fraction of the power dissipated. If \( \text{Re}k_\perp \gg \text{Im}k_\perp \), then \( \text{Re}\lambda_1 \gg \text{Im}\lambda_1 \) is also true. The relative ordering of the other parameters is determined by the different situations of interest. We solve the dispersion relation for real and imaginary parts of \( \lambda_1 \) within the given ordering scheme. They then give the real and imaginary parts of \( k_\perp \). The calculations were performed with the aid of MACSYMA.

i) Small \( k_\parallel \)

We consider the case where \( \delta < \beta_1, \eta \). Numerically we have seen that g.o. is applicable for the FAW in region 1 after the FAW is decoupled from the KEW and before the damping profile enters regions 2 and 3. The FAW decouples from the KEW approximately when the argument of the \( Z \)-function takes a value of 1 at the KEW separation distance \( x = x_m \) given by (6). The decoupling criteria is given by

\[
k_\parallel \rho_1 \geq \frac{18\beta_1^3\eta + \beta_1^4}{4(\eta + \beta_1)}.
\]  

(7)

We approximate \( \text{Im}k_\perp(x) \) with a Gaussian. The peak of the profile can be found by expanding the \( Z \) function for small argument. We impose \( \zeta \ll \beta_1, \eta \), but we do not order \( \zeta \) with respect to \( \delta \). This gives

\[
\text{Re}k_\perp \rho_1 \simeq \sqrt{\frac{4}{3}\beta_1},
\]  

(8)
\[
\text{Im} k_\perp \rho_1 \simeq \frac{\pi k_{I0}(9\eta^2 + 24\beta_1 \eta + 16\beta_1^2)}{(9\eta^2 + 48\beta_1 \eta + 64\beta_1^2)\zeta^2 + \sqrt{\pi \delta}(12\eta + 32\beta_1)\zeta + 24\sqrt{\pi \beta_1 \eta + 9\pi \eta^2 + 16\pi \beta_1^2}}(9)
\]

where
\[
k_{I0} \simeq \alpha \left( \frac{4\beta_1 \delta}{9\eta + 12\beta_1} + \frac{192\eta + 64\beta_1 \beta_1^4}{243\eta + 324\beta_1 \delta} \right), \quad (10)
\]

and
\[
\delta = 2\sqrt{\frac{2}{\pi}} k_{\parallel} \rho_1, \quad \alpha = \frac{1}{2\text{Re} k_{\perp} \rho_1}.
\]

The first term in (10) is due to ion cyclotron damping and the second term is the remnant coupling to the KEW. The maximum of (9) is \(k_{I0}\) and it occurs at
\[
\zeta_{\text{peak}} = \frac{-2\sqrt{\pi \delta}}{3\eta + 8\beta_1}.
\]

The argument of the Gaussian is determined by matching to the tail of the damping profile. Expanding in the asymptotic regime of the Z-function gives for \(\eta = 0\)
\[
\text{Im} k_\perp \rho_1 \simeq \alpha \frac{\pi}{3} \frac{\zeta^2}{\delta^3} e^{-\zeta^2}, \quad (12)
\]

and for \(\eta > 2\beta_1\) the result is
\[
\text{Im} k_\perp \rho_1 \simeq \alpha \frac{\pi}{27} \left( \frac{3\eta + 2\beta_1}{\eta + 2\beta_1} \right) \frac{\beta_1^2}{\eta} \frac{\zeta^2}{\delta^3} e^{-\zeta^2/4}. \quad (13)
\]

In a two ion-species plasma the damping is due predominantly to the minority. This is manifested by the factor of 1/4 in the argument of the exponential in (13) which is a consequence of the fact that the Z-function argument of the minority is a factor of two less than that of the majority. With these considerations we let \(\text{Im} k_\perp = k_{I0} e^{-\zeta^2}\) for \(\eta = 0\), and \(\text{Im} k_\perp = k_{I0} e^{-\zeta^2/4}\) for \(\eta > 2\beta_1\). Using (7) and (10) in (2) gives
\[
\mu = \frac{\pi}{16} \sqrt{\frac{3}{\beta_1}} \frac{R_0 \delta}{\rho_1} \left( \frac{\delta}{3} + \frac{16\beta_1^4}{81\delta} \right), \quad \text{for } \eta = 0, \quad (14)
\]

and
\[
\mu = \frac{\pi}{8} \sqrt{\frac{3}{\beta_1}} \frac{R_0 \delta}{\rho_1} \left( \frac{4\beta_1 \delta}{9\eta + 12\beta_1} + \frac{192\eta + 64\beta_1 \beta_1^4}{243\eta + 324\beta_1 \delta} \right), \quad \text{for } \eta > 2\beta_1. \quad (15)
\]

The fraction of the power dissipated given by (3) for low field incidence is plotted in fig. 9 for the parameters of fig. 6 and compared to the numerical solution of the fourth order
differential equation[1]. It is difficult to make a direct comparison since (3) only gives
the damping on the FAW for a single pass but any part of the FAW that is reflected
is also damped. In order to compensate for this the values for the reflection coefficients
from the numerical results were used to compute the damping on the reflected wave using
\[ P_{d_{\text{ref}}} = P_d R / (1 - P_d), \]
where \( R \) is the reflection coefficient. The reflection corrected
power dissipated fits the numerical results quite well. The maximum \( k_\parallel \) for which the
geometrical optics approximation still holds is indicated on the graph. In terms of the
dispersion relation this is where the damping profile begins to infringe on region 3. The
FAW \( R_{\text{en}} \) varies rapidly here so geometrical optics is not applicable. Deciding on when
the damping profile enters region 3 is somewhat arbitrary. The criterion chosen is to let
the exponent in (12) and (13) take a value of 4 at the low field edge of region 3 for \( \eta = 0 \)
and at the low field edge of region 2 for \( \eta > 2 \beta_1 \). Using the results from Appendix 1 we obtain

\[ k_\parallel \rho_1 \lesssim \frac{2 - \sqrt{3}}{2\sqrt{2}} \beta_1, \quad \text{for } \eta = 0, \]  
\[ k_\parallel \rho_1 \lesssim \frac{1}{4\sqrt{2}} \eta, \quad \text{for } \eta > 2 \beta_1. \]  

ii) Large \( k_\parallel \)

In this regime we consider the case where \( \beta_1, \eta < \delta << 1 \). In section II. B we found
that g.o. is applicable in the large \( k_\parallel \) regime when the FAW and the IBW are decoupled.
The criteria for decoupling is chosen to be satisfied when the peak of the damping region
reaches the high field side of region 3. The peak of the profile is approximately located
at the position where the \( Z \)-function argument is 1. Using the results of Appendix 1 we obtain

\[ k_\parallel \rho_1 \gtrsim \frac{2 + \sqrt{3}}{\rho_1} \beta_1, \quad \text{for } \eta = 0, \]  
\[ k_\parallel \gtrsim \frac{1}{4\sqrt{2}\rho_1} \left( 3\eta + 4\beta_1 + 2\sqrt{3}(\beta_1\eta + \beta_1^2) \right), \quad \text{for } \eta > 2 \beta_1. \]

These criteria require a larger \( k_\parallel \) for decoupling than the \( k_\parallel \) required by previous criteria[7].
Full decoupling is considered to be achieved only when the fast wave is varying slowly
enough for geometrical optics to apply. The decoupling is well described by allowing the peak damping to coincide with the high field side of region 3. However, the approximate dispersion relation from where these criteria were calculated ignore $x/R_0$ with respect to unity. The error arising from this approximation leads to an enlargement of region 3 towards the high field side[11]. This discrepancy in (17b) begins to become apparent for $\eta$ larger than 0.03. To account for this a new criterion is used to replace (17b) for $\eta > 0.03$. This criterion is satisfied when the peak damping reaches the center of region 3. This then yields

$$k_{\parallel 1} = \frac{1}{\sqrt{2}} \left( \frac{3}{2} \eta + \beta_1 \right), \quad (17c)$$

which is similar to that in [7]. The criteria (17a), (17b) and (17c) should only be used as rough guidelines to when the expressions are valid.

Proceeding with a similar method as before the profile of the imaginary part of the FAW is found. For small arguments of the Z function

$$\text{Re} \, k_{\perp} \simeq \frac{256\beta_1^2 - 32\pi\delta^2\beta_1 - 3\pi^2\delta^4}{128\beta_1 + 24\pi\delta^2}, \quad (18)$$

$$\text{Im} \, k_{\perp} \simeq \frac{4\pi(\kappa_1 + \kappa_2)\delta^3}{2\sqrt{\pi}(16\beta_1 + 3\eta)\zeta + 4\pi\delta^3}, \quad (19)$$

where $\kappa_1$ and $\kappa_2$ are large expressions and given in full in Appendix 2. If $\delta^2/\beta_1$ is small the expressions can be expanded giving $\kappa_1 \simeq 2\beta_1^2/\delta + O(\delta^2/\beta_1)$ and $\kappa_2 \simeq \eta\beta_1/\delta + O(\delta^2/\beta_1)$. These expressions lose validity when $k_{\parallel}$ is so large that $\delta^2$ becomes larger than $\beta_1$. At such large $k_{\parallel}$, the topology of the dispersion relation changes and $k_{\parallel}$ begins to dominate $k_{\perp}$ invalidating the approximation.

In the large $k_{\parallel}$ regime the peak of the profile does not lie in the small $\zeta$ regime. It is located at $x = -L$ where the length scale $L$ is of the order max$(\beta_1, \eta)$. However, we find using (19) that

$$\left| \frac{\text{Im} k'_{\perp}}{\text{Im} k_{\perp}} \right|_{z=0} \simeq \frac{2(16\beta_1 + 3\eta)}{\pi\delta^2} \max(\beta_1, \eta) << 1. \quad (20)$$
The $\text{Im} k_\perp$ profile is very flat and slowly varying so its $x = 0$ value can be substituted for the actually peak value. For large $\zeta$ we obtain

$$\text{Im} k_\perp \approx \alpha \beta_1 \eta e^{-\zeta^2/\delta^2} + 2\beta_2^2 e^{-\zeta^2}. \quad (21)$$

The profile is modeled with

$$\text{Im} k_\perp \approx \alpha(\kappa_1 e^{-\zeta^2/\delta^2} + \kappa_2 e^{-\zeta^2}). \quad (22)$$

From this we obtain

$$\mu = \pi R_0 \alpha \delta \left( \frac{\kappa_1}{2} + \frac{\kappa_2}{4} \right). \quad (23)$$

In most cases $\delta^2/\beta_1$ is not very small so the full expression for $\kappa_1$ and $\kappa_2$ should be used in order to get accurate numerical results. This is applicable to both single and double ion species plasmas. $\kappa_1 = 0$ for a single species case. The fraction of the power dissipated for low field incidence is plotted for large $k_\parallel$ and compared to fourth order computations[1] in figs. 9 and 10. The theory compares very well with the numerical results for large $k_\parallel$.

**IV. CONCLUSION**

We have derived the conditions under which the damping along the FAW can be obtained by using the geometrical optics approximations. Furthermore, we have also derived an expression for the damping. There are basically two distinct regimes for the validity of g.o. and these regimes are determined by the parameters describing the plasma. We distinguish these regimes conveniently in the $k_\parallel$-space. One regime is for small $k_\parallel$'s where the mode-conversion region is distinctly separated from the cyclotron resonance layer. The other regime is for larger $k_\parallel$'s where the FAW and the IBW are essentially decoupled. The two regimes are separated by a narrow range of $k_\parallel$'s where mode-conversion plays an important role. The division of the ranges in $k_\parallel$-space is important because, essentially, the $k_\parallel$'s of the FAW propagating in the plasma are determined by the antenna exciting the waves at the edge. Thus, knowing the domain of high effective damping, one can design the appropriate antennas to insure maximum transfer of energy from the FAW to the plasma particles. Moreover, given the formulas that we have derived, it is easy to determine the optimum plasma conditions required to insure maximum possible damping.

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APPENDIX 1.

The dispersion relation given by (5) without the KEW is solved. For $\eta > 2\beta_1$ the dispersion plot is shown in fig. 5, and is the traditional ion ion dispersion relation. However for $\eta < 2\beta$ the dispersion plot looks more like a modified single species plot. See figs. 3 and 4. Consider first $\eta < 2\beta_1$. The plot can be divided into 4 regions defined by:

1. $-\frac{\eta}{2} R_0 \leq x,$
2. $x_- \leq x \leq -\frac{\eta}{2} R_0,$
3. $x_+ \leq x \leq x_-,$
4. $x \leq x_+,$

where

$$x_\pm = -\left(\frac{43 + 3972 + (247 + 323 1 + 972 + 123 1 + 41^2)}{4}\right) R_0.$$ 

Define also the following

$$Q = \frac{\sqrt{16x^2 + (24\eta + 32\beta_1)x + 9\eta^2 + 12\beta_1\eta + 4\beta_1^2}}{3\rho_1^2},$$  
$$R = \frac{2\beta_1 - 3\eta - 4x}{3\rho_1^2},$$  
$$A = \frac{\sqrt{R + \sqrt{R^2 + Q^2}}}{\sqrt{2}},$$  
$$B = \frac{\sqrt{R - \sqrt{R^2 + Q^2}}}{\sqrt{2}}.$$ 

$x_\pm$ define the high and low field sides of the coupling region. The ion ion cutoff $x/R_0 = -\eta/2$ is carried over from the cold dispersion relation. The roots region by region are given in the table below.

<table>
<thead>
<tr>
<th>Region</th>
<th>Re $k_{FW}$</th>
<th>Im $k_{FW}$</th>
<th>Re $k_{IBW}$</th>
<th>Im $k_{IBW}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\sqrt{Q + R}$</td>
<td>0</td>
<td>0</td>
<td>$\sqrt{Q - R}$</td>
</tr>
<tr>
<td>2</td>
<td>$\sqrt{Q + R}$</td>
<td>0</td>
<td>$\sqrt{R - Q}$</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>$A$</td>
<td>$Q/(2A)$</td>
<td>$A$</td>
<td>$-Q/(2A)$</td>
</tr>
<tr>
<td>4</td>
<td>$\sqrt{R - Q}$</td>
<td>0</td>
<td>$\sqrt{R + Q}$</td>
<td>0</td>
</tr>
</tbody>
</table>
For $\eta > 2\beta_1$ define another parameter

$$x_0 = (\frac{-3}{4} \eta + \frac{\beta}{2})R_0.$$  

$R$ is 0 at $x = x_0$. The different regions are given by:

(1) $\frac{-\eta}{2} R_0 \leq x$,

(2) $x_- \leq x \leq \frac{-\eta}{2} R_0$,

(3) $x_0 \leq x < x_-$,

(4) $x_+ \leq x \leq x_0$,

(5) $x \leq x_+.$

The roots by region are in the table below.

<table>
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<tr>
<th>Region</th>
<th>Re $k_{FW}$</th>
<th>Im $k_{FW}$</th>
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</tr>
</thead>
<tbody>
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<td>1</td>
<td>$\sqrt{Q + R}$</td>
<td>0</td>
<td>0</td>
<td>$\sqrt{Q - R}$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$\sqrt{-(Q + R)}$</td>
<td>0</td>
<td>$\sqrt{Q - R}$</td>
</tr>
<tr>
<td>3</td>
<td>$B$</td>
<td>$Q/(2B)$</td>
<td>$B$</td>
<td>$-Q/(2B)$</td>
</tr>
<tr>
<td>4</td>
<td>$A$</td>
<td>$Q/(2A)$</td>
<td>$A$</td>
<td>$-Q/(2A)$</td>
</tr>
<tr>
<td>5</td>
<td>$\sqrt{R - Q}$</td>
<td>0</td>
<td>$\sqrt{R + Q}$</td>
<td>0</td>
</tr>
</tbody>
</table>
APPENDIX 2.

The imaginary part at $x = 0$ for $\delta^2 < \beta_1, \eta < \delta$ is given by

$$\text{Im} \ k_\perp = \alpha(\kappa_1 + \kappa_2),$$

where

$$\kappa_1 = \frac{(216\pi^3 \delta^8 - 1152\pi^2 \beta_1 \delta^4 - 6144\pi \beta_1^2 \delta^2 + 32768\beta_1^3 \eta \beta_1)}{216\pi^3 \delta^7 + 3456\pi^2 \beta_1 \delta^5 + 18432\pi \beta_1^2 \delta^3 + 32768\beta_1^3 \delta}
\approx \frac{\beta_1 \eta}{\delta} \left[ 1 - \frac{6}{16} y + \frac{9}{64} y^2 - \frac{81}{2048} y^3 + \frac{81}{8192} y^4 - \cdots \right],$$

$$\kappa_2 = \frac{-27\pi^4 \beta_1 \delta^8 + 4608\pi^2 \beta_1^3 \delta^4 - 32768\pi \beta_1^4 \delta^2 + 65536\beta_1^5}{216\pi^3 \delta^7 + 3456\pi^2 \beta_1 \delta^5 + 18432\pi \beta_1^2 \delta^3 + 32768\beta_1^3 \delta}
\approx \frac{2\beta_1^2}{\delta} \left[ 1 - \frac{17}{16} y + \frac{9}{16} y^2 - \frac{27}{128} y^3 + \frac{135}{2048} y^4 - \cdots \right],$$

and

$$y = \frac{\pi \delta^2}{\beta_1}.$$
REFERENCES

2. M. Brambilla, Private communication.
FIGURES

1. The dispersion relation near the second harmonic resonance where $x$ has been normalized to the major radius and $k_{\perp}$ to the majority Larmor radius. The plasma parameters are: $n_1 = 8 \times 10^{14} \text{cm}^{-3}$, $T = 2 \text{KeV}$ and $B = 2.4 \text{T}$.

2. Same as figure 1. except very near to the second harmonic.

3. $n_{\perp}$ vs. $x$ for $\eta = 0$ and plasma parameters: $R_0 = 1.75 \text{m}$, $n_1 = 6 \times 10^{14} \text{cm}^{-3}$, $T = 7 \text{KeV}$, and $B = 10 \text{T}$ giving $\beta_1 = 0.02$. The real parts are given by the solid lines and the imaginary parts by the dashed lines. The figure is divided into 4 regions. The FAW and IBW are labeled wherever it is simple to do so.

4. Same as figure 3. except for $\eta = 0.01$.

5. Same as figure 3. except for $\eta = 0.1$.

6. The real and imaginary parts of $n_{\perp}$ vs. $x$ for PLT type parameters: $R_0 = 3.0 \text{m}$, $n_1 = 4 \times 10^{13} \text{cm}^{-3}$, $T = 5 \text{KeV}$, and $B = 4 \text{T}$. $\eta = 0.05$ and $n_{||} = 3$. (a) The real part of $n_{\perp}$. (b) The imaginary part of $n_{\perp}$.

7. Same as figure 6. except with $n_{||} = 9$.

8. Same as figure 6. except with $n_{||} = 15$.

9. $P_d$ vs $k_{||}$ for the same parameters as figure 6. Present theory; dashed line. Numerical results[1]; solid line. The lines are drawn within their regions of validity.

10. Same as figure 9. except for $\eta = 0$. For small $k_{||}$ the region of validity is indistinguishable from zero.
Figure 1
Figure 2
Figure 3
figure 5
Figure 6(a)
figure 8(a)
Figure 9