HAMiltonian Chaos
In WAVE-PARTicle Interactions

A. K. Ram and A. Bers

October 1994

Plasma Fusion Center
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139 USA

This work was supported by NASA Grant No. NAGW-2048, by DOE
Grant No. DE-FG02-91ER-54109, and by NSF Grant No. ECS-88-
22475. Reproduction, translation, publication, use and disposal, in
whole or part, by or for the United States Government is permitted.

To be published in Physics of Space Plasmas (1993), SPI Confer-
ence Proceedings and Reprint Series, Number 13, T. Chang and J. R.
# HAMILTONIAN CHAOS IN WAVE-PARTICLE INTERACTIONS

A. K. Ram and A. Bers

## TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>1</td>
</tr>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>I. Motion of a Charged Particle in an Electrostatic Wave</td>
<td>2</td>
</tr>
<tr>
<td>II. Chirikov-Taylor Map or the Standard Map</td>
<td>5</td>
</tr>
<tr>
<td>III. Some Properties of the Standard Map</td>
<td>7</td>
</tr>
<tr>
<td>IV. Numerical Results of the Standard Map</td>
<td>7</td>
</tr>
<tr>
<td>V. Properties of Chaotic Motion</td>
<td>11</td>
</tr>
<tr>
<td>A. Evolution of Particle Orbits</td>
<td>11</td>
</tr>
<tr>
<td>B. Separation of Orbits of Initially Nearby Particles</td>
<td>12</td>
</tr>
<tr>
<td>C. Frequency Spectrum of Particle Orbits</td>
<td>12</td>
</tr>
<tr>
<td>VI. Effect of an Electrostatic Wave on a Charged Particle in a Magnetic Field</td>
<td>14</td>
</tr>
<tr>
<td>VII. Conclusions – Global Descriptions</td>
<td>19</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>20</td>
</tr>
<tr>
<td>References</td>
<td>20</td>
</tr>
<tr>
<td>Appendix A: Particle Motion in a Plane Wave</td>
<td>21</td>
</tr>
<tr>
<td>Appendix B: Action-Angle Variables for the Nonlinear Oscillator</td>
<td>22</td>
</tr>
<tr>
<td>Appendix C: Derivation of the Standard Map</td>
<td>24</td>
</tr>
<tr>
<td>Figures</td>
<td>25</td>
</tr>
</tbody>
</table>
HAMILTONIAN CHAOS IN WAVE-PARTICLE INTERACTIONS

Abhay K. Ram, Abraham Bers
Plasma Fusion Center and Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge, MA 02139 U.S.A.

ABSTRACT

Wave-particle interactions are an integral part of plasma dynamics in space and laboratory plasmas. In this tutorial we discuss the motion of charged particles under the influence of waves when the particle dynamics is assumed to be dissipation-free so that it can be described by a Hamiltonian. Distinguishing features between chaotic and coherent particle motion are illustrated pictorially using paradigmatic models describing the interaction of particles with prescribed waves.

INTRODUCTION

The microdynamics of high temperature plasmas are generally described by wave-particle interactions on time-scales for which collisions can be neglected. The waves may be internally generated by some plasma instability or can be coupled into the plasma from an external electromagnetic source. In the case of a plasma microinstability, one saturation mechanism for the instability arises from the resonant interaction of its unstable fields with the charged particles in the plasma: the particles gain energy and/or momentum from these fields and, as a consequence, their distribution functions are modified. In the case of externally coupled waves, as occurs for heating plasmas or driving currents in plasmas, charged particles interacting resonantly with these waves similarly remove energy and/or momentum from the wave and in this process their distribution functions are also modified. It is well-known that the interaction of plasma particles with randomly phased waves can lead to irreversible changes in the particle's distribution function (quasilinear dynamics in the random phase approximation). The importance of Hamiltonian chaos is that the interaction of coherent waves with plasma particles can also lead to an essentially irreversible change in the distribution function of the particles. In general, a modification to the distribution function of the particles may lead to changes in the wave propagation properties of the plasma, thereby requiring a self-consistent treatment of wave-particle interactions. However, in numerous situations of physical interest the waves interact with a small number of energetic particles so that the bulk of the distribution function, which determines the wave propagation properties, is essentially unaffected.
In this tutorial we describe some characteristic nonlinear aspects of wave-particle interactions in which the waves are assumed to be coherent and completely prescribed \textit{ab initio}, and the particle motion is described by a Hamiltonian (i.e. collisionless dynamics).

The motion of charged particles interacting with coherent electromagnetic waves in a plasma can exhibit a rich and interesting behavior. In particular, for appropriate conditions, the motion of a particle can become chaotic. In this tutorial we describe important characteristics of this chaotic motion and compare them with the case when the particle motion is not chaotic, i.e. when it is predictable. In assuming that the waves are prescribed, their amplitude, frequency, and wavelengths are given and do not evolve with time. The interaction between the charged particles and the waves is taken to be dissipation-free so that there are no other sinks or sources of energy or momentum. Thus, the interaction is conservative and the motion of a particle can be described by a Hamiltonian. A very important aspect of such systems is that volume elements in phase space are conserved, i.e. the flow in phase space is incompressible. This is a consequence of Liouville's theorem. Hence, there do not exist any phase-space attractors – strange or otherwise – and one is forced to study the entire phase space spanned by the Hamiltonian system. While this may seem like a daunting task there exist criteria which determine regions of phase space where, for instance, the motion of particles becomes chaotic.

In the description of particle dynamics that follows, we have tried to avoid getting involved in detailed mathematical analysis. Rather, we illustrate properties of Hamiltonian dynamics pictorially, based on the computations of actual particle orbits, and outline some of the mathematical aspects that help in understanding these nonlinear dynamics. We have chosen the so-called standard map as a paradigm to illustrate in detail properties of chaotic dynamics. In addition, we discuss the motion of particles in a magnetic field being acted upon by an electrostatic wave propagating transverse to the magnetic field. This is of relevance to a large variety of laboratory and space plasmas. While we are not aware of any prior work which has the detailed pictorial illustrations given in this tutorial, there do exist two excellent publications [1–3] to which we refer the reader for more mathematical details.

I. MOTION OF A CHARGED PARTICLE IN AN ELECTROSTATIC WAVE

The one-dimensional motion of a charged particle of mass $m$ and charge $e$ in a plane electrostatic travelling wave of amplitude $E$, wave number $k$ and frequency $\omega$ is given by the Lorentz equation:

$$\frac{dx}{dt} = v$$

(1a)
\[ \frac{dv}{dt} = \frac{eE}{m} \sin(kx - \omega t) \]  
\[ \text{(1b)} \]

where \( x \) and \( v \) are the position and velocity of the particle, respectively, and \( t \) is time. In the frame moving with the phase velocity of the wave, \( \omega/k \), the above equations become:

\[ \frac{dq}{d\tau} = p \]  
\[ \frac{dp}{d\tau} = \alpha \sin(q) \]  
\[ \text{(2a)} \] 
\[ \text{(2b)} \]

where \( \tau = \omega t \) is the normalized time, \( p = kv/\omega - 1 \) is the particle velocity in the wave frame normalized to the phase velocity, \( q = kx - \omega t \) is the position of the particle in the wave frame normalized to the wave number, and \( \alpha = eEk/(mw^2) \) is the normalized amplitude of the wave (the square of the particle's bounce frequency near the bottom of the potential well of the wave divided by the square of the wave frequency). These equations of motion are identical to those of a nonlinear oscillator whose solutions are well-known [Appendix A]. From (2a) and (2b) it can be easily shown that the energy \( H(q,p) = p^2/2 + \alpha \cos(q) \) (which is the Hamiltonian for the above equations of motion) is a constant of the motion. Thus, for a given initial condition, the motion of a particle will lie on a surface of constant \( H(q,p) \) with the constant being determined by the initial conditions. The projection of such surfaces onto the \( q \)-\( p \) plane are shown in Fig. 1. The phase space for the motion of the particles can be divided into two regions: \( |H| < \alpha \), and \( H > \alpha \). The surface \( H = \alpha \) dividing the two regions is known as the "separatrix". The phase-space region given by \( |H| \leq \alpha \) corresponds to particles trapped in the wave potential, while the region given by \( H > \alpha \) corresponds to untrapped, or passing, particles (Fig. 1). The trapping width \( \Delta p_{tr} = 2\sqrt{\alpha} \) gives half the extent in velocity space of the trapped particle region (Fig. 1).

For a particle that is initially at \( p_0 = 0 \) and \( q_0 = \pi \), we see from Eqs. (2a, 2b), that it is unaffected by the wave. This is a particle at the bottom of the wave potential. This point in phase space is known as a "fixed" point. Consider a particle in the vicinity of this fixed point with \( q = \pi + \bar{q}, p = \bar{p} \), where \( |\bar{q}|, |\bar{p}| \ll 1 \). Then the constant energy surface on which this particle moves is given by:

\[ \frac{\bar{p}^2}{2\alpha} + \frac{\bar{q}^2}{2} = \text{constant} \]  
\[ \text{(3)} \]

where the constant on the right hand side is given by the initial conditions. The above equation describes an ellipse in phase space and, consequently,
the fixed \((q_0, p_0)\) is known as an "elliptic" fixed point. Solving Eqs. (2a, 2b) in the vicinity of the fixed point \((q_0, p_0)\) shows that motion is oscillatory with a frequency of \(\sqrt{\alpha}\). Thus, a particle in the vicinity of this fixed point will remain in its vicinity for subsequent times. This is apparent from Fig. 1.

From Eqs. (2a, 2b) we find that a particle that is initially at \(p_1 = 0\) and \(q_1 = 0\) will also not be affected by the wave for subsequent times. Hence, this is also a fixed point. Consider a particle in the vicinity of this fixed point with \(q = \bar{q}, p = \bar{p}\), where \(|\bar{q}|, |\bar{p}| \ll 1\). Then the constant energy surface on which this particle moves is given by:

\[
\frac{\bar{p}^2}{2\alpha} - \frac{\bar{q}^2}{2} = \text{constant} \tag{4}
\]

where the constant on the right hand side is given by the initial conditions. This is an equation for a hyperbola and, consequently, the fixed point at \((q, p) = (0, 0)\) is known as a "hyperbolic" fixed point. The asymptotes of this hyperbola are: \(p/\sqrt{\alpha} = \pm \bar{q}\). Solving Eqs. (2a, 2b) in the vicinity of the hyperbolic fixed point give:

\[
\bar{q} = c_0 e^{\sqrt{\alpha}t} + c_1 e^{-\sqrt{\alpha}t} \tag{5a}
\]

\[
\bar{p} = c_0 \sqrt{\alpha} e^{\sqrt{\alpha}t} + c_1 \sqrt{\alpha} e^{-\sqrt{\alpha}t} \tag{5b}
\]

where \(c_0\) and \(c_1\) are constants determined by the initial conditions. These results show that, for small displacements of a particle away from the hyperbolic fixed point, along the asymptote \(\bar{p}/\sqrt{\alpha} = \bar{q}\) the particle moves at an initially exponential rate away from the hyperbolic fixed point. This asymptote is known as the "unstable manifold". For small displacements of a particle away from the hyperbolic fixed point along the asymptote \(\bar{p}/\sqrt{\alpha} = -\bar{q}\), the particle approaches the hyperbolic fixed point at an exponentially slow rate. This asymptote is known as the "stable manifold". The motion of a particle in the vicinity of the hyperbolic fixed point and the existence of the two manifolds is apparent from Fig. 1. Furthermore, Fig. 1 also shows that the stable and unstable manifolds overlap completely.

For periodic systems of the type discussed above, it is useful to define another set of coordinates — the action (I)-angle (\(\theta\)) coordinates [4], where \(0 \leq \theta < 2\pi\). For the nonlinear oscillator these coordinates are given in Appendix B. The advantage of this new coordinate system is that the Hamiltonian can then be expressed as a function of action only, \(H = H(I)\), while the frequency of the nonlinear oscillations \(\Omega\) is given by the time derivative of the angle, i.e. \(d\theta/dt = \Omega\). If we define a phase-space cylinder whose radius is \(I\), azimuthal coordinate is \(\theta\), and axial coordinate
is time, then the orbit of a particle with action $I$ winds helically along the surface of this cylinder. The orbits of different particles with different actions will lie on surfaces of concentric cylinders. The time taken by the particle to execute one azimuthal rotation is $2\pi/\Omega$, so that the oscillation frequency of the particle is the azimuthal frequency of the motion. If we assume that time is periodic, with period $T$, then it is convenient to define a phase-space torus whose minor radius is $I$, poloidal angle is $\theta$, and toroidal angle is $\phi = 2\pi t/T$. The particle's orbit then winds along the surface of this torus. If we consider the intersection of this torus with a plane at fixed $\phi$, then the orbit of the particle will intersect this plane every time it completes a toroidal cycle around the torus. If for a given particle $\Omega T/(2\pi) = r/s$ where $r$ and $s$ are any integers, i.e. the ratio of the particle's poloidal frequency to the toroidal frequency is a rational number, then the change in $\theta$ per toroidal orbit is $2\pi r/s$. Thus, after $s$ toroidal orbits the particle will return to the same value of $\theta$. The particle will then intersect with the plane at fixed $\phi$ at only $s$ number of points. Such tori are known as rational tori. The ratio of the two frequencies is known as the winding number. If the winding number is an irrational number, the intersection of the particle's orbit with the constant $\phi$ plane will eventually form a closed curve (a circle). Such tori are known as irrational tori. The set of discrete points formed on the constant $\phi$ surface whenever the particle's orbit intersects it forms the so-called Poincaré surface-of-section. The study of this two-dimensional phase-space surface is equivalent to studying the motion of a particle in the full three-dimensional phase space [1]. Generalization of these ideas to higher dimensional periodic systems can be easily done [1].

II. CHIRIKOV-TAYLOR MAP OR THE STANDARD MAP

The one-dimensional motion of a charged particle in an infinite set of plane electrostatic waves, where all the waves have the same wave number and wave amplitude but whose frequencies are integer multiples of a fundamental frequency, is given by:

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = \frac{eE}{m} \sum_{n=-\infty}^{\infty} \sin(kx - n\omega t)$$

The corresponding Hamiltonian:

$$H(x, v, t) = \frac{v^2}{2} + \frac{eE}{mk} \sum_{n=-\infty}^{\infty} \cos(kx - n\omega t)$$

5
is periodic in time with period $T = 2\pi/\omega$. By using Fourier series analysis, it is easy to show that Eq. (6b) can be re-expressed as:

$$\frac{dv}{dt} = \frac{eE}{m} T \sin(kx) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

(8)

where $\delta$ is the Dirac delta function. Eq. (8) shows that in this interaction the particle receives a periodic kick in time, with period $T$, and with the strength of the kick being a function of its position. Using the new variables $(\theta, I, \tau)$ where $\theta = kx$ ($0 \leq \theta < 2\pi$) is the angle, $I = kTv$ is the action, and $\tau = t/T$ is the normalized time, Eqs. (6a) and (8) become:

$$\frac{d\theta}{d\tau} = I$$

(9a)

$$\frac{dI}{d\tau} = K \sin(\theta) \sum_{n=-\infty}^{\infty} \delta(\tau - n)$$

(9b)

where $K = kT^2(eE/m)$ is the normalized amplitude. The periodic nature of the interaction leads to the following mapping equations [Appendix C]:

$$\theta_{n+1} = \theta_n + I_n + K \sin(\theta_n) \quad \text{(mod } 2\pi)$$

(10a)

$$I_{n+1} = I_n + K \sin(\theta_n)$$

(10b)

where $(\theta_n, I_n)$ are the angle and the action of the particle just before it receives its $n$-th kick (i.e. just before time $nT$). The mapping equations (10a) and (10b) are known as the Chirikov-Taylor map [5, 6] or, more commonly, the standard map. This map is a paradigm for twist maps and has been studied extensively in the literature [7]. The mapping equations produce a Poincaré surface-of-section [1, 2] of the motion of the particle so that the motion, which occurs in three phase-space dimensions (namely, $\theta$, $I$, and $\tau$, with $\tau$ assumed to be periodic with period 1), can be analyzed and viewed in two phase-space dimensions $(\theta, I)$. The mapping equations give a set of discrete points formed by the intersection of the particle's orbit with a constant $\tau$ plane. Moreover, the mapping equations are much easier to set up and evolve accurately on a computer than the differential equations from which they are derived.

For unperturbed motion, i.e. $K = 0$, a particle's orbit lies on a torus of minor radius $I$ (constant of the motion), poloidal angle $\theta$, and toroidal angle $\phi = 2\pi \tau$. From Eq. (9a) the time taken by the particle to go one poloidal cycle is $2\pi/I$, so that the poloidal frequency of the particle's orbit is $I$. The toroidal frequency, in our normalized units, is $2\pi$ (which is the same as the normalized frequency of the kicks). Thus, the ratio
of the poloidal to the toroidal frequency for a particle is \( I/(2\pi) \). The unperturbed phase space for the particles are nested rational \((I/(2\pi) = r/s, r \text{ and } s \text{ any integers})\) and irrational tori.

III. SOME PROPERTIES OF THE STANDARD MAP

If in Eqs. (10a, 10b) we replace \( I \) by \( I + 2\pi \), the mapping equations remain invariant. So the standard map is periodic in \( I \) with period \( 2\pi \). Thus, when looking at a surface-of-section we need to display \( I \) between \(-\pi \) and \( \pi \) only. The rest of phase space will look exactly the same.

Consider a particle whose initial angle \( \theta^{(1)}_0 \) and action \( I^{(1)}_0 \) are such that \( I^{(1)}_1 = I^{(1)}_0 \) and \( \theta^{(1)}_1 = \theta^{(1)}_0 \mod 2\pi \) (i.e. after a kick there is no change in the particle's action and angle). Here the subscript denotes the mapping iteration index and the superscript denotes the number of iterates required to come back to the initial values of action and angle (modulo \( 2\pi \)). Then, from Eqs. (10a, 10b), we note that \( \theta^{(1)}_0 = 0 \) or \( \pi \), and \( I^{(1)}_0 = 2m\pi \) for \( m \) any integer. The points represented by these action-angle values are known as fixed points (since they do not change their values after any number of kicks). Thus, the standard map has two sets of fixed points: \((\theta^{(1)}_1, I^{(1)}_1) = (0, 2m\pi) \) and \((\theta^{(1)}_1, I^{(1)}_1) = (\pi, 2m\pi) \). In analogy with the fixed points of the nonlinear oscillator, the stability of these fixed points can be easily determined (see e.g. [2]). We find that the first set of fixed points corresponds to hyperbolic fixed points and the motion in the vicinity of these fixed points is unstable. The second set of fixed points corresponds to elliptic fixed points provided \( K < 4 \) and the motion in the vicinity of these fixed points is stable. However, for \( K > 4 \) the second set of fixed points also become hyperbolic. The fixed points having the properties discussed here are known as “first order” fixed points. An s-th order fixed point, \((\theta^{(s)}_0, I^{(s)}_0) \), satisfies the relations: \( \theta^{(s)}_n = \theta^{(s)}_0 \mod 2\pi \), \( I^{(s)}_n = I^{(s)}_0 \) for \( n = s \) and not for any other integer \( 0 < n < s \). An s-th order fixed point corresponds to a particle whose orbit lies on a rational torus with \( I/(2\pi) = r/s \) for \( r \) any integer.

There is another set of points associated with the standard map that are of interest. Consider the case when the amplitude of the wave is an integer multiple of \( 2\pi \), i.e. \( K = 2\pi k \) (\( k \) being a positive integer) and a particle’s initial angle is \( \pi/2 \) and its initial action is \( 2\pi l \) (\( l \) being any integer). Then after each subsequent kick the particle’s action will increase by \( 2\pi k \). If the initial angle was \( 3\pi/2 \) the particle’s action will decrease by \( 2\pi k \) after each kick. These are known as “accelerator modes.”

IV. NUMERICAL RESULTS OF THE STANDARD MAP

In plotting the numerical results for the standard map we need to
show only the range $-\pi \leq I < \pi$. The angle $\theta$, which describes the poloidal motion of a particle on the phase-space torus, will be plotted as a linear coordinate with period $2\pi$. So, clearly, all values of $I$ at $\theta = 2\pi$ are identical to those at $\theta = 0$. Thus, the unperturbed (i.e. $K = 0$) motion of a particle will form a set of discrete points at a fixed $I$ ranging in $\theta$ from 0 to $2\pi$. Different particles will correspond to different sets of initial conditions.

Fig. 2 displays the surface-of-section when $K = 0.1$ for a number of different initial conditions. It is easy to identify the orbit of a given particle as most of the orbits form almost closed curves. Whereas all the particles’ orbits were along straight lines of constant $I$ for $K = 0$, we see that, for $K = 0.1$, there is a dramatic modification to the orbits which are around $I = 0$. However, outside this region the orbits are small deviations away from straight lines of constant $I$. The behavior near $I = 0$ can be easily understood. From Eqs. (6a, 6b), we see that, for small amplitudes, $\alpha = E\kappa/(m\omega^2) \ll 1$, the motion of a particle resonant with a given wave (labelled $n$), i.e. a particle whose velocity is near $n\omega/k \ (n = 0, \pm 1, \pm 2, \ldots)$, is most affected by that wave. In other words, a particle which is nearly at rest in the frame of one of the plane waves will be most affected by that wave. The effect of other waves will essentially phase mix to give a very small perturbation. However, as seen in the first section, particles whose velocities are near the phase velocity of a plane wave are trapped and the motion corresponds to a nonlinear oscillator. Indeed, Fig. 2 shows exactly the nonlinear oscillator behavior, as in Fig. 1, in the region around $I = 0$. The trapping width for any of the plane waves is $(\Delta v)^n_{tr} = 2\sqrt{\alpha}(\omega/k)$, or in terms of the normalized action-angle variables: $(\Delta I)^n_{tr} = 2\sqrt{K}$. A quick check of Fig. 2 shows that the trapping width is exactly what would be expected from this simple argument. Moreover, the location of the first order hyperbolic and elliptic fixed points at $I = 0, \theta = 0$ and $I = 0, \theta = \pi$, respectively, are exactly as indicated in the previous section.

The consequences of two very important theorems are on display in Fig. 2. The first theorem, referred to as the Poincaré-Birkhoff theorem [8], states that on a rational torus corresponding to the unperturbed motion, with winding number $r/s$, there will remain $2ls \ (l = 1, 2, \ldots)$ number of fixed points after a small perturbation. Since for the unperturbed motion $I/(2\pi) = 0$ is a rational surface with $s = 1$, the number of fixed points observed in Fig. 2 at $I = 0$ are in agreement with the Poincaré-Birkhoff theorem. The similar breakup of other rational tori are not visible in Fig. 2 as their trapping widths are very small. The second theorem, referred to as the KAM theorem (named after Kolmogorov, Arnold, and Moser [9-11]) states that unperturbed irrational tori which are sufficiently far away from the nearest unperturbed rational tori are stable to small
perturbations (details of the theorem and the conditions for its validity can be found in [2]). Fig. 2 shows many such tori which have been slightly modified from their unperturbed forms. Such tori are referred to as KAM tori or KAM surfaces.

Figure 3 shows the results of the standard map for $K = 0.5$. Even though the amplitude of the perturbation is large, the breakup of the rational tori with $s = 2$ at $I = \pm 2\pi/2 = \pm \pi$ and with $s = 3$ at $I = \pm 2\pi/3$, in agreement with the Poincaré-Birkhoff theorem, can be clearly identified. Furthermore, even though the amplitude is greater than that required for the rigorous validity of the KAM theorem, there continue to be KAM surfaces which persist at this large amplitude. In this figure additional features are clearly discernable near $I = 0$ and $\theta = 0$. A magnification of the region marked by the box is shown in Fig. 4. Here we notice the richness and the complicated behavior of the motion in the vicinity of the hyperbolic fixed point. There exist a series of high-order fixed points, both elliptic and hyperbolic, with the nonlinear oscillator type of motion occurring in the vicinity of elliptic points. Thus, as is apparent, the phase-space picture is self-similar with the phase-space structure of Fig. 3 being repeated on a finer scale. The nearly uniformly dense region surrounding the hyperbolic fixed point at $I = 0$ and $\theta = 0$ and extending along the separatrix of the first order fixed point (seen in Fig. 3) is due to a single particle. The usual phase-space tori are completely destroyed in this region and the motion in the vicinity of the hyperbolic fixed point is very complex. This motion is referred to as “stochastic” or “chaotic” motion and we shall discuss detailed properties of this type of motion later. Figs. 3 and 4 show regions of local stochasticity where no periodic motion is discernable and the motion of the particle is no longer restricted to the surface of a torus.

In section I, it was shown that there emanate stable and unstable manifolds from a hyperbolic fixed point. While these manifolds overlapped for the nonlinear oscillator, small perturbations tend to separate out these manifolds. A manifold cannot intersect itself, otherwise the solutions to the equations of motion would not be unique for a given set of initial conditions. However, the stable and unstable manifolds can intersect each other. This intersection point in phase space is known as a “homoclinic” point. (The intersection point of a stable manifold from one hyperbolic fixed point with an unstable manifold from a different hyperbolic fixed point is known as a “heteroclinic” point.) A homoclinic point is not a fixed point of the motion, and if there exists at least one homoclinic point, then there must be an infinite number of them. The density of homoclinic points increases as one approaches the hyperbolic fixed point. This, along with the requirement that Hamiltonian systems satisfy Liouville’s theorem, leads to the extremely complicated behavior
in the vicinity of hyperbolic fixed points and along the separatrix. Details of this complexity are beyond the scope of this article. Interested readers are referred to [1, 2] and references therein. Fig. 4 gives evidence of the complicated interaction not only between stable and unstable manifolds of the hyperbolic fixed point at \((I = 0, \theta = 0)\) but also between different hyperbolic fixed points generated through the breakup of rational tori.

Upon comparing Fig. 2 with Fig. 3 we notice that, as the amplitude of the perturbation is increased, more KAM tori are destroyed with the concurrent appearance of more fixed points. KAM surfaces form barriers in phase space which do not allow, for example, intersection of stable and unstable manifolds from hyperbolic points on different sides of the KAM surface. Thus, particle orbits, which can wander in phase along these manifolds (as seen in Fig. 4), are prohibited from crossing those parts of phase space where KAM tori exist. In other words, the existence of KAM tori stops the phase-space motion of certain particles from becoming globally stochastic where these particles have access to the entire range of action. One is inevitably led to ask the question: at what amplitude is the last KAM surface destroyed? A simple way to answer this question is as follows. Each rational torus \(I = 2\pi n (n = 0, \pm 1, \pm 2, \ldots)\) corresponds to a particle velocity being equal to the phase velocity of the \(n\)-th plane wave in Eq. (6b), i.e. the particle is in resonance with the \(n\)-th plane wave. These tori are also called resonant tori. The trapping width, as discussed above, associated with the resonant tori is approximately given by \((\Delta I)^{r}_{n} = 2\sqrt{K}\) where each resonant torus is treated independently of other resonant tori (i.e. each plane wave is treated independently of other plane waves). Consider two such neighboring tori \(I\) and \(I+1\). If the sum of the trapping width of each of these tori is equal to the separation between the tori, i.e. the trapping widths of neighboring resonant tori overlap, then it is intuitively clear that there cannot exist any KAM surface between these tori. This criterion can be expressed as: \((\Delta I)^{r}_{I} + (\Delta I)^{r}_{I+1} \geq (\Delta I)_{I,I+1}\) where \((\Delta I)_{I,I+1}\) is the separation in action between two neighboring resonant tori. This is known as the simple Chirikov resonance overlap criterion. For the standard map this condition gives \(K \geq \pi^2/4\). This condition is modified significantly due to the presence of secondary resonances (corresponding to rational tori besides those at \(I = 2\pi n\)) as seen in Fig. 2. Detailed numerical calculations show that the last KAM surface is destroyed for \(K \approx 0.9716\) [12]. Thus, the simple resonance overlap criterion is too pessimistic. Once the last KAM surface is destroyed we have a transition from local to global stochasticity. The portrait of the standard map at \(K = 1\) in Fig. 5 shows a very rich phase-space structure. Notice that remnants of the resonant tori at \(I = 0, \pm 2\pi/3, \pm \pi\) still remain. Thus, the motion of many particles remains coherent for values of \(K\) just exceeding the one that destroys
the last KAM surface. The majority of initial conditions do not lead to chaotic dynamics until $K$ is closer to, or greater than, the values of the Chirikov condition.

V. PROPERTIES OF CHAOTIC MOTION

There are distinct differences between coherent and chaotic motion besides those that appear in the surface-of-section plots discussed above. For motion occurring in three phase-space dimensions ($\theta$, $I$, and $r$, being the phase-space variables for the standard map), a Poincaré surface-of-section is two-dimensional so that the entire dynamics can be visualized easily. However, in cases when the phase-space dimensions are greater than three, a Poincaré surface-of-section does not allow for simple visualization so that regions of chaotic and coherent motion may not be easily discernable. In this section, using the standard map as an example, we will discuss some of the distinguishing features between coherent and chaotic motion that are applicable to dynamical systems with phase-space dimensions greater than three.

A. Evolution of Particle Orbits

Figure 6 shows regions of coherent and chaotic motion in the surface-of-section plot for the standard map when $K = 1.5$. The coherent part of the phase space is dominated by the island surrounding the first order elliptic fixed point discussed earlier. Consider a set of initial conditions, given by the circle in Fig. 7a, located inside this island for $K = 1.5$. After 100 iterates of the standard map this circle is mapped into the curve shown in the same figure. A magnified view of the tip, given in Fig. 7b, clearly shows that the curve obtained after 100 iterates is a closed curve. The area enclosed by this curve is the same as the area of the circle. This is a consequence of Liouville's theorem. Next consider a set of initial conditions, given by the circle in Fig. 8a and having the same area as the circle in Fig. 7a, located in the chaotic part of the phase space of Fig. 6. The mapping of this curve after four iterates of the map and after seven iterates of the map is shown in the same figure. Clearly, the initial circle is mapping into a complicated curve after seven iterates. This curve is thinner and longer than the circle and, as expected, encloses an area which is the same as the area of the circle. The complicated feature of this curve is exemplified in Fig. 8b where a magnified view of a part of the curve shows multiple sheets associated with the curve. These sheets are generated by a stretching and folding over of the original circle of initial conditions. After 100 iterates of the map, the initial circle of initial conditions maps into the picture shown
in Fig. 8c. Clearly, considerable stretching and folding has taken place in order to produce the observed surface-of-section. The initial conditions are now spread out over the chaotic region of Fig. 6. The large scale regions which are empty correspond to coherent motion. A comparison of Fig. 8c with Fig. 7a shows a dramatic difference in the evolution of particle trajectories between the coherent and the chaotic regions.

B. Separation of Orbits of Initially Nearby Particles

Consider two particles which are initially separated in action by $1.0 \times 10^{-8}$ and have the same $\theta$. If both of these particles are located in the coherent part of phase space (i.e. lying initially within the circle of Fig. 7a), the separation in action between these particles as function of the number of iterates of the map is as shown in Fig. 9 for $K = 1.5$. The actions of the two particles remain close to each other over an extended period of time. However, if the two particles are located in the chaotic part of phase space (i.e. lying initially within the circle of Fig. 8a), then the difference in the actions of the particles increases exponentially with time for early times. This is shown in Fig. 10 for two values of $K$. The separation is faster for larger amplitudes of the waves and the separation in action saturates at earlier times as the amplitude is increased. The exponential increase in the separation of initially nearby orbits in phase space indicates dynamics with sensitive dependence on initial conditions; this is the essential feature which leads to loss of information in chaotic systems. Small errors in measurements at some initial time will grow exponentially with time leading to unpredictability.

C. Frequency Spectrum of Particle Orbits

An orbit of a particle in phase space can be decomposed into a frequency spectrum leading to additional information about the dynamics. In particular, periodic, quasiperiodic, and chaotic properties of the orbit can be determined from the spectrum analysis. In order to illustrate the connection between a particle’s orbit and its frequency spectrum we Fourier analyze the action of a particle obtained from the standard map.

Fig. 11a is the surface-of-section for a single particle started in the coherent part of phase space for $K = 1.5$. The particle’s orbit initially started off on one island returns to that island after every six iterates of the standard map. The sixth-order islands shown in Fig. 11a are easily discernable in Fig. 6 where they surround the primary island. The discrete set of actions generated by Eq. (10b), leading to Fig. 11a, are Fourier analyzed to give the frequency spectrum, $I(\nu)$ as a function of the frequency $\nu$, shown in Fig. 11b. Since the mapping equations generate
a value for the action after one (normalized) unit of time, the frequency spectrum is bound to lie in the interval \([-0.5, 0.5]\). However, the spectrum is symmetric around \(\nu = 0\) so we only show the spectrum in the interval \([0, 0.5]\). There are two primary frequencies which determine the frequency spectrum of orbit. The first frequency corresponds to the particle jumping from island to island in the surface-of-section and the other frequency corresponds to the motion of the particle in any given island. The former frequency is obviously \(1/6\) and dominates the spectrum in Fig. 11b. The latter frequency has its main peak near \(\nu = 0.2\). The rest of the spectrum is essentially composed of harmonics and sums and differences of these frequencies. The motion of the particle is clearly periodic.

Fig. 12a shows the orbit of a particle started near the separatrix of the sixth-order islands of Fig. 6 for \(K = 1.5\). The corresponding frequency spectrum, apart from the dominating peak at \(\nu = 1/6\) corresponding to the sixth-order island, is a broadband spectrum and shown in Fig. 12b. The spectrum corresponding to the motion of the particle around any given island is broadband, composed of many peaks, and centered around \(\nu = 0\). The jumping of the particle from island to island in the surface-of-section shifts the spectrum to lie near \(\nu = 1/6\), maintaining the features of being broadband and composed of many peaks.

Fig. 13a shows the surface-of-section for the orbit of a single particle, for \(K = 1.5\), which is initially located in the chaotic part of phase space. It is worth noting that this picture is essentially the same as generated in Fig. 8c. One aspect which is intuitively clear from Fig. 13a but difficult to prove is that the motion of a single particle in chaotic phase space will eventually come arbitrarily close to any given point in the connected chaotic phase space. This property forms a basis for the *ergodic hypothesis* [2]. The frequency spectrum for this particle, Fig. 13b, is broadband covering the entire frequency range. The peak at \(\nu = 1/6\) is a consequence of the fact that the particle, during its travels in the chaotic phase space, spends a lot of time near the sixth-order chain of islands discussed above. This is obvious in Fig. 13a where the darker part of phase space is near the sixth-order islands.

Figs. 14a and 14b show the surface-of-section for a single particle in chaotic phase space and the corresponding frequency spectrum, respectively, for \(K = 4.5\). The spectrum is more uniformly broadband than in Fig. 13b. Fig. 14a shows that the fixed point at \(I = 0, \theta = \pi\) which was elliptic for \(K = 1.5\) (Fig. 6) has now become hyperbolic. This transition in the character of a fixed point has been discussed earlier. A comparison of Figs. 13a and 14a, and of Figs. 13b and 14b, shows that as the amplitude \(K\) is increased a larger part of phase space becomes chaotic and the frequency spectrum becomes more uniformly broadband, i.e. more "noiselike."
In summary, we have shown in this section that, initially, nearby particles in phase space will diverge exponentially if these particles are located in the chaotic phase space. Also, the frequency spectrum of an orbit of a particle located in chaotic phase space will be broadband. The trajectory of a single particle will wander through the entire connected region of chaos.

VI. EFFECT OF AN ELECTROSTATIC WAVE ON A CHARGED PARTICLE IN A MAGNETIC FIELD

In space plasmas and laboratory plasmas, static magnetic fields play an important role in determining the dynamics of charged particles. Electrostatic waves, which either exist in the plasma or are externally imposed, can interact with the charged particles and significantly modify their trajectories. An example of such electrostatic waves is lower hybrid waves whose frequencies typically lie in the range of the ion-plasma frequency or below. In auroral plasmas, lower hybrid waves are generated by energetic electrons which are injected into the suprathermal region from the plasma sheet. It has been shown that these lower hybrid waves, propagating across the magnetic field, interact with ions ($H^+$ and $O^+$) of ionospheric origin and accelerate them upwards along the magnetic field lines [13]. In laboratory plasmas, lower hybrid waves are excited by means of waveguide grills at the edge of the plasma. The interaction of lower hybrid waves with ions has been of considerable interest and has been extensively studied [14-18]. Below we describe some of the interesting features of the motion of an ion in the presence of lower-hybrid type waves.

Consider the motion of an ion in a constant magnetic field, being acted upon by an electrostatic wave propagating across the magnetic field. The Hamiltonian for the motion of the ion is:

$$H(x, v; t) = \frac{v^2}{2} + \frac{\Omega^2 v^2}{2} + \frac{Q E_0}{Mk} \cos(k x - \omega_0 t)$$ (11)

where the magnetic field is assumed to be along the $\hat{z}$-direction, the electrostatic wave is propagating along the $\hat{x}$-direction, $x$ and $v$ are the position and velocity (along $\hat{x}$), respectively, of an ion of mass $M$ and charge $Q$, $E_0$ is the electrostatic field amplitude with $\omega$ and $k$ being the frequency and wave vector (along $\hat{z}$), respectively, and $\Omega = Q B_0 / M$ is the ion cyclotron frequency in a magnetic field of strength $B_0$. Upon normalizing $\omega t \rightarrow \tau$, $k x \rightarrow q$, $kv/\omega \rightarrow p$, $\Omega/\omega \rightarrow \omega_0$, the above Hamiltonian can be transformed to:

$$H(q, p; \tau) = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} + \epsilon \cos(q - \tau)$$ (12)
where $\epsilon = Q E_0 k / (M \omega^2)$. The unperturbed Hamiltonian describes the motion of a simple pendulum. A canonical transformation to the action-angle \((I - \theta)\) variables of the simple pendulum:

\[
I = \frac{\omega_0}{2} \left( q^2 + \frac{p^2}{\omega_0^2} \right) ; \quad \theta = \tan^{-1} \left( \frac{\omega_0 \theta}{p} \right) \tag{13}
\]

yields the Hamiltonian:

\[
H(\theta, I; \tau) = \omega_0 I + \epsilon \cos \left( \sqrt{\frac{2I}{\omega_0}} \sin \theta - \tau \right) \\
= \omega_0 I + \epsilon \sum_{n=-\infty}^{\infty} J_n \left( \sqrt{\frac{2I}{\omega_0}} \right) \cos (n\theta - \tau) \tag{14}
\]

where \(J_n\) is the Bessel’s function of the \(n\)-th order. The unperturbed Hamiltonian, as expected, is just a function of the action, and the action as defined in Eq. (13) is a positive quantity. The Hamiltonian is periodic in \(\tau\) with period \(2\pi\). A surface-of-section plot is obtained by numerically integrating the equations of motion corresponding to the above Hamiltonian and plotting the action-angle coordinates of the orbit after every \(2\pi\) interval in \(\tau\). There are two distinct cases which are of interest and will be discussed below: the on-resonance case when \(1/\omega_0 = \) integer (i.e. the wave frequency is a harmonic of the cyclotron frequency), and the off-resonance case when \(1/\omega_0\) is close to an integer.

For the case of \(1/\omega_0 = 5\), corresponding to an on-resonance case, and \(\epsilon = 10^{-2}\), a plot of the surface-of-section is shown in Fig. 15a. The hyperbolic and the elliptic fixed points can be easily identified in the figure and it is interesting to note that neighboring islands, vertically or horizontally, have a common separatrix. Furthermore, all the separatrices are connected in such a way that a particle started on any given separatrix can, in principle, move to arbitrary values of the action along the separatrices. This phase-space picture does not change in its essential characteristics for any amplitude \(\epsilon < 10^{-2}\), however infinitesimally small the amplitude becomes, provided \(\epsilon > 0\). Thus, unlike the standard map where the trapping width corresponding to the primary island (island of order 1) increased in size as the square-root of the amplitude for small amplitudes of the perturbation, the trapping width corresponding to the primary islands remains essentially independent of amplitude for small amplitudes. There are no KAM surfaces separating the primary islands. The Chirikov resonance overlap criterion would not seem to apply since the neighboring islands overlap, sharing a common separatrix, for arbitrarily small amplitudes. The phase-space plot in Fig. 15a can be easily understood.
Consider a canonical transformation of the Hamiltonian in Eq. (14) to a new set of canonical variables \((\psi, J)\) which are defined as: \(\psi = m\theta - \tau\), \(J = I/m\), where \(m\) is an integer. The transformed Hamiltonian is:

\[
\overline{H}(\psi, J; \tau) = (m\omega_0 - 1)J + \epsilon J_m \left( \sqrt{\frac{2mJ}{\omega_0}} \right) \cos(\psi) + \epsilon \sum_{n = -\infty}^{\infty} J_n \left( \sqrt{\frac{2mJ}{\omega_0}} \right) \cos \left\{ \frac{n}{m} \psi + \left( \frac{n - m}{m} \right) \tau \right\}
\]

(15)

In the on-resonance case, there exists an integer \(m\) such that \(m\omega_0 = 1\). Since the terms under the summation sign depend explicitly on time the average effect on the motion of the particle can be ignored for small amplitudes. The leading order Hamiltonian in this case is just \(\overline{H}_0(\psi, J; \tau) = \epsilon J_m \cos(\psi)\), and the corresponding equations of motion are:

\[
\frac{d\psi}{d\tau} = \epsilon \frac{m}{\sqrt{2J}} J'_m \left( m\sqrt{2J} \right) \cos(\psi) \quad (16a)
\]

\[
\frac{dJ}{d\tau} = \epsilon J_m \left( m\sqrt{2J} \right) \sin(\psi) \quad (16b)
\]

where the prime denotes a derivative with respect to the argument. There are two sets of fixed points of Eqs. (16a) and (16b), given by \((\psi_h, J_h)\) and \((\psi_e, J_e)\), which satisfy, respectively, the following equations:

\[
\cos(\psi_h) = 0 \quad \text{and} \quad J_m \left( m\sqrt{2J_h} \right) = 0 \quad (17a)
\]

\[
\sin(\psi_e) = 0 \quad \text{and} \quad J'_m \left( m\sqrt{2J_e} \right) = 0 \quad (17b)
\]

It is easy to show that the first set of conditions, Eq. (17a), corresponds to hyperbolic fixed points while the second set of conditions, Eq. (17b), corresponds to elliptic fixed points. Let us define \(j_{m,s}\) and \(j'_{m,s}\) \((s = 1, 2, \ldots)\) to be such that \(J_m(j_{m,s}) = 0\) and \(J'_m(j'_{m,s}) = 0\), respectively. Then, for a given \(s\), hyperbolic fixed points are located at \(J_{h,s} = j_{m,s}^2/(2m^2)\) and \(\psi_{h,s} = (2l + 1)\pi/2 \quad (l = 0, 1, 2, \ldots)\), and elliptic points are located at \(J_{e,s} = j_{m,s}^2/(2m^2)\) and \(\psi_{e,s} = l\pi \quad (l = 0, 1, 2, \ldots)\). Since a surface-of-section for an orbit is obtained by determining the action-angle coordinates of a particle every \(2\pi\) steps in \(\tau\), and since the angle coordinate \(\theta\) is periodic with period \(2\pi\), then, from the transformation equations
relating \((\psi, J)\) to \((\theta, I)\), the hyperbolic and elliptic points will be located at \(I_{h,s} = J_{m,s}^2/(2m)\), \(\theta_{h,s} = (2l + 1)\pi/(2m)\), and \(I_{e,s} = J_{m,s}^2/(2m)\), \(\theta_{e,s} = l\pi/m\), for \(s = 1, 2, \ldots\) and \(l = 0, 1, \ldots, 2m - 1\). Thus, for any given \(s\) there will be \(2m\) hyperbolic and \(2m\) elliptic fixed points.

For \(m = 5\) the first couple of zeros of the Bessel’s function and its derivative are [19]: \(j_5,1 \approx 8.77, j_5,2 \approx 12.34, j_5',1 \approx 6.42\) and \(j_5',2 \approx 10.52\). Then the hyperbolic points are at \(I_{h,1} \approx 7.69\) and \(I_{h,2} \approx 15.22\). For each of these actions the hyperbolic points are at \(\theta = \pi/10, 3\pi/10, 5\pi/10, \ldots, 19\pi/10\). The elliptic points are at \(I_{e,1} \approx 4.12\) and \(I_{e,2} \approx 15.23\). For each of these actions the elliptic points are at \(\theta = 0, \pi/5, 2\pi/5, \ldots, 9\pi/5\). These locations of the fixed points are in good agreement with those in Fig. 15a that are determined by numerically integrating the complete equations of motion.

The simple analysis given above breaks down as the amplitude of the perturbation is increased since the terms under the summation sign in Eq. (15) cannot be ignored. For \(1/\omega_0 = 5\) and \(\epsilon = 5 \times 10^{-2}\) the surface-of-section plot is shown in Fig. 15b. From this figure it is apparent that a chaotic layer in phase space develops around the separatrices. This chaotic layer grows in thickness as the amplitude of the perturbation is increased. Also, for a fixed amplitude, the thickness increases with decreasing action. This is known as web stochasticity and has been discussed in the literature [20]. As the amplitude is increased the connected region of chaos expands. However, unlike the standard map which is periodic in action, the chaotic phase space is always bounded in action for any finite amplitude. This is apparent from Fig. 15b where we note that the regions of large action are not chaotic while the regions of smaller action are chaotic.

In the off-resonance case the wave frequency is not a harmonic of the cyclotron frequency. Let us consider the slightly off-resonance case when \(\omega_0 = (1 - \delta_m)/m\) where \(m\) is an integer and \(|\delta_m| \ll 1\). Using arguments similar to those for the on-resonance case, the leading order Hamiltonian for small amplitudes is:

\[
H_0(\psi, J; \tau) = -\delta_m J + \epsilon J_m \left(m\sqrt{2J}\right) \cos(\psi) \tag{18}
\]

where in the argument of the Bessel’s function we have made use of the assumption that \(|\delta_m| \ll 1\). The corresponding equations of motion are:

\[
\frac{d\psi}{d\tau} = -\delta_m + \epsilon \frac{m}{\sqrt{2J}} J_m' \left(m\sqrt{2J}\right) \cos(\psi) \tag{19a}
\]

\[
\frac{dJ}{d\tau} = \epsilon J_m \left(m\sqrt{2J}\right) \sin(\psi) \tag{19b}
\]
For $\epsilon = 0$, the leading order Hamiltonian in Eq. (18) is not zero, and from Eq. (19b) $J$ is a constant of the motion. Thus, the surface-of-section will be made up of lines of constant $I$. Furthermore, for $\epsilon = 0$, there are no fixed points for Eqs. (19a) and (19b). There is a threshold value of $\epsilon$ for which there will exist fixed points of these equations. The conditions for the existence of fixed points $(\psi_h, J_h)$ and $(\psi_e, J_e)$ is:

$$J_m \left( m \sqrt{2J_h} \right) = 0 \quad \text{and} \quad \epsilon - \frac{m}{\sqrt{2J_h}} J'_m \left( m \sqrt{2J_h} \right) \cos(\psi_h) = \delta_m \quad (20a)$$

$$\psi_e = l\pi \quad \text{and} \quad (-1)^l \epsilon \frac{m}{\sqrt{2J_e}} J'_m \left( m \sqrt{2J_e} \right) = \delta_m \quad (20b)$$

where $l = 0, 1, 2, \ldots$. It can be easily shown that, if the above conditions are satisfied, Eq. (20a) will give the hyperbolic fixed points and Eq. (20b) will give the elliptic fixed points. From Eq. (20a) the condition for the existence of a hyperbolic fixed point is:

$$\epsilon \geq \left| \frac{\delta_m \xi_h}{m^2 J'_m(\xi_h)} \right| \quad (21)$$

where $\xi_h = m \sqrt{2J_h}$ satisfies $J(\xi_h) = 0$. From Eq. (20b) the condition for the existence of elliptic fixed points at a given $\xi_e = m \sqrt{2J_e}$ is that:

$$\epsilon \geq (-1)^l \frac{\delta_m \xi_e}{m^2 J'_m(\xi_e)} \quad (22)$$

Since the local maxima of $|J'(\xi)|$ occur for those $\xi$ where $J(\xi) = 0$, Eqs. (21) and (22) give the same condition for the threshold amplitude for the existence of hyperbolic and elliptic fixed points. One can easily show from the above expressions that the threshold amplitude increases for increasing action $I$ and decreases for increasing harmonic number $m$.

As an example of a slightly off-resonance case we choose $\omega_0 = 1/5.05$, $m = 5$, so that $\delta_m \approx 0.0099$. From the conditions in Eqs. (21) and (22) we find that the threshold amplitude for the existence of elliptic (corresponding to $l$ being an even integer) and hyperbolic fixed points is $\epsilon_{th} \approx 0.0134$. Elliptic fixed points (corresponding to $l$ being an odd integer) have a threshold amplitude of $\epsilon_{th} \approx 0.0136$. At this amplitude there appear another set of hyperbolic fixed points. This is borne out by numerical integration of the exact equations of motion. Figs. 16a and 16b show the surface-of-section plots for $\epsilon = 10^{-2}$, which is just below the threshold amplitude for the appearance of fixed points, and $\epsilon = 2.5 \times 10^{-2}$, respectively. It should be noted that the phase-space islands are not connected to each other as in the on-resonance case. As
the amplitude is increased, fixed points appear at larger values of the action while there appear regions of chaos for lower values of the action. This is evident in Fig. 16c which gives the surface-of-section for $\epsilon = 5 \times 10^{-2}$. The appearance of chaos is similar in fashion to the on-resonance case discussed earlier. Chaos appears near the separatrix and this region widens in phase space as the amplitude is increased. The chaotic region is always bounded in action.

VII. CONCLUSIONS – GLOBAL DESCRIPTIONS

We have illustrated features of chaotic particle motion in prescribed coherent fields and conditions for the existence of such motion using models which are paradigms of nonlinear wave-particle dynamics. When there is a region in phase space that is chaotic, an initial distribution of particles, which is localized in action in the chaotic part of phase space, will spread out to cover the entire region of connected chaos in action. This was shown for the standard map, in Figs. 8a and 8c, where a localized circle of initial conditions evolved to fill up the entire chaotic phase space. It was also shown that nearby orbits diverged exponentially, thereby making it difficult/impossible to keep track of an orbit of a particle accurately in the chaotic phase space, thus rendering the interaction irreversible.

Based upon the above, one may attempt to describe the statistical evolution of a distribution of particles when the dynamics is chaotic. If an initial distribution of particles is localized in action near small values of the action in chaotic phase space, they will, on the average, gain energy and/or momentum from the wave as they spread to cover higher values of the actions in the chaotic phase space. Particles which stick to KAM surfaces will, on the average, not gain energy and/or momentum from the waves. In the chaotic phase space, the particle dynamics appears as if the particles are acted upon by random forces. In such circumstances one attempts to describe the evolution of a distribution function through the Fokker-Planck equation, where the chaotic dynamics is accounted for by a diffusion coefficient. Although this is very useful as a global description, there are many uncertainties and limitations associated with such a description of Hamiltonian chaos. In cases of physical interest, the chaotic phase space is bounded, as in Figs. 15b and 16c, and there are regions of coherent motion imbedded in regions of chaotic phase space. Also, in chaotic phase space there persist long-time correlations, i.e. the orbits of particles do not become randomized or ergodic on a short time scale (e.g. if the standard map represented Brownian motion, then the correlation time would be one iterate of the map – this is clearly not the case since, based on the computational results in Figs. 8a and 10, correlations exist over many iterates of the map). These issues complicate the derivation of
the Fokker-Planck description and the associated evaluation of the diffusion coefficient. Some aspects of these issues remain unresolved. A study of the Fokker-Planck description and the diffusion coefficient is beyond the scope of this tutorial. However, the reader is referred to [2, 21] for details and further discussion. Finally, we want point out that in some chaotic dynamics of wave-particle interactions one encounters “chaotic streaming” [22, 23] for which a Fokker-Planck description is inadequate [24, 25].

ACKNOWLEDGEMENTS

This work was supported by NASA Grant No. NAGW-2048, by DOE Grant No. DE-FG02-91ER-54109, and by NSF Grant No. ECS-88-22475.

REFERENCES

APPENDIX A: PARTICLE MOTION IN A PLANE WAVE

Consider a particle which at time $t = 0$ has $q = q_0$ and $p = p_0$. Since $H(q_0, p_0) = p_0^2/2 + \alpha \cos(q_0) \equiv H_0$ is a constant of the motion, i.e. the particle’s $q$ and $p$ at some time $t$ are such that $p^2/2 + \alpha \cos(q) = H_0$, using Eq. (2a) we get:

$$\int_{q_0}^{q} dq' \frac{\sqrt{H_0 - \alpha \cos(q')}}{\sqrt{H_0 - \alpha \cos(q_0)}} = \sqrt{2} \int_{0}^{t} dt'$$  \hspace{1cm} (A1)

The integral on the left-hand side can be evaluated analytically [26, 27] for the case of trapped and passing particles.

Trapped Particles

Since $|H_0| \leq \alpha$ for trapped particles, $|q| \leq \cos^{-1}(H_0/\alpha)$. The maximum value of $|q|$ corresponds to the turning point of the trapped particle. Without loss of generality, we can set $q_0 = \pi$. Then the solution of Eq. (A1) is:

$$q = 2 \cos^{-1} \left[ \kappa \sin \left( K(\kappa^2) + \sqrt{\alpha} t \right) \right]$$  \hspace{1cm} (A2)

where $\kappa^2 = (H_0 + \alpha)/(2\alpha)$, $K(\kappa^2)$ is the complete elliptic integral of the first kind, and $\sin$ is the sine amplitude Jacobian elliptic function [19].
(Note that the complete elliptic integral of the first kind will be referred to as $K(\kappa^2)$ in order to distinguish it from the amplitude of the standard map.) From Eq. (A2) it readily follows that:

$$ p = -2\kappa\sqrt{\alpha} \cn(K(\kappa^2) + \sqrt{\alpha} t \mid \kappa^2) $$

where $\cn$ is the cosine amplitude Jacobian elliptic function [19].

**Passing Particles**

For untrapped particles $H_0 > \alpha$ and we assume that $q_0 = 0$. Then, Eq. (A1) gives:

$$ q = 2\cos^{-1} \left[ \sn \left( K(\beta^2) + \frac{\sqrt{\alpha}}{\beta} t \mid \beta^2 \right) \right] $$

where $\beta = 1/\kappa$. Using Eq. (2a):

$$ p = -2\sqrt{\alpha} \beta \dn \left( K(\beta^2) + \frac{\sqrt{\alpha}}{\beta} t \mid \beta^2 \right) $$

where $\dn$ is the delta amplitude Jacobian elliptic function [19].

**APPENDIX B: ACTION-ANGLE VARIABLES FOR THE NONLINEAR OSCILLATOR**

The action-angle variables can be obtained by solving the Hamilton-Jacobi equation [4]. Here we give the results without going into details of the Hamilton-Jacobi theory.

The action variable is defined as:

$$ I = \frac{1}{2\pi} \oint dq \ p = \frac{1}{\sqrt{2\pi}} \oint \sqrt{H_0 - \alpha \cos(q)} $$

where the integral is over one orbit for a trapped particle, and from 0 to $2\pi$ for the passing particle. The action is a constant of the motion (being a function of $H_0$ only) and the Hamiltonian can be expressed as a function of $I$ only. The angle coordinate is given by:

$$ \theta = \frac{1}{\sqrt{2}} \frac{\partial H_0}{\partial I} \int_{q_0}^q dq' \frac{1}{\sqrt{H_0 - \alpha \cos(q)}} $$

where $q_0$ is the initial coordinate of the particle. As in Appendix A we treat passing and trapped particles separately.
Trapped Particles

The integral in Eq. (B1), tabulated in [27], is evaluated by appropriately accounting for the turning points of the trapped particle. This gives:

\[ I = \frac{8}{\pi} \sqrt{\alpha} \left[ (\kappa^2 - 1) K(\kappa^2) + E(\kappa^2) \right] \tag{B3} \]

where \( E(\kappa^2) \) is the complete elliptic integral of the second kind [19]. The action is a function of the Hamiltonian only. This relationship can be inverted to express the Hamiltonian as a function of the action only. Thus, the action is a constant of the motion.

Since all trapped particles pass through the point \( q = \pi \), the integral in Eq. (B2) for the angle is evaluated by setting \( q_0 = \pi \). Again, this integral is tabulated in [27] and we find:

\[ \theta = \frac{\pi}{2K(\kappa^2)} F(\eta, \kappa^2) \tag{B4} \]

where \( F \) is the elliptic integral of the first kind [19], and \( \kappa \sin(\eta) = \cos(q/2) \). The frequency of oscillation, \( \Omega_{tr} \), for the trapped particles is given by the time derivative of Eq. (B4). This is equivalent to taking the derivative of the Hamiltonian with respect to the action [4]. Thus:

\[ \Omega_{tr} \equiv \frac{d\theta}{dt} = \frac{dH_0}{dI} = \frac{\pi \sqrt{\alpha}}{2 K(\kappa^2)} . \tag{B5} \]

Passing Particles

The action is:

\[ I = \frac{4}{\pi} \sqrt{\alpha \kappa E(\beta^2)} \tag{B6} \]

where \( \beta \) is defined in Appendix A.

The angle \( \theta \) is obtained from Eq. (B2) by setting \( q_0 = 0 \). The resulting integral is tabulated (see [27]) so that:

\[ \theta = \frac{\pi}{K(\beta^2)} F(\xi, \beta^2) \tag{B7} \]

where \( \sin(\xi) = \sin(q/2)/\cos(\eta) \). The frequency of oscillation for the passing particles, where a single oscillation corresponds to going from \( q = 0 \) to \( q = 2\pi \), is:

\[ \Omega_p \equiv \frac{d\theta}{dt} = \frac{dH_0}{dI} = \frac{\pi \kappa \sqrt{\alpha}}{K(\beta)} . \tag{B8} \]
APPENDIX C: DERIVATION OF THE STANDARD MAP

Here we solve Eqs. (9a) and (9b) and derive the standard mapping
equations. Let $\theta_{n-\epsilon}$ and $I_{n-\epsilon}$ be the angle and action of the particle,
respectively, at time $\tau_0 = n - \epsilon$ (where $\epsilon \ll 1$). Thus, $\tau_0$ is the time
just before the particle receives its $n$-th kick. The action of the particle
$I(\tau)$ at a subsequent time $\tau$, where $n < \tau < (n + 1) - \epsilon$, is obtained by
integrating Eq. (9b):

$$I(t) - I_{n-\epsilon} = K \sin(\theta_n) \quad (C1)$$

where $\theta_n = \theta(n)$ is the angle of the particle at time $n$. Then, from Eq.
(9a), the angle of the particle at time $t$ is:

$$\theta(t) - \theta_{n-\epsilon} = \left( I_{n-\epsilon} + K \sin(\theta_n) \right) (\tau - \tau_0) \quad (C2)$$

If $\theta_{n+1-\epsilon}$ and $I_{n+1-\epsilon}$ are the angle and action of the particle, respectively,
at time $\tau = (n + 1) - \epsilon$, then, in the limit $\epsilon \to 0$ Eqs. (C1) and (C2)
reduce to the mapping equations (7a) and (7b).
Figure 1. Surfaces of constant energy for the nonlinear oscillator showing the trapping width, $p_{tr}$, trapped motion, untrapped motion, and the separatrix.
Figure 2. Surface-of-section for $K = 0.1$ showing a first order phase-space island of trapping width $2\sqrt{K}$ centered at $I = 0$. 
Figure 3. Surface-of-section for $K = 0.5$ showing the existence of higher order islands.
Figure 4. A magnified view of the rectangular region marked in Fig. 3 near the hyperbolic fixed point at $\theta = 0, I = 0$. 
Figure 5. Surface-of-section for $K = 1$. This $K$ is slightly greater than the critical $K_c \approx 0.9716$, for which the last KAM surface is destroyed.
Figure 7a. Phase-space portrait of a set of particles, in the coherent part of phase space, at \( n = 0 \) and at \( n = 100 \) for \( K = 1.5 \).
Figure 7b. A magnified view of the box shown in Fig. 7a.
Figure 8a. Phase-space portrait of a set of particles, in the chaotic part of phase space, at \( n = 0 \), \( n = 4 \), and \( n = 7 \).

Initial Conditions
Figure 8b. A magnified view of the box shown in Fig. 8a.
Figure 8c. Phase-space portrait at $n = 100$ for the set of particles shown in Fig. 8a.
Figure 9. The magnitude of the difference in action of two particles versus $n$ ($K = 1.5$), located in the coherent part of phase space.
Figure 10. Same as Fig. 9, except that the particles are initially located in the chaotic part of phase space.
Figure 11a. Surface-of-section for a single particle ($K = 1.5$).
Figure 11b. Frequency spectrum of the orbit in Fig. 11a.
Figure 12a. Surface-of-section for a single particle ($K = 1.5$).
Figure 13a. Surface-of-section for a single particle ($X = 1.5$).
Figure 13b. Frequency spectrum of the orbit in Fig. 13a.
Figure 14a. Surface-of-section for a single particle ($\kappa = 4.5$).
Figure 14b. Frequency spectrum of the orbit in Fig. 14a.
Figure 15a. Surface-of-section for $\omega_0 = 1/5$ and $\epsilon = 10^{-2}$. 
Figure 15b. Surface of section for $\omega_0 = 1/5$ and $\epsilon = 5 \times 10^{-2}$. 
Figure 16a. Surface-of-section for $\omega_0 = 1/5.05$ and $\epsilon = 10^{-2}$. 
Figure 16b. Surface-of-section for $\omega_0 = 1/5.05$ and $\epsilon = 2.5 \times 10^{-2}$. 
Figure 16c. Surface-of-section for $\omega_0 = 1/5.05$ and $\epsilon = 5 \times 10^{-2}$. 