New Mechanisms of Ion Energization by Multiple Electrostatic Waves in a Magnetized Plasma

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It is shown that with *two or more* electrostatic waves propagating across a uniform and steady-state magnetic field, ions with arbitrarily low initial energies can be nonlinearly accelerated to very high energies. This acceleration of ions requires an appropriate choice of wave amplitudes, frequencies, and wavenumbers. The new nonlinear energization mechanism is analyzed and shown to involve both coherent and chaotic dynamics in a new way.

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The interaction between electrostatic waves and charged particles in a magnetized plasma plays a fundamental role in laboratory, space, and astrophysical plasmas. The dynamics of particles in the presence of these waves is generally nonlinear, and this wave-particle interaction is a paradigm for nonlinear Hamiltonian dynamics. The waves in the plasma can be internally generated, due to an instability, or excited by coupling to an external source. In the test particle approach to the study of nonlinear particle dynamics, the plasma particles interacting with the waves are assumed to be a small fraction of the total population and, thus, they can be assumed to have no effect on the propagation characteristics of the waves. Consequently, in such wave-particle interactions the wave properties, once prescribed, remain the same. The interaction of ions in a magnetized plasma with a single, plane electrostatic wave, in the lower-hybrid frequency range (above the ion cyclotron frequency), and propagating across the ambient magnetic field, has been studied in considerable detail analytically [1-4] and experimentally [5-7]. It is found that if the wave amplitude exceeds a threshold values, then the dynamical phase space is divided into two distinct regions - chaotic and coherent. The orbit of an ion can only be in one of the two regions. In the chaotic region, which is bounded in energy from above and below, an ion gains energy in approximately an ergodic fashion. In the coherent region the ion energy remains approximately a constant.

In this letter we show that the situation becomes dramatically different when ions interact with two or more electrostatic waves having appropriately chosen frequencies and wavenumbers. We analytically deduce, and numerically confirm, a new phenomenon of nonlinear, coherent energization by which ions, with arbitrarily low initial energies, started off in (what for a single wave would be) the coherent part of phase space can be energized into the chaotic domain, where they quickly gain much more energy than they would from a single wave of similar amplitude. For an appropriate choice of the wave parameters, the waves can also be used to remove energy from the ions by this nonlinear process. The new nonlinear energization or de-energization mechanism has the potential for use in a wide range of important physical problems: understanding the energization of ionospheric ions to the magnetosphere; new means of plasma heating with two or more lower-hybrid waves; ion accelerators driven by multiple waves; de-energization of α-particles in a fusion plasma. Details of these will be presented elsewheres.

We study the dynamics of an ion of mass m and charge q in a uniform magnetic field $\vec{B} = B_0\hat{z}$, and being perturbed by a discrete spectrum of electrostatic waves $\vec{E} = \hat{x}\sum_{i=1}^{N} E_i \sin(k_i x - \omega_i t + \varphi_i)$. We normalize time to Ω^{-1}, where $\Omega = qB_0/m$ is the cyclotron angular frequency, and length to k_1^{-1}. This defines the dimensionless variables $X = k_1 x$ and $\tau = \Omega t$. In the absence of an electric field perturbation ($\vec{E} = 0$), we can define action-angle variables for the dynamics. The normalized action is $I = X^2/2 + \dot{X}^2/2$, where $\dot{X} = dX/d\tau$. The angle θ is defined by $X = \rho\sin\theta$, $\dot{X} = \rho\cos\theta$, where $\rho = \sqrt{2I}$ is the normalized Larmor radius. In the variables $(I, \Phi = \theta - \tau)$ the dynamics of an ion in the electrostatic fields of the waves is defined by the Hamiltonian

$$H = \sum_{i=1}^{N} (\varepsilon_i/\kappa_i)\cos[\kappa_i \rho \sin(\Phi + \tau) - \nu_i \tau + \varphi_i] \quad (1)$$

where $\kappa_i = k_i/k_1$, $\nu_i = \omega_i/\Omega$, and $\varepsilon_i = (k_1 q E_i)/(m\Omega^2)$.

We first describe the dynamics of a low energy ion, i.e. we restrict ourselves to $\rho \leq \rho_l = \min(\nu_i - \sqrt{\varepsilon_i})$. For one wave, such an ion's orbit lies in the coherent part of phase space [1,2,8]. For multiple waves, the ion's orbit can be very accurately described by an integrable Hamiltonian. Therefore, as long as $\rho \leq \rho_l$ the ion's dynamics will be referred

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to as "coherent". Unlike in the case of one wave, for multiple waves separated in frequency by an amount close to an integer multiple of the cyclotron frequency, and nearly the same wavenumbers, such a low energy ion can be energized to high energies, and possibly into the chaotic domain, by the waves. This is illustrated in Figure 1, obtained by a numerical integration of the exact orbit from (1), which shows that a low-energy ion can be *coherently energized by multiple electrostatic waves.* In the presence of a single wave the energy of this ion would remain at its initial value. The ion dynamics for $\rho \leq \rho_l$ can be accurately described by a second order (in ε_i) perturbation analysis performed on (1), as is illustrated in Fig. 1. The details of this analytical analysis, given elsewheres [9], are too lengthy to be included in this letter. We begin the study with the case of two waves. Using the formalism of Lie transform [10], we carry out a canonical perturbation analysis of (1) to second order in ε_i. We define a (near identity) canonical transformation $(I, \Phi) \rightarrow (\tilde{I}, \tilde{\Phi})$, in the phase-space region corresponding to $\rho \leq \rho_l$ for which (1) is transformed into a new Hamiltonian $\tilde{H}$ given by:

$$\tilde{H} = \varepsilon_1^2 F_1 + \varepsilon_2^2 F_2 + \delta \varepsilon_1 \varepsilon_2 S_\kappa cos[(\nu_1 - \nu_2)\tilde{\Phi} + \varphi] \tag{2}$$

where $\kappa = k_2/k_1$, $\delta = 1$ if $\nu_1 - \nu_2$ is an integer and zero otherwise, $\varphi = \varphi_1 - \varphi_2$, F_1, F_2, and S_κ are functions of $\tilde{\rho}$, $F_2(\tilde{\rho}) = (1/2\kappa\tilde{\rho}) \sum_{m=-\infty}^{+\infty} [m J_m(\kappa\tilde{\rho}) J_m'(\kappa\tilde{\rho})] / \; [\nu_2 - m]$, where the prime denotes the derivative with respect to the argument of the function. F_1 is obtained from F_2 by replacing ν_2 by ν_1 and κ by 1, and $S_\kappa(\tilde{\rho}) = (1/2\tilde{\rho}) \sum_{m=-\infty}^{+\infty} [m J_m(\tilde{\rho}) J'_{\nu_2 - \nu_1 + m}(\kappa\tilde{\rho})]/[\nu_1 - m] +$ $(1/2\kappa\tilde{\rho}) \sum_{m=-\infty}^{+\infty}$ $[m J_m(\kappa\tilde{\rho}) J'_{\nu_1 - \nu_2 + m}(\tilde{\rho})]/[\nu_2 - m]$. Eq. (2) is valid when neither ν_1 nor ν_2 is an integer (the generalization to the case of integer ν_1 and/or ν_2 is straightforward [9]). Since $\tilde{H}$ is time independent, the ion orbits are obtained by solving $\tilde{H} = const$. These orbits are found to be close to those obtained from the Hamiltonian (1). Thus, by solving $\tilde{H} = const$ we can determine the dependence of ion energization on the characteristic parameters of the two waves.

From (2), if $\delta = 0$, i.e. if $\nu_1 - \nu_2$ is not an integer, then $\tilde{H} = const$ yields that $\tilde{\rho} = const$, and the ion is not energized by the waves. (Actually the ion can be energized even if $\nu_1 - \nu_2$ is not exactly an integer. For $\nu_1 - \nu_2 = n + \delta\nu$, where n is an integer and $0 < \delta\nu << 1$, the Hamiltonian (2) no longer gives a good approximation to the ion's orbit [11]. The condition on $\delta\nu$ to accelerate ions requires that $\varepsilon_1\varepsilon_2 S_\kappa$ be at least comparable to $\delta\nu\rho^2/2n$ [9,11]. For such $\delta\nu$, the results described below are still valid [9,11].)

If $\nu_1 - \nu_2$ is an integer, the energization of an ion will increase as $\varepsilon_1\varepsilon_2 S_\kappa(\tilde{\rho})$ is increased compared to $\varepsilon_1^2 F_1(\tilde{\rho}) + \varepsilon_2^2 F_2(\tilde{\rho})$. Since $(\varepsilon_1\varepsilon_2)/(\varepsilon_1^2 + \varepsilon_2^2)$ is a maximum when $\varepsilon_1 = \varepsilon_2$, the acceleration of ions is a maximum when the two waves have the same amplitude.

From (2) we note that the phases of the two waves play no role in the acceleration mechanism. So, without loss of generality, we can assume that $\varphi_1 = \varphi_2 = 0$. For $\varepsilon_1 = \varepsilon_2 = \varepsilon$ and $\delta = 1$, (2) becomes

$$\tilde{H} = \varepsilon^2 \left\{ F_1(\tilde{\rho}) + F_2(\tilde{\rho}) + S_\kappa(\tilde{\rho}) \cos[(\nu_1 - \nu_2)\tilde{\Phi}] \right\}. \tag{3}$$

The orbits of (3) are independent of ε. Thus, there is no threshold value for the amplitude of the waves for which the ions become energized. However, the time needed to energize an ion is proportional to ε^{-2}.

Solving $\tilde{H} = \varepsilon^2 const$ yields $\cos[(\nu_1 - \nu_2)\tilde{\Phi}] = [const - F_1(\tilde{\rho}) - F_2(\tilde{\rho})]/S_\kappa(\tilde{\rho})$. This implies that if the initial value of the ion Larmor radius, ρ_0, lies in between two zeros of S_κ, ρ_1^* and ρ_2^*, $\rho_1^* \leq \rho_2^*$, then the ion Larmor radius will always remain in between ρ_1^* and ρ_2^*. In particular, ρ_2^* is an upper bound of the maximum value of the ion Larmor radius. Thus, by varying κ, we can choose the level of energization of an ion. The first zero of S_κ, and, consequently the energy gained by an ion, tends to decrease with $\mid \kappa - 1 \mid$. For κ such that S_κ does not have any zero in $]0, \rho_l]$, an ion can reach values of ρ larger than ρ_l where its dynamics can no longer be considered as coherent if $\varepsilon > \varepsilon_{th} \simeq \min(\nu_i^{2/3}/4)$. In this case, we will say that the ion has reached the chaotic part of phase space.

If $\mid \nu_1 - \nu_2 \mid \leq 2$ an ion may reach the chaotic phase space regardless of its initial Larmor radius. However, for $\mid \nu_1 - \nu_2 \mid \geq 3$ there is a threshold in the initial ion Larmor radius for it to access the chaotic phase space. When $\rho \rightarrow 0$, S_κ scales as $\rho^{|\nu_1 - \nu_2|}$ while F_1 and F_2 scale as ρ^2, whatever ν_1 and ν_2. Therefore, for $\mid \nu_1 - \nu_2 \mid \geq 3$ and with $\rho \rightarrow 0$, $\varepsilon_1\varepsilon_2 S_\kappa$ becomes negligible compared to $\varepsilon_1^2 F_1 + \varepsilon_2^2 F_2$. Then, $\tilde{H} = const$ yields $\rho \simeq const$ and there is no energization of an ion. Thus, by varying the wave frequencies we can choose the ion population that is to be energized. In an experiment where the wave characteristics can be chosen, this gives considerable control on the energization of ions.

For more than two waves, a second order perturbation analysis still gives very accurate results in the region $\rho \leq \rho_l$. Since the Hamiltonian obtained from a second-order perturbation analysis involves the nonlinear interaction of pairs of waves, the results obtained for two waves can be easily generalized to an arbitrary number of waves [9]. In particular, it can be shown that for more than two waves there can be acceleration even if all the normalized wave frequencies are not separated by an amount close to an integer, provided that the energy of the waves with frequency differences close to an integer is comparable to that of the other waves [9].

We now address the case when an ion has accessed the chaotic part of phase space, i.e. we consider $\varepsilon > \varepsilon_{th}$ and those ions with $\rho \geq \rho_l$. In the case of one wave an ion cannot be energized beyond $\rho_{\max} = (2/\pi)^{1/3}(4\varepsilon\nu)^{2/3}$, the upper bound of the stochastic domain [1,2]. However, with more than one wave, we find that an ion can be energized beyond this limit. This illustrated in Fig. 2 where an ion reaches $\rho \simeq 48$. For a single wave, this ion would reach $\rho_{\max} \simeq 20.5$. Thus, the maximum energy gained by an ion is nearly 5 times larger than what it would gain from a single wave. Moreover, as can be seen in Fig. 2, the phase space at the high energies is not covered ergodically. We show that for ν_i=integer, an ion orbit actually remains close to the orbits found from a first order perturbation analysis performed on (1); thus, the maximum energy that can be gained by ions can be analytically estimated.

We begin this analytical study with the case of two waves whose frequencies are an integer multiple of the ion-cyclotron frequency, having the same wave numbers, and frequencies such that $\nu_2 = \nu_1 + 1$. Then, to first order in ε_i, we find that:

$$\tilde{\tilde{H}} = \varepsilon_1 J_{\nu_1} \cos(\nu_1 \tilde{\Phi} + \varphi_1) + \varepsilon_2 J_{\nu_1+1} \cos[\nu_2 \tilde{\Phi} + \varphi_2] \tag{4}$$

where the argument of J_{ν_1} and J_{ν_1+1} is $\tilde{\rho}$. A large argument expansion of the Bessel's functions yields $\tilde{\tilde{H}} \simeq \sqrt{2/(\pi\rho)}\{\varepsilon_1 \cos(R)\cos(\nu_1\Phi + \varphi_1) + \varepsilon_2 \sin(R)\cos[(\nu_1+1)\Phi + \varphi_2]\}$, where $R = \tilde{\rho} - \nu_1\pi/2 - \pi/4$. Solving for $\tilde{\tilde{H}} = 0$ yields

$$\tan(R) = -\frac{\varepsilon_1}{\varepsilon_2} \frac{\cos[\nu_1\tilde{\Phi} + \varphi_1]}{\cos[(\nu_1+1)\tilde{\Phi} + \varphi_2]} \tag{5}$$

It is sufficient to solve (5) in an interval of length π in $\tilde{\Phi}$, since changing $\tilde{\Phi}$ by $\tilde{\Phi} + \pi$ only changes the sign of the right-hand side of (5). Any semi-open interval, J, of length π contains exactly ν_1 zeros and $\nu_1 + 1$ poles, $(p_1, p_2, ..., p_{\nu_1+1})$, of the right-hand side of (5). Hence, for any relevant phase realizations, it is always possible to choose the interval J such that in any interval $]p_i, p_{i+1}[$ there is always one, and only one, zero of the right hand side of (5). Therefore, over each interval $]p_i, p_{i+1}[$, the right-hand side of (5) varies continuously from $-\infty$ to $+\infty$ (or from $+\infty$ to $-\infty$), which implies that R, and thus $\tilde{\rho}$, increases (or decreases) by π. Hence, over the entire interval J, $\tilde{\rho}$ increases (or decreases) by $\nu_1\pi$. An orbit solution of $\tilde{\tilde{H}} = 0$ has thus an extension of the order of $\nu_1\pi$ along the ρ axis. It is important to note that this extension does not depend on the amplitudes of the waves. Thus, an ion can be accelerated by two on-resonance waves, regardless of the amplitudes of the waves.

In the case of Fig. 2, $\nu_1\pi \simeq 28$, and the initial value of ρ is 8. From our analysis, we expect the maximum value of ρ to be close about 36. This is less than the value in Fig. 2 by a factor of about 25%. There are two reasons for this discrepancy. First, the large argument expansion of the Bessel functions lacks sufficient accuracy; a numerical solution of $\tilde{\tilde{H}} = const$, without making the large argument expansion, yields a maximum value of ρ close to 42. Second the orbit shown in Fig. 2 is clearly not a coherent orbit, and it is thus not similar to any orbit of $\tilde{\tilde{H}}$, even though it always remains close to orbits of $\tilde{\tilde{H}}$. Actually, solving $\tilde{\tilde{H}} = const$ yields not one, but a whole set of orbits, as shown in Fig. 3. For the dynamics defined by (1), some stochastic layers form about these orbits, which allows some of them to connect to each other. In the case of Fig. 2, four orbits are connected which allows the ion to reach a maximum value of ρ close to 48 instead of the value close to 42 predicted by $\tilde{\tilde{H}} = const$. Thus, even though an ion orbit from (1) is not exactly the same as the orbit from $\tilde{\tilde{H}}$, we can still obtain a good estimate of the maximum energy gained by an ion from the study of $\tilde{\tilde{H}}$. This is valid as long as the wave amplitudes are small enough so that ρ_{max} for a single wave is small compared to $\nu_1\pi$. When this is not the case, the maximum energies that an ion can attain in the case of two waves is approximately the same as ρ_{max} for one wave.

If the waves have different wavenumbers, then the first order orbits still have an extent along ρ-axis of the order of $\nu_1\pi$ provided that they do not have any vertical tangent when ρ varies from 0 to $\nu_1\pi$ [9]. This happens if the ratio of the two wavenumbers differ from unity by an amount less than $1/(2\nu_1)$ [9]. Then the energy gained by an ion is the same as when the wavenumbers are equal. If the ratio of the wavenumbers is very different from unity then the ensuing phenomena are too rich to be described in detail here, but are reported in [9]. The main result is that in most cases the energy gained by an ion is of the same order as when the wavenumbers are the same.

For an arbitrary discrete spectrum, the enhanced acceleration illustrated in Figs. 2 and 3 persists provided that the energy of the waves whose normalized frequencies are close to an integer are comparable to that of the other waves [9].

In conclusion, we have shown that, by the mechanism of nonlinear, coherent energization, electrostatic waves can accelerate ions whose initial energies are below the chaotic part of phase space. The dependence of the nonlinear, coherent acceleration on the characteristic parameters of the wave spectrum was derived using a second-order perturbation analysis. We also have shown that when an ion has accessed the chaotic part of phase space, the maximum energy attained can be much higher in the case of multiple waves than in the case of one wave. The ion energization is essentially independent of the amplitude of the waves: over a wide range of amplitudes, it corresponds to an increase of $k_\perp r_L \sim \pi\omega/\Omega$, where $k_\perp$ is a typical wave number, r_L is the ion Larmor radius, ω is a typical frequency and Ω is the ion cyclotron frequency. The analysis of new nonlinear acceleration mechanisms presented here provides a theoretical basis for understanding the transverse acceleration of ionospheric ions [12] in lower-hybrid solitary structures that are observed in the auroral ionosphere [13,14]. Details of this and other applications will be given elsewheres.

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FIGURE CAPTIONS

Fig. 1: Solid line: Poincaré section of the dynamics defined by (1) for six waves with frequencies such that $\nu_1 = 30$, $\nu_2 = 29$, $\nu_3 = 31.75$, $\nu_4 = 31.25$, $\nu_5 = 30.5$, $\nu_6 = 29.25$, wave numbers such that $\kappa_1 = 1$, $\kappa_2 = 0.98$, $\kappa_3 = 1$, $\kappa_4 = 1$, $\kappa_5 = 1$, $\kappa_6 = 0.98$, having all the same amplitudes $\varepsilon = 4.5$ and all having zero initial phase: $\varphi_i = 0$. The initial condition is $\rho(0) = 5$, $\Phi(0) = \pi/2$. Dashed line: Orbit obtained from second order perturbation theory for the same parameters as the solid line.

Fig. 2: Poincaré section of the dynamics defined by the Hamiltonian (1) in the case on 2 on-resonance waves, such that $\kappa_1 = \kappa_2 = 1$, $\nu_1 = 9$, $\nu_2 = 10$, $\varepsilon_1 = \varepsilon_2 = 3.24$. The initial condition is $\rho(0) = 8$ and $\Phi(0) = 0.94\pi$.

Fig. 3: Orbits solution of $\tilde{\tilde{H}} = const$ for the same parameters as in Fig. 2.

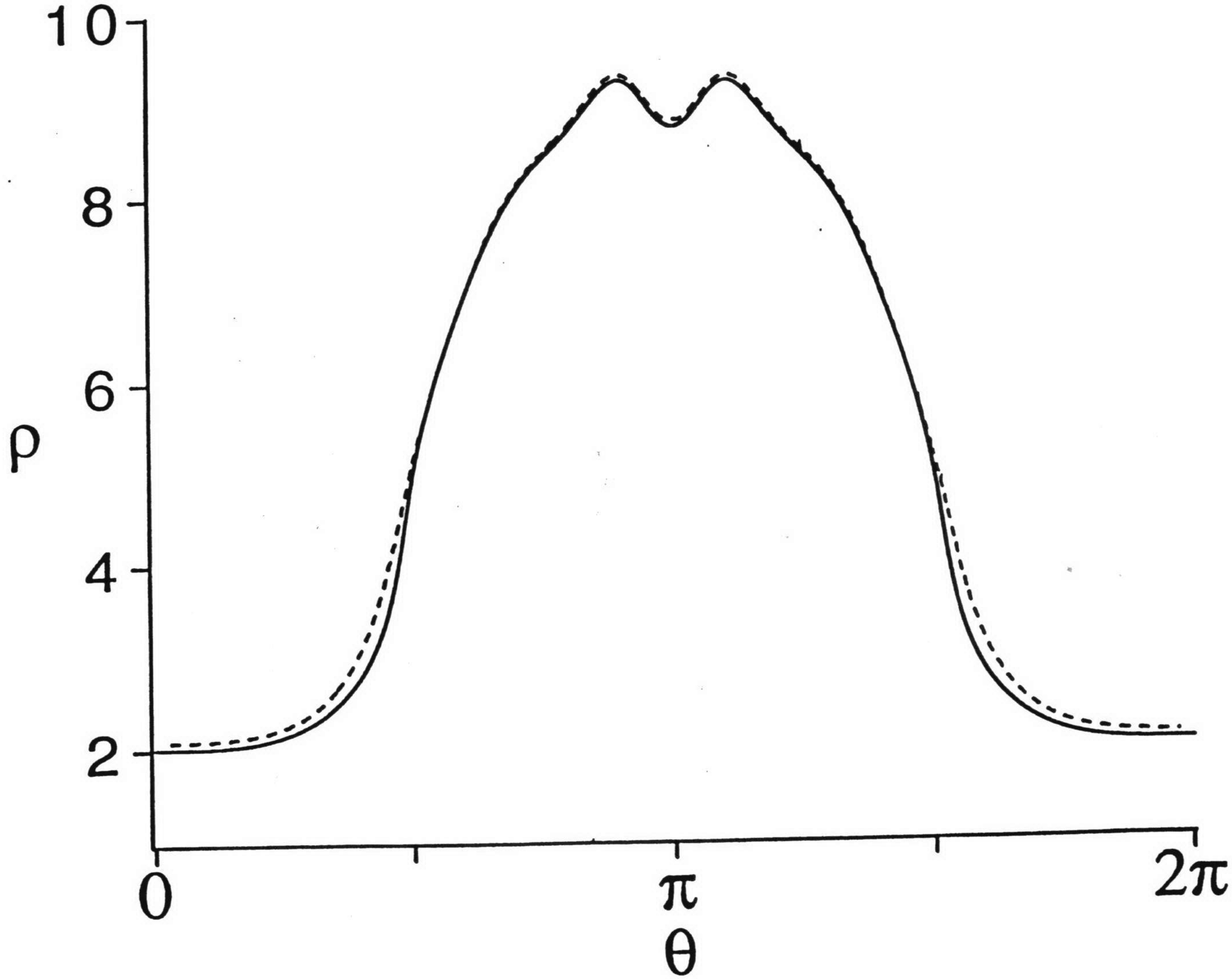

FIGURE 1

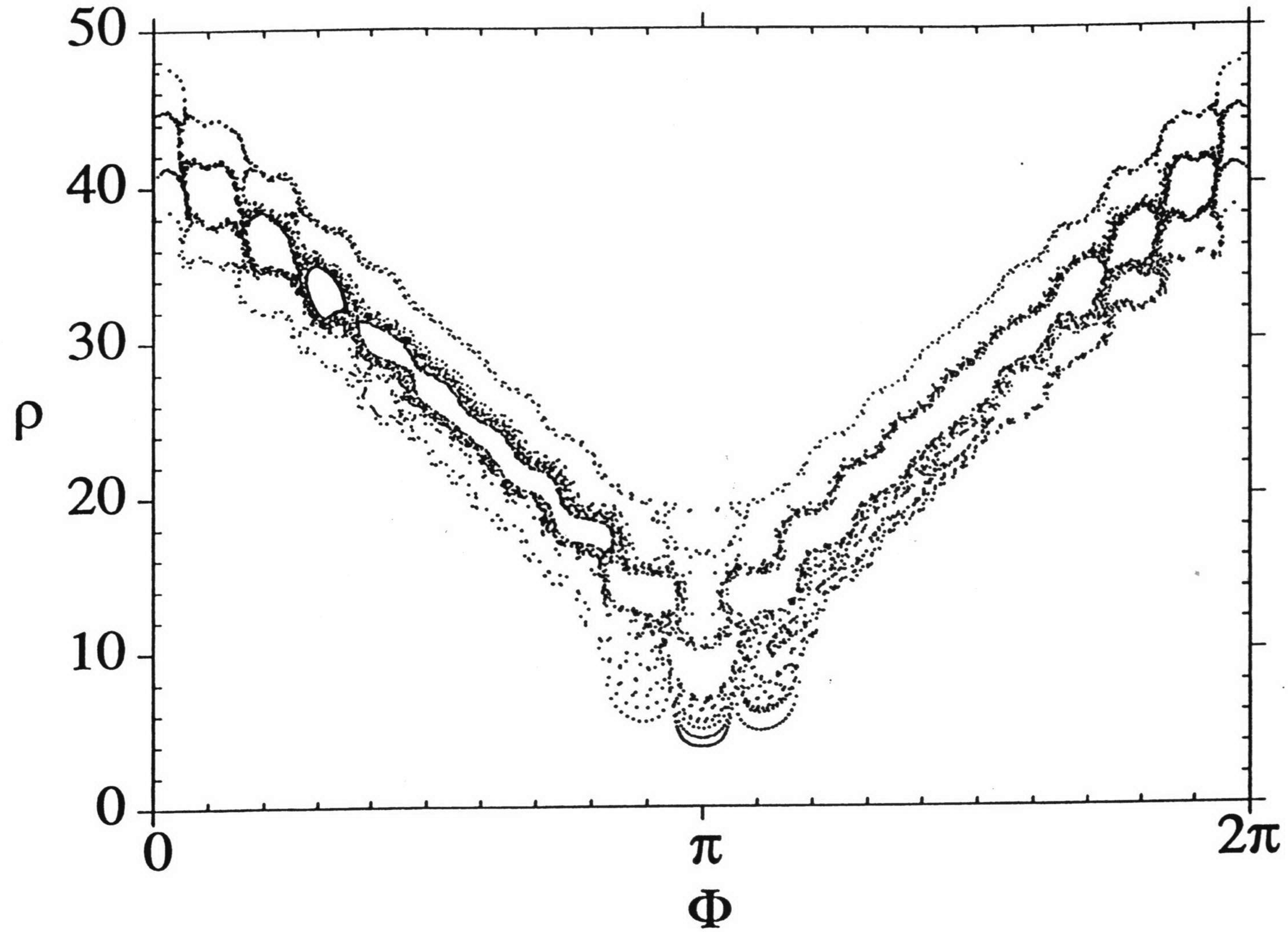

FIGURE 2

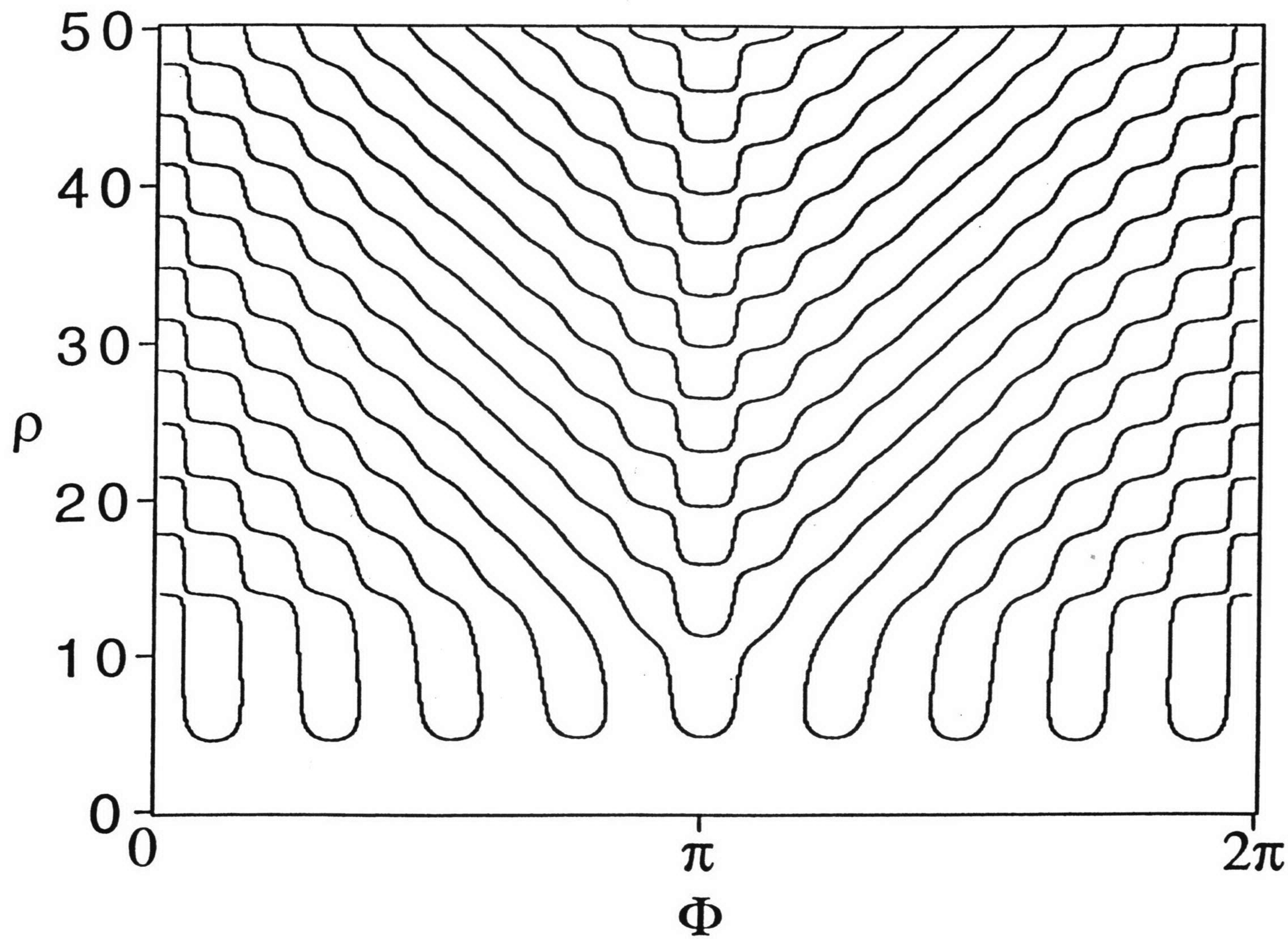

FIGURE 3