Alpha Particle Losses From Toroidicity
Induced Alfvén Eigenmodes, Part I:
Phase-Space Topology of Energetic Particle
Orbits in Tokamak Plasma

C. T. Hsu and D. J. Sigmar

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Plasma Fusion Center
Massachusetts Institute of Technology
Cambridge, MA 02139 USA

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ALPHA PARTICLE LOSSES FROM TOROIDICITY

INDUCED ALFVÉN EIGENMODES

Part I: Phase-Space Topology of Energetic Particle Orbits in Tokamak Plasma

C. T. HSU and D. J. SIGMAR

Massachusetts Institute of Technology, Plasma Fusion Center, 167 Albany Street, NW16-260, Cambridge, MA 02139 USA, Tel: (617) 253-2470

Abstract

Phase space topology of energetic particles in tokamak plasma with arbitrary shape of cross section is studied based upon the guiding center theory. Important phase space boundaries such as prompt loss boundary, trapped passing boundary, and other boundaries between classes of nonstandard orbits (e.g. pinch and stagnation orbits) are studied. This phase space topology information is applied to the study of anomalous phase space diffusion due to finite amplitude Alfvén wave fluctuations of energetic particles. The separatrix between trapped and circulating particles contributes dominantly to the losses.

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I. Introduction

The presence of large-banana-width fast ions in present and future tokamak devices is becoming increasingly common as a result of auxiliary heating and in the near future, due to fusion reactions (e.g. TFTR\textsuperscript{1} and JET\textsuperscript{2}). The significance of the existence of these particles is two-fold: (i) if confined, their energy can be transferred to the background plasma; (ii) they may have destabilizing effects on the background plasma instabilities such as high-$n$ ballooning,\textsuperscript{3} fishbone,\textsuperscript{4} and shear-Alfvén instabilities.\textsuperscript{5} In particular, the low-$n$ TAE mode (Toroidicity Induced Alfvén Eigenmode) has been found to be strongly destabilized in the presence of energetic particles, both analytically\textsuperscript{5,6} and experimentally.\textsuperscript{7} Corresponding $\alpha$-particles losses due to TAE-modes have been found in a Hamiltonian guiding center Monte Carlo study.\textsuperscript{8}

The fundamental difficulty in studying the loss process of energetic particles is due to the large banana width which is a fundamental step size of the radial diffusion. When this step size becomes comparable with the radial scale length, the radial diffusion becomes undefinable. However, there exist several constants of motion (COMs) which change only slightly under the influence of small disturbances such as magnetic field ripple, electromagnetic waves, and collisions. Therefore, in a properly chosen COM phase space, diffusion analysis is possible.\textsuperscript{9}

There have been several previous analyses of various COMs suitable for energetic particles. However, COMs such as the adiabatic invariant $\oint v_||dS$, action $\oint P_\phi d\theta$, and the maximum radial position of an orbit,\textsuperscript{10} $\psi_m$, all exhibit strong discontinuity near the trapped-passing boundary. This means a large step size of these COMs occurs when particles are crossing this boundary. Nonetheless, in Refs. 8, 11 it has been shown that the crossing of this transition layer plays an important role during the transport process.

In an axisymmetric system, the particle guiding center trajectory in $(\psi_p, \theta)$ configuration space can be described through first order in gyroradius\textsuperscript{12} by the COMs $(P_\varphi, \mu, E; \sigma)$ via the conservation law of the toroidal angular momentum $P_\varphi$. Here, $E$ is the particle energy, $\mu$ the magnetic moment, and $\sigma \equiv \frac{v_||}{|v_||}$. $\psi_p$ is the poloidal flux variable and $\theta$ a
poloidal angle. This set of COMs is found appropriate for studying the transport process of energetic particles not only because of their continuity in phase space but also because they are natural Hamiltonian variables compatible with the Hamiltonian guiding center particle code (ORBIT)\textsuperscript{13} used in Ref. 8.

Before a systematic scheme of utilizing the COMs for studying transport can be constructed, various important boundaries such as the loss-orbit boundary, and trapped-passing boundary in the phase space of these COMs have to be thoroughly analyzed. In addition, the existence of non-standard orbits especially near the trapped-passing boundary can importantly modify the transport behavior; and the more energetic the particles are, the more important is the population of these non-standard orbits. The main task of this work is thus to study the phase space topology of energetic particles in the \((P_\varphi, \lambda)\) plane \((\lambda \equiv \mu B_0 / E)\) with fixed energy, and to study the corresponding non-standard orbits and their regions of occupancy in phase space.

Finally, in order to give an example, the analysis of phase space topology is utilized, in collaboration with numerical studies using the Monte Carlo ORBIT code, to briefly study the alpha particle transport under the influences of TAE modes. For a detailed study of the \(\alpha\)-particle response due to TAE modes, we refer the reader to Ref. 14 which represents Part II of the present investigation.

II. Orbit-Equilibrium Plot in the \((h, \psi_p)\) Plane and Analysis of Orbit Topology

We start with the toroidal momentum equation normalized by \(m\Omega_0 R_0^2\)

\[
\dot{P}_\varphi = g\rho_0 \sigma \sqrt{(1 - \phi)h^2 - \lambda h - \psi_p}.
\] (1)

Explicitly, time scale is normalized by \(\Omega_0\) and length scale by \(R_0\) throughout the paper. Here \(m\) is particle mass, \(\Omega_0 = \frac{ZeB_0}{mc}\) the gyrofrequency, \(R_0\) the major radius of the magnetic axis,

\[
\dot{P}_\varphi \equiv \frac{P_\varphi}{m\Omega_0 R_0^2}, \quad \psi_p \equiv -\frac{A_\varphi}{B_0 R_0}
\]
(note that $\psi_p = 0$ at the magnetic axis and $A_\varphi \equiv A \cdot R^2 \nabla \varphi$ is the toroidal component of the vector potential),

$$h = h(\psi_p, \theta) \equiv \frac{B_0}{B}, \quad \rho_0 \equiv \sqrt{\frac{2E}{m}} / \Omega_0 R_0, \quad \phi(\psi_p) \equiv \frac{Z e \Phi}{E}$$

the normalized electrostatic potential, and $g(\psi_p) \equiv \frac{B R^2 \nabla \varphi}{B_0 R_0}$ is the normalized toroidal magnetic field. (In a low beta equilibrium, $g =$constant). Also, the magnetic coordinates $(\psi_p, \theta, \varphi)$ for general toroidal equilibrium as given in White and Chance$^{13}$ are adopted. For further details, we refer the reader to Ref. 13. Note that the caret on $P_\varphi$ will be omitted henceforth.

It is obvious from Eq. (1) that a particle orbit can be most conveniently studied in the $(h, \psi_p)$ plane.$^{10}$ For given $(P_\varphi, \lambda, \rho_0)$, a particle orbit appears to be an “upward” quasi-hyperbolic curve in the $(h, \psi_p)$ plane, due to Eq. (1), (cf. Fig. 1). For $g =$ constant this curve will be exactly hyperbolic. Note that the other solution of Eq. (1), which has $h \leq 0$, is unphysical. The $h - \psi_p$ plot was first introduced and utilized to study orbit topology by Rome and Peng$^{10}$ in which some of the orbit characteristics discussed in this paper have already been studied. Their choice of COM’s included the maximum $\psi_p$ of an orbit as the third COM. However, to be self-contained, we briefly discuss the $h - \psi_p$ plot in this section.

It is important to note that the extremum point on the curve occurs at

$$h = \frac{\lambda}{(1 - \phi(\psi_p = -P_\varphi))} \quad \text{and} \quad \psi_p = -P_\varphi.$$

The physical significance of this point, at which the particle experiences minimum $h$ (i.e. maximum $B$) is that the parallel velocity vanishes at this point – the banana tip. This also implies that in the $(\psi_p, \theta)$ configuration space, the banana tip occurs at the point where the particle trajectory is tangential with the constant $B$ curve. In Fig. 1, showing the $(h, \psi_p)$ plane, the trajectory on the left hand side of the banana tip represents counter-going trajectories and on the right hand side co-going trajectories.

To complete the description of the particle guiding center orbit in the $(h, \psi_p)$ plane, one needs to also impose the plasma equilibrium curves in the same $(h, \psi_p)$ plane, as given
in Fig. 2. The plasma domain is confined inside curves $h_{\text{max}}(\psi_p)$, $h_{\text{min}}(\psi_p)$, and $\psi_{pb}$ where $h_{\text{max}}(\psi_p)$ and $h_{\text{min}}(\psi_p)$ correspond to the maximum and minimum values of $h \equiv \frac{B_0}{B}$ on a given flux surface labeled by $\psi_p$, respectively, and $\psi_{pb}$ corresponds to $\psi_p$ at the plasma boundary. The orbit curve on the $(h, \psi_p)$ plane is physically meaningful only inside the plasma domain set by the equilibrium curves.

Another important physical significance of $h_{\text{min}}(\psi_p)$ and $h_{\text{max}}(\psi_p)$ can be understood from the equation of radial excursion, which is basically due to $\nabla B$ drift, i.e.\textsuperscript{13}

$$\psi_p = \frac{g \rho_0^2}{Dh^2} \left( (1 - \phi)h - \frac{\lambda}{2} \right) \frac{\partial h}{\partial \theta}. \quad (2)$$

Here

$$D \equiv \rho_\parallel [g' I' - Ig'] + I + gq,$$

$$\rho_\parallel = \rho_0 \sigma \sqrt{(1 - \phi)h^2 - \lambda h}.$$ 

$I$ is a measure of poloidal magnetic field, $q$ is the safety factor, prime denotes derivative with respect to $\psi_p$, and $\rho_\parallel \equiv v_\parallel / \Omega$. One notices that extremum values of $\psi_p$ on the orbit correspond to the points where $\frac{\partial h}{\partial \theta} \bigg|_{\psi_p} = 0$, at $\theta = \theta_c(\psi_p)$. In a typical tokamak equilibrium, $\theta_c = (0, \pi)$; however, in a bean-shaped tokamak equilibrium there may exist one more poloidal angle $0 < \theta_c < \pi$ at which $\frac{\partial h}{\partial \theta} \bigg|_{\psi_p} = 0$. This extra $\theta_c$ corresponds to an extra magnetic well centered at $\theta = \pi$, and thus leads to an inverse trapped orbit\textsuperscript{13} centered at $\theta = \pi$. This bean-shaped equilibrium, although interesting, will not be considered in the rest of this paper, for simplicity.

Many important properties of the particle orbit can be obtained by using the orbit-equilibrium plot. For instance, the particle is trapped if the tip ($h = \lambda/(1 - \phi), \psi_p = -P_\psi$) is inside of the plasma domain; and the orbit is a prompt loss orbit if the orbit curve intersects with $\psi_p = \psi_{pb}$ in between points $U, V$ in Fig. 2. In the rest of this section, we will study some interesting properties of orbit topology using the orbit-equilibrium plot. More details will be discussed in the next section.
A. Inner Counter-Passing and Magnetic Axis Encircling Trapped Orbits

A given set of \((E, \lambda, P_\phi)\), while corresponding to one orbit curve in the \((h, \psi_p)\) plane, may correspond to two distinct orbits in \((\psi_p, \theta)\) configuration space if the orbit curve is intersected by the equilibrium curve \(h = h_{\text{min}}(\psi_p)\). In this case, the inner portion (which has smaller \(\psi_p\)) of the orbit curve corresponds to the inner counter-passing (ICP) orbit, while the outer one may correspond to either a co-passing or a trapped orbit. As shown in Fig. 3a, part of the counter-going portion of the orbit curve is intersected by \(h_{\text{min}}(\psi_p)\) outside the plasma domain, while the banana tip point is still inside the plasma domain. Hence, one finds one ICP orbit and one magnetic axis encircling (henceforth “encircling”) or “encircling”-trapped (ET) orbit (cf. Fig. 3b). Here, passing refers to an orbit whose \(v||\) never vanishes (i.e., enclosing the tokamak axis of symmetry), while encircling refers to an orbit whose \(\frac{d\theta}{dt}\) never vanishes (i.e., enclosing the magnetic axis). It will be shown later that ICP and ET orbits are very important in the transport process of energetic particles since they can change into each other quite easily via a small disturbance and introduce large radial excursions.

B. Stagnation Point

It is interesting to consider the limiting case when the orbit curve becomes tangential with the extremum equilibrium curve \(h_m(\psi_p) \equiv h(\psi_p, \theta_c)\). Using Eq. (1), this limiting case yields

\[
\frac{dh_m}{d\psi_p} = \frac{(P_\phi + \psi_p)(1 - g'/g(P_\phi + \psi_p))}{g^2 \rho_0^2((1 - \phi)h_m - \lambda/2)}.
\]

By eliminating \(P_\phi\) using Eq. (1), Eq. (3a) can also be written as

\[
\frac{dh_m}{d\psi_p} = \frac{\sigma \sqrt{(1 - \phi)h_m^2 - \lambda h_m - \rho_0 g'((1 - \phi)h_m^2 - \lambda h_m)}}{g \rho_0((1 - \phi)h_m - \lambda/2)}.
\]

Before further studying the physical significance of Eq. (3b), it is useful to write down the equation of \(\theta\) evolution

\[
\dot{\theta} = -\frac{\rho_0^2}{2Dh^2g} \frac{\partial}{\partial \psi_p} \left( g^2 \left[ (1 - \phi)h^2 - \lambda h \right] \right) + \frac{\rho_0}{Dh^2} \sigma \sqrt{(1 - \phi)h^2 - \lambda h},
\]

(4)
where the first term and the second term on the RHS of Eq. (4) are due to the $\theta$-component of $v_d$ and $v_\parallel$, respectively.

It is then straightforward to see that the solution of Eq. (3b) satisfies $\dot{\theta} = \dot{\psi}_p = 0$, according to Eqs. (4) and (2). That is, for given $E, \lambda$, and magnetic field equilibrium, Eq. (3b) leads to solution $\psi_p = \psi_{ps}$, and $(\psi_{ps}, \theta_c)$ corresponds to the "stagnation points" at which the orbit projection in $(\psi_p, \theta)$ configuration space remains at a point while $v_\parallel = v_\varphi \neq 0$. One also notices that the vanishing of $\dot{\theta}$ is due to the cancelation between $v_\parallel \theta$ and $v_d \theta$, rather than the vanishing of $v_\parallel$, whether the particle is energetic or not. Recall that the value of $\theta_c$ is $\theta$ corresponding to $h_m(\psi_p)$ used in Eq. (3b) (also see the discussion near Eq. (2)). This implies that in a bean shaped plasma, there may exist a stagnation point which does not lie on the equatorial plane.

From a direct inspection of the orbit equilibrium plot in the $(h, \psi_p)$ plane, one notices that an orbit curve tangential with $h_{max}(\psi_p)/(h_{min}(\psi_p))$ is co-going (counter-going), due to the fact that $\frac{dh_{max}}{d\psi_p} > 0$, $\frac{dh_{min}}{d\psi_p} < 0$. For the co-going case, with given $E$, each $\lambda$ leads to only one stagnation point $\psi_{ps}$. For the counter-going case, however, each $\lambda$ may lead to two $\psi_{ps}$ corresponding to two types of stagnation points. For type I (type II), the orbit curve is above (beneath) the $h_{min}(\psi_p)$ curve near the stagnation point. It is also obvious from the inspection that $\psi_{ps}$ of type II corresponds to an isolated stagnation (IS) orbit which, with a small change of $(E, P_\varphi, \lambda)$, will change into an orbit circling in the vicinity of the stagnation point.

On the other hand, the orbit curve containing a type I stagnation point refers to a "pinch" point, which acts like a separatrix and is not isolated. It is important to note that the so-called pinch orbit is actually not one orbit, but contains an ICP, an ET, and a stagnation orbit, where ICP and ET orbits will pinch at the stagnation point and stay there to become a stagnation orbit. A small change of $(E, P_\varphi, \lambda)$ can thus turn this stagnation orbit into an ICP orbit, an ET orbit, or a fat banana orbit. Hence, an IS point is an O-point while a pinch point is an X-point (also cf. section III.A and Fig. 4). Clearly, the co-going stagnation orbit is an IS orbit.

In summary, it has been found in this section that orbit and equilibrium curves in the
(h, ψp) plane can be most conveniently utilized to study the properties of particle orbits. One finds (1) an orbit curve is physically meaningful only inside the plasma domain defined by equilibrium; (2) a particle is trapped if its banana tip

\[
h = \lambda / \left(1 - \phi(\psi_p = -P_\phi)\right), \psi_p = -P_\phi
\]

falls inside the plasma domain; (3) there exists an inner counter-passing orbit (ICP) if part of the orbit curve is intersected by the \(h_{\text{min}}(\psi_p)\) curve outside the plasma domain; and (4) extremum values of \(\psi_p\) occur at the intersection of an orbit curve with the \(h_{\text{min}}(\psi_p)\) or \(h_{\text{max}}(\psi_p)\) curve; (5) the stagnation point occurs at the tangency point of the \(h_m(\psi_p)\) curve and the orbit curve.

In the next section, the information developed in this section will be used to determine various physical boundaries in the phase space (λ, P_φ) plane, for fixed energy. Also, more details about various nonstandard orbit topologies will be given. The formulation Eqs. (1)-(4) does not exclude the effects of an electrostatic potential \(\phi(\psi_p)\) or bean-shaped equilibrium, but in the numerical results shown in the figures these effects were omitted for simplicity. Also, the important effects of sheared radial electric field\(^{15}\) (i.e. finite \(\phi''(\psi_p)\)) to the orbit topology are not discussed.

III. Topology in (λ, P_φ) Phase Space

Clearly, the full phase space should be a three dimensional (3-D) domain containing coordinates \(E, \lambda, P_\phi\). However, by fixing the energy \(E\), the phase space projection in the (λ, P_φ) plane provides more transparent physics insight. In section III.A, various boundaries which divide the (λ, P_φ) phase space into regions of distinct orbit characteristics will be studied. The governing equations for these boundaries will be derived and then solved numerically using a CIT (Compact Ignition Tokamak)\(^{16}\) equilibrium as an example. In addition, during the derivation of these boundaries, many types of nonstandard orbits will be discussed. By setting the particle energy \(E = 3.5\) MeV, α particle results are obtained and plotted in Fig. 5.
To make the illustration of phase space regions occupied by these nonstandard orbits more transparent, an enhanced case in which the particle energy is set to be \( E = 14.7 \text{ MeV} \) (e.g. a problem from a D-\(^3\)He fusion reaction) will be studied and plotted in Fig. 6 using the same CIT equilibrium. We note that Fig. 5 and 6 are topologically similar. Actually, except for non-standard equilibria or strong effects\(^{15}\) of the electric sheath potential \( \phi \), this topology will be typical. Furthermore, the phase space topology of orbits on a local flux surface labeled by fixed \( \psi_p \) will be studied in subsection III.B. The discussions of global and local phase space follows next.

A. Global Phase Space Topology

In this subsection, the phase space of all orbits inside the plasma region is studied. First, it is worth mentioning that although each orbit occupies only one phase space point, one phase space point in the global phase space may not uniquely correspond to only one orbit, as described in subsection II.A. Thus, the "trapped-passing boundary" type of description is not fully encompassing. Also, several kinds of non-standard orbits will be found and analyzed. Note that the figures of non-standard orbits are obtained numerically from the ORBIT code with \((E, \lambda, P_\phi)\) determined from Fig. 6 and the analysis given in this subsection.

We start with the following physically meaningful curves in the phase space as shown in Fig. 6 derived from an orbit-equilibrium plot in \((h, \psi_p)\). The corresponding non-standard orbits will be shown as well.

(i) Curve \( \text{AD}_1\text{G} \) (or \( \text{BD}_2\text{F} \)): due to orbits passing through \((h_m(\psi_{pb}), \psi_{pb})\), and thus defined by

\[
P_\phi = \sigma g(\psi_{pb})\rho_0 \sqrt{h_m(\psi_{pb})[(1 - \phi(\psi_{pb}))h_m(\psi_{pb}) - \lambda] - \psi_{pb}}, \tag{5}
\]

with \( h_m = h_{\text{max}} \) (or \( h_{\text{min}} \)), and \( \sigma = +1 \) (or \(-1\)).

(ii) Curve \( \text{D}_1\text{J} \) (or \( \text{D}_2\text{J} \)) due to orbits whose tips lie on \( h_m(\psi_p) \) (or equivalently, tips at \( \theta = \theta_c \)), and thus defined by

\[
\lambda = (1 - \phi(\psi_p = -P_\phi))h_m(\psi_p = -P_\phi) \tag{6}
\]
with \( h_m = h_{\text{max}} \) (or \( h_{\text{min}} \)). Phase space points on this curve correspond to D-orbits satisfying \( v_\parallel = \dot{\theta} = 0 \) at the tips. Recall that typically \( h = h_{\text{max}} \) (or \( h_{\text{min}} \)) at \( \theta_c = 0 \) (or \( \pi \)). Hence, a D-orbit on curve \( D_1J \) (or \( D_2J \)) has its tip at \( \theta_c = 0 \) (or \( \pi \)), and is thus non-axis encircling (axis encircling) [see Fig. 7a (7b)]. Note that both kinds of D-orbits are co-going.

(iii) Curve EK (or CNM): due to orbit curves tangential with \( h_m(\psi_p) \), and thus defined by Eq. (3a) and (3b) with \( h_m = h_{\text{max}} \) (or \( h_{\text{min}} \)). Such orbits contain the stagnation point. Curve EK describes co-passing stagnation orbits [see Fig. 4a], curve CN describes counter-passing stagnation orbits [see Fig. 4b], and curve NM describes pinch orbits [see Fig. 4c] as explained in subsection II.B.

(iv) Curve HQJL: due to orbits passing through the magnetic axis, and thus defined by

\[
P_\varphi = \sigma g(0)\rho_0 \sqrt{h_0[(1 - \phi(0))h_0 - \lambda]},
\]

where \( h_0 \equiv h(\psi_p = 0) \). Phase space points of co-passing, counter-passing, and trapped axis-orbits lie on curves JL, JQ, and QJ, respectively [see Figs. 6 and 8]. One distinct feature is that the banana width of an axis-orbit is non-zero which seemingly contradicts simple extrapolation from standard neoclassical theory. Indeed, by assuming large-aspect-ratio circular geometry and neglecting effects of \( \phi \), Eqs. (1) and (7) yield that the width of trapped and passing axis-orbits are \( (qR^2\rho_0)^{2/3} \) and \( q\rho_0 \), respectively. This will then strongly modify the radial scaling of transport coefficients near the magnetic axis. One important consequence is the existence of a nonzero parallel current near the axis\(^17\) which can be a candidate for the seed current\(^18\) necessary for a bootstrap current tokamak without external current-drive.

It is now straightforward to describe the phase space regions (as in Fig. 6) of various types of orbits, including the nonstandard orbits.

(1) Trapped region: bounded by curve \( D_1JD_2D_1 \). Also note that the trapped orbits beneath the pinch-orbit line (CQ) or the axis-orbit line (QJ) are encircling (Fig. 9a). Recall that “encircling” refers to magnetic axis encircling.
(2) Co-passing region: bounded by $D_1JD_2FKD_1$. Above lines $D_1J$ or $JL$, the orbits are non-encircling (Fig. 9b).

(3) Counter-passing region: bounded by $D_1AMNCD_1$. Beneath line $HJ$, the orbits are non-encircling (Fig. 9c).

It is clear that phase space is bounded by $AD_1EK$ outside which no orbit exists. Also, when $NM$ intersects with $D_2F$ (e.g., at point $R$ in Fig. 6) there is no orbit beneath MRF.

Furthermore, it is interesting to study the loss-orbit boundary — one of the most important boundaries in phase space. In Fig. 6, the boundary extends between points (i) PT for trapped particles, (ii) EP and TG for co-passing particles, and (iii) BD$_2$CS for counter-passing particles. Note that the line CS is particularly important because, across it, a well confined ICP orbit turns into a fat prompt-loss trapped-orbit. This can also be understood from the fact that the phase space points on line CS correspond to an X-type stagnation point as described in section II.B.

B. Local Phase Space Topology

The phase space studied in the last subsection includes all mono-energy particles in the whole plasma region. It is however equally interesting to study the topology of a phase space region representing particles at a local position in configuration space.

More explicitly, the local phase space corresponds to all $(E, \lambda, P_\varphi)$ satisfying Eq. (1) with fixed $(\psi_p, \theta)$. For each given energy, this leads to a downward parabolic curve centered at $P_\varphi = -\psi_p$. The local $(\lambda, P_\varphi)$ phase space domain corresponding to all particles on one flux surface $\psi_p$ thus lies between the two downward parabolic curves which correspond to $h_{\text{max}}$ and $h_{\text{min}}$.

Using the analysis given in Sec. II and III.A, it is straightforward to find the local phase space topology. In Fig. 10, a local phase space for $E = 14.7$ MeV $\alpha$-particles in a CIT equilibrium is shown. It is important to note that in local phase space, each point corresponds uniquely to one orbit. The phase space points on the right (left) of $P_\varphi = -\psi_p$
correspond to orbits which are co-going (counter-going) when intersecting with the flux surface \( \psi_p \).

The boundary between trapped and counter-passing (co-passing) is the pinch-orbit curve EF (axis encircling D-orbit curve GI). The region for encircling trapped orbits is bounded by FGIKF, and the region for non-encircling co-passing orbits lies above line HJO. There exists a co-passing stagnation orbit on line HJD and in between HJO. On the other hand, the existence of a counter-passing non-encircling or stagnation orbit in the local flux surface clearly prefers smaller \( \psi_p \) and larger energy. Typically, a co-passing orbit exists except for a surface very near the magnetic surface while counter-passing orbits exist only if the particle is energetic enough and lies very near the magnetic axis.

One important application of the local phase space is to carry out the flux surface average of some arbitrary moment \( \int dv F \), at \( \psi_p \), i.e.

\[
\langle \int dv F \rangle \equiv \frac{\int_{\psi_p} \psi_p d\theta d\phi J \int dv F}{\int_{\psi_p} d\theta d\phi J},
\]

where \( J \equiv |\nabla \phi \times \nabla \psi_p \cdot \nabla \theta|^{-1} \) is the Jacobian in \((\psi_p, \theta, \phi)\) space. With our normalization, \( J = (I + gg)h^2 \). For details, we refer the reader to Ref. 13.

For an up/down symmetric toroidally axisymmetric guiding center particle system, one can in general write

\[
F = F(E, \mu, \psi_p, \theta; \sigma)
\]

which can be rewritten as

\[
F = F(E, \lambda, \psi_p, P_\phi; \sigma_p).
\]

Here, \( \sigma_p \equiv \frac{\dot{\psi}_p}{|\dot{\psi}_p|} \) is a label of up/down asymmetry because

\[
\begin{align*}
\sigma_p &= 1 & \text{for} & & 0 < \theta < \pi \\
\sigma_p &= -1 & \text{for} & & -\pi < \theta < 0.
\end{align*}
\]

The transformation \((\psi_p, \theta) \rightarrow (\psi_p, P_\phi)\) requires the additional label \( \sigma_p \) because \( P_\phi = P_\phi(E, \lambda, \psi_p, h) \) and \( h \) can only represent the up/down symmetric part of the \( \theta \) dependence. Consequently, the local phase space shown in Fig. 10 only represents the upper plane, i.e.
$0 \leq \theta \leq \pi$. On the other hand, since each local phase space point is unique, the label $\sigma$ is not needed.

After lengthy but straightforward manipulation (see Appendix C), one arrives at

$$\langle \int d\nu F \rangle = \frac{\pi \Omega_0^3 R_0^3 (I + gq)}{V'} \int dE \sum_{\sigma_p} \int_{\psi_p} dP_\varphi d\lambda \frac{dP_\varphi d\lambda}{D|\psi_p|} F.$$  

(8c)

Here $\int_{\psi_p} dP_\varphi d\lambda$ integrates over the local ($\lambda, P_\varphi$) phase space as given in Fig. 10,

$$V' \equiv V'(\psi_p) \equiv \int_{\psi_p} d\theta d\varphi J,$$

and $D$ is defined after Eq. (2). Note that all the phase space values $E, P_\varphi, \lambda$ are normalized as in Sec. I. By letting $F = \dot{\psi}_p f$ to calculate the radial flux, it is remarkable to see immediately from Eq. (8c) that only the up/down asymmetric portion of the distribution function $f(\sigma_p)$ will generate a net radial flux.

It is also interesting to discuss the relevant bounce average with constant ($E, \lambda, P_\varphi$). Consider the drift kinetic equation

$$\dot{\psi}_p \frac{\partial f}{\partial \psi_p} \bigg|_{P_\varphi, \lambda, E} = C(f) + S$$  

(9a)

with $C(f)$ the collision operator and $S$ the (fusion) source term. The solubility condition is

$$\{C(f)\}_b + \{S\}_b = 0.$$  

(9b)

Here, the curly brackets describe the bounce average

$$\{F\}_b \equiv \sum_{\sigma_p} \int_{\psi_n}^{\psi_m} d\psi_p \frac{d\psi_p}{|\psi_p|} F$$  

(9c)

with integration along a constant ($P_\varphi, E, \lambda$) trajectory, where $\psi_n \equiv \psi_n(E, P_\varphi \lambda)$ and $\psi_m \equiv \psi_m(E, P_\varphi \lambda)$ are the minimum and maximum $\psi_p$ of the orbit with a given ($E, P_\varphi \lambda$), respectively. Again it is straightforward to see that because the collision operator is up/down symmetric, an up/down asymmetric source term leads to an up/down asymmetric distribution function, which then leads to radial diffusion.
IV. Application to Energetic Particle Transport due to Alfvén Modes

The topic of Toroidal Alfvén Eigenmodes (TAE) destabilized by energetic particles has recently received intensive attention in fusion research. Energetic particles losses due to the TAE mode have been studied numerically, analytically, and experimentally. In this section we demonstrate the usage of the preceding phase space analysis for studying α transport process. For a fully detailed discussion of the α-particle response to TAE modes we refer readers to Ref. 14 which is Part II of the present work.

The particle dynamics is studied based upon the ORBIT code with implementation of a CIT equilibrium and \( n = 1, m = (0, 1, 2, 3) \) TAE modes from NOVA-K code implemented. For general ingredients of the ORBIT code, we refer readers to Ref. 13.

In the first study, 512 α particles with \( E_0 = 3.52 \) MeV and randomly sampled initial pitch \( \xi_0 = v_\parallel / v \), initial angles \( \theta_0, \varphi_0 \), and an initial box-shaped radial-profile with \( \psi_{p0} \leq 0.7\psi_{pb} \) are launched. Prompt-loss orbits are deliberately screened out from the initial population using the phase space topology boundaries discussed in the present paper in order to isolate the physics of particle loss due to TAE modes.

After 1000 transit times, the lost particles are counted, and their initial and final phase space positions are plotted in Figs. 11a,b. Results are as follows:

(i) For small mode amplitude, only particles are lost whose initial phase space positions are near the loss boundary (hereafter this will be called near boundary loss). The total loss is found proportional to \( \alpha \equiv \frac{\delta A_\|}{Br_0} \), and all losses are found to occur within 100 transit times. \( \delta A_\| \) is the Alfvén perturbation vector potential.

(ii) For large enough mode amplitude, particles whose initial phase space positions are far from the loss boundary can also be lost. The total number lost keeps growing with time and is proportional to \( (\delta A_\|)^2 \). This implies the existence of a diffusive loss process.

By investigating for both cases the time series of the loss and the initial and final phase space position of lost particles for different mode amplitudes, one can distinguish the near boundary loss from the diffusive loss.
To analyze the diffusion, a further study is made launching 512 particles with the same initial \((E, P_\varphi)\) chosen to be far from the loss boundary and randomly sampled initial \(\theta, \varphi,\) and pitch \(\frac{v_\parallel}{v}\). Over 1000 transit times, the time series of the quantity

\[
\langle (\Delta P_\varphi)^2 \rangle = \langle P_\varphi^2(t) \rangle - \langle P_\varphi(t) \rangle^2
\]

which measures the broadening of the phase space distribution, is studied. Here, \(\langle \ldots \rangle\) denotes an ensemble average. Note that as \(\langle (\Delta P_\varphi)^2 \rangle\) grows continuously for periods much longer than the transit time, its growth rate is a good measure of the diffusion coefficient in phase space. One finds that

1. with small amplitude, \(\langle (\Delta P_\varphi)^2 \rangle\) will immediately saturate at a very small level during the initial transient (cf. Fig. 12a);

2. for increasing mode amplitude, a continuous linear growth of \(\langle (\Delta P_\varphi)^2 \rangle\) is observed after the initial transient (cf. Fig. 12b).

3. In the small amplitude case, the saturation level of \(\langle (\Delta P_\varphi)^2 \rangle\) is linear with \(\hat{\alpha}\), while in the longer amplitude case, the slope of \(\langle (\Delta P_\varphi)^2 \rangle\) vs. \(t\) is proportional to \(\hat{\alpha}^2\). Indeed, one finds that \(D_{pp} \propto \left(\frac{\delta B}{B}\right)^2\) where

\[
D_{pp} = \lim_{\Delta t \gg 1} \frac{\langle (\Delta P_\varphi)^2 \rangle}{2 \Delta t}
\]

(cf. Fig. 13). This demonstrates diffusion.

The above observations can be understood from a simple physics picture of nonlinear dynamics\(^{21}\). There are two types of motion in the phase space: (1) the regular motion constrained on KAM surfaces, and (2) the stochastic motion when there is no well behaved KAM surface. Under the influence of the modes, KAM surfaces are subject to a distortion proportional to \(\hat{\alpha}\), as can be seen from

\[
\frac{dP_\varphi}{dt} \propto \left( v_\parallel - \frac{\omega}{k_\parallel} \right) \frac{\hat{\alpha}}{\hbar}
\]

(cf. Appendix B.) (For the TAE, the resonance condition is \(\omega - k_\parallel v_\parallel - k_\perp v_D = 0\) where \(v_\parallel, v_D\) are the guiding center velocities.) This regular distortion of the KAM surface thus
leads to the transient growth of \(((\Delta P_r)^2\). In addition, when the distorted KAM surfaces intersect with the plasma loss boundary, trajectories on these KAM surfaces are subject to boundary loss. Clearly, orbits moving on other KAM surfaces will stay well confined unless other randomization process such as collisions are superimposed, but collisions are not considered in this paper.

For each set of mode numbers (corresponding to action-conjugated angles), there usually exists one resonant KAM surface which is subject to the largest distortion among neighboring KAM surfaces. As the mode amplitude increases, the two neighboring resonant KAM surfaces start to get closer and finally overlap with each other. Without overlapping, the resonant KAM surfaces are regular and the phase space trajectory is constrained on each isolated KAM surface, and no diffusion can occur. We refer more detailed discussions to Ref. 14, (i.e. Part II), and 21-25 and Appendix A (in which one also sees that poloidal harmonic sideband resonant overlap can occur with only one \((n, m)\) in configuration space).

On the other hand, when overlap occurs, the KAM surface topology is destroyed and the trajectory becomes stochastic, producing diffusion. The diffusion coefficient in phase space can be numerically calculated in the stochastic region through \(D_{pp}\) which is a function of \((E, P_\phi, \mu)\). \(D_{pp}\) was defined in (10).

In order to construct the surface of section for the phase space \((E, \mu, P_\phi, \psi, \theta, \varphi)\) we launch a class of particles having the same constant of motion under the influence of the finite amplitude perturbation, i.e., \(\mu = (1/2)mv^2/B, C = E - \omega_n P_\phi\). (This constant follows from \(\frac{dE}{dt} = -\frac{\partial H}{\partial t}, \frac{dP_e}{dt} = \frac{\partial H}{\partial \varphi}\) where \(H\) includes the perturbation.) Without this choice, well behaved KAM surfaces from different classes of particles may appear to cross each other in a given surface of section plot, although no real overlapping between the KAM surfaces occurs. For a given \((\psi, \theta, \varphi, \mu)\) and \(C\), the equation \(E - \frac{\omega_n}{n} P_\phi = C\) leads to

\[
\frac{\rho_p^2}{2h} + \frac{\mu}{h} + \frac{\tilde{\phi}}{\hbar} - (g\rho_p - \psi_p + g\tilde{\alpha})\frac{\omega_n}{n} = C
\]

which has two roots for \(\rho_p\left(=\frac{\rho_p}{\Omega}\right)\). This can complicate the appearance of the map. The surface of the section plot shown in Figs. 14a,b,c is produced by 32 alpha trajectories having the same value of \(C\) but with different energies.
We prescribe various values of \( \lambda \equiv \frac{\mu B_0}{E_0(3.5 \text{MeV})} \) (or equivalently \( (v_\parallel/v)_\theta=0 \)) and study the two dimensional (2-D) map in \( P_\varphi - \varphi \) space. Here \( P_\varphi = g(\rho_\parallel + \tilde{\alpha}) - \psi_p \) is the toroidal canonical momentum (and a measure of the particles radial position) and \( \varphi = (\varphi - \frac{\omega_0}{n} t)/2\pi \) of the time dependent Hamiltonian. First we consider particles with the same \( \mu \) and \( C \equiv E - \frac{\mu}{n} P_\varphi \) (thus, initial \( E, P_\varphi \) are different, and \( C \) is chosen so that \( (\frac{1}{2}mv^2)_{\max} \approx 3.5 \text{ MeV} \)), \( \lambda = 0.7, \sigma \equiv \frac{v_\parallel}{|v_\parallel|} = -1 \) (i.e., a counter-going orbit near the trapped/passing boundary), and with \( \lambda = 0 \).

The onset of stochasticity is demonstrated in Figs. 14a,b,c. Depending on \( \lambda(\equiv \mu B_0/E) \) the threshold for \( \tilde{\alpha} \) varies. When \( \tilde{\alpha} = 2 \times 10^{-3} \) (a large value for \( \delta A_\parallel \)) the threshold is exceeded. Figure 14a shows orbits with \( \lambda = 0.7 \) which describes passing alphas near the passing/trapped separatrix with relatively large orbit width (cf. Fig. 9 of Part II, Ref. 14). The TAE resonance is producing strong chaos. Figure 14b shows that at this amplitude even well circulating alphas away from the separatrix show stochasticity in the inner region, while well behaved KAM surfaces still persist in the outer region. Figure 14c reveals that for small \( \delta A_\parallel (\tilde{\alpha} = 2 \times 10^{-4}) \) the outer region of the system is below the stochastic threshold, even for \( \lambda = 0.7 \). The fact that the outer region is below stochastic onset while the inner region is above, is due to the radial profile of the mode as shown in Ref. 14. Indeed, a series of detailed runs shows that the stochastic onsets are \( (\delta B/B) = 5 \times 10^{-4} \) for transition orbits \( (\lambda = 0.7) \) and \( (\delta B/B) \approx 2 \times 10^{-3} \) for well passing orbits \( (\lambda = 0) \). These results are in good agreement with the analytic prediction\(^{25} \) and with a recent independent numerical work using model map equations based on the assumption that the orbit width is much greater than the mode width.\(^{26} \) Note that in Ref. 25 a stochastic onset for a trapped orbit is given as \( \Delta_s \equiv \frac{\delta B}{B} = \left( \frac{r}{R_{\text{m,n,q}}} \right)^{3/2} (2\rho q') \). Here \( n \) is the toroidal mode number, \( \rho \) is the Larmor radius. Taking \( \rho_\alpha = 5 \text{ cm}, \frac{r}{R} = \frac{1}{8}, R = 2 \text{ m}, n = 1 \), one finds \( \Delta_s \lesssim 10^{-3} \).

In the stochastic region, the diffusion coefficient in a phase space region far from the boundaries (such as KAM surfaces or loss boundary) can be analytically estimated by quasilinear theory\(^{24} \) which predicts a \( \alpha^2 \) scaling. Consequently, if the phase space is fully stochastic, the loss rate should also scale as \( \alpha^2 \). A fundamental case is analyzed in [22,26,27].
A complete understanding of the diffusive loss process for tokamak orbits is not simple. This is due to the different stochastic threshold for each \((E, \mu)\) and the radial profile of the TAE modes. Typically, the \(n = 1\) TAE mode peaks in the radial core region. Therefore, it is possible that the inner region is stochastic while KAM surfaces still persist intact in the outer region preventing the diffusion from continuing toward the loss boundary near the plasma boundary \(\psi_p = \psi_{pb}\). \(P_\phi = g(\rho_\| + \tilde{\alpha}) - \psi_p\) is mainly a measure of radial particle position since the first term is of the order of the poloidal Larmor radius for most alphas.

Furthermore, we emphasize that since different \(\mu\) values correspond to different stochastic thresholds, it is possible that for a given peak mode amplitude, some classes of particles in the system are confined while others are subject to stochastic diffusion loss.

Finally, it is interesting to compare the theoretical loss boundary of Figs. 5-6, with the particle simulation results of Figs. 11a,b. The analytic result is borne out in this simulation and one notes that the trapped-passing boundary layer plays an important role in the diffusion process. This is due to the large orbit width in this region and large radial excursion during the crossing of the phase space trajectory over this boundary layer as discussed in sections II and III.

V. Summary

The phase space topology of energetic particles in a general tokamak equilibrium has been studied with particular emphasis on delineating the particle loss boundaries. Various types of phase space boundaries and non-standard orbits have been found and analyzed. In addition, the phase space topology for certain spatially localized particles has been analyzed. An example is given in Fig. 8 which shows all large \(\alpha\) orbits intersecting the magnetic axis. (This study should be extended to quantitatively determine the \(\alpha\) seed current for the bootstrap current.) It is found that in this phase space, the up/down symmetry label \(\sigma_p = \psi_p/|\psi_p|\), instead of the usual pitch label \(\sigma = v_\|/|v_\|\), is needed.

The results of the phase space topology analysis are then utilized in a numerical guiding center following code to study the \(\alpha\) particles response to the toroidal Alfvén
eigenmodes. Above a certain amplitude collisionless diffusion is found due to stochasticity even for one toroidal mode number $n$ (which couples toroidally to several poloidal mode numbers $m$). This stochastic threshold depends on the particle pitch angle and for one TAE amplitude some alphas are stochastic while others are on regular orbits. Furthermore, the phase space region near the trapped-passing boundary layer is found to be particularly important for transport of energetic particles to the loss boundaries. A fuller treatment of this problem will be presented in Ref. 14 as Part II of the present paper.
Acknowledgments

We thank Prof. John R. Cary for valuable discussions of our stochastic maps and scrutiny of the Hamiltonian guiding center orbit integrator, R. White for the ORBIT code, and C. Z. Cheng for providing the TAE mode structure for this work which was performed under Grant No. DE-FG02-91ER-54109.
Appendix A

Resonance Condition of Particle Wave Interaction

Here we briefly study the resonant interaction between particles and waves which can be best understood in the “action-angle” \((J, \theta)\) phase space.\textsuperscript{21,22,23} This includes a discussion of the resonant condition “sideband” coupling in a toroidal system.

Consider the Hamiltonian

\[
H = H_0 + \mathcal{H}
\]  

where \(H_0\) and \(\mathcal{H}\) are the unperturbed and perturbed Hamiltonian respectively. Without \(\mathcal{H}\), a particle orbit has several constants of motion. This leads to the existence of certain canonical transformations \((p, x) \rightarrow (J, \theta)\) such that the unperturbed Hamiltonian \(H_0\) can be expressed as

\[
H_0 = H_0(J)
\]

and the unperturbed orbit is described by

\[
\begin{aligned}
\frac{dJ}{dt} &= -\nabla_J H_0 = 0 \\
\frac{d\theta}{dt} &= -\nabla_\theta H_0 \equiv \omega_0(J)
\end{aligned}
\]

i.e., \(J = \text{const.}\) and \(\theta = \omega_0 t = \text{const.}\)

The action-angle variables for charged particles in an axisymmetric system can be obtained through a canonical transformation \((P, X) \rightarrow (P_\varphi, J_\theta, \mu, \hat{\varphi}, \hat{\theta}, \hat{\theta}_g)\) using the generating function\textsuperscript{23}

\[
G = \mu \hat{\theta}_g + P_{\varphi} \hat{\varphi} + \int_0^\theta d\theta' \psi(\theta, \mu, P_{\varphi}, J_\theta).
\]

Here, the angles are derived from

\[
\hat{\varphi} = \frac{\partial G}{\partial P_{\varphi}}, \hat{\theta}_g = \frac{\partial G}{\partial \mu}, \hat{\theta} = 2\pi \int_0^\theta \frac{d\theta'}{\hat{\theta}'},
\]

and the action

\[
J_\theta \equiv \int \frac{d\theta}{2\pi} \psi_T(\theta, \mu, P_{\varphi}, E)
\]

measures the total toroidal flux enclosed by an orbit. Now, consider the perturbed field described by \(\mathcal{H} \ll H_0\). The equations of motion become

\[
\begin{aligned}
\frac{dJ}{dt} &= -\nabla_J \mathcal{H} \\
\frac{d\theta}{dt} &= \omega_0(J) + \nabla_\theta \mathcal{H}
\end{aligned}
\]
It is therefore useful to express \( \vec{H} \) in terms of \( \mathbf{J} \) and \( \theta \), i.e.,

\[
\vec{H} = \sum_k \vec{H}_k(\mathbf{J}) e^{i(k \cdot \theta - \omega t)}
\]  

(A.8)

Resonant interaction occurs when

\[
\omega = k \cdot \frac{d\theta}{dt} \simeq k \cdot \omega(\mathbf{J})
\]  

(A.9)

is satisfied, and \( \mathbf{J} \) undergoes secular change. In the context of drift kinetic Hamiltonian theory for low frequency waves \( (\omega \ll \Omega) \), the action-angle phase space reduces to four dimensional (4-D), i.e., \( (P, J_\theta, \dot{\varphi}, \dot{\theta}) \) while \( \mu \) is constant. One thus has

\[
k \cdot \theta = \dot{n} \varphi - \dot{m} \dot{\theta}
\]

and the resonance condition Eq. (A.9) becomes

\[
\omega = n \omega_\varphi - m \omega_\dot{\theta}.
\]  

(A.10)

Consider a well untrapped particle with pitch angle variable

\[
\lambda \equiv \frac{\mu B_0}{E} \ll 1
\]

One has

\[
\dot{\theta} \simeq \theta, \dot{\varphi} \simeq \varphi
\]

\[
\omega_\theta \simeq \sigma \mu_0 \kappa / q_0, \quad \omega_\varphi \simeq q_0 \omega_\theta
\]  

(A.11)

where

\[
\mu_0 \equiv 2\sqrt{\varepsilon_0 \lambda E_0}
\]

\[
\kappa = \kappa(\mathbf{J}) \equiv \frac{1 - \phi_0 - \lambda / h_{\text{min}}}{2\varepsilon_0 \lambda}
\]

and \( \varepsilon_0 \equiv \frac{R_0}{\kappa_\text{max}}, \phi_0, h_{\text{min}}, q_0 \) are all evaluated at \( \psi_p = -P_\varphi \). The resonance condition is thus described by

\[
\omega \simeq (\dot{n} - \frac{\dot{m}}{q_0}) \sigma \mu_0 \kappa
\]  

(A.12)
For each \((\hat{n}, \hat{m})\) and given \((E_0, \lambda; \sigma)\), Eq. (A.12) leads to a solution for \(P_\varphi\) which corresponds to a resonant surface of section in the \((P_\varphi, \varphi)\) plane. It is interesting to study the role of \(\sigma \equiv \frac{\nu_{||}}{\nu_{\parallel}}\). Using the fact that \(\omega, \mu_0, \kappa > 0\) and \(q \gg 1\) for most of the domain in phase space, one sees that for \((\hat{n}, \hat{m}) = (1, 1)\), only co-passing particles can be resonant with the wave and for \((\hat{n}, \hat{m}) = (1, 2)\) or \((1, 3)\), counter-passing particles are resonant more easily.

Finally, the sideband coupling can be seen as follows. In practice, the perturbed field is expressed in configuration space, i.e., \(\hat{H}\) can be expressed as

\[
\hat{H} = \sum H_{nm}(P_\varphi, \lambda, \psi_p)e^{i(n\varphi - m\theta - \omega t)}.
\]

Therefore, \(\hat{H}_{\hat{n}\hat{m}}(J)\) in Eq. (A.8) can be evaluated from \(\hat{H}_{\hat{n}\hat{m}}(J) = \oint \frac{d\theta}{2\pi} \sum H_{nm}(P_\varphi, \lambda, J, \psi_p)e^{i(n\varphi - \hat{n}\varphi)}e^{i(m\theta - \hat{m}\theta)}\). By considering the well untrapped particles as described in Eq. (A.11) and using the fact that

\[
\psi_p = \psi_p(P_\varphi, \lambda, J, \theta)
\]

one finds that each \((n, m)\) can lead to \(\hat{H}_{n,m+1}\) and \(\hat{H}_{n,m-1}\) which corresponds to "sideband coupling." This implies that one set of modenumbers \((n, m)\) in a configuration can lead to two sideband resonant interactions corresponding to \(\hat{n} = n, \hat{m} = m \pm 1\) and thus can lead to resonant overlap. For a given \(\hat{n}, \hat{m}, \mu_0, C \equiv E - \frac{\omega}{\hat{n}}P_\varphi\), Eq. (A.12) can be solved to give the resonant surface in the \((P_\varphi, \varphi)\) surface of section. The distance between two resonant surfaces can thus be calculated. Furthermore, using Eq. (B.3) and the bounce time for a particle trapped in the wave, one can calculate \(\Delta P_\varphi\) on a given resonant surface. The KAM surfaces of section overlap when \(\Delta P_\varphi\) exceeds the distance between two neighboring resonant surfaces, and thus lead to stochasticity. For details, we refer the reader to Ref. 25.
Appendix B

Evaluation of \( \frac{dP_\varphi}{dt} \)

Starting with the Hamiltonian

\[
H = \frac{(P_\varphi + \psi_p - g\tilde{\alpha})^2}{2g^2h^2} + \frac{\lambda}{h} - \phi, \quad (B.1)
\]

one finds that

\[
\frac{dP_\varphi}{dt} = -\frac{\partial H}{\partial \varphi} = \frac{P_\varphi + \psi_p - g\tilde{\alpha}}{gh^2} \frac{\partial \tilde{\alpha}}{\partial \varphi} \frac{\partial \phi}{\partial \varphi} = \frac{v_\parallel}{h} \frac{\partial \tilde{\alpha}}{\partial \varphi} \frac{\partial \phi}{\partial \varphi} \quad (B.2)
\]

For ideal MHD waves, one has \( E_\parallel = i\omega \tilde{\alpha} / h - k_\parallel \phi = 0 \). Equation (B.2) thus yields

\[
\frac{dP_\varphi}{dt} = \left( v_\parallel - \frac{\omega}{k_\parallel} \right) \frac{1}{h} \left( \frac{\partial}{\partial \varphi} \tilde{\alpha} \right) \quad (B.3)
\]

At resonance,

\[
\frac{dP_\varphi}{dt} \approx -i \frac{k_\perp \cdot \mathbf{V}_d n_\perp}{k_\parallel} \tilde{\alpha}.
\]

Here, \( n \) is the toroidal mode number. Thus, at each resonant kick, \( \Delta P_\varphi \propto q\rho_\alpha n \left( \frac{\delta B}{B} \right) \).

Consequently,

\[
\langle (\Delta P_{\varphi})^2 \rangle \propto q^2 \rho_\alpha^2 n^2 \left( \frac{\delta B}{B} \right)^2 \propto \frac{E_\alpha}{I_p^2} n^2 \left( \frac{\delta B}{B} \right)^2
\]

Here, \( I_p \) is the plasma current.
Appendix C

Determination of \( \int d \psi F \)

First, in an axisymmetric system, \( d \phi \) can be omitted from Eq. (8a) and

\[
\int_{\psi_p} \int d \theta J \int d \psi F
\]

\[
= \int_{\psi_p} \int d \theta J \left[ \pi \Omega^2 R^3 \sum_{\sigma} \int_0^\infty dE \rho_0 \int_0^h \frac{d \lambda}{\sqrt{h^2 - \lambda^2}} F \right]. \tag{C.1}
\]

From Eqs. (1) and (2) one finds

\[
\frac{\partial \theta}{\partial \psi} \bigg|_{E, \psi, \psi_p} = \left( \frac{\sigma \rho_0}{\sqrt{h^2 - \lambda^2}} \right)^{-1} = \frac{\sqrt{h^2 - \lambda^2} \sigma}{\psi_p} = \frac{\sqrt{h^2 - \lambda^2} \sigma \psi_p}{Dh^2 \rho_0} \tag{C.2}
\]

Using Eq. (C.2) and Fig. 10 one finds

\[
\sum_{\sigma} \int_{\psi_p} d \theta = \int_0^\pi d \theta + \sum_{\sigma} \int_{-\pi}^0 d \theta
\]

\[
= \left[ \left( \int_{P_3}^{P_1} \frac{d \psi}{|\psi_p|} - \int_{P_2}^{P_1} \frac{d \psi}{|\psi_p|} \right) + \left( \int_{P_3}^{P_1} \frac{d \psi}{|\psi_p|} + \int_{P_2}^{P_1} \frac{d \psi}{|\psi_p|} \right) \right]
\]

\[
= \sum_{\sigma_p} \int_{P_1}^{P_4} \frac{d \psi}{|\psi_p|} \frac{\sqrt{h^2 - \lambda^2}}{Dh^2 \rho_0} \tag{C.3}
\]

where

\[
P_1 = P_\psi(\theta = 0, \sigma = -1), \quad P_2 = P_\psi(\theta = \pi, \sigma = -1)
\]

\[
P_3 = P_\psi(\theta = \pi, \sigma = -1), \quad P_4 = P_\psi(\theta = 0, \sigma = 1).
\]

Combining Eqs. (C.1)–(C.3), and using \( J = (I + \sigma q)h^2 \), one obtains

\[
\int d \theta J \int d \psi F
\]

\[
= \pi \Omega^2 R^3 \sum_{\sigma_p} \int_0^\infty dE \int_{\psi_p} \frac{d \psi}{|\psi_p|} \frac{d \lambda}{\psi_p} \left( \frac{I + \sigma q}{D} \right) F(P_\psi, \lambda, \psi; \sigma_p) \tag{C.4}
\]

where \( D \) is defined after Eq. (2). Typically, \( \rho_\parallel (gI' - Ig') \ll I + \sigma q \), thus \( I + \sigma q/D \approx 1 \), thus

\[
\int d \theta J \int d \psi F \approx \pi \Omega^2 R^3 \sum_{\sigma_p} \int_0^\infty dE \int_{\psi_p} \frac{d \psi}{|\psi_p|} \frac{d \lambda}{\psi_p} F(P_\psi, \lambda, \psi; \sigma_p)
\]

Also note that energy \( E \) and canonical angular momentum \( P_\psi \) are both normalized as in Section II.
References


16R. R. Parker, G. Bateman, P. L. Colestock, H. P. Furth, R. J. Goldston, W. A. Houlberg,


Figure Captions

Fig. 1  Particle guiding center trajectory in the \((h, \psi_p)\) plane, from Eq. (1), at constant \(\dot{P}_\phi\).

Fig. 2  The dashed line shows background plasma equilibrium, from \(h(\psi_p, \theta) = B_0/B\) at const. \(\theta\); the solid line shows particle trajectory. \((\theta = 0 \text{ at outboard.})\)

Figs. 3  Co-existence of inner counter-passing (ICP) and axis encircling-trapped (ET) orbits (1) in the \((h, \psi_p)\) plane (Fig. 3a), and in the \((\psi_p, \theta)\) plane (Fig. 3b).

Figs. 4  Stagnation orbits Fig. 4a: co-passing, Fig. 4b: counter-passing, and Fig. 4c: pinch orbit.

Fig. 5  Global phase-space topology of energetic particle with \(E_0 = 3.52\) MeV.

Fig. 6  Global phase-space topology of energetic particle with \(E_0 = 14.7\) MeV.

Figs. 7  Illustration of two types of D-orbits: Fig. 7a: non-axis encircling, Fig. 7b: axis encircling.

Fig. 8  Axis orbits passing through magnetic axis for various values of \(\frac{v_{//}}{v}\) at the magnetic axis.

Figs. 9  Illustration of counter-intuitive orbits (Fig. 9a) axis encircling trapped orbit, (Fig. 9b) non-axis encircling co-passing orbit, (Fig. 9c) non-axis encircling counter-passing orbit.

Fig. 10  Local phase-space topology of energetic particle with \(E_0 = 14.7\) MeV.

Figs. 11  Phase-space positions of particles lost due to TAE modes: Fig. 11a initial position, Fig. 11b final position.

Figs. 12  \(\langle(\Delta P_\phi)^2\rangle\) vs. \(t/\tau_{\text{transit}}\): Fig. 12a with wave amplitude \(\tilde{\alpha} = 2 \times 10^{-4}\); Fig. 12b with wave amplitude \(\tilde{\alpha} = 2 \times 10^{-3}\).

Fig. 13  \(D_{pp} = \frac{\langle(\Delta P_\phi)^2\rangle}{2\Delta t} \text{ vs. } (\delta B/B)^2/c^2\), with \(c = 2 \times 10^{-3}\). Here \(P_\phi\) is normalized against \(m_\alpha \Omega_\alpha R_0^2\) and \(\Delta t\) normalized against \(\tau_{\text{transit}}\).

Figs. 14  Orbit Poincaré plots \(\frac{P_\phi}{\psi_{pp}}\) vs. \((\varphi - \frac{\omega_n t}{2\pi})\): Fig. 14a with \(\tilde{\alpha} = 2 \times 10^{-3}; \frac{\mu B_0}{E_0} = 0.7\); Fig. 14b with \(\tilde{\alpha} = 2 \times 10^{-3}; \frac{\mu B_0}{E_0} = 0\); Fig. 14c with \(\tilde{\alpha} = 2 \times 10^{-4}; \frac{\mu B_0}{E_0} = 0.7\).
Figure 2
Figure 3a